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Eigenvarieties for reductive groups

By Eric Urban

Abstract

We develop the theory of overconvergent cohomology introduced by G. Stevens, and we use it to give a construction of eigenvarieties associated to any reductive group G over $\mathbb{Q}$ such that $G(\mathbb{R})$ has discrete series. We prove that the so-called eigenvarieties are equidimensional and generically flat over the weight space.

Contents

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0. **Introduction**

The theory of p-adic families of automorphic forms has known many developments since the original breakthrough of H. Hida in the early eighties who constructed p-adic families of ordinary modular cusp eigenforms. His results were further extended to finite slope modular forms by Coleman using the p-adic spectral theory of the Atkin U_p operator and the construction by Coleman-Mazur of the eigencurve brought a more geometric and global aspect to the theory.

Although Hida's original approach as well as Coleman's strategy were built on Katz' theory of p-adic and overconvergent modular forms, the cohomological[1] method, whose idea is originally due to Shimura, using cohomology of arithmetic subgroups to study congruences between Hecke eigenvalues led several authors following H. Hida in the ordinary case and G. Stevens in the finite slope case to construct families of Hecke eigensystems for reductive groups G over $\mathbb{Q}$ whose archimedean part $G(\mathbb{R})$ is compact modulo center. For example,

[1] The original geometric approach has also been also generalized by Hida to all Shimura varieties of PEL type and also in some nonordinary case by other authors. See, for example, [Kas04], [Kas06], [KL05], [SU06b].

see [Buz04], [Che04], [Eme06]. However, the extension of these techniques to more general reductive groups was hindered by the difficulty to handle the torsion of the cohomology of the corresponding arithmetic subgroups.

In this paper, we bypass this difficulty for groups G such that $G^{\mathrm{der}}(\mathbb{R})$ satisfies the Harish-Chandra condition[2] (i.e., contains a compact Cartan subgroup) and construct p-adic families of automorphic (cuspidal) representations for G. In particular, we construct eigenvarieties associated to such groups whose points are in bijection with automorphic representations having nontrivial Euler-Poincaré characteristics, and we prove that it is equidimensional of the expected dimension. The main difference between our approach and other authors' strategies is to work with the total cohomology instead of studying the cohomology separately in each degree. More precisely, we work with some kind of "perfect complex" that computes the cohomology of the corresponding arithmetic groups with coefficients in a family of certain p-adic distribution spaces. Moreover, instead of trying to construct a universal module that interpolates the cohomology when the weight of the system of coefficients varies, we rather make a p-adic analytic interpolation of the trace of the Hecke operators acting on the total cohomology. Our approach is somehow similar to Wiles' idea of constructing families of Galois representations by the use of pseudo-representations. We give two applications of the p-adic analyticity property with respect to the weight of these traces. One concerns the construction of eigenvarieties and therefore of p-adic families of automorphic representations. We actually give an axiomatic treatment of the construction of eigenvarieties from the existence of p-adic analytic families of traces of Hecke operators. The other one is to derive a p-adic trace formula in geometric terms very similar to the one by Arthur-Selberg.

For groups G such that $G(\mathbb{R})$ do not satisfy the Harish-Chandra condition, like GL_n with $n > 2$, our method provides evidences that the presence of torsion is related to the vanishing of the Euler-Poincaré characteristic of the π-isotypical component of the cohomology when π is a cuspidal representation of $G(\mathbb{A})$. In fact, for $\mathrm{GL}_{3/\mathbb{Q}}$,[3] Ash and Stevens have noticed that certain cuspidal representations do not lie in a p-adic family, and they came up with a conjecture that vaguely says that a cuspidal representation can be deformed p-adically in a family of *classical* cuspidal representations if and only if it is essentially self-dual. According to Langlands' philosophy, all such representations should come from orthogonal or symplectic groups, and the "if" part of this conjecture would follow from our result applied to these classical groups. Surprisingly, it

[2]For example, it applies to any unitary or symplectic groups over a totally real field.

[3]For $G = \mathrm{GL}(2)_{/K}$ with K imaginary quadratic, Calegari and Mazur have remarked the same phenomena.

turns out[4] that our method can be modified in order to give a proof of this without assuming Langlands' transfer principle (see §5.6).

After I gave several lectures on this work, I learned that M. Koike and L. Clozel had worked on a somehow similar approach in the past, similar in the sense that they also proved some continuity statement with respect to the weight of the trace of the Hecke operators (see Clozel's unpublished manuscript [Clo93] and Koike's papers [Koi75], [Koi76]) by using the explicit form of the Selberg trace formula. Our approach differs from theirs in two ways. Firstly, we do not use an explicit form of the trace formula to prove the analyticity statement, and we use p-adic spectral theory on the Banach spaces defined by the locally analytic induction of p-adic characters. Secondly, our trace is a trace of a compact operator acting on a complex of p-adic Banach spaces while theirs is the usual classical trace computed by Arthur-Selberg. As a byproduct, we have obtained a p-adic trace formula. We hope to give some applications of it in the future.

To illustrate our work, we now give a description of a special case of our main result on the existence of p-adic families of automorphic representations. We want to stress the fact that we have constructed here deformations of automorphic representations rather than deformations of automorphic forms as is usually done in the literature. It has the advantage of giving better control over the information at the ramified places. This fact is very useful for applications such as those in [SU06a], [SU06b].

We let G be a reductive group such that $G(\mathbb{R})$ has discrete series. Let $S_G(K)$ be the locally symmetric space associated to G and a neat open subgroup K of the finite adelic points of G. Let π be an automorphic representation of $G(\mathbb{A})$ occurring in the cohomology of $S_G(K)$ with the system of coefficients $\mathbb{V}^\vee_{\lambda^{\mathrm{alg}}}(\mathbb{C})$ the dual of the irreducible algebraic representation of highest weight λ^{alg} with respect to some Borel pair. Such a representation will be called automorphic and cohomological of weight λ^{alg}. It can be seen (and we will see such a representation in this way) as a representation of the Hecke algebra of the $\mathbb{Q}$-valued locally constant functions on $G(\mathbb{A}_f)$ with compact support that we denote $C_c^\infty(G(\mathbb{A}_f),\mathbb{Q})$.

We now fix a prime p. For simplicity let us assume in this introduction that G splits over $\mathbb{Q}_p$ and let (B,T) be a Borel pair. Let R^+ be the set of the corresponding positive roots. We denote by I an Iwahori subgroup of $G(\mathbb{Q}_p)$ in good position with respect to (B,T). More generally, we denote I_m the

[4] I am grateful to D. Vogan for drawing my attention to the fact that the twisted Euler characteristic by a Cartan type involution is not trivial for most of the cohomological representations.

Iwahori subgroup of depth m defined by

$$I_m := \{g \in G(\mathbb{Z}_p) | g \pmod{p^m} \in B(\mathbb{Z}/p^m\mathbb{Z})\}.$$

We consider the monoid

$$T^- := \{t \in T(\mathbb{Q}_p) || \alpha(t)|_p \leq 1 \ \forall \alpha \in R^+\},$$

and we denote $T^{--} \subset T^-$ as the sub-monoid of elements t such that the inequalities in the above definition are strict. Then put $\Delta^- := I.T^-.I$ and $\Delta^{--} := I.T^{--}.I$. We consider $\mathcal{U}_p$ the local Hecke algebra[5] at p generated over $\mathbb{Z}_p$ by the characteristic function of the double classes $u_t := I_m t I_m$. Let $K^p \subset G(\mathbb{A}_f^p)$ be an open compact subgroup of the finite adelic points of G away from p. We now consider

$$\mathcal{H}_p(K^p) := C_c^\infty(K^p \backslash G(\mathbb{A}_f^p)/K^p, \mathbb{Q}_p) \otimes \mathcal{U}_p \subset C_c^\infty(G(\mathbb{A}_f), \mathbb{Q}_p),$$

where the left-hand side of the tensor product denotes the locally constant functions on $G(\mathbb{A}_f)$ with compact support which are bi-invariant by K^p. We call *finite slope representations* of G of level K^p the finite-dimensional representations of $\mathcal{H}_p(K^p)$ over a complete field extension of $\mathbb{Q}_p$ such that, for each $t \in T^-$, the double class u_t acts as an invertible automorphism. In particular, if such a representation is irreducible, then the u_t's act by multiplication by nonzero scalars.

Let π be a cohomological automorphic representation. Let us choose $K^p \subset G(\mathbb{A}_f^p)$ and some sufficiently deep pro-p-Iwahori subgroup I'_m such that $\pi_f^{K^p.I'_m} \neq 0$. We consider the action of $I_m/I'_m \cong T(\mathbb{Z}/p^m\mathbb{Z})$ on this subspace and choose a character ε of this group such that the ε-isotypical component of $\pi_f^{K^p.I'_m}$ is nontrivial. Then we have an action of $\mathcal{H}_p(K^p)$ on $\pi_f^{K^p.I'_m} \otimes \varepsilon^{-1}$. An irreducible constituent of the representation of $\mathcal{H}_p(K^p)$ acting on this space will be called a *p-stabilization* of π. The finite slope irreducible representations of $\mathcal{H}_p(K^p)$ obtained in this way will be called *finite slope classical automorphic* representations of weight $\lambda = \lambda^{\mathrm{alg}}\varepsilon$ that we consider to be a p-adic valued continuous character of $T(\mathbb{Z}_p)$. If the original π is cuspidal, then a p-stabilization of it will be called cuspidal too.

Let S be a finite set of primes such that the open subgroup $K^p \subset G(\mathbb{A}_f^p)$ is maximal hyperspecial away from S. Then denote by $R_{S,p}$ the abstract Hecke algebra defined as the sub-algebra of $\mathcal{H}_p(K^p)$ of $\mathbb{Z}_p$-valued locally constant functions with compact support contained in $G(\mathbb{A}_f^{S\cup\{p\}}).\Delta^-$. Note that $R_{S,p}$ is contained in the center of $\mathcal{H}_p(K^p)$. Therefore, the action of $R_{S,p}$ on a finite

[5] It is an easy fact that the structure of this algebra does not depend upon m. Hence we have dropped m from the notation.

slope representation σ of $\mathcal{H}_p(K^p)$ is given by multiplication by a character θ_σ of $R_{S,p}$.

Let π_0 be a finite slope irreducible cuspidal representation of $G(\mathbb{A})$ of weight λ_0. We normalize the $R_{S,p}$-action so that the double class u_t acts by multiplying the standard action by $|\lambda_0(t)|_p^{-1}$, and we denote by θ_0 the corresponding character of $R_{S,p}$.

We say that θ_0 is not critical with respect to λ_0 if, for some $t \in T^{--}$, we have

$$v_p(\theta_0(u_t)) < (\lambda_0(H_\alpha) + 1)v_p(\alpha(t))$$

for each simple root α, with H_α denoting the corresponding co-root.

Finally, we introduce the weight space. Let $Z_p = Z_p(K^p)$ be the p-adic closure of the image of $Z(\mathbb{Q}) \cap K^p.T(\mathbb{Z}_p)$ in $T(\mathbb{Z}_p)$ for some K^p neat and hyperspecial maximal away from S. It easy to see that Z_p does not depend on K^p but only on S. If G is $\mathbb{Q}$-split, then Z_p is trivial; otherwise, its rank is related to the rank of some units and also some Leopoldt defect. Our weight space is the rigid analytic space $\mathfrak{X} = \mathfrak{X}_{K^p}$ defined over $\mathbb{Q}_p$ such that $\mathfrak{X}(\overline{\mathbb{Q}}_p) = \mathrm{Hom}_{\mathrm{cont}}(T(\mathbb{Z}_p)/Z_p, \overline{\mathbb{Q}}_p^\times)$. Then we have the following theorem (see §5.4.3).

THEOREM. *Assume that θ_0 is not critical with respect to λ_0 and that the algebraic part of λ_0 is dominant regular. Then, there exists*

(1) *an affinoid open neighborhood $\mathfrak{U} \subset \mathfrak{X}$ of λ_0;*
(2) *a generically flat finite cover $\mathfrak{V}$ of $\mathfrak{U}$ with structural morphism $\mathbf{w}$;*
(3) *a homomorphism $\theta_{\mathfrak{V}}$ from $R_{S,p}$ to the ring $\mathcal{O}(\mathfrak{V})$ of analytic functions on $\mathfrak{V}$;*
(4) *a distribution character $I_{\mathfrak{V}} : \mathcal{H}_p(K^p) \to \mathcal{O}(\mathfrak{V})$;*
(5) *a point $y_0 \in \mathfrak{V}(\overline{\mathbb{Q}}_p)$ above λ_0;*
(6) *a Zariski dense subset $\Sigma \subset \mathfrak{V}(\overline{\mathbb{Q}}_p)$ such that $\mathbf{w}(y)$ is algebraic regular dominant for all $y \in \Sigma$;*
(7) *for each $y \in \Sigma$, a nonempty set Π_y of finite slope irreducible automorphic representation of weight $\lambda_y = \mathbf{w}(y) = \lambda_y^{\mathrm{alg}}\varepsilon_y$;*

satisfying the following:

(i) *The specialization of $\theta_{\mathfrak{V}}$ at y_0 is equal to θ_0.*
(ii) *For any $y \in \Sigma$, the specialization θ_y of $\theta_{\mathfrak{V}}$ at y is a character occurring in the representation of $R_{S,p}$ in $\pi^{I_m K^p}$ for all $\pi \in \Pi_y$.*
(iii) *For each $y \in \Sigma$, the specialization I_y of $I_{\mathfrak{V}}$ at y satisfies*

$$I_y(f) = \sum_{\sigma \in \Pi_y} m(\pi, \lambda) I_\pi(f)$$

for all $f \in \mathcal{H}_p(K^p)$*, where* $m(\pi,\lambda)$ *is the Euler-Poincaré characteristic of* π *in* $H^*(\widetilde{S}_G, \mathbb{V}_\lambda^\vee)$ *defined as*

$$m(\pi,\lambda) := \sum_i (-1)^i \dim_{\mathbb{C}} \mathrm{Hom}_{\mathcal{H}_p(K^p)}(\pi^{K^p I_m}, H^i_{\mathrm{cusp}}(S_G(K^p I_m), \mathbb{V}_\lambda^\vee)),$$

where $f \mapsto I_\pi(f)$ *is the trace distribution* $f \mapsto \mathrm{tr}(\pi(f))$ *defined on* $\mathcal{H}_p(K^p)$.

Moreover, the distribution character $I_{\mathfrak{Y}}$ *can be chosen such that* Π_y *is a singleton for a Zariski dense subset of* Σ.

In a recent preprint, Ash-Stevens have constructed[6] all we need to construct the total eigenvariety for any reductive group (see [AS08]). By total, we mean that the points occurring in that eigenvariety are in a ono-to-one correspondence with systems of Hecke eigenvalues of finite slope occurring in the cohomology with coefficients in a Banach space of p-adic distributions (and not only those having a nontrivial Euler-Poincaré characteristic). However, their result does not give any information about the dimension of the irreducible components. Our construction gives a complete description of the components of the eigenvariety having the same dimension as weight space. When the group is anisotropic, it is exactly the set of Hecke eigensystems having a nontrivial Euler-Poincaré characteristic. In general, we describe it as a union of cuspidal and various Eisenstein components. In Section 5.4 of the present paper, we present a conjecture for the dimension of the irreducible components passing through a given point of the eigenvariety in terms of the multiplicities of the corresponding system of Hecke eigenvalues in the various degrees of the cohomology. It is also worth noticing that these components do not always[7] contain a Zariski dense subset of classical points (i.e., attached to a classical automorphic representation). I hope this paper will convince the reader that the good framework to study p-adic automorphic representations and their families is the one of the derived category.

