Holomorphic factorization of mappings into $\mathrm{SL}_n(\mathbb{C})$

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Abstract

We solve Gromov's Vaserstein problem. Namely, we show that a null-homotopic holomorphic mapping from a finite dimensional reduced Stein space into $\mathrm{SL}_n(\mathbb{C})$ can be factored into a finite product of unipotent matrices with holomorphic entries.

Contents

1. Introduction

It is standard material in a Linear Algebra course that the group $\mathrm{SL}_m(\mathbb{C})$ is generated by elementary matrices $E + \alpha e_{ij}$, $i \neq j$, i.e., matrices with 1's on the diagonal and all entries outside the diagonal are zero, except one entry. Equivalently, every matrix $A \in \mathrm{SL}_m(\mathbb{C})$ can be written as a finite product of upper and lower diagonal unipotent matrices (in interchanging order). The same question for matrices in $\mathrm{SL}_m(R)$ where R is a commutative ring instead of the field $\mathbb{C}$ is much more delicate. For example, if R is the ring of complex valued functions (continuous, smooth, algebraic or holomorphic) from a

space X, the problem amounts to finding for a given map $f : X \to \mathrm{SL}_m(\mathbb{C})$ a factorization as a product of upper and lower diagonal unipotent matrices

$$f(x) = \begin{pmatrix} 1 & 0 \\ G_1(x) & 1 \end{pmatrix} \begin{pmatrix} 1 & G_2(x) \\ 0 & 1 \end{pmatrix} \cdots \begin{pmatrix} 1 & G_N(x) \\ 0 & 1 \end{pmatrix},$$

where the G_i are maps $G_i : X \to \mathbb{C}^{m(m-1)/2}$.

Since any product of (upper and lower diagonal) unipotent matrices is homotopic to a constant map (multiplying each entry outside the diagonals by $t \in [0,1]$ we get a homotopy to the identity matrix), one has to assume that the given map $f : X \to \mathrm{SL}_m(\mathbb{C})$ is homotopic to a constant map or as we will say *null-homotopic*. In particular this assumption holds if the space X is contractible.

This very general problem has been studied in the case of polynomials of n variables. For $n = 1$, i.e., $f : \mathbb{C} \to \mathrm{SL}_m(\mathbb{C})$ a polynomial map (the ring R equals $\mathbb{C}[z]$) it is an easy consequence of the fact that $\mathbb{C}[z]$ is an Euclidean ring that such f factors through a product of upper and lower diagonal unipotent matrices. For $m = n = 2$, the following counterexample was found by Cohn [Coh66]: the matrix

$$\begin{pmatrix} 1 - z_1 z_2 & z_1^2 \\ -z_2^2 & 1 + z_1 z_2 \end{pmatrix} \in \mathrm{SL}_2(\mathbb{C}[z_1, z_2])$$

does not decompose as a finite product of unipotent matrices.

For $m \geq 3$ (and any n), it is a deep result of Suslin [Sus77] that any matrix in $\mathrm{SL}_m(\mathbb{C}[\mathbb{C}^n])$ decomposes as a finite product of unipotent (and equivalently elementary) matrices. More results in the algebraic setting can be found in [Sus77] and [GMV94]. For a connection to the Jacobian problem on $\mathbb{C}^2$, see [Wri78].

In the case of continuous complex valued functions on a topological space X the problem was studied and partially solved by Thurston and Vaserstein [TV86] and then finally solved by Vaserstein [Vas88]; see Theorem 2.2.

It is natural to consider the problem for rings of holomorphic functions on Stein spaces, in particular on $\mathbb{C}^n$. Explicitly this problem was posed by Gromov in his groundbreaking paper [Gro89] where he extends the classical Oka-Grauert theorem from bundles with homogeneous fibers to fibrations with elliptic fibers, e.g., fibrations admitting a dominating spray (for definition, see 3.1). In spite of the above mentioned result of Vaserstein he calls it the

VASERSTEIN PROBLEM (see [Gro89, §3.5.G]). *Does every holomorphic map* $\mathbb{C}^n \to \mathrm{SL}_m(\mathbb{C})$ *decompose into a finite product of holomorphic maps sending* $\mathbb{C}^n$ *into unipotent subgroups in* $\mathrm{SL}_m(\mathbb{C})$?

Gromov's interest in this question comes from the question about s-homotopies (s for spray). In this particular example the spray on $\mathrm{SL}_m(\mathbb{C})$ is that

coming from the multiplication with unipotent matrices. Of course one cannot use the upper and lower diagonal unipotent matrices only to get a spray (there is no submersivity at the zero section!), there need to be at least one more unipotent subgroup to be used in the multiplication. Therefore the factorization in a product of upper and lower diagonal matrices seems to be a stronger condition than to find a map into the iterated spray, but since all maximal unipotent subgroups in $\mathrm{SL}_m(\mathbb{C})$ are conjugated and the upper and lower diagonal matrices generate $\mathrm{SL}_m(\mathbb{C})$, these two problems are in fact equivalent. We refer the reader for more information on the subject to Gromov's above mentioned paper.

The main result of this paper is a complete positive solution of Gromov's Vaserstein problem, namely we prove

MAIN THEOREM (see Theorem 2.3). *Let X be a finite dimensional reduced Stein space and $f\colon X \to \mathrm{SL}_m(\mathbb{C})$ be a holomorphic mapping that is null-homotopic. Then there exist a natural number K and holomorphic mappings $G_1, \ldots, G_K\colon X \to \mathbb{C}^{m(m-1)/2}$ such that f can be written as a product of upper and lower diagonal unipotent matrices*

$$f(x) = \begin{pmatrix} 1 & 0 \\ G_1(x) & 1 \end{pmatrix} \begin{pmatrix} 1 & G_2(x) \\ 0 & 1 \end{pmatrix} \cdots \begin{pmatrix} 1 & G_K(x) \\ 0 & 1 \end{pmatrix}$$

for every $x \in X$.

The method of proof is an application of the Oka-Grauert-Gromov-principle to certain stratified fibrations. The existence of a topological section for these fibrations we deduce from Vaserstein's result.

We need the principle in its strongest form suggested by Gromov, completely proven by Forstnerič and Prezelj [FP01]; see Theorem 3.6 and also Forstnerič [For10, Th. 8.3]. After the Gromov-Eliashberg embedding theorem for Stein manifolds (see [EG92], [Sch97]) this is to our knowledge the second time this holomorphic h-principle has an application which goes beyond the classical results of Grauert, Forster and Rammspott [Gra58], [Gra57b], [Gra57a], [For71], [FR68b], [FR68a], [FR66b], [FR66a].

The paper is organized as follows. In Section 2 we introduce the fibration and give an overview of the proof. In the next section we explain how the Oka-Grauert-Gromov-principle is used in the proof. In Sections 4 and 5 we prove all technical details referred to in earlier sections. In the last section we comment on the number of matrices needed in the multiplication.

The results of the present paper have been announced in *Comptes Rendus* [IK08], communicated by Misha Gromov. The authors like to thank him for his interest in the subject. Also the authors thank Franc Forstnerič, Josip Globevnik, Marco Slapar, Erik Løw and Erlend Fornæss Wold for valuable

discussions on the subject over the years. Especially we thank Franc Forstnerič for including an extra section into his paper [For10] to provide us with the exact version of h-principle we need. A special thank goes to Wilberd van der Kallen for bringing the work of Vaserstein to our attention.

2. **Statement of the result and overview of the proof**

All complex spaces considered in this paper will be assumed reduced, and we will not repeat this every time. We call a complex space X finite dimensional if its smooth part $X \setminus X^{\text{sing}}$ has finite dimension. Note that this does not imply that they have finite embedding dimension.

