Annals of Mathematics **175** (2012), 1–21
http://dx.doi.org/10.4007/annals.2012.175.1.1

Nonhomogeneous locally free actions of the affine group

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Abstract

We classify smooth locally free actions of the real affine group on closed orientable three-dimensional manifolds up to smooth conjugacy. As a corollary, there exists a nonhomogeneous action when the manifold is the unit tangent bundle of a closed surface with a hyperbolic metric.

1. Introduction

Let GA be the group of orientation preserving affine transformations of the real line. In 1979, Ghys [5] showed that any volume-preserving locally free GA-action on a closed three-dimensional manifold is smoothly conjugate to a homogeneous action. It is natural to ask whether the assumption on an invariant volume is removable or not, and it was open for almost thirty years. In this paper, we give a classification of locally free GA-actions on closed orientable three-dimensional manifolds up to smooth conjugacy. As a corollary, there *exists* an action with no invariant volume.

Let G be a Lie group and M a manifold. We say that a C^∞ right G-action $\rho : M \times G \to M$ is *locally free* if the isotropy subgroup $\{g \in G \mid \rho(p,g) = p\}$ is discrete in G for any $p \in M$. By $\mathcal{O}_\rho(p)$, we denote the orbit $\{\rho(p,g) \mid g \in G\}$ of $p \in M$. If the action ρ is locally free, the decomposition $\mathcal{O}_\rho = \{\mathcal{O}_\rho(p) \mid p \in M\}$ forms a C^∞ foliation on M. We say that two right G-actions ρ_1 and ρ_2 on manifolds M_1 and M_2 are C^∞ *conjugate* if there exists an isomorphism σ of G and a C^∞ diffeomorphism H from M_1 to M_2 such that $H(\rho_1(p,g)) = \rho_2(H(p), \sigma(g))$ for any $g \in G$ and $p \in M_1$. The map H is called *a* C^∞ *conjugacy* between ρ_1 and ρ_2.

There are two classical examples of locally free GA-actions on closed three-dimensional manifolds. Let $\mathrm{PSL}(2,\mathbb{R})$ be the group of orientation preserving projective transformations of the real projective line $\mathbb{R}P^1$ and $\widetilde{\mathrm{PSL}}(2,\mathbb{R})$ its universal covering group. The group $\widetilde{\mathrm{PSL}}(2,\mathbb{R})$ contains a closed subgroup H

Partially supported by JSPS PostDoctoral Fellowships for Research Abroad and Grant-in-Aid for Encouragement of Young Scientists (B) (No.19740085).

which is isomorphic to GA, e.g., the group of elements which fix $\infty \in \mathbb{R}P^1 = \mathbb{R} \cup \{\infty\}$. Fix an isomorphism $i : \mathrm{GA} \to H$. For each cocompact lattice Γ, *the standard* GA-*action* ρ_Γ on $\Gamma\backslash\widetilde{\mathrm{PSL}}(2,\mathbb{R})$ is given by $\rho_\Gamma(\Gamma g, h) = \Gamma(g \cdot i(h))$. A similar construction gives another locally free action ρ_{solv} when we start with a three-dimensional solvable subgroup

$$\mathrm{Solv} = \{h_{t,x,y} : (u,v) \mapsto (e^{-t}u + x, e^t v + y) \mid t, x, y \in \mathbb{R}\}$$

of the group of affine transformations of $\mathbb{R}^2$. In this case, GA is identified with the subgroup $\{h_{t,x,0} \mid t, x \in \mathbb{R}\}$. We say that a GA-action is *homogeneous* if it is C^∞ conjugate to one of the above examples.

As mentioned at the beginning, Ghys showed the rigidity of volume-preserving actions in his doctoral thesis [5].

THEOREM 1.1 ([5]; see also [7]). *Let M be a closed, connected, oriented, and three-dimensional manifold. If a C^∞ locally free action of* GA *on M admits a continuous invariant volume form, then it is homogeneous.*

The following problem is quite natural, but was open for thirty years.

Problem 1.2 ([7, p. 525]; see also [2, p. 2], [17, p. 1841]). Remove the assumption on an invariant volume, or construct a C^∞ locally free action of GA on a closed three-dimensional manifold which is not homogeneous.

There are several partial solutions to the problem. We say that two actions ρ_1 and ρ_2 on manifolds M_1 and M_2 are C^∞ *orbit equivalent* if there exists a C^∞ diffeomorphism H from M_1 to M_2 such that $H(\mathcal{O}_{\rho_1}(p)) = \mathcal{O}_{\rho_2}(H(p))$ for any $p \in M_1$.

THEOREM 1.3 ([7]). *Any C^∞ locally free* GA-*action on a three-dimensional rational homology sphere preserves a continuous volume. In particular, it is a homogeneous action.*

THEOREM 1.4 ([12]). *Let Γ be a cocompact lattice of* Solv. *Then, any C^∞ locally free action of* GA *on* $\Gamma\backslash\mathrm{Solv}$ *is homogeneous.*

THEOREM 1.5 ([1]). *Let M be a closed, connected, oriented, and three-dimensional manifold and ρ a C^∞ locally free action of* GA *on M. Then, ρ is C^∞ orbit equivalent to a homogeneous action. In particular, M is diffeomorphic to* $\Gamma\backslash\widetilde{\mathrm{PSL}}(2,\mathbb{R})$ *or* $\Gamma\backslash\mathrm{Solv}$ *with a cocompact lattice* Γ.

We will give a proof of Theorem 1.4 in Appendix A since it has not been published.

Let Γ be a cocompact lattice of $\widetilde{\mathrm{PSL}}(2,\mathbb{R})$. We denote the quotient space $\Gamma\backslash\widetilde{\mathrm{PSL}}(2,\mathbb{R})$ by M_Γ. Let $\mathcal{A}_\Gamma$ be the set of C^∞ locally free actions of GA on $\Gamma\backslash\widetilde{\mathrm{PSL}}(2,\mathbb{R})$ whose orbit foliation coincides with that of the standard GA-action. By the above results, it is sufficient to classify actions in $\mathcal{A}_\Gamma$ up to C^∞ conjugacy for the complete classification of locally free GA-actions on closed, orientable, and three-dimensional manifold.[1]

Main Theorem. *There exists an open subset Δ_Γ of $H^1(M_\Gamma, \mathbb{R})$ and a family $(\rho_a)_{a \in \Delta_\Gamma}$ of actions $\mathcal{A}_\Gamma$ such that*

(i) *any $\rho \in \mathcal{A}_\Gamma$ is C^∞ conjugate to ρ_a for some $a \in \Delta_\Gamma$, and*
(ii) *$a = a'$ if and only if ρ_a is C^∞ conjugate to $\rho_{a'}$ by a diffeomorphism homotopic to the identity.*

In particular, if M_Γ is not a rational homology sphere, then $\mathcal{A}_\Gamma$ contains nonhomogeneous actions.

We will not prove the smoothness of the family $(\rho_a)_{\alpha \in \Delta_\Gamma}$ in this paper. It needs more work; it will be shown in a forthcoming paper.

The organization of this paper is the following. In Section 2, we associate a GA-action $\check{\rho}_\omega$ with any given closed one-form ω on $\Gamma \backslash \widetilde{\mathrm{PSL}}(2, \mathbb{R})$ satisfying a mild condition. The action is not homogeneous if the cohomology class of ω is nonzero. As a corollary, we obtain the existence of nonhomogeneous actions when $\Gamma \backslash \widetilde{\mathrm{PSL}}(2, \mathbb{R})$ is not a rational homology sphere. In Section 3, we define a natural parametrization $\bar{a}_\Gamma$ of actions and determine its image Δ_Γ. In fact, we show that any action in $\mathcal{A}_\Gamma$ is C^∞ conjugate to an action in the family $(\check{\rho}_\omega)$ constructed in Section 2. In Appendix A, we give a proof of Theorem 1.4 and discuss the regularity of the unstable foliation of the Anosov flow associated with a nonhomogeneous action.

Acknowledgements. The author wrote the first version of this article when he stayed at Unité de Mathématiques Pures et Appliquées, École Normale Supérieure de Lyon. He would like to thank the members of UMPA, especially Professor Étienne Ghys, for their warm hospitality. The author is also grateful to Professor Shigenori Matsumoto and an anonymous referee for many suggestions to improve the paper. At last, I would like to thank my wife, Junko, for encouragements.

2. **Nonhomogeneous actions**

In this section, we construct a nonhomogeneous GA-action. More precisely, for any given closed one-form ω on $M_\Gamma = \Gamma \backslash \widetilde{\mathrm{PSL}}(2, \mathbb{R})$ with a mild condition, we construct a C^ω action $\check{\rho}_\omega$ on a manifold diffeomorphic to M_Γ such that $\det D\check{\rho}_\omega$ is controlled by ω (Theorem 2.7). The action $\check{\rho}_\omega$ is not homogeneous if the cohomology class of ω is nonzero (Proposition 2.8). As a corollary, if $H^1(M_\Gamma)$ is nontrivial, then M_Γ admits a C^ω nonhomogeneous action.

[1] We can also classify actions on nonorientable manifolds by taking an orientable double cover.

The main ingredient of the construction is deformation of a codimension-one Anosov system along the strong stable foliation. It is invented by E. Cawley [3] in order to describe the moduli space of two-dimensional Anosov diffeomorphisms. Here, we apply her technique to codimension-one Anosov flows with a C^ω invariant splitting. It produces another Anosov flow with the C^ω strong stable foliation whose rate of contraction is constant. Such a flow induces a locally free GA-action naturally.

In Section 2.1, we recall the definitions of an Anosov flow and the Margulis measure associated with it. Deformation of Anosov flow along the strong stable foliation is done in Section 2.2. Finally, we construct nonhomogeneous actions associated with a closed one-form in Section 2.3.