We now give a brief description of the content of the different sections. In Section 1, we introduce the basic notation, and we give a brief account of the cohomolgy of arithmetic groups with algebraic coefficients. Especially, we recall the important results of Borel-Wallach, Saper, Li-Schwermer and Franke. In Section 2, we introduce the formalism of the derived category of complexes of Banach spaces and the spectral theory of compact operators acting on them. In Section 3, we define the locally p-adic analytic induction

[6]They have actually constructed local pieces of the total eigenvariety. A global construction has been recently carried out by Z. Xiang in [Xia12].

[7]See the $\mathrm{GL}(3)_{/\mathbb{Q}}$-example of Ash-Pollack-Stevens [APS] or the $\mathrm{GL}(2)_{/K}$-case with K imaginary quadratic case considered by Calegari-Mazur [CM09].

spaces "à la Ash-Stevens-Hida" that will be used to define what we have called the "overconvergent cohomology." In Section 4, we study this cohomology and we prove that the trace of the compact operators u_t acting on it is an analytic function of the weight $\lambda \in \mathfrak{X}(\overline{\mathbb{Q}}_p)$. We also introduce the notion of effective finite slope character distribution, which is a linear functional on the Hecke algebras $\mathcal{H}'_p(K^p)$ that behaves like the trace of compact operator acting on a p-adic Banach space. In Section 5, we make an axiomatic construction of the eigenvariety associated to analytic families of effective finite slope distribution, and we apply it to the analytic families of finite slope distribution constructed in the previous chapter. In Section 6, we establish some p-adic trace formulae interpolating Arthur's and Franke's trace formulae.

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1. **Cohomology of arithmetic groups**

1.1. *Notation and conventions.*

1.1.1. *General notation.* We denote respectively by $\mathbb{Q}, \mathbb{R}$ and $\mathbb{C}$ the fields of rational, real and complex numbers. For any prime p, $\mathbb{Q}_p$ is the field of p-adic numbers, and we denote by $\mathbb{C}_p$ the p-adic closure of an algebraic closure $\overline{\mathbb{Q}}_p$ of $\mathbb{Q}_p$. Throughout this paper, we denote by $\overline{\mathbb{Q}}$ an algebraic closure of $\mathbb{Q}$ and we fix the embeddings ι_∞ and ι_p of $\overline{\mathbb{Q}}$ into $\mathbb{C}$ and $\mathbb{C}_p$ respectively. Any number field M will be considered as a subfield of $\overline{\mathbb{Q}}$.

We denote by $|\cdot|_p$ the non-archimedean norm of $\mathbb{C}_p$ normalized by $|p|_p = p^{-1}$. We denote by $\mathbb{A}$ the adèle ring over $\mathbb{Q}$, and we fix a decomposition such that $\mathbb{A} = \mathbb{A}_f \times \mathbb{R}$. If M is a number field, we will also denote $\mathbb{A}_M = \mathbb{A} \otimes M$ and, for each place v of M, we write M_v for the completion of M at v.

Sometimes we denote by $|X|$ or by $\#X$ the cardinal of a finite set X.

[8]Centre de Recerca Matematica (Bellaterra, Spain).

For any algebraic group H defined over $\mathbb{Q}$, we set $H_f = H(\mathbb{A}_f)$ and $H_\infty = H(\mathbb{R})$. We denote by H_∞^+ the connected component of H_∞ containing the identity. Whenever we put the superscript $+$ to a given group, it will mean that we consider the subgroup of the elements whose infinity part belongs to the connected component of this infinity part.

1.1.2. *Locally symmetric spaces and reductive groups.* We let G be a connected reductive group over $\mathbb{Q}$. We denote by $Z = Z_G$ the center of G. We let P_0 be a minimal parabolic subgroup defined over $\mathbb{Q}$ and a Levi decomposition $P_0 = M_0.N_0$, and we denote by $\mathcal{P}_G$ the set of standard parabolic subgroup P of G defined over $\mathbb{Q}$ (i.e., those containing P_0). We also denote $\mathcal{L}_G$ as the set of standard Levi subgroup (i.e., containing M_0). For any Levi subgroup M, we will denote by $\mathcal{W}_M^0$ its (rational) Weyl group; i.e., $\mathcal{W}_M^0 := N_M(M_0)/M_0$. (Here $N_M(M_0)$ denotes the normalizator of M_0 in M.)

We let K_∞ be a maximal compact subgroup of $G_\infty := G(\mathbb{R})$ and put $C_\infty^G = K_\infty.Z_\infty$. We denote by G_∞^+ the identity component of G_∞. We also write $G_\infty^1 \subset G_\infty^+$ for the kernel of the map from G_∞ onto the connected component of the $\mathbb{R}$-split part of the co-center $G_\infty/G_\infty^{\mathrm{der}}$ of G_∞.

We let $K_m = K_{\max} \subset G(\mathbb{A}_f)$ be a maximal compact subgroup and dg a Haar measure of G_f such that

$$\int_{K_{\max}} dg = 1.$$

If $K \subset G_f$ is a measurable set, we write

$$\mathrm{Meas}(K) = \mathrm{Meas}(K, dg) := \int_K dg.$$

For any open compact subgroup $K \subset G_f$, we consider the locally symmetric space

$$S_G(K) := G(\mathbb{Q})\backslash G(\mathbb{A})/K.C_\infty^G.$$

Let $\mathcal{H}_G = G_\infty^+/C_\infty^G \cap G_\infty^+$. Throughout this paper, we assume that the strong approximation theorem applies to G^{der}. In particular, for any open compact subgroup $K \subset G_f$), we have a finite decomposition

$$G(\mathbb{A}) = \bigsqcup_i G(\mathbb{Q}) \times G_\infty^+ \times g_i.K. \tag{1}$$

For any $x \in G(\mathbb{A}_f)$ and any open subgroup $K \subset G(\mathbb{A}_f)$, we denote by $\Gamma(x, K)$ the image of $xKx^{-1} \cap G(\mathbb{Q})^+$ in $G^{\mathrm{ad}}(\mathbb{Q}) = G(\mathbb{Q})/Z_G(\mathbb{Q})$. Then we have

$$S_G(K) \cong \bigsqcup \Gamma_i\backslash\mathcal{H}_G,$$

with $\Gamma_i := \Gamma(g_i, K)$. We say that K is neat if none of the $\Gamma_i := \Gamma(g_i, K)$ contains finite order elements. In that case, $S_G(K)$ is a smooth real analytic variety, and for each connected component, the universal covering morphism is étale.

1.2. *Local systems and cohomology.* We will be interested in the cohomology of local systems on $S_G(K)$. They are defined by representations of the fundamental group of each connected component. Below, we recall different equivalent constructions of these local systems.

1.2.1. *First definition.* If M is a $\mathbb{Q}$-vector space equipped with an action of $G(\mathbb{Q})$, we denote by $\widetilde{M}$ the local system defined as the sheaf of locally constant sections of the cover

$$\widetilde{M} := G(\mathbb{Q})\backslash(G(\mathbb{A}) \times M)/K.C_\infty \to G(\mathbb{Q})\backslash G(\mathbb{A})/K.C_\infty = S_G(K),$$

with left action of $G(\mathbb{Q})$ and right action of $K.C_\infty$ defined by the formula

$$\gamma.(g, m).k := (\gamma g k, \gamma.m)$$

for any $\gamma \in G(\mathbb{Q})$, $g \in G(\mathbb{A})$, $k \in K.C_\infty$ and $m \in M$.

Let $Z_K := Z_G(\mathbb{Q}) \cap K$. Then a necessary condition for $\widetilde{M}$ to be nontrivial is that

$$Z_K \text{ acts trivially on } M. \tag{2}$$

When this condition is satisfied, the subgroups $\Gamma(x, K)$ act on M. Then for each irreducible component $\Gamma_i\backslash\mathcal{H}$, the local system is defined by

$$\widetilde{M}_i := \Gamma_i\backslash(\mathcal{H}_G \times M) \to \Gamma_i\backslash\mathcal{H}_G,$$

and the isomorphism $\widetilde{M} \cong \sqcup_i \widetilde{M}_i$ is induced by the maps $(\gamma.g_i.g_\infty.k, m) \mapsto (g_\infty, \gamma^{-1}.m)$.

1.2.2. *Second definition.* If M is a left K-module, we can define also the local system

$$\widetilde{M}_K := G(\mathbb{Q})\backslash G(\mathbb{A}) \times M/K.C_\infty \to G(\mathbb{Q})\backslash G(\mathbb{A})/K.C_\infty = S_G(K),$$

with left action of $G(\mathbb{Q})$ and right action of $K.C_\infty$ given by the formula

$$\gamma.[g, m].k := [\gamma g k, k^{-1}.m].$$

When the action of K or $G(\mathbb{Q})$ on M in the definitions (1.2.1) and (1.2.2) above extends in a compatible way to an action of $G(\mathbb{A})$, then the two defined local systems $\widetilde{M}$ and $\widetilde{M}_K$ coincide by the isomorphism $(g, m) \mapsto [g, g^{-1}.m]$. Again the sheaf $\widetilde{M}$ is nontrivial only if condition (2) is satisfied.

1.2.3. Let v be a place of $\mathbb{Q}$ such that the action of K factorizes through the projection $K \mapsto K_v$, with K_v the image of K into $G(\mathbb{Q}_v)$. Assume that the image of g_i in $G(\mathbb{Q}_v)$ be trivial. Then we can describe the local system $\widetilde{M}$ on each irreducible component by

$$\widetilde{M}_i := \widetilde{M}|_{\Gamma_i\backslash\mathcal{H}_G} \cong \Gamma_i\backslash(\mathcal{H}_G \times M) \to \Gamma_i\backslash\mathcal{H}_G$$

Here the left action of Γ_i is defined by $\gamma.[z,m] = [\gamma.z, \gamma.m]$ for all $z \in \mathcal{H}_G$, $m \in M$ and $\gamma \in \Gamma_i$, and the isomorphism between $\widetilde{M}_i$ and $\Gamma_i \backslash (\mathcal{H}_G \times M)$ is defined by

$$[\gamma.g_i.g_\infty.k, m] \mapsto [\gamma.g_\infty, k.m].$$

1.2.4. *Cohomology.* We write $H^\bullet(S_G(K), \mathcal{F})$ and $H_c^\bullet(S_G(K), \mathcal{F})$ respectively for the cohomology and cohomology with compact support of a local system $\mathcal{F}$ on $S_G(K)$. We also denote by $H_!^\bullet(S_G(K), \mathcal{F})$ the image of the canonical map from $H_c^\bullet(S_G(K), \mathcal{F})$ into $H^\bullet(S_G(K), \mathcal{F})$. When $\mathcal{F} = \widetilde{M}$, we will sometimes drop the symbol $\sim$ from the notation; thus we write $H^\bullet(S_G(K), M)$. When $\widetilde{M}$ is defined by an action of $G(\mathbb{Q})$, we also write $H_?^\bullet(\widetilde{S}_G, M)$ for the projective limit of the $H_?^\bullet(S_G(K), M)$ over all the open compact subgroups $K \subset G(\mathbb{A}_f)$.

We also recall that the decomposition into connected components induces a canonical isomorphism

$$H^\bullet(S_G(K), M) = \oplus_i H^\bullet(\Gamma_i \backslash \mathcal{H}_G, \widetilde{M}_i) \cong \oplus_i H^\bullet(\Gamma_i, M). \tag{3}$$

1.2.5. *Hecke operators.* We now briefly recall some equivalent definitions of the action of the Hecke operators on the cohomology.

Assume that $x \in G(\mathbb{A}_f)$. Then the right translation by x defines an isomorphism $R_x : S_G(K) \to S_G(x^{-1}Kx)$, and we expect that it induces a map

$$H^\bullet(S_G(x^{-1}Kx), M) \to H^\bullet(S_G(K), M). \tag{4}$$

This is indeed so if $\widetilde{M}$ is defined by a representation of $G(\mathbb{Q})$ (i.e., as in definition (1.2.1)). In that case, the definition above induces a left action of $G(\mathbb{A}_f)$ on $H^\bullet(S_G, M)$ such that there is a canonical isomorphism

$$H^\bullet(\widetilde{S}_G, M)^K \cong H^\bullet(S_G(K), M),$$

where the left-hand side of the above equation denotes the K-invariants of $H^\bullet(S_G, M)$. If M is a $\mathbb{Q}$-vector space, we can therefore define an action of the Hecke algebra $C_c^\infty(G(\mathbb{A}_f), \mathbb{Q})$ on $H^\bullet(S_G, M)$ from our choice of an Haar measure on $G(\mathbb{A}_f)$.

When M is not a representation of $G(\mathbb{Q})$ but is a left K-module, we need to make some additional assumptions on M. Let $\Delta_f \subset G(\mathbb{A}_f)$ be a semi-group containing K and let us assume further that M is equipped with a Δ_f-left action extending the action of K. Then for any $x \in \Delta_f$, the map $m \mapsto x.m$ induces canonically a map $R_x^* \widetilde{M}_{x^{-1}Kx} \to \widetilde{M}_K$. We therefore have a map in cohomology

$$H^\bullet(S_G(x^{-1}Kx), M) \to H^\bullet(S_G(K), M). \tag{5}$$

In each case, this enables us to define a left action of the double cosets KxK. Moreover, these actions coincide when the definitions (1.2.1) and (1.2.2) are compatible as explained at the end of Section 1.2.2. We can also compare this action with the definition (1.2.3). We write $g_i x = \gamma_{x,i} g_{j_i} h g_\infty$ with $h \in K$. To the double class $\Gamma_i \gamma_{x,i}^{-1} \Gamma_{j_i}$, we can associate the map $H^\bullet(\Gamma_i \backslash \mathcal{H}_G, M) \to H^\bullet(\Gamma_{j_i} \backslash \mathcal{H}_G, M)$. In view of the isomorphism (3), the action of the double coset KxK can equivalently be seen as

$$[KxK] = \oplus_i [\Gamma_i \gamma_{x,i}^{-1} \Gamma_{j_i}]. \tag{6}$$

1.3. *Cohomology and automorphic forms.* In this section, we review some of the important known results establishing a link between the cohomology of arithmetic groups and automorphic forms. The study of this relationship has been originally undertaken by A. Borel and N. Wallach [BW00] and developed by many authors including G. Harder, J. Schwermer, J.-S. Li, and the theory has culminated with the results of J. Franke [Fra98] who proved Borel's conjecture.

1.3.1. *Algebraic representations of G.* We recall here the definition and construction of some irreducible and algebraic representations of G. Let $F \subset \overline{\mathbb{Q}}$ be the smallest splitting field for G. We let $(B_{/F}, T_{/F})$ be a Borel pair contained in the pair $(P_{0/F}, M_{0/F}$. For all field L containing F, we denote respectively by $N_{/L}$, $B^-_{/L}$ and $N^-_{/L}$ the unipotent radical of $B_{/L}$, the Borel subgroup opposite to $B_{/L}$ and the unipotent radical of $B^-_{/L}$. For any algebraic dominant weight λ of $G_{/L}$ with respect to the Borel pair $(B_{/L}, T_{/L})$, we denote by $\mathbb{V}^G_\lambda(L)$ the irreducible algebraic representation over L of $G_{/L}$ of highest weight λ defined as the set of algebraic functions $G_{/L} \mapsto \mathbb{A}^1_{/L}$, such that

$$f(n^- tg) = \lambda(t) f(g)$$

for all $n^- \in N^-(L)$, $t \in T(L)$ and $g \in G(L)$. When G is clear from the context, we usually drop it from the notation and write $\mathbb{V}_\lambda(L)$.

We are interested in the cohomology of $\mathbb{V}^\vee_\lambda(\mathbb{C})$ because it can be interpreted in terms of automorphic forms. We will recall below the main results about this fact.