We introduce the following notation. Let n and K be natural numbers. If K is odd, write $Z_K \in \mathbb{C}^{n(n-1)/2}$ as

$$Z_K = (z_{21,K}, \ldots, z_{ji,K}, \ldots, z_{n(n-1),K})$$

for $1 \leq i < j \leq n$. For K even, write

$$Z_K = (z_{12,K}, \ldots, z_{ji,K}, \ldots, z_{(n-1)n,K})$$

for $1 \leq j < i \leq n$. Now define $M_k \colon \mathbb{C}^{n(n-1)/2} \to \mathrm{SL}_n(\mathbb{C})$ as

$$M_{2l}(Z_{2l}) = \begin{pmatrix} 1 & z_{12,2l} & \cdots & z_{1n,2l} \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & z_{(n-1)n,2l} \\ 0 & \cdots & 0 & 1 \end{pmatrix}$$

and

$$M_{2l-1}(Z_{2l-1}) = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ z_{21,2l-1} & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ z_{n1,2l-1} & \cdots & z_{n(n-1),2l-1} & 1 \end{pmatrix}.$$

Remark 2.1. In the proofs of our results we will study products

$$M_1(Z_1)^{-1} \cdots M_K(Z_K)^{-1}.$$

This is done for purely technical reasons. Using automorphisms of $\mathbb{C}^{n(n-1)/2}$ sending a unipotent matrix to its inverse, this is equivalent to a product $M_1(X_1) \cdots M_K(X_K)$ in new coordinates $(X_1, \ldots, X_K)$.

As pointed out in the introduction Vaserstein constructed the factorization in the case of continuous mappings. Namely, he proved

THEOREM 2.2 (see [Vas88, Th. 4]). *For any natural number n and an integer $d \geq 0$ there is a natural number K such that for any finite dimensional normal topological space X of dimension d and null-homotopic continuous mapping $f \colon X \to \mathrm{SL}_n(\mathbb{C})$, the mapping can be written as a finite product of no more than K unipotent matrices. That is, one can find continuous mappings $F_l \colon X \to \mathbb{C}^{n(n-1)/2}$, $1 \leq l \leq K$ such that $f(x) = M_1(F_1(x)) \cdots M_K(F_K(x))$ for every $x \in X$.*

Here the dimension of the topological space is meant to be the covering dimension (which for normal second countable topological spaces is anyhow the same as the large and small inductive dimension; see [HW41]).

We recall the statement of the main result of this paper.

THEOREM 2.3. *Let X be a finite dimensional reduced Stein space and $f \colon X \to \mathrm{SL}_n(\mathbb{C})$ be a holomorphic mapping that is null-homotopic. Then there exist a natural number K and holomorphic mappings $G_1, \ldots, G_K \colon X \to \mathbb{C}^{n(n-1)/2}$ such that*

$$f(x) = M_1(G_1(x)) \cdots M_K(G_K(x))$$

for every $x \in X$.

We have the following corollary which in particular solves Gromov's Vaserstein problem.

COROLLARY 2.4. *Let X be a finite dimensional reduced Stein space that is topologically contractible and $f \colon X \to \mathrm{SL}_n(\mathbb{C})$ be a holomorphic mapping. Then there exist a natural number K and holomorphic mappings $G_1, \ldots, G_K \colon X \to \mathbb{C}^{n(n-1)/2}$ such that*

$$f(x) = M_1(G_1(x)) \cdots M_K(G_K(x))$$

for every $x \in X$.

By the definition of the Whitehead K_1-group of a ring (see [Ros94, p. 61]), this implies.

COROLLARY 2.5. *Let X be a finite dimensional reduced Stein space that is topologically contractible and denote by $\mathcal{O}(X)$ the ring of holomorphic functions on X. Then $SK_1(\mathcal{O}(X))$ is trivial and the determinant induces an isomorphism $\det \colon K_1(\mathcal{O}(X)) \to \mathcal{O}(X)^\star$.*

The strategy for proving Theorem 2.3 is as follows. Define $\Psi_K \colon (\mathbb{C}^{n(n-1)/2})^K \to \mathrm{SL}_n(\mathbb{C})$ as

$$\Psi_K(Z_1, \ldots, Z_K) = M_1(Z_1)^{-1} \cdots M_K(Z_K)^{-1}.$$

We want to show the existence of a holomorphic map

$$G = (G_1, \ldots, G_K) \colon X \to (\mathbb{C}^{n(n-1)/2})^K$$

such that

$$
\begin{array}{ccc}
 & & (\mathbb{C}^{n(n-1)/2})^K \\
 & {}^{G}\nearrow & \downarrow {\scriptstyle \Psi_K} \\
X & \xrightarrow{\quad f \quad} & \mathrm{SL}_n(\mathbb{C})
\end{array}
$$

is commutative. The result by Vaserstein shows the existence of a continuous map such that the diagram above is commutative.

We will prove Theorem 2.3 using the Oka-Grauert-Gromov principle for sections of holomorphic submersions over X. One candidate submersion would be the pull-back of $\Psi_K \colon (\mathbb{C}^{n(n-1)/2})^K \to \mathrm{SL}_n(\mathbb{C})$. It turns out that Ψ_K is not a submersion at all points in $(\mathbb{C}^{n(n-1)/2})^K$. It is a surjective holomorphic submersion if one removes a certain subset from $(\mathbb{C}^{n(n-1)/2})^K$. Unfortunately the fibers of this submersion are quite difficult to analyze and we therefore elect to study

$$
\begin{array}{ccc}
 & & (\mathbb{C}^{n(n-1)/2})^K \\
 & {}^{F}\nearrow & \downarrow {\scriptstyle \pi_n \circ \Psi_K} \\
X & \xrightarrow{\quad \pi_n \circ f \quad} & \mathbb{C}^n \setminus \{0\},
\end{array}
$$

where we define the projection $\pi_n \colon \mathrm{SL}_n(\mathbb{C}) \to \mathbb{C}^n \setminus \{0\}$ to be the projection of a matrix to its last row:

$$
\pi_n\left(\begin{pmatrix} z_{11} & \cdots & z_{1n} \\ \vdots & \ddots & \vdots \\ z_{n1} & \cdots & z_{nn} \end{pmatrix} \right) = (z_{n1}, \ldots, z_{nn}).
$$

However, even the map $\Phi_K = \pi_n \circ \Psi_K \colon (\mathbb{C}^{n(n-1)/2})^K \to \mathbb{C}^n \setminus \{0\}$ is not submersive everywhere. We have the following three results about that map which will be proved in later sections.

LEMMA 2.6. *The mapping* $\Phi_K = \pi_n \circ \Psi_K \colon (\mathbb{C}^{n(n-1)/2})^K \to \mathbb{C}^n \setminus \{0\}$ *is a holomorphic submersion exactly at points* $Z = (Z_1, \ldots, Z_K) \in (\mathbb{C}^{n(n-1)/2})^K \setminus S_K$ *where for* $K \geq 2$,

$$
S_K = \left(\bigcap_{1 \leq 2j+1 < K} \left\{ (Z_1, \ldots, Z_K) \in (\mathbb{C}^{n(n-1)/2})^K : z_{n1,2j+1} = \cdots = z_{n(n-1),2j+1} = 0 \right\} \right)
$$

$$
\cap \left(\bigcap_{1 \leq 2j < K} \left\{ (Z_1, \ldots, Z_K) \in (\mathbb{C}^{n(n-1)/2})^K : z_{1n,2j} = \cdots = z_{(n-1)n,2j} = 0 \right\} \right);
$$

that is, the entries in the last row of each lower triangular matrix and the entries in the last column of each upper triangular matrix are 0, *except for the K-th matrix where no conditions are imposed.*

LEMMA 2.7. *The mapping $\Phi_K = \pi_n \circ \Psi_K \colon (\mathbb{C}^{n(n-1)/2})^K \setminus S_K \to \mathbb{C}^n \setminus \{0\}$ is surjective when $K \geq 3$.*

PROPOSITION 2.8. *Let X be a finite dimensional reduced Stein space and $f \colon X \to \mathrm{SL}_n(\mathbb{C})$ be a null-homotopic holomorphic map. Assume that there exists a natural number K and a continuous map $F \colon X \to (\mathbb{C}^{n(n-1)/2})^K \setminus S_K$ such that*

$$
\begin{array}{ccc}
 & & (\mathbb{C}^{n(n-1)/2})^K \setminus S_K \\
 & \overset{F}{\nearrow} & \Big\downarrow {\scriptstyle \pi_n \circ \Psi_K} \\
X & \xrightarrow[\pi_n \circ f]{} & \mathbb{C}^n \setminus \{0\}
\end{array}
$$

is commutative. Then there exists a holomorphic map $G \colon X \to (\mathbb{C}^{n(n-1)/2})^K \setminus S_K$, homotopic to F via continuous maps $F_t \colon X \to (\mathbb{C}^{n(n-1)/2})^K \setminus S_K$, such that the diagram above is commutative for all F_t.