2.1. *Anosov flows and the Margulis measure.* Let Φ be a C^1 flow on a closed manifold M without stationary points. Its differential defines a flow $D\Phi$ on the tangent bundle TM of M. Let $T\Phi$ be the one-dimensional subbundle of TM which is tangent to the orbits of Φ. The flow Φ is called *Anosov* if there exists a continuous $D\Phi$-invariant splitting $TM = T\Phi \oplus E^{ss} \oplus E^{uu}$ and $T > 0$ such that

$$\max\left\{\|D\Phi^T|_{E^{ss}(p)}\|, \|D\Phi^{-T}|_{E^{uu}(p)}\|\right\} < 1/2$$

for any $p \in M$, where $\|\cdot\|$ is the operator norm with respect to a Riemannian metric of M. The above splitting is unique for Φ and is called *the Anosov splitting* of Φ. The $D\Phi$-invariant subbundles E^{ss}, E^{uu}, $E^s = T\Phi \oplus E^{ss}$, and $E^u = T\Phi \oplus E^{uu}$ of TM generate continuous foliations $\mathcal{F}^{ss}$, $\mathcal{F}^{uu}$, $\mathcal{F}^s$, and $\mathcal{F}^u$. They are called *the strong stable*, *strong unstable*, *weak stable*, and *weak unstable foliations*, respectively.

By $\mathrm{Per}(\Phi)$, we denote the union of periodic orbits of Φ. For $p \in \mathrm{Per}(\Phi)$, let $\tau(p;\Phi)$ be the (minimal) period of p. We define a loop $\gamma(p;\Phi) : [0,1]\to M$ by $\gamma(p;\Phi)(t) = \Phi^{t\cdot\tau(p;\Phi)}(p)$. Put

$$\begin{aligned}
J(p;\Phi) &= \log\left|\det D\Phi_p^{\tau(p;\Phi)}\right|,\\
J^s(p;\Phi) &= \log\left|\det\left(D\Phi^{\tau(p;\Phi)}|_{E^{ss}(p)}\right)\right|,\\
J^u(p;\Phi) &= \log\left|\det\left(D\Phi^{\tau(p;\Phi)}|_{E^{uu}(p)}\right)\right|.
\end{aligned}$$

Remark that J, J^s, and J^u does not depend on the choice of the Riemannian metric.

When two points $p, q \in M$ are sufficiently close to each other, we can define the holonomy map $\tilde{h}_{pq}$ of $\mathcal{F}^{uu}$ from a neighborhood $\tilde{U}_{pq}$ of p in $\mathcal{F}^s(p)$ to $\mathcal{F}^s(q)$. We say that a flow is *topologically transitive* if it admits a dense orbit.

THEOREM 2.1 (Margulis [16]). *If Φ is a topologically transitive Anosov flow, then there exists a family $(\tilde{\mu}_p)_{p\in M}$ which has the following properties*:

(i) *$\tilde{\mu}_p$ is a locally finite Borel measure on $\mathcal{F}^s(p)$ which is nonatomic and positive on each nonempty open subset of $\mathcal{F}^s(p)$.*

(ii) *$\tilde{\mu}_p = \tilde{\mu}_q$ if $\mathcal{F}^s(p) = \mathcal{F}^s(q)$.*

(iii) *The family $(\tilde{\mu}_p)_{p \in M}$ is holonomy invariant; i.e., if p and q are sufficiently close to each other, then $\tilde{\mu}_q(\tilde{h}_{pq}(A)) = \tilde{\mu}_p(A)$ for any Borel subset A of $\tilde{U}_{pq}$.*

(iv) *There exists $\lambda > 0$ such that*

$$\tilde{\mu}_p(\Phi^t(A)) = e^{-\lambda t} \cdot \tilde{\mu}_p(A)$$

for any $p \in M$, $t \in \mathbb{R}$, and any Borel subset A of $\mathcal{F}^s(p)$.

As mentioned in Chapter 3 of [16], the family $(\tilde{\mu}_p)_{p \in M}$ induces a family $(\mu_p)_{p \in M}$ of measures on leaves of $\mathcal{F}^{ss}$ such that

$$\tilde{\mu}_p(\{\Phi^t(q) \mid (q,t) \in \tilde{A} \times (0, \varepsilon)\}) = (1 - e^{-\lambda \varepsilon}) \cdot \mu_p(\tilde{A})$$

for any small Borel set $\tilde{A} \subset \mathcal{F}^{ss}(p)$ and any small $\varepsilon > 0$. We call the family $(\mu_p)_{p \in M}$ *the Margulis measure* of Φ along $\mathcal{F}^{ss}$. It satisfies the following properties:

- For any small Borel subset $\tilde{A} \subset \mathcal{F}^s(p)$ of the form

$$\tilde{A} = \{\Phi^t(p') \mid (p', t) \in A \times [\eta_-(p'), \eta_+(p')]\},$$

  where A is a Borel subset of $\mathcal{F}^{ss}(p)$ and η_- and η_+ are measurable functions on A,

$$\tilde{\mu}_p(\tilde{A}) = \int_A \{e^{-\lambda \cdot \eta_-(p')} - e^{-\lambda \cdot \eta_+(p')}\} d\mu_p(p'). \tag{1}$$

- For any $p \in M$, $t \in \mathbb{R}$, and any small Borel subset A of $\mathcal{F}^{ss}(p)$,

$$\mu_{\Phi^t(p)}(\Phi^t(A)) = e^{-\lambda t} \cdot \mu_p(A). \tag{2}$$

When two points p and q in M are sufficiently close to each other, we can define the holonomy map h_{pq} of $\mathcal{F}^u$ from a neighborhood U_{pq} of p in $\mathcal{F}^{ss}(p)$ to $\mathcal{F}^{ss}(q)$.

Proposition 2.2. *Suppose that the flow Φ and the foliations $\mathcal{F}^{ss}$, $\mathcal{F}^{uu}$, $\mathcal{F}^s$, and $\mathcal{F}^u$ are real-analytic. Then, for any given $p \in M$, the Radon-Nikodym derivative $\frac{d(\mu_q \circ h_{pq})}{d\mu_p}(p')$ is a C^ω function of $q \in M$ and $p' \in U_{pq}$.*

Proof. Since $\mathcal{F}^s$ is subfoliated by $\mathcal{F}^{ss}$, there exists a neighborhood U of p in M and a C^ω function η on $U \times U$ such that $\eta(p', p') = 0$ and $\Phi^{\eta(p',q)}(\tilde{h}_{p'q}(p')) = h_{p'q}(p')$ for any $(p', q) \in U \times U$.

Take a small Borel set $A \subset U \cap U_{pq}$ and a small positive number $\varepsilon > 0$. Since Φ preserves the foliation $\mathcal{F}^{uu}$, we have $\tilde{h}_{pq} \circ \Phi^t = \Phi^t \circ \tilde{h}_{pq}$ for any $t \in [0, \varepsilon]$.

The equation implies

$$\begin{aligned}(1-e^{-\lambda\varepsilon})\mu_q(h_{pq}(A)) &= \tilde{\mu}_q(\{\Phi^{t+\eta(p',q)}\circ\tilde{h}_{pq}(p') \mid (p',t)\in A\times(0,\varepsilon)\})\\ &= (\tilde{\mu}_q\circ\tilde{h}_{pq})(\{\Phi^{t+\eta(p',q)}(p') \mid (p',t)\in A\times(0,\varepsilon)\})\\ &= \tilde{\mu}_p(\{\Phi^{t+\eta(p',q)}(p') \mid (p',t)\in A\times(0,\varepsilon)\})\\ &= \int_A\{e^{-\lambda\cdot\eta(p',q)}-e^{-\lambda(\eta(p',q)+\varepsilon)}\}d\mu_p(p').\end{aligned}$$

In particular,

$$(\mu_q\circ h_{pq})(A)=\int_A e^{-\lambda\eta(p',q)}d\mu_p(p');$$

hence

$$\frac{d(\mu_q\circ h_{pq})}{d\mu_p}(p')=\exp(-\lambda\cdot\eta(p',q))$$

for any $p'\in\mathcal{F}^{ss}(p)$ sufficiently close to p. The right-hand side is a C^ω function of (p',q). □

2.2. *Deformation of codimension-one Anosov flows.* In this section, we deform an Anosov flow into another Anosov flow with constant contraction.

Proposition 2.3. *Let M be an n-dimensional closed manifold and Φ a C^ω topologically transitive Anosov flow on M. Suppose that Φ admits the C^ω Anosov splitting and the strong stable foliation is one-dimensional and orientable. Then, there exists another C^ω Anosov flow $\check{\Phi}$ on a C^ω closed manifold $\check{M}$, a C^ω vector field Y^s on $\check{M}$, a homeomorphism $H: M\to\check{M}$, and a constant $\lambda>0$ which satisfy the following properties:*

- *$H\circ\Phi^t=\check{\Phi}^t\circ H$ for any $t\in\mathbb{R}$.*
- *$J^u(H(p);\check{\Phi})=J^u(p;\Phi)$ for any $p\in\mathrm{Per}(\Phi)$.*
- *Y^s generates the strong stable foliation of $\check{\Phi}$ and $D\check{\Phi}^t(Y^s)=e^{-\lambda t}\cdot Y^s$ for any $t\in\mathbb{R}$. In particular, $J^s(\check{p};\check{\Phi})=-\lambda\cdot\tau(\check{p};\check{\Phi})$ for any $p\in\mathrm{Per}(\check{\Phi})$.*

The rest of this section is devoted to the proof. We follow Cawley's idea in [3], where she deformed two-dimensional Anosov diffeomorphisms. Although the deformed diffeomorphism is of class C^1 in her case, we obtain a C^ω flow in our case because of the regularity of the invariant splitting and the constant contraction along $\mathcal{F}^{ss}$ with respect to the Margulis measure.