1.3.2. *$(\mathfrak{g}, K_\infty)$-cohomology.* Let $\mathfrak{g} := \mathrm{Lie}_\mathbb{R}(G^1_\infty)$ and $\mathfrak{k} := \mathrm{Lie}_\mathbb{R}(K_\infty)$. For any $(\mathfrak{g}, K_\infty)$-module H, one denotes its $(\mathfrak{g}, K_\infty)$-cohomology by $H^\bullet(\mathfrak{g}, K_\infty; H)$. The reader can consult [BW00] for the definitions and basic properties of this notion. Let ω_λ be the character of Z_∞ acting on $\mathbb{V}_\lambda(\mathbb{C})$. We denote by $C^\infty(G(\mathbb{Q}) \backslash G(\mathbb{A})/K, \omega_\lambda)$ the space of C^∞-functions ϕ on $G(\mathbb{Q}) \backslash G(\mathbb{A})/K$ such that $\phi(gz) = \omega_\lambda(z)\phi(g)$ for all $g \in G(\mathbb{A})$ and $z \in Z_\infty$. This is a $(\mathfrak{g}, K_\infty)$-module. By noticing that the tangent space at the origin of $S_G(K)$ is canonicaly isomorphic to $\mathfrak{g}/\mathfrak{k}$ and that multiplication by $g \in G(\mathbb{A})$ gives an isomorphism between the tangent space at the origine and the one at the class of g, it is not difficult and classical to obtain an isomorphism

$$\begin{aligned} H^\bullet(\mathfrak{g}, K_\infty; C^\infty(G(\mathbb{Q})\backslash G(\mathbb{A})/K, \omega_\lambda) \otimes \mathbb{V}_\lambda^\vee(\mathbb{C})) &\overset{\delta}{\cong} H^\bullet_{\mathrm{dR}}(S_G(K), \mathbb{V}_\lambda^\vee(\mathbb{C})) \\ &\cong H^\bullet(S_G(K), \mathbb{V}_\lambda^\vee(\mathbb{C})). \end{aligned}$$

1.3.3. *Cuspidal and L^2-cohomology.* Let $L^2(G(\mathbb{Q})\backslash G(\mathbb{A}), \omega_\lambda)$ be the space of square integrable functions on $G(\mathbb{Q})\backslash G(\mathbb{A})$ which tranform by ω_λ under the action of Z_∞. There is a natural action of $G(\mathbb{A})$ on these spaces. Moreover, we have a decomposition

$$L^2(G(\mathbb{Q})\backslash G(\mathbb{A}), \omega_\lambda) = L^2_d(G(\mathbb{Q})\backslash G(\mathbb{A}), \omega_\lambda) \oplus L^2_{\mathrm{cont}}(G(\mathbb{Q})\backslash G(\mathbb{A}), \omega_\lambda),$$

where L^2_{cont} and L^2_{d} denote respectively the continuous and discrete spectrum of L^2 for the action of $G(\mathbb{A})$. We also denote by $L^2_{\mathrm{cusp}}(G(\mathbb{Q})\backslash G(\mathbb{A}), \omega_\lambda)$the subspace of automorphic functions $\phi \in L^2(G(\mathbb{Q})\backslash G(\mathbb{A}), \omega_\lambda)$ satisfying

$$\int_{N_P(\mathbb{A}/N_P(\mathbb{Q}))} \phi(ng)dn = 0$$

for the unipotent radical N_P of any $\mathbb{Q}$-rational parabolic subgroup P of G. We have $L^2_{\mathrm{cusp}} \subset L^2_d$. Notice also that there is a spectral decomposition:

$$L^2_d(G(\mathbb{Q})\backslash G(\mathbb{A}), \omega_\lambda) = \bigoplus_\pi V_\pi^{m(\pi)},$$

where (π, V_π) runs in a set of irreducible representation with central character at infinity ω_λ. Similarly, we have

$$L^2_{\mathrm{cusp}}(G(\mathbb{Q})\backslash G(\mathbb{A}), \omega_\lambda), = \bigoplus_\pi V_\pi^{m_{\mathrm{cusp}}(\pi)}$$

where now the representations π are irreducible and cuspidal. In the above decomposition, $m(\pi)$ and $m_{\mathrm{cusp}}(\pi)$ are nonnegative integer denoting respectively the multiplicities of π in L^2_{d} and L^2_{cusp}.

For an irreducible representation π as above, let V_π^{fin} be the subspace of vectors that generate a finite dimensional vector space under the action of K_∞. It can be shown that, under the decomposition above, V_π^{fin} is contained in $C^\infty(G(\mathbb{Q})/G(\mathbb{A}), \omega_\lambda)$. (In fact, more generally, if π_∞ is an admissible representation of G_∞, then $\pi_\infty^{\mathrm{fin}}$ is a $(\mathfrak{g}, K_\infty)$-module.) One defines the cuspidal cohomology as

$$H^\bullet_{\mathrm{cusp}}(S_G(K), \mathbb{V}_\lambda^\vee(\mathbb{C})) := \bigoplus_{\pi \subset L^2_{\mathrm{cusp}}} H^\bullet(\mathfrak{g}, K_\infty; (V_\pi^{\mathrm{fin}})^K \otimes \mathbb{V}_\lambda^\vee(\mathbb{C}))$$

and the square integrable cohomology as

$$H^\bullet_2(S_G(K), \mathbb{V}_\lambda^\vee(\mathbb{C})) := \bigoplus_{\pi \subset L^2_d} H^\bullet(\mathfrak{g}, K_\infty; (V_\pi^{\mathrm{fin}})^K \otimes \mathbb{V}_\lambda^\vee(\mathbb{C})).$$

By a theorem of A. Borel, both inject via δ in the cohomology of $S_G(K)$. Thus we have the following inclusions:

$$H^\bullet_{\mathrm{cusp}}(S_G(K),\mathbb{V}_\lambda^\vee(\mathbb{C}))\subset H^\bullet_!(S_G(K),\mathbb{V}_\lambda^\vee(\mathbb{C}))\subset H^\bullet_2(S_G(K),\mathbb{V}_\lambda^\vee(\mathbb{C})).$$

For $? = \emptyset, !, \mathrm{cusp}$ or 2, we put

$$H^\bullet_?(\widetilde{S}_G,\mathbb{V}_\lambda^\vee(\mathbb{C})) := \varinjlim_{K\subset G_f} H^\bullet_?(S_G(K),\mathbb{V}_\lambda^\vee(\mathbb{C})).$$

It has a natural action of G_f. Any irreducible representation π has a decomposition $\pi=\pi_f\otimes\pi_\infty$, where π_f and π_∞ are respectively representations of G_f and G_∞. We now recall which π_∞ intervene in the cohomology.

1.3.4. *L-packet at infinity.* For each dominant weight λ, one defines the "cohomological " packet Π_λ as the set of essentially unitary representation π_∞ of $G(\mathbb{R})$ of central character ω_λ and such that

$$H^\bullet(\mathfrak{g},K_\infty;\pi_\infty^{\mathrm{fin}}\otimes\mathbb{V}_\lambda^\vee(\mathbb{C}))\neq 0.$$

A cuspidal representation π is said to be cohomological if its archimedean component π_∞ belongs to Π_λ for some weight λ (that we call its cohomological weight).

1.3.5. *The Harish-Chandra condition.* We say that G_∞ satisfies the Harish-Chandra condition if the compact rank of G_∞^{der} equals its semi-simple rank. In other words, G_∞^{der} contains a compact Cartan subgroup. In that case, it is known by the foundational work of Harish-Chandra that G_∞ has discrete series representations (i.e., with square integrable matrix coefficients). In that case, the dimension of $S_G(K)$ is even, and we write d_G for half its real dimension.

Assume that G_∞ satisfies the Harish-Chandra condition. Then, if λ is dominant and regular, it is known that

(VZ): The representations in Π_λ are in the discrete series (i.e., can be realized in $L^2(G_\infty,\omega_\lambda)$). Moreover, $\pi_\infty\in\Pi_\lambda$ has cohomology only in degree d_G, and it has dimension 1.

(HC): The set Π_λ is in bijection with $\{w.(\lambda+\rho)-\rho\,;w\in\mathcal{W}_{G_\infty}/\mathcal{W}_{K_\infty}\}$.

(VZ) is due to Vogan and Zuckerman and (HC) is due to Harish-Chandra. Even if G_∞ does not satisfies the Harish-Chandra condition, the representations in Π_λ are known to be tempered[9] when λ is regular. By a theorem of N. Wallach, it is known that if π is such that π_∞ is tempered, then the respective multiplicities $m_{\mathrm{cusp}}(\pi)$ and $m(\pi)$ of π in L^2_{cusp} and in L^2_d are equal. An immediate consequence of these facts is the following (well-known) proposition.

[9]This means that the matrix coefficients of this representations belong to $L^{2+\varepsilon}(G_\infty/Z_\infty)$ for all $\varepsilon>0$.

Proposition 1.3.6. *Assume that G_∞ satisfies the Harish-Chandra condition and that λ is regular*; *then*

(i) *We have*

$$H^q_{\mathrm{cusp}}(\widetilde{S}_G, \mathbb{V}_\lambda^\vee(\mathbb{C})) = H^q_{(2)}(\widetilde{S}_G, \mathbb{V}_\lambda^\vee(\mathbb{C})),$$

and these groups vanish except for $q = d_G$.

(ii) *For $q = d_G$, we have an isomorphism of G_f*

$$H^{d_G}_{\mathrm{cusp}}(\widetilde{S}_G, W_\lambda^\vee(\mathbb{C})) \cong \bigoplus_\pi H^{d_G}(\mathfrak{g}, K_\infty; \pi_\infty^{\mathrm{fin}} \otimes \mathbb{V}_\lambda) \otimes \pi_f^{\oplus m(\pi)} \cong \bigoplus_\pi \pi_f^{\oplus m(\pi)}.$$

Here $\pi = \pi_f \otimes \pi_\infty$ runs in the set of cuspidal representations such that $\pi_\infty \in \Pi_\lambda$.

1.4. *Franke's trace formula.* When the group G is not anisotropic, the cohomology of $S_G(K)$ is not reduced to its cuspidal or square integral part. In the case of GL(2), G. Harder has proved that it can be decomposed into its cuspidal part and Eisenstein part [Har87]. This result has been generalized by him [Har93] and J. Schwermer [Sch83] in many other cases including the rank one case and when λ is regular. The question has been settled in fairly good generality thanks to the work of J. Franke [Fra98] and works built on it. See, for example, the results of J.-S. Li and J. Schwermer [LS04]. It also results from the works of J. Franke that there is a trace formula for the cohomology $H^\bullet(\widetilde{S}_G, \mathbb{V}_\lambda^\vee(\mathbb{C}))$ in terms of trace of the square integral cohomology of the Levi subgroup of G. This formula becomes quite simple when the weight λ is regular.

1.4.1. *Eisenstein classes.* Recall that $\mathcal{P}_G$ and $\mathcal{L}_G$ denote respectively the set of the standard parabolic subgroups of G and the standard Levi subgroups of G. We also write $\mathcal{L}_G^c$ for the subset of $\mathcal{L}_G$ of Levi's satisfying the Harish-Chandra condition. For $P \in \mathcal{P}_G$ with Levi decomposition $P = MN$, we denote by ρ_P the modulus function associated to P. Recall that it is defined by

$$\rho_P(m) := \det(m; \mathfrak{n})^{1/2}$$

with $\mathfrak{n} := \operatorname{Lie} N$ for $m \in M$ acting on $\mathfrak{n}$ via the adjoint representation. Let

$$\mathcal{W}^M := \{w \in \mathcal{W}_G \mid w^{-1}(\alpha) \in R^+, \ \forall \alpha \in R^+ \cap R_M\}.$$

This is the set of representatives of the cosets $\mathcal{W}_G/\mathcal{W}_M$ of minimal length. Here R^+ is the set of positive roots with respect to $(B_{/L}, T_{/L})$, and R_M is the set of root for the pair $(M_{/L}, T_{/L})$. Then we have the so-called Kostant decomposition

$$H^q(\mathfrak{n}, \mathbb{V}_\lambda(\mathbb{C})) := \bigoplus_{\substack{w \in \mathcal{W}^M \\ l(w) = q}} \mathbb{V}^M_{w(\lambda+\rho_P)-\rho_P}(\mathbb{C}),$$

where $\mathbb{V}_\mu^M$ denotes the irreducible algebraic representation of M of highest weight μ. From the definition of $\mathcal{W}^M$, $w(\lambda + \rho_P) - \rho_P$ is a dominant weight with respect to the pair $(B_{/L} \cap M_{/L}, T_{/L})$. For further use, let us remind the reader that we have the relation

$$w * \lambda := w(\lambda + \rho_B) - \rho_B = w(\lambda + \rho_P) - \rho_P \tag{7}$$

for all $w \in \mathcal{W}^M$. Since $H^{\dim \mathfrak{n}}(\mathfrak{n}, \mathbb{C}) = \mathbb{C}(-2\rho_P)$, by Poincaré duality the Kostant decomposition gives

$$H^q(\mathfrak{n}, \mathbb{V}_\lambda^\vee(\mathbb{C})) := \bigoplus_{\substack{w \in \mathcal{W}^M \\ l(w) = \dim \mathfrak{n} - q}} \mathbb{V}^M_{w(\lambda+\rho_P)+\rho_P}(\mathbb{C})^\vee.$$

Let $R_P \subset X^*(Z_M)$ be the set of roots for the pair $(Z_M, \mathfrak{n})$, and denote by $R_P^\vee \subset X_*(Z_M)$ the corresponding set of coroots. Then we put

$$\mathcal{W}^M_{\mathrm{Eis}} := \{w \in \mathcal{W}^M, w^{-1}(\beta^\vee) \in R^+, \forall \beta \in R_P\}.$$

From the definition, we see that if λ is regular, then $(w * \lambda)|_{Z_M \cap G^{\mathrm{der}}}$ is R_P-dominant if and only if $w \in \mathcal{W}^M_{\mathrm{Eis}}$. From results of Harder, Schwermer-Li and Franke, it is known that for $w \in \mathcal{W}^M_{\mathrm{Eis}}$ and for regular λ, the Eisenstein series associated to a cohomology class in $H^\bullet(S_M(K_M), \mathbb{V}^M_{w*\lambda}(\mathbb{C})^\vee)$ is in the domain of convergence; therefore it defines an Eisenstein class in $H^\bullet(S_G(K), \mathbb{V}^\vee_\lambda(\mathbb{C}))$. When the weight λ is regular, these authors have furthermore proved that the cohomolgy can be expressed in terms of cuspidal and Eisenstein classes. One formulation of this fact is stated in the next theorem.

For any function $f \in C_c^\infty(G(\mathbb{A}_f))$ and any $M \in \mathcal{L}_G$, we denote its constant term $f_M \in C_c^\infty(M(\mathbb{A}_f))$. Recall that it is defined for any $m \in M(\mathbb{A}_f)$ by

$$f_M(m) := \int_{K_m \times N(\mathbb{A}_f)} f(k^{-1} m n k) dn dk,$$

where $P = MN$ is the Levi decomposition of the unique standard parabolic subgroup of G of Levi subgroup M. This definition is relevant in view of the following (standard) formula:

$$\mathrm{tr}\Big(f : \mathrm{Ind}_{P(\mathbb{A}_f)}^{G(\mathbb{A}_f)} \sigma\Big) = \mathrm{tr}(f_M : \sigma). \tag{8}$$

The induced representation[10] here is the set of smooth functions on $G(\mathbb{A}_f)$ such that $\phi(pg) = \sigma(p).\phi(g)$ for all $p \in P(\mathbb{A}_f)$ and $g \in G(\mathbb{A}_f)$.

[10]Notice here that we *do not* consider the unitary induction. The main reason is that the nonunitary induction preserves rationality and integrality of the Hecke eigenvalues.