Proof of Theorem 2.3. We use induction on n. Note that the result is obvious for $n = 1$. Suppose that Proposition 2.8 is valid for n and Theorem 2.3 for $n - 1$. Put $\Phi_K = \pi_n \circ \Psi_K$. We can find a continuous map $F \colon X \to (\mathbb{C}^{n(n-1)/2})^K \setminus S_K$ for some natural number K such that $f(x) = \Psi_K(F(x))$. Indeed, since a finite dimensional Stein space is finite dimensional as a topological space, the Vaserstein result (Theorem 2.2) gives us a map $(F_1, \ldots, F_{K'})$ into $\left(\mathbb{C}^{n(n-1)/2} \right)^{K'}$. Abusing notation slightly one sees (use Lemma 2.6) that $F = (F_1, \ldots, F_{K'}, (0, \ldots, 1), (0, \ldots, 0), (0, \ldots, -1))$ gives a map from X into $\left(\mathbb{C}^{n(n-1)/2} \right)^{K'+3} \setminus S_{K'+3}$, and putting $K = K' + 3$ we have $f(x) = \Psi_K(F(x))$. It follows that $\Psi_K(F(x))f(x)^{-1} = E_n$. Using Proposition 2.8 we know that F is homotopic to a holomorphic map G such that

$$
\Phi_K(F(x)) = \pi_n(f(x)) = \Phi_K(G(x));
$$

that is, the last rows of the matrices $\Psi_K(F(x))$ and $\Psi_K(G(x))$ are equal. Therefore

$$
\Psi_K(G(x))f(x)^{-1} =
\begin{pmatrix}
\widetilde{f}_1^1(x) & \cdots & \widetilde{f}_1^{n-1}(x) & h_1^n(x) \\
\vdots & \ddots & \vdots & \vdots \\
\widetilde{f}_{n-1}^1(x) & \cdots & \widetilde{f}_{n-1}^{n-1}(x) & h_{n-1}^n(x) \\
0 & \cdots & 0 & 1
\end{pmatrix},
$$

where all entries are holomorphic. We see that

$$f_{n-1}(x) = \begin{pmatrix} \widetilde{f}_1^1(x) & \cdots & \widetilde{f}_1^{n-1}(x) \\ \vdots & \ddots & \vdots \\ \widetilde{f}_{n-1}^1(x) & \cdots & \widetilde{f}_{n-1}^{n-1}(x) \end{pmatrix}$$

defines a holomorphic map $f_{n-1}\colon X \to \mathrm{SL}_{n-1}(\mathbb{C})$. The homotopy

$$\Psi_K(F_t(x))f(x)^{-1}$$

consists of matrices having the last row equal to $(0,\ldots,0,1)$, and therefore the $(n-1) \times (n-1)$ left upper corner of the matrices $\Psi_K(F_t(x))f(x)^{-1}$ are in $\mathrm{SL}_{n-1}(\mathbb{C})$ for all t. Since for $t = 0$ it is the identity matrix, the map $f_{n-1}\colon X \to \mathrm{SL}_{n-1}(\mathbb{C})$ is null-homotopic.

We use the induction hypotheses to write f_{n-1} as a product of unipotent matrices with holomorphic entries. That is, there exists $\widetilde{K}$, a holomorphic map $\widetilde{G}\colon X \to (\mathbb{C}^{(n-1)(n-2)/2})^{\widetilde{K}} \setminus S_{\widetilde{K}}$ such that

$$\widetilde{f}(x) = M_1(\widetilde{G}_1(x))\cdots M_{\widetilde{K}}(\widetilde{G}_{\widetilde{K}}(x)).$$

Hence we have

$$\begin{pmatrix} & & & -h_1^n(x) \\ & E_{n-1} & & \vdots \\ & & & -h_{n-1}^n(x) \\ 0 & \cdots & 0 & 1 \end{pmatrix} \Psi_K(G(x))f(x)^{-1} = \begin{pmatrix} & & & 0 \\ & \widetilde{f}(x) & & \vdots \\ & & & 0 \\ 0 & \cdots & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} & & & 0 \\ M_1(\widetilde{G}_1(x)) & & \vdots \\ & & & 0 \\ 0 & \cdots & 0 & 1 \end{pmatrix} \cdots \begin{pmatrix} & & & 0 \\ M_{\widetilde{K}}(\widetilde{G}_{\widetilde{K}}(x)) & & \vdots \\ & & & 0 \\ 0 & \cdots & 0 & 1 \end{pmatrix},$$

and the result follows by induction.   □

In order to complete the proof of the theorem we need to establish Proposition 2.8, Lemma 2.6, and 2.7.

3. Stratified sprays

We will introduce the concept of a spray associated with a holomorphic submersion following [Gro89] and [FP02]. First we introduce some notation and terminology. Let $h\colon Z \to X$ be a holomorphic submersion of a complex manifold Z onto a complex manifold X. For any $x \in X$ the fiber over x of this submersion will be denoted by Z_x. At each point $z \in Z$ the tangent space T_zZ contains *the vertical tangent space* $VT_zZ = \ker Dh$. For holomorphic vector bundles $p\colon E \to Z$ we denote the zero element in the fiber E_z by 0_z.

Definition 3.1. Let $h\colon Z \to X$ be a holomorphic submersion of a complex manifold Z onto a complex manifold X. A spray on Z associated with h is a triple (E, p, s), where $p\colon E \to Z$ is a holomorphic vector bundle and $s\colon E \to Z$ is a holomorphic map such that for each $z \in Z$, we have

(i) $s(E_z) \subset Z_{h(z)}$,
(ii) $s(0_z) = z$, and
(iii) the derivative $Ds(0_z)\colon T_{0_z}E \to T_zZ$ maps the subspace $E_z \subset T_{0_z}E$ surjectively onto the vertical tangent space VT_zZ.

Remark 3.2. We will also say that the submersion admits a spray. A spray associated with a holomorphic submersion is sometimes called a (fiber) dominating spray.

One way of constructing dominating sprays, as pointed out by Gromov, is to find finitely many $\mathbb{C}$-complete vector fields that are tangent to the fibers and span the tangent space of the fibers at all points in Z. One can then use the flows φ_j^t of these vector fields V_j to define $s\colon Z \times \mathbb{C}^N \to Z$ via $s(z, t_1, \ldots, t_N) = \varphi_1^{t_1} \circ \cdots \circ \varphi_N^{t_N}(z)$ which gives a spray.

Definition 3.3. Let X and Z be complex spaces. A holomorphic map $h\colon Z \to X$ is said to be a submersion if for each point $z_0 \in Z$, it is locally equivalent via a fiber preserving biholomorphic map to a projection $p\colon U \times V \to U$, where $U \subset X$ is an open set containing $h(z_0)$ and V is an open set in some $\mathbb{C}^d$.

We will need to use stratified sprays which are defined as follows.

Definition 3.4. We say that a submersion $h\colon Z \to X$ admits stratified sprays if there is a descending chain of closed complex subspaces $X = X_m \supset \cdots \supset X_0$ such that each stratum $Y_k = X_k \setminus X_{k-1}$ is regular and the restricted submersion $h\colon Z|_{Y_k} \to Y_k$ admits a spray over a small neighborhood of any point $x \in Y_k$.

Remark 3.5. We say that the stratification $X = X_m \supset \cdots \supset X_0$ is associated with the stratified spray and vice versa.

In [FP01] (see also [For10, Th. 8.3]), the following theorem is proved.

THEOREM 3.6. *Let X be a Stein space with a descending chain of closed complex subspaces $X = X_m \supset \cdots \supset X_0$ such that each stratum $Y_k = X_k \setminus X_{k-1}$ is regular. Assume that $h\colon Z \to X$ is a holomorphic submersion which admits stratified sprays associated with the stratification then any continuous section $f_0\colon X \to Z$ such that $f_0|_{X_0}$ is holomorphic can be deformed to a holomorphic section $f_1\colon X \to Z$ by a homotopy that is fixed on X_0.*

LEMMA 3.7. *The holomorphic submersions* $\Phi_K \colon (\mathbb{C}^{n(n-1)/2})^K \backslash S_K \to \mathbb{C}^n \backslash \{0\}$, *for* $K \geq 3$, *admit stratified sprays.*

This lemma will be established in Section 5. Assuming it true for the moment we prove Proposition 2.8.

