Rational points near manifolds and metric Diophantine approximation

By Victor Beresnevich

Dedicated to Maurice Dodson

Abstract

This work is motivated by problems on simultaneous Diophantine approximation on manifolds, namely, establishing Khintchine and Jarník type theorems for submanifolds of $\mathbb{R}^n$. These problems have attracted a lot of interest since Kleinbock and Margulis proved a related conjecture of Alan Baker and V. G. Sprindžuk. They have been settled for planar curves but remain open in higher dimensions. In this paper, Khintchine and Jarník type divergence theorems are established for arbitrary analytic nondegenerate manifolds regardless of their dimension. The key to establishing these results is the study of the distribution of rational points near manifolds — a very attractive topic in its own right. Here, for the first time, we obtain sharp lower bounds for the number of rational points near nondegenerate manifolds in dimensions $n > 2$ and show that they are ubiquitous (that is uniformly distributed).

Contents

1. **Introduction**

Let $\mathcal{M}$ be a bounded smooth manifold in $\mathbb{R}^n$. Given $Q > 1$ and $\varepsilon > 0$, let

$$N(Q, \varepsilon) \;=\; \#\Big\{ \mathbf{p}/q \in \mathbb{Q}^n \,:\, 1 \le q \le Q, \; \mathrm{dist}(\mathbf{p}/q, \mathcal{M}) \le \varepsilon \Big\},$$

where $\#S$ is the cardinality of a set S, $\mathbf{p} \in \mathbb{Z}^n$, $q \in \mathbb{Z}$, $\mathrm{dist}(\mathbf{r}, \mathcal{M}) = \inf_{\mathbf{y} \in \mathcal{M}} |\mathbf{r} - \mathbf{y}|$ and $|\cdot|$ is the Euclidean norm on $\mathbb{R}^n$. Thus, $N(Q, \varepsilon)$ counts rational points with bounded denominator lying 'ε-near' $\mathcal{M}$. The following intricate problem will be our main concern.

Problem 1.1. Estimate $N(Q, \varepsilon)$ for a 'generic' smooth manifold $\mathcal{M}$.

Our study of Problem 1.1 is motivated by open problems on simultaneous Diophantine approximation on manifolds; see Section 2. However, the interest to the distribution of rational points near manifolds is not limited to these problems; see, e.g., [28], [46]. In this paper a sharp lower bound on $N(Q, \varepsilon)$ is established when ε is bounded below by some naturally occurring function of Q. To begin with, we briefly review the state of the art.

Planar curves. The first general estimates for $N(Q, \varepsilon)$ are due to Huxley [33], [32]. In particular, he proved that for any curve $\mathcal{M}$ in $\mathbb{R}^2$ with curvature bounded between positive constants, $N(Q, \varepsilon) \ll \varepsilon Q^{3+\theta}$ for $\varepsilon \gg Q^{-2}$, where $\theta > 0$ is arbitrary and "$\ll$" is the Vinogradov symbol. Huxley's estimate was the only general result until Vaughan and Velani remarkably removed the θ-term from Huxley's estimate [54]. On the other hand, Dickinson, Velani and the author [8] obtained the complementary bound $N(Q, \varepsilon) \gg \varepsilon Q^3$ for $\varepsilon \gg Q^{-2}$. Consequently, the theory for planar curves is reasonably complete.

Higher dimensions. Very little is known. Effectively, there are only rather crude bounds on $N(Q, \varepsilon)$ obtained via Khintchine's transference principle [17] and estimates for topological products of planar curves [18, §§4.4.2, 5.4.4]. In this paper we investigate the distribution of rational points near arbitrary analytic nondegenerate submanifold of $\mathbb{R}^n$ for all $n > 1$. Analytic nondegenerate manifolds are natural to consider as they run through Diophantine approximation and beyond. Recall that a connected analytic submanifold $\mathcal{M}$ of $\mathbb{R}^n$ is *nondegenerate* if $\mathcal{M}$ is not contained in a proper affine subspace of $\mathbb{R}^n$. If $\mathcal{M}$ is immersed by an analytic map $\bar{\xi} = (\xi_1, \ldots, \xi_n) : U \to \mathbb{R}^n$ defined on a ball $U \subset \mathbb{R}^d$, then $\mathcal{M}$ is nondegenerate if and only if the functions $1, \xi_1, \ldots, \xi_n$ are linearly independent over $\mathbb{R}$.

Throughout $m = \mathrm{codim}\,\mathcal{M} \ge 1$. Then we have the following obvious 'volume based'

(1.1)   HEURISTIC ESTIMATE: $N(Q, \varepsilon) \asymp \varepsilon^m Q^{n+1}$,

where $\asymp$ means both $\ll$ and $\gg$. In order to gain some insight into when the heuristic estimate (1.1) could potentially be true we now consider the following two counterexamples.

Example 1.2. Let $\mathcal{M} = \{(x_1, \ldots, x_n) \in \mathbb{R}^n : x_1^2 + x_2^2 = 3\}$. Obviously, $\mathcal{M}$ is nondegenerate. It is readily verified that $\mathcal{M} \cap \mathbb{Q}^n = \varnothing$. Further, if $\varepsilon = o(Q^{-2})$ and Q is large enough, then the rational points contributing to $N(Q, \varepsilon)$ must lie on $\mathcal{M}$, resulting in $N(Q, \varepsilon) = 0$ for $\varepsilon = o(Q^{-2})$. This example can be extended to submanifolds of any codimension by using Pyartli's slicing technique [48]. The next example is of a different nature.

Example 1.3. Let

$$\mathcal{M} = \{(x_1, \ldots, x_{d-1}, x_d, x_d^2, \ldots, x_d^{m+1}) \in \mathbb{R}^n : \max_{1 \leq i \leq d} |x_i| < 1\},$$

where $d \geq 2$. Clearly $\mathcal{M}$ is nondegenerate and bounded. Given a positive integer $q \leq Q$, the rational points $\mathbf{p}/q$ with $\mathbf{p} = (p_1, \ldots, p_{d-1}, 0, \ldots, 0) \in \mathbb{Z}^n$ obviously lie on $\mathcal{M}$. The number of such points is $\asymp Q^d$, thus implying $N(Q, \varepsilon) \gg Q^d$ regardless of the size of ε. The latter is significantly larger than the heuristic estimate (1.1) unless $\varepsilon \gg Q^{-(m+1)/m}$.

In this paper we shall show that the condition $\varepsilon \gg Q^{-(m+1)/m}$ is sufficient to prove the heuristic lower bound for $N(Q, \varepsilon)$. Also we shall see in Section 7 that this condition can be significantly relaxed when $\mathcal{M}$ is a curve. The results will be presented in a form convenient for the applications in metric Diophantine approximation that we have in mind; see Section 2. Furthermore, the form of their presentation reveals the distribution of rational points in question, which is far more delicate than simply counting.

We will naturally and nonrestrictively work with manifolds $\mathcal{M}$ locally. Then, in view of the Implicit Function Theorem, this allows us to represent $\mathcal{M}$ by Monge parameterisations. Therefore without loss of generality, we can assume that

$$(1.2) \quad \mathcal{M} := \left\{(x_1, \ldots, x_d, f_1(\mathbf{x}), \ldots, f_m(\mathbf{x})) \in \mathbb{R}^n : \mathbf{x} = (x_1, \ldots, x_d) \in U\right\},$$

where U is an open subset of $\mathbb{R}^d$ and $\mathbf{f} = (f_1, \ldots, f_m) : U \to \mathbb{R}^m$ is a map. Here and elsewhere $d = \dim \mathcal{M}$ and $m = \operatorname{codim} \mathcal{M}$. The distribution of rational points near the manifold (1.2) is then conveniently described in terms of the set

$$\mathcal{R}^\delta(Q, \psi, B) := \left\{(q, \mathbf{a}, \mathbf{b}) \in \mathbb{N} \times \mathbb{Z}^d \times \mathbb{Z}^m : \begin{array}{c} \mathbf{a}/q \in B, \ \delta Q < q \leq Q \\ |q\mathbf{f}(\mathbf{a}/q) - \mathbf{b}|_\infty \leq \psi \\ \gcd(q, \mathbf{a}, \mathbf{b}) = 1 \end{array} \right\},$$

where $Q > 1$, $\psi \geq 0$, $\delta \geq 0$, $B \subset U$ and $|\cdot|_\infty$ denotes the supremum norm. Also define

$$\Delta^\delta(Q, \psi, B, \rho) := \bigcup_{(q, \mathbf{a}, \mathbf{b}) \in \mathcal{R}^\delta(Q, \psi, B)} B\big(\mathbf{a}/q, \rho\big),$$

where $B(\mathbf{x}, \rho)$ denotes a ball centred at $\mathbf{x}$ of radius ρ. Roughly speaking, the set $\Delta^\delta(Q, \psi, B, \rho)$ indicates which part of the manifold can be covered by balls of radius $\asymp \rho$ centered at the rational points of interest. The following key result of this paper shows that this part is substantial for a suitable choice of parameters. In what follows μ_d denotes d-dimensional Lebesgue measure.

THEOREM 1.4. *Let the manifold* (1.2) *be analytic and nondegenerate and let* $B_0 \subset U$ *be a compact ball. Then there are absolute positive constants* k_0, ρ_0 *and* δ_0 *depending on* B_0 *only with the following property. For any ball* $B \subset B_0$ *there are positive constants* $C_0 = C_0(B)$ *and* $Q_0 = Q_0(B)$ *such that for all* $Q \geq Q_0$ *and all* ψ *satisfying*

$$(1.3) \qquad\qquad C_0 Q^{-1/m} < \psi < C_0^{-1},$$

we have

$$(1.4) \qquad\qquad \mu_d\left(\Delta^{\delta_0}(Q, \psi, B, \rho) \cap B\right) \geq k_0\, \mu_d(B),$$

where $\rho := \rho_0 \times (\psi^m Q^{d+1})^{-1/d}$.

COROLLARY 1.5. *Let* $\mathcal{M}$ *and* B_0 *be as in Theorem* 1.4. *Then, there are constants* δ_0 *and* $k_1 > 0$ *such that for any ball* $B \subset B_0$, *there exist* $Q_0 > 0$ *and* $C_0 > 0$ *such that for all* $Q \geq Q_0$ *and all* ψ *satisfying* (1.3), *we have that*

$$(1.5) \qquad N^{\delta_0}(Q, \psi, B) := \#\mathcal{R}^{\delta_0}(Q, \psi, B) \geq k_1 \psi^m Q^{d+1} \mu_d(B).$$

Proof of Corollary 1.5. For any $\mathbf{r} \in \mathbb{R}^d$ we obviously have that $\mu_d(B(\mathbf{r}, \rho) \cap B) \leq V_d \rho^d$, where V_d is the volume of a d-dimensional ball of radius 1. Therefore, the r.h.s.[1] of (1.4) is bounded above by $N^{\delta_0}(Q, \psi, B) V_d \rho^d$. By (1.4), we get that $N^{\delta_0}(Q, \psi, B) \geq V_d^{-1} \rho^{-d} k_0 \mu_d(B)$. Substituting the value of ρ from Theorem 1.4 into the last inequity completes the proof. $\qquad\square$

Remark 1.6. It is clear that every rational point $(\mathbf{a}/q, \mathbf{b}/q)$ arising from $\mathcal{R}^{\delta_0}(Q, \psi, B)$ lies within the distance $\varepsilon = \psi(\delta_0 Q)^{-1}$ from $\mathcal{M}$. Thus, $N(Q, \varepsilon) \geq N^{\delta_0}(Q, \varepsilon\delta_0 Q, B_0)$. By Corollary 1.5, we get the lower bound $N(Q, \delta) \gg \varepsilon^m Q^{n+1}$ valid for $\varepsilon \gg Q^{-(m+1)/m}$ consistent with (1.1).

Remark 1.7. In the case of hypersurfaces $m = 1$. Therefore, the condition $\varepsilon \gg Q^{-(m+1)/m}$ transforms into $\varepsilon \gg Q^{-2}$. This is the same as for planar curves [8]. It tells us that rational points with denominator $q \leq Q$ can get

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[1]Throughout the paper 'r.h.s.' means '*right-hand side*' and 'l.h.s.' means '*left-hand side.*'

CONST$\times Q^{-2}$ close to an arbitrary analytic nondegenerate hypersurface. In fact, in view of Example 1.2 this is generically best possible!

Remark 1.8. In the case of planar curves the lower bound (1.5) has already been established in [8, Th. 6]. However, in that paper the constant k_1 happens to dependent on B, while in this paper k_1 is uniform.

2. **Diophantine approximation on manifolds**

In this section we apply Theorem 1.4 to simultaneous Diophantine approximation on manifolds. Traditionally, problems on the proximity of rational points to points in $\mathbb{R}^n$ assume finding optimal relations between the accuracy of approximation and the 'height' of approximating rational points $\mathbf{p}/q$. In our case, the latter is measured by q while the former is measured by ψ/q. Therefore, throughout this section $\psi : \mathbb{N} \to \mathbb{R}^+$ will be regarded as a decreasing function referred to as an *approximation function*, where $\mathbb{R}^+ = (0, +\infty)$. Given $\tau > 0$, the approximation function $q \mapsto q^{-\tau}$ will be denoted by $\psi_\tau(q)$.

The point $\mathbf{y} \in \mathbb{R}^n$ is called ψ-*approximable* if there are infinitely many $q \in \mathbb{N}$ satisfying

$$\text{(2.1)} \qquad \qquad \|q\mathbf{y}\| < \psi(q),$$

where $\|q\mathbf{y}\|$ denotes the distance of $q\mathbf{y}$ from $\mathbb{Z}^n$ with respect to the sup-norm $|\cdot|_\infty$. Throughout, $\mathcal{S}_n(\psi)$ denotes the set of ψ-approximable points in $\mathbb{R}^n$.

By Dirichlet's theorem (see, e.g., [50]), $\mathcal{S}_n\big(\psi_{1/n}\big) = \mathbb{R}^n$. The point $\mathbf{y} \in \mathbb{R}^n$ such that $\mathbf{y} \in \mathcal{S}_n(\psi_\tau)$ for some $\tau > 1/n$ is called *very well approximable*. If $\mathbf{y} \notin \mathcal{S}_n(\psi_\tau)$ for any $\tau > 1/n$, then $\mathbf{y}$ is called *not very well approximable* or *extremal*.[2] A relatively easy consequence of the Borel-Cantelli lemma is that almost all points in $\mathbb{R}^n$ are extremal; see, e.g., [18]. The property of extremality is fundamental in Diophantine approximation. For example, Roth's celebrated theorem establishes nothing but the extremality of irrational algebraic numbers. Within this paper we will be dealing with problems that go back to the profound conjecture of Mahler [44] that almost all points on the Veronese curves $(x, \ldots, x^n)$ are extremal. The problem was studied in depth for over 30 years and eventually settled in full by Sprindžuk in 1964 (see [51]) who also stated the following general conjecture [53].

CONJECTURE (Sprindžuk). *Any analytic nondegenerate submanifold of $\mathbb{R}^n$ is extremal.*

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[2]The terminology of (*not*) *very well-approximable points* was first used by Schmidt [50]. The terminology of *extremal points* was first used by Koksma [43] and later reinvigorated by Sprindžuk [52] with the notion of extremal manifolds. In some literature extremal points are referred to be *Diophantine* or to be of a certain *Diophantine type*.

Formally a differentiable manifold $\mathcal{M} \subset \mathbb{R}^n$ is called *extremal* if almost all points of $\mathcal{M}$ (with respect to the induced Lebesgue measure on $\mathcal{M}$) are extremal. For $n = 2$, the conjecture is a consequence of Schmidt's theorem [49] and for $n = 3$, it has been proved by Bernik and the author [4]. The full conjecture (with the analyticity assumption dropped) has been established by Kleinbock and Margulis in the tour de force [41] and later re-established in [3] using different techniques. The work of Kleinbock and Margulis has also dealt with the far more delicate multiplicative case known as the Baker-Sprindžuk conjecture and led to a surge of activity that led to establishing the extremality of various classes of manifolds and sets; see, for example, [38], [39], [40], [42].

The following two major problems now arise (see, e.g., [8, §1] or [6, §6]):

Problem 2.1. To develop a Khintchine type theory for $\mathcal{S}_n(\psi) \cap \mathcal{M}$.

Problem 2.2. To develop a Hausdorff measure theory for $\mathcal{S}_n(\psi) \cap \mathcal{M}$.

The goal of Problem 2.1 is a metric theory of $\mathcal{S}_n(\psi) \cap \mathcal{M}$ with ψ being a general approximation function, not just $\psi_\tau(q) = q^{-\tau}$ associated with extremality. The goal of Problem 2.2 is to determine the 'size' of $\mathcal{S}_n(\psi) \cap \mathcal{M}$ via Hausdorff measure and dimension.

Before we proceed with the more detailed discussion of the above problems, it is worth mentioning that there are dual versions of Problems 2.1 and 2.2. In the dual case the approximating objects are rational hyperplanes rather than rational points. The problems in the dual case are much more tractable and progress has been significantly better. In particular, the dual version of Problem 2.1 has been fully settled [3], [6], [19] and very deep answers regarding the dual version of Problem 2.2 found [5], [7], [16], [22], [24]. However, as we shall see, Problems 2.1 and 2.2 (nondual) have more or less been understood only in $\mathbb{R}^2$.

2.1. *Khintchine type theory.* Let $\mathcal{M} \subset \mathbb{R}^n$ be a manifold. If for any approximation function $\psi : \mathbb{N} \to \mathbb{R}^+$ such that

$$(2.2) \qquad \sum_{q \in \mathbb{Z}} \psi(q)^n$$

converges *almost no* point on $\mathcal{M}$ is ψ-approximable, then $\mathcal{M}$ is called of *Khintchine type for convergence.* In turn, $\mathcal{M}$ is called of *Khintchine type for divergence* if for any approximation function ψ such that the sum (2.2) diverges, *almost all* points on $\mathcal{M}$ are ψ-approximable. This terminology represents a zero-one law and has been introduced in [18] to acknowledge the fundamental contribution of Khintchine who discovered this beautiful law in the case $\mathcal{M} = \mathbb{R}^n$ [35], [37]. We now discuss the state of the art for proper submanifolds of $\mathbb{R}^n$.

Planar curves ($n = 2$). The story has begun with the pioneering work [15] of Bernik who showed that the parabola (x, x^2) is of Khintchine type for convergence. Subsequently, working towards a conjecture of Alan Baker, Mashanov has established a multiplicative analogue of Bernik's result [45]. There has been no progress with planar curves since then, until Dickinson, Velani and the author have shown that any $C^{(3)}$ nondegenerate planar curve is of Khintchine type for divergence [8] and subsequently Vaughan and Velani have established that any $C^{(2)}$ nondegenerate planar curve is of Khintchine type for convergence [54]. See also [1], [9], [10] for further progress.

Higher dimensions ($n > 2$). In this case the Khintchine type theory also exists but is rather bizarre. Bernik [13], [14] has shown that the manifolds in $\mathbb{R}^{mk}$ given as the cartesian product of m nondegenerate curves in $\mathbb{R}^k$ are of Khintchine type for convergence if $m \geq k$ and for divergence if $k = 2$ and $m \geq 4$. Dodson, Rynne and Vickers [25], [26] have found Khintchine type manifolds satisfying certain curvature conditions. However, these conditions significantly constrain the dimension of the manifolds and completely rule out curves. For example, the Khintchine type manifolds of [25], [26] assume that $d = \dim \mathcal{M} \geq \max\{2, \sqrt{2n} - \frac{3}{2}\}$ for convergence and $d \geq \frac{3}{4}(n + 5)$ and $n \geq 19$ for divergence. Thus, the simplest example of a Khintchine type manifold for divergence could only be an 18-dimensional manifolds in $\mathbb{R}^{19}$. It should be noted that Dodson, Rynne and Vickers established their divergence Khintchine type theorem in the quantitative form. Assuming a condition on ψ which implies that $\mathcal{S}_n(\psi) = \mathbb{R}^n$, Harman [31] has obtained a quantitative result for Veronese curves and manifolds that are known to be of Khintchine type for convergence. Recently Gorodnik and Shah [30] have obtained a Khintchine type theorem for approximation by integer points $(p_1, \ldots, p_n, q) \in \mathbb{Z}^{n+1}$ lying on a quadratic variety of the form $x_1^2 \pm \cdots \pm x_d^2 - y^2 = 1$ to their limit points (with respect to the projective distance) lying on the unit sphere $\mathrm{S}^n \subset \mathbb{R}^n$. For general manifolds admitting group actions, Ghosh, Gorodnik and Nevo [29] also consider related Diophantine problems involving approximation by rational points. The Khintchine type theory for curves in dimensions $n > 2$ is simply nonexistent. However, in view of Pyartli's slicing technique [48], curves underpin the whole theory. The following result of this paper covers arbitrary nondegenerate analytic curves as well as arbitrary nondegenerate analytic submanifolds of $\mathbb{R}^n$.

THEOREM 2.3. *For any $n \geq 2$, any nondegenerate analytic submanifold of $\mathbb{R}^n$ is of Khintchine type for divergence.*

Classical case. In order to illustrate the statement of Theorem 2.3, let us consider the following classical problem on rational approximations to consecutive powers of a real number. That is, we consider the inequality

$$(2.3) \qquad \max \left\{ \|qx\|, \|qx^2\|, \dots, \|qx^n\| \right\} < \psi(q).$$

Since the consecutive powers of x are real analytic functions of x which, together with 1, are linearly independent over $\mathbb{R}$, Theorem 2.3 implies the following

COROLLARY 2.4. *Given any monotonic $\psi : \mathbb{N} \to \mathbb{R}^+$ such that the sum (2.2) diverges, for almost all $x \in \mathbb{R}$ inequality (2.3) has infinitely many solutions $q \in \mathbb{N}$.*

In 1925 Khintchine [36] established such a statement in the special case when $\psi(q) = cq^{-1/n}$ with arbitrary but fixed $c > 0$. The latter has been generalised by R. C. Baker [2] to smooth manifolds but the same class of approximation functions. Corollary 2.4 is thus the first improvement on that result of Khintchine in the period of over 80 years. It obviously contains Khintchine's result and is believed to be best possible. In fact, a folk conjecture suggests that for almost all $x \in \mathbb{R}$ there are only finitely many $q \in \mathbb{N}$ satisfying (2.3) provided that the sum (2.2) converges.

2.2. *Hausdorff dimension and measure theory.* Problem 2.2 throws up a few surprises. For example, unlike the dual case the dimension of $\mathcal{S}_n(\psi) \cap \mathcal{M}$ happens to depend on the arithmetic properties of $\mathcal{M}$. To grasp the ideas consider the following popular example. Let $\mathcal{C}_r$ be the circle $x^2 + y^2 = r$. It is easily verified that if $r \in \mathbb{N}$, $\tau > 1$ and $\psi(q) = \psi_\tau(q) = q^{-\tau}$, then all the rational points implicit in (2.1) must lie on $\mathcal{C}_r$ for sufficiently large q. For the unit circle $\mathcal{C}_1$, these points are parameterised by Pythagorean triples and well understood. As a result

$$(2.4) \qquad \dim \mathcal{S}_2(\psi_\tau) \cap \mathcal{C}_1 = \frac{1}{\tau + 1} \qquad \text{for } \tau > 1,$$

where dim stands for Hausdorff dimension. The fact (2.4) has been established in two complementary papers by Melnichuk [47] and Dickinson-Dodson [23]. On the other hand, it is easily seen that $\mathcal{C}_3 \cap \mathbb{Q}^2 = \varnothing$. Consequently,

$$(2.5) \qquad \dim \mathcal{S}_2(\psi_\tau) \cap \mathcal{C}_3 = 0 \qquad \text{for } \tau > 1.$$

Thus, scaling $\mathcal{C}_1$ by $\sqrt{3}$ completely changes the character of the set of ψ_τ-approximable points lying on it. Luckily, this cannot happen if $\tau < 1$. In fact, as shown in [8],

$$(2.6) \qquad \dim \mathcal{S}_2(\psi_\tau) \cap \mathcal{C} = \frac{2 - \tau}{\tau + 1} \qquad \text{when } 1/2 \leq \tau < 1$$

for all $C^{(3)}$ curves $\mathcal{C}$ in $\mathbb{R}^2$ nondegenerate everywhere except possibly on a set of Hausdorff dimension $\leq \frac{2-\tau}{\tau+1}$. The Hausdorff dimension of $\mathcal{S}_2(\psi) \cap \mathcal{C}$ has also been found in [8] for general approximation functions ψ. Furthermore, an analogue of Jarník's theorem [34] has been established in [8] and [54] which provides a complete picture of the s-dimensional Hausdorff measure of $\mathcal{S}_2(\psi)$ $\cap \mathcal{C}$; see [8], [54] for details.