THEOREM 1.4.2 (Franke). *Assume that λ is regular and $f \in C_c^\infty(G(\mathbb{A}_f))$; then we have*

$$\operatorname{tr}(f : H^\bullet(\widetilde{S}_G, \mathbb{V}_\lambda^\vee(\mathbb{C}))) = \sum_{M \in \mathcal{L}_G} (-1)^{\dim \mathfrak{n}_M} \cdot \sum_{w \in \mathcal{W}^M_{\mathrm{Eis}}} (-1)^{l(w)} \operatorname{tr}(f_M : H^\bullet_{\mathrm{cusp}}(\widetilde{S}_M, \mathbb{V}^M_{w(\lambda+\rho_P)+\rho_P}(\mathbb{C}))^\vee).$$

Proof. This follows from formula (2) on page 266 of the paper of Franke [Fra98]. □

2. Spectral theory on p-adic Banach spaces

The aim of this section is to set up some basic facts about the spectral theory for the derived category of Banach modules over a Banach algebra. We define what we call the finite slope cohomology attached to complexes of p-adic Banach spaces equipped with completely continuous (or compact) operators. Then we extend it to p-adic Fréchet spaces. We do not pretend any originality in this section, but it gives a convenient frame for the theory developed in the fourth section.

2.1. *Perfect complexes of Banach spaces.* We introduce the notion of perfect complexes of Banach spaces that will be suitable to our spectral theory. It arises naturally as the theory of perfect complexes in the Grothendieck theory of Lefschetz trace formula in étale cohomology.

2.1.1. In this section, A will be a topologically finitely generated Banach $\mathbb{Q}_p$-algebra. Recall that it is a $\mathbb{Q}_p$ algebra equipped with an ultrametric norm $|.|_A$ satisfying

$$|ab|_A \leq |a|_A \cdot |b|_A \text{ for all } a, b \in A$$

and that we can normalize such that $|x|_A = |x|_p$ for all $x \in \mathbb{Q}_p$. We will assume $(A, |.|)$ satisfies the Hypothesis **M** of [Col97, A1]. We write $A^0 := \{a \in A \mid |a| \leq 1\}$ and $m := \{a \in A \mid |a| < 1\} \subset A^0$. Then we furthermore assume that A is semi-simple in the sense of [Col97, A1], which means A^0/m is a field and that the norm on it induced by that of A is multiplicative. We again refer to the reference [Col97] for the notions and the basic properties of Banach A-modules. We call a Fréchet A-module a topological $\mathbb{Q}_p$-vector space V equipped with a continuous A-module structure, which is topologically isomorphic to a projective limit of Banach A-modules. A Fréchet A-module will be said to be compact if the transition maps of the projective limit are completely continuous.

2.1.2. *The categories $\mathcal{B}an_A$ and* Fre_A. For any $\mathbb{Q}_p$-Banach algebra A, we consider the category $\mathcal{B}an_A$ (respectively Fre_A) whose objects are the Banach (respectively Fréchet) A-modules, and the homomorphisms are the continuous A-linear maps. These categories are obviously additive but not abelian. However, they are exact in the sense of Quillen, the short exact sequences being those exact in the category of A-modules. Moreover, $\mathcal{B}an_A$ satisfies[11] the axiom 1.3.0 of [Lau83]. Every morphism $f \in \mathrm{Hom}(E, F)$ has a kernel and a coimage (and therefore an image and a cokernel). One can check easily that

$$\mathrm{Ker}\, f = f^{-1}(0),\ \mathrm{Im}\, f = \overline{f(E)},\ \mathrm{Coker}\, f = F/\overline{f(E)} \text{ and } \mathrm{Coim}\, f = E/f^{-1}(0).$$

Recall that a morphism f is said to be strict if and only if it induces an isomorphism from $\mathrm{Coim}\, f$ onto $\mathrm{Im}\, f$. This means here that $f(E)$ is closed in F. Equivalently, an epimorphim is strict if it is surjective as a morphism of A-module, and a monomorphism is strict if its image is closed.

2.1.3. Let $\mathcal{C}(\mathcal{B}an_A)$ be the category of complexes of Banach A-modules in which the homomorphisms are the maps of degree 0 between complexes. We denote by $\mathcal{K}(\mathcal{B}an_A)$ the triangulated category of complexes of objects of $\mathcal{B}an_A$ "modulo homotopy." We say that a complex is acyclic if it is exact at every degree and denote by $\mathcal{K}^{\emptyset}(\mathcal{B}an_A)$ the full subcategory of acyclic complexes. This category is "épaisse," and, following Deligne, it is therefore possible to consider the derived category $D(\mathcal{B}an_A)$ as the quotient category $\mathcal{K}(\mathcal{B}an_A)/\mathcal{K}^{\emptyset}(\mathcal{B}an_A)$. The objects of $D(\mathcal{B}an_A)$ are the same as those of $\mathcal{K}(\mathcal{B}an_A)$. Recall that if $M^\bullet, N^\bullet \in \mathrm{Ob}(D(\mathcal{B}an_A))$, then a homomorphism from $M^\bullet$ to $N^\bullet$ is a triple $(P^\bullet, s, f)$, where $s \in \mathrm{Hom}_{\mathcal{K}(\mathcal{B}an_A)}(P^\bullet, N^\bullet)$ and $f \in \mathrm{Hom}_{\mathcal{K}(\mathcal{B}an_A)}(P^\bullet, M^\bullet)$ such that s is a quasi-isomorphism. The latter means that the cone of s is acyclic.

Recall the following definition.

Definition 2.1.4. Let A be a Banach p-adic algebra over a p-adic field L and M a Banach A-module. One says that M is an orthonormalizable A-module or is orthonormalizable over A if there exists a countable family $B = (m_i)_{i\in I} \in M^I$ of elements of norm ≤ 1 such that for any $m \in M$ there is a unique family $(a_i)_{i\in I} \in A^I$ satisfying $\lim_i a_i = 0$ and $m = \sum_{i\in I} a_i.m_i$. We call B an orthonormal basis of M.

For any Banach space N over L, we denote by N° the lattice of elements of norm ≤ 1. We recall the following lemma.

LEMMA 2.1.5. *Let M be a Banach A-module and $a \in A$ such that $|a| < 1$. If there exists a subset $B \subset M$ such that the image of B in $M^\circ/a.M^\circ$ is a*

[11]The category $\mathcal{B}an_A$ is quasi-abelian in the sense of [Sch98]. This means that the *pull-back* (resp. push-forward) of any strict epimorphism (resp. any strict monomorphism) is still a strict epimorphism (resp. strict monomorphism). This fact can be easily verified as in [Sch98, Prop. 3.2.4].

basis of this module over $A^\circ/a.A^\circ$, then B is an orthonormal basis of N. In particular any Banach space over L is orthonormalizable on L.

2.1.6. *Projective Banach A-modules.* An object P of $\mathcal{B}an_A$ is said to be projective if, for any strict epimorphism $f : N \to M$ and any map $g : P \to M$, there exists a map h from P to N so that the following diagram is commutative:

$$\begin{tikzcd} & P \arrow[dl, dashed, "h"'] \arrow[d, "g"] \\ N \arrow[r, two heads, "f"'] & M. \end{tikzcd}$$

A complex $M^\bullet$ is called perfect if it is bounded and if M^q is projective for all q. We denote $\mathcal{K}_{\mathrm{pf}}(\mathcal{B}an_A)$ as the full subcategory of $\mathcal{K}(\mathcal{B}an_A)$ of perfect complexes.

LEMMA 2.1.7. *A Banach A-module is projective* (*in the category $\mathcal{B}an_A$*) *if and only it is a direct factor of an orthonormalizable A-module.*

Proof. Let us prove that an orthonormalizable P is projective. Let N, M, f and g be as above. Since f is strict, it induces an isomorphism of $\mathbb{Q}_p$-Banach spaces from $N/\operatorname{Ker} f$ onto M. Then we choose a $\mathbb{Q}_p$-direct factor M' of $\operatorname{Ker} f$ inside N so that f induces an isomorphism of $\mathbb{Q}_p$-Banach spaces from M' onto M. If $(e_i)_i$ is an A-basis of P, let $h_i \in M'$ for each i be such that $f(h_i) = g(e_i)$ for all i. Since g is continuous and $f|_{M'}$ is an isomorphism of $\mathbb{Q}_p$-Banach spaces, we know that the family $(h_i)_i$ is bounded. Therefore there exists a continuous A-linear map from P to N such that $h(e_i) = h_i$ for all i and $f \circ h = g$. What remains to be proved is formal and left to the reader. $\square$

LEMMA 2.1.8. *Let $M^\bullet$ and $N^\bullet$ be two complexes of Banach A-modules. Assume that $M^\bullet$ is perfect*; *then the following canonical morphism is an isomorphism*

$$\operatorname{Hom}_{\mathcal{K}(\mathcal{B}an_A)}(M^\bullet, N^\bullet) \to \operatorname{Hom}_{D(\mathcal{B}an_A)}(M^\bullet, N^\bullet).$$

Proof. This is a special case of [Lau83, Cor. 2.2.3] in which the space X is reduced to one point, the fibred category C over X is $\mathcal{B}an_A$ and C_0 is the full subcategory whose objects are the projective objects of $\mathcal{B}an_A$. $\square$

2.1.9. *Category of perfect complexes.* By $D_{\mathrm{pf}}(\mathcal{B}an_A)$ we denote the image of $\mathcal{K}_{\mathrm{pf}}(\mathcal{B}an_A)$ in $D(\mathcal{B}an_A)$. From the lemma above, it follows that $D_{\mathrm{pf}}(\mathcal{B}an_A)$ is a full subcategory of $D(\mathcal{B}an_A)$. If $A = L$ is a finite extension of $\mathbb{Q}_p$, every Banach space over L is orthonormalizable. Therefore, $D_{\mathrm{pf}}(\mathcal{B}an_L) = D(\mathcal{B}an_L)$.

2.2. *Compact operators.*

2.2.1. *Fredholm determinant.* Let M be an orthonormalizable Banach A-module and u be a compact (or completely continuous) operator acting on M.

By definition, recall that this means there exists a sequence of projective and finitely generated Banach A-modules $(M_i)_i$ such that $u|_{M_i}$ converges to u when $i \to \infty$. One can define its Fredholm determinant by

$$P_M(u, X) = \det(1 - X.u|M) := \lim_{i\to\infty} \det(1 - X.u_i|M_i),$$

where for all i, we have written u_i for the composite of $u|_{M_i}$ with the projection onto M_i. This definition is independent of the choice of the sequence $(M_i)_i$; cf. [Col97]. It extends easily to projective Banach modules.

2.2.2. *Trace versus determinant.* The first coefficient of the series $\det(1 - X.u|M)$ is the opposite of the trace $\mathrm{tr}(u; M)$. On the other hand, one can recover the Fredholm series from the trace map. There exists a universal sequence of polynomials[12] $Q_n(X_1, \ldots, X_n) \in \mathbb{Q}[X_1, \ldots, X_n]$ for $n \geq 1$ such that we have

$$P_M(u, X) = \det(1 - X.u|M) = \sum_{n=1}^{\infty} \mathrm{tr}(Q_n(u, u^2, \ldots, u^n); M)X^n.$$

These polynomials are well known; for instance, $Q_1(X_1) = -X_1$, $Q_2(X_1, X_2) = \frac{1}{2}(X_1^2 - X_2)$, etc.

From this, one can see that many properties of the Fredholm determinant follow from the corresponding properties of the trace map. In particular, if we have an A-valued linear map t from an ideal of a sub-algebra of $\mathrm{End}_A(M)$, then one can define the associated formal series for any element u of this ideal by

$$P_t(u, X) := \sum_{n=1}^{\infty} t(Q_n(u, u^2, \ldots, u^n))X^n.$$

2.2.3. We consider the category $\mathcal{B}an_A^*$ whose objects are pairs (M, u_M) with $M \in \mathrm{Ob}(\mathcal{B}an_A)$. Let u_M be an A-linear compact operator on M. A morphism $(M, u_M) \to (N, u_N)$ is given by a morphism $M \xrightarrow{f} N$ in $\mathcal{B}an_A$ such that $f \circ u_M = u_N \circ f$. This category is also exact and has a kernel and a cokernel. A sequence is exact if its image under the forgetful functor $\mathcal{B}an_A \to \mathcal{B}an_A^*$ is an exact sequence in $\mathcal{B}an_A$. We can define similarly the categories $\mathcal{C}(\mathcal{B}an_A^*)$, $\mathcal{K}(\mathcal{B}an_A^*)$, $\mathcal{K}_{\mathrm{pf}}(\mathcal{B}an_A^*)$, $D(\mathcal{B}an_A^*)$ and $D_{\mathrm{pf}}(\mathcal{B}an_A^*)$.

LEMMA 2.2.4. *Let M, N and P be three projective Banach A-modules fitting in an exact sequence in the category $\mathcal{B}an_A^*$:*

$$0 \to (N, u_N) \xrightarrow{f} (M, u_M) \xrightarrow{g} (P, u_P) \to 0.$$

[12] The Q_n's are sometimes called Newton polynomials.

Then we have

$$\det(1 - X.u_M|M) = \det(1 - X.u_N|N) \cdot \det(1 - X.u_P|P).$$

Proof. This follows easily from [Col97, Lemma A.2.4]. □

2.2.5. *Definition.* Let $(M^\bullet, u^\bullet) \in \mathrm{Ob}(\mathcal{C}(\mathcal{B}an_A^*))$ be such that $M^\bullet$ is a perfect complex. Then we put

$$\det(1 - X.u^\bullet|M^\bullet) := \prod_q \det(1 - X.u^q|M^q)^{(-1)^q}.$$

It follows from Lemma 2.2.4 that $\det(1 - X.u^\bullet|M^\bullet) = 1$ if $M^\bullet$ is acyclic.

2.2.6. Let $M^\bullet \in \mathrm{Ob}(D_{\mathrm{pf}}(\mathcal{B}an_A))$. A homomorphism $\mathrm{Hom}_{D(\mathcal{B}an_A)}(M^\bullet, M^\bullet)$ is said to be compact (or completely continuous) if it has a representative $u^\bullet \in \mathrm{Hom}_{\mathcal{K}(\mathcal{B}an_A)}(M^\bullet, M^\bullet)$ (i.e., in its homotopy class) such that $(M^\bullet, u^\bullet) \in \mathrm{Ob}(\mathcal{C}(\mathcal{B}an_A^*))$.

Lemma 2.2.7. *The operator $u^\bullet$ is compact if and only if there exists a sequence of operators of finite rank $(u_n^\bullet)_n$ in $\mathrm{Hom}_{D(\mathcal{B}an_A)}(M^\bullet, M^\bullet)$ such that $\lim_{n\to\infty} u_n^q = u^q$ for all q. Equivalently, there are finite rank projection operators $e_n^\bullet \in \mathrm{Hom}_{D(\mathcal{B}an_A)}(M^\bullet, M^\bullet)$ such that $\lim_{n\to\infty} e_n^q \circ u^q = u^q$ for all q.*

Proof. This follows easily from [Col97, A.1.6]. □

Lemma 2.2.8. *Let $u^\bullet$ and $v^\bullet$ be two A-linear compact operators on a perfect complex $M^\bullet$. If $u^\bullet$ and $v^\bullet$ are homotopically equivalent, then* $\det(1 - X.u^\bullet|M^\bullet) = \det(1 - X.v^\bullet|M^\bullet)$.