Proof of Proposition 2.8. Assume that K is a natural number so that there exists a continuous map $F \colon X \to (\mathbb{C}^{n(n-1)/2})^K \setminus S_K$ such that

$$
\begin{array}{ccc}
 & & (\mathbb{C}^{n(n-1)/2})^K \setminus S_K \\
 & \overset{F}{\nearrow} & \downarrow \Phi_K \\
X & \xrightarrow[\pi_n \circ f]{} & \mathbb{C}^n \setminus \{0\}
\end{array}
$$

is commutative. Put $Y = (\mathbb{C}^{n(n-1)/2})^K \setminus S_K$ and $p = \pi_n \circ f$. Define the pull-back of $(Y, \Phi_K, \mathbb{C}^n \setminus \{0\})$ via $p \colon X \to \mathbb{C}^n \setminus \{0\}$ as $(p^\star Y, p^\star \Phi_K, X)$ where

$$
p^\star Y = \{(x, Z) \in X \times Y; p(x) = \Phi_K(Z)\}
$$

and $p^\star \Phi_K(x, Z) = x$. Using that Φ_K is a holomorphic submersion we see that $p^\star \Phi_K$ is a holomorphic submersion. The continuous mapping F defines a continuous section

$$
p^\star F(x) = (x, F(x))
$$

of $(p^\star Y, p^\star \Phi_K, X)$. We need to show that $(p^\star Y, p^\star \Phi_K, X)$ admits stratified sprays. Let $\mathbb{C}^n \setminus \{0\} = V_m \supset \cdots \supset V_0$ be the stratification of $\mathbb{C}^n \setminus \{0\}$ corresponding to the stratified spray of $(Y, \Phi_K, \mathbb{C}^n \setminus \{0\})$. Define $X_j = p^{-1}(V_j)$ for $0 \leq j \leq m$. These complex subspaces need to be stratified in order for us to apply Theorem 3.6. Define $X_{0,i} = X_{0,i-1}^{\mathrm{sing}}$ for $i \geq 1$ and $X_{0,0} = X_0$. This defines a stratification of X_0 since $X_{0,J} = \emptyset$ when $J > L$ for some L, since the singularity set of reduced complex spaces has strictly lower dimension than the space itself. We continue by putting $X_{j,0} = X_j$ and $X_{j,i} = X_{j,i-1}^{\mathrm{sing}} \cup X_{j-1}$ for $i \geq 1$. Since X is finite dimensional, this gives a stratification of X. Now the result follows by Theorem 3.6. $\qquad \square$

4. Proof of Lemma 2.7 and 2.6

Recall that

$$
S_K = \left(\bigcap_{1 \leq 2j+1 < K} \left\{ (Z_1, \ldots, Z_K) \in (\mathbb{C}^{n(n-1)/2})^K : z_{n1,2j+1} = \cdots = z_{n(n-1),2j+1} = 0 \right\} \right)
$$

$$
\cap \left(\bigcap_{1 \leq 2j < K} \left\{ (Z_1, \ldots, Z_K) \in (\mathbb{C}^{n(n-1)/2})^K : z_{1n,2j} = \cdots = z_{(n-1)n,2j} = 0 \right\} \right).
$$

We begin by proving Lemma 2.7.

Proof of Lemma 2.7. First note that the set S_K is invariant under the automorphism in $(\mathbb{C}^{n(n-1)/2})^K$ replacing $M_1(Z_1)^{-1}\cdots M_K(Z_K)^{-1}$ with $M_1(X_1)$ $\cdots M_K(X_K)$. Also note that if

$$(\mathbb{C}^{n(n-1)/2})^K \setminus S_K \ni (X_1,\ldots,X_K) \mapsto \pi_n\left(M_1(X_1)\cdots M_K(X_K)\right) \in \mathbb{C}^n \setminus \{0\}$$

is surjective, then

$$(\mathbb{C}^{n(n-1)/2})^{K+J} \setminus S_{K+J} \ni (X_1,\ldots,X_{K+J})$$
$$\mapsto \pi_n\left(M_1(X_1)\ldots M_{K+J}(X_{K+J})\right) \in \mathbb{C}^n \setminus \{0\}$$

is surjective when $J \geq 0$, since

$$S_{K+J} \subset \{(X_1,\ldots,X_{K+J}); (X_1,\ldots,X_K) \in S_K\}.$$

Therefore is enough to show the lemma for

$$\pi_n\left(M_1(X_1)M_2(X_2)M_3(X_3)\right).$$

First

$$\pi_n\left(M_1(X_1)M_2(X_2)\right)$$

$$= \pi_n\left(\begin{pmatrix} 1 & 0 & \ldots & 0 \\ x_{21,1} & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ x_{n1,1} & \ldots & x_{n(n-1),1} & 1 \end{pmatrix}\begin{pmatrix} 1 & x_{12,2} & \ldots & x_{1n,2} \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & x_{(n-1)n,2} \\ 0 & \ldots & 0 & 1 \end{pmatrix}\right)$$

$$= \left(x_{n1,1}\ ,\ x_{n2,1} + x_{n1,1}x_{12,2}\ ,\ \ldots\ ,\ x_{n(n-1),1}\right.$$

$$\left. + \sum_{j=1}^{n-2} x_{nj,1}x_{j(n-1),2}\ , 1 + \sum_{j=1}^{n-1} x_{nj,1}x_{jn,2}\right).$$

It is clear that we can map onto the set

$$\{(a_1,\ldots,a_n) \in \mathbb{C}^n \setminus \{0\}; a_1 \neq 0\}.$$

To map onto $\mathbb{C}^n \setminus \{0\}$ we need to use a third matrix. Consider matrices $M_1(X_1)$ and $M_2(X_2)$ such that

$$\pi_n(M_1(X_1)M_2(X_2)) = (1, a_2, \ldots, a_n).$$

For such matrices we have

$$\pi_n\left(M_1(X_1)M_2(X_2)M_3(X_3)\right)$$

$$= \pi_n\left(\begin{pmatrix} \star & \cdots & \cdots & \star \\ \vdots & \ddots & \ddots & \vdots \\ \star & \cdots & \cdots & \star \\ 1 & a_2 & \cdots & a_n \end{pmatrix} \begin{pmatrix} 1 & 0 & \cdots & 0 \\ x_{21,3} & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ x_{n1,3} & \cdots & x_{n(n-1),3} & 1 \end{pmatrix}\right)$$

$$= \left(1 + \sum_{j=2}^n a_j x_{j1,3}\ ,\ a_2 + \sum_{j=3}^n a_j x_{j2,3}\ ,\ \ldots\ ,\ a_n\right).$$

We can choose $X_3, a_2, \ldots, a_n$ freely to produce any vector in $\mathbb{C}^n \setminus \{0\}$. Note that we cannot produce 0 since this would force $a_2 = \cdots = a_n = 0$. $\square$

We turn to the proof of Lemma 2.6.