Let $\mathcal{F}^{ss}$ and $\mathcal{F}^u$ be the strong stable foliation and the weak unstable foliation of Φ. We denote the coordinate system of $\mathbb{R}^n$ by $(x_1,\dots,x_{n-1},y)$. For each $p\in M$, we take an open neighborhood U_p of p and a C^ω diffeomorphism

$\varphi_p : U_p \to (-1,1)^n$ such that $\varphi_p(p) = (0,0)$ and

$$\varphi_p^{-1}(\{x\} \times (-1,1)) \subset \mathcal{F}^{ss}(\varphi_p^{-1}(x,y)),$$
$$\varphi_p^{-1}((-1,1)^{n-1} \times \{y\}) \subset \mathcal{F}^{u}(\varphi_p^{-1}(x,y))$$

for any $(x,y) \in \mathbb{R}^{n-1} \times \mathbb{R}$. Let $\pi_p : U_p \to \varphi_p^{-1}((-1,1)^{n-1} \times 0)$ be the projection along $\mathcal{F}^{ss}$. For $q \in U_p$, let $h_{pq} : (\mathcal{F}^{ss}|_{U_p})(p) \to (\mathcal{F}^{ss}|_{U_p})(q)$ be the holonomy map along $\mathcal{F}^u$. They satisfy

$$\pi_p \circ \varphi_p^{-1}(x,y) = \varphi_p^{-1}(x,0),$$
$$h_{pq} \circ \varphi_p^{-1}(0,y) = \varphi_p^{-1}(x_q,y)$$

for any $(x,y) \in (-1,1)^{n-1} \times (-1,1)$, where $\varphi_p(q) = (x_q, y_q)$.

Fix an orientation of $\mathcal{F}^{ss}$. Let $(\mu_p)_{p \in M}$ be the Margulis measure of Φ along $\mathcal{F}^{ss}$. For an oriented interval $I \subset \mathcal{F}^{ss}(p)$, we put $\nu(I) = \mu_p(I)$ if the orientations of I and $\mathcal{F}^{ss}(p)$ coincide or $\nu(I) = -\mu_p(I)$ otherwise. For $p \in M$ and $q_-, q_+ \in \mathcal{F}^{ss}(p)$, let $[q_-, q_+]$ be the oriented interval in $\mathcal{F}^{ss}(p)$ which connects q_- and q_+ and whose positive end point is q_+.

Lemma 2.4. *For any given $y_1, y_2 \in (-1,1)$, the function*

$$x \mapsto \nu([\varphi_p^{-1}(x,y_1), \varphi_p^{-1}(x,y_2)])$$

from $(-1,1)^{n-1}$ to $\mathbb{R}$ is real-analytic.

Proof. Without loss of generality, we may assume that $y_1 < y_2$. Put $q(x) = \varphi_p^{-1}(x,0)$ for $x \in (-1,1)^{n-1}$. Then,

$$\begin{aligned} |\nu([\varphi_p^{-1}(x,y_1), \varphi_p^{-1}(x,y_2)])| &= \mu_{q(x)}(\varphi_p^{-1}(x \times [y_1,y_2])) \\ &= (\mu_{q(x)} \circ h_{pq(x)})(\varphi_p^{-1}(0 \times [y_1,y_2])) \\ &= \int_{\varphi_p^{-1}(0 \times [y_1,y_2])} \frac{d(\mu_{q(x)} \circ h_{pq(x)})}{d\mu_p}\, d\mu_p. \end{aligned}$$

By Proposition 2.2, the last term is real-analytic with respect to x. □

We define a function $\check{y}_p : U_p \to \mathbb{R}$ and a map $\check{\varphi}_p : U_p \to \mathbb{R}^n$ by

$$\check{y}_p(q) = \nu([\pi_p(q), q]),$$
$$\check{\varphi}_p(q) = (x, \check{y}_p(q))$$

for $q \in U_p$ with $\varphi_p(q) = (x,y)$. Since μ_p is locally finite, nonatomic, and positive on any nonempty open subset of $\mathcal{F}^{ss}(p)$, the map $\check{\varphi}_p$ is a homeomorphism onto an open subset of $\mathbb{R}^n$. Remark that

$$\pi_q \circ \check{\varphi}_p^{-1}(x,y) = \varphi_p^{-1}(x, y_q)$$

for any $q \in U_p$ with $\varphi_p(q) = (x_q, y_q)$ and any $(x,y) \in \check{\varphi}_p(U_p \cap U_q)$.

LEMMA 2.5. *For $p \in M$ and $q \in U_p$, the map $\check{\varphi}_q \circ \check{\varphi}_p^{-1}$ is real-analytic. The differential $D(\check{\varphi}_q \circ \check{\varphi}_p^{-1})$ preserves the vector field $(\partial/\partial y)$.*

Proof. Since φ_p and φ_q are adapted to the pair $(\mathcal{F}^u, \mathcal{F}^{ss})$ of foliations, there exist C^ω coordinate transformations ψ and ψ' such that

$$\varphi_q \circ \varphi_p^{-1}(x,y) = (\psi(x), \psi'(y)).$$

For $p' \in U_p \cap U_q$ with $(x,y) = \check{\varphi}_p(p')$, we have

$$\begin{aligned}\check{\varphi}_q \circ \check{\varphi}_p^{-1}(x,y) &= (\psi(x), \nu([\pi_q(p'), p']))\\ &= (\psi(x), \nu([\pi_p(p'), p']) - \nu([\pi_p(p'), \pi_q(p')]))\\ &= (\psi(x), y - \nu([\varphi_p^{-1}(x,0), \varphi_p^{-1}(x, y_q)])),\end{aligned}$$

where $\varphi_p(q) = (x_q, y_q)$. By Lemma 2.4, the last term is real-analytic with respect to (x,y). The above equation also implies that $D(\check{\varphi}_q \circ \check{\varphi}_p^{-1})$ preserves $(\partial/\partial y)$. □

The lemma implies that $(\check{\varphi}_p)_{p\in M}$ defines a C^ω structure on M. By $\check{M}$, we denote the manifold M endowed with this C^ω structure. Let $H : M \to \check{M}$ be the identity map as a set. It is a homeomorphism but is not a diffeomorphism in general since $\check{\varphi}_p \circ \varphi_p^{-1}$ is continuous but may not be smooth. We define a continuous foliation $\check{\mathcal{F}}^u$ on $\check{M}$ by $\check{\mathcal{F}}^u(H(p)) = H(\mathcal{F}^u(p))$.

LEMMA 2.6. *Each leaf of $\check{\mathcal{F}}^u$ is a C^ω immersed manifold and the restriction of H to a leaf is a C^ω diffeomorphism.*

Proof. For $p \in M$, $\varphi_p((-1,1)^{n-1} \times 0)$ is a neighborhood of p in $\mathcal{F}^u(p)$ and $\check{\varphi}_p((-1,1)^{n-1} \times 0)$ is a neighborhood of $H(p)$ in $\check{\mathcal{F}}^u(H(p))$. The latter implies that $\check{\mathcal{F}}^u(H(p))$ is a C^ω immersed manifold. Since $\check{\varphi}_p \circ \varphi_p^{-1}(x,0) = (x,0)$, the restriction of H to $\mathcal{F}^u(p)$ is a C^ω diffeomorphism. □

Let $TM = T\Phi \oplus E^{ss} \oplus E^{uu}$ be the Anosov splitting of Φ. By the above lemma, the subbundle $\check{E}^{uu} = DH(E^{uu})$ is well defined. Since $D(\check{\varphi}_q \circ \check{\varphi}_p^{-1})$ preserves the vector field $\partial/\partial y$, we can define a C^ω vector field Y^s on $\check{M}$ by $Y^s = (D\check{\varphi}_p)^{-1}(\partial/\partial y)$ on U_p. We also define a flow $\check{\Phi}$ on $\check{M}$ by $\check{\Phi}^t = H \circ \Phi^t \circ H^{-1}$. Recall that there exists $\lambda > 0$ such that $\mu_{\Phi^t(p)} \circ \Phi^t = e^{-\lambda t}\mu_p$ for any $p \in M$ and $t \in \mathbb{R}$.

Now, we check that the quadruple $(\check{\Phi}, H, Y^s, \lambda)$ satisfies the conditions in Proposition 2.3.

Proof of Proposition 2.3. Fix $p \in M$. Since Φ preserves the foliations $\mathcal{F}^u$ and $\mathcal{F}^{ss}$, there exists a C^ω local flow Ψ_p on $(-1,1)^{n-1}$, a neighborhood $V_p \subset U_p$ of p, and $\varepsilon > 0$ such that $\varphi_p \circ \Phi^t \circ \varphi_p^{-1}(x,y) = (\Psi_p^t(x), y)$ for any $(x,y) \in \varphi_p(V_p)$ and $t \in (-\varepsilon, \varepsilon)$. Take $q \in V_p$ and $t \in (-\varepsilon, \varepsilon)$. Put $(x,y) = \check{\varphi}_p(q)$

and $(x', y') = \check{\varphi}_p \circ \check{\Phi}^t(q)$. Then, $x' = \Psi_p^t(x)$ and

$$y' = \nu([\pi_p(\Phi^t(q)), \Phi^t(q)]) = \nu(\Phi^t([\pi_p(q), q])) = e^{-\lambda t}\nu([\pi_p(q), q]) = e^{-\lambda t} y.$$

Therefore,

$$\check{\varphi}_p \circ \check{\Phi}^t \circ \check{\varphi}_p^{-1}(x, y) = (\Psi_p^t(x), e^{-\lambda t} y).$$

This implies that $\check{\Phi}$ is a C^ω flow and $D\check{\Phi}^t(Y^s) = e^{-\lambda t} \cdot Y^s$. Since $DH|_{E^u}$ is well defined and $DH(E^{uu}) = \check{E}^{uu}$, the flow $D\check{\Phi}$ preserves $\check{E}^{uu}$ and there exists a C^∞ norm $\|\cdot\|$ on $T\check{M}$ and $T > 0$ such that $\|D\check{\Phi}^{-T}|_{\check{E}^{uu}(p)}\| < 1/2$ for any $p \in \check{M}$. Hence, $\check{\Phi}$ is an Anosov flow and its Anosov splitting is $T\check{M} = T\check{\Phi} \oplus \mathbb{R}Y^s \oplus \check{E}^{uu}$. Since $DH|_{E^{uu}}$ is well defined, $J^u(H(q); \check{\Phi}) = J^u(q; \Phi)$ for any $q \in \mathrm{Per}(\Phi)$. □