Higher dimensions. Khintchine's transference principle [50] can be used to deduce bounds on $\dim \mathcal{S}_n(\psi_\tau) \cap \mathcal{M}$ from the much better understood dual case. Although the bounds obtained this way are rather crude, until recently nothing else was known. In [27] Drutu established a comprehensive theory for nondegenerate rational quadrics in $\mathbb{R}^n$ when the approximating rational points lie on quadrics. In particular, her results include (2.4) and (2.5) as two special cases. More recently Budarina and Dickinson [21] have investigated $\mathcal{S}_n(\psi_\tau) \cap \mathcal{M}$ for hypersurfaces $\mathcal{M}$ in $\mathbb{R}^n$ parameterised by the forms $x_1^d + \cdots + x_{n-1}^d$ of degree $d < \log n$, the exponent τ being large and the approximating rational points being lying on $\mathcal{M}$. However, except for planar curves, the approximating rational points always lie on the manifold. In view of this, Theorem 2.5 below appears to be the first general result concerning Problem 2.2 in dimensions $n > 2$.

Let $\mathcal{H}^s$ denote s-dimensional Hausdorff measure. In order to state the result we now introduce the *exponent of* ψ also known as the lower order of $1/\psi$ at infinity:

$$\tau(\psi) := \liminf_{q \to \infty} \frac{-\log \psi(q)}{\log q}.$$

THEOREM 2.5. *Let $\mathcal{M}$ be a nondegenerate analytic submanifold of $\mathbb{R}^n$, $d = \dim \mathcal{M}$ and $m = \operatorname{codim} \mathcal{M}$. Thus, $d + m = n$. Let $\psi : \mathbb{N} \to \mathbb{R}^+$ be a monotonic function such that $q\psi(q)^m \to \infty$ as $q \to \infty$. Then for any $s \in \left(\frac{m}{m+1}d, d\right)$,*

$$(2.7) \qquad \mathcal{H}^s(\mathcal{S}_n(\psi) \cap \mathcal{M}) = \infty \qquad if \qquad \sum_{q=1}^{\infty} q^n \left(\frac{\psi(q)}{q}\right)^{s+m} = \infty.$$

Consequently, if $\tau = \tau(\psi)$ satisfies $1/n < \tau < 1/m$, then

$$(2.8) \qquad\qquad \dim \mathcal{S}_n(\psi) \cap \mathcal{M} \geq s_0 := \frac{n+1}{\tau+1} - m.$$

We shall see in Section 7 that for nondegenerate analytic curves $(d = 1)$ Theorem 2.5 holds for $s \in (d/2; d)$. It is also possible to obtain the version of Theorem 2.5 that would incorporate generalised Hausdorff measures. We opt to omit further details which can be easily recovered using the ideas of [8, §8.1] where the case $n = 2$ is considered.

2.3. *Proof of Theorems* 2.3 *and* 2.5. The proof below generalises the arguments given in Sections 3, 6, and 7 of [8] to higher dimensions.

Note 1. Within Theorem 2.5 it suffices to establish (2.7), for (2.8) follows from (2.7).

Proof. By the definition of $\tau(\psi)$, for any $\varepsilon > 0$ there are infinitely many q such that $\psi(q) \geq q^{-\tau-\varepsilon}$. Since ψ is monotonic, $\psi(2^t) \geq 2^{-(t+1)(\tau+\varepsilon)}$ for $t \in \mathbb{Z}$ satisfying $2^t \leq q \leq 2^{t+1}$. Therefore, there are infinitely many $t \in \mathbb{N}$ such that $\psi(2^t) \geq 2^{-(t+1)(\tau+\varepsilon)}$. Hence, on taking $s = \frac{n+1}{\tau+1+\varepsilon} - m$ with $\varepsilon > 0$, one verifies that $2^{t(n+1)}(\psi(2^t)2^{-t})^{s+m} \geq 2^{-(n+1)}$. The latter holds for infinitely many t and implies that $\sum_{t=1}^{\infty} 2^{t(n+1)}(\psi(2^t)2^{-t})^{s+m} = \infty$. Due to the monotonicity of ψ this further implies that the sum in (2.7) diverges and therefore, by (2.7), $\mathcal{H}^s(\mathcal{S}_n(\psi) \cap \mathcal{M}) = \infty$. By the definition of Hausdorff dimension, we deduce that $\dim \mathcal{S}_n(\psi) \cap \mathcal{M} \geq s = \frac{n+1}{\tau+1+\varepsilon} - m$, whence (2.8) readily follows. $\square$

Note 2. The condition

(2.9) $$\lim_{q\to\infty} q\psi(q)^m = \infty,$$

which is a part of Theorem 2.5, can be assumed in the proof of Theorem 2.3.

Proof. To verify (2.9) consider the monotonic function

$$\psi_1(q) = \max\{q^{-2/(2n-1)}, \psi(q)\}.$$

Then the divergence of (2.2) implies $\sum_{q=1}^{\infty} \psi_1(q)^n = \infty$. Obviously $\mathcal{S}_n(\psi_1) = \mathcal{S}_n(\psi) \cup \mathcal{S}_n(2/(2n-1))$. Since $2/(2n-1) > 1/n$ and every nondegenerate submanifold of $\mathbb{R}^n$ is extremal we obviously have that the set $\mathcal{M} \cap \mathcal{S}_n(2/(2n-1))$ has zero measure on $\mathcal{M}$. Hence $\mathcal{M} \cap \mathcal{S}_n(\psi_1)$ and $\mathcal{M} \cap \mathcal{S}_n(\psi)$ are of the same measure and ψ can be replaced with ψ_1, which satisfies (2.9). $\square$

Note 3. In view of the metric nature of Theorems 2.3 and 2.5, it is enough to consider a sufficiently small neighborhood of an arbitrary point on $\mathcal{M}$. Therefore, by the Implicit Function Theorem, without loss of generality we can assume that $\mathcal{M}$ is of the Monge form (1.2) and that the functions $f_1, \ldots, f_m$ are Lipschitz; that is, for some $c_1 \geq 1$,

(2.10) $$\max_{1\leq l\leq m} |f_l(\mathbf{x}) - f_l(\mathbf{x}')| \leq c_1|\mathbf{x} - \mathbf{x}'|_\infty \quad \text{for all } \mathbf{x}, \mathbf{x}' \in U.$$

Note 4. Let $\mathcal{S}_\mathbf{f}(\psi)$ be the set of $\mathbf{x} \in U$ such that $(\mathbf{x}, \mathbf{f}(\mathbf{x})) \in \mathcal{S}_n(\psi)$. Obviously, $\mathcal{S}_\mathbf{f}(\psi)$ is the orthogonal projection of $\mathcal{S}_n(\psi) \cap \mathcal{M}$ onto $\mathbb{R}^d$. By (2.10), $\mathcal{S}_\mathbf{f}(\psi)$ and $\mathcal{S}_n(\psi) \cap \mathcal{M}$ are related by a bi-Lipschitz map and therefore $\mathcal{S}_\mathbf{f}(\psi)$ is of full Lebesgue measure in U if and only if $\mathcal{S}_n(\psi) \cap \mathcal{M}$ is of full induced Lebesgue measure on $\mathcal{M}$; see [18, §1.5.1]. Further, recall that d-dimensional Lebesgue measure is comparable to $\mathcal{H}^d$. Therefore, to prove Theorem 2.3 it

suffices to show that for every compact ball B_0 in U,

$$(2.11) \qquad \mathcal{H}^d(\mathcal{S}_{\mathbf{f}}(\psi) \cap B_0) = \mathcal{H}^d(B_0) \quad \text{if} \quad \sum_{q=1}^{\infty} \psi(q)^n = \infty.$$

Similarly one can show that Theorem 2.5 follows on showing that

$$(2.12) \qquad \mathcal{H}^s(\mathcal{S}_{\mathbf{f}}(\psi) \cap B_0) = \mathcal{H}^s(B_0) \quad \text{if} \quad \sum_{q=1}^{\infty} q^n \left(\frac{\psi(q)}{q}\right)^{s+m} = \infty$$

holds for every compact ball B_0 in U and $s \in (md/(m+1), d)$. Note that for $s < d$, $\mathcal{H}^s(B_0) = \infty$. Also note that in the case $s = d$, (2.12) is simply (2.11).

Upshot. On establishing (2.12) for $s \in \left(\frac{m}{m+1}d, d\right]$ and ψ satisfying (2.9) we prove Theorems 2.3 and 2.5.

Ubiquitous systems. In what follows we will use the ubiquitous systems technique. The notion of ubiquity introduced below is equivalent to that of [7] in the setting that is now to be described. Let B_0 be a ball in $\mathbb{R}^d$ and $\mathcal{R} := (R_\alpha)_{\alpha \in J}$ be a family of points R_α in B_0 (usually called *resonant points*) indexed by a countable set J. Let $\beta : J \to \mathbb{R}^+ : \alpha \mapsto \beta_\alpha$ be a function on J, which attaches a 'weight' β_α to points R_α. For $t \in \mathbb{N}$, define $J(t) := \{\alpha \in J : \beta_\alpha \leq 2^t\}$ and assume that $J(t)$ is always finite.

Definition 2.6. Let $\rho : \mathbb{R}^+ \to \mathbb{R}^+$ be a function such that $\lim_{t \to \infty} \rho(t) = 0$. The system $(\mathcal{R}; \beta)$ is called *locally ubiquitous in B_0 relative to ρ* if there is an absolute constant $k_0 > 0$ such that for any ball $B \subset B_0$,

$$(2.13) \qquad \liminf_{t \to \infty} \mu_d\Big(\bigcup_{\alpha \in J(t)} B\big(R_\alpha, \rho(2^t)\big) \cap B\Big) \geq k_0 \, \mu_d(B).$$

Here as before μ_d denotes Lebesgue measure in $\mathbb{R}^d$ and $B(\mathbf{x}, r)$ denotes the ball in $\mathbb{R}^d$ centred at $\mathbf{x}$ of radius r. The function ρ is referred to as *ubiquity function*.

Given a function $\Psi : \mathbb{R}^+ \to \mathbb{R}^+$, let

$$\Lambda_{\mathcal{R}}(\Psi) := \{\mathbf{x} \in B_0 : |\mathbf{x} - R_\alpha|_\infty < \Psi(\beta_\alpha) \text{ holds for infinitely many } \alpha \in J\}.$$

The following lemma follows from Corollaries 2, 4 and 5 from [7]. In the case $d = 1$ a simplified proof of Lemma 2.7 is given in [8, Ths. 9 and 10]; see also [11].

LEMMA 2.7. *Let $\Psi : \mathbb{R}^+ \to \mathbb{R}^+$ be a monotonic function such that for some $\lambda < 1$, $\Psi(2^{t+1}) \leq \lambda\Psi(2^t)$ holds for t sufficiently large. Let $(\mathcal{R}, \beta)$ be a locally ubiquitous system in B_0 relative to ρ. Then for any $s \in (0, d]$,*

$$(2.14) \qquad \mathcal{H}^s\big(\Lambda_{\mathcal{R}}(\Psi)\big) = \mathcal{H}^s(B_0) \quad \text{if} \quad \sum_{t=1}^{\infty} \frac{\Psi(2^t)^s}{\rho(2^t)^d} = \infty.$$

Proof of Theorems 2.3 *and* 2.5 *modulo Theorem* 1.4. Recall that our goal is to establish (2.12) for $s \in (md/(m + 1), d]$ and approximation functions ψ satisfying (2.9), where B_0 is an arbitrary nonempty compact ball in U. Therefore, for the rest of this section we fix such a B_0. Also recall that the map $\mathbf{f}$ which arises from (1.2) satisfies the Lipschitz condition (2.10). We can also assume that $\lim_{q \to \infty} \psi(q) = 0$ as otherwise $\mathcal{S}_n(\psi) = \mathbb{R}^n$ and there is nothing to prove.

We first construct a ubiquitous system relevant to our main goal. Let ρ_0 and δ_0 be the same as in Theorem 1.4. Define the ubiquity function $\rho(q) = \rho_0 \times (\psi(q)^m q^{d+1})^{-1/d}$ and the sequence $\mathcal{R} := \{\mathbf{a}/q\}_{(q,\mathbf{a}) \in J}$ of resonant points in B_0, where

$$J := \Big\{ (q, \mathbf{a}) \in \mathbb{N} \times \mathbb{Z}^d \, : \, \mathbf{a}/q \in B_0, \ \max_{1 \leq l \leq m} \|q f_l(\mathbf{a}/q)\| \leq \tfrac{1}{2}\psi(q) \Big\}.$$

For $\alpha = (q, \mathbf{a}) \in J$ define $\beta_\alpha := q$. We prove the following

LEMMA 2.8. *Assume that Theorem* 1.4 *holds. Then, with* B_0, $\mathcal{R}$, β *and* ρ *as above, the system* $(\mathcal{R}, \beta)$ *is locally ubiquitous in* B_0 *relative to* ρ.

Proof. First of all, by (2.9), $\rho(q) \to 0$ as $q \to \infty$. We now verify (2.13) for the specific choice of $\mathcal{R}$, β and ρ we have made. Obviously $J(t)$ consists of $(q, \mathbf{a}) \in J$ such that $q \leq Q := 2^t$. Fix an arbitrary ball $B \subset B_0$ and consider the union in (2.13). This union contains
(2.15)
$$\bigcup_{\delta_0 Q \leq q \leq Q} \ \bigcup_{\mathbf{a} \in \mathbb{Z}^d \, : \, (q,\mathbf{a}) \in J} B\big(\mathbf{a}/q, \rho(Q)\big) \cap B \ \supset \Delta^{\delta_0}\big(Q, \tfrac{1}{2}\psi(Q), B, \rho(Q)\big) \cap B \, ,$$

where $\Delta^{\delta_0}(\,\cdot\,)$ is the set defined in Section 1 and appearing in Theorem 1.4. By (2.9) and the assumption $\lim_{q \to \infty} \psi(q) = 0$, conditions (1.3) are met for sufficiently large Q and therefore, by Theorem 1.4, the μ_d-measure of the sets in (2.15) is at least $k_0 \mu_d(B)$. Therefore (2.13) is fulfilled and the proof is complete. $\qquad\square$

In the next two statements we establish a relation between $\Lambda_{\mathcal{R}}(\Psi)$ and $\mathcal{S}_{\mathbf{f}}(\psi)$ and an analogue of (2.12) in terms of $\Lambda_{\mathcal{R}}(\Psi)$.

LEMMA 2.9. *Let* $\Psi(q) = \psi(q)/(2c_1 q)$, *where* c_1 *arises from* (2.10) *and let* B_0, $\mathcal{R}$, β *and* ρ *be as in Lemma* 2.8. *Then* $\Lambda_{\mathcal{R}}(\Psi) \subset \mathcal{S}_{\mathbf{f}}(\psi)$.

Proof. Assume that $\mathbf{x} = (x_1, \ldots, x_d) \in \Lambda_{\mathcal{R}}(\Psi)$. Then

$$(2.16) \qquad\qquad |\mathbf{x} - \mathbf{a}/q|_\infty < \Psi(q) = \psi(q)/(2c_1 q)$$

for infinitely many $(q, \mathbf{a}) \in \mathbb{N} \times \mathbb{Z}^d$ such that

$$(2.17) \qquad\qquad \max_{1 \leq l \leq m} |q f_l(\mathbf{a}/q) - b_l| \leq \tfrac{1}{2}\psi(q)$$

for some $\mathbf{b} = (b_1, \ldots, b_m) \in \mathbb{Z}^m$. By the triangular inequality,

$$(2.18) \qquad |f_l(\mathbf{x}) - b_l/q| \quad \leq \quad |f_l(\mathbf{x}) - f_l(\mathbf{a}/q)| + |f_l(\mathbf{a}/q) - b_l/q|$$

$$\overset{(2.10)}{\leq} \quad c_1|\mathbf{x} - \mathbf{a}/q|_\infty + |f_l(\mathbf{a}/q) - b_l/q|$$

$$\overset{(2.16)\&(2.17)}{<} \quad c_1 \cdot \psi(q)/(2c_1 q) + \frac{1}{2}\psi(q)/q = \psi(q)/q.$$

Since (2.16) and (2.18) hold for infinitely many q, we have that $(\mathbf{x}, \mathbf{f}(\mathbf{x})) \in \mathcal{S}_n(\psi)$; that is $\mathbf{x}$ belongs to $\mathcal{S}_{\mathbf{f}}(\psi)$. Therefore, $\Lambda_{\mathcal{R}}(\Psi) \subset \mathcal{S}_{\mathbf{f}}(\psi)$. $\qquad \square$

LEMMA 2.10. *Assume that Theorem* 1.4 *holds. Let* $\Psi(q) = \psi(q)/(2c_1 q)$, *where* c_1 *arises from* (2.10) *and let* B_0, $\mathcal{R}$, β *and* ρ *be as in Lemma* 2.8. *Then*

$$(2.19) \qquad \mathcal{H}^s\big(\Lambda_{\mathcal{R}}(\Psi)\big) = \mathcal{H}^s(B_0) \quad if \quad \sum_{q=1}^\infty q^n \left(\frac{\psi(q)}{q}\right)^{s+m} = \infty.$$

Proof. Since ψ is decreasing, $\Psi(2^{t+1}) \leq \lambda \Psi(2^t)$ with $\lambda = 1/2$. Further, using the explicit form for Ψ and ρ verify that

$$\sum_{t=1}^\infty \frac{\Psi(2^t)^s}{\rho(2^t)^d} \asymp \sum_{t=1}^\infty \frac{\psi(2^t)^s 2^{-st}}{\psi(2^t)^{-m} 2^{-(d+1)t}} \asymp \sum_{t=1}^\infty \left(\frac{\psi(2^t)}{2^t}\right)^{s+m} 2^{(n+1)t}.$$

In view of the monotonicity of ψ, the latter sum diverges if and only if $\sum_{q=1}^\infty q^n \left(\frac{\psi(q)}{q}\right)^{s+m}$ diverges. Hence, by Lemmas 2.7 and 2.8, we get (2.19). $\qquad \square$

We are now able to complete the proof of Theorems 2.3 and 2.5. Recall that we have to establish (2.12). Let Ψ, B_0, $\mathcal{R}$, β and ρ be as in Lemma 2.10. By the monotonicity of $\mathcal{H}^s$, $\mathcal{H}^s(\mathcal{S}_{\mathbf{f}}(\psi) \cap B_0) \leq \mathcal{H}^s(B_0)$. Therefore, to establish (2.12) it suffices to show that $\mathcal{H}^s(\mathcal{S}_{\mathbf{f}}(\psi) \cap B_0) \geq \mathcal{H}^s(B_0)$ provided that the sum in (2.12) diverges. In view of Lemma 2.9, this follows from (2.19), and the proof of Theorems 2.3 and 2.5 modulo Theorem 1.4 is thus complete. $\qquad \square$

3. Some auxiliary geometry

The distance of a rational point from a manifold is conveniently studied using the notion of projective distance (due to H. and J. Weyl [55]) which involves exterior and interior products. These classical and well-established topics are now briefly recalled. The overview below is mostly taken from [50] and [56]. We will use the standard embedding of $\mathbb{R}^n$ into the real projective space $\mathbb{P}^n$. Given $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n$, the point $\boldsymbol{x} = (\lambda, \lambda x_1, \ldots, \lambda x_n) \in \mathbb{R}^{n+1}$ with $\lambda \neq 0$ will be referred to as the *homogeneous coordinates* of $\mathbf{x}$.

3.1. *Exterior product and projective distance.* Throughout, $\wedge^p(\mathbb{R}^{n+1})$ denotes the p-th exterior power of $\mathbb{R}^{n+1}$ and "$\wedge$" denotes the exterior product. If $p \le n+1$ and $\boldsymbol{e}_0, \dots, \boldsymbol{e}_n$ is a basis of $\mathbb{R}^{n+1}$, then the multivectors

$$(3.1) \qquad \boldsymbol{e}_I := \wedge_{i \in I} \, \boldsymbol{e}_i, \qquad I \in C(n+1, p)$$

form a basis of $\wedge^p(\mathbb{R}^{n+1})$, where $C(n+1, p)$ denotes the set of all subsets of $\{0, \dots, n\}$ of cardinality p. The following well-known formula (see [56, p. 38]) expresses the exterior product of vectors $\boldsymbol{x}_i = \sum_{j=0}^n x_{i,j} \, \boldsymbol{e}_j \in \mathbb{R}^{n+1}$ $(1 \le i \le p)$ in terms of the basis (3.1):

$$(3.2) \qquad \wedge_{i=1}^p \, \boldsymbol{x}_i = \sum_{I = \{i_1 < \cdots < i_p\} \in C(n+1, p)} \det \left(x_{j, i_k} \right)_{1 \le j, k \le p} \boldsymbol{e}_I.$$

Recall that the exterior product is *alternating*, that is $\boldsymbol{u} \wedge \boldsymbol{v} = -\boldsymbol{v} \wedge \boldsymbol{u}$ so that $\boldsymbol{v} \wedge \boldsymbol{v} = 0$. Further, let $\boldsymbol{u} \cdot \boldsymbol{v}$ denote the standard inner product of $\boldsymbol{u}, \boldsymbol{v} \in \mathbb{R}^{n+1}$. Then, there is a uniquely defined *inner product* on $\wedge^p(\mathbb{R}^{n+1})$ such that

$$(3.3) \qquad (\boldsymbol{v}_1 \wedge \cdots \wedge \boldsymbol{v}_p) \cdot (\boldsymbol{u}_1 \wedge \cdots \wedge \boldsymbol{u}_p) = \det \left(\boldsymbol{v}_i \cdot \boldsymbol{u}_j \right)_{1 \le i, j \le p}$$

for any $\boldsymbol{v}_1, \dots, \boldsymbol{v}_p, \boldsymbol{u}_1, \dots, \boldsymbol{u}_p \in \mathbb{R}^{n+1}$. Furthermore, if $\boldsymbol{e}_0, \dots, \boldsymbol{e}_n$ is an orthonormal basis, then so is (3.1). Often (3.3) is referred to as the *Laplace identity* [50, p. 105]. The Euclidean norm on $\wedge^p(\mathbb{R}^{n+1})$ induced by (3.3) will be denoted by $|\cdot|$. By (13) in [56, p. 49],

$$(3.4) \qquad |\boldsymbol{u} \wedge \boldsymbol{v}| \le |\boldsymbol{u}| \, |\boldsymbol{v}| \qquad \text{if } \boldsymbol{u} \text{ or } \boldsymbol{v} \text{ is decomposable.}$$

Recall that a multivector $\boldsymbol{u}$ is *decomposable* if $\boldsymbol{u} = \boldsymbol{u}_1 \wedge \cdots \wedge \boldsymbol{u}_p$ for some $\boldsymbol{u}_1, \dots, \boldsymbol{u}_p \in \mathbb{R}^{n+1}$. Finally, given $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$,

$$d_p(\mathbf{x}, \mathbf{y}) \;=\; \frac{|\boldsymbol{x} \wedge \boldsymbol{y}|}{|\boldsymbol{x}| \, |\boldsymbol{y}|}$$

is called the *projective distance* between $\mathbf{x}$ and $\mathbf{y}$. Obviously $d_p(\mathbf{x}, \mathbf{y})$ is well defined. It is known that $d_p(\mathbf{x}, \mathbf{y}) = \sin \varphi(\boldsymbol{x}, \boldsymbol{y})$, where $\varphi(\boldsymbol{x}, \boldsymbol{y})$ denotes the acute angle between $\boldsymbol{x}$ and $\boldsymbol{y}$; see (3.13) below. In particular, this angular property of d_p implies that $d_p(\mathbf{x}, \mathbf{y})$ is a metric. Furthermore, d_p is locally comparable to the euclidean norm since

$$(3.5) \qquad d_p(\mathbf{x}, \mathbf{y}) \;\le\; |\mathbf{x} - \mathbf{y}| \;\le\; \sqrt{1 + |\mathbf{x}|^2} \, \sqrt{1 + |\mathbf{y}|^2} \, d_p(\mathbf{x}, \mathbf{y})$$

for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$. To see that (3.5) is true take $\boldsymbol{x} = (1, x_1, \dots, x_n)$ and $\boldsymbol{y} = (1, y_1, \dots, y_n)$. Then the l.h.s. of (3.5) is proved as follows:

$$d_p(\mathbf{x}, \mathbf{y}) = \frac{|\boldsymbol{x} \wedge \boldsymbol{y}|}{|\boldsymbol{x}| \, |\boldsymbol{y}|} = \frac{|(\boldsymbol{x} - \boldsymbol{y}) \wedge \boldsymbol{y}|}{|\boldsymbol{x}| \, |\boldsymbol{y}|} \overset{(3.4)}{\le} \frac{|\boldsymbol{x} - \boldsymbol{y}| \, |\boldsymbol{y}|}{|\boldsymbol{x}| \, |\boldsymbol{y}|} = \frac{|\boldsymbol{x} - \boldsymbol{y}|}{|\boldsymbol{x}|} \le |\boldsymbol{x} - \boldsymbol{y}| = |\mathbf{x} - \mathbf{y}|.$$

On the other hand, $|\mathbf{x} - \mathbf{y}| \le |\boldsymbol{x} \wedge \boldsymbol{y}| = \sqrt{1 + |\mathbf{x}|^2} \, \sqrt{1 + |\mathbf{y}|^2} \, d_p(\mathbf{x}, \mathbf{y})$, where the first inequality is a consequence of (3.2).