Proof of Lemma 2.6. We begin with the base case $K = 2$. Let

$$(P_{1,1}(Z_1), \ldots, P_{n,1}(Z_1)) = \pi_n\left(\begin{pmatrix} 1 & 0 & \cdots & 0 \\ z_{21,1} & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ z_{n1,1} & \cdots & z_{n(n-1),1} & 1 \end{pmatrix}^{-1}\right)$$

and

$$(P_{1,2}(Z_1, Z_2), \ldots, P_{n,2}(Z_1, Z_2))$$

$$= \pi_n\left(\begin{pmatrix} 1 & 0 & \cdots & 0 \\ z_{21,1} & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ z_{n1,1} & \cdots & z_{n(n-1),1} & 1 \end{pmatrix}^{-1} \begin{pmatrix} 1 & z_{12,2} & \cdots & z_{1n,2} \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & z_{(n-1)n,2} \\ 0 & \cdots & 0 & 1 \end{pmatrix}^{-1}\right).$$

We get relations by studying

$$\begin{pmatrix} \star & \cdots & \star \\ \vdots & \ddots & \vdots \\ \star & \cdots & \star \\ P_{1,1}(Z_1) & \cdots & P_{n,1}(Z_1) \end{pmatrix}$$

$$= \begin{pmatrix} \star & \cdots & \star \\ \vdots & \ddots & \vdots \\ \star & \cdots & \star \\ P_{1,2}(Z_1, Z_2) & \cdots & P_{n,2}(Z_1, Z_2) \end{pmatrix} \begin{pmatrix} 1 & z_{12,2} & \cdots & z_{1n,2} \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & z_{(n-1)n,2} \\ 0 & \cdots & 0 & 1 \end{pmatrix}.$$

They are

$$P_{1,2}(Z_1, Z_2) = P_{1,1}(Z_1)$$
$$P_{2,2}(Z_1, Z_2) = P_{2,1}(Z_1) - z_{12,2}P_{1,2}(Z_1, Z_2)$$

$$\vdots$$

(1) $$P_{k,2}(Z_1, Z_2) = P_{k,1}(Z_1) - \sum_{j=1}^{k-1} z_{jk,2}P_{j,2}(Z_1, Z_2)$$

$$\vdots$$

$$P_{n,2}(Z_1, Z_2) = P_{n,1}(Z_1) - \sum_{j=1}^{n-1} z_{jn,2}P_{j,2}(Z_1, Z_2).$$

We need to establish at which points $dP_{1,2} \wedge dP_{2,2} \wedge \cdots \wedge dP_{n,2} = 0$. Note that $P_{n,1} \equiv 1$. Also note that $dP_{1,1} \wedge \cdots \wedge dP_{n-1,1}$ never vanishes since $dz_{21,1} \wedge \cdots \wedge dz_{n1,1} \wedge \cdots \wedge dz_{n(n-1),1}$ never vanishes, and $P_{1,1}, \ldots, P_{n-1,1}$ are components of an automorphism of $\mathbb{C}^{n(n-1)/2}$ induced by $M_1(Z_1) \mapsto M_1(Z_1)^{-1}$. Set $\Omega_2 = dP_{1,2} \wedge \cdots \wedge dP_{n,2}$. We will show that this form is zero if and only if $P_{1,2}(Z_1, Z_2) = \cdots = P_{n-1,2}(Z_1, Z_2) = 0$. Indeed, when we plug in

$$dP_{k,2}(Z_1, Z_2) = dP_{k,1}(Z_1) - \sum_{j=1}^{k-1} P_{j,2}(Z_1, Z_2)\, dz_{jk,2} - \sum_{j=1}^{k-1} z_{jk,2}\, dP_{j,2}(Z_1, Z_2)$$

to calculate Ω_2 we see that the terms in the last sum do not contribute to the result. Next one sees that Ω_2 contains summands of the form

$$(P_{k,2})^{n-k}\, dP_{1,1} \wedge \cdots \wedge dP_{k,1} \wedge dz_{k(k+1),2} \wedge \cdots \wedge dz_{kn,2},$$

and these are the only summands containing the wedge $dz_{k(k+1),2} \wedge \cdots \wedge dz_{kn,2}$. This implies that at points where Ω_2 vanishes we have

$$P_{1,2}(Z_1, Z_2) = \cdots = P_{n-1,2}(Z_1, Z_2) = 0$$

and all other summands involve products of these functions as coefficients. Thus Ω_2 vanishes if and only if $P_{1,2}(Z_1, Z_2) = \cdots = P_{n-1,2}(Z_1, Z_2) = 0$.

From (1) we see that this is equivalent to $P_{1,1}(Z_1) = \cdots = P_{n-1,1}(Z_1) = 0$. Note that $P_{1,1}, \ldots, P_{n-1,1}$ are components of the automorphism of $\mathbb{C}^{n(n-1)/2}$ induced by $M_1(Z_1) \mapsto M_1(Z_1)^{-1}$. Since this automorphism fixes $S_2 = \{z_{n1,1} = \cdots = z_{n(n-1),1} = 0\}$, we conclude that Φ_2 is submersive exactly at points outside S_2. In order to make our induction step we need some further properties. Note that at points where $P_{1,2} = \cdots = P_{n-1,2} = 0$, that is in S_2, we have

$$dP_{1,2} \wedge \cdots \wedge dP_{n-1,2} = dP_{1,1} \wedge \cdots \wedge dP_{n-1,1} \neq 0.$$

We also have $dP_{n,1} \equiv 0$ since $P_{n,1}(Z_1) \equiv 1$.

We now consider K odd. Our induction assumptions are besides the description of the nonsubmersivity set S_{K-1} the following:

If K is odd and $dP_{1,K-1} \wedge \cdots \wedge dP_{n,K-1} = 0$, then

(I_{K-1}): $dP_{n,K-2} = 0$,

(II_{K-1}): $dP_{1,K-1} \wedge \cdots \wedge dP_{n-1,K-1} \neq 0$, and

(III_{K-1}): $P_{j,K-1} = 0$ for $1 \leq j \leq n-1$.

We now describe the induction step from $K-1$ to K when K is odd. Doing similar calculations as for $K = 2$ we get the relations

$$
\begin{aligned}
(2) \\
P_{n,K}(Z_1, \ldots, Z_K) &= P_{n,K-1}(Z_1, \ldots, Z_{K-1}) \\
P_{n-1,K}(Z_1, \ldots, Z_K) &= P_{n-1,K-1}(Z_1, \ldots, Z_{K-1}) - z_{n(n-1),K} P_{n,K}(Z_1, \ldots, Z_K) \\
&\vdots \\
P_{k,K}(Z_1, \ldots, Z_K) &= P_{k,K-1}(Z_1, \ldots, Z_{K-1}) - \sum_{j=k+1}^{n} z_{jk,K} P_{j,K}(Z_1, \ldots, Z_K) \\
&\vdots \\
P_{1,K}(Z_1, \ldots, Z_K) &= P_{1,K-1}(Z_1, \ldots, Z_{K-1}) - \sum_{j=2}^{n} z_{j1,K} P_{j,K}(Z_1, \ldots, Z_K).
\end{aligned}
$$

We see that

$$
\begin{aligned}
\Omega_K &= dP_{n,K} \wedge dP_{n-1,K} \wedge \cdots \wedge dP_{1,K} \\
&= dP_{n,K-1} \wedge \cdots \wedge dP_{1,K-1} + \text{ terms involving } dz_{jl,K}.
\end{aligned}
$$

At points where Ω_K vanishes, the form $\Omega_{K-1} = dP_{n,K-1} \wedge \cdots \wedge dP_{1,K-1}$ must vanish since it involves no terms in $dz_{jl,K}$. By the induction hypotheses this forces

$$
Z \in \{(Z_1, \ldots, Z_K); (Z_1, \ldots, Z_{K-1}) \in S_{K-1}\}.
$$

A calculation shows that at these points

$$
P_{n,K-1}(Z_1, \ldots, Z_{K-1}) = P_{n,K}(Z_1, \ldots, Z_K) = 1.
$$

Plugging in

$$
\begin{aligned}
dP_{k,K}(Z_1, \ldots, Z_K) = dP_{k,K-1}(Z_1, \ldots, Z_{K-1}) &- \sum_{j=k+1}^{n} P_{j,K}(Z_1, \ldots, Z_K)\, dz_{jk,K} \\
&- \sum_{j=k+1}^{n} z_{jk,K}\, dP_{j,K}(Z_1, \ldots, Z_K)
\end{aligned}
$$

to calculate Ω_K we see that the terms in the last sum do not contribute to the result. We also find a term of the following form:

$$
(P_{n,K})^{n-1}\, dP_{n,K} \wedge dz_{n(n-1),K} \wedge \cdots \wedge dz_{nk,K} \wedge \cdots \wedge dz_{n1,K}.
$$

Since $P_{n,K} = 1$ at these points, we see that we must have $dP_{n,K} = dP_{n,K-1} = 0$ in order for Ω_K to vanish, and this obviously implies that Ω_K vanishes. We have

$$P_{n,K-1} = P_{n,K-2} - \sum_{j=1}^{n-1} z_{jn,K-1} P_{j,K-1}$$

and

$$dP_{n,K-1} = dP_{n,K-2} - \sum_{j=1}^{n-1} P_{j,K-1}\, dz_{jn,K-1} - \sum_{j=1}^{n-1} z_{jn,K-1}\, dP_{j,K-1}.$$

By the induction assumption at points where Ω_{K-1} vanishes, we have $dP_{n,K-2} = 0$ and $P_{j,K-1} = 0$ for $1 \leq j \leq n-1$. Therefore

$$dP_{n,K-1} = -\sum_{j=1}^{n-1} z_{jn,K-1}\, dP_{j,K-1}$$

at these points. Moreover, by the induction hypotheses, $dP_{1,K-1} \wedge \cdots \wedge dP_{n-1,K-1} \neq 0$, meaning that $dP_{1,K-1}, \ldots, dP_{n-1,K-1}$ are linearly independent at these points, which implies $z_{jn,K-1} = 0$ for $1 \leq j \leq n-1$. Therefore the mapping is nonsubmersive exactly in S_K.