2.3. *Construction of a nonhomogeneous action.* Fix a cocompact lattice Γ of $\widetilde{\mathrm{PSL}}(2, \mathbb{R})$. We denote the quotient space $\Gamma\backslash\widetilde{\mathrm{PSL}}(2, \mathbb{R})$ by M_Γ and the Lie algebra of $\widetilde{\mathrm{PSL}}(2, \mathbb{R})$ by $\mathfrak{sl}_2(\mathbb{R})$. The Lie algebra $\mathfrak{sl}_2(\mathbb{R})$ is naturally identified with the Lie algebra of trace-free real 2×2-matrices. It is generated by three elements:

$$X = \begin{bmatrix} 1/2 & 0 \\ 0 & -1/2 \end{bmatrix}, \quad S = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad U = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}.$$

Let $(X^t)_{t \in \mathbb{R}}$, $(S^t)_{t \in \mathbb{R}}$, and $(U^t)_{t \in \mathbb{R}}$ be the one-parameter subgroups of $\widetilde{\mathrm{PSL}}(2, \mathbb{R})$ corresponding to X, S, and U, respectively. Since $[X, S] = S$, the Lie subalgebra spanned by X and S generates a subgroup of $\widetilde{\mathrm{PSL}}(2, \mathbb{R})$ which is isomorphic to GA. *In the rest of the paper, we identify this group with* GA. Under the identification, the standard GA-action ρ_Γ on M_Γ is written as $\rho_\Gamma(\Gamma g, h) = \Gamma(gh)$.

The elements X, S, and U of $\mathfrak{sl}_2(\mathbb{R})$ can be identified with the left invariant vector fields on $\widetilde{\mathrm{PSL}}(2, \mathbb{R})$. They naturally induce vector fields X_Γ, S_Γ, and U_Γ on M_Γ. We define a flow Φ_Γ on M_Γ by $\Phi_\Gamma^t(\Gamma g) = \Gamma(gX^t)$. It is generated by X_Γ and satisfies $D\Phi_\Gamma^t(S_\Gamma) = e^{-t} \cdot S_\Gamma$ and $D\Phi_\Gamma^t(U_\Gamma) = e^t \cdot U_\Gamma$ for any $t \in \mathbb{R}$. Hence, Φ_Γ is an Anosov flow with the Anosov splitting $TM_\Gamma = T\Phi \oplus \mathbb{R}S_\Gamma \oplus \mathbb{R}U_\Gamma$ and

$$\tau(p; \Phi_\Gamma) = J^u(p; \Phi_\Gamma) = -J^s(p; \Phi_\Gamma)$$

for any $p \in \mathrm{Per}(\Phi_\Gamma)$. For a loop γ in M_Γ, we denote its homology class by $[\gamma]$. The following properties of Φ_Γ are well known and easy to prove:

- Φ_Γ is topologically transitive.
- There exists $p_0 \in \mathrm{Per}(\Phi_\Gamma)$ such that $[\gamma(p_0; \Phi_\Gamma)] = 0$.
- The set $\{[\gamma(p; \Phi_\Gamma)] \mid p \in \mathrm{Per}(\Phi_\Gamma)\}$ spans $H_1(M_\Gamma, \mathbb{R})$.
- For any $p \in \mathrm{Per}(\Phi_\Gamma)$, there exists $p' \in \mathrm{Per}(\Phi_\Gamma)$ such that $\tau(p, \Phi_\Gamma) = \tau(p', \Phi_\Gamma)$ and $[\gamma(p'; \Phi_\Gamma)] = -[\gamma(p; \Phi_\Gamma)]$.

We say that two flows Ψ_1 and Ψ_2 on manifolds M_1 and M_2 are *topologically equivalent* if there exists a homeomorphism $H : M_1 \to M_2$ which sends orbits of Ψ_1 to those of Ψ_2 and preserves the orientation of orbits. The map H is called *a topological equivalence* between Ψ_1 and Ψ_2.

Now, we are ready to construct a GA-action associated with a closed one-form.

THEOREM 2.7. *Let ω be a C^ω closed one-form on M_Γ with $1+\omega(X_\Gamma) > 0$. Then, there exists a C^ω manifold M_ω, a C^ω locally free* GA*-action $\check{\rho}_\omega$ on M_ω, a homeomorphism $H : M_\Gamma \to M_\omega$, and a constant $\lambda > 0$ such that*

(i) *the flow $\Phi_{\check{\rho}_\omega}$ defined by $\Phi^t_{\check{\rho}_\omega}(\check{p}) = \check{\rho}_\omega(\check{p}, X^t)$ is an Anosov flow on M_ω;*
(ii) *H is a topological equivalence between Φ_Γ and $\Phi_{\check{\rho}_\omega}$; and*
(iii) *for any $p \in \mathrm{Per}(\Phi_\Gamma)$,*

$$J^u(H(p); \Phi_{\check{\rho}_\omega}) = \tau(p; \Phi_\Gamma),$$
$$-J^s(H(p); \Phi_{\check{\rho}_\omega}) = \tau(H(p); \Phi_{\check{\rho}_\omega}) = \lambda \cdot \{\tau(p; \Phi_\Gamma) + \langle \omega, \gamma(p; \Phi_\Gamma)\rangle\}.$$

Proof. Fix a C^ω closed one-form ω such that $1 + \omega(X_\Gamma) > 0$. The vector field $X_\omega = (1+\omega(X_\Gamma))^{-1} X_\Gamma$ generates an Anosov flow Φ_ω whose orbit is same as Φ_Γ.[2] Then,

$$\tau(p; \Phi_\omega) = \tau(p; \Phi_\Gamma) + \langle \omega, \gamma(p; \Phi_\Gamma)\rangle$$

for any $p \in \mathrm{Per}(\Phi_\Gamma)$. Put $S_\omega = S_\Gamma - \omega(S_\Gamma) \cdot X_\omega$ and $U_\omega = U_\Gamma - \omega(U_\Gamma) \cdot X_\omega$. By a direct calculation, we have

$$[S_\omega, X_\omega] = (1 + \omega(X_\Gamma))^{-1} \cdot (-S_\Gamma + \{X_\Gamma \cdot \omega(S_\Gamma) - S_\Gamma \cdot \omega(X_\Gamma)\} X_\omega).$$

Since $d\omega = 0$,[3] it follows that $[S_\omega, X_\omega] = -(1 + \omega(X_\Gamma))^{-1} S_\omega$. Similarly, we have $[U_\omega, X_\omega] = (1 + \omega(X_\Gamma))^{-1} U_\omega$. These equations imply that Φ_ω admits the C^ω Anosov splitting $TM_\Gamma = T\Phi_\omega \oplus \mathbb{R}S_\omega \oplus \mathbb{R}U_\omega$.

By Proposition 2.3, there exists a C^ω Anosov flow $\check{\Phi}_\omega$ on a C^ω manifold M_ω, a homeomorphism $H : M_\Gamma \to M_\omega$, a C^ω vector field Y^s on M_ω, and a constant $\lambda > 0$ such that

- $\check{\Phi}^t_\omega \circ H = H \circ \Phi^t_\omega$ and $D\check{\Phi}^t_\omega(Y^s) = e^{-\lambda t} \cdot Y^s$ for any $t \in \mathbb{R}$.
- $J^u(H(p); \check{\Phi}_\omega) = J^u(p; \Phi_\omega)$ for any $p \in \mathrm{Per}(\Phi_\omega)$.

Remark that

$$-J^s(H(p); \check{\Phi}_\omega) = \lambda \cdot \tau(H(p); \check{\Phi}_\omega) = \lambda \cdot \tau(p; \Phi_\omega) = \lambda \cdot \{\tau(p; \Phi_\Gamma) + \langle \omega, \gamma(p; \Phi_\Gamma)\rangle\}$$

for any $p \in \mathrm{Per}(\Phi_\omega)$. Since Φ_ω is a time-change of Φ_Γ, we also have $\mathrm{Per}(\check{\Phi}_\omega) = H(\mathrm{Per}(\Phi_\omega)) = H(\mathrm{Per}(\Phi_\Gamma))$ and

$$J^u(H(p); \check{\Phi}_\omega) = J^u(p; \Phi_\Gamma) = \tau(p; \Phi_\Gamma)$$

for any $p \in \mathrm{Per}(\Phi_\Gamma)$.

Let $\Phi_{\check{\rho}_\omega}$ be the flow defined by $\Phi^t_{\check{\rho}_\omega} = \check{\Phi}^{t/\lambda}_\omega$ and Ψ_ω the C^ω flow generated by Y^s. Then, $\Phi^t_{\check{\rho}_\omega} \circ \check{\Psi}^x_\omega = \check{\Psi}^{e^{-t}x}_\omega \circ \Phi^t_{\check{\rho}_\omega}$ for any $x, t \in \mathbb{R}$. Since $\Phi_{\check{\rho}_\omega}$ is a constant

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[2]The flow Φ_Γ is essentially same as the flow investigated in [8] and [19].