3.2. *Interior product and Hodge duality.* In what follows " $\cdot$ " will denote the *interior product* of multivectors. For $\boldsymbol{u} \in \wedge^p(\mathbb{R}^{n+1})$ and $\boldsymbol{v} \in \wedge^q(\mathbb{R}^{n+1})$ the latter is defined as follows. Assume that $p \geq q$ and consider the two linear functions on $\wedge^{p-q}(\mathbb{R}^{n+1})$ given by

$$\boldsymbol{x} \mapsto \boldsymbol{u} \cdot (\boldsymbol{v} \wedge \boldsymbol{x}) \qquad \text{and} \qquad \boldsymbol{x} \mapsto (\boldsymbol{x} \wedge \boldsymbol{v}) \cdot \boldsymbol{u}.$$

Since $\wedge^{p-q}(\mathbb{R}^{n+1})$ is euclidean, there are unique $(p-q)$-vectors, which will be denoted by $\boldsymbol{u} \cdot \boldsymbol{v}$ and $\boldsymbol{v} \cdot \boldsymbol{u}$, such that $(\boldsymbol{u} \cdot \boldsymbol{v}) \cdot \boldsymbol{x} = \boldsymbol{u} \cdot (\boldsymbol{v} \wedge \boldsymbol{x})$ and $\boldsymbol{x} \cdot (\boldsymbol{v} \cdot \boldsymbol{u}) = (\boldsymbol{x} \wedge \boldsymbol{v}) \cdot \boldsymbol{u}$ for all $\boldsymbol{x} \in \wedge^{p-q}(\mathbb{R}^{n+1})$. The multivectors $\boldsymbol{u} \cdot \boldsymbol{v}$ and $\boldsymbol{v} \cdot \boldsymbol{u}$ are called the *interior products* of $\boldsymbol{u}$ and $\boldsymbol{v}$, and $\boldsymbol{v}$ and $\boldsymbol{u}$ respectively. It is easily seen that $\boldsymbol{v} \cdot \boldsymbol{u} = (-1)^{q(p-q)}\boldsymbol{u} \cdot \boldsymbol{v}$ and that in the case $p = q$ the interior product is simply the inner product (3.3). The definition of interior product readily implies that

$$(3.6) \qquad \boldsymbol{a} \cdot (\boldsymbol{b} \wedge \boldsymbol{c}) = (\boldsymbol{a} \cdot \boldsymbol{b}) \cdot \boldsymbol{c} \qquad \text{and} \qquad (\boldsymbol{c} \wedge \boldsymbol{b}) \cdot \boldsymbol{a} = \boldsymbol{c} \cdot (\boldsymbol{b} \cdot \boldsymbol{a})$$

if $\boldsymbol{a} \in \wedge^p(\mathbb{R}^{n+1})$, $\boldsymbol{b} \in \wedge^q(\mathbb{R}^{n+1})$, $\boldsymbol{c} \in \wedge^r(\mathbb{R}^{n+1})$ with $p \geq q + r$; see (5) and (6) in [56, p. 43].

Let $\mathbf{e}_0, \ldots, \mathbf{e}_n$ be the standard basis of $\mathbb{R}^{n+1}$ and $\boldsymbol{i} = \mathbf{e}_0 \wedge \mathbf{e}_1 \wedge \cdots \wedge \mathbf{e}_n \in \wedge^{n+1}(\mathbb{R}^{n+1})$. By " $\perp$ " we will denote the Hodge star operator which is defined by

$$(3.7) \qquad\qquad\qquad\qquad \boldsymbol{u}^\perp := \boldsymbol{i} \cdot \boldsymbol{u}.$$

Note that the multivector $\boldsymbol{u} \in \wedge^p(\mathbb{R}^{n+1})$ is decomposable if and only if $\boldsymbol{u}^\perp \in \wedge^{n+1-p}(\mathbb{R}^{n+1})$ is decomposable; see Lemma 11A in [56, p. 48]. The map (3.7) is obviously linear. Also

$$(3.8) \qquad (\boldsymbol{v}^\perp)^\perp = (-1)^{(n+1-p)p}\boldsymbol{v} \qquad \text{for any } \boldsymbol{v} \in \wedge^p(\mathbb{R}^{n+1}).$$

The latter, known as the Hodge duality, follows from (2) in [56, p. 49] but can also be easily verified for basis vectors and then extended by linearity. Obviously $\boldsymbol{v} \mapsto \boldsymbol{v}^\perp$ is a one-to-one correspondence between $\wedge^p(\mathbb{R}^{n+1})$ and $\wedge^{n+1-p}(\mathbb{R}^{n+1})$. Also, an easy consequence of (3.6) and (3.8) is that the Hodge operator is an isometry, that is $|\boldsymbol{v}^\perp| = |\boldsymbol{v}|$ for any $\boldsymbol{v} \in \wedge^p(\mathbb{R}^{n+1})$. Also the Hodge operator conveniently relates the interior and exterior products. Indeed, let $\boldsymbol{u} \in \wedge^p(\mathbb{R}^{n+1})$ and $\boldsymbol{v} \in \wedge^q(\mathbb{R}^{n+1})$. Then using (3.6) readily gives

$$(3.9) \qquad\qquad \boldsymbol{v}^\perp \cdot \boldsymbol{u} = (\boldsymbol{v} \wedge \boldsymbol{u})^\perp \qquad \text{if} \qquad p + q \leq n + 1.$$

Since the Hodge operator is an isometry, this relation implies that

$$(3.10) \qquad\qquad |\boldsymbol{v}^\perp \cdot \boldsymbol{u}| = |\boldsymbol{v} \wedge \boldsymbol{u}| \qquad \text{if} \qquad p + q \leq n + 1.$$

3.3. *Relations between multivectors and subspaces of $\mathbb{R}^{n+1}$*. Throughout, $\mathcal{V}(\boldsymbol{v}_1, \ldots, \boldsymbol{v}_r)$ denotes the vector space spanned by vectors $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_r$. Also, given a multivector $\boldsymbol{w} \in \wedge(\mathbb{R}^{n+1})$, let $\mathcal{V}(\boldsymbol{w})$ be the linear subspace of $\mathbb{R}^{n+1}$ given by

$$\mathcal{V}(\boldsymbol{w}) := \{\boldsymbol{x} \in \mathbb{R}^{n+1} : \boldsymbol{w} \wedge \boldsymbol{x} = \mathbf{0}\}.$$

LEMMA 3.1. *If $\boldsymbol{u}_1, \ldots, \boldsymbol{u}_p \in \mathbb{R}^{n+1}$ are linearly independent, then $\mathcal{V}(\boldsymbol{u}_1 \wedge \cdots \wedge \boldsymbol{u}_p) = \mathcal{V}(\boldsymbol{u}_1, \ldots, \boldsymbol{u}_p)$. Furthermore if $\boldsymbol{u}, \boldsymbol{v} \in \wedge^p(\mathbb{R}^{n+1})$ are nonzero decomposable multivectors, then $\mathcal{V}(\boldsymbol{u}) = \mathcal{V}(\boldsymbol{v}) \iff \boldsymbol{u} = \theta \boldsymbol{v}$ for some $\theta \neq 0$.*

For details see Lemma 6B and Lemma 6C in [50, pp. 104–105]. Lemma 3.1 gives a one-to-one correspondence between nonzero decomposable p-vectors taken up to a constant multiple and linear subspaces in $\mathbb{R}^{n+1}$ of dimension p. The latter is known as a *Grassmann manifold* and will be denoted by $\mathrm{Gr}_p(\mathbb{R}^{n+1})$. Thus $\mathrm{Gr}_p(\mathbb{R}^{n+1})$ is embedded into $\mathbb{P}(\wedge^p(\mathbb{R}^{n+1}))$ and so is equipped with a natural topology induced from $\mathbb{P}(\wedge^p(\mathbb{R}^{n+1}))$ with respect to which it is obviously compact. Naturally, through the above correspondence the elements of $\mathrm{Gr}_p(\mathbb{R}^{n+1})$ can be thought of as unit decomposable p-vectors taken up to sign.

The following lemma gives a convenient way of expressing orthogonal subspaces via the Hodge operator and justifies the notation for the operator that we use within this paper. In what follows $W^\perp$ denotes the linear subspace of $\mathbb{R}^{n+1}$ orthogonal to $W \subset \mathbb{R}^{n+1}$.

LEMMA 3.2. *Let $\boldsymbol{u} \in \wedge^p(\mathbb{R}^{n+1})$ be a nonzero decomposable multivector. Then*

$$(3.11) \qquad \mathcal{V}(\boldsymbol{u}^\perp) = \mathcal{V}(\boldsymbol{u})^\perp = \{\boldsymbol{v} \in \mathbb{R}^{n+1} : \boldsymbol{u} \cdot \boldsymbol{v} = 0\}.$$

Proof. Take any orthogonal basis $\boldsymbol{e}_1, \ldots, \boldsymbol{e}_p$ of $\mathcal{V}(\boldsymbol{u})$ such that $\boldsymbol{u} = \boldsymbol{e}_1 \wedge \cdots \wedge \boldsymbol{e}_p$. This is possible in view of Lemma 3.1. If $\boldsymbol{v} \in \mathbb{R}^{n+1}$ is orthogonal to $\mathcal{V}(\boldsymbol{u})$ then, using (3.3) it is easy to see that $\boldsymbol{u} \cdot (\boldsymbol{v} \wedge \boldsymbol{x}) = 0$ for any decomposable $\boldsymbol{x} \in \wedge^{p-1}(\mathbb{R}^{n+1})$. On the other hand, if $\boldsymbol{v} \in \mathbb{R}^{n+1}$ is not orthogonal to $\mathcal{V}(\boldsymbol{u})$, say $\boldsymbol{e}_1 \cdot \boldsymbol{v} \neq 0$, then, by (3.3), $\boldsymbol{u} \cdot (\boldsymbol{v} \wedge \boldsymbol{e}_2 \wedge \cdots \wedge \boldsymbol{e}_p) = \boldsymbol{e}_1 \cdot \boldsymbol{v} \neq 0$. The upshot is that $\boldsymbol{u} \cdot (\boldsymbol{v} \wedge \boldsymbol{x})$ vanishes identically for all $\boldsymbol{x} \in \wedge^{p-1}(\mathbb{R}^{n+1})$ if and only if $\boldsymbol{v} \in \mathcal{V}(\boldsymbol{u})^\perp$. By the definition of interior product, this precisely means that $\boldsymbol{u} \cdot \boldsymbol{v} = 0$ if and only if $\boldsymbol{v} \in \mathcal{V}(\boldsymbol{u})^\perp$. The latter establishes the r.h.s. of (3.11). Finally, by (3.10), $\boldsymbol{u} \cdot \boldsymbol{v} = 0$ if and only if $\boldsymbol{u}^\perp \wedge \boldsymbol{v} = 0$. The latter implies the l.h.s. of (3.11). $\qquad\square$

LEMMA 3.3. *Let $\boldsymbol{u} \in \wedge^p(\mathbb{R}^{n+1})$ and $\boldsymbol{v} \in \wedge^q(\mathbb{R}^{n+1})$ be decomposable. Then $\mathcal{V}(\boldsymbol{u}) \cap \mathcal{V}(\boldsymbol{v}) = \varnothing$ if and only if $\boldsymbol{u} \wedge \boldsymbol{v} \neq 0$. Consequently, if $\boldsymbol{u} \wedge \boldsymbol{v} \neq 0$, then $\mathcal{V}(\boldsymbol{u}) \oplus \mathcal{V}(\boldsymbol{v}) = \mathcal{V}(\boldsymbol{u} \wedge \boldsymbol{v})$. Also if $p \geq q$ and $\boldsymbol{u} \cdot \boldsymbol{v} \neq \mathbf{0}$, then $\mathcal{V}(\boldsymbol{u} \cdot \boldsymbol{v}) = \mathcal{V}(\boldsymbol{u}) \cap \mathcal{V}(\boldsymbol{v}^\perp)$.*

Proof. The condition $\mathcal{V}(\boldsymbol{u}) \cap \mathcal{V}(\boldsymbol{v}) = \varnothing$ means that the sum $\mathcal{V}(\boldsymbol{u}) + \mathcal{V}(\boldsymbol{v})$ is direct, which is equivalent to $\boldsymbol{u} \wedge \boldsymbol{v} \neq 0$. The equality $\mathcal{V}(\boldsymbol{u}) \oplus \mathcal{V}(\boldsymbol{v}) = \mathcal{V}(\boldsymbol{u} \wedge \boldsymbol{v})$ is then a consequence of Lemma 3.1. Finally, by (3.9), $\boldsymbol{u} \cdot \boldsymbol{v} = \pm(\boldsymbol{u}^{\perp} \wedge \boldsymbol{v})^{\perp}$. Then, by Lemmas 3.2 and 3.3, $\mathcal{V}(\boldsymbol{u} \cdot \boldsymbol{v}) = \mathcal{V}(\boldsymbol{u}^{\perp} \wedge \boldsymbol{v})^{\perp} = (\mathcal{V}(\boldsymbol{u}^{\perp}) \oplus \mathcal{V}(\boldsymbol{v}))^{\perp} = \mathcal{V}(\boldsymbol{u}^{\perp})^{\perp} \cap \mathcal{V}(\boldsymbol{v})^{\perp} = \mathcal{V}(\boldsymbol{u}) \cap \mathcal{V}(\boldsymbol{v}^{\perp})$. $\qquad\square$

The following lemma is easily established using the Laplace identity (3.3).

LEMMA 3.4. *Let $\boldsymbol{u} \in \wedge^p(\mathbb{R}^{n+1})$ and $\boldsymbol{v} \in \wedge^q(\mathbb{R}^{n+1})$ be decomposable and $p + q \leq n + 1$. If $\mathcal{V}(\boldsymbol{u}) \perp \mathcal{V}(\boldsymbol{v})$, then $|\boldsymbol{u} \wedge \boldsymbol{v}| = |\boldsymbol{u}| \, |\boldsymbol{v}|$.*

3.4. *Multivectors and projections.* There are various relations between exterior/interior product and projections of vectors in $\mathbb{R}^{n+1}$ onto subspaces. The properties we are particularly interested in are summarized as

LEMMA 3.5. *Let $\boldsymbol{u} \in \mathbb{R}^{n+1}$, $\boldsymbol{v} \in \wedge^p(\mathbb{R}^{n+1})$ with $1 \leq p \leq n$ be decomposable and let π denote the orthogonal projection from $\mathbb{R}^{n+1}$ onto $\mathcal{V}(\boldsymbol{v})$. Then*

$$(3.12) \qquad |\boldsymbol{v} \wedge \boldsymbol{u}| = |\boldsymbol{v}| \cdot |\boldsymbol{u} - \pi\boldsymbol{u}| \qquad and \qquad |\boldsymbol{v} \cdot \boldsymbol{u}| = |\boldsymbol{v}| \cdot |\pi\boldsymbol{u}|.$$

Furthermore, $|\boldsymbol{v}|^2 \pi\boldsymbol{u} = \pm\, \boldsymbol{v} \cdot (\boldsymbol{v} \cdot \boldsymbol{u})$, where the sign is either $+$ or $-$.

Proof. Fix an orthogonal basis $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_p$ of $\mathcal{V}(\boldsymbol{v})$ such that $\boldsymbol{v} = \boldsymbol{v}_1 \wedge \cdots \wedge \boldsymbol{v}_p$. Let $\boldsymbol{u}' = \boldsymbol{u} - \pi\boldsymbol{u}$. Obviously $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_p, \boldsymbol{u}'$ is an orthogonal system. Also, since $\pi\boldsymbol{u} \in \mathcal{V}(\boldsymbol{v})$, by Lemma 3.1, $\boldsymbol{v} \wedge \pi\boldsymbol{u} = \boldsymbol{0}$. Therefore, $\boldsymbol{v} \wedge \boldsymbol{u} = \boldsymbol{v} \wedge \boldsymbol{u}'$. Now applying (3.3) gives

$$|\boldsymbol{v} \wedge \boldsymbol{u}|^2 = |\boldsymbol{v}_1 \wedge \cdots \wedge \boldsymbol{v}_p \wedge \boldsymbol{u}'|^2 \overset{(3.3)}{=} |\boldsymbol{v}_1|^2 \ldots |\boldsymbol{v}_p|^2 \, |\boldsymbol{u}'|^2 \overset{(3.3)}{=} |\boldsymbol{v}|^2 \, |\boldsymbol{u} - \pi\boldsymbol{u}|^2.$$

This establishes the l.h.s. of (3.12). Further, notice that $\boldsymbol{u} - \pi\boldsymbol{u}$ is the orthogonal projection of $\boldsymbol{u}$ onto $\mathcal{V}(\boldsymbol{v}^{\perp}) = \mathcal{V}(\boldsymbol{v})^{\perp}$. Therefore, the r.h.s. of (3.12) follows on applying (3.10) to the l.h.s. of (3.12), when $\boldsymbol{v}$ is replaced by $\boldsymbol{v}^{\perp}$. The final identity of the lemma is very well known and easy when $p = 1$. We consider $p \geq 2$. First, notice that $\boldsymbol{u} \wedge \pi\boldsymbol{u} = \boldsymbol{u} \wedge (\boldsymbol{u} - \boldsymbol{u}') = -\boldsymbol{u} \wedge \boldsymbol{u}'$ and that $\boldsymbol{v} \cdot \boldsymbol{u}' = 0$; see Lemma 3.2. Therefore, $(\boldsymbol{v} \cdot \boldsymbol{u}) \cdot \pi\boldsymbol{u} \overset{(3.6)}{=} \boldsymbol{v} \cdot (\boldsymbol{u} \wedge \pi\boldsymbol{u}) = -\boldsymbol{v} \cdot (\boldsymbol{u} \wedge \boldsymbol{u}') = \boldsymbol{v} \cdot (\boldsymbol{u}' \wedge \boldsymbol{u}) \overset{(3.6)}{=} (\boldsymbol{v} \cdot \boldsymbol{u}') \cdot \boldsymbol{u} = 0$. Hence, by Lemma 3.2, $\pi\boldsymbol{u} \perp \mathcal{V}(\boldsymbol{v} \cdot \boldsymbol{u})$. Also, since π is the projection onto $\mathcal{V}(\boldsymbol{v})$, we have that $\pi\boldsymbol{u} \perp \mathcal{V}(\boldsymbol{v})^{\perp} = \mathcal{V}(\boldsymbol{v}^{\perp})$. Therefore, $\pi\boldsymbol{u} \perp \mathcal{V}(\boldsymbol{v} \cdot \boldsymbol{u}) + \mathcal{V}(\boldsymbol{v}^{\perp})$. By Lemma 3.3, the space $\mathcal{V}(\boldsymbol{v} \cdot \boldsymbol{u})$ is a subspace of $\mathcal{V}(\boldsymbol{v})$ and so is orthogonal to $\mathcal{V}(\boldsymbol{v}^{\perp})$. Then, the sum $\mathcal{V}(\boldsymbol{v} \cdot \boldsymbol{u}) + \mathcal{V}(\boldsymbol{v}^{\perp})$ is direct and, by Lemma 3.3, it is equal to $\mathcal{V}(\boldsymbol{v}^{\perp} \wedge (\boldsymbol{v} \cdot \boldsymbol{u}))$. The latter space is readily seen to have codimension 1. Theretofore, the relation $\pi\boldsymbol{u} \perp \mathcal{V}(\boldsymbol{v} \cdot \boldsymbol{u}) + \mathcal{V}(\boldsymbol{v}^{\perp})$ implies that $\pi\boldsymbol{u} \| (\boldsymbol{v}^{\perp} \wedge (\boldsymbol{v} \cdot \boldsymbol{u}))^{\perp} \overset{(3.9)}{=} \pm\boldsymbol{v} \cdot (\boldsymbol{v} \cdot \boldsymbol{u})$. Finally, since the Hodge operator is an isometry,

$$|\boldsymbol{v}|^2 \cdot |\pi\boldsymbol{u}| = |\boldsymbol{v}^{\perp}| \cdot |\boldsymbol{v}| \cdot |\pi\boldsymbol{u}| \overset{(3.12)}{=} |\boldsymbol{v}^{\perp}| \cdot |\boldsymbol{v} \cdot \boldsymbol{u}| \overset{\text{Lemma 3.4}}{=} |\boldsymbol{v}^{\perp} \wedge (\boldsymbol{v} \cdot \boldsymbol{u})| \overset{(3.10)}{=} |\boldsymbol{v} \cdot (\boldsymbol{v} \cdot \boldsymbol{u})|$$

and the identity $|\boldsymbol{v}|^2 \pi\boldsymbol{u} = \pm\, \boldsymbol{v} \cdot (\boldsymbol{v} \cdot \boldsymbol{u})$ now readily follows. $\qquad\square$

Given two lines ℓ_1 and ℓ_2 in $\mathbb{R}^{n+1}$ through the origin, let $\varphi(\ell_1, \ell_2)$ denote the acute angle between ℓ_1 and ℓ_2. Further, given a linear subspace L of $\mathbb{R}^{n+1}$ of dimension p and a line ℓ through the origin, the angle $\varphi(\ell, L)$ between L and ℓ is defined to be $\inf_{\ell' \in L} \varphi(\ell, \ell')$, where the infimum is taken over over lines ℓ' in L through the origin. It is well known that $\varphi(\ell, L)$ is the angle between ℓ and the orthogonal projection of ℓ onto L. Thus, if $\boldsymbol{u}$ is a directional vector of ℓ and π denotes the orthogonal projection onto L, then $\sin \varphi(\ell, L) = |\boldsymbol{u}|^{-1}|\boldsymbol{u} - \pi\mathbf{u}|$. Further, if $\boldsymbol{v} \in \wedge^p(\mathbb{R}^{n+1})$ is a Grassmann representative of L, that is $L = \mathcal{V}(\boldsymbol{v})$, then, by Lemma 3.5,

$$(3.13) \qquad \sin \varphi(\ell, L) = \frac{|\boldsymbol{v} \wedge \boldsymbol{u}|}{|\boldsymbol{v}|\,|\boldsymbol{u}|} \stackrel{(3.10)}{=} \frac{|\boldsymbol{v}^{\perp} \cdot \boldsymbol{u}|}{|\boldsymbol{v}|\,|\boldsymbol{u}|}.$$

The following lemma is a consequence of the fact that the angle between a line ℓ and a plane L_1 is not bigger than the angle between this line ℓ and any other plane $L_2 \subset L_1$.

LEMMA 3.6. *Let $\boldsymbol{v} \in \wedge^p(\mathbb{R}^{n+1})$ be a nonzero decomposable multivector and $\boldsymbol{u} \in \mathbb{R}^{n+1}$. Then for any nonzero $\boldsymbol{w} \in \mathcal{V}(\boldsymbol{v})$,*

$$\frac{|\boldsymbol{w} \cdot \boldsymbol{u}|}{|\boldsymbol{w}|} \le \frac{|\boldsymbol{v} \cdot \boldsymbol{u}|}{|\boldsymbol{v}|}.$$

Proof. In view of (3.10),

$$(3.14) \qquad \frac{|\boldsymbol{w} \cdot \boldsymbol{u}|}{|\boldsymbol{w}|} \le \frac{|\boldsymbol{v} \cdot \boldsymbol{u}|}{|\boldsymbol{v}|} \qquad \Longleftrightarrow \qquad \frac{|\boldsymbol{v}^{\perp} \wedge \boldsymbol{u}|}{|\boldsymbol{v}|\,|\boldsymbol{u}|} \ge \frac{|\boldsymbol{w}^{\perp} \wedge \boldsymbol{u}|}{|\boldsymbol{w}|\,|\boldsymbol{u}|}.$$

Obviously $L_2 := \mathcal{V}(\boldsymbol{v}^{\perp}) \subset L_1 := \mathcal{V}(\boldsymbol{w}^{\perp})$. Let $\ell := \mathcal{V}(\boldsymbol{u})$. Therefore, by (3.13), the l.h.s. of (3.14) is equivalent to $\sin \varphi(\ell, L_2) \ge \sin \varphi(\ell, L_1)$. The latter is obvious in view of the fact that $L_2 \subset L_1$. The proof is thus complete. $\qquad \square$

4. Detecting rational points near a manifold

In this section we describe the mechanism for investigating the distribution of rational points near manifolds.