To pass from K to $K+1$, that is from odd to even, we like to establish (IV_K) and (V_K) below. We have already established (IV_K) above. To prove (V_K) look at

$$\begin{aligned}
&dP_{1,K} \wedge \cdots \wedge dP_{n-1,K} \\
&= dP_{1,K-1} \wedge \cdots \wedge dP_{n-1,K-1} + \text{ terms involving } dz_{jl,K} \neq 0.
\end{aligned}$$

The nonvanishing of the left-hand side at points in S_K follows from (II_{K-1}). This concludes the induction step from $K-1$ to K for odd K. However, let us for future use explicitly state that

(3) $dP_{n,K}$ vanishes at all points in S_K.

Consider the case K even. Our induction assumption are besides the description of the nonsubmersivity set S_{K-1} the following:

If K is even and $dP_{1,K-1} \wedge \cdots \wedge dP_{n,K-1} = 0$, then

(IV_{K-1}): $dP_{1,K-1} \wedge \cdots \wedge dP_{n-1,K-1} \neq 0$ and
(V_{K-1}): $P_{n,K-1} = 1$.

We have

$$P_{1,K} = P_{1,K-1}$$
$$P_{2,K} = P_{2,K-1} - z_{12,K}P_{1,K}$$

(4) $$\vdots$$

$$P_{n,K} = P_{n,K-1} - \sum_{j=1}^{n-1} z_{jn,K}P_{j,K}.$$

We see that

$$\Omega_K = dP_{1,K} \wedge \cdots \wedge dP_{n,K} =$$
$$= dP_{1,K-1} \wedge \cdots \wedge dP_{n,K-1} + \text{ terms involving } dz_{jl,K}.$$

At points where Ω_K vanishes, the form $\Omega_{K-1} = dP_{1,K-1} \wedge \cdots \wedge dP_{n,K-1}$ must vanish since it involves no terms in $dz_{jl,K}$. By the induction hypotheses this forces

$$Z \in \widetilde{S}_{K-1} = \{(Z_1, \ldots, Z_K); (Z_1, \ldots, Z_{K-1}) \in S_{K-1}\}.$$

We will show that Ω_K vanishes at points in $\widetilde{S}_{K-1}$ if and only if $P_{1,K}(Z_1, \ldots, Z_K)$ $= \cdots = P_{n-1,K}(Z_1, \ldots, Z_K) = 0$. Indeed, when we plug in

$$dP_{k,K} = dP_{k,K-1} - \sum_{j=1}^{k-1} P_{j,K}\, dz_{jk,K} - \sum_{j=1}^{k-1} z_{jk,K}\, dP_{j,K}$$

to calculate Ω_K we see that the terms in the last sum do not contribute to the result. Next one sees that Ω_K contains summands of the form

$$(P_{k,K})^{n-k}\, dP_{1,K-1} \wedge \cdots \wedge dP_{k,K-1} \wedge dz_{k(k+1),K} \wedge \cdots \wedge dz_{kn,K},$$

and these are the only summands containing the wedge $dz_{k(k+1),K} \wedge \cdots \wedge dz_{kn,K}$. Using ($IV_{K-1}$) we see that for all $1 \le k \le n-1$, the wedge $dP_{1,K-1} \wedge \cdots \wedge dP_{k,K-1}$ never vanishes on $\widetilde{S}_{K-1}$. Therefore on $\widetilde{S}_{K-1}$ the vanishing of Ω_K implies $P_{1,K} = \cdots = P_{n-1,K} = 0$. All other summands involve products of these functions as coefficients. Thus Ω_K vanishes if and only if $Z \in \widetilde{S}_{K-1}$ and $P_{1,K} = \cdots = P_{n-1,K} = 0$. Inspecting (4) we see that at these points $P_{j,K} = P_{j,K-1} = 0$ for $1 \le j \le n-1$ and $P_{n,K} = P_{n,K-1}$. By (V_{K-1}) we have $P_{n,K} = P_{n,K-1} = 1$.

Going back one step to $K-1$ we have

$$P_{n,K-1} = P_{n,K-2}$$
$$P_{n-1,K-1} = P_{n-1,K-2} - z_{n(n-1),K-1}P_{n,K-1}$$

$$\vdots$$

$$P_{1,K-1} = P_{1,K-2} - \sum_{j=2}^{n} z_{j1,K-1}P_{j,K-1},$$

and since $P_{j,K-2} = 0$ for $1 \leq j \leq n-1$ at points in $\widetilde{S}_{K-2} \supset \widetilde{S}_{K-1}$, we see that we must have

$$P_{j,K} = P_{j,K-1} = -z_{nj,K-1} = 0$$

for $1 \leq j \leq n-1$ at the points we are considering. This implies that the mapping is submersive exactly at points outside S_K.

Finally we need justify the induction assumptions (I_K), (II_K), and (III_K). We have already established (III_K). We also see that

$$dP_{1,K} \wedge \cdots \wedge dP_{n-1,K} = dP_{1,K-1} \wedge \cdots \wedge dP_{n-1,K-1} + \text{ terms involving } dz_{jl,K},$$

and the first term is nonvanishing on $\widetilde{S}_{K-1}$ by (IV_{K-1}) (and thus on S_K). This establishes (II_K). By (3) we know that $dP_{n,K-1}$ vanishes identically on $\widetilde{S}_{K-1}$ and therefore on S_K, and this is (I_K). This concludes our induction step and the lemma follows. $\square$

Remark 4.1. Let us make some observations about the sets S_K and how they are situated in relation to the fibers of the mapping. First when K is even one sees that the image of S_K using the map $\Phi_K \colon \left(\mathbb{C}^{n(n-1)/2}\right)^K \to \mathbb{C}^n \setminus \{0\}$ is $(0, \ldots, 0, 1)$ so the points in S_K are all contained in $\Phi^{-1}(\{(0, \ldots, 0, 1)\})$. When K is odd the image of S_K is $\{(z_1, \ldots, z_n) \in \mathbb{C}^n \setminus \{0\}; z_n = 1\}$.

5. **Proof of Lemma 3.7**

Definition 5.1. We say that a polynomial $p(x_1, \ldots, x_n) \in \mathbb{C}[\mathbb{C}^n]$ is no more than linear in x_k if there exist two polynomials $\widetilde{p}, \widetilde{q} \in \mathbb{C}[\mathbb{C}^n]$ both independent of x_k such that

$$p = x_k \widetilde{p} + \widetilde{q}.$$

We need the following two lemmata.

LEMMA 5.2. *Let* $p(x_1, \ldots, x_n) \in \mathbb{C}[\mathbb{C}^n]$ *be a polynomial which is no more than linear in each variable. Then the vector fields*

$$V_{ij,p} = \frac{\partial p}{\partial x_i} \frac{\partial}{\partial x_j} - \frac{\partial p}{\partial x_j} \frac{\partial}{\partial x_i}$$

for $1 \leq i < j \leq n$ *are globally integrable on* $\mathbb{C}^n$.