Proof. From the remark made in Section 2.2.2, it suffices to show that if $u^\bullet$ and $v^\bullet$ are homotopically equivalent, then they have the same trace. For this, it is sufficent to treat the case where $v^\bullet = 0$. Then there are operators $k^q : M^q \to M^{q-1}$ such that $u^q = d^{q-1} \circ k^q + k^{q+1} \circ d^q$ for all q with d^q denoting the differential of the complex $M^\bullet$ in degree q. Let $(e_n^\bullet)_n$ be as in the previous lemma such that $\lim_{n\to\infty} e_n^q \circ u^q = u^q$ for all q. Let us put $k_n^q := e_n^{q-1} \circ k^q \circ e_n^q$. Since $e_n^{q+1} \circ d^q = d^q \circ e_n^q$, we have $\mathrm{tr}(e_n^q \circ u^q) = \mathrm{tr}(e_n^q \circ u^q \circ e_n^q) = \mathrm{tr}(e_n^q \circ d^{q-1} \circ k^q \circ e_n^q + e_n^q \circ k^{q+1} \circ d^q \circ e_n^q) = \mathrm{tr}(d^{q-1} \circ k_n^q + k_n^{q+1} \circ d^q)$. This implies that $\mathrm{tr}(e_n^\bullet \circ u^\bullet) := \sum_q (-1)^q \mathrm{tr}(e_n^q \circ u^q) = 0$; therefore passing to the limit, we have $\mathrm{tr}(u^\bullet) = 0$. □

2.2.9. For any $M^\bullet \in \mathrm{Ob}(D_{\mathrm{pf}}(\mathcal{B}an_A))$ and any compact A-linear operator $u^\bullet$ of $M^\bullet$, it follows from the previous lemma that the Fredholm determinant $\det(1 - X.u^\bullet|M^\bullet)$ does not depend on the homotopy class of $u^\bullet$. For any compact homomorphism $u \in \mathrm{Hom}_{D_{\mathrm{pf}}(\mathcal{B}an_A)}(M^\bullet, M^\bullet)$, the Fredholm determinant of u is therefore (well) defined as the Fredholm determinant of any of its representatives in its homotopy class.

Corollary 2.2.10. *Let $(u_N^\bullet, u_M^\bullet, u_P^\bullet)$ be the morphism of a distinguished triangle of $D_{pf}(\mathcal{B}an_A)$*

$$\begin{array}{ccccccc} N^\bullet & \longrightarrow & M^\bullet & \longrightarrow & P^\bullet & \longrightarrow & N^\bullet[1] \\ \downarrow{\scriptstyle u_N^\bullet} & & \downarrow{\scriptstyle u_M^\bullet} & & \downarrow{\scriptstyle u_P^\bullet} & & \downarrow{\scriptstyle u_N^\bullet[1]} \\ N^\bullet & \longrightarrow & M^\bullet & \longrightarrow & P^\bullet & \longrightarrow & N^\bullet[1] \end{array}$$

such that $u_N^\bullet, u_M^\bullet$ and $u_P^\bullet$ are compact. Then we have

$$\det(1 - X.u_M^\bullet | M^\bullet) = \det(1 - X.u_N^\bullet | N^\bullet) \cdot \det(1 - X.u_P^\bullet | P^\bullet).$$

Proof. Since this triangle is distinguished, $(P^\bullet, u_P^\bullet)$ is homotopic to the cone of $(N^\bullet, u_N^\bullet) \to (M^\bullet, u_M^\bullet)$. The corollary follows from this observation and the previous lemma. □

2.3. *Spectral decompositions.*

2.3.1. *Slope decompositions.* We recall here a notion[13] introduced by Ash-Stevens [AS97] in their work on GL_n. Let $L/\mathbb{Q}_p$ be a finite extension. A polynomial $Q(X) \in L[X]$ of degree $d \in \mathbb{Z}_{\geq 0}$ is said to be of slope $\leq h$ if $Q(0) \in O_L^\times$ and if the roots of $Q^*(X) := X^d Q(1/X)$ in $\overline{\mathbb{Q}}_p$ have their slope (i.e., p-adic valuation) less or equal to h.

Let M be a vector space over L, and let u be a (continuous) linear endomorphism of the vector space M. We do not require M to be equipped with a p-adic topology. A $\leq h$-slope decomposition of M with respect to u is a direct sum decompostion $M := M_1 \oplus M_2$ such that

(1) M_1 and M_2 are stable under the action of u.
(2) M_1 is finitely dimensional over L.
(3) The polynomial $\det(1 - X.u|M_1)$ is of slope $\leq h$.
(4) For any polynomial Q of slope $\leq h$, the restriction of $Q^*(u)$ to M_2 is an invertible endomorphism of M_2.

Lemma 2.3.2. *Let M and M' be two L-vector spaces. Let u and u' be two endomorphisms of M and M' respectively. Let $M = M_1 \oplus M_2$ and $M' = M'_1 \oplus M'_2$ be $\leq h$-slope decompositions of M and M' with respect to u and u' respectively. Let f be a* (*continuous*) *L-linear map from M to M' satisfying $f \circ u = u' \circ f$. Then f maps respectively M_1 and M_2 into M'_1 and M'_2* (*i.e., $f(M_i) \subset M'_i$ for $i = 1, 2$*).

Proof. Let $Q(X) := \det(1 - X.u|M_1)$ and $Q'(X) := \det(1 - X.u'|M'_1)$. Let $z \in M_1$ and write $f(z) = x + y$ with $x \in M'_1$ and $y \in M'_2$. Since $Q^*(u).z = 0$,

[13]Our definition is actually stronger than theirs since they do not require that M_1 has a stable direct factor.

we must have $Q^*(u').x = -Q^*(u').y = 0$. By hypothesis, $Q^*(u')$ is invertible on M'_2. Therefore $y = 0$ and $f(z) \in M'_1$. Let now $z \in M_2$. Since $Q'^*(u)$ is invertible on M_2, there exists $w \in M_2$ such that $z = Q'^*(u).w$. Write $f(w) = x + y$ with $x \in M'_1$ and $y \in M'_2$. Since $Q'^*(u').x = 0$, we must have $f(z) = Q'^*(u').f(w) = Q'^*(u').y \in M'_2$. □

Corollary 2.3.3. *Let M, u and h be as in* 2.3.1. *Then we have uniqueness of the $\leq h$-slope decomposition of M. We will write $M^{\leq h}$ for M_1 and $M^{>h}$ for M_2.*

Proof. Apply the previous lemma for $M = M'$ and $f = \mathrm{id}_M$. □

Corollary 2.3.4. *Let M,M', u, u' and f be as in the previous lemma and suppose there is a map $\phi : M' \to M$ such that we have a commutative diagram*

$$\begin{tikzcd} M \arrow[r, "f"] \arrow[d, "u"] & M' \arrow[d, "u'"] \arrow[dl, "\phi"'] \\ M \arrow[r, "f"] & M'. \end{tikzcd}$$

Assume that M and M' have $\leq h$-slope decompositions with respect to u and u'. Then f induces an isomorphism between $M^{\leq h}$ and $M'^{\leq h}$.

Proof. By Lemma 2.3.2, f induces a map $M^{\leq h} \to M'^{\leq h}$. We want to show that it is an isomorphism. From the definition of the $\leq h$-slope decomposition, the restrictions of u and u' to respectively $M^{\leq h}$ and $M'^{\leq h}$ are invertible. Since we have $\phi \circ f = u$ and $f \circ \phi = u'$, this easily implies that f induces an isomorphism from $M^{\leq h}$ onto $M'^{\leq h}$. □

Corollary 2.3.5. *Let M, u and h be as in* 2.3.1 *and $N \subset M$ a subspace stable under the action by u. Assume that M has a $\leq h$-slope decomposition. Then N has $\leq h$-slope decomposition if and only if M/N does. When this is the case, we have*

- $(M/N)^{\leq h} = M^{\leq h}/N^{\leq h}$ *and* $N^{\leq h} = N \cap M^{\leq h}$.
- $(M/N)^{>h} = M^{>h}/N^{>h}$ *and* $N^{>h} = N \cap M^{>h}$.

Proof. Assume that N has a slope decomposition. (The other case is left to the reader.) From Lemma 2.3.2, we have $N^{\leq h} \subset M^{\leq h}$ and $N^{>h} \subset M^{>h}$. Hence $N^{\leq h} = N \cap M^{\leq h}$ and $N^{>h} = N \cap M^{>h}$; therefore $\bar{M} = M/N = \bar{M}_1 \oplus \bar{M}_2$ with $\bar{M}_1 := M^{\leq h}/N^{\leq h}$ and $\bar{M}_2 := M^{>h}/N^{>h}$. The first three conditions of 2.3.1 are obviously satisfied. If Q is a polynomial of slope $\leq h$, then $Q^*(u)$ induces an isomorphism of $M^{>h}$ and of $N^{>h}$. It therefore induces an isomorphism on the quotient $\bar{M}_2$ which proves the fourth condition. □

2.3.6. *Finite slope part of M for an operator u.* Let M and u be as in Section 2.3.1. Let $h' \in \mathbb{Q}$ with $h' \geq h$. Then if M has a $\leq h'$-slope decomposition with respect to u, it has a $\leq h$-slope decomposition, and we have a u-stable decomposition $M^{\leq h'} = M^{\leq h} \oplus M^{>h, \leq h'}$ such that $M^{>h} = M^{>h,\leq h'} \oplus M^{>h'}$. We will say that M has a slope decomposition with respect to u if for an increasing sequence h_n of rationals going to infinity (and therefore for all such sequences), M has a $\leq h_n$-slope decomposition for all h_n. Then we put

$$\det(1 - X.u) := \lim_{n\to\infty} \det(1 - X.u|M^{\leq h_n})$$

for $(h_n)_n$ any sequence of rationals going to infinity. It is straightforward to check that this sequence is convergent in $L[[X]]$ and that the limit does not depend on the sequence (h_n). If M has a slope decomposition with respect to u, we sometimes write M_{fs} to denote the inductive limit over n of the $M^{\leq h_n}$'s. We call it the finite slope part of M. The space M_{fs} obviously has a slope decomposition and $M_{\mathrm{fs}}^{\leq h} = M^{\leq h}$ for all h.

2.3.7. Let now $N \subset M$ be a u-stable subspace of M. Assume that N has a slope decomposition. Then, by Corollary 2.3.5, so does M/N and we have

$$\det(1 - X.u|M) = \det(1 - X.u|N).\det(1 - X.u|(M/N)).$$

THEOREM 2.3.8. *Let A be a Banach $\mathbb{Q}_p$-algebra, M be a projective Banach A-module and u be a compact A-linear operator of M. Then $P(X,u) := \det(1-X.u|M)$ is an entire power series with coefficient in A (i.e., $\in A\{\{T\}\}$).*

If we have a prime decomposition $P(X,u) = Q(X)S(X)$ in $A\{\{X\}\}$ with Q a polynomial such that $Q(0) = 1$ and $Q^*(0)$ is invertible in A, then there exists an entire power series $R_Q(T) \in TA\{\{T\}\}$ whose coefficients are polynomials in the coefficients of Q and S, and we have a decomposition of M

$$M = N_u(Q) \oplus F_u(Q)$$

of closed sub A-modules such that

(1) *The projector on $N_u(Q)$ is given by $R_Q(u)$.*
(2) *$Q^*(u)$ annihilates $N_u(Q)$.*
(3) *$Q^*(u)$ is invertible on $F_u(Q)$.*

If, moreover, A is noetherian, then $N_u(Q)$ is projective of rank r and

$$\det(1 - X.u|N_u(Q)) = Q(X).$$

Proof. This follows directly from the results of Serre's and Coleman's works and their proofs and some generalizations by Buzzard; see [Col97], [Ser62], [Buz04] □

If $A = L =$ is a finite extension of $\mathbb{Q}_p$, then an immediate consequence of the previous result is that M has a $\leq h$-slope decomposition for any $h \in \mathbb{Q}$.

Moreover, the definitions of $\det(1 - X.u|M)$ of Sections 2.3.6 and 2.2 coincide. The following proposition is not really needed but it shows that taking the Hausdorff quotient is not harmful when considering the finite slope part for a given operator.

PROPOSITION 2.3.9. *Let* $N \overset{j}{\hookrightarrow} M$ *a continuous injection of* L*-Banach spaces* (*we do not assume the image is closed*). *Let* u_N *and* u_M *be respectively compact endomorphisms of* N *and* M *such that* $j \circ u_N = u_M \circ j$. *Then* $M/j(N)$ *has slope decompositions with respect to* $u_{M/N} = u_M \pmod{j(N)}$, *and*

$$\det(1 - X.u_M) = \det(1 - X.u_N).\det(1 - X.u_{M/N}).$$

Let $\overline{j(N)}$ *be the closure of* $j(N)$ *inside* M. *Then* u *induces a compact operator* $\tilde{u}_{M/N}$ *on the Hausdorf quotient* $\widetilde{M/N} = M/\overline{j(N)}$ *of* M/N, *and we have*

$$\det(1 - X.u_{M/N}) = \det(1 - X.\tilde{u}_{M/N}).$$

Proof. By Serre's theorem (Theorem 2.3.8 with $A = L$), N has a slope decomposition with respect to u_N, and therefore so does $j(N)$ with respect to u_M since j is injective and $j \circ u_N = u_M \circ j$. The first formula follows from 2.3.7. We now prove the second part of the proposition. Notice first that

$$\overline{j(N)} = \overline{j(N)^{\leq h}} \oplus \overline{j(N)^{>h}}. \tag{9}$$

Indeed let $z_n = x_n + y_n$ be a sequence of N with $x_n \in N^{\leq h}$ and $y_n \in N^{>h}$ such that $j(z_n)$ is converging in M. To prove our claim, it suffices to show, for example, that $j(y_n)$ converges in $M^{>h}$. Let $Q = \det(1 - X.u|M^{\leq h})$. Then the sequence $Q^*(u).j(z_n) = Q^*(u).j(y_n)$ must converge. Since $Q^*(u)$ is a continuous isomorphism of the Banach subspace $M^{>h}$, it is bi-continuous by the open mapping theorem. Therefore, y_n converges to y and we have proved (9). From the fact that $M^{\leq h}$ and $M^{>h}$ are closed and the inclusion $j(N)^{\leq h} \subset M^{\leq h}$ and $j(N)^{>h} \subset M^{>h}$, we easily deduce that (9) is the $\leq h$-slope decomposition of $\overline{j(N)}$. Since $N^{\leq h}$ is finite-dimensional, we have $j(N^{\leq h}) = \overline{j(N^{\leq h})}$, and, from the decomposition (9), this is equal to the $\leq h$-slope part of $\overline{j(N)}$. Therefore, $\det(1 - X.u_N|N^{\leq h}) = \det(1 - X.u_M|\overline{j(N)}^{\leq h})$. This equality for all h implies that $\det(1 - X.u_N) = \det(1 - X.u_M|\overline{j(N)})$, and the last claim follows from the first equality for the lemma applied to $j(N) \subset M$ and $\overline{j(N)} \subset M$. $\square$

2.3.10. *Fredholm determinant for complexes revisited.* We assume that $M^\bullet$ is a perfect complex of Banach vector spaces over L a finite extension of $\mathbb{Q}_p$. Let $u^\bullet : M^\bullet \to M^\bullet$ be a continuous endomorphism of $M^\bullet$ such that, for all q, the operator $u^q \in \operatorname{End}_L(M^q)$ is a compact operator.

Let us denote by d^q the differential homomorphism of the resolution from M^q to M^{q+1}. By definition, this homomorphism is continuous, and therefore $\operatorname{Ker} d^q$ is a Banach subspace of M^q. However, $\operatorname{Im} d^{q-1} \subset \operatorname{Ker} d^q$ is not necessarily closed, and therefore $H^q(M^\bullet) = \mathrm{Ker}\, d^q/d^{q-1}(M^{q-1})$ might not be Hausdorff. We denote by $\widetilde{H}^q(M^\bullet)$ its maximal Hausdorff quotient. We have $\widetilde{H}^q(M^\bullet) = \mathrm{Ker}\, d^q/\,\mathrm{Im}\, d^{q-1} = \mathrm{Ker}\, d^q/\overline{d^{q-1}(M^{q-1})}$.