4.1. *Local geometry near a manifold.* Let $\mathcal{M}$ be a $C^{(2)}$ manifold of the Monge form (1.2). For $\mathbf{x} = (x_1, \ldots, x_d) \in U$, let $\mathbf{y} = \mathbf{y}(\mathbf{x})$ be the point $(\mathbf{x}, \mathbf{f}(\mathbf{x})) \in \mathcal{M}$. We will use the lifting of $\mathcal{M}$ into $\mathbb{R}^{n+1}$ given by

$$(4.1) \qquad \boldsymbol{y}(\mathbf{x}) = (1, \mathbf{y}(\mathbf{x})) = (1, \mathbf{x}, \mathbf{f}(\mathbf{x}))$$

which represents the projective embedding of $\mathbf{y}(\mathbf{x})$. Further, consider the following maps:

$$(4.2) \qquad \boldsymbol{g} : U \to \wedge^m(\mathbb{R}^{n+1}) \ : \ \mathbf{x} \mapsto \left(\boldsymbol{y}(\mathbf{x}) \wedge \partial_1 \boldsymbol{y}(\mathbf{x}) \wedge \cdots \wedge \partial_d \boldsymbol{y}(\mathbf{x})\right)^{\perp}$$

and

$$(4.3) \qquad \boldsymbol{u} : U \to \wedge^d(\mathbb{R}^{n+1}) \ : \ \mathbf{x} \mapsto \big(\boldsymbol{y}(\mathbf{x}) \wedge \boldsymbol{g}(\mathbf{x})\big)^\perp,$$

where $\partial_i := \partial/\partial x_i$. The vectors $\boldsymbol{y}(\mathbf{x}), \partial_1\boldsymbol{y}(\mathbf{x}), \ldots, \partial_d\boldsymbol{y}(\mathbf{x})$ are linearly independent because $\boldsymbol{y}(\mathbf{x})$ has the Monge form. Hence $\boldsymbol{g}(\mathbf{x}) \neq \mathbf{0}$. Also, by Lemma 3.2, $\boldsymbol{y}(\mathbf{x}) \perp \mathcal{V}(\boldsymbol{g}(\mathbf{x}))$. Therefore, $\boldsymbol{y}(\mathbf{x}) \wedge \boldsymbol{g}(\mathbf{x}) \neq 0$ further implying $\boldsymbol{u}(\mathbf{x}) \neq \mathbf{0}$.

Convention. In order to simplify notation, we will write $\boldsymbol{g}_\mathbf{x}$, $\boldsymbol{u}_\mathbf{x}$ and $\boldsymbol{y}_\mathbf{x}$ for $\boldsymbol{g}(\mathbf{x})$, $\boldsymbol{u}(\mathbf{x})$ and $\boldsymbol{y}(\mathbf{x})$ respectively and drop the subscript $\mathbf{x}$ whenever there is no risk of confusion. It is useful to keep in mind the following geometric nature of $\boldsymbol{g}$ and $\boldsymbol{u}$. The homogeneous equations $\boldsymbol{g} \cdot \boldsymbol{z} = 0$ and $\boldsymbol{u} \cdot \boldsymbol{z} = 0$ with respect to $\boldsymbol{z} = (z_0, z_1, \ldots, z_n)$ define the tangent and transversal planes to $\mathcal{M}$ respectively. Furthermore, $|\boldsymbol{g} \cdot \boldsymbol{r}| \, |\boldsymbol{g}|^{-1}|\boldsymbol{r}|^{-1}$ (resp. $|\boldsymbol{u} \cdot \boldsymbol{r}| \, |\boldsymbol{u}|^{-1}|\boldsymbol{r}|^{-1}$) is the projective distance of $\boldsymbol{r}$ from the tangent (resp. transversal) plane; see (3.13).

LEMMA 4.1. *For every* $\mathbf{x} \in U$ *we have that* $\mathbb{R}^{n+1} = \mathcal{V}(\boldsymbol{g}) \oplus \mathcal{V}(\boldsymbol{u}) \oplus \mathcal{V}(\boldsymbol{y})$ *is a decomposition of* $\mathbb{R}^{n+1}$ *into pairwise orthogonal subspaces.*

Proof. Recall the convention that $\boldsymbol{g} = \boldsymbol{g}_\mathbf{x}$, $\boldsymbol{u} = \boldsymbol{u}_\mathbf{x}$ and $\boldsymbol{y} = \boldsymbol{y}_\mathbf{x}$. Fix an $\mathbf{x} \in U$. Let $\boldsymbol{t} := \partial_1\boldsymbol{y}(\mathbf{x}) \wedge \cdots \wedge \partial_d\boldsymbol{y}(\mathbf{x})$. Then, by (4.2), $\boldsymbol{g} = (\boldsymbol{y} \wedge \boldsymbol{t})^\perp$. Then, using Lemmas 3.1 and 3.2, we get that $\mathcal{V}(\boldsymbol{g}) = \mathcal{V}((\boldsymbol{y} \wedge \boldsymbol{t})^\perp) = \mathcal{V}(\boldsymbol{y} \wedge \boldsymbol{t})^\perp \subset \mathcal{V}(\boldsymbol{y})^\perp$. It follows that $\mathcal{V}(\boldsymbol{g}) \perp \mathcal{V}(\boldsymbol{y})$. It is similarly established that $\mathcal{V}(\boldsymbol{u}) \perp \mathcal{V}(\boldsymbol{y})$ and $\mathcal{V}(\boldsymbol{g}) \perp \mathcal{V}(\boldsymbol{u})$. Thus, the subspaces $\mathcal{V}(\boldsymbol{g})$, $\mathcal{V}(\boldsymbol{u})$ and $\mathcal{V}(\boldsymbol{y})$ are pairwise orthogonal and so their sum is direct. Moreover, using Lemma 3.1 one readily finds the dimension of each of the subspaces, resulting in $\dim \mathcal{V}(\boldsymbol{g}) \oplus \mathcal{V}(\boldsymbol{u}) \oplus \mathcal{V}(\boldsymbol{y}) = n + 1$. Therefore, $\mathbb{R}^{n+1} = \mathcal{V}(\boldsymbol{g}) \oplus \mathcal{V}(\boldsymbol{u}) \oplus \mathcal{V}(\boldsymbol{y})$. $\square$

Lemma 4.1 provides a natural choice for local coordinates akin to the Frenet frame. The following Lemma 4.2 estimates the projective distance of a point $\mathbf{r} \in \mathbb{R}^n$ from $\boldsymbol{y} \in \mathcal{M}$ in terms of the projective distance of $\mathbf{r}$ from the tangent and transversal planes.

LEMMA 4.2. *For any* $\boldsymbol{r} \in \mathbb{R}^{n+1}$ *and any* $\mathbf{x} \in U$,

$$(4.4) \qquad \frac{|\boldsymbol{y} \wedge \boldsymbol{r}|}{|\boldsymbol{y}|} \leq \frac{|\boldsymbol{g} \cdot \boldsymbol{r}|}{|\boldsymbol{g}|} + \frac{|\boldsymbol{u} \cdot \boldsymbol{r}|}{|\boldsymbol{u}|}.$$

Proof. Let $\boldsymbol{r}_g$, $\boldsymbol{r}_u$ and $\boldsymbol{r}_y$ be the orthogonal projections of $\boldsymbol{r}$ onto $\mathcal{V}(\boldsymbol{g})$, $\mathcal{V}(\boldsymbol{u})$ and $\mathcal{V}(\boldsymbol{y})$ respectively. Then, by Lemma 4.1, $\boldsymbol{r} = \boldsymbol{r}_g + \boldsymbol{r}_u + \boldsymbol{r}_y$ and therefore $\boldsymbol{r} - \boldsymbol{r}_y = \boldsymbol{r}_g + \boldsymbol{r}_u$. By Lemma 3.5,

$$(4.5) \qquad |\boldsymbol{y} \wedge \boldsymbol{r}| \cdot |\boldsymbol{y}|^{-1} = |\boldsymbol{r} - \boldsymbol{r}_y| \leq |\boldsymbol{r}_g| + |\boldsymbol{r}_u|.$$

Again, by Lemma 3.5, $|\boldsymbol{g} \cdot \boldsymbol{r}| = |\boldsymbol{g}| \cdot |\boldsymbol{r}_g|$ and $|\boldsymbol{u} \cdot \boldsymbol{r}| = |\boldsymbol{u}| \cdot |\boldsymbol{r}_u|$. Substituting $|\boldsymbol{r}_g|$ and $|\boldsymbol{r}_u|$ from the latter equalities into (4.5) gives (4.4). $\square$

Lemma 4.2 is in general sharp as (4.4) can be reversed with some positive constant. Nevertheless, the distance of $\boldsymbol{r}$ from $\mathcal{M}$ rather than from a particular point $\mathbf{y}$ on $\mathcal{M}$ can be estimated in a more efficient way. This relies on the fact that the tangent plane deviates from a $C^{(2)}$ manifold with a quadratic error. A similar idea is explored by Elkies [28] in his algorithm for computing rational points near manifolds. Before we state the next result, recall that given a ball $B = B(\mathbf{x}, r)$ and $\lambda > 0$, $\lambda B := B(\mathbf{x}, \lambda r)$ and $\overline{B}$ is the closure of B.

LEMMA 4.3. *Let $\mathcal{M}$ be a $C^{(2)}$ manifold of the form (1.2) and B_0 be a ball of radius $r_{B_0} < \infty$ such that $2\overline{B}_0 \subset U$. Then there is a constant $C > 1$ depending on B_0 only satisfying the following property. For any $\boldsymbol{r} \in \mathbb{R}^{n+1}$ and $\mathbf{x} \in B_0$ such that*

$$(4.6) \qquad \frac{|\boldsymbol{g}_{\mathbf{x}} \cdot \boldsymbol{r}|}{|\boldsymbol{g}_{\mathbf{x}}|\,|\boldsymbol{r}|} < \delta \qquad and \qquad \frac{|\boldsymbol{u}_{\mathbf{x}} \cdot \boldsymbol{r}|}{|\boldsymbol{u}_{\mathbf{x}}|\,|\boldsymbol{r}|} < \varepsilon$$

for some positive δ and ε satisfying

$$(4.7) \qquad \varepsilon^2 \leq \delta \leq \varepsilon \leq \varepsilon_0 := \frac{\min\{1, r_{B_0}\}}{2d(n+1)(C+1)^2},$$

there is a point $\mathbf{x}' \in 2B_0$ such that

$$(4.8) \qquad \frac{|\boldsymbol{y}_{\mathbf{x}'} \wedge \boldsymbol{r}|}{|\boldsymbol{y}_{\mathbf{x}'}|\,|\boldsymbol{r}|} \leq K\,\delta, \qquad where \; K = 14(n+1)^3(C+1)^5 d^2.$$

Proof of Lemma 4.3. Without loss of generality we will assume that $|\boldsymbol{r}| = 1$. Since $2\overline{B}_0 \subset U$, there is a constant $C > 1$ such that

$$(4.9) \qquad\qquad\qquad 2\overline{B}_0 \subset [-C, C]^d$$

and

$$(4.10) \qquad \sup_{\mathbf{x} \in 2\overline{B}_0} \max \left\{ |f_l(\mathbf{x})|, \max_{1 \leq i \leq d} |\partial_i f_l(\mathbf{x})|, \max_{1 \leq i,j \leq d} |\partial_i \partial_j f_l(\mathbf{x})| \right\} \leq C$$

for $1 \leq l \leq m$, where ∂_i means differentiating by x_i and the functions f_l arise from (1.2).

Step 1. At this step we express $\boldsymbol{r}$ as a linear combination of $\boldsymbol{y}, \partial_1 \boldsymbol{y}, \ldots, \partial_d \boldsymbol{y}$ plus an error term. Let $\boldsymbol{r_g}$, $\boldsymbol{r_u}$ and $\boldsymbol{r_y}$ be the orthogonal projections of $\boldsymbol{r}$ onto $\mathcal{V}(\boldsymbol{g})$, $\mathcal{V}(\boldsymbol{u})$ and $\mathcal{V}(\boldsymbol{y})$ respectively. By Lemma 3.5 and the assumption $|\boldsymbol{r}| = 1$, inequalities (4.6) imply that

$$(4.11) \qquad\qquad |\boldsymbol{r_g}| < \delta \qquad and \qquad |\boldsymbol{r_u}| < \varepsilon.$$

Also, by Lemma 4.2, inequalities (4.6) imply that $|\boldsymbol{y}|^{-1}|\boldsymbol{y} \wedge \boldsymbol{r}| < \delta + \varepsilon$. By (3.3), we have the identity $|\boldsymbol{y} \wedge \boldsymbol{r}|^2 = |\boldsymbol{y}|^2|\boldsymbol{r}|^2 - |\boldsymbol{y} \cdot \boldsymbol{r}|^2$. Since $|\boldsymbol{r}| = 1$, the latter implies

$$0 \leq 1 - \frac{|\boldsymbol{y} \cdot \boldsymbol{r}|}{|\boldsymbol{y}|} \leq 1 - \left(\frac{|\boldsymbol{y} \cdot \boldsymbol{r}|}{|\boldsymbol{y}|} \right)^2 = \left(\frac{|\boldsymbol{y} \wedge \boldsymbol{r}|}{|\boldsymbol{y}|} \right)^2 \leq (\delta + \varepsilon)^2 \overset{(4.7)}{\leq} 4\delta.$$

The latter inequality together with the fact that $|\boldsymbol{y}|^{-1}|\boldsymbol{y} \cdot \boldsymbol{r}| = |\boldsymbol{r_y}|$ implied by Lemma 3.5, shows that for some $\eta \in \{-1, 1\}$,

$$(4.12) \qquad \boldsymbol{r_y} = \eta|\boldsymbol{y}|^{-1}\boldsymbol{y} + \boldsymbol{w}_0 \qquad \text{with} \qquad |\boldsymbol{w}_0| \leq 4\delta.$$

By (4.2) and Lemma 3.1, we see that the vectors $\boldsymbol{y} = \boldsymbol{y}(\mathbf{x}), \partial_1\boldsymbol{y}(\mathbf{x}), \ldots, \partial_d\boldsymbol{y}(\mathbf{x})$ form a basis of $\mathcal{V}(\boldsymbol{g}^\perp)$. By Lemmas 3.1 and 4.1, we have that $\mathcal{V}(\boldsymbol{u}) \subset \mathcal{V}(\boldsymbol{g}^\perp)$. Therefore, since $\boldsymbol{r_u} \in \mathcal{V}(\boldsymbol{u})$, there are real numbers $\lambda_0, \ldots, \lambda_d$ such that

$$(4.13) \qquad \boldsymbol{r_u} = \lambda_0\boldsymbol{y}(\mathbf{x}) + \sum_{i=1}^{d} \lambda_i\partial_i\boldsymbol{y}(\mathbf{x}).$$

Since $\boldsymbol{y}$ is of the Monge form,

$$(4.14) \qquad \boldsymbol{r_u} = (\lambda_0, \lambda_1 + x_1\lambda_0, \ldots, \lambda_d + x_d\lambda_0, *, \ldots, *),$$

where $*$ stands for a real number. By (4.9), (4.14) and the r.h.s. of (4.11),

$$(4.15) \qquad |\lambda_0| < \varepsilon \qquad \text{and} \qquad |\lambda_i| < (C+1)\varepsilon \qquad (1 \leq i \leq d).$$

On plugging the expressions for $\boldsymbol{r_y}$ and $\boldsymbol{r_u}$ given by (4.12) and (4.13) into the identity $\boldsymbol{r} = \boldsymbol{r_g} + \boldsymbol{r_u} + \boldsymbol{r_y}$ and applying the l.h.s. of (4.11), we get

$$(4.16) \qquad \boldsymbol{r} = \lambda_0^*\boldsymbol{y}(\mathbf{x}) + \sum_{i=1}^{d} \lambda_i\partial_i\boldsymbol{y}(\mathbf{x}) + \boldsymbol{w}_1 \qquad \text{with} \quad |\boldsymbol{w}_1| \leq 5\delta,$$

where $\lambda_0^* = \eta|\boldsymbol{y}(\mathbf{x})|^{-1} + \lambda_0$.

Step 2. At this step we define the point $\mathbf{x}'$. By (4.9) and (4.10), $|\boldsymbol{y}(\mathbf{x})|^{-1} \geq (n+1)^{-1}C^{-1}$. On the other hand, by (4.7) and (4.15), $|\lambda_0| \leq \frac{1}{2}(n+1)^{-1}C^{-1}$. Therefore, $|\lambda_0^*| \geq \frac{1}{2}(n+1)^{-1}C^{-1}$ or equivalently

$$(4.17) \qquad\qquad\qquad |\lambda_0^*|^{-1} \leq 2(n+1)C.$$

Further, define $\lambda_i^* = \lambda_i/\lambda_0^*$ for $i = 1, \ldots, d$. Inequalities (4.15) and (4.17) imply that

$$(4.18) \qquad\qquad |\lambda_i^*| \leq 2\varepsilon(n+1)(C+1)^2 \qquad (1 \leq i \leq d).$$

Dividing (4.16) by λ_0^* and applying (4.17) to estimate the remainder term gives

$$(4.19) \quad \lambda_0^{*-1}\boldsymbol{r} = \boldsymbol{y}(\mathbf{x}) + \sum_{i=1}^{d} \lambda_i^*\partial_i\boldsymbol{y}(\mathbf{x}) + \boldsymbol{w}_2 \qquad \text{with} \quad |\boldsymbol{w}_2| \leq 10\delta(n+1)C.$$

Now define $\mathbf{x}' = \mathbf{x} + \boldsymbol{\lambda}^*$, where $\boldsymbol{\lambda}^* = (\lambda_1^*, \ldots, \lambda_d^*)$. Conditions (4.7) and (4.18) ensure that $|\boldsymbol{\lambda}^*| \leq r_{B_0}$. Therefore, since $\mathbf{x} \in B_0$, $\mathbf{x}' \in 2B_0$.

Step 3. At this step we verify (4.8). By (4.9), (4.10), (4.18) and Taylor's formula, we get

$$(4.20) \qquad \Big| \boldsymbol{y}(\mathbf{x}') - \boldsymbol{y}(\mathbf{x}) - \sum_{i=1}^{d} \lambda_i^* \partial_i \boldsymbol{y}(\mathbf{x}) \Big| \leq 4\varepsilon^2 (n+1)^3 (C+1)^5 d^2.$$

Further, using (4.7), (4.19) and (4.20) we get

$$(4.21) \qquad |\boldsymbol{y}_{\mathbf{x}'} - \lambda_0^{*-1} \boldsymbol{r}| \leq \delta \Big(10(n+1)C + 4(n+1)^3(C+1)^5 d^2 \Big) \leq K\delta.$$

From (4.1), $|\boldsymbol{y}_{\mathbf{x}'}| \geq 1$. Therefore, using $|\boldsymbol{r}| = 1$, we obtain

$$\frac{|\boldsymbol{y}_{\mathbf{x}'} \wedge \boldsymbol{r}|}{|\boldsymbol{y}_{\mathbf{x}'}|\,|\boldsymbol{r}|} \leq |\boldsymbol{y}_{\mathbf{x}'} \wedge \boldsymbol{r}| = |(\boldsymbol{y}_{\mathbf{x}'} - \lambda_0^{*-1}\boldsymbol{r}) \wedge \boldsymbol{r}| \overset{(3.4)}{\leq} |(\boldsymbol{y}_{\mathbf{x}'} - \lambda_0^{*-1}\boldsymbol{r})| \cdot |\boldsymbol{r}| \overset{(4.21)}{\leq} K\delta.$$

This establishes (4.8) and thus completes the proof of Lemma 4.3. $\square$

4.2. *Good "cells" near a manifold.* Let ψ_*, Q_* and κ be positive parameters. In practice, Q_* and ψ_* will be proportional to Q and ψ respectively. Further, for every $\mathbf{x} \in U$, consider the system

$$(4.22) \qquad \frac{|\boldsymbol{g}_{\mathbf{x}} \cdot \boldsymbol{r}|}{|\boldsymbol{g}_{\mathbf{x}}|} < \psi_*, \qquad \frac{|\boldsymbol{u}_{\mathbf{x}} \cdot \boldsymbol{r}|}{|\boldsymbol{u}_{\mathbf{x}}|} < (\psi_*^m Q_*)^{-\frac{1}{d}}, \qquad \frac{|\boldsymbol{y}_{\mathbf{x}} \cdot \boldsymbol{r}|}{|\boldsymbol{y}_{\mathbf{x}}|} \leq \kappa\, Q_*,$$

where $\boldsymbol{r} \in \mathbb{R}^{n+1}$. Obviously the set of $\boldsymbol{r} \in \mathbb{R}^{n+1}$ satisfying (4.22) is a convex body symmetric about the origin. Then as a consequence of Minkowski's theorem for convex bodies one has

LEMMA 4.4. *Let v_d denote the volume of a ball of diameter 1 in $\mathbb{R}^d$ and $\kappa_0 := (v_d v_m)^{-1}$. Then, for any $\kappa \geq \kappa_0$, all $\psi_*, Q_* > 0$ and every $\mathbf{x} \in U$, there is an integer point $\boldsymbol{r} \in \mathbb{Z}^{n+1} \smallsetminus \{0\}$ satisfying (4.22).*

The convex body (4.22) in $\mathbb{R}^{n+1}$ is essentially a set of homogeneous coordinates of points that lies in a certain *"cell"* near $\mathbf{y}(\mathbf{x}) \in \mathcal{M}$. Clearly, the bigger the $|\boldsymbol{r}|$, the smaller the projective distance of $\boldsymbol{r}$ from the tangent and transversal planes to $\mathcal{M}$. (Note however that $|\boldsymbol{r}| \ll Q$ in any case.) Then, using Lemma 4.3 one can efficiently estimate the distance of $\boldsymbol{r}$ from $\mathcal{M}$. In order to give a formal statement we introduce the following sets. Let $\mathcal{B}_{\mathbf{f}}(Q_*, \psi_*, \kappa)$ be the set of $\mathbf{x} \in U$ such that there is an $\boldsymbol{r} \in \mathbb{Z}^{n+1} \smallsetminus \{0\}$ satisfying (4.22). Further, let $\mathcal{G}_{\mathbf{f}}(Q_*, \psi_*, \kappa) = U \setminus \mathcal{B}_{\mathbf{f}}(Q_*, \psi_*, \kappa)$. We will restrict $\mathbf{y}$ to $\mathcal{G}_{\mathbf{f}}(Q_*, \psi_*, \kappa)$ for some suitably chosen κ. This has the benefit of minimizing the distance of $\boldsymbol{r}$ from $\mathcal{M}$.

THEOREM 4.5. *Let $\mathcal{M}$ be a $C^{(2)}$ submanifold given by (1.2) and let B be a ball of radius $r_B < \infty$ such that $2\overline{B} \subset U$. Then there is an explicit constant $c_0 > 2$ such that for any choice of positive numbers ψ_*, Q_*, κ such that $\kappa < 1$,*

$$(4.23) \qquad\qquad Q_* \geq \max \Big\{ \frac{c_0}{\kappa^2}, \frac{c_0^2}{\kappa^4 r_B} \Big\}$$

and

$$(4.24) \qquad \kappa^{-\frac{d}{2n-d}} \, Q_*^{-\frac{d+2}{2n-d}} \ \leq \ \psi_* \ \leq \ 1,$$

we have the inclusion

$$B \cap \mathcal{G}_{\mathbf{f}}(Q_*, \psi_*, \kappa) \ \subset \ \Delta^{\delta_0}(Q, \psi, 2B, \rho),$$

where $\rho := c_0 \kappa^{-2} \big(\psi_*^m Q_*^{d+1} \big)^{-\frac{1}{d}}, \ \ \psi = c_0^3 \kappa^{-2} \psi_*, \ \ Q = c_0 Q_* \ \ and \ \ \delta_0 = \kappa c_0^{-2}.$

Before establishing Theorem 4.5 we shall give a formal proof of Lemma 4.4.