Proof. Note that $\partial p / \partial x_i$ is no more than linear in x_j and independent of x_i since p is no more than linear in each variable separately. Hence the vector field $\partial p / \partial x_i (\partial / \partial x_j)$ is globally integrable and independent of x_i. Similarly the vector field $\partial p / \partial x_j (\partial / \partial x_i)$ is globally integrable and independent of x_j. Therefore the vector fields $V_{ij,p}$ are globally integrable. $\square$

LEMMA 5.3. *Let* $p(x_1, \ldots, x_n) \in \mathbb{C}[\mathbb{C}^n]$ *and*

$$F_p(c) = \{X = (x_1, \ldots, x_n) \in \mathbb{C}^n; p(X) = c\}$$

be the fiber of p over the value c. Then the vector fields

$$V_{ij,p} = \frac{\partial p}{\partial x_i}\frac{\partial}{\partial x_j} - \frac{\partial p}{\partial x_j}\frac{\partial}{\partial x_i}, \ 1 \le i < j \le n$$

span the tangent space of $F_p(c)$ at all smooth points $X \in F_p(c)$ (i.e., those points where dp does not vanish.)

Proof. We have

$$V_{ij,p}(p - c) = \frac{\partial p}{\partial x_i}\frac{\partial p}{\partial x_j} - \frac{\partial p}{\partial x_j}\frac{\partial p}{\partial x_i} \equiv 0$$

so the vector fields are tangential to $F_p(c)$. We need to show that

$$\dim \mathrm{span}\,(V_{ij,p}; 1 \le i < j \le n) = n - 1.$$

But this is obvious since at points where $dp \neq 0$, at least one of the components, say $\partial p/\partial x_n$, is nonzero and then

$$\dim \mathrm{span}\,(V_{in,p}; 1 \le i \le n - 1) = n - 1. \qquad \square$$

Proof of Lemma 3.7. In order to construct a spray we will produce globally integrable vector fields that span the tangent spaces of the fibers of Φ_K. These fibers are given by n polynomial equations in $Kn(n - 1)/2$ variables. It is difficult to produce globally integrable vector fields that leave these polynomials invariant. The main goal of our proof will be to reduce, on each stratum individually, these polynomial equations to essentially a single polynomial equation. This polynomial equation will be no more than linear in each variable, and therefore Lemmas 5.2 and 5.3 will provide us with the desired integrable fields.

Recall that we have the relations

$$
\begin{aligned}
P_{1,K} &= P_{1,K-1} \\
P_{2,K} &= P_{2,K-1} - z_{12,K}P_{1,K} \\
&\vdots \\
P_{k,K} &= P_{k,K-1} - \sum_{j=1}^{k-1} z_{jk,K}P_{j,K} \\
&\vdots \\
P_{n,K} &= P_{n,K-1} - \sum_{j=1}^{n-1} z_{jn,K}P_{j,K}
\end{aligned}
$$

(5)

when K is even and

$$P_{n,K} = P_{n,K-1}$$
$$P_{n-1,K} = P_{n-1,K-1} - z_{n(n-1),K}P_{n,K}$$
$$\vdots$$

(6)
$$P_{k,K} = P_{k,K-1} - \sum_{j=k+1}^{n} z_{jk,K}P_{j,K}$$

$$\vdots$$

$$P_{1,K} = P_{1,K-1} - \sum_{j=2}^{n} z_{j1,K}P_{j,K}$$

when K is odd.

Lets make the following

Observation $(\star)$. From (5) and (6) one easily deduces by induction that the map $\Phi_K = (P_{1,K}, \ldots, P_{n,K})$ has polynomial entries that are no more than linear in each variable. Using (5) one sees that $P_{1,K}$ is independent of Z_K, $P_{2,K}$ depends only on $Z_1, \ldots, Z_{K-1}, z_{12,K}$, and in general $P_{k,K}$ depends only on $Z_1, \ldots, Z_{K-1}$ and $z_{ij,K}$ for $1 \leq i < j \leq k$ when K is even. When K is odd one concludes, using (6), that $P_{k,K}$ depends only on $Z_1, \ldots, Z_{K-1}$ and $z_{ij,K}$ for $k \leq j < i \leq n$.

We will begin by considering the case K even. Here we will stratify $\mathbb{C}^n \backslash \{0\}$ as

- $V_n = \mathbb{C}^n \setminus \{0\}$,
- $V_{n-k} = \{(z_1, \ldots, z_n) \in \mathbb{C}^n \backslash \{0\}; z_1 = \cdots = z_k = 0\}$ when $1 \leq k \leq n-1$, and
- $V_0 = \emptyset$.

First consider a fiber over a point $a = (a_1, \ldots, a_n) \in V_n \setminus V_{n-1}$. Here we have

$$P_{1,K} = a_1 \neq 0$$
$$P_{2,K} = P_{2,K-1} - z_{12,K}P_{1,K} = a_2$$

$$\vdots$$

$$P_{n,K} = P_{n,K-1} - \sum_{j=1}^{n-1} z_{jn,K}P_{j,K} = a_n.$$

Put $Z_K = (Z'_K, Z''_K)$, where $Z''_K = (z_{12,K}, \ldots, z_{1n,K})$ and Z'_K consists of the other variables in Z_K. We see that the fiber $\Phi_K^{-1}(a)$ is biholomorphic to (it is a graph over)

$$B_n(a) = \{Z = (Z_1, \ldots, Z'_K) \in (\mathbb{C}^{n(n-1)/2})^{K-1} \times \mathbb{C}^{(n-1)(n-2)/2}; P_{1,K}(Z) = a_1\}$$

(remember that $P_{1,K}$ is independent of Z_K) since

$$z_{12,K} = \frac{P_{2,K-1} - a_2}{a_1}$$

$$\vdots$$

$$z_{1n,K} = \frac{P_{n,K-1} - a_n - \sum_{j=2}^{n-1} z_{jn,K}a_j}{a_1}.$$

This also shows that the fibration $\Phi_K \colon \Phi_K^{-1}(V_n \setminus V_{n-1}) \to V_n \setminus V_{n-1}$ is biholomorphic to the fibration

$$\{(Z_1, \ldots, Z_K', a) \in \mathbb{C}^M \times (V_n \setminus V_{n-1}); P_{1,K}(Z) = a_1\}$$

$$\downarrow {\scriptstyle (Z_1,\ldots,Z_K',a) \mapsto a}$$

$$V_n \setminus V_{n-1},$$

where $\mathbb{C}^M = (\mathbb{C}^{n(n-1)/2})^{K-1} \times \mathbb{C}^{(n-1)(n-2)/2}$. The vector fields $V_{ij,P_{1,K}}$, where i, j run through all pairs of variables in $\mathbb{C}^M$, are globally integrable and span the tangent space of each individual fiber by Lemmas 5.2 and 5.3 (since the fibers of Φ_K over $V_n \setminus V_{n-1}$ are smooth and biholomorphic to $\{P_{1,K}(Z) = a_1\}$; see Remark 4.1). This gives us the vector fields needed to conclude that the restricted submersion over $V_n \setminus V_{n-1}$ admits a spray.

Next let us study the fiber over a point $a = (0, a_2, \ldots, a_n) \in V_{n-1} \setminus V_{n-2}$. Here the relations for the fiber are

$$P_{1,K} = P_{1,K-1} = 0$$
$$P_{2,K} = P_{2,K-1} - z_{12,K}P_{1,K} = a_2 \neq 0$$

$$\vdots$$

$$P_{n,K} = P_{n,K-1} - \sum_{j=1}^{n-1} z_{jn,K}P_{j,K} = a_n.$$

Since $P_{1,K} = 0$, the system is equivalent to

$$P_{1,K} = P_{1,K-1} = 0$$
$$P_{2,K} = P_{2,K-1} = a_2 \neq 0$$

$$\vdots$$

$$P_{n,K} = P_{n,K-1} - \sum_{j=2}^{n-1} z_{jn,K}P_{j,K} = a_n,$$

and $z_{12,K}, \ldots, z_{1n,K}$ are free variables. As in the case above we can (using $a_2 \neq 0$) solve the last $n-2$ equations for the variables $z_{23,K}, \ldots, z_{2n,K}$, namely

$$z_{23,K} = \frac{P_{3,K-1} - a_3}{a_2}$$

$$\vdots$$

$$z_{2n,K} = \frac{P_{n,K-1} - a_n - \sum_{j=3}^{n-1} z_{jn,K} a_j}{a_2}.$$

Put $Z_K = (Z'_K, Z''_K)$ where $Z''_K = (z_{12,K}, \ldots, z_{1n,K}, z_{23,K}, \ldots, z_{2n,K})$ and Z'_K consists of the other variables in Z_K. We see that the fiber $\Phi_K^{-1}(a)$ is biholomorphic to

$$B_{n-1}(a) = \{Z = (Z_1, \ldots, Z'_K) \in (\mathbb{C}^{n(n-1)/2})^{K-1} \times \mathbb{C}^{(n-2)(n-3)/2};$$

$$P_{1,K}(Z) = 0, P_{2,K}(Z) = a_2\} \times \mathbb{C}^{n-1}_{(z_{12,K}, \ldots, z_{1n,K})}.$$