On the other hand, since d^q is continuous and commutes with u^q, $\mathrm{Ker}(d^q)$ is a Banach space with action of u^q. It therefore has a slope decomposition, and this implies that $M^q/\,\mathrm{Ker}(d^q) \cong \mathrm{Im}(d^q)$ has a slope decomposition. This is true for all q's; therefore $H^q(M^\bullet) = \mathrm{Ker}(d^q)/\,\mathrm{Im}(d^{q-1})$ has a slope decomposition. Let us write $H^q_{\mathrm{fs}}(M^\bullet)$ for its finite slope part. By the proposition above and its proof, we can therefore deduce that

$$H^q_{\mathrm{fs}}(M^\bullet) = \widetilde{H}^q(M^\bullet)_{\mathrm{fs}}.$$

Corollary 2.3.11. *With the notation as above, we have*

$$\det(1 - X.u^\bullet|M^\bullet) = \prod_q \det(1 - X.u^q|\widetilde{H}^q(M^\bullet))^{(-1)^q}.$$

Proof. We put $M = \mathrm{Ker}\, d^q$, $N = M^{q-1}/\,\mathrm{Ker}\, d^{q-1}$ with its structure of quotient Banach space and let j be the continuous injective homomorphism from N into M induced by d^q so that $j(N) = \mathrm{Im}\, d^{q-1}$. By the previous proposition, we have

$$\det(1 - X.u^q|H^q(M^\bullet)) = \det(1 - X.u^q|\widetilde{H}^q(M^\bullet)) = \frac{\det(1 - X.u^q|\,\mathrm{Ker}\, d^q)}{\det(1 - X.u^q|\,\mathrm{Im}(d^{q-1})}.$$

On the other hand, from the exact sequence $0 \to \mathrm{Ker}\, d^q \to M^q \to \mathrm{Im}\, d^q \to 0$, by the previous proposition we have

$$\det(1 - X.u^q|M^q) = \det(1 - X.u^q|\,\mathrm{Ker}\, d^q).\det(1 - X.u^q|\,\mathrm{Im}\, d^q).$$

Our claim follows now easily by making the alternate product of the two previous equalities and rearranging the terms. ☐

2.3.12. *Generalization to compact p-adic Fréchet spaces or p-adic nuclear spaces.* Recall that a p-adic topological vector space V is called a compact p-adic Fréchet space if it is the projective limit of p-adic Banach spaces V_n such that the transition maps $V_n \to V_m$ for $n > m$ are completely continuous. An endomorphism u of V will be said to be compact if it is continuous and if for each n we have a commutative diagram of continuous maps

$$\begin{tikzcd} V \arrow[r] \arrow[d, "u"] & V_{n-1} \arrow[d, "u'_n"] \\ V \arrow[r] & V_n, \end{tikzcd}$$

where the horizontal arrows are the canonical projection maps coming from the projective limit. By composing u'_n with the transition map $V_n \to V_{n-1}$, we get an endomorphism u_n of the Banach space V_n which is completely continuous.

It is easy to check that u_n is uniquely determined by u and that we have the following commutative diagram for any n:

$$\begin{tikzcd} V \arrow[r] \arrow[d, "u"] & V_n \arrow[r] \arrow[d, "u_n"'] & V_{n-1} \arrow[d, "u_{n-1}"] \arrow[dl, "u'_n"] \\ V \arrow[r] & V_n \arrow[r] & V_{n-1}. \end{tikzcd}$$

If the V_n are projective Banach space on some Banach algebra A, then we put

$$\operatorname{tr}(u;V) := \operatorname{tr}(u_n;V_n),$$

and it is clear from the above diagram that this definition is independent of n. Similarly, we can write a slope decomposition for V as we did for p-adic Banach spaces. More generally, it is easy to see that all the previous discussion on compact operators on complexes of p-adic Banach spaces extends *mutatis mutandis* to the case of compact operators on complexes of compact Fréchet spaces. We will take this fact for granted in the following sections and apply the results explicitly stated for Banach spaces to the case of compact p-adic Fréchet spaces without further notice. We need to state the following lemma.

LEMMA 2.3.13. *Let u be a compact operator on a compact Fréchet space V over a finite extension of $\mathbb{Q}_p$. For any $h \geq 0$, there is a $\leq h$-slope decomposition of V with respect to u. Moreover, we have*

$$V^{\leq h} \cong V_n^{\leq h}$$

for all n where for each n, $V_n = V_n^{\leq h} \oplus V_n^{>h}$ is the slope decomposition of V_n with respect to u_n. This fact holds as well for compact maps between complexes of compact Fréchet spaces and the induced slope decomposition on their cohomology.

Proof. By Lemma 2.3.2, the projection $V_m \to V_n$ for $n > m$ induces projections on the $\leq h$-slope decompositions of V_m and V_n with respect to u_m and u_n. This implies that $V^{\leq h} = \varprojlim_n V_n^{\leq h}$ and $V^{>h} = \varprojlim_n V_n^{>h}$ are well defined and provide a $\leq h$-slope decomposition for V. We are now left to prove that the projections $V_n^{\leq h} \to V_m^{\leq h}$ are actually isomorphisms. By an induction argument, it is sufficient to prove this for any $m > n$. But for $m = n'$ as in the diagram above, this follows from Corollary 2.3.4. The result holds clearly when one replaces V by a complex of compact Fréchet spaces and this implies the result for the cohomology by the arguments used in this section. $\square$

3. p-adic overconvergent coefficients

Let L be a finite extension of $\mathbb{Q}_p$. The goal of this section is to define the system of *overconvergent*[14] coefficients that will be used to interpolate the cohomology of the local systems $\mathbb{V}_\lambda(L)$. As in the original GL(2)-situation carried over by H. Hida in the ordinary case and by G. Stevens in general, the idea to do this is to replace $\mathbb{V}_\lambda(L)$ by a system of coefficients which does not depend on the weight but is endowed with an action[15] of G which does depend on the weight. When $G = \mathrm{GL}(n)$, one recovers the construction due to A. Ash and G. Stevens in [AS97] which they recently generalized in [AS08].

3.1. *Basic notations and definitions.*

3.1.1. *Algebraic data.* In this section, G is a connected reductive group over $\mathbb{Q}_p$ that we suppose to be quasi-split.[16] We let T be a maximal torus of G and let B be a Borel subgroup containing T. We choose a finite Galois extension $F/\mathbb{Q}_p$ such that $G_{/F}$ is split. Then $(B_{/F}, T_{/F})$ defines a Borel pair of $G_{/F}$. We denote by N the unipotent radical of B and B^- (resp. N^-) the opposite Borel subgroup (resp. opposite unipotent radical). We will use gothic letters to denote the corresponding Lie algebra over $\mathbb{Q}_p$, $\mathfrak{g}, \mathfrak{t}, \mathfrak{b}, \mathfrak{b}^-, \mathfrak{n}, \mathfrak{n}^-$.

We set the lattice of algebraic weights $X^*(T) := \mathrm{Hom}_{gp}(T_{/F}, \mathbb{G}_{m/F}) \hookrightarrow \mathrm{Hom}_F(\mathfrak{t}_{/F}, F)$ and algebraic co-weights $X_*(T) := \mathrm{Hom}(\mathbb{G}_{m/F}, T_{/F})$, and we denote by $(.,.)$ the canonical pairing on $X^*(T) \otimes X_*(T)$. We let $R^+ \subset X^*(T)$ be the set of positive roots with respect to $(B,T)_{/F}$. For each root $\alpha \in R$, we denote H_α (resp. $\alpha^\vee$) the corresponding coroot in $\mathfrak{t}_{/F}$ (resp. in $X_*(T)$; recall that $(\alpha, \alpha^\vee) = \alpha(H_\alpha) = 2$. For each root α, we also choose a basis X_α of $\mathfrak{g}_\alpha = \{X \in \mathfrak{g}_{/F} \mid \mathrm{ad}(t).X = \alpha(t)X \quad \forall t \in T\}$ such that $[X_\alpha, X_{-\alpha}] = H_\alpha$. A weight $\lambda \in X^*(T)$ is called dominant with respect to B if $\lambda(H_\alpha) \geq 0$ for all positive root α, and we write $X^*(T)^+$ for the cone of dominant weights.

We denote by $\mathcal{W}$ the Weyl group of the pair $(G_{/F}, T_{/F})$ acting on $T_{/F}$ and therefore on the lattices $X^*(T)$ and $X_*(T)$. It is generated by elements of order two called simple reflexions s_α for α running among the simple roots of R. The action on the weights is given by the well-known formula $s_\alpha(\lambda) = \lambda - \lambda(H_\alpha)\alpha$ for any $\alpha \in R$. We will also need the notion of length of an element of $\mathcal{W}$. It can be defined as the smallest integer l such that there is a decomposition

[14] This terminology is nonstandard. It is used as a reminiscence of the notion of overconvergent modular forms.

[15] Not an action of G in fact but of some sub-semi-group of it containing an Iwahori subgroup.

[16] This assumption is certainly unnecessary if one wants to use the language of Bruhat-Tits' buildings.

$w = s_{\alpha_1} \cdots s_{\alpha_l}$, where the α_i's are simple roots. Such a decomposition is called a reduced decomposition.

3.1.2. *Iwahori subgroups and semi-groups.* We let $I \subset G(\mathbb{Q}_p)$ be an Iwahori subgroup in good position with respect to B. By this, we mean that we have fixed compatible integral models for G, B, T, N, N^- over $\mathbb{Z}_p$ such that $I = I_1$, where for every integer $m \geq 1$, we have put

$$I_m := \{g \in G(\mathbb{Z}_p) \mid g \mod p^m \in B(\mathbb{Z}/p^m\mathbb{Z})\},$$
$$I'_m := \{g \in G(\mathbb{Z}_p)) \mid g \mod p^m \in N(\mathbb{Z}/p^m\mathbb{Z})\}.$$

We have $I_m/I'_m \cong T(\mathbb{Z}/p^m\mathbb{Z})$. Recall that we have the Iwahori decomposition

$$I_m = I_m^-.T(\mathbb{Z}_p).N(\mathbb{Z}_p)$$

with $I_m^- := I_m \cap N^-(\mathbb{Q}_p)$. We consider T^+ as the set of elements $t \in T(\mathbb{Q}_p)$ such that

$$t^{-1}.N(\mathbb{Z}_p).t \subset N(\mathbb{Z}_p)$$

and $T^{++} \subset T^+$ as the set of elements t such that

$$\bigcap_{i \geq 1} t^{-i}.N(\mathbb{Z}_p).t^i = \{1\},$$

and we put $\Delta_m^+ = I_m T^+ I_m$ and $\Delta_m^{++} = I_m T^{++} I_m$. We will drop the index m from the notation when it is equal to 1. Using the Iwahori decomposition, it is straightforward to see that any element $g \in \Delta_m^+$ has a unique decomposition

$$g = n_g^- t_g n_g^+ \qquad \text{with} \quad n_g^- \in I_m^-,\ t_g \in T^+ \text{ and } n_g^+ \in N(\mathbb{Z}_p). \tag{10}$$

The sets T^+ and T^{++} are sub-semi-groups of $T(\mathbb{Q}_p)$, and we clearly have $T(\mathbb{Z}_p) \subset T^+$. Since $T(\mathbb{Q}_p)/T(\mathbb{Z}_p)$ is isomorphic to a sum of copies of $\mathbb{Z}$, we may choose a splitting[17] ξ of the canonical projection $T(\mathbb{Q}_p) \to T(\mathbb{Q}_p)/T(\mathbb{Z}_p)$ which induces an isomorphism of groups

$$T(\mathbb{Q}_p) \cong T(\mathbb{Z}_p) \times T(\mathbb{Q}_p)/T(\mathbb{Z}_p).$$

We will also write ξ for the composite $T(\mathbb{Q}_p) \to T(\mathbb{Q}_p)/T(\mathbb{Z}_p) \to T(\mathbb{Q}_p)$. Notice that $\xi(T^+) \subset T^+$ since $\xi(t)t^{-1} \in T(\mathbb{Z}_p)$ for all $t \in T(\mathbb{Q}_p)$. If T is split over an unramified extension of $\mathbb{Z}_p$, we can (and do) choose ξ so that for any algebraic character $\lambda^{\mathrm{alg}} \in X^*(T)$, we have

$$\lambda^{\mathrm{alg}}(\xi(t)) = |\lambda^{\mathrm{alg}}(t)|_p^{-1} \tag{11}$$

for all $t \in T(\mathbb{Q}_p)$.

[17]This choice is similar to the choice of uniformizing elements in the definition of Hecke operators at places dividing p in [Hid88].

3.1.3. *The left $*$-action of I and Δ^+ on some p-adic spaces.* We consider the following p-adic cells:

$$\Omega_0 := I_1^- T(\mathbb{Z}_p)\backslash I \subset B^-(\mathbb{Q}_p)\backslash G(\mathbb{Q}_p),$$
$$\Omega_1 := I_1^- \backslash I \subset N^-(\mathbb{Q}_p)\backslash G(\mathbb{Q}_p).$$

By the Iwahori decomposition, we have $\Omega_0 \cong N(\mathbb{Z}_p)$ and $\Omega_1 \cong B(\mathbb{Z}_p)$. For any element $x \in G(\mathbb{Q}_p)$, let us denote by $[x]$ its class in $B^-(\mathbb{Q}_p)\backslash G(\mathbb{Q}_p)$. Then the right translation by I defines an action on Ω_0 by $[x] * g = [xg]$. We can extend this action into an action of Δ^+. For this it is convenient to introduce the p-adic spaces

$$\Omega_0^+ := I_1^- T(\mathbb{Z}_p)\backslash \Delta^+ \subset B^-(\mathbb{Q}_p)\backslash G(\mathbb{Q}_p),$$
$$\Omega_1^+ := I_1^- \backslash \Delta^+ \subset N^-(\mathbb{Q}_p)\backslash G(\mathbb{Q}_p).$$

We have a retraction $\mathbf{s} :\ \Omega_0^+ \to \Omega_0$ for the natural inclusion $\Omega_0 \subset \Omega_0^+$ defined by $[g] \mapsto \mathbf{s}([g]) := [\xi(t_g)^{-1} g]$, where the element t_g is defined via the decomposition (10) and ξ is the section defined in the previous paragraph. Since $\xi(t_g) = 1$ when $g \in I$, this is a well-defined retraction of the inclusion $\Omega_0 \subset \Omega_0^+$. Then the action[18] of $g \in \Delta^+$ on Ω_0 is defined by

$$[x] * g := \mathbf{s}([xg]) = [\xi(t_g)^{-1} xg].$$

We define the action of I and Δ^+ on Ω_1 using the same retraction of $\Omega_1 \subset \Omega_1^+$.

Lemma 3.1.4. *The above formula gives a well-defined left action of the monoid Δ^+ on Ω_i that extends the natural action of I.*

Proof. One needs to show that $[x] * gg' = ([x] * g) * g'$ for any $g, g' \in \Delta^+$. This follows from the fact that $t_{gg'} = t_g t_{g'}$, which is easily checked using the Iwahori decomposition. Indeed we have $[x] * gg' = [\xi(t_{g'} t_g)^{-1} xgg'] = [\xi(t_g^{-1}) xg] * g' = ([x] * g) * g'$. Moreover, if $g \in I$, then $\xi(t_g) = 1$. Therefore the action of Δ^+ restricted to I is the same as the right translation by I. $\square$

3.2. *Local analytic induction.*

3.2.1. *Analytic functions.* We now recall some definitions and some facts on locally analytic functions. We will call a p-adic space any topological space X which is isomorphic[19] to an open subset of $\mathbb{Q}_p^r$ for some r. We will always identify such a space with an open subset of $\mathbb{Q}_p^r$, and the definitions below will not depend on this identification. In the examples we will consider later, X

[18] The reader should keep in mind now and in all this paper that the $*$-action depends of the splitting character ξ.

[19] defined up to a (locally) $\mathbb{Q}_p$-algebraic isomorphism

will be either a compact open subset of $G(\mathbb{Q}_p)$ or Ω_0 that we will identify with $N(\mathbb{Z}_p)(\subset G(\mathbb{Q}_p))$ via the Iwahori decomposition.