Proof of Lemma 4.4. Fix an arbitrary $\mathbf{x} \in U$. Obviously our goal is to show that there is an $\boldsymbol{r} \in \mathbb{Z}^{n+1} \smallsetminus \{0\}$ satisfying (4.22). Recall that $B = \{\boldsymbol{r} \in \mathbb{R}^{n+1} : (4.22) \text{ holds}\}$ is a convex body in $\mathbb{R}^{n+1}$ symmetric about the origin. By Lemmas 3.5 and 4.1, B is the direct sum of $B_{\boldsymbol{g}}$, $B_{\boldsymbol{u}}$ and $B_{\boldsymbol{y}}$, where the latter are the orthogonal projections of B onto the subspaces $\mathcal{V}(\boldsymbol{g})$, $\mathcal{V}(\boldsymbol{u})$ and $\mathcal{V}(\boldsymbol{y})$ respectively. Furthermore, $B_{\boldsymbol{g}}$ is a ball in $\mathcal{V}(\boldsymbol{g})$ of radius ψ_*, $B_{\boldsymbol{u}}$ is a ball in $\mathcal{V}(\boldsymbol{u})$ of radius $(\psi_*^m Q_*)^{-\frac{1}{d}}$, and $B_{\boldsymbol{y}}$ is a ball in $\mathcal{V}(\boldsymbol{y})$ of radius κQ_*. Since $\dim \mathcal{V}(\boldsymbol{g}) = m$, $\dim \mathcal{V}(\boldsymbol{u}) = d$ and $\dim \mathcal{V}(\boldsymbol{y}) = 1$ (Lemma 3.1),

$$(4.25) \qquad \mathrm{Vol}(B_{\boldsymbol{g}}) = 2^m \psi_*^m v_m,$$

$$\mathrm{Vol}(B_{\boldsymbol{u}}) = 2^d \Big((\psi_*^m Q_*)^{-\frac{1}{d}} \Big)^d v_d,$$

$$\mathrm{Vol}(B_{\boldsymbol{y}}) = 2\kappa Q_*.$$

Since the subspaces $\mathcal{V}(\boldsymbol{g})$, $\mathcal{V}(\boldsymbol{u})$ and $\mathcal{V}(\boldsymbol{y})$ are orthogonal, $\mathrm{Vol}(B) = \mathrm{Vol}(B_{\boldsymbol{g}}) \times \mathrm{Vol}(B_{\boldsymbol{u}}) \times \mathrm{Vol}(B_{\boldsymbol{y}})$. The latter together with (4.25) implies that $\mathrm{Vol}(B) = 2^{n+1} \kappa v_m v_d$. If $\kappa > (v_m v_d)^{-1}$, then $\mathrm{Vol}(B) > 2^{n+1}$ and, by Minkowski's theorem for convex bodies [50, §4.1], B contains a nonzero integer point $\boldsymbol{r} = \boldsymbol{r}_\kappa$. This proves the lemma when $\kappa > (v_m v_d)^{-1}$. Finally notice that the integer points $\boldsymbol{r}_\kappa$ with $\kappa_0 < \kappa < \kappa_0 + 1$ are contained in a bounded set. Therefore there are only finitely many of these points. It follows that there is a sequence (κ_i) with $\kappa_i > \kappa_0$ and $\kappa_i \to \kappa_0$ as $i \to \infty$ such that the points $\boldsymbol{r}_{\kappa_i}$ are the same and equal to, say, $\boldsymbol{r}'$. This point is easily seen to satisfy (4.22) with $\kappa = \kappa_0$. $\qquad \square$

Proof of Theorem 4.5. Since $2\overline{B} \subset U$, there is a constant $C > 1$ such that (4.9) and (4.10) are fulfilled. We will assume that $\kappa < \kappa_0$ as otherwise, by Lemma 4.4, there is nothing to prove. Let ψ_*, Q_* and κ satisfy the conditions of Theorem 4.5. Take any $\mathbf{x} \in B \cap \mathcal{G}_{\mathbf{f}}(Q_*, \psi_*, \kappa)$. Our goal is to show that

$$(4.26) \qquad \mathbf{x} \in B(\mathbf{a}/q, \rho) \ \text{ for some } \ (q, \mathbf{a}, \mathbf{b}) \in \mathcal{R}^{\delta_0}(Q, \psi, 2B).$$

The constant c_0 is defined to absorb various other constants appearing in the proof. More precisely, we set

$$(4.27) \ \ c_0 := \max \Big\{ \varepsilon_0^{-2}; \ \ \kappa_0 + 1; \ \ 16C^2(n+1)^4; \ \ 6K(\kappa_0+1)(n+1)^2 C^2 \Big\},$$

where $\varepsilon_0 = \min\{1, r_B\}(4d(n+1)C)^{-1}$ and $K = 14(n+1)^3(C+1)^5 d^2$ are the constants appearing in Lemma 4.3 and κ_0 is as in Lemma 4.4. By Lemma 4.4, $(4.22)_{\kappa=\kappa_0}$ has a solution $\boldsymbol{r} = (r_0, r_1, \ldots, r_n) \in \mathbb{Z}^{n+1} \smallsetminus \{0\}$. Without loss of generality we can assume that $\gcd(r_0, r_1, \ldots, r_n) = 1$ and that $r_0 \geq 0$. We set $q = r_0$, $\mathbf{a} = (r_1, \ldots, r_d)$ and $\mathbf{b} = (r_{d+1}, \ldots, r_n)$. Obviously $\gcd(q, \mathbf{a}, \mathbf{b}) = 1$. For the rest of the proof we show that $(q, \mathbf{a}, \mathbf{b})$ is the required point, that is (4.26) is satisfied for this choice of $(q, \mathbf{a}, \mathbf{b})$.

Step 1: *bounds on* $|\boldsymbol{r}|$. Let $\boldsymbol{r_g}$, $\boldsymbol{r_u}$ and $\boldsymbol{r_y}$ be the orthogonal projections of $\boldsymbol{r}$ onto $\mathcal{V}(\boldsymbol{g})$, $\mathcal{V}(\boldsymbol{u})$ and $\mathcal{V}(\boldsymbol{y})$. By $(4.22)_{\kappa=\kappa_0}$ and Lemma 3.5,

$$(4.28) \qquad |\boldsymbol{r_g}| < \psi_*, \qquad |\boldsymbol{r_u}| < (\psi_*^m Q_*)^{-1/d} \qquad \text{and} \qquad |\boldsymbol{r_y}| \leq \kappa_0 Q_*.$$

By Lemma 4.1, $\boldsymbol{r_g}$, $\boldsymbol{r_u}$ and $\boldsymbol{r_y}$ are pairwise orthogonal. Therefore, $|\boldsymbol{r}|^2 = |\boldsymbol{r_g}|^2 + |\boldsymbol{r_u}|^2 + |\boldsymbol{r_y}|^2$. The latter together with (4.28) gives

$$(4.29) \qquad |\boldsymbol{r}|^2 < \psi_*^2 + (\psi_*^m Q_*)^{-2/d} + \kappa_0^2 Q_*^2.$$

Using the l.h.s. of (4.24) and the fact that $\kappa < 1$, one readily verifies that

$$(4.30) \qquad (\psi_*^m Q_*)^{-1/d} < Q_*^{1/2}.$$

By the r.h.s. of (4.24), $\psi_* < 1$. Then (4.29) implies that $|\boldsymbol{r}|^2 < 1 + Q_* + \kappa_0^2 Q_*^2 \leq (\kappa_0^2 + 1)Q_*^2$. The latter inequality is due to (4.23). Hence $|\boldsymbol{r}| < (\kappa_0 + 1)Q_*$. Further, notice that the fact that $\mathbf{x} \in \mathcal{G}_{\mathbf{f}}(Q_*, \psi_*, \kappa)$ ensures that (4.22) does not have a solution in $\mathbb{Z}^{n+1} \smallsetminus \{0\}$. This is only possible if $|\boldsymbol{y}|^{-1}|\boldsymbol{y} \cdot \boldsymbol{r}| \geq \kappa Q_*$. Therefore, $|\boldsymbol{r}| \geq \kappa Q_*$, whence

$$(4.31) \qquad \kappa Q_* \leq |\boldsymbol{r}| \leq (\kappa_0 + 1)Q_*.$$

Step 2: *bounds on* $|r_0|$. We now show the first inequality of the following relations:

$$(4.32) \qquad |r_0| \geq \frac{\kappa Q_*}{2(n+1)C} \overset{(4.27)}{\geq} \frac{\kappa Q_*}{c_0}.$$

Assume the contrary. Then, by (4.31), there is an $i_0 \in \{1, \ldots, n\}$ such that $|r_{i_0}| \geq \kappa(n+1)^{-1}Q_*$. Let $\boldsymbol{y} = (1, y_1, \ldots, y_n)$. Observe that the expression $r_{i_0} - r_0 y_{i_0}$ is one of the coordinates of $\boldsymbol{y} \wedge \boldsymbol{r}$ in the standard basis. Therefore,
$$(4.33)$$
$$|\boldsymbol{y} \wedge \boldsymbol{r}| \geq |r_{i_0} - r_0 y_{i_0}| \geq |r_{i_0}| - |r_0 y_{i_0}| \geq \frac{\kappa Q_*}{n+1} - \frac{\kappa Q_*}{2(n+1)C} \times C = \frac{\kappa Q_*}{2(n+1)}.$$

Here we used the fact that $|y_{i_0}| \leq C$ implied by (4.9) and (4.10). In order to derive a contradiction we now obtain an upper bound for $|\boldsymbol{y} \wedge \boldsymbol{r}|$. By Lemma 4.2 and (4.22), $\frac{1}{|\boldsymbol{y}|}|\boldsymbol{y} \wedge \boldsymbol{r}| \leq \psi_* + (\psi_*^m Q_*)^{-1/d}$. Further, by (4.24) and (4.30), we get $\frac{1}{|\boldsymbol{y}|}|\boldsymbol{y} \wedge \boldsymbol{r}| < 1 + Q_*^{1/2} < 2Q_*^{1/2}$. The latter together with (4.9) and (4.10) gives $|\boldsymbol{y} \wedge \boldsymbol{r}| < 2C(n+1)Q_*^{1/2}$. Combining the latter with (4.33) implies that

$Q_*^{1/2} < 4C(n+1)^2/\kappa$. In view of (4.23) and (4.27), the latter inequality is contradictory, thus establishing (4.32).

Step 3: *completion of the proof.* We will first use Lemmas 4.3 with

$$(4.34) \qquad \delta = \frac{\psi_*}{\kappa \, Q_*} \qquad \text{and} \qquad \varepsilon = \frac{(\psi_*^m Q_*)^{-\frac{1}{d}}}{\kappa \, Q_*}.$$

Therefore, we assume that $\delta \leq \varepsilon$, and we begin by verifying (4.6) and (4.7).

Obviously, (4.22) and (4.31) imply (4.6). Further, the l.h.s. of (4.24) implies that $\varepsilon^2 \leq \delta$; this is the first inequality of (4.7). The second inequality of (4.7), that is $\delta \leq \varepsilon$, is simply assumed. Finally, by (4.30), $\varepsilon \leq (\kappa \, Q_*^{1/2})^{-1}$. By (4.23) and (4.27), $(\kappa \, Q_*^{1/2})^{-1} \leq \varepsilon_0$ and hence $\varepsilon \leq \varepsilon_0$; this shows the last inequality of (4.7). Thus, Lemma 4.3 is applicable and therefore, by (4.8), there is a point $\mathbf{x}' \in 2B$ such that $d_p(\mathbf{y}_{\mathbf{x}'}, \mathbf{r}) \leq K\delta$, where $\mathbf{r} = (r_1/r_0, \ldots, r_n/r_0)$. Also, by Lemma 4.2 together with (4.6), we get $d_p(\mathbf{y}_{\mathbf{x}}, \mathbf{r}) \leq 2\varepsilon$. Thus, using (4.34) we obtain that

$$(4.35) \qquad d_p(\mathbf{y}_{\mathbf{x}'}, \mathbf{r}) \leq K \frac{\psi_*}{\kappa Q_*} \qquad \text{and} \qquad d_p(\mathbf{y}_{\mathbf{x}}, \mathbf{r}) \leq 2 \frac{(\psi_*^m Q_*)^{-\frac{1}{d}}}{\kappa Q_*}.$$

We have shown the validity of (4.35) under the assumption that $\delta \leq \varepsilon$. However, note that (4.35) also holds when $\delta > \varepsilon$. Indeed, we simply set $\mathbf{x}' = \mathbf{x}$. Then (4.35) is an easy consequence of (4.6), Lemma 4.2 and the fact that $K > 2$.

By (4.9) and (4.10),

$$(4.36) \qquad |\mathbf{y}_{\mathbf{x}'}| \leq nC \quad \text{and} \qquad |\mathbf{y}_{\mathbf{x}}| \leq nC.$$

Also, by (4.31) and (4.32),

$$(4.37) \qquad |\mathbf{r}| \leq \frac{|\boldsymbol{r}|}{|r_0|} \leq \frac{(\kappa_0 + 1)Q_*}{\frac{\kappa Q_*}{2(n+1)C}} = \frac{2(n+1)(\kappa_0 + 1)C}{\kappa}.$$

Recall that the euclidean and projective distances are locally comparable; see (3.5). Then, by (4.36), (4.37) and (3.5), the l.h.s. of (4.35) implies that

$$(4.38) \qquad |\mathbf{r} - \mathbf{y}_{\mathbf{x}'}| \leq \left(\frac{2(\kappa_0 + 1)(n+1)C}{\kappa} + 1\right) (nC + 1) \, K \frac{\psi_*}{\kappa \, Q_*}$$

$$\leq \frac{3K(\kappa_0 + 1)(n+1)^2 C^2}{\kappa^2} \frac{\psi_*}{Q_*} \leq \frac{c_0}{2\kappa^2} \frac{\psi_*}{Q_*} < \frac{c_0}{\kappa^2} \frac{\psi_*}{Q_*},$$

and similarly the r.h.s. of (4.35) implies that

$$(4.39) \qquad |\mathbf{r} - \mathbf{y}_{\mathbf{x}}| < \frac{c_0}{\kappa^2} (\psi_*^m Q_*^{d+1})^{-\frac{1}{d}} = \rho.$$

Trivially, (4.39) implies that $|\mathbf{a}/q - \mathbf{x}| < \rho$, that is $\mathbf{x} \in B(\mathbf{a}/q, \rho)$ whence the l.h.s. of (4.26) holds. Also, by (4.23), $\rho \leq r_B$ and therefore $\mathbf{a}/q \in 2B$. Further,

using the triangle inequality, the Mean Value Theorem and (4.10), we get

$$|f_l(\mathbf{a}/q) - b_l/q| \le |f_l(\mathbf{a}/q) - f_l(\mathbf{x}')| + |f_l(\mathbf{x}') - b_l/q|$$

$$\le C|\mathbf{a}/q - \mathbf{x}'| + |f_l(\mathbf{x}') - b_l/q| \le C|\mathbf{r} - \mathbf{y}_{\mathbf{x}'}| \overset{(4.38)}{\le} C\,\frac{c_0}{\kappa^2}\,\frac{\psi_*}{Q_*}.$$

This implies that $|qf_l(\mathbf{a}/q) - b_l| \overset{(4.31)}{<} (\kappa_0 + 1)C\,c_0\kappa^{-2}\psi_* < c_0^3\kappa^{-2}\psi_* = \psi$. Trivially, (4.31) and (4.32) give $\delta_0 Q \le q \le Q$. Thus, $(q, \mathbf{a}, \mathbf{b}) \in \mathcal{R}^{\delta_0}(Q, \psi, 2B)$ and the r.h.s. of (4.26) is established. This completes the proof of Theorem 4.5.
□

4.3. *Uniform version of Theorem 4.5.* Within Theorem 4.5 the constant c_0 depends on B. Now restricting B to lie in a compact ball $B_0 \subset U$ gives the following version of Theorem 4.5 in which c_0 is independent of B.

THEOREM 4.6. *Let $\mathcal{M}$ be a $C^{(2)}$ submanifold given by (1.2) and let B_0 be a compact subset of U. Then there is a constant $c_0 = c_0(B_0) > 1$ such that for any choice of positive numbers ψ_*, Q_*, κ satisfying $\kappa < 1$, (4.24) and*

$$(4.40) \qquad\qquad Q_* \ge 4c_0^2\kappa^{-4}$$

for any ball $B \subset B_0$, we have that

$$(4.41) \qquad\qquad \tfrac{1}{2}B \cap \mathcal{G}_{\mathbf{f}}(Q_*, \psi_*, \kappa) \subset \Delta^{\delta_0}(Q, \psi, B, \rho)\,,$$

where $\rho := c_0\kappa^{-2}\big(\psi_*^m Q_*^{d+1}\big)^{-\frac{1}{d}}$, $\psi = c_0\kappa^{-2}\psi_*$, $Q = c_0 Q_*$ and $\delta_0 = \kappa c_0^{-1}$.

Proof. Since $B_0 \subset U$ and U is open, for every $\mathbf{x} \in B_0$ there is a ball $B_{\mathbf{x}}$ centred at $\mathbf{x}$ such that $2B_{\mathbf{x}} \subset U$. The collection of balls $\{B_{\mathbf{x}} : \mathbf{x} \in B_0\}$ is obviously a cover of B_0. Since B_0 is compact, there is a finite subcover $\mathcal{C} = \{B_1, \dots, B_t\}$. Any $B_i \in \mathcal{C}$ satisfies the conditions of Theorem 4.5. Let $c_{0,i}$ be the constant c_0 arising from Theorem 4.5 when $B = B_i$. Set

$$c_0 = \frac{\max_{1 \le i \le t} c_{0,i}^3}{\min\big\{1, \min_{1 \le i \le t} r_{B_i}\big\}}\,,$$

where r_{B_i} is the radius of B_i. Let ψ_*, Q_* and κ satisfy the conditions of Theorem 4.6. Then, by the choice of c_0 and by Theorem 4.5, it is readily seen that

$$(4.42) \qquad\qquad B_0 \cap \mathcal{G}_{\mathbf{f}}(Q_*, \psi_*, \kappa) \subset \Delta^{\delta_0}(Q, \psi, U, \rho)\,,$$

with $\rho := c_0\kappa^{-2}\big(\psi_*^m Q_*^{d+1}\big)^{-\frac{1}{d}}$, $\psi = c_0\kappa^{-2}\psi_*$, $Q = c_0 Q_*$ and $\delta_0 = \kappa c_0^{-1}$. Now, let $B \subset B_0$ be a ball. Trivially, if $\mathbf{a}/q \notin B$, then $(1 - \rho)B \cap B(\mathbf{a}/q, \rho) = \varnothing$. By (4.40), $\rho < 1/2$. Therefore, $\tfrac{1}{2}B \subset (1 - \rho)B$ and $\tfrac{1}{2}B \cap B(\mathbf{a}/q, \rho) = \varnothing$ if $\mathbf{a}/q \notin B$. Therefore, (4.41) is implied by (4.42) and the proof is complete.   □

5. **Integer points in 'random' parallelepipeds**

5.1. *Main problem and result.* By Minkowski's theorem on linear forms, any parallelepiped Π in $\mathbb{R}^k$ symmetric about the origin contains a nonzero integer point provided that the volume of Π is bigger than 2^k. The latter condition is in general best possible, though Π might contain a nonzero integer point otherwise. Suppose $\Pi(\mathbf{x})$ is a smooth family of parallelepipeds of small volume, where $\mathbf{x} \in B$, a ball in $\mathbb{R}^d$. In this section we consider the following

Problem 5.1. What is the probability that $\Pi(\mathbf{x})$ contains a nonzero integer point?

As we shall see in Section 6, answering the question of Problem 5.1 is absolutely crucial to achieving our main goal — establishing Theorem 1.4. To avoid ambiguity, the parallelepipeds $\Pi(\mathbf{x})$ will be given by the system of inequalities

$$(5.1) \qquad \left| \sum_{j=1}^{k} g_{i,j}(\mathbf{x})\, a_j \right| \leq \theta_i \quad (1 \leq i \leq k),$$

where $g_{i,j} : U \to \mathbb{R}$ are some functions of $\mathbf{x}$ defined on an open subset U of $\mathbb{R}^d$, $a_1, \ldots, a_k$ are real variables and $\bar{\theta} = (\theta_1, \ldots, \theta_k)$ is a fixed k-tuple of positive numbers. We will naturally assume that the matrix $G(\mathbf{x}) := (g_{i,j}(\mathbf{x}))_{1 \leq i,j \leq k}$ is nonsingular for every $\mathbf{x} \in U$. Thus $G : U \to \mathrm{GL}_k(\mathbb{R})$. The above family of parallelepipeds Π is therefore determined by the map G and the vector of parameters $\bar{\theta}$. Further, define the set

$$\mathcal{A}(G, \bar{\theta}) \stackrel{\text{def}}{=} \{ \mathbf{x} \in U : \exists\, \mathbf{a} = (a_1, \ldots, a_k) \in \mathbb{Z}^k \smallsetminus \{\mathbf{0}\} \text{ satisfying } (5.1)\}.$$

Problem 5.1 restated in terms of G and $\bar{\theta}$ can now be formalized as follows: *given a ball $B \subset U$, what is the probability that a random $\mathbf{x} \in B$ belongs to $B \cap \mathcal{A}(G, \bar{\theta})$?*

In this section we introduce a characteristic of G which enables us to produce an effective bound on the measure of $\mathcal{A}(G, \bar{\theta})$ for arbitrary analytic maps G. The characteristic is computable for various natural classes of G and is indeed computable for the maps G arising from the applications we have in mind.

As before let $\bar{\theta} = (\theta_1, \ldots, \theta_k)$ be the k-tuple of positive numbers and let θ be given by

$$(5.2) \qquad \theta^k = \theta_1 \cdots \theta_k.$$

Thus, θ is the geometric mean value of $\theta_1, \ldots, \theta_k$. Given $\mathbf{x} \in U$ and a linear subspace V of $\mathbb{R}^k$ with $\operatorname{codim} V = r$, $1 \leq r < k$, we define the number

$$(5.3) \quad \Theta_{\bar{\theta}}(\mathbf{x}, V) := \min \left\{ \theta^{-r} \prod_{i=1}^{r} \theta_{j_i} \; : \; \begin{array}{l} (j_1, \ldots, j_r) \in C(k, r) \text{ such that} \\ V \oplus \mathcal{V}\big(\mathbf{g}_{j_1}(\mathbf{x}), \ldots, \mathbf{g}_{j_r}(\mathbf{x})\big) = \mathbb{R}^k \end{array} \right\},$$

where $\mathcal{V}\big(\mathbf{g}_{j_1},\ldots,\mathbf{g}_{j_r}\big)$ is a vector subspace of $\mathbb{R}^k$ spanned by $\mathbf{g}_{j_1},\ldots,\mathbf{g}_{j_r}$ and $C(k,r)$ denotes the set of all subsets of $\{1,\ldots,k\}$ of cardinality r. Obviously, since $G(\mathbf{x}) \in \mathrm{GL}_k(\mathbb{R})$, the set in the r.h.s. of (5.3) is not empty and thus $\Theta_{\overline{\theta}}(\mathbf{x}, V)$ is well defined and positive. We will be interested in the local behavior of $\Theta_{\overline{\theta}}(\mathbf{x}, V)$ in a neighborhood a point $\mathbf{x}_0$ by looking at

$$(5.4) \quad \widehat{\Theta}_{\overline{\theta}}(\mathbf{x}_0, V) := \liminf_{\mathbf{x} \to \mathbf{x}_0} \Theta_{\overline{\theta}}(\mathbf{x}, V) \qquad \text{and} \qquad \widehat{\Theta}_{\overline{\theta}}(\mathbf{x}_0) := \sup_V \widehat{\Theta}_{\overline{\theta}}(\mathbf{x}_0, V),$$

where the latter supremum is taken over all linear subspaces V of $\mathbb{R}^k$ with $1 \le \operatorname{codim} V < k$. The number $\widehat{\Theta}_{\overline{\theta}}(\mathbf{x}_0)$ will be referred to as the $\overline{\theta}$-*weight of* G *at* $\mathbf{x}_0$. The following statement represents the main result of this section.

THEOREM 5.2 (Random parallelepipeds theorem). *Let U be an open subset of $\mathbb{R}^d$, $G : U \to \mathrm{GL}_k(\mathbb{R})$ be an analytic map and $\mathbf{x}_0 \in U$. Then there is a ball $B_0 \subset U$ centred at $\mathbf{x}_0$ and constants $K_0, \alpha > 0$ such that for any ball $B \subset B_0$, there is a constant $\delta = \delta(B) > 0$ such that for any k-tuple $\overline{\theta} = (\theta_1, \ldots, \theta_k)$ of positive numbers,*

$$(5.5) \qquad \mu_d\big(B \cap \mathcal{A}(G, \overline{\theta})\big) \le K_0 \left(1 + \sup_{\mathbf{x} \in B} \widehat{\Theta}_{\overline{\theta}}(\mathbf{x})^\alpha / \delta^\alpha \right) \theta^\alpha \, \mu_d(B) \,.$$

5.2. *Auxiliary statements.* We will derive Theorem 5.2 from a general result due to Kleinbock and Margulis. This will require translating the problem into the language of lattices. We proceed with further notation. Given a lattice $\Lambda \subset \mathbb{R}^k$, let $\delta(\Lambda) := \min_{\mathbf{v} \in \Lambda \smallsetminus \{0\}} |\mathbf{v}|_\infty$. Thus, δ is a map on the space of lattices. Then the set $\mathcal{A}(G, \overline{\theta})$ can be straightforwardly rewritten using this δ-map as follows:

$$\mathcal{A}(G, \overline{\theta}) := \big\{ \mathbf{x} \in U \, : \, \delta(\operatorname{diag}(\overline{\theta})^{-1} G(\mathbf{x}) \mathbb{Z}^k) \le 1 \big\},$$

where $\operatorname{diag}(\overline{\theta})$ denotes the diagonal $k \times k$ matrix with $\overline{\theta}$ on the diagonal. In order to see this simply multiply the i-th inequality of (5.1) by θ_i^{-1}. Then it is readily seen that the fact $\mathbf{x} \in \mathcal{A}(G, \overline{\theta})$ is equivalent to the existence of $\mathbf{a} \in \mathbb{Z}^k \smallsetminus \{0\}$ such that $|\operatorname{diag}(\overline{\theta})^{-1} G(\mathbf{x}) \mathbf{a}|_\infty \le 1$. The latter is obviously the same as saying that the lattice $\operatorname{diag}(\overline{\theta})^{-1} G\mathbb{Z}^k$ has a nonzero vector of norm ≤ 1, that is $\delta(\operatorname{diag}(\overline{\theta})^{-1} G(\mathbf{x}) \mathbb{Z}^k) \le 1$.