This system of two equations can be reduced to one equation by going back one step and using the last equation of (6) which says

$$P_{1,K-1} = \left(P_{1,K-2} - \sum_{j=3}^{n} z_{j1,K-1} P_{j,K-1} \right) - z_{21,K-1} P_{2,K-1} = 0.$$

It allows us to solve for $z_{21,K-1}$:

$$z_{21,K-1} = \frac{\left(P_{1,K-2} - \sum_{j=3}^{n} z_{j1,K-1} P_{j,K-1} \right)}{a_2}.$$

From Observation ($\star$) we see that

$$\left(P_{1,K-2} - \sum_{j=3}^{n} z_{j1,K-1} P_{j,K-1} \right)$$

does not depend on $z_{21,K-1}$. Putting

$$\mathbb{C}^M = (\mathbb{C}^{n(n-1)/2})^{K-2} \times \mathbb{C}^{n(n-1)/2-1} \times \mathbb{C}^{(n-2)(n-3)/2}$$

and

$$X = (Z_1, \ldots, Z_{K-2}, \ldots, \widehat{z}_{21,K-1}, \ldots, Z'_K) \in \mathbb{C}^M,$$

we have shown that the fibration $\Phi_K \colon \Phi_K^{-1}(V_{n-1} \setminus V_{n-2}) \to V_{n-1} \setminus V_{n-2}$ is biholomorphic to the fibration

$$\{(X, a) \in \mathbb{C}^M \times (V_{n-1} \setminus V_{n-2}); P_{2,K-1}(X) = a_2\} \times \mathbb{C}^{n-1}_{(z_{12,K}, \ldots, z_{1n,K})}$$

$$\downarrow {\scriptstyle (X, z_{12,K}, \ldots, z_{1n,K}, a) \mapsto a}$$

$$V_{n-1} \setminus V_{n-2},$$

and the spray is constructed as above.

For general k we proceed analogously. Using $a_1 = \cdots = a_{k-1} = 0$ we get free variables $z_{ij,K}$ for $1 \le i \le k-1$ and $i < j \le n$, and the first k equations become

$$P_{1,K-1} = \cdots = P_{k-1,K-1} = 0$$

and

$$P_{k,K-1} = a_k.$$

We now solve the last $n-k$ equations for the variables $z_{k(k+1),K}, \ldots, z_{kn,K}$ using that $a_k \ne 0$. Then we go one step back and use the last $k-1$ equations of (6) to rewrite our first $k-1$ equations. This allow us to solve for the variables

$$z_{k1,K-1}, z_{k2,K-1}, \ldots, z_{k(k-1),K-1}.$$

Also the variables $z_{ij,K-1}$, for $1 \le j \le k-2$ and $j < i \le k-1$, become free variables. This shows that the fibration $\Phi_K \colon \Phi_K^{-1}(V_{n-k+1} \setminus V_{n-k}) \to V_{n-k+1} \setminus V_{n-k}$ is biholomorphic to the fibration

$$\{(X,a) \in \mathbb{C}^M \times (V_{n-k+1} \setminus V_{n-k}); P_{k,K-1}(X) = a_k\} \times \mathbb{C}^N_w$$

$$\downarrow {\scriptstyle (X,w,a) \mapsto a}$$

$$V_{n-k+1} \setminus V_{n-k}$$

for appropriate N. The globally integrable vector fields $V_{ij,P_{k,K}}$ where i,j run through all pairs of variables in $\mathbb{C}^M$ together the fields $\partial/\partial w_l$ where l runs over all variables in $\mathbb{C}^N_w$ give the spray on the (smooth part, i.e., when S_K is removed, of the) fibers over $V_{n-k+1} \setminus V_{n-k}$. Remembering Remark (4.1) we have smooth fibers over all strata except for the very last stratum where we have a nonsmooth fiber over $(0, \ldots, 0, 1)$.

When K is odd the method is basically the same. Only here the stratification is $V_n = \mathbb{C}^n \setminus \{0\}$, $V_{n-k} = \{(z_1, \ldots, z_n) \in \mathbb{C}^n \setminus \{0\}; z_{n-k+1} = \cdots = z_n = 0\}$ when $1 \le k \le n-1$, and $V_0 = \emptyset$. On $V_{n-k+1} \setminus V_{n-k}$ one show that the fibers are biholomorphic to $\{P_{n-k+1,K-1} = a_{n-k+1}\}$ times free variables. Note that the singular fibers are contained in the first stratum $V_n \setminus V_{n-1}$; see Remark 4.1. $\square$

Remark 5.4. We do not know whether there is a possibly finer stratification so that the restricted submersions are locally trivial fiber bundles. In some cases we see from the explicit form of the polynomials that we have local triviality. We have not been able to decide this in all cases.

6. On the number of factors

A natural question to ask is how the number of factors needed in the factorization depends on the space X and the map f. In the algebraic setting there is no such uniform bound as proved by van der Kallen in [vdK82]. However in the holomorphic setting (exactly as in the topological setting) it is easy to see that there is an upper bound depending only on the dimension of the space $X (= m)$ and the size of the matrix $(= n)$.

It follows from Vaserstein's result (Theorem 2.2) that there exists a uniform bound K depending on the dimension of the space X $(= m)$ and the size of the matrix $= n$ such that the fibration

$$
\begin{array}{c}
p^\star Y \\
\downarrow {\scriptstyle p^\star \Phi_K} \\
X
\end{array}
$$

from the proof of Proposition 2.8 has a topological section and hence a holomorphic section. Going through the induction over the size of the matrix as in the proof of Theorem 2.3 we conclude that there is a uniform bound even in the holomorphic case.

Another way to prove the existence of such a uniform bound is the following. Suppose it would not exist, i.e., for all natural numbers i there are Stein spaces X_i of dimension m and holomorphic maps $f_i \colon X_i \to \mathrm{SL}_n(\mathbb{C})$ such that f_i does not factor over a product of less than i unipotent matrices. Set $X = \cup_{i=1}^\infty X_i$ the disjoint union of the spaces X_i and $F \colon X \to \mathrm{SL}_n(\mathbb{C})$ the map that is equal to f_i on X_i. By our main result F factors over a finite number of unipotent matrices. Consequently all f_i factor over the same number of unipotent matrices which contradicts the assumption on f_i.

Thus we proved

THEOREM 6.1. *There is a natural number K such that for any reduced Stein space X of dimension m and any null-homotopic holomorphic mapping $f \colon X \to \mathrm{SL}_n(\mathbb{C})$ there exist holomorphic mappings $G_1, \ldots, G_K \colon X \to \mathbb{C}^{n(n-1)/2}$ such that*

$$
f(x) = M_1(G_1(x)) \cdots M_K(G_K(x))
$$

for every $x \in X$.

Let us denote by $K_{\mathcal{C}}(m, n)$ the number of matrices needed to factorize any null-homotopic map from a Stein space of dimension m into $\mathrm{SL}_n(\mathbb{C})$ by continuous triangular matrices and the number needed in the holomorphic case by $K_{\mathcal{O}}(m, n)$. We do not know these numbers. We know that the Cohn example can be factored as four matrices with continuous entries but if one wants to factor it using matrices with holomorphic entries, one needs five matrices. It is natural to ask the following question.

PROBLEM 6.2. *How are the numbers $K_{\mathcal{C}}(m, n)$ and $K_{\mathcal{O}}(m, n)$ exactly related? Obviously $K_{\mathcal{C}}(m, n) \leq K_{\mathcal{O}}(m, n)$.*

Examining our proof in the case $n = 2$ one easily deduces the estimate $K_{\mathcal{O}}(m, 2) \leq K_{\mathcal{C}}(m, 2) + 4$. At least for the case $n = 2$ we believe the answer to the above question can be found. Some more precise results on the number of factors are obtained by the authors in [IK].

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