[3]In the construction of $\check{\rho}_\omega$, this is the only place we use $d\omega = 0$.

time-change of $\check{\Phi}_\omega$, we have $\tau(q;\Phi_{\check{\rho}_\omega}) = \lambda\cdot\tau(q;\check{\Phi}_\omega)$, $J^u(q;\Phi_{\check{\rho}_\omega}) = J^u(q;\check{\Phi}_\omega)$, and $J^s(q;\Phi_{\check{\rho}_\omega}) = J^s(q;\check{\Phi}_\omega)$ for any $q\in \mathrm{Per}(\check{\Phi}_\omega)$. Hence,

$$J^u(H(p);\Phi_{\check{\rho}_\omega}) = \tau(p;\Phi_\Gamma),$$
$$-J^s(H(p);\Phi_{\check{\rho}_\omega}) = \tau(H(p);\Phi_{\check{\rho}_\omega}) = \lambda\cdot\{\tau(p;\Phi_\Gamma) + \langle\omega,\gamma(p;\Phi_\Gamma)\rangle\}$$

for any $p\in\mathrm{Per}(\Phi_\Gamma)$. Now, a locally free GA action $\check{\rho}_\omega$ on M_ω defined by $\check{\rho}_\omega(\check{p}, X^tS^x) = \check{\Psi}^x_\omega\circ\Phi^t_{\check{\rho}_\omega}(\check{p})$ satisfies the required conditions. □

PROPOSITION 2.8. *If $[\omega]\neq 0$, then the action $\check{\rho}_\omega$ is not homogeneous.*

Proof. Let H and λ be the homeomorphism and the constant in Theorem 2.7. The equations in Theorem 2.7 imply that

$$J(H(p);\Phi_{\check{\rho}_\omega}) = (1-\lambda)\cdot\tau(p;\Phi_\Gamma) - \lambda\cdot\langle\omega,\gamma(p;\Phi_\Gamma)\rangle.$$

If $[\omega]\neq 0$, then there exists $p_1, p_2\in\mathrm{Per}(\Phi_\Gamma)$ such that $\tau(p_1,\Phi_\Gamma) = \tau(p_2,\Phi_\Gamma)$ and $\langle\omega,\gamma(p_1,\Phi_\Gamma)\rangle = -\langle\omega,\gamma(p_2;\Phi_\Gamma)\rangle\neq 0$. Hence, at least one of $J(H(p_1);\Phi_{\check{\rho}_\omega})$ or $J(H(p_2);\Phi_{\check{\rho}_\omega})$ is nonzero. This implies that $\check{\rho}_\omega$ admits no invariant volume. Hence, $\check{\rho}_\omega$ is not homogeneous. □

It is known that any homeomorphism between C^ω closed three-dimensional manifolds is homotopic to a C^ω diffeomorphism. Therefore, we obtain

COROLLARY 2.9. *If $H^1(M_\Gamma,\mathbb{R})$ is nontrivial, there exists a C^ω nonhomogeneous locally free* GA*-action on M_Γ.*

3. **Classification of actions**

In this section, we prove the main theorem. After we review some known results on conjugacy between Anosov flows in Section 3.1, we introduce a natural map $\bar{a}_\Gamma : \mathcal{A}_\Gamma\to H^1(M_\Gamma,\mathbb{R})$ in Section 3.2. It was originally introduced by Ghys [7] as an obstruction to being a homogeneous action. We will see that the map $\bar{a}_\Gamma$ classifies actions up to C^∞ conjugacy homotopic to the identity. In Section 3.3, we determine the image Δ_Γ of $\bar{a}_\Gamma$. In fact, the actions constructed in Section 2 induce a family $(\rho_a)_{a\in\Delta_\Gamma}$ in $\mathcal{A}_\Gamma$ such that $\bar{a}_\Gamma(\rho_a) = a$.

3.1. *Conjugacy between Anosov flows.* In this section, we review some known results on conjugacy between Anosov flows and give a criterion for C^∞ conjugacy between locally free GA-actions.

Let Φ_1 and Φ_2 be flows on manifolds M_1 and M_2, respectively. We say that a homeomorphism $H : M_1\to M_2$ is *a topological conjugacy* if $H\circ\Phi_1^t = \Phi_2^t\circ H$ for any $t\in\mathbb{R}$. When H is a C^r diffeomorphism, it is called *a C^r conjugacy.* A continuous function $\alpha : M\times\mathbb{R}\to\mathbb{R}$ is called a *cocycle* over Φ_1 if it satisfies $\alpha(p,0) = 0$ and $\alpha(p,s+t) = \alpha(p,s) + \alpha(\Phi_1^s(p),t)$ for any $p\in M_1$ and $s,t\in\mathbb{R}$.

THE LIVSCHITZ THEOREM ([14]). *Let Φ be a C^2 topologically transitive Anosov flow on a closed manifold M and α be a Hölder continuous cocycle* *over* Φ. *If* $\alpha(p,\tau(p;\Phi)) = 0$ *for any* $p \in \mathrm{Per}(\Phi)$, *then there exists a Hölder continuous function* β *such that*

$$\alpha(p,t) = \beta \circ \Phi^t(p) - \beta(p)$$

for any $p \in M$ *and* $t \in \mathbb{R}$. *It is unique up to the constant term.*

The following results are applications of the Livschitz theorem.

THEOREM 3.1 (cf. [13, Th. 19.2.9]). *Let* Φ_1 *and* Φ_2 *be* C^∞ *topologically transitive Anosov flows on closed manifolds* M_1 *and* M_2, *respectively. Suppose that a topological equivalence* H *between* Φ_1 *and* Φ_2 *satisfies* $\tau(p;\Phi_1) = \tau(H(p);\Phi_2)$ *for any* $p \in \mathrm{Per}(\Phi_1)$. *Then, there exists a Hölder continuous function* β *on* M_1 *such that the map* $H_1 : M_1 \to M_2$ *defined by* $H_1(p) = \Phi_2^{\beta(p)}(H(p))$ *is a topological conjugacy between* Φ_1 *and* Φ_2.

Proof. As Theorem 19.1.5 of [13], we can replace a topological equivalence H by a bi-Hölder one. Then, there exists a Hölder continuous cocycle $\alpha : M_1 \times \mathbb{R} \to \mathbb{R}$ over Φ_1 such that $H(\Phi_1^t(p)) = \Phi_2^{\alpha(p,t)}(H(p))$ for any $p \in M$ and $t \in \mathbb{R}$. The assumption implies that $\alpha(p,\tau(p;\Phi_1)) = \tau(H(p);\Phi_2) = \tau(p;\Phi_1)$ for any $p \in \Phi_1$. By the Livschitz theorem, there exists a Hölder continuous function β on M_1 such that $\alpha(p,t) = t + \beta(p) - \beta \circ \Phi_1(p)$. Let $H_1 : M_1 \to M_2$ be a continuous map defined by $H_1(p) = \Phi_2^{\beta(p)}(H(p))$. We can show that $H_1 \circ \Phi_1^t = \Phi_2^t \circ H_1$ by a direct computation. The equation implies that H_1 is locally injective on each orbit of Φ_1. Hence, H_1 is a covering map. Since the mapping degree of H_1 is one, it is a homeomorphism. □

We denote the topological entropy of a flow Ψ by $h_{\mathrm{top}}(\Psi)$.

PROPOSITION 3.2. *Let* Φ *be a* C^2 *Anosov flow. Suppose that there exists* $\lambda > 0$ *such that* $J^u(p;\Phi) = \lambda \cdot \tau(p;\Phi)$ *for any* $p \in \mathrm{Per}(\Phi)$. *Then,* $h_{\mathrm{top}}(\Phi) = \lambda$.

Proof. It follows from several standard facts in ergodic theory of hyperbolic systems. Fix a Riemannian metric on M. It is known that the subbundle E^{uu} is Hölder continuous (see [13, Th. 19.1.6]). By the Livschitz theorem, there exists a Hölder continuous function β such that $\log|\det D\Phi^t|_{E^{uu}(p)}| = \lambda t + \beta \circ \Phi^t(p) - \beta(p)$ for any $p \in M$ and $t \in \mathbb{R}$. Hence, the Lyapunov exponent along E^{uu} is λ at any point. The entropy of Φ-invariant measure is not greater than λ by Ruelle's inequality, and the entropy of the SRB measure is λ by Pesin's formula. Hence, the variational principle implies that $h_{\mathrm{top}}(\Phi) = \lambda$. □

De la Llave and Moriyón characterized C^∞ conjugacy between three-dimensional Anosov flows by the coincidence of J^s and J^u.

THEOREM 3.3 ([15]). *Let* Φ_1 *and* Φ_2 *be* C^∞ *topologically transitive Anosov flows on three-dimensional closed manifolds* M_1 *and* M_2, *respectively. Suppose that a topological conjugacy* H *between* Φ_1 *and* Φ_2 *satisfies* $J^s(p;\Phi_1) =$

$J^s(H(p);\Phi_2)$ and $J^u(p;\Phi_1) = J^u(H(p);\Phi_2)$ for any $p \in \mathrm{Per}(\Phi_1)$. Then, H is a C^∞ diffeomorphism.

As an application of the above results, we give a criterion of C^∞ conjugacy between actions. For a locally free GA-action ρ on a closed three-dimensional manifold, we define a flow Φ_ρ by $\Phi_\rho^t(p) = \rho(p, X^t)$.

PROPOSITION 3.4 ([7, p. 518]). *The flow Φ_ρ is topologically transitive.*

THEOREM 3.5 ([1]). *The flow Φ_ρ is Anosov.*

We call Φ_ρ *the Anosov flow associated with ρ.*

PROPOSITION 3.6. *Let ρ_1 and ρ_2 be locally free* GA *actions on closed three-dimensional manifolds M_1 and M_2. Let Φ_{ρ_i} be the Anosov flow associated with ρ_i. Suppose that there exists a topological equivalence $H : M_1 \to M_2$ between Φ_{ρ_1} and Φ_{ρ_2}, and a constant $\lambda > 0$ such that $J^u(H(p);\Phi_{\rho_2}) = J^u(p;\Phi_{\rho_1})$ and $\tau(H(p);\Phi_{\rho_2}) = \lambda \cdot \tau(p;\Phi_{\rho_1})$ for any $p \in \mathrm{Per}(\Phi_{\rho_1})$. Then, ρ_1 is C^∞ conjugate to ρ_2 by a diffeomorphism homotopic to H.*

Proof. First, we show that Φ_{ρ_1} and Φ_{ρ_2} are topologically conjugate. Put $\Psi^t = \Phi_{\rho_2}^{\lambda t}$. Then, H is a topological equivalence between Φ_{ρ_1} and Ψ. Moreover, $J^u(H(p);\Psi) = J^u(p;\Phi_{\rho_1})$ and $\tau(H(p);\Psi) = \tau(p;\Phi_{\rho_1})$ for any $p \in \mathrm{Per}(\Phi_{\rho_1})$. By Theorem 3.1, there exists a Hölder continuous function β on M_1 such that the map $H_1 : M_1 \to M_2$ defined by $H_1(p) = \Psi^{\beta(p)}(H(p))$ is a topological conjugacy between Φ_{ρ_1} and Ψ. In particular, $h_{\mathrm{top}}(\Phi_{\rho_1}) = h_{\mathrm{top}}(\Psi) = \lambda \cdot h_{\mathrm{top}}(\Phi_{\rho_2})$. On the other hand, $h_{\mathrm{top}}(\Phi_{\rho_1}) = h_{\mathrm{top}}(\Phi_{\rho_2}) = 1$ by Proposition 3.2. Therefore, $\lambda = 1$, and hence, $\Psi = \Phi_{\rho_2}$. This implies that H_1 is a topological conjugacy between Φ_{ρ_1} and Φ_{ρ_2}, and $J^u(H_1(p);\Phi_{\rho_2}) = J^u(p;\Phi_{\rho_1})$ for any $p \in \mathrm{Per}(\Phi_{\rho_1})$.