The map δ obviously satisfies the property that $\delta(x\Lambda) = x\delta(\Lambda)$ for any lattice Λ and any positive scalar x. Therefore, multiplying $\delta(\operatorname{diag}(\overline{\theta})^{-1} G(\mathbf{x}) \mathbb{Z}^k) \le 1$ through by θ (see (5.2) for the definition of θ), we get the equivalent inequality $\delta(g_{\mathbf{t}} G(\mathbf{x}) \mathbb{Z}^k) \le \theta$, where $g_{\mathbf{t}} = \operatorname{diag}\{t_1, \ldots, t_k\}$ and

$$(5.6) \qquad\qquad\qquad t_i := \theta/\theta_i \quad (1 \le i \le k) \,.$$

Note that $\det g_{\mathbf{t}} = 1$. To sum up,

$$(5.7) \qquad \mathcal{A}(G, \overline{\theta}) = \{\mathbf{x} \in U : \delta\big(h(\mathbf{x}) \mathbb{Z}^k\big) \le \theta\}, \qquad \text{where } h(\mathbf{x}) = g_{\mathbf{t}} G(\mathbf{x}).$$

As we have mentioned above the proof of Theorem 5.2 will be based on a result due to Kleinbock and Margulis. In order to state this result we recall various definitions from [41]. Let U be an open subset of $\mathbb{R}^d$, $f : U \to \mathbb{R}$ be a continuous function and let $C, \alpha > 0$. The function f is called (C, α)-*good on* U if for any open ball $B \subset U$ the following is satisfied:

$$(5.8) \qquad \forall \varepsilon > 0 \qquad \mu_d\Big\{\mathbf{x} \in B : |f(\mathbf{x})| < \varepsilon \sup_{\mathbf{x} \in B} |f(\mathbf{x})|\Big\} \;\leq\; C \varepsilon^\alpha \, \mu_d(B).$$

Given a $\lambda > 0$ and a ball $B = B(\mathbf{x}_0, r) \subset \mathbb{R}^d$ centred at $\mathbf{x}_0$ of radius r, λB will denote the ball $B(\mathbf{x}_0, \lambda r)$. Further, $\mathcal{C}(\mathbb{Z}^k)$ will denote the set of all nonzero complete sublattices of $\mathbb{Z}^k$. An integer lattice $\Lambda \subset \mathbb{Z}^k$ is called *complete* if it contains all integer points lying in the linear space generated by Λ. Given a lattice $\Lambda \subset \mathbb{R}^k$ and a basis $\mathbf{w}_1, \ldots, \mathbf{w}_r$ of Λ, the multivector $\mathbf{w}_1 \wedge \cdots \wedge \mathbf{w}_r$ is uniquely defined up to sign since any two basis of Λ are related by a unimodular transformation. Therefore, the following *height* function on the set of nonzero lattices is well defined:

$$(5.9) \qquad \qquad \|\Lambda\| \overset{\mathrm{def}}{=} |\mathbf{w}_1 \wedge \cdots \wedge \mathbf{w}_r|_\infty,$$

where $|\cdot|_\infty$ denotes the supremum norm on $\wedge(\mathbb{R}^k)$. The following result due to Kleinbock and Margulis appears as Theorem 5.2 in [41].

THEOREM KM . *Let $d, k \in \mathbb{N}$, $C, \alpha > 0$ and $0 < \rho \leq 1/k$ be given. Let B be a ball in $\mathbb{R}^d$ and $h : 3^k B \to \mathrm{GL}_k(\mathbb{R})$ be given. Assume that for any $\Lambda \in \mathcal{C}(\mathbb{Z}^k)$,*

    (i) *the function $\mathbf{x} \mapsto \|h(\mathbf{x})\Lambda\|$ is (C, α)-good on $3^k B$, and*
    (ii) $\sup_{\mathbf{x} \in B} \|h(\mathbf{x})\Lambda\| \geq \rho$.

Then there is a constant N_d depending on d only such that for any $\varepsilon > 0$ one has

$$\mu_d\Big\{\mathbf{x} \in B : \delta\big(h(\mathbf{x})\mathbb{Z}^k\big) \leq \varepsilon\Big\} \leq kC(3^d N_d)^k \left(\frac{\varepsilon}{\rho}\right)^\alpha \mu_d(B).$$

Before we proceed with the proof of Theorem 5.2, let us recall some auxiliary statements about (C, α)-good functions.

LEMMA 5.3 ([41, Lemma 3.1]). *Let $U \subset \mathbb{R}^d$ be open and $C, \alpha > 0$. If $f_1, \ldots, f_m$ are (C, α)-good functions on U and $\lambda_1, \ldots, \lambda_m \in \mathbb{R}$, then $\max_i |\lambda_i f_i|$ is a (C, α)-good function on U.*

LEMMA 5.4 ([38, Cor. 3.3]). *Let $\mathbf{f} = (f_1, \ldots, f_m)$ be a real analytic map from a connected open subset U of $\mathbb{R}^d$ to $\mathbb{R}^m$. Then for any point $\mathbf{x}_0 \in U$, there is a ball $B(\mathbf{x}_0) \subset U$ centred at $\mathbf{x}_0$ and constants $C, \alpha > 0$ such that any function $\alpha_0 + \sum_{i=1}^m \alpha_i f_i$ with $\alpha_0, \ldots, \alpha_m \in \mathbb{R}$ is (C, α)-good on $B(\mathbf{x}_0)$.*

Also, for the purpose of establishing Theorem 5.2, we now prove the following technical statement that translates the definition of $\widehat{\Theta}_{\bar{\theta}}(\mathbf{x}_0)$ into the

language of exterior algebra. Within this section we refer to Section 3 assuming that $n + 1 = k$.

LEMMA 5.5. *Let* $r \in \{1, \ldots, k-1\}$ *and* $\mathbf{x}_0 \in U$. *Then for any ball* $B \subset U$ *centred at* $\mathbf{x}_0$, *for any nonzero decomposable multivector* $\boldsymbol{v} \in \wedge^r(\mathbb{R}^k)$ *there is a* $J \in C(k, r)$ *and* $\mathbf{x} \in B$ *such that*

$$(5.10) \qquad\qquad \theta^{-r} \prod_{j \in J} \theta_j \leq \widehat{\Theta}_{\overline{\theta}}(\mathbf{x}_0)$$

and

$$(5.11) \qquad\qquad \bigwedge_{j \in J} \mathbf{g}_j(\mathbf{x}) \cdot \boldsymbol{v} \neq \mathbf{0}.$$

Proof. Let $\boldsymbol{v} \in \wedge^r(\mathbb{R}^k)$ be a decomposable multivector with $1 \leq r < k$. Define $V = \mathcal{V}(\boldsymbol{v}^{\perp})$, a vector subspace of $\mathbb{R}^k$. By Lemma 3.1, codim $V = r$. Observe that for a fixed $\overline{\theta}$, the function $\Theta_{\overline{\theta}}(\mathbf{x}, V)$ of $\mathbf{x}$ takes discrete values. Then, using (5.4) it is easy to see that for any ball $B \subset U$ centred at $\mathbf{x}_0$, there is an $\mathbf{x} \in B$ such that $\Theta_{\overline{\theta}}(\mathbf{x}, V) \leq \widehat{\Theta}_{\overline{\theta}}(\mathbf{x}_0)$. By the definition of $\Theta_{\overline{\theta}}(\mathbf{x}, V)$, there is a $J = \{j_1, \ldots, j_r\} \in C(k, r)$ satisfying (5.10) such that $V \oplus \mathcal{V}\big(\mathbf{g}_{j_1}(\mathbf{x}), \ldots, \mathbf{g}_{j_r}(\mathbf{x})\big) = \mathbb{R}^k$, that is since $V = \mathcal{V}(\boldsymbol{v}^{\perp})$, $\mathcal{V}(\boldsymbol{v}^{\perp}) \oplus \mathcal{V}\big(\mathbf{g}_{j_1}(\mathbf{x}), \ldots, \mathbf{g}_{j_r}(\mathbf{x})\big) = \mathbb{R}^k$, whence, by Lemma 3.3, $\boldsymbol{v}^{\perp} \wedge \big(\wedge_{j \in J} \mathbf{g}_j(\mathbf{x})\big) \neq \mathbf{0}$. Finally,

$$\mathbf{0} \neq |\boldsymbol{v}^{\perp} \wedge \big(\bigwedge_{j \in J} \mathbf{g}_j(\mathbf{x})\big)| \stackrel{(3.10)}{=} |(\boldsymbol{v}^{\perp})^{\perp} \cdot \bigwedge_{j \in J} \mathbf{g}_j(\mathbf{x})| \stackrel{(3.8)}{=} |\boldsymbol{v} \cdot \bigwedge_{j \in J} \mathbf{g}_j(\mathbf{x})|. \qquad \square$$

5.3. *Proof of Theorem* 5.2. By (5.2) and (5.6), we obviously have that $\prod_{i=1}^{k} t_i = 1$. Therefore, $\det g_{\mathbf{t}} = 1$ and

$$(5.12) \qquad\qquad \det G(\mathbf{x}) = \det h(\mathbf{x}),$$

where h is given by (5.7). Therefore, $h(\mathbf{x})$ is a map from U to $\mathrm{GL}_k(\mathbb{R})$.

Our next goal is to verify conditions (i) and (ii) of Theorem KM for the specific choice of h made by (5.7). Fix a $\Gamma \in \mathcal{C}(\mathbb{Z}^k)$. Let $r = \dim \Gamma > 0$. Fix a basis of Γ, say $\mathbf{w}_1, \ldots, \mathbf{w}_r \in \mathbb{Z}^k$. Then $h(\mathbf{x})\mathbf{w}_1, \ldots, h(\mathbf{x})\mathbf{w}_r$ is a basis of the lattice $h(\mathbf{x})\Gamma$. By definition (5.9),

$$\|h(\mathbf{x})\Gamma\| = |h(\mathbf{x})\mathbf{w}_1 \wedge \cdots \wedge h(\mathbf{x})\mathbf{w}_r|_{\infty}.$$

Given an $l \in \{1, \ldots, r\}$, it is readily seen that the coordinates of $h(\mathbf{x})\mathbf{w}_l$ are equal to $t_i \mathbf{g}_i(\mathbf{x})\mathbf{w}_l$, $i = \overline{1, k}$. Therefore, by (3.2), for every $I = \{i_1 < \cdots < i_r\} \subset \{1, \ldots, k\}$, the I-coordinate of

$$(5.13) \qquad\qquad h(\mathbf{x})\mathbf{w}_1 \wedge \cdots \wedge h(\mathbf{x})\mathbf{w}_r \in \wedge^r(\mathbb{R}^k)$$

in the standard basis equals

$$(5.14) \qquad \det\left(t_{i_j}\mathbf{g}_{i_j}(\mathbf{x})\mathbf{w}_l\right)_{1\le j,l\le r} = \left(\prod_{j=1}^{r}t_{i_j}\right)\det\left(\mathbf{g}_{i_j}(\mathbf{x})\mathbf{w}_l\right)_{1\le j,l\le r}$$

$$\overset{(3.3)}{=}\left(\prod_{j=1}^{r}t_{i_j}\right)\left(\wedge_{j=1}^{r}\mathbf{g}_{i_j}(\mathbf{x})\right)\cdot\left(\wedge_{l=1}^{r}\mathbf{w}_l\right).$$

Since G is analytic, the coordinate functions of $\wedge_{j=1}^{r}\mathbf{g}_{i_j}(\mathbf{x})$ are analytic. Let $f_1,\ldots,f_M$ be the collection of these functions taken over all possible choices of r and I. Note that this is a finite collection of analytic functions. Obviously, (5.14) is a linear combination of $f_1,\ldots,f_M$. By Lemma 5.4, there is a ball B_0 centred at $\mathbf{x}_0$ and positive C and α such that (5.14) (regarded as a function of $\mathbf{x}$) is (C,α)-good on $3^k B_0$ for any choice of r and I. If B_0 is sufficiently small then, by the continuity of G, we can also ensure the conditions

$$(5.15) \qquad |\det G(\mathbf{x})| \ge \frac{1}{2}|\det G(\mathbf{x}_0)| \qquad \text{for all } \mathbf{x}\in B_0$$

and

$$(5.16) \qquad \max_{1\le j\le k}\sup_{\mathbf{x}\in B_0}\left|\mathbf{g}_j(\mathbf{x})\right| < \infty.$$

Take any ball $B\subset B_0$. Since every coordinate function of $h(\mathbf{x})\Gamma$ is (C,α)-good on $3^k B$, by Lemma 5.3, the map $\mathbf{x}\mapsto\|h(\mathbf{x})\Gamma\|$ is (C,α)-good on $3^k B$. This verifies condition (i) of Theorem KM. We proceed with establishing condition (ii). This splits into two cases.

Case $r < k$. Let $C'(k,r)$ be the subset of $C(k,r)$ consisting of $I = \{i_1 < \cdots < i_r\}\subset\{1,\ldots,k\}$ such that

$$(5.17) \qquad \theta^{-r}\prod_{j=1}^{r}\theta_{i_j} \le \widehat{\Theta} := \sup_{\mathbf{x}\in B}\widehat{\Theta}_{\overline{\theta}}(\mathbf{x}).$$

It is readily seen that $C'(r,k)$ is nonempty. By (5.14) and (5.17), for any $I\in C'(r,k)$ we get that

$$\|h(\mathbf{x})\Gamma\| \ge \left(\prod_{i\in I}t_i\right)\left|\left(\wedge_{i\in I}\mathbf{g}_i(\mathbf{x})\right)\cdot\left(\wedge_{l=1}^{r}\mathbf{w}_l\right)\right| \overset{(5.17)}{\ge} \frac{1}{\widehat{\Theta}}\left|\left(\wedge_{i\in I}\mathbf{g}_i(\mathbf{x})\right)\cdot\left(\wedge_{l=1}^{r}\mathbf{w}_l\right)\right|.$$

Taking the supremum over all $\mathbf{x}\in B$ and then taking the maximum over all $I\in C'(r,k)$ gives

$$(5.18) \qquad \sup_{\mathbf{x}\in B}\|h(\mathbf{x})\Gamma\| \ge \frac{1}{\widehat{\Theta}}\max_{I\in C'(r,k)}\sup_{\mathbf{x}\in B}\left|\left(\bigwedge_{i\in I}\mathbf{g}_i(\mathbf{x})\right)\cdot\left(\bigwedge_{l=1}^{r}\mathbf{w}_l\right)\right|.$$

Now, since $\mathbf{w}_l$ are integer points, $\wedge_{l=1}^{r}\mathbf{w}_l$ has integer coordinates. Because $\mathbf{w}_1,\ldots,\mathbf{w}_r$ are linearly independent, $\wedge_{l=1}^{r}\mathbf{w}_l$ is nonzero and therefore $|\wedge_{l=1}^{r}\mathbf{w}_l|$

≥ 1. Dividing the r.h.s. of (5.18) by $|\wedge_{l=1}^r \mathbf{w}_l|$ gives

$$(5.19) \qquad \sup_{\mathbf{x} \in B} \|h(\mathbf{x})\Gamma\| \geq \frac{1}{\widehat{\Theta}} \max_{I \in C'(r,k)} \sup_{\mathbf{x} \in B} \left| \left(\bigwedge_{i \in I} \mathbf{g}_i(\mathbf{x}) \right) \cdot \frac{\mathbf{w}_1 \wedge \cdots \wedge \mathbf{w}_r}{|\mathbf{w}_1 \wedge \cdots \wedge \mathbf{w}_r|} \right|.$$

The multivector $\boldsymbol{u} = |\mathbf{w}_1 \wedge \cdots \wedge \mathbf{w}_r|^{-1} \mathbf{w}_1 \wedge \cdots \wedge \mathbf{w}_r$ is unit and decomposable. Thus, taking the infimum in (5.19) over all $\boldsymbol{u} \in \mathrm{Gr}_r(\mathbb{R}^k)$, that is over all unit decomposable r-vectors $\boldsymbol{u}$ taken up to sign, gives

$$(5.20) \qquad \sup_{\mathbf{x} \in B} \|h(\mathbf{x})\Gamma\| \geq \frac{1}{\widehat{\Theta}} \inf_{\boldsymbol{u} \in \mathrm{Gr}_r(\mathbb{R}^k)} \max_{I \in C'(r,k)} \sup_{\mathbf{x} \in B} \left| \left(\bigwedge_{i \in I} \mathbf{g}_i(\mathbf{x}) \right) \cdot \boldsymbol{u} \right|.$$

Our next goal is to show that the constant in the r.h.s. of (5.20) is positive. To this end, consider the following functions of $\mathbf{x} \in B$ and $\boldsymbol{u} \in \wedge^r(\mathbb{R}^k)$ given by

$$(5.21) \ \ M_{r,I}(\boldsymbol{u}, \mathbf{x}) = \left| \left(\bigwedge_{i \in I} \mathbf{g}_i(\mathbf{x}) \right) \cdot \boldsymbol{u} \right| \qquad \text{and} \qquad M_{B,r,I}(\boldsymbol{u}) = \sup_{\mathbf{x} \in B} M_{r,I}(\boldsymbol{u}, \mathbf{x}).$$

For every fixed $\mathbf{x}$, the function $M_{r,I}(\boldsymbol{u}, \mathbf{x})$ is the absolute value of a function linear in $\boldsymbol{u}$. Therefore, using (5.16) one readily gets that $M_{r,I}(\boldsymbol{u}, \mathbf{x})$ is uniformly continuous in $\boldsymbol{u}$. Henceforth, $M_{B,r,I}(\boldsymbol{u})$ is continuous. To prove this formally fix any $\boldsymbol{u}_0 \in \wedge^r(\mathbb{R}^k)$ and any $\varepsilon > 0$. Then there is an $\eta > 0$ such that for all $\boldsymbol{u} \in \wedge^r(\mathbb{R}^k)$ satisfying $|\boldsymbol{u} - \boldsymbol{u}_0| < \eta$,

$$(5.22) \qquad |M_{r,I}(\boldsymbol{u}, \mathbf{x}) - M_{r,I}(\boldsymbol{u}_0, \mathbf{x})| < \varepsilon/2 \qquad \text{for all } \mathbf{x} \in B.$$

By definition (5.21), there is $\mathbf{x}_0 \in B$ such that $M_{B,r,I}(\boldsymbol{u}_0) < M_{r,I}(\boldsymbol{u}_0, \mathbf{x}_0) + \varepsilon/2$. Therefore,

$$M_{B,r,I}(\boldsymbol{u}_0) < M_{r,I}(\boldsymbol{u}_0, \mathbf{x}_0) + \varepsilon/2 \overset{(5.22)}{\leq} M_{r,I}(\boldsymbol{u}, \mathbf{x}_0) + \varepsilon \overset{(5.21)}{\leq} M_{B,r,I}(\boldsymbol{u}) + \varepsilon.$$

Similarly we show the complementary inequality, namely that $M_{B,r,I}(\boldsymbol{u}_0) > M_{B,r,I}(\boldsymbol{u}) - \varepsilon$. Therefore, $|M_{B,r,I}(\boldsymbol{u}) - M_{B,r,I}(\boldsymbol{u}_0)| < \varepsilon$ for all $\boldsymbol{u}$ satisfying $|\boldsymbol{u} - \boldsymbol{u}_0| < \eta$. This proves the continuity of $M_{B,r,I}(\boldsymbol{u})$. Further, define

$$M_{B,r}(\boldsymbol{u}) := \max_{I \in C'(r,k)} M_{B,r,I}(\boldsymbol{u}).$$

This is also a continuous function of $\boldsymbol{u}$ as the maximum of a finite number of continuous functions. By Lemma 5.5 and the definition of $M_{B,r}(\boldsymbol{u})$, $M_{B,r}(\boldsymbol{u}) > 0$ for all decomposable multivectors $\boldsymbol{u} \in \wedge^r(\mathbb{R}^k)$. Since the Grassmannian $\mathrm{Gr}_r(\mathbb{R}^k)$ is compact and $M_{B,r}(\boldsymbol{u})$ is continuous, there is a $\boldsymbol{u}_{B,r} \in \mathrm{Gr}_r(\mathbb{R}^k)$ such that

$$\inf_{\boldsymbol{u} \in \mathrm{Gr}_r(\mathbb{R}^k)} M_{B,r}(\boldsymbol{u}) = M_{B,r}(\boldsymbol{u}_{B,r}) > 0.$$

Thus, (5.20) implies that

$$\sup_{\mathbf{x} \in B} \|h(\mathbf{x})\Gamma\| \geq \frac{1}{\widehat{\Theta}} M_{B,r}(\boldsymbol{u}_{B,r}) \geq \frac{1}{\widehat{\Theta}} M_B$$

for any $\Gamma \in \mathcal{C}(\mathbb{Z}^k)$ with $\dim \Gamma < k$, where $M_B = \min\limits_{1 \le r < k} M_{B,r}(\boldsymbol{u}_{B,r}) > 0$.

Case $r = k$. Now we assume that $\dim \Gamma = k$. Since Γ is complete, $\Gamma = \mathbb{Z}^k$ and therefore the standard basis of $\mathbb{R}^k$, say $\mathbf{e}_1, \ldots, \mathbf{e}_k$, is also a basis of Γ. Therefore, (5.13) is exactly $\pm \det h(\mathbf{x})$. Further,

$$\sup_{\mathbf{x} \in B} \|h(\mathbf{x})\Gamma\| = \sup_{\mathbf{x} \in B} |\det h(\mathbf{x})| \overset{(5.12)}{=} \sup_{\mathbf{x} \in B} |\det G(\mathbf{x})| \overset{(5.15)}{\ge} \frac{1}{2}|\det G(\mathbf{x}_0)| > 0.$$

Final step. The upshot of the above discussion is that for any $\Gamma \in \mathcal{C}(\mathbb{Z}^k)$,

$$(5.23) \qquad \sup_{\mathbf{x} \in B} \|h(\mathbf{x})\Gamma\| \ge \min\left\{\frac{1}{k}, \frac{1}{2}|\det G(\mathbf{x}_0)|, \frac{M_B}{\widehat{\Theta}}\right\} = \rho > 0.$$

This verifies condition (ii) of Theorem KM. Further, using the trivial inequality $\min\{|x|, |y|, |z|\}^{-1} \le |x|^{-1} + |y|^{-1} + |z|^{-1}$ we deduce from (5.23) that

$$(5.24) \qquad \rho^{-\alpha} \le k^\alpha + 2^\alpha |\det G(\mathbf{x}_0)|^{-\alpha} + \left(\frac{\widehat{\Theta}}{M_B}\right)^\alpha$$

$$\le (2k)^\alpha + (2k)^\alpha |\det G(\mathbf{x}_0)|^{-\alpha} + \left(\frac{\widehat{\Theta}}{M_B}\right)^\alpha$$

$$= (2k)^\alpha \left(1 + |\det G(\mathbf{x}_0)|^{-\alpha}\right)\left(1 + \left(\frac{\widehat{\Theta}}{\delta}\right)^\alpha\right),$$

where $\delta = \delta(B)$ is implied by (5.24). By (5.7) and Theorem KM (with $\varepsilon = \theta$), we now obtain (5.5) with $K_0 = (2k)^\alpha k C (3^d N_d)^k \left(1 + |\det G(\mathbf{x}_0)|^{-\alpha}\right)$. Obviously, K_0 is independent of B. The proof of Theorem 5.2 is thus complete.

5.4. *Hierarchic families of parallelepipeds.* It is in general possible but not straightforward to give bounds on the $\bar{\theta}$-weight of G. In this subsection we introduce a condition on G that enables us to give a clear-cut estimate for $\widehat{\Theta}_{\bar{\theta}}(\mathbf{x})$ and produce an interesting corollary of Theorem 5.2. Let B be a ball in U. We will say that G is *hierarchic on B* if for any vector subspace V of $\mathbb{R}^k$ of $\operatorname{codim} V = r$, the set

$$(5.25) \qquad \left\{\mathbf{x} \in B \,:\, V \oplus \mathcal{V}\big(\mathbf{g}_1(\mathbf{x}), \ldots, \mathbf{g}_r(\mathbf{x})\big) = \mathbb{R}^k\right\}$$

is dense in B.