Let L be a finite extension of $\mathbb{Q}_p$. A continuous function on such a space, $f : X \to L$, is said to be L-analytic if it can be expressed as a converging power series

$$f(x_1, \ldots, x_r) = \sum_{n_1,\ldots,n_r} \alpha_{n_1,\ldots,n_r}(x_1 - a_1)^{n_1} \cdots (x_r - a_r)^{n_r} \tag{12}$$

for all $(x_1, \ldots, x_r) \in X$, where $\alpha_{n_1,\ldots,n_r} \in L$ for some $\underline{a} = (a_1, \ldots, a_r) \in X$. Of course, it is algebraic if almost all the $\alpha_{n_1,\ldots,n_r}$'s are zero. Basic examples of analytic functions are given by the *logarithm* or *exponential* functions respectively defined on p-adic Lie groups and Lie algebras.

For any integer $n \in \mathbb{Z}$, we will say that $f :\ X \to L$ is n-locally L-analytic if X can be covered by disks of radius p^{-n} over which f is L-analytic. It is said to be locally L analytic if it is n-locally L-analytic for some n. We usually denote by $\mathcal{A}(X, L)$ the space of locally L-analytic functions on X and by $\mathcal{A}_n(X, L) \subset \mathcal{A}(X, L)$ those that are n-locally analytic. Of course, we have

$$\mathcal{A}(X, L) = \bigcup_{n \geq 0} \mathcal{A}_n(X, L).$$

If X is *compact*, then each $\mathcal{A}_n(X, L)$ is a p-adic Banach space equipped with the *sup* norm

$$\|f\|_n := \mathrm{Sup}_{\underline{a}} |\alpha_{n_1,\ldots,n_r}(\underline{a})|_p p^{-n \sum_{i=1}^r n_i},$$

where the $\underline{a} = (a_1, \ldots, a_r)$'s run through the set of centers of disks of radius p^{-n} inside X and where the $\alpha_{n_1,\ldots,n_r}(\underline{a})$'a are the coefficients of the Taylor expansion at the point $\underline{a}$ described as in the expression (12). The L-vector space $\mathcal{A}(X, L)$ is then naturally equipped with the inductive limit topology of the $\mathcal{A}_n(X, L)$'s. We have the well-known elementary lemma.

Lemma 3.2.2. *Assume that X is compact. Then the inclusions $\mathcal{A}_n(X, L) \subset \mathcal{A}_{n+1}(X, L)$ are completely continuous.*

Proof. Since X is compact, it easy to reduce this statement to the compactness of the restriction map $\mathcal{A}_n(p^n \mathbb{Z}_p^r) \to \mathcal{A}_{n+1}(p^{n+1} \mathbb{Z}_p^r)$. This fact is an elementary exercise left to the reader. □

Let $\mathcal{D}(X, L)$ (resp. $\mathcal{D}_n(X, L)$) be the continuous L-dual of $\mathcal{A}(X, L)$ (resp. $\mathcal{A}_n(X, L)$). The space $\mathcal{D}(X, L)$ is called the space of L-valued distribution on X. Then $\mathcal{D}(X, L)$ is the projective limit over n of the $\mathcal{D}_n(X, L)$. An immediate corollary of the previous lemma is

Corollary 3.2.3. *If X is compact, then $\mathcal{D}(X, L)$ is a compact Fréchet space.*

3.2.4. *Weights.* A (p-adic) *weight* is a continuous (for the p-adic topology) group homomorphism

$$\lambda: \ T(\mathbb{Z}_p) \to \overline{\mathbb{Q}}_p^\times.$$

A weight is called algebraic if it can be obtained as the composite

$$T(\mathbb{Z}_p) \hookrightarrow T(F) \stackrel{\lambda^{\mathrm{alg}}}{\to} F^\times \subset \overline{\mathbb{Q}}_p^\times$$

for some $\lambda^{\mathrm{alg}} \in X^*(T)$. In fact, we can see that any weight is locally analytic.

Lemma 3.2.5. *Let $L \subset \overline{\mathbb{Q}}_p$ be a finite extension of F. Any continuous p-adic character $\lambda \in \mathrm{Hom}_{\mathrm{cont}}(T(\mathbb{Z}_p), L^\times)$ is n-locally L-analytic for some $n \geq 0$. We sometimes denote by n_λ the smallest n for which this is the case.*

Proof. This is a special case of Lemma 3.4.6. Let $\mathfrak{t} := \mathrm{Lie}\ T_{/\mathbb{Z}_p}$. Then $H \mapsto \log(\lambda(\exp(H))$ is well defined if H is sufficiently close to 0 in $\mathfrak{t}$. It is $\mathbb{Z}_p$-linear and so defines an element $\lambda^{\mathrm{an}} \in \mathrm{Hom}(\mathfrak{t}, L) = X^*(T) \otimes L$. Then we have $\lambda(t) = \exp(\lambda^{\mathrm{an}}(\log(t))$ if t is sufficiently close to 1 in $T(\mathbb{Z}_p)$. In particular, λ is analytic on a neighborhood of 1 and it is therefore locally analytic. □

Notice that λ is n-locally algebraic for some n if and only if $\lambda^{\mathrm{an}} \in X^*(T)$. In that case, we write λ^{alg} for the corresponding algebraic character, and we have a decomposition

$$\lambda = \lambda^{\mathrm{alg}}.\varepsilon$$

with ε a finite order character factorizing through $T(\mathbb{Z}_p/p^n\mathbb{Z})$. We will say in that case that ε of λ is of level p^n. We will say that λ is arithmetic if it is locally algebraic and if λ^{alg} is dominant (i.e., $\lambda^{\mathrm{alg}} \in X^*(T)^+$).

3.2.6. *Locally analytic induction spaces.* Let $L \subset \overline{\mathbb{Q}}_p$ be a finite extension of $\mathbb{Q}_p$ and λ be an L-valued weight (i.e., $\lambda \in \mathrm{Hom}_{\mathrm{cont}}(T(\mathbb{Z}_p), L^\times)$). Then we denote by $\mathcal{A}_\lambda(L) \subset \mathcal{A}(I, L)$ the space of locally L-analytic function on I such that

$$f(n^- tg) = \lambda(t) f(g)$$

for all $n^- \in I_1^-$, $t \in T(\mathbb{Z}_p)$ and $g \in I$. This space is a closed subspace of the topological space $\mathcal{A}_\lambda(I, L)$ and is therefore a compact inductive limit of L-Banach spaces. By the Iwahori decomposition, we see that we have a canonical linear homeomorphism $\psi_\lambda : \mathcal{A}_\lambda(L) \cong \mathcal{A}(\Omega_0, L)$ via the map $f \mapsto \psi_\lambda(f)$, where $\psi_\lambda(f)$ is defined by

$$\psi_\lambda(f)([g]) := f(n_g), \qquad \forall g \in I.$$

Therefore, we see that $\mathcal{A}_\lambda(L)$ satisfies the topological properties of $\mathcal{A}(\Omega_0, L)$. In particular, its L-dual, which we denote $\mathcal{D}_\lambda(L)$, is a compact Fréchet space. Our space $\mathcal{A}_\lambda(L)$ is equipped with a continuous left action of I defined by $(g.f)(h) := f(hg)$ for all $g, h \in I$. It is easy to check that this action is continuous. We therefore inherit a (dual) continuous right action of I on $\mathcal{D}_\lambda(L)$.

Since elements of f are left invariant by I^-, we also have a natural inclusion $\mathcal{A}_\lambda(L) \subset \mathcal{A}(\Omega_1, L)$. If $f \in \mathcal{A}_\lambda(L)$, then let us write $\tilde{f}$ the corresponding element in $\mathcal{A}(\Omega_1, L)$. Then we have $\tilde{f}([g]) := f(g)$. We have a natural action of $T(\mathbb{Z}_p)$ on $\mathcal{A}(\Omega_1, L)$. If $\phi \in \mathcal{A}(\Omega_1, L)$ and $t \in T(\mathbb{Z}_p)$, we set

$$(t.\phi)([g]) := \phi([tg]).$$

It is easy to check that it is well defined and that $\phi = \tilde{f}$ for some $f \in \mathcal{A}_\lambda(L)$ if and only if $t.\phi = \lambda(t)\phi$. In other words, we have an identification

$$\mathcal{A}_\lambda(L) = \mathcal{A}(\Omega_1, L)[\lambda] := \{\phi \in \mathcal{A}(\Omega_1, L) |\ t.\phi = \lambda(t)\phi\}.$$

We now consider the $*$-action of Δ^+ on these spaces. For any $g \in \Delta^+$ and $\phi \in \mathcal{A}(\Omega_1, L)$, we define $g * \phi$ by

$$(g * \phi)([x]) := \phi([x] * g)$$

for all $[x] \in \Omega_1$.

Lemma 3.2.7. *The $*$-action of Δ^+ commutes with the natural action of $T(\mathbb{Z}_p)$ on $\mathcal{A}(\Omega_1, L)$. In particular, $\mathcal{A}_\lambda(L)$ is stable by the $*$-action of Δ^+. Moreover, for any $g \in I$, the $*$-action of I on $\mathcal{A}_\lambda(L)$ coincides with the natural left action*

$$(g * \tilde{f})([h]) = f(hg)$$

for all $g, h \in I$.

Proof. Let $x \in I$, $t \in T(Z_p)$ and $g \in \Delta^+$. Then we have

$$\begin{aligned}(t.(g * \phi))([x]) &= (g * \phi)([tx]) = \phi([\xi(t_g)^{-1}txg]) \\ &= \phi([t\xi(t_g)^{-1}xg]) = (t.\phi)([\xi(t_g)^{-1}xg]) = (g * (t.\phi))([x])\end{aligned}$$

This obviously implies that $\mathcal{A}_\lambda(L)$ is stable by the $*$-action. Let us check the last point now. Indeed we have $[h] * g = [hg]$ if $g \in I$; therefore $(g * \tilde{f})([h]) = \tilde{f}([h] * g) = \tilde{f}([hg])$. □

We consider the dual right action of Δ^+ on $\mathcal{D}_\lambda(L)$. This $*$-action is a very important ingredient of the theory developed in this paper. In particular, the following lemma (although easy) is crucial.[20]

Lemma 3.2.8. *If $\delta \in \Delta^{++}$, then the right $*$-action of δ defines a compact operator on the compact Fréchet space $\mathcal{D}_\lambda(L)$.*

[20]In other words, this lemma is saying that the action of δ improves the local analyticity.

Proof. We may suppose that $\delta \in T^{++}$ since the action of I is continuous. Let us put $\mathcal{A}_{m,\lambda}(L) := \psi_\lambda^{-1}(\mathcal{A}_m(\Omega_0, L))$. To prove the lemma, we check that the $*$-action of δ on $\mathcal{A}_{m,\lambda}(L)$ factorizes through the natural inclusion $\mathcal{A}_{m-1,\lambda}(L) \subset \mathcal{A}_{m,\lambda}(L)$. Since this one is completely continuous by Lemma 3.2.2, this will prove our claim. If $f \in \mathcal{A}_{m,\lambda}(L)$, then

$$\psi_\lambda(\delta * f)([n]) = \psi_\lambda(f)([\xi(\bar{\delta})^{-1}n\delta]) = \lambda(\xi(\bar{\delta})^{-1}\delta)\psi_\lambda(f)([\delta^{-1}n\delta]) \tag{13}$$

for all $n \in N(\mathbb{Z}_p)$. We make a choice of a basis of $\mathfrak{n}$ which gives a system of coordinates of $N(\mathbb{Z}_p)$ that we denote by $x_1(n), \ldots, x_r(n) \in \mathbb{Z}_p$ for all $n \in N(\mathbb{Z}_p)$. Let T_δ be the matrix of the action of $Ad(\delta^{-1})$ on $N(\mathbb{Z}_p)$. Since $\delta \in T^{++}$, the p-adic limit of $(T_\delta)^k$ is 0 when $k \to \infty$. Therefore the entries of the matrix T_δ are divisible by p. Now if $f(x_1, \ldots, x_r)$ is an analytic function of radius of convergence p^{-m}, this implies that $f((x_1, \ldots, x_r)^t T_\delta)$ has a radius of convergence at least p^{-m+1}. In view of the identity (13) above, this implies that our claim follows. □

3.2.9. *Locally algebraic induction.* Let λ be an arithmetic weight with the decomposition $\lambda = \lambda^{\mathrm{alg}}.\varepsilon$ with ε a finite order character of $T(\mathbb{Z}_p)$ of conductor p^m and $\lambda^{\mathrm{alg}} \in X^*(T)^+$. We then denote by $V_\lambda(L)$ the subset of functions $f \in \mathcal{A}_\lambda(L)$ which are locally L-algebraic.

Let us assume that L contains F. Recall that $B^-_{/L}$ is the Borel subgroup opposite to $B_{/L}$. Then we write $\left(\mathrm{Ind}_{B^-_{/L}}^{G_{/L}}\lambda\right)^{\mathrm{alg}}$ for the set of L-algebraic functions $f : G_{/L} :\to \mathbb{A}^1_{/L}$ such that

$$f(bg) = \lambda^{\mathrm{alg}}(b)f(g)$$

for all $b \in B^-(L)$ and $g \in G(L)$, where λ^{alg} is seen as a character of $B^-_{/L}$ via the canonical projection $B^-_{/L} \to T_{/L}$. One defines an action of $G(L)$ on this induction by right translation: $(g.f)(h) := f(gh)$. As we have recalled in the first section of this paper, we have

$$\mathbb{V}_{\lambda^{\mathrm{alg}}}(L) = \left(\mathrm{Ind}_{B^-_{/L}}^{G_{/L}}\lambda\right)^{\mathrm{alg}}$$

as the irreducible algebraic representation of $G(L)$ of highest weight λ^{alg} with respect to the Borel pair $(B_{/L}, T_{/L})$.

Let $L(\varepsilon)$ be the one-dimensional representation of I_m given by the character

$$I_m \mapsto B(\mathbb{Z}/p^m\mathbb{Z}) \to T(\mathbb{Z}/p^m\mathbb{Z}) \xrightarrow{\varepsilon} L^\times.$$

Then we have a canonical injection of I_m-left module

$$\mathbb{V}_{\lambda^{\mathrm{alg}}}(\varepsilon, L) := \mathbb{V}_{\lambda^{\mathrm{alg}}}(L) \otimes_L L(\varepsilon) \hookrightarrow V_\lambda(L)$$

given by $f \otimes 1 \mapsto f_\varepsilon$ with

$$f_\varepsilon(g) := \varepsilon(t_g) f(n_g^- t_g n_g). \tag{14}$$

We have $\mathbb{V}_{\lambda^{\mathrm{alg}}}(\varepsilon, L) = V_\lambda(L) \cap \mathcal{A}_m(I, L)$ (i.e., the elements of $\mathbb{V}_{\lambda^{\mathrm{alg}}}(\varepsilon, L)$ are those that are m-locally L-algebraic).

For a later use, let us record here that the $*$-action of Δ_m^+ and the twisted algebraic action on $V_\lambda(L)$ are related by the relation

$$\delta * f = \lambda(\xi(t_\delta))^{-1}(\delta \cdot f). \tag{15}$$

We will see below how to check that an element of $\mathcal{A}_\lambda(L)$ actually belongs to $\mathbb{V}_{\lambda^{\mathrm{alg}}}(L)$. We first need to define an action of $\mathcal{W}_G$ on the locally algebraic character. For any $\lambda = \lambda^{\mathrm{alg}}.\varepsilon$ and any $w \in \mathcal{W}_G$, we write $w * \lambda$ for the character given by

$$t \mapsto t^{w(\lambda^{\mathrm{alg}}+\rho)-\rho}\varepsilon(t).$$

We consider now the left l-action of I on $\mathcal{A}(I, L)$ defined by

$$l(h).f(g) := f(h^{-1}g) \qquad \forall h, g \in I.$$

This action is L-analytic and therefore induces an action of the Lie algebra $\mathfrak{g}_{/F}$. We have the following proposition.