Since $J^s(H_1(p);\Phi_{\rho_2}) = -\tau(H_1(p);\Phi_{\rho_2})$ and $J^s(p;\Phi_{\rho_1}) = -\tau(p;\Phi_{\rho_1})$, we have $J^s(H_1(p);\Phi_{\rho_2}) = J^s(p;\Phi_{\rho_1})$ for any $p \in \mathrm{Per}(\Phi_{\rho_1})$. By Theorem 3.3, the conjugacy H_1 is a C^∞ diffeomorphism.

Let X_i be the vector field generating Φ_{ρ_i} and S_i be the vector field generating the flow $(\rho_i(\cdot, S^x))_{x\in\mathbb{R}}$ for each $i = 1, 2$. Since $[S_i, X_i] = -S_i$, the vector field S_i is tangent to the strong stable foliation of Φ_{ρ_i}. Since the C^∞ conjugacy H_1 preserves the strong stable foliation, there exists a C^∞ function g on M_2 such that $g(p) \neq 0$ and $DH_1(S_1)(p) = g(p) \cdot S_2(p)$ for any $p \in M_2$. Then,

$$g \cdot S_2 = DH_1(S_1) = DH_1([X_1, S_1]) = [X_2, g \cdot S_2] = (g + X_2 g) \cdot S_2,$$

and hence, $X_2 g = 0$. Since the flow Φ_{ρ_2} is topologically transitive, the function g is constant with a nonzero value c. Now, we have

$$\rho_2(H_1(p), X^t S^{cx}) = H_1(\rho_1(p, X^t S^x))$$

for any $t, x \in \mathbb{R}$ and $p \in M_\Gamma$. Hence, ρ_1 is C^∞ conjugate to ρ_2. □

3.2. *The map* $\bar{a}_\Gamma$. In this section, we introduce a natural parametrization of GA-actions up to C^∞ conjugacy homotopic to the identity map.

In [7], Ghys showed that the Jacobian of ρ is controlled by a closed one-form.

THEOREM 3.7 ([7, Ch. IV]). *Let M be a closed oriented three-dimensional manifold with a fixed volume and ρ a C^∞ locally free* GA*-action on M. Then, there exists a C^∞ closed one-form ω_ρ on M such that*

$$(\rho^g)^*\omega_\rho - \omega_\rho = -d\left(\log\det D\rho^g\right) \tag{3}$$

for any $g \in \mathrm{GA}$, where ρ^g is a diffeomorphism on M given by $\rho^g(p) = \rho(p, g)$.

Fix a cocompact lattice Γ of $\widetilde{\mathrm{PSL}}(2,\mathbb{R})$ and put $M_\Gamma = \Gamma\backslash\widetilde{\mathrm{PSL}}(2,\mathbb{R})$. The manifold M_Γ is orientable and admits the standard volume induced from the Haar measure on $\widetilde{\mathrm{PSL}}(2,\mathbb{R})$. Let ρ_Γ be the standard GA-action on M_Γ and $\mathcal{F}_\Gamma$ the orbit foliation of ρ_Γ. Recall that the flow Φ_Γ on M_Γ is defined by $\Phi_\Gamma^t(\Gamma g) = \Gamma(gX^t)$. It is the Anosov flow associated with ρ_Γ. The following proposition characterizes an Anosov flow whose weak stable foliation is $\mathcal{F}_\Gamma$.

PROPOSITION 3.8. *Let Φ be an Anosov flow on M_Γ such that its weak stable foliation coincides with $\mathcal{F}_\Gamma$. Then, there exists a topological equivalence between Φ_Γ and Φ which is isotopic to the identity and preserves each leaf of $\mathcal{F}_\Gamma$.*

Proof. Let Σ be a closed surface with a hyperbolic metric and $S^1\Sigma$ be its unit tangent bundle. In Sections 3 and 4 of [6], Ghys proved that if the weak stable foliation of an Anosov flow on $S^1\Sigma$ is transverse to the fibers of the fibration $S^1\Sigma \to \Sigma$, then there exists a topological equivalence between the Anosov flow and the geodesic flow on $S^1\Sigma$ which is homotopic to the identity.[4] His proof works well even for our case by taking a finite covering of $S^1\Sigma$. Hence, there exists a topological equivalence H between Φ_Γ and Φ which is homotopic to the identity.

Let $\tilde{H}$, $\tilde{\Phi}$, and $\tilde{\Phi}_\Gamma$ be the lifts of H, Φ, and Φ_Γ to $\widetilde{\mathrm{PSL}}(2,\mathbb{R})$. We denote the lift of $\mathcal{F}_\Gamma$ to $\widetilde{\mathrm{PSL}}(2,\mathbb{R})$ by $\tilde{\mathcal{F}}_\Gamma$. Since $\tilde{H}$ maps the $\tilde{\Phi}_\Gamma$-orbit of x to the $\tilde{\Phi}$-orbit of $\tilde{H}(x)$ for any $x \in \widetilde{\mathrm{PSL}}(2,\mathbb{R})$, the Hausdorff distance between these orbits is bounded. This implies that $\tilde{H}(x)$ is contained in $\tilde{\mathcal{F}}_\Gamma(x)$. Therefore, H preserves each leaf of $\mathcal{F}_\Gamma$. □

Recall that $\mathcal{A}_\Gamma$ is the space of C^∞ locally free GA-action on M_Γ whose orbit foliation is $\mathcal{F}_\Gamma$. By $\mathbb{R}_+$, we denote the set of positive real numbers.

[4]Ghys did not mentioned that the topological equivalence he constructed is homotopic to the identity. However, this fact follows easily from his proof.

PROPOSITION 3.9. *Let ρ be an action in $\mathcal{A}_\Gamma$, H a topological equivalence between Φ_Γ and Φ_ρ which preserves each leaf of $\mathcal{F}_\Gamma$, and ω_ρ a closed one-form given in Theorem 3.7. Then, there exists a unique pair $(a, \lambda) \in H^1(M_\Gamma, \mathbb{R}) \times \mathbb{R}_+$ such that*

$$\tau(H(p); \Phi_\rho) = \lambda \cdot \{\tau(p; \Phi_\Gamma) + \langle a, \gamma(p; \Phi_\Gamma)\rangle\} \tag{4}$$

for any $p \in \mathrm{Per}(\Phi_\Gamma)$.

Proof. Let X_ρ be the vector field generating Φ_ρ. We define a function $\xi : M_\Gamma \to \mathbb{R}$ by $\xi(p) = (d/dt) \log \det(D\Phi^t_\rho)_p|_{t=0}$. By definition (3) of ω_ρ,

$$\begin{aligned}\omega_\rho(X_\rho) \circ \Phi^t_\rho - \omega_\rho(X_\rho) &= -X_\rho[\log \det D\Phi^t_\rho] \\ &= -\xi \circ \Phi^t_\rho(p) + \xi(p)\end{aligned}$$

for any $p \in M_\Gamma$. In particular, $\xi + \omega_\rho(X_\rho)$ is a Φ_ρ-invariant function. Since Φ_ρ is topologically transitive, there exists $\delta \in \mathbb{R}$ such that $\xi + \omega_\rho(X_\rho) = \delta$. This implies that

$$J(q; \Phi_\rho) = \delta \cdot \tau(q; \Phi_\rho) - \langle[\omega_\rho], \gamma(q; \Phi_\rho)\rangle$$

for any $q \in \mathrm{Per}(\Phi_\rho)$. The map H is homotopic to the identity map and preserves each leaf of $\mathcal{F}_\Gamma$. Hence, $[\gamma(H(p); \Phi_\rho)] = [\gamma(p; \Phi_\Gamma)]$ and $J^u(H(p); \Phi_\rho) = J^u(p; \Phi_\Gamma) = \tau(p; \Phi_\Gamma)$ for any $p \in \mathrm{Per}(\Phi_\Gamma)$. Since Φ_ρ is the Anosov flow associated with ρ, we have $J^s(H(p); \Phi_\rho) = -\tau(H(p); \Phi_\rho)$. These equations imply that

$$(1 + \delta) \cdot \tau(H(p); \Phi_\rho) = \tau(p; \Phi_\Gamma) + \langle[\omega_\rho], \gamma(p; \Phi_\Gamma)\rangle$$

for any $p \in \mathrm{Per}(\Phi_\Gamma)$. Since $[\gamma(p_0; \Phi_\Gamma)] = 0$ for some $p_0 \in \mathrm{Per}(\Phi_\Gamma)$, we have $1 + \delta > 0$. Therefore, $[\omega_\rho]$ satisfies equation (4) for $\lambda = (1 + \delta)^{-1}$.