LEMMA 5.6. *If $G : U \to \operatorname{GL}_k(\mathbb{R})$ is hierarchic on a ball $B_0 \subset U$, then for any k-tuple $\bar{\theta} = (\theta_1, \ldots, \theta_k)$ of positive numbers and any $\mathbf{x}_0 \in U$,*

$$(5.26) \qquad \widehat{\Theta}_{\bar{\theta}}(\mathbf{x}_0) \le \widetilde{\Theta} := \max_{1 \le r \le k-1} \frac{\theta_1 \cdots \theta_r}{\theta^r}.$$

Proof. Fix any $\mathbf{x}_0 \in U$. In order to prove (5.26) it suffices to show that $\widehat{\Theta}_{\bar{\theta}}(\mathbf{x}_0, V) \le \widetilde{\Theta}$ for every subspace V of $\mathbb{R}^k$ with $\operatorname{codim} V = r \in \{1, \ldots, k-1\}$. Since the set (5.25) is dense in U, $\theta^{-r} \prod_{j=1}^r \theta_j$ belongs to the set in the r.h.s. of (5.3) for points $\mathbf{x} \in U$ arbitrarily close to $\mathbf{x}_0$. This means that $\Theta(\mathbf{x}, V) \le$

$\theta^{-r} \prod_{j=1}^{r} \theta_j \leq \widetilde{\Theta}$ for points $\mathbf{x}$ arbitrarily close to $\mathbf{x}_0$. Therefore, by (5.4), $\widehat{\Theta}_{\bar{\theta}}(\mathbf{x}_0, V) \leq \widetilde{\Theta}$ and the proof is complete. □

The following example of hierarchic maps will be utilized to sharpen Theorem 1.4 is Section 7.

LEMMA 5.7. *Let* $G = \left(g_j^{(i-1)} \right)_{1 \leq i,j \leq k} : U \to \mathrm{GL}_k(\mathbb{R})$ *be the Wronski matrix of analytic linearly independent over* $\mathbb{R}$ *functions* $g_1, \ldots, g_k : U \to \mathbb{R}$ *defined on an interval* U *in* $\mathbb{R}$. *Then* G *is hierarchic on* U.

Proof. Recall the well-known fact that r analytic functions of a real variable are linearly dependent if and only if their Wronskian is identically zero; see, for example, [20]. Let $\mathbf{g} = (g_1, \ldots, g_k)$. Take any nontrivial vector subspace V of $\mathbb{R}^k$ with codim $V = r \leq k-1$. We will verify that the set (5.25) is dense in U by showing that its complement is countable. Let $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_r$ be a basis of $V^{\perp}$. Define $\boldsymbol{v} := \boldsymbol{v}_1 \wedge \cdots \wedge \boldsymbol{v}_r$. Then, by Lemma 3.2, $V = \mathcal{V}(\boldsymbol{v}^{\perp})$.

Let $S(V)$ denote the complement of the set (5.25). Obviously, the point x belongs to $S(V)$ if and only if $V \cap \mathcal{V}(\mathbf{g}(x) \wedge \cdots \wedge \mathbf{g}^{(r)}(x)) \neq \varnothing$. By Lemma 3.3, this is equivalent to $\boldsymbol{v}^{\perp} \wedge (\mathbf{g}(x) \wedge \cdots \wedge \mathbf{g}^{(r)}(x)) = 0$ and, by (3.10) and the fact that $\boldsymbol{v} = \boldsymbol{v}_1 \wedge \cdots \wedge \boldsymbol{v}_r$, this further gives

$$(5.27) \qquad (\boldsymbol{v}_1 \wedge \cdots \wedge \boldsymbol{v}_r) \cdot (\mathbf{g}(x) \wedge \cdots \wedge \mathbf{g}^{(r)}(x)) = 0.$$

By the Laplace identity (3.3), the latter is exactly the Wronskian of the functions $\eta_i(x) = \mathbf{g}(x) \cdot \boldsymbol{v}_i$. Since $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_r$ are linearly independent vectors, the functions $\eta_1, \ldots, \eta_r$ $(1 \leq i \leq r)$ are linearly independent over $\mathbb{R}$. Therefore, the Wronskian of $\eta_1, \ldots, \eta_r$ is not identically zero and, as an analytic function, it can vanish only on a countable subset of U. Therefore, the set $S(V)$ is at most countable and the proof is complete. □

In view of Lemmas 5.6 and 5.7, specializing Theorem 5.2 to the Wronski matrix gives

THEOREM 5.8. *Let* $g_1, \ldots, g_k$ *be a collection of real analytic linearly independent over* $\mathbb{R}$ *functions defined on an interval* $U \subset \mathbb{R}$. *Let* $x_0 \in U$ *be a point such that the Wronskian* $W(g_1, \ldots, g_k)(x_0) \neq 0$. *Then there is an interval* I_0 *centred at* x_0 *and positive constants* K_0 *and* α *satisfying the following property. For any interval* $J \subset I_0$, *there is a constant* $\delta = \delta(J)$ *such that for any positive* $\theta_1, \ldots, \theta_k$, *the set*

$$\left\{ x \in J : \begin{array}{l} \exists \, (a_1, \ldots, a_k) \in \mathbb{Z}^k \smallsetminus \{0\} \quad satisfying \\ |a_1 g_1^{(i)}(x) + \cdots + a_k g_k^{(i)}(x)| < \theta_i \quad \forall \, i = 1, \ldots, k \end{array} \right\}$$

has Lebesgue measure at most $K_0 \big(1 + (\delta^{-1} \widetilde{\Theta})^{\alpha} \big) \theta^{\alpha} |J|$, *where* $|J|$ *is the length of* J,

$$\theta = (\theta_1 \cdots \theta_k)^{1/k} \qquad and \qquad \widetilde{\Theta} := \max_{1 \leq r \leq k-1} \frac{\theta_1 \cdots \theta_r}{\theta^r}.$$

The following even more explicit estimate for $\widetilde{\Theta}$ is now given.

LEMMA 5.9. *Let* $\theta_1 \leq \theta_2 \leq \cdots \leq \theta_{k-1} \leq \theta_k$. *Then* $\widetilde{\Theta} \leq (\theta_{k-1}/\theta_k)^{1/k} \leq 1$.

Proof. By definition, there is an $r < k$ such that $\widetilde{\Theta} = \theta_1 \cdots \theta_r/\theta^r$. Raise the latter equation to the power k and substitute $\theta_1 \cdots \theta_k$ for θ^k. This way we obtain

$$\widetilde{\Theta}^k = \frac{\theta_1^k \cdots \theta_r^k}{\theta_1^r \cdots \theta_k^r} = \frac{\overbrace{\theta_1 \cdots \theta_1}^{k} \cdot \ldots \cdot \overbrace{\theta_r \cdots \theta_r}^{k}}{\underbrace{\theta_1 \cdots \theta_1}_{r} \cdot \ldots \cdot \underbrace{\theta_k \cdots \theta_k}_{r}} .$$

Obviously the numerator and the denominator of the above fraction have the same number of factors. Also, by the conditions of the lemma, any factor in the numerator is not bigger than the corresponding factor in the denominator in the same place. This gives that $\widetilde{\Theta}^k \leq \theta_r/\theta_k$. Furthermore, since $r < k$, $\theta_r \leq \theta_{k-1}$ and so $\widetilde{\Theta}^k \leq \theta_{k-1}/\theta_k$, whence the lemma readily follows. $\square$

6. The proof of main result: Theorem 1.4

6.1. *Localisation and outline of the proof.* Using standard covering arguments we establish the following lemma, which allows us to impose a convenient condition on B_0 while establishing Theorem 1.4.

LEMMA 6.1. *Let* $\mathcal{C}$ *be a collection of nonempty compact balls contained in* U *such that* $U = \bigcup_{B_0 \in \mathcal{C}} \frac{1}{2} B_0^\circ$, *where* B_0° *denotes the interior of* B_0. *Then the validity of the statement of Theorem* 1.4 *for all* $B_0 \in \mathcal{C}$ *implies the validity of the statement of Theorem* 1.4 *for arbitrary compact ball* B_0 *in* U.

Proof. Fix an arbitrary compact ball $B_0 \subset U$. Since $\{\frac{1}{2}B^\circ : B \in \mathcal{C}\}$ is an open cover of B_0, there is a finite subcollection of $\mathcal{C}$, say $\mathcal{C}_0 = \{B_{0,1}, \ldots, B_{0,N}\}$, such that

(6.1) $$B_0 \subset \bigcup_{i=1}^{N} \tfrac{1}{2} B_{0,i}.$$

We may assume that every ball in this subcollection is of positive radius. For $i = 1, \ldots, N$, let $k_{0,i}$, $\rho_{0,i}$ and $\delta_{0,i}$ be the constants k_0, ρ_0 and δ_0 arising from Theorem 1.4 when $B_0 = B_{0,i}$. Also let r_0 be the radius of the smallest ball in $\mathcal{C}_0$. Clearly, r_0 is positive. Define

$$\rho_0 = \max_{1 \leq i \leq N} \rho_{0,i}, \quad \delta_0 = \min_{1 \leq i \leq N} \delta_{0,i}, \quad k_0 = \min_{1 \leq i \leq N} k_{0,i},$$

and take any ball $B \subset B_0$. Note that verifying (1.4) for some suitable choice of C_0 and Q_0 would complete the proof of Lemma 6.1. This splits into two cases.

Case (i). Assume that $r(B)$, the radius of B, satisfies $r(B) \leq \frac{1}{4}r_0$. By (6.1) and the inclusion $B \subset B_0$, there is a $B_{0,i} \in \mathcal{C}_0$ such that $\frac{1}{2}B_{0,i} \cap B \neq \varnothing$. Then, since $r(B_{0,i}) \geq r_0$ and $r(B) \leq \frac{1}{4}r_0$, $B \subset B_{0,i}$ and the validity of (1.4) becomes obvious.

Case (ii). Assume that $r(B) > \frac{1}{4}r_0$. In this case the idea is to pack B with sufficiently many disjoint balls of radius $\leq \frac{1}{4}r_0$ and apply Case (i) to each of these balls. The formal procedure is as follows.

Let $\mathcal{C}' = \{B_1, \ldots, B_M\}$ be a maximal collection of pairwise disjoint balls centred in $\frac{1}{2}B$ of common radius $r(B_i) = \frac{1}{8}r_0$. The existence of $\mathcal{C}'$ is readily seen. Obviously $\mathcal{C}'$ is nonempty and, by construction, any ball $B_i \in \mathcal{C}'$ is contained in B. Let $\mathbf{x} \in \frac{1}{2}B$. By the maximality of $\mathcal{C}'$, the ball $B(\mathbf{x}, \frac{1}{8}r_0)$ may not be pairwise disjoint with all the balls in $\mathcal{C}'$. Therefore, $\mathbf{x} \in 2B_i$ for some $B_i \in \mathcal{C}'$. It follows that $\frac{1}{2}B \subset \bigcup_{i=1}^{M} 2B_i$. Hence,

$$(6.2) \qquad 2^{-d}\mu_d(B) = \mu_d(\tfrac{1}{2}B) \leq \sum_{i=1}^{M} \mu_d(2B_i) = 2^d \sum_{i=1}^{M} \mu_d(B_i).$$

Since every $B_i \in \mathcal{C}'$ is of radius $< \frac{1}{4}r_0$, we are within Case (i). This means that there exist constants $C_{0,i} > 0$ and $Q_{0,i} > 0$ such that for all $Q \geq Q_{0,i}$ and all ψ satisfying the inequalities $C_{0,i}Q^{-1/m} < \psi < C_{0,i}^{-1}$,

$$(6.3) \qquad \mu_d\big(\Delta^{\delta_0}(Q, \psi, B_i, \rho) \cap B_i\big) \geq k_0\, \mu_d(B_i).$$

Now define $C_0 = \max_{1 \leq i \leq M} C_{0,i}$, $Q_0 = \max_{1 \leq i \leq M} Q_{0,i}$. Then (6.3) holds whenever (1.3) is satisfied and $Q > Q_0$. Using the disjointness of balls in $\mathcal{C}'$ and the fact that $\bigcup_{i=1}^{M} B_i \subset B$ we get from (6.3) that

$$\mu_d\big(\Delta^{\delta_0}(Q, \psi, B, \rho) \cap B\big) \geq \sum_{i=1}^{M} \mu_d\big(\Delta^{\delta_0}(Q, \psi, B_i, \rho) \cap B_i\big)$$

$$\overset{(6.3)}{\geq} \sum_{i=1}^{M} k_0\, \mu_d(B_i) \overset{(6.2)}{\geq} 4^{-d} k_0\, \mu_d(B).$$

This shows (1.4) with k_0 replaced by $4^{-d}k_0$ and thus completes the proof. $\qquad\square$

Outline of the proof of Theorem 1.4. Recall that $\mathcal{M}$ is a nondegenerate analytic submanifold of $\mathbb{R}^n$ given by (1.2) and let B_0 be a compact ball in U. By Lemma 6.1, B_0 is assumed to be a sufficiently small ball. The proof contains the following three steps.

(i) Firstly, to establish (1.4) take any ball B in B_0. In view of Theorem 4.6, namely inclusion (4.41), (1.4) follows on showing that for sufficiently large Q_*,

$$(6.4) \qquad \mu_d(\tfrac{1}{2}B \cap \mathcal{G}(Q_*, \psi_*, \kappa)) \gg \mu_d(B).$$

(ii) In order to establish (6.4), for each $\mathbf{x} \in B_0$ we circumscribe a parallelepiped (5.1) around the body (4.22). This way the complement

of $\mathcal{G}_{\mathbf{f}}(Q_*, \psi_*, \kappa)$ becomes embedded into the set $\mathcal{A}(G, \bar{\theta})$ appearing in Theorem 5.2, thus giving

$$(6.5) \qquad \tfrac{1}{2}B \setminus \mathcal{A}(G, \bar{\theta}) \subset \mathcal{G}_{\mathbf{f}}(Q_*, \psi_*, \kappa) \cap \tfrac{1}{2}B.$$

(iii) On applying Theorem 5.2 we will obtain that $\mu_d(\tfrac{1}{2}B \cap \mathcal{A}(G, \bar{\theta})) \leq \tfrac{1}{2}\mu_d(\tfrac{1}{2}B)$. In view of the embedding (6.5) it will further imply (6.4) and complete the task.

We now proceed with the details of the proof.

6.2. *G and $\bar{\theta}$.* Let $\boldsymbol{g}, \boldsymbol{u}, \boldsymbol{y}$ be given by (4.1)–(4.3). Since $\mathcal{M}$ is analytic, $\boldsymbol{y}$ is analytic. Further, the coordinate functions of $\boldsymbol{g}$ and $\boldsymbol{u}$ are obviously polynomials of analytic functions and thus are analytic.

LEMMA 6.2. *Let $\boldsymbol{g}, \boldsymbol{u}, \boldsymbol{y}$ be as above. Then for every point $\mathbf{x}_0 \in U$ there is a ball $B_0 \subset U$ centred at $\mathbf{x}_0$ and an analytic map $G : B_0 \to \mathrm{GL}_{n+1}(\mathbb{R}^{n+1})$ with rows $\mathbf{g}_1, \ldots, \mathbf{g}_{n+1}$ such that for every $\mathbf{x} \in B_0$,*

$$(6.6) \qquad |\mathbf{g}_i(\mathbf{x})| \leq 1 \quad \text{for all } i = 1, \ldots, n+1$$

and

$$(6.7) \qquad \mathbf{g}_i(\mathbf{x}) \in \mathcal{V}\big(\boldsymbol{g}(\mathbf{x})\big) \quad \text{for} \quad i = 1, \ldots, m,$$

$$\mathbf{g}_i(\mathbf{x}) \in \mathcal{V}\big(\boldsymbol{u}(\mathbf{x})\big) \quad \text{for} \quad i = m+1, \ldots, n,$$

$$\mathbf{g}_i(\mathbf{x}) \in \mathcal{V}\big(\boldsymbol{y}(\mathbf{x})\big) \quad \text{for} \quad i = n+1.$$

Proof. Fix any basis $\mathbf{g}_1(\mathbf{x}_0), \ldots, \mathbf{g}_{n+1}(\mathbf{x}_0)$ of $\mathbb{R}^{n+1}$ with $|\mathbf{g}_i(\mathbf{x}_0)| \leq 1/2$ for all $i = 1, \ldots, n+1$ such that $(6.7)_{\mathbf{x}=\mathbf{x}_0}$ is satisfied. Define

$$(6.8) \qquad \mathbf{g}_i(\mathbf{x}) := \frac{1}{|\boldsymbol{g}(\mathbf{x})|^2} \, \boldsymbol{g}(\mathbf{x}) \cdot \big(\boldsymbol{g}(\mathbf{x}) \cdot \mathbf{g}_i(\mathbf{x}_0)\big) \qquad \text{for } i = 1, \ldots, m,$$

$$\mathbf{g}_i(\mathbf{x}) := \frac{1}{|\boldsymbol{u}(\mathbf{x})|^2} \, \boldsymbol{u}(\mathbf{x}) \cdot \big(\boldsymbol{u}(\mathbf{x}) \cdot \mathbf{g}_i(\mathbf{x}_0)\big) \qquad \text{for } i = m+1, \ldots, n,$$

$$\mathbf{g}_i(\mathbf{x}) := \frac{1}{|\boldsymbol{y}(\mathbf{x})|^2} \, \boldsymbol{y}(\mathbf{x}) \cdot \big(\boldsymbol{y}(\mathbf{x}) \cdot \mathbf{g}_i(\mathbf{x}_0)\big) \qquad \text{for } i = n+1.$$

By Lemma 3.5, $\mathbf{g}_i(\mathbf{x})$ is (up to sign) the orthogonal projection of $\mathbf{g}_i(\mathbf{x}_0)$ onto $\mathcal{V}(\boldsymbol{g}(\mathbf{x}))$ for $i \in \{1, \ldots, m\}$, onto $\mathcal{V}(\boldsymbol{u}(\mathbf{x}))$ for $i \in \{m+1, \ldots, n\}$ and $\mathcal{V}(\boldsymbol{y}(\mathbf{x}))$ for $i = n+1$. Obviously, the maps $\mathbf{g}_i$ given by (6.8) are well defined and analytic. Also, by continuity,

$$(6.9) \qquad \bigwedge_{i=1}^{n+1} \mathbf{g}_i(\mathbf{x}) \to \pm \bigwedge_{i=1}^{n+1} \mathbf{g}_i(\mathbf{x}_0) \qquad \text{as} \quad \mathbf{x} \to \mathbf{x}_0.$$

Since $\mathbf{g}_1(\mathbf{x}_0), \ldots, \mathbf{g}_{n+1}(\mathbf{x}_0)$ are linearly independent, the r.h.s. of (6.9) is non-zero. Therefore, there is a neighborhood B_0 of $\mathbf{x}_0$ such that for all $\mathbf{x} \in B_0$, the l.h.s. of (6.9) is nonzero. This proves that G is nonsingular. In view of the continuity of $\mathbf{g}_i$ and the condition $|\mathbf{g}_i(\mathbf{x}_0)| \leq 1/2$, we have $|\mathbf{g}_i(\mathbf{x})| \leq 1$ for all $i = 1, \ldots, n+1$ provided that B_0 is small enough. $\qquad\square$

LEMMA 6.3. *Let G and B_0 arise from Lemma 6.2 and ψ_*, Q_*, κ be any positive numbers. Let*

$$(6.10) \qquad\qquad \theta_1 = \cdots = \theta_m = \psi_*,$$

$$\theta_{m+1} = \cdots = \theta_n = (\psi_*^m Q_*)^{-1/d},$$

$$\theta_{n+1} = \kappa Q_*.$$

Then (6.5) is satisfied, where $\overline{\theta} = (\theta_1, \ldots, \theta_{n+1})$.

Proof. Observe that (6.5) is equivalent to $\frac{1}{2}B \setminus \mathcal{G}_{\mathbf{f}}(Q_*, \psi_*, \kappa) \subset \frac{1}{2}B \cap \mathcal{A}(G, \overline{\theta})$. By definition, for every point $\mathbf{x} \in \frac{1}{2}B \setminus \mathcal{G}_{\mathbf{f}}(Q_*, \psi_*, \kappa)$ there is a nonzero integer solution $\boldsymbol{r}$ to the system (4.22). Using (6.6) and Lemmas 3.6 and 6.2 in an obvious manner implies that (5.1) is satisfied when $(a_1, \ldots, a_k)$ is identified with $\boldsymbol{r}$. This exactly means that $\mathbf{x} \in \mathcal{A}(G, \overline{\theta})$ and completes the proof. $\qquad\square$

We now estimate the $\overline{\theta}$-weight of G for the above G and $\overline{\theta}$. See Section 5.1 for its definition.

LEMMA 6.4. *Let $\mathcal{M}$ be a nondegenerate analytic manifold given by (1.2). Let G and B_0 arise from Lemma 6.2 and let $\overline{\theta}$ be given by (6.10). Let κ, ψ_* and Q_* satisfy the conditions of Theorem 4.6 and let*

$$(6.11) \qquad\qquad C_* Q_*^{-1/m} \leq \psi_* \leq C_*^{-1}$$

for some $C_* > 1$. Then for any $\mathbf{x}_0 \in B_0$,

$$(6.12) \qquad\qquad \widehat{\Theta}_{\overline{\theta}}(\mathbf{x}_0) \leq (\kappa C_*)^{-1/(n+1)}.$$

Proof. By the definitions of θ and $\overline{\theta}$, i.e., by (5.2) and (6.10),

$$(6.13) \qquad\qquad \theta = \kappa^{1/(n+1)}.$$

Further, using inequalities (6.11) and the assumption $C_* > 1$, it is readily seen that

$$(6.14) \qquad\qquad \theta_i \leq 1 \qquad (1 \leq i \leq n).$$

Fix any point $\mathbf{x}_0 \in B_0$ and any vector subspace V of $\mathbb{R}^{n+1}$ with codim $V = r \in \{1, \ldots, n\}$. Since $\mathcal{M}$ is nondegenerate, for every ball $B(\mathbf{x}_0) \subset B_0$ centred at $\mathbf{x}_0$ there is a point $\mathbf{x} \in B(\mathbf{x}_0)$ such that $\boldsymbol{y} = \boldsymbol{y}(\mathbf{x}) \notin V^{\perp}$. That is $\mathcal{V}(\boldsymbol{y}) \not\subset V^{\perp}$. The latter is easily seen to be equivalent to $\mathcal{V}(\boldsymbol{y})^{\perp} \not\supset V$. By Lemma 4.1 and by (6.7),

we see that the first n rows of G, which are simply the vectors $\mathbf{g}_1(\mathbf{x}), \ldots, \mathbf{g}_n(\mathbf{x})$, form a basis of $\mathcal{V}(\boldsymbol{y})^\perp$. Thus, $V \not\subset \mathcal{V}(\mathbf{g}_1(\mathbf{x}), \ldots, \mathbf{g}_n(\mathbf{x}))$, and therefore

$$(6.15) \qquad \dim\big(V + \mathcal{V}(\mathbf{g}_1(\mathbf{x}), \ldots, \mathbf{g}_n(\mathbf{x}))\big) > \dim \mathcal{V}\big(\mathbf{g}_1(\mathbf{x}), \ldots, \mathbf{g}_n(\mathbf{x})\big) = n.$$

The latter implies that the l.h.s. of (6.15) is equal to $n+1$. Hence there is a sub-collection $J = \{j_1 < \cdots < j_r\} \subseteq \{1, \ldots, n\}$ satisfying $V \oplus \mathcal{V}\big(\mathbf{g}_{j_1}(\mathbf{x}), \ldots, \mathbf{g}_{j_r}(\mathbf{x})\big) = \mathbb{R}^{n+1}$. By (5.3),

$$
(6.16) \qquad \Theta_{\overline{\theta}}(\mathbf{x}, V) \quad \leq \quad \theta^{-r} \prod_{i=1}^r \theta_{j_i} \overset{(6.13)}{=} \kappa^{-r/(n+1)} \prod_{i=1}^r \theta_{j_i}
$$

$$
\overset{(6.14)}{\leq} \quad \kappa^{-r/(n+1)} \max_{1 \leq i \leq r} \theta_{j_i}
$$

$$
\overset{(6.10)}{\leq} \quad \kappa^{-r/(n+1)} \max\{\psi_*, (\psi_*^m Q_*)^{-1/d}\}
$$

$$
\overset{(6.11)}{\leq} \quad \kappa^{-r/(n+1)} \max\{C_*^{-1}, C_*^{-1/d}\}
$$

$$
\overset{C_* > 1,\, \kappa < 1}{\leq} \quad \kappa^{-1/(n+1)} C_*^{-1/(n+1)} = (\kappa C_*)^{-1/(n+1)}.
$$

Recall that $B(\mathbf{x}_0)$ can be made arbitrarily small so that $\mathbf{x}$ can be made arbitrary close to $\mathbf{x}_0$. Therefore, in view of the definition (5.4) of $\widehat{\Theta}_{\overline{\theta}}(\mathbf{x}_0, V)$, (6.16) implies that $\widehat{\Theta}_{\overline{\theta}}(\mathbf{x}_0, V) \leq (\kappa C_*)^{-1/(n+1)}$. Finally, since V is arbitrary nontrivial subspace of $\mathbb{R}^{n+1}$, we obtain (6.12). □

6.3. *Completion of the proof of Theorem* 1.4. We now proceed to the final phase of the proof of Theorem 1.4. Let $\mathbf{x}_0 \in U$ be an arbitrary point. Let B_0 be a ball centred at $\mathbf{x}_0$ arising from Lemma 6.2. We may assume without loss of generality that B_0 is compact. Further, shrink B_0 if necessary to ensure that Theorem 5.2 is applicable. Next, let B be an arbitrary ball in B_0 and let ψ and Q satisfy the conditions of Theorem 1.4, where C_0 and Q_0 are sufficiently large constants.

Let $c_0 = c_0(B_0) > 1$ be the constant arising from Theorem 4.6 and let K_0, α and $\delta = \delta(\frac{1}{2}B)$ be the constants arising from Theorem 5.2. Set

$$(6.17) \qquad\qquad\qquad \kappa = (4K_0)^{-\frac{n+1}{\alpha}}.$$

Obviously, $0 < \kappa < 1$ and is independent of B. Define

$$(6.18) \quad \psi_* = \kappa^2 c_0^{-1} \psi, \quad Q_* = c_0^{-1} Q, \quad \delta_0 = \kappa c_0^{-1}, \quad \rho = c_0 \kappa^{-2} \big(\psi_*^m Q_*^{d+1}\big)^{-\frac{1}{d}}.$$

It is easily verified that $1/m \leq (d+2)/(2n-d)$. Therefore, (1.3) implies that

$$(6.19) \qquad\qquad\qquad C_0 Q^{-(d+2)/(2n-d)} < \psi < C_0^{-1}.$$

Then, using (6.18) and (6.19) one readily verifies (4.24) and (4.40) provided that C_0 and Q_0 are sufficiently large. Therefore, Theorem 4.6 is applicable and

so (4.41) is satisfied. Further, let $\overline{\theta} = (\theta_1, \ldots, \theta_{n+1})$ be given by (6.10) and G be as in Lemma 6.2. Then, by Lemma 6.3 and (4.41), we obtain

$$(6.20) \qquad \tfrac{1}{2}B \setminus \mathcal{A}(G, \overline{\theta}) \ \subset \ \tfrac{1}{2}B \cap \mathcal{G}_{\mathbf{f}}(Q_*, \psi_*, \kappa) \ \subset \ \Delta^{\delta_0}(Q, \psi, B, \rho) \cap B.$$

Since Theorem 5.2 is applicable, by (5.5), we get

$$(6.21) \qquad \mu_d\left(\tfrac{1}{2}B \cap \mathcal{A}(G, \overline{\theta})\right) \leq K_0 \left(1 + \widehat{\Theta}_{\overline{\theta}}(\tfrac{1}{2}B)^{\alpha}/\delta^{\alpha}\right) \theta^{\alpha} \mu_d(\tfrac{1}{2}B),$$

where $\widehat{\Theta}_{\overline{\theta}}(\tfrac{1}{2}B) := \sup_{\mathbf{x} \in \frac{1}{2}B} \widehat{\Theta}_{\overline{\theta}}(\mathbf{x})$. By (1.3) and (6.18), (6.11) holds with

$$(6.22) \qquad\qquad\qquad\qquad C_* = C_0 \kappa^2 c_0^{-1-1/m}.$$

Clearly $C_* > 1$ if C_0 is sufficiently large. Then, by Lemma 6.4, we get

$$(6.23) \qquad\qquad\qquad \widehat{\Theta}_{\overline{\theta}}(\tfrac{1}{2}B) \leq (\kappa\, C_*)^{-1/(n+1)} \leq \delta$$

provided that C_0 and respectively C_* is sufficiently large. Recall by (6.13) that $\theta = \kappa^{1/(n+1)}$. Then, by (6.21) and (6.23), we get that

$$(6.24) \qquad \mu_d\left(\tfrac{1}{2}B \cap \mathcal{A}(G, \overline{\theta})\right) \leq 2K_0 \kappa^{\alpha/(n+1)} \mu_d(\tfrac{1}{2}B) \overset{(6.17)}{=} \tfrac{1}{2}\mu_d(\tfrac{1}{2}B).$$

Combining (6.24) with (6.20) gives $\mu_d\left(\Delta^{\delta_0}(Q, \psi, B, \rho) \cap B\right) \geq \tfrac{1}{2}\mu_d(\tfrac{1}{2}B) = 2^{-d-1}\mu_d(B)$, thus establishing (1.4) with $k_0 = 2^{-d-1}$ and $\rho_0 = (c_0^{n+d+1}\kappa^{-2n})^{1/d}$. The latter constant is easily deducted from (6.18) and is absolute. This completes the proof of Theorem 1.4.

7. Further theory for curves

In this section we relax condition (1.3) in the case of curves. Namely, the exponent $\frac{1}{m} = \frac{1}{n-1}$ will be replaced by $\frac{3}{2n-1}$. The latter allows us to widen the range of s for which Theorem 2.5 is applicable by the factor of $\frac{n}{2}$.

7.1. *Statement of results.* Given an analytic map $\boldsymbol{y} = (y_0, y_1, \ldots, y_n) : U \to \mathbb{R}^{n+1}$, where $U \subset \mathbb{R}$ is an interval, let $W_{\boldsymbol{y}}(x)$ denote the Wronskian of $y_0, y_1, \ldots, y_n$.

THEOREM 7.1. *Let $d = 1$ and the curve* (1.2) *satisfies $W_{\boldsymbol{y}}(x) \neq 0$ for all $x \in U$, where $\boldsymbol{y}$ as in* (4.1). *Then Theorem* 1.4 *and consequently Corollary* 1.5 *remain valid if* (1.3) *is replaced by*

$$(7.1) \qquad\qquad\qquad C_0 Q^{-3/(2n-1)} < \psi < C_0^{-1}.$$

Recall that the analytic curve $\mathcal{M}$ is nondegenerate if and only if the functions $1, y_1, \ldots, y_n$ are linearly independent over $\mathbb{R}$. Equivalently, $W_{\boldsymbol{y}}(x)$ is not identically zero. As $\boldsymbol{y} = (1, y_1, \ldots, y_n)$ is analytic, the Wronskian $W_{\boldsymbol{y}}(x)$ is analytic too. The nondegeneracy of $\mathcal{M}$ then implies that $W_{\boldsymbol{y}}(x) \neq 0$ everywhere except possibly on a countable set consisting of isolated points. Therefore, the

condition "$W_{\boldsymbol{y}}(x) \neq 0$ for all $x \in U$" imposed in the statement of Theorem 7.1 is not particularly restrictive if compared to nondegeneracy.

THEOREM 7.2. *Let $\mathcal{M}$ be a nondegenerate analytic curve in $\mathbb{R}^n$. Let $\psi : \mathbb{N} \to \mathbb{R}^+$ be a monotonic function such that $q\psi(q)^{(2n-1)/3} \to \infty$ as $q \to \infty$. Then for any $s \in (\frac{1}{2}, 1)$,*

$$(7.2) \qquad \mathcal{H}^s(\mathcal{S}_n(\psi) \cap \mathcal{M}) = \infty \qquad if \qquad \sum_{q=1}^{\infty} q^n \Big(\frac{\psi(q)}{q}\Big)^{s+n-1} = \infty.$$

Consequently, if $\tau = \tau(\psi)$ satisfies $1/n < \tau < 3/(2n-1)$, then

$$(7.3) \qquad \dim \mathcal{S}_n(\psi) \cap \mathcal{M} \geq \frac{n+1}{\tau+1} - (n-1).$$

The proof of Theorem 7.2 can be obtained by making minor and indeed obvious modifications to the proof of Theorem 2.5. Below we consider the proof of Theorem 7.1 only.

7.2. *Dual map.* Let the analytic map $\boldsymbol{y}$ be given by (4.1). The map $\boldsymbol{z} : U \to \mathbb{R}^n$ given by

$$(7.4) \qquad \boldsymbol{z}(x) = \Big(\boldsymbol{y}(x) \wedge \boldsymbol{y}'(x) \wedge \cdots \wedge \boldsymbol{y}^{(n-1)}(x)\Big)^{\perp}$$

will be called *dual* to $\boldsymbol{y}$. Obviously, every coordinate function of $\boldsymbol{z}$ is a polynomial expression of coordinate functions of $\boldsymbol{y}$ and their derivatives. Therefore, $\boldsymbol{z}$ is analytic. The following statement describes $\boldsymbol{z}$ via a system of linear differential equations.

LEMMA 7.3. *Let $\boldsymbol{y}$ and $\boldsymbol{z}$ be as above. Then*

$$(7.5) \qquad \begin{cases} \boldsymbol{z}^{(j)}(x) \cdot \boldsymbol{y}^{(i)}(x) & = & 0, & 0 \leq i+j \leq n-1 \\ \boldsymbol{z}^{(j)}(x) \cdot \boldsymbol{y}^{(i)}(x) & = & (-1)^j W_{\boldsymbol{y}}(x), & i+j = n. \end{cases}$$

Proof. By (7.4) and Lemma 3.2, we immediately get that $\boldsymbol{z}(x) \cdot \boldsymbol{y}^{(j)}(x) = 0$ for all $j \in \{0, \ldots, n-1\}$. On differentiating the latter equations we obviously obtain the first set of equation in (7.5). Now we compute $\boldsymbol{z}(x) \cdot \boldsymbol{y}^{(n)}(x)$:

$$\boldsymbol{z}(x) \cdot \boldsymbol{y}^{(n)}(x) \overset{(7.4)}{=} \left(\boldsymbol{y}(x) \wedge \boldsymbol{y}'(x) \wedge \cdots \wedge \boldsymbol{y}^{(n-1)}(x) \right)^{\perp} \cdot \boldsymbol{y}^{(n)}(x)$$

$$\overset{(3.7)}{=} \left(\boldsymbol{i} \cdot \left(\boldsymbol{y}(x) \wedge \boldsymbol{y}'(x) \wedge \cdots \wedge \boldsymbol{y}^{(n-1)}(x) \right) \right) \cdot \boldsymbol{y}^{(n)}(x)$$

$$\overset{(3.6)}{=} \left(\boldsymbol{e}_0 \wedge \boldsymbol{e}_1 \wedge \cdots \wedge \boldsymbol{e}_n \right) \cdot \left(\boldsymbol{y}(x) \wedge \cdots \wedge \boldsymbol{y}^{(n-1)}(x) \wedge \boldsymbol{y}^{(n)}(x) \right)$$

$$\overset{(3.3)}{=} \det \left(\boldsymbol{e}_i \cdot \boldsymbol{y}^{(j)}(x) \right)_{0 \leq i,j \leq n} = \det \left(y_i^{(j)}(x) \right)_{0 \leq i,j \leq n} = W_{\boldsymbol{y}}(x) \,,$$

where $\boldsymbol{e}_0, \ldots, \boldsymbol{e}_n$ is the standard basis of $\mathbb{R}^{n+1}$. This shows the $j = 0$ equation of the second set of equations in (7.5). Then we proceed by induction. Assume that the second set of inequalities of (7.5) holds for $j = j_0 \leq n - 1$. Then differentiating $\boldsymbol{z}^{(j_0)}(x) \cdot \boldsymbol{y}^{(n-j_0-1)}(x) = 0$, we get

$$
\begin{aligned}
0 &= \boldsymbol{z}^{(j_0+1)}(x) \cdot \boldsymbol{y}^{(n-j_0-1)}(x) + \boldsymbol{z}^{(j_0)}(x) \cdot \boldsymbol{y}^{(n-j_0)}(x) \\
&= \boldsymbol{z}^{(j_0+1)}(x) \cdot \boldsymbol{y}^{(n-j_0-1)}(x) + (-1)^{j_0} W_{\boldsymbol{y}}(x) \,.
\end{aligned}
$$

This implies (7.5) for $j = j_0 + 1$ and thus completes the proof of Lemma 7.3. □

LEMMA 7.4. *Let $\boldsymbol{y}$ and $\boldsymbol{z}$ be as above. Then for all x, $|W_{\boldsymbol{z}}(x)| \geq |W_{\boldsymbol{y}}(x)|^n$.*

Proof. By (7.5) and (3.3), it is easy to see that the inner product in $\wedge^{n+1}(\mathbb{R}^{n+1})$

$$(7.6) \qquad (\boldsymbol{z}(x) \wedge \boldsymbol{z}'(x) \wedge \cdots \wedge \boldsymbol{z}^{(n)}(x)) \cdot (\boldsymbol{y}(x) \wedge \boldsymbol{y}'(x) \wedge \cdots \wedge \boldsymbol{y}^{(n)}(x))$$

is the determinant of an $(n+1) \times (n+1)$ triangle matrix with $\pm W_{\boldsymbol{y}}(x)$ on the diagonal and is equal to $(-1)^{[n/2]} W_{\boldsymbol{y}}(x)^{n+1}$. Further, recall that

$$|\boldsymbol{y}(x) \wedge \cdots \wedge \boldsymbol{y}^{(n)}(x)| = |W_{\boldsymbol{y}}(x)| \qquad \text{and} \qquad |\boldsymbol{z}(x) \wedge \cdots \wedge \boldsymbol{z}^{(n)}(x)| = |W_{\boldsymbol{z}}(x)|.$$

Then, applying the Cauchy-Schwarz inequality to the inner product (7.6) gives

$$\left| W_{\boldsymbol{y}}(x) \right|^{n+1} \leq |\boldsymbol{z}(x) \wedge \cdots \wedge \boldsymbol{z}^{(n)}(x)| \cdot |\boldsymbol{y}(x) \wedge \cdots \wedge \boldsymbol{y}^{(n)}(x)| = |W_{\boldsymbol{z}}(x)| \cdot |W_{\boldsymbol{y}}(x)|,$$

further implying $|W_{\boldsymbol{z}}(x)| \geq |W_{\boldsymbol{y}}(x)|^n$. □

7.3. *Proof of Theorem* 7.1. Clearly, Lemma 6.1 can be used in the context of Theorem 7.1. Then we can assume that B_0 is a sufficiently small interval centred at an arbitrary point x_0 in U. We may assume without loss of generality that B_0 is compact. Further, shrink B_0 if necessary to ensure that Theorem 5.2 is applicable. Next, let B be an arbitrary interval in B_0 and let ψ and Q satisfy the conditions of Theorem 7.1, where C_0 and Q_0 are sufficiently large constants. Furthermore, in view of Theorem 1.4, without loss of generality we may assume that

$$(7.7) \qquad\qquad C_0 Q^{-3/(2n-1)} < \psi < Q^{-1/n}.$$

Let $\boldsymbol{z}$ be dual to $\boldsymbol{y}$ (see §7.2). Since B_0 is compact, there is a constant $K_1 > 1$ such that

$$(7.8) \qquad |\boldsymbol{z}^{(i)}(x)| \le K_1 \qquad \text{for all } x \in B_0 \text{ and all } i \in \{0, \dots, n\}.$$

Let $c_0 = c_0(B_0) > 1$ be the constant arising from Theorem 4.6 and let K_0, α and $\delta = \delta(\frac{1}{2}B)$ be the constants arising from Theorem 5.8. Set

$$(7.9) \qquad \kappa = (2K_1(4K_0)^{\frac{1}{\alpha}})^{-n-1}.$$

Obviously, $0 < \kappa < 1$ and is independent of B. Define ψ_*, Q_*, δ_0 and ρ by (6.18) assuming that $d = 1$ and $m = n - 1$.

Then, using (6.18) and (7.7) one readily verifies (4.24) and (4.40) provided that C_0 and Q_0 are sufficiently large. Therefore, Theorem 4.6 is applicable and so (4.41) is satisfied.

Take any point $\mathbf{x} \in \frac{1}{2}B \setminus \mathcal{G}_{\mathbf{f}}(Q_*, \psi_*, \kappa)$. Then, by definition, there is a nonzero integer solution $\boldsymbol{r}$ to system (4.22). Observe that $\boldsymbol{g} = (\boldsymbol{y} \wedge \boldsymbol{y}')^{\perp}$. Then, by (7.5) and Lemma 3.2, we get $\boldsymbol{z}^{(i)} \in \mathcal{V}(\boldsymbol{g})$ for $i = 0, \dots, n-2$. Therefore, by Lemma 3.6, (4.22) implies that

$$(7.10) \qquad |\boldsymbol{z}^{(i)}(x) \cdot \boldsymbol{r}| \le K_1 \psi_* \qquad \text{for } i = 0, \dots, n-2.$$

Again, by (7.5) and Lemma 3.2, $\boldsymbol{z}^{(n-1)} \in \mathcal{V}(\boldsymbol{y}^{\perp})$. Therefore, we get

$$(7.11) \quad |\boldsymbol{z}^{(n-1)}(x) \cdot \boldsymbol{r}| \overset{\text{Lemma 3.6}}{\le} |\boldsymbol{z}^{(n-1)}(x)| \, |\boldsymbol{y}(x)^{\perp} \cdot \boldsymbol{r}| \, |\boldsymbol{y}(x)^{\perp}|^{-1}$$

$$\overset{(7.8)}{\le} \quad K_1 \, |\boldsymbol{y}(x)^{\perp} \cdot \boldsymbol{r}| \, |\boldsymbol{y}(x)^{\perp}|^{-1}$$

$$\overset{(3.10)}{\le} \quad K_1 \, |\boldsymbol{y}(x) \wedge \boldsymbol{r}| \, |\boldsymbol{y}(x)|^{-1} \overset{(4.1)}{\le} K_1 \, |\boldsymbol{y}(x) \wedge \boldsymbol{r}|.$$

Here we have also used the fact that the Hodge operator is an isometry. Using (6.18) and the r.h.s. of (7.7), we get that $\psi_* < (\psi_*^{n-1} Q_*)^{-1}$. Therefore, applying Lemma 4.2 and (4.22) to (7.11) further gives

$$(7.12) \qquad |\boldsymbol{z}^{(n-1)}(x) \cdot \boldsymbol{r}| \le K_1 \, (\psi_* + (\psi^{n-1} Q_*)^{-1}) \le 2K_1 (\psi^{n-1} Q_*)^{-1}.$$

Finally, arguing the same way as in Step 1 of the proof of Theorem 4.5 (see §4.2), one easily verifies that $|\boldsymbol{r}|^2 \le 1 + Q_* + \kappa^2 Q_*^2$, whence $|\boldsymbol{r}| \le 2\kappa Q_*$ when Q_* is sufficiently large. Now we trivially get

$$(7.13) \qquad |\boldsymbol{z}^{(n)}(x) \cdot \boldsymbol{r}| \le |\boldsymbol{z}^{(n)}(x)| \, |\boldsymbol{r}| \le 2K_1 \kappa Q_*.$$

Let G be the Wronski matrix of the dual map $\boldsymbol{z}$. For $W_{\boldsymbol{y}}(x) \ne 0$ for all $x \in U$, by Lemma 7.4, $W_{\boldsymbol{z}}(x) \ne 0$ for all $x \in U$, that is $G : U \to \mathrm{GL}_{n+1}(\mathbb{R})$. Inequalities (7.10), (7.12) and (7.13) are equivalent to $x \in \mathcal{A}(G, \bar{\theta})$ with
(7.14)
$$\theta_1 = \cdots = \theta_{n-1} = K_1 \psi_*, \qquad \theta_n = 2K_1(\psi_*^{n-1} Q_*)^{-1}, \qquad \theta_{n+1} = 2K_1 \kappa Q_*.$$

Thus we have shown that $\frac{1}{2}B \setminus \mathcal{G}_{\mathbf{f}}(Q_*, \psi_*, \kappa) \subset \frac{1}{2}B \cap \mathcal{A}(G, \overline{\theta})$. Hence, $\frac{1}{2}B \setminus \mathcal{A}(G, \overline{\theta}) \subset \frac{1}{2}B \cap \mathcal{G}_{\mathbf{f}}(Q_*, \psi_*, \kappa)$. By (4.41),

$$(7.15) \qquad \frac{1}{2}B \setminus \mathcal{A}(G, \overline{\theta}) \subset \Delta^{\delta_0}(Q, \psi, B, \rho) \cap B.$$

By the r.h.s. of (7.7), $\theta_i \leq \theta_{i+1}$ for all $i = 1, \ldots, n$. Further, by the l.h.s. of (7.7), $\theta_n \ll Q^{1/2}$. Further, $\theta_{n+1} \asymp Q$. Then, by Lemma 5.9, $\widetilde{\Theta} \ll Q^{-1/(2n+2)} < \delta$ for sufficiently large Q. Theorem 5.8, (7.9) and (7.14) imply that $\mu_d(\frac{1}{2}B \cap \mathcal{A}(G, \overline{\theta})) \leq 2K_0\theta^\alpha \leq \frac{1}{2}\mu_d(\frac{1}{2}B)$. Combining this with (7.15) gives the required result.

8. Final comments

In view of the results of this paper, establishing upper bounds for $N(Q, \varepsilon)$ becomes a very topical problem. Unfortunately, nondegeneracy alone is not enough to reverse (1.5). A counterexample can be constructed by considering the manifolds

$$\mathcal{M}_k = \Big\{ (x_1, \ldots, x_{d-1}, x_d, x_d^{k+1}, x_d^{k+2}, \ldots, x_d^{k+m}) : \max_{1 \leq i \leq d} |x_i| < 1 \Big\}.$$

Nevertheless, requiring that for every $\mathbf{x} \in U$ there exists $l \in \{1, \ldots, m\}$ such that Hess $f_l(\mathbf{x}) \neq 0$ is possibly enough to reverse (1.4), where Hess $f(\mathbf{x})$ denotes the Hessian matrix of $f(\mathbf{x})$. Any progress with this would have obvious implications for the theory of Diophantine approximation on manifolds, where the following two conjectures are now of extremely high interest.

CONJECTURE 8.1. *Any analytic nondegenerate submanifold of $\mathbb{R}^n$ is of Khintchine type for convergence.*

CONJECTURE 8.2. *Let $\mathcal{M}$ be a nondegenerate analytic submanifold of $\mathbb{R}^n$, $d = \dim \mathcal{M}$ and $m = \operatorname{codim} \mathcal{M}$. Let $\psi : \mathbb{N} \to \mathbb{R}^+$ be a monotonic function. If $\frac{m}{m+1}d < s < d$, then*

$$(8.1) \qquad \mathcal{H}^s(\mathcal{S}_n(\psi) \cap \mathcal{M}) = 0 \qquad if \qquad \sum_{q=1}^{\infty} q^n \Big(\frac{\psi(q)}{q}\Big)^{s+m} < \infty.$$

In the case of $\mathcal{M} = \mathbb{R}^n$, Conjecture 8.2 together with Theorem 2.5 coincides with Jarník's theorem, or rather the modern version of Jarník's theorem; see [7]. Therefore, Conjecture 8.2 can be regarded as a Jarník-type theorem for convergence for manifolds. In turn, Theorem 2.5 can be regarded as a Jarník-type theorem for divergence for manifolds. Conjecture 8.2 combined with Theorem 2.5 would also imply that (2.8) is an equality.

If Theorem 4.5 was used to its 'full potential,' then one would be able to prove (2.7) for $s \in (d/2, d)$. This naturally suggests the following problem: *Describe analytic nondegenerate manifolds $\mathcal{M}$ for which (2.7) and/or (8.1) hold for $s \in (d/2, d)$.*

Note that within this paper there are two instances when (2.7) is established for $s \in (d/2, d)$: hypersurfaces and curves. It is quite possible that for these types of manifolds (8.1) also holds for $s \in (d/2, d)$. However, note that for manifolds other than curves and hypersurfaces, establishing (8.1) for $s \in (d/2, d)$ is in general impossible unless extra constraints are added. This can be shown by considering $\mathcal{M}$ as in Example 1.3.

The main results of this paper are established in the case of analytic manifolds. However, within this paper the analyticity assumption is only used in establishing Theorem 5.2. More precisely, the analyticity assumption is used to verify condition (i) of Theorem KM. A natural challenging question is then *to what extent the analyticity assumption can be relaxed within Theorem 5.2 and consequently within all the main results of this paper.*

Recall that the analyticity assumption is not present in the planar curves results [8], [54]. Even though, there is a minor disagreement in the smoothness conditions imposed in convergence and divergence results: the divergence results deal with $C^{(3)}$ nondegenerate planar curve only. In general, the non-degeneracy of planar curves requires $C^{(2)}$. This raises the following intriguing question: *Are $C^{(2)}$ nondegenerate planar curves of Khintchine type for divergence?*[3]

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University of York, Heslington, York, England
E-mail: victor.beresnevich@york.ac.uk