We show the uniqueness of the pair (a, λ). Suppose that (a, λ) satisfies the required condition. Then,

$$\langle a - [\omega_\rho], \gamma(p; \Phi_\Gamma)\rangle = (\lambda^{-1} - (1 + \delta))\tau(H(p); \Phi_\omega)$$

for any $p \in \mathrm{Per}(\Phi_\Gamma)$. Take $p_0, \dots, p_k \in \mathrm{Per}(\Phi_\Gamma)$ such that $[\gamma(p_0; \Phi_\Gamma)] = 0$ and $[\gamma(p_1; \Phi_\Gamma)], \dots, [\gamma(p_k; \Phi_\Gamma)]$ span $H_1(M_\Gamma, \mathbb{R})$. The evaluation of the above equation at $[\gamma(p_0; \Phi_\Gamma)]$ implies that $\lambda = (1+\delta)^{-1}$. The evaluation at $[\gamma(p_1; \Phi_\Gamma)], \dots,$ $[\gamma(p_k; \Phi_\Gamma)]$ implies that $a = [\omega_\rho]$. □

By the above proposition, the action ρ determines the class $[\omega_\rho]$ uniquely. We define maps $\bar{a}_\Gamma : \mathcal{A}_\Gamma \to H^1(M_\Gamma, \mathbb{R})$ and $\bar{\lambda}_\Gamma : \mathcal{A}_\Gamma \to \mathbb{R}_+$ so that $(\bar{a}_\Gamma(\rho), \bar{\lambda}_\Gamma(\rho))$ is the pair in Proposition 3.9.

PROPOSITION 3.10. *For $\rho_1, \rho_2 \in \mathcal{A}_\Gamma$, $\bar{a}_\Gamma(\rho_1) = \bar{a}_\Gamma(\rho_2)$ if and only if ρ_1 is C^∞ conjugate to ρ_2 by a diffeomorphism homotopic to the identity map.*

Proof. It is easy to see that $\bar{a}_\Gamma(\rho_1) = \bar{a}_\Gamma(\rho_2)$ if two actions are C^∞ conjugate by a diffeomorphism homotopic to the identity map.

Suppose that $\bar{a}_\Gamma(\rho_1) = \bar{a}_\Gamma(\rho_2)$. By Proposition 3.8, there exists a topological equivalence $H_i : M_\Gamma \to M_\Gamma$ between Φ_Γ and Φ_{ρ_i} which is homotopic to the identity and which preserves each leaf of $\mathcal{F}_\Gamma$. Then, $J^u(H_1(p); \Phi_{\rho_1}) = J^u(H_2(p); \Phi_{\rho_2}) = J^u(p; \Phi_\Gamma)$ for any $p \in \mathrm{Per}(\Phi_\Gamma)$. By Proposition 3.9, the assumption $\bar{a}_\Gamma(\rho_1) = \bar{a}_\Gamma(\rho_2)$ implies

$$\bar{\lambda}_\Gamma(\rho_1)^{-1} \cdot \tau(H_1(p); \Phi_{\rho_1}) = \bar{\lambda}_\Gamma(\rho_2)^{-1} \cdot \tau(H_2(p); \Phi_{\rho_2}).$$

Applying Proposition 3.6 to ρ_1 and ρ_2, we obtain a diffeomorphism H homotopic to $H_2 \circ H_1^{-1}$ which conjugates ρ_1 to ρ_2. Since H_1 and H_2 are homotopic to the identity map, so H is. □

3.3. *The image of* $\bar{a}_\Gamma$. We define a subset Δ_Γ of $H^1(M_\Gamma, \mathbb{R})$ by

$$\Delta_\Gamma = \left\{ a \in H^1(M_\Gamma, \mathbb{R}) \;\middle|\; \sup_{p \in \mathrm{Per}(\Phi_\Gamma)} \frac{|\langle a, \gamma(p; \Phi_\Gamma)\rangle|}{\tau(p; \Phi_\Gamma)} < 1 \right\}. \tag{5}$$

We will show that the image of $\bar{a}_\Gamma$ is Δ_Γ.

First, we show that each $a \in \Delta_\Gamma$ admits a nice representative.

Lemma 3.11. *A cohomology class* $a \in H^1(M_\Gamma, \mathbb{R})$ *is contained in* Δ_Γ *if and only if it admits a* C^ω *representative such that* $1 + \omega(X_\Gamma) > 0$.

Proof. The "if" part of the lemma is trivial. We show the "only if" part. By $\mathcal{M}(\Phi_\Gamma)$ and $\mathcal{M}_{\mathrm{per}}(\Phi_\Gamma)$, we denote the set of Φ_Γ-invariant probability measures and its subset consisting of measures supported on a periodic orbit, respectively.

Take a C^ω closed one-form ω_0 on M_Γ which represents $a \in \Delta_\Gamma$. Then, there exists $\varepsilon > 0$ such that

$$\int 1 + \omega(X_\Gamma) d\mu = 1 + \frac{\langle a, \gamma(p; \Phi)\rangle}{\tau(p; \Phi_\Gamma)} \geq \varepsilon$$

for any $\mu \in \mathcal{M}_{\mathrm{per}}(\Phi_\Gamma)$. It is known that $\mathcal{M}(\Phi_\Gamma)$ coincides with the convex hull of $\mathcal{M}_{\mathrm{per}}(\Phi_\Gamma)$; e.g., see Lemma 2.4 of [9]. Hence, the integral of the function $(1 + \gamma(X_\Gamma))$ with respect to any $\mu \in \mathcal{M}(\Phi_\Gamma)$ is positive. By Lemma 2.5 of [9], there exists a C^∞ function g_0 on M_Γ such that $1 + \omega(X_\Gamma) + X_\Gamma g_0 > 0$. Since M_Γ is compact, we may approximate g_0 by a C^ω function g such that $1 + \omega(X_\Gamma) + X_\Gamma g > 0$. The C^ω one-form $\omega + dg$ satisfies the required condition. □

Corollary 3.12. Δ_Γ *is a convex open subset of* $H^1(M_\Gamma, \mathbb{R})$.

Next, we construct a family $(\rho_a)_{a \in \Delta_\Gamma}$ in $\mathcal{A}_\Gamma$ such that $\bar{a}_\Gamma(\rho_a) = a$ for any $a \in \Delta_\Gamma$. Essentially, it is a corollary of the construction of nonhomogeneous actions in Section 2.

PROPOSITION 3.13. *For any* $a \in \Delta_\Gamma$, *there exists* $\rho_a \in \mathcal{A}_\Gamma$ *such that* $\bar{a}_\Gamma(\rho_a) = a$.

Proof. Fix $a \in \Delta_\Gamma$ and a C^ω representative ω of a such that $1+\omega(X_\Gamma) > 0$. By Theorem 2.7, there exists a C^ω manifold M_ω, a C^ω locally free GA-action $\check{\rho}_\omega$ on M_ω, a topological equivalence H between Φ_Γ and $\Phi_{\check{\rho}_\omega}$, and a constant $\lambda > 0$ such that

$$\begin{aligned} J^u(H(p); \Phi_{\check{\rho}_\omega}) &= \tau(p; \Phi_\Gamma), \\ \tau(H(p); \Phi_{\check{\rho}_\omega}) &= \lambda \cdot \{\tau(p; \Phi_\Gamma) + \langle [\omega], \gamma(p; \Phi_\Gamma)\rangle\} \end{aligned}$$

for any $p \in \mathrm{Per}(\Phi_\Gamma)$.

Let $\mathcal{F}_\omega$ be the orbit foliation of $\check{\rho}_\omega$. Since any locally free GA-action on a closed oriented manifold is C^∞ orbit equivalent to a homogeneous action, there exists a homogeneous action ρ_h whose orbit foliation is $\mathcal{F}_\omega$. By Proposition 3.8, $\Phi_{\check{\rho}_\omega}$ is topologically equivalent to Φ_{ρ_h} by a homeomorphism $H_1 : M_\omega \to M_\omega$ which is homotopic to the identity map and preserves each leaf of $\mathcal{F}_\omega$. Then,

$$\tau(H_1 \circ H(p); \Phi_{\rho_h}) = J^u(H_1 \circ H(p); \Phi_{\rho_h}) = J^u(H(p); \Phi_{\check{\rho}_\omega}) = \tau(p; \Phi_\Gamma)$$

for any $p \in \mathrm{Per}(\Phi_\Gamma)$. Since both ρ_Γ and ρ_h are homogeneous, it implies that $J^u(H_1 \circ H(p); \Phi_{\rho_h}) = J^u(p; \Phi_{\rho_\Gamma})$ and $J^s(H_1 \circ H(p); \Phi_{\rho_h}) = J^s(p; \Phi_{\rho_\Gamma})$. By Proposition 3.6, there exists a C^∞ diffeomorphism $H_\omega : M_\Gamma \to M_\omega$ which conjugates ρ_Γ and ρ_h.

We define a GA-action ρ_a on M_Γ by $\rho_a(p, g) = H_\omega^{-1}(\check{\rho}_\omega(H_\omega(p), g))$. Since H_ω sends leaves of $\mathcal{F}_\Gamma$ to those of $\mathcal{F}_\omega$, ρ_a is an element of $\mathcal{A}_\Gamma$. It is easy to check that $H_\omega^{-1} \circ H$ is a topological equivalence between Φ_Γ and Φ_{ρ_a} which is homotopic to the identity map and preserves each leaf of $\mathcal{F}_\Gamma$. Moreover, we have

$$\tau(H_\omega^{-1} \circ H(p); \Phi_{\rho_a}) = \tau(H(p); \Phi_{\check{\rho}_\omega}) = \lambda \cdot \{\tau(p; \Phi_\Gamma) + \langle [\omega], \gamma(p; \Phi_\Gamma)\rangle\}$$

for any $p \in \mathrm{Per}(\Phi_\Gamma)$. Therefore, $\bar{a}(\rho_a) = [\omega] = a$. □

Combined with Propositions 3.10 and 3.13, the following proposition completes the proof of the main theorem.

PROPOSITION 3.14. *The image of* $\bar{a}_\Gamma$ *is contained in* Δ_Γ.

Proof. Take $\rho \in \mathcal{A}_\Gamma$. Put $a = \bar{a}_\Gamma(\rho)$ and $\lambda = \bar{\lambda}_\Gamma(\rho)$. Let H be a topological equivalence between Φ_Γ and Φ_ρ which preserves each leaf of $\mathcal{F}_\Gamma$. Then, there exists $\varepsilon > 0$ such that $\varepsilon\lambda \cdot \tau(p; \Phi_\Gamma) \leq \tau(H(p); \Phi_\rho)$ for any $p \in \mathrm{Per}(\Phi_\Gamma)$. Since $\tau(H(p); \Phi_\rho) = \lambda \cdot \{\tau(p; \Phi_\Gamma) + \langle a, \gamma(p; \Phi_\Gamma)\rangle\}$, we have

$$\langle a, \gamma(p; \Phi_\Gamma)\rangle \geq -(1-\varepsilon)\tau(p; \Phi_\Gamma)$$

for any $p \in \mathrm{Per}(\Phi_\Gamma)$.

For any given $p \in \mathrm{Per}(\Phi_\Gamma)$, there exists $p' \in \mathrm{Per}(\Phi_\Gamma)$ such that $\tau(p'; \Phi_\Gamma) = \tau(p; \Phi_\Gamma)$ and $[\gamma(p'; \Phi_\Gamma)] = -[\gamma(p; \Phi_\Gamma)]$. The above inequality for p' implies that $\langle a, \gamma(p; \Phi_\Gamma)\rangle \leq (1-\varepsilon)\tau(p; \Phi_\Gamma)$. Therefore, $a = \bar{a}_\Gamma(\rho) \in \Delta_\Gamma$. □

Appendix A. Locally free actions on solvable manifolds

In this section, we give a proof of the following unpublished result by Ghys.

THEOREM A.1. *Let M be a closed three-dimensional manifold whose fundamental group is solvable. Then, any C^∞ locally free action of* GA *on M is homogeneous.*

Proof. Here, we present a shorter proof using a rigidity result due to Matsumoto and Mitsumatsu [17]. Fix a C^∞ locally free action ρ on M. Let $\mathcal{F}_\rho$ be the orbit foliation of ρ. By Plante's theorem [18], the Anosov flow associated with ρ is topologically equivalent to the suspension flow of a hyperbolic toral automorphism A. Let M_A be the mapping torus of A. Since the orbit foliation of ρ admits no closed leaves, a classification result by Ghys and Sergiescu [4] implies that $\mathcal{F}_\rho$ is C^∞ diffeomorphic to the orbit foliation $\mathcal{F}_A$ of a homogeneous GA-action ρ_A on M_A. In [17], Matsumoto and Mitsumatsu proved that any locally free GA-action on M_A whose orbit foliation is $\mathcal{F}_A$ is C^∞ conjugate to ρ_A. Therefore, ρ is C^∞ conjugate to the homogeneous action ρ_A. □

Appendix B. Regularity of the unstable foliation

In this section, we characterize homogeneous actions by the regularity of the unstable foliation of the associated Anosov flow.

THEOREM B.1. *Let Γ be a cocompact lattice of $\widetilde{\mathrm{PSL}}(2,\mathbb{R})$ and ρ be an action in $\mathcal{A}_\Gamma$. If the weak unstable foliation of the Anosov flow associated with ρ is a C^2 foliation, then ρ is C^∞ conjugate to the standard* GA-*action on $M_\Gamma = \Gamma\backslash\widetilde{\mathrm{PSL}}(2,\mathbb{R})$.*

For a cocompact lattice Γ of $\widetilde{\mathrm{PSL}}(2,\mathbb{R})$, let ρ_Γ be the standard GA-action on M_Γ and Φ_Γ the associated Anosov flow, i.e., $\rho(\Gamma g, h) = \Gamma(gh)$ and $\Phi_\gamma^t(\Gamma g) = \Gamma(gX^t)$. For an Anosov flow Φ, we denote the stable foliation and the unstable foliation of Φ by $\mathcal{F}_\Phi^s$ and $\mathcal{F}_\Phi^u$. To simplify notation, we put $\mathcal{F}_\Gamma^s = \mathcal{F}_{\Phi_\Gamma}^s$ and $\mathcal{F}_\Gamma^u = \mathcal{F}_{\Phi_\Gamma}^u$. Remark that $\mathcal{F}_\Gamma^s$ is the orbit foliation of the standard action ρ_Γ. For an Anosov flow Φ on M_Γ and $\gamma \in \Gamma$, we put

$$\mathrm{Per}_\gamma(\Phi) = \{p \in M_\Gamma \mid p = \Gamma g,\ \gamma g = \tilde{\Phi}^T(g) \text{ for some } g \in \widetilde{\mathrm{PSL}}(2,\mathbb{R}), T > 0\},$$

where $\tilde{\Phi}$ is the lift of Φ to $\widetilde{\mathrm{PSL}}(2,\mathbb{R})$.

The hyperbolic plane $\mathbb{H}^2$ admits a natural isometric action of $\widetilde{\mathrm{PSL}}(2,\mathbb{R})$. We define the *translation length* $L(g)$ of $g \in \widetilde{\mathrm{PSL}}(2,\mathbb{R})$ by

$$L(g) = \inf_{z \in \mathbb{H}^2} d_{\mathbb{H}^2}(z, gz),$$

where $d_{\mathbb{H}^2}$ is the distance induced by the hyperbolic metric.

Fix a cocompact lattice Γ of $\widetilde{\mathrm{PSL}}(2,\mathbb{R})$ and an action $\rho \in \mathcal{A}_\Gamma$. Let Φ be the Anosov flow associated with ρ. Suppose that $\mathcal{F}^u_\Phi$ is a C^2 foliation.

LEMMA B.2. *There exists an injective homomorphism $\sigma : \Gamma \to \widetilde{\mathrm{PSL}}(2,\mathbb{R})$ such that*

$$L(\sigma(\gamma)) = \bar{\lambda}_\Gamma(\rho)\,\{L(\gamma) + \langle \bar{a}_\Gamma(\rho), \gamma \rangle\}$$

for any $\gamma \in \Gamma$ with $\mathrm{Per}_\gamma(\Phi_\Gamma) \neq \emptyset$.

Proof. By a classification of three-dimensional Anosov flows with the C^2 Anosov splitting due to Ghys [11], [10], there exists a cocompact lattice Γ' of $\widetilde{\mathrm{PSL}}(2,\mathbb{R})$ and a C^2 diffeomorphism $H : M_\Gamma \to M_{\Gamma'}$ such that H sends leaves of $\mathcal{F}^u_\Phi$ to those of $\mathcal{F}^u_{\Gamma'}$. We define a flow Ψ on $M_{\Gamma'}$ by $\Psi^t = H \circ \Phi^t \circ H^{-1}$. Remark that $\mathcal{F}^u_\Psi$ coincides with $\mathcal{F}^u_{\Gamma'}$. The lift of H to $\widetilde{\mathrm{PSL}}(2,\mathbb{R})$ induces an isomorphism $\sigma : \Gamma \to \Gamma'$ such that $H(\mathrm{Per}_\gamma(\Phi)) = \mathrm{Per}_{\sigma(\gamma)}(\Psi)$ for any $\gamma \in \Gamma$.

By Proposition 3.8, there exists a topological equivalence $H_\Gamma : M_\Gamma \to M_\Gamma$ between Φ_Γ and Φ which preserves each leaf of $\mathcal{F}^s_\Gamma$. Similarly, there exists a topological equivalence $H_{\Gamma'} : M_{\Gamma'} \to M_{\Gamma'}$ between $\Phi_{\Gamma'}$ and Ψ which preserves each leaf of $\mathcal{F}^u_{\Gamma'}$.

It is well known and easy to see that $\tau(p;\Phi_\Gamma) = J^u(p;\Phi_\Gamma) = L(\gamma)$ for any $\gamma \in \Gamma$ and $p \in \mathrm{Per}_\gamma(\Phi_\Gamma)$. Similarly, $J^s(p';\Phi_{\Gamma'}) = -L(\gamma')$ for any $\gamma' \in \Gamma'$ and $p' \in \mathrm{Per}_{\gamma'}(\Phi_{\Gamma'})$. Since $\gamma(H_{\Gamma'}(p');\Psi)$ is freely homotopic to $\gamma(p';\Phi_{\Gamma'})$ in $\mathcal{F}^u_{\Gamma'}(p')$, we have $J^s(H_{\Gamma'}(p');\Psi) = J^s(p';\Phi_{\Gamma'}) = -L(\gamma')$ for any $\gamma' \in \Gamma'$ and $p' \in \mathrm{Per}_{\gamma'}(\Phi_{\Gamma'})$. Moreover, since H is a C^2 diffeomorphism and Φ is the Anosov flow associated with ρ, we have

$$\tau(H_\Gamma(p);\Phi) = -J^s(H_\Gamma(p);\Phi) = -J^s(H \circ H_\Gamma(p);\Psi) = L(\sigma(\gamma))$$

for any $\gamma \in \Gamma$ and $p \in \mathrm{Per}_\gamma(\Phi_\Gamma)$. Now, the required equation follows from the definition of $(\bar{a}_\Gamma, \bar{\lambda}_\Gamma)$ and the above equations. □

Now we prove Theorem B.1. Take $\gamma \in \Gamma$ with $\mathrm{Per}_\gamma(\Phi_\Gamma) \neq \emptyset$. There exists $g \in \widetilde{\mathrm{PSL}}(2,\mathbb{R})$ and $T > 0$ such that $\gamma g = g X^T$. Put $g' = g \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$. Then, $\gamma^{-1} g' = g' X^T$. Hence, $\mathrm{Per}_{\gamma^{-1}}(\Phi_\Gamma)$ is nonempty. Since $L(g^{-1}) = L(g)$ for any $g \in \widetilde{\mathrm{PSL}}(2,\mathbb{R})$, Lemma B.2 implies that $\langle \bar{a}_\Gamma(\rho), \gamma \rangle = 0$. It is known that the set

$$\{[\gamma] \in H_1(M_\Gamma, \mathbb{R}) \mid \gamma \in \Gamma, \mathrm{Per}_\gamma(\Phi) \neq \emptyset\}$$

spans $H_1(M_\Gamma, \mathbb{R})$. Therefore, $\bar{a}_\Gamma(\rho) = 0$. This implies that ρ is a homogeneous action.

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(Received: February 28, 2007)
(Revised: September 14, 2011)

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