Proposition 3.2.11. *Let λ be a weight and α be a simple root. Assume that $\lambda^{\mathrm{an}}(H_\alpha) = N \in \mathbb{Z}_{\geq -1}$. Then there exists an intertwining (for the $*$-action of I) map Θ_α*

$$\mathcal{A}_\lambda(L) \to \mathcal{A}_{s_\alpha * \lambda}(L)$$

defined by

$$\Theta_\alpha(f) = l(X_\alpha)^{N+1}.f.$$

Moreover, we have

$$\Theta_\alpha(g * f) = \xi(t_g)^{s_\alpha * \lambda - \lambda} t * \Theta_\alpha(f)$$

for all $g \in \Delta^+$.

Proof. The fact that Θ_α commutes with the action of I is clear from its definition. We will need to prove that it lands in $\mathcal{A}_{\lambda\alpha^{-N-1}}(L)$. Let $f \in \mathcal{A}_\lambda(L)$. First notice that

$$l(H_\alpha).f = -\lambda^{\mathrm{an}}(H_\alpha) f \tag{16}$$

since $l(t).f = \lambda(t)^{-1} f$ for any $t \in T(\mathbb{Z}_p)$. Now put $f_1 := \Theta_\alpha(f) = l(X_\alpha)^{N+1}.f$. One first sees that $f_1(n^- g) = f_1(g)$ for any $n^- \in I^-$ and any $g \in I$. In order to prove this claim, we first check that $l(X_{-\beta}).f_1 = 0$ for any simple root β. If $\beta \neq -\alpha$, then this is due to the fact that in this case we have $[X_{-\beta}, X_\alpha] = 0$. If $\beta = -\alpha$, we use the following relation in the enveloping algebra of $\mathfrak{g}$:

$$[X_{-\alpha}, X_\alpha^{i+1}] = -(i+1) X_\alpha^i (H_\alpha + i),$$

which is valid for any integer $i \geq 0$. Now $l(X_{-\alpha}).f = 0$ since $f(n_-.g) = f(g)$ for any $n^- \in I^-$. Therefore,

$$l(X_{-\alpha}).f_1 = l(X_{-\alpha}.X_\alpha^{N+1}).f = l([X_{-\alpha}, X_\alpha^{N+1}]).f.$$

Thus, $l(X_{-\alpha}).f_1 = 0$ by the relation above for $i = N = \lambda^{\mathrm{an}}(H_\alpha)$ and relation (16). This implies that $l(X).f_1 = 0$ for all $X \in \mathrm{Lie}(N^-_{/L})$ since the $X_{-\alpha}$'s generate $\mathrm{Lie}(N^-_{/L})$ as a Lie algebra. This easily implies that $f_1(n^- g) = f_1(g)$ for all $n^- \in I^-$ and $g \in I$ (for instance one can use the log and exponential map). Now, for any $t \in T(\mathbb{Z}_p)$, we have

$$l(t.X_\alpha^{N+1}).f = \alpha(t)^{N+1} l(X_\alpha^{N+1}.t).f = \alpha^{N+1}(t)\lambda^{-1}(t) l(X_\alpha^{N+1}).f.$$

Therefore $l(t).f_1 = \lambda^{-1}\alpha^{N+1}(t) f_1$ and $f_1 \in \mathcal{A}_{\lambda\alpha^{-N-1}}(L)$. We conclude by noticing that $s_\alpha * \lambda = \lambda\alpha^{-N-1}$. □

We have the following proposition.

PROPOSITION 3.2.12. *Let λ be a locally algebraic dominant weight. Then we have a canonical isomorphism*

$$V_\lambda(L) \cong \mathcal{A}_\lambda(L) \cap \bigcap_{\alpha \in \Phi} \mathrm{Ker}(\Theta_\alpha),$$

where Φ stands for the set of simple roots for the Borel pair $(B_{/F}, T_{/F})$.

Proof. We give an elementary proof here. Notice that this proposition is a refinement[21] of the exactness in degree 1 of the locally analytic BGG complex (see next paragraph).

We first prove $V_\lambda(L) \subset \bigcap_{\alpha\in\Phi} \mathrm{Ker}(\Theta_\alpha)$. Let $f \in V_\lambda(L)$. We may assume without loss of generality that f is actually algebraic. Then, for each simple root α, $\Theta_\alpha(f)$ belongs to the algebraic induction from B^- to G of the algebraic character $s_\alpha * \lambda$. Therefore, $\Theta_\alpha(f) = 0$ since the weight $s_\alpha(\lambda^{\mathrm{alg}} + \rho) - \rho$ is no longer dominant. So the first inclusion is proved.

We now prove the opposite inclusion. Take $f \in \bigcap_{\alpha\in\Phi} \mathrm{Ker}(\Theta_\alpha)$. Since f is locally L-analytic, the restriction of f to a neighborhood U of the identity is an analytic function which is invariant by left translation by elements in $U \cap N^-(F)$. Now since $f \in \mathrm{Ker}(\Theta_\alpha)$, for all $g \in U$, the function $n \mapsto f(ng)$ is a locally polynomial function on N_α for N_α the image by the exponential map (which is an algebraic map on $\mathfrak{n}$) of a neighborhood of 0 in $F.X_\alpha \subset \mathfrak{n}_{/F}$. Now since the N_α's for $\alpha \in \Phi$ generate a neighborhood of the identity in $N(F)$, the restriction of f to this neighborhood will be algebraic. Because λ is locally algebraic by assumption, we deduce that f is algebraic on a neighborhood of

[21]Since it gives a precise form of the differential maps in degree 1 of the BGG complex that could be explicitly described as it was done in [BGG75].

the identity in $B(F)$. This implies that f is locally algebraic as claimed since $N^-.B$ is Zariski dense in G. $\square$

3.2.13. *Integral structure.* Recall that O_L is the ring of integers of L. We denote by $\mathcal{A}(I, O_L)$ the O_L-submodule of $\mathcal{A}(I, L)$ consisting of functions f taking values in O_L. For any weight $\lambda \in \mathfrak{X}(O_L)$, we consider the topological O_L-module $\mathcal{A}_\lambda(O_L) := \mathcal{A}_\lambda(L) \cap \mathcal{A}(I, O_L)$ and its O_L-dual $\mathcal{D}_\lambda(O_L)$. If λ is arithmetic, then we can also define $V_\lambda(O_L) := V_\lambda(L) \cap \mathcal{A}(I, O_L)$. From the definition of the $*$-action, it is clear that these O_L-submodules are stable under the action of Δ^+. With regard to formula (15), this gives important information about the p-divisibility of the usual left action. The $*$-action can then be viewed as the optimal normalization of the usual left action on $\mathbb{V}_{\lambda^{\mathrm{alg}}}(L)$ that preserves the integrality.

3.3. *The locally analytic* BGG-*resolution.* The fact that there should be a locally analytic version of the BGG-resolution is an outcome of a conversation with M. Harris.

3.3.1. Let L be a finite extension of F and let λ be an L-valued arithmetic weight. The purpose of this paragraph is to define a bounded complex of Fréchet spaces $C^\bullet_\lambda(L)$ in terms of the spaces of distributions we have defined previously, which gives a resolution of $V^\vee_\lambda(L) := \mathrm{Hom}_{\mathrm{cont}}(\mathbb{V}_\lambda(L), L)$. Such a resolution is nowadays well known as the Berstein-Gelfand-Gelfand complex when one replaces respectively *distributions* by *Verma modules* and $V^\vee_\lambda(L)$ by $\mathbb{V}^\vee_\lambda(L)$. We just have to adapt the usual theory to the locally L-analytic context.

3.3.2. Let X be a p-adic space as in Section 3.2.1. We have defined the space $\mathcal{A}(X, L)$ of locally L-analytic functions on X. More generally, we denote by $\mathcal{A}^i(X, L)$ the space of locally L-analytic differential i-forms on X. Then $\mathcal{A}^0(X, L) = \mathcal{A}(X, L)$. We also denote by $LC(X, L)$ the space of locally constant L-valued functions on X. We have exterior differential maps d_i from $\mathcal{A}^i(X, L)$ to $\mathcal{A}^{i+1}(X, L)$. Clearly, $LC(X, L)$ is the kernel of d_0. More generally, it is easy to check that we have a locally L-analytic version of the Poincaré lemma.

Lemma 3.3.3. *Let d be the dimension of X. Then the following sequence is exact*:

$$0 \to LC(X, L) \to \mathcal{A}(X, L) \xrightarrow{d_0} \mathcal{A}^1(X, L) \xrightarrow{d_1} \mathcal{A}^2(X, L) \xrightarrow{d_2} \cdots \to \mathcal{A}^d(X, L) \to 0.$$

Proof. The corresponding Poincaré lemma for the sheaf of locally analytic differential forms can be proved in the same way as in the classical case of real analytic differential forms. The local analyticity condition is important here since integration may decrease the radius of convergence (on closed disks) of

p-adic power series. The exact sequence now follows from the Poincaré lemma since a p-adic space is completely discontinuous. The details are left to the reader and can probably be found in the literature. $\square$

3.3.4. We will apply this lemma when $X = \Omega_0 = B^-(\mathbb{Z}_p) \cap I \backslash I$. In that case, we have a natural action of I on $\mathcal{A}^i(\Omega_0, L)$. If $\omega \in \mathcal{A}^i(\Omega_0, L)$, $g \in I$ and $X_1, \dots, X_i$ belong to the tangent space of Ω_0 at $x \in \Omega_0$, then

$$(g.\omega)(x)(X_1, \dots, X_i) := \omega(x * g)(X_i * g, \dots, X_i * g),$$

where $X \mapsto X * x$ is the map between the tangent spaces $T_{\Omega_0,x}$ and $T_{\Omega_0,x*g}$ of Ω_0 at respectively x and $x * g$ induced by the map $x \mapsto x * g$. It is then straightforward to check that the differential maps d^i are equivariant for the action of I

3.3.5. *Analytic induction of B^--representations.* For any finite-dimensional analytic L-representation V of $B^-(\mathbb{Z}_p) \cap I$, we denote by $\mathcal{A}(V)$ the space of locally analytic functions f from I to V such that $f(b^- g) = b^-.f(g)$ for any $g \in B^-(\mathbb{Z}_p) \cap I$. Let $\mathfrak{g}$ and $\mathfrak{b}^-$ be the F-Lie algebras of G and B^- respectively. They are respectively equipped with the adjoint action of $G(F)$ and $B^-(F)$. We can therefore consider $\mathfrak{g}/\mathfrak{b}^-$ as a representation of $B^-(\mathbb{Z}_p) \cap I$. This action is F-algebraic.

LEMMA 3.3.6. *For any integer i between 0 and d, we have a canonical I-equivariant isomorphism*

$$\mathcal{A}^i(\Omega_0, L) \cong \mathcal{A}(\bigwedge^i (\mathfrak{g}/\mathfrak{b}^-)^* \otimes_F L),$$

where $*$ stands for F-dual.

Proof. Let $\omega \in \mathcal{A}^i(\Omega_0, L)$. We consider the function f_ω on I taking values in $\bigwedge^i (\mathfrak{g}/\mathfrak{b}^-)^* \otimes_F L$ defined by

$$f_\omega(g)(X_1 \wedge \cdots \wedge X_i) = \omega([g])(X_1 * g, \dots, X_i * g),$$

where we have identified $T_{\Omega_0,[id]}$ with $\mathfrak{g}/\mathfrak{b}^-$. It is now easy to check that the map $\omega \mapsto f_\omega$ is I-equivariant and that it defines an isomorphism. $\square$

3.3.7. For simplicity, we now assume that λ is an algebraic dominant weight. Let $\mathcal{A}^i_\lambda := \mathcal{A}^i(\Omega_0, F) \otimes_F \mathbb{V}_\lambda(L)$. Tensoring the exact sequence of Lemma 3.3.3 for $X = \Omega_0$ by $\mathbb{V}_\lambda(L)$ gives the I-equivariant exact sequence

$$0 \to \mathbb{V}_{\lambda^{\mathrm{alg}}}(L) \to \mathcal{A}^0_\lambda \stackrel{d_0}{\to} \mathcal{A}^1_\lambda \stackrel{d_1}{\to} \mathcal{A}^2_\lambda \stackrel{d_2}{\to} \cdots \to \mathcal{A}^d_\lambda \to 0. \tag{17}$$

On the other hand, we have the I-equivariant isomorphism

$$\begin{aligned} \mathcal{A}^i_\lambda &= \mathcal{A}^i(\Omega_0, L) \otimes_F \mathbb{V}_\lambda(L) \\ &\cong \mathcal{A}(\wedge^i (\mathfrak{g}/\mathfrak{b}^-)^*) \otimes \mathbb{V}_\lambda \cong \mathcal{A}(\wedge^i (\mathfrak{g}/\mathfrak{b}^-)^* \otimes \mathbb{V}_\lambda(L)). \end{aligned} \tag{18}$$

Now, we remark that as an algebraic representation of B^-, we have a stable filtration of $(\mathfrak{g}/\mathfrak{b}^-)^* \otimes_F \mathbb{V}_\lambda(L)$,

$$0 = F_0 \subset F_1 \subset F_2 \subset \cdots \subset F_r = (\mathfrak{g}/\mathfrak{b}^-)^* \otimes_F \mathbb{V}_\lambda(L),$$

such that for all $j \geq 1$, F_j/F_{j-1} is one-dimensional over L with action of B^- given by some algebraic character ξ_j. We denote by $S_i(\lambda)$ the set of these characters. By the isomorphism (18), the above filtration induces a stable filtration on $\mathcal{A}^i_\lambda$ with graded pieces isomorphic to $\mathcal{A}_{\xi_j}(L)$ for $\xi_j \in S_i(\lambda)$.

3.3.8. *Infinitesimal characters.* Recall that $\mathfrak{t} := \text{Lie } T$. Harish-Chandra has defined a homomorphism ϑ from $Z(\mathfrak{g})$, the center of the universal algebra of $\mathfrak{g}$, into $U(\mathfrak{t})$, the universal algebra of $\mathfrak{t}$. Recall that the natural action of the Weyl group W_G on $\mathfrak{t}$ induces an automorphism of $U(\mathfrak{t})$. Then the Harish-Chandra homomorphism ϑ induces an isomorphism

$$Z(\mathfrak{g}) \cong U(\mathfrak{t})^{W_G}.$$

For a given algebraic character ξ of T, let $d\xi$ be the corresponding character of $U(\mathfrak{t})$. We set $\chi_\xi := d\xi \circ \vartheta$. For any irreducible algebraic representation W of $U(\mathfrak{g})$, the induced action of $Z(\mathfrak{g})$ on W is given by a character χ_W, called the infinitesimal character of W. For any representation space W of $U(\mathfrak{g})$ and any character χ of $Z(\mathfrak{g})$, we denote by W_χ the χ-generalized eigenspace of W. Similarly, if now W is a locally analytic representation of I, then differentiation yields an action of $U(\mathfrak{g})$ on W. It can be easily seen that W_χ is stable under the I-action and that the functor $W \mapsto W_\chi$ is exact. It is well known that we have $\chi_{\mathbb{V}_\lambda} = \chi_\lambda$. Similarly, one can remark that for any character $\lambda \in X^*(T)$, the infinitesimal character of $\mathcal{A}_\lambda(L)$ is given by χ_λ.

3.3.9. *Construction of the locally analytic* BGG *resolution.* Let us write for short $\mathcal{A}^i_{\lambda,\chi_\lambda} := (\mathcal{A}^i_\lambda)_{\chi_\lambda}$. By applying the exact functor $W \mapsto W_\chi$ to the exact sequence (17), we have the exact sequence

$$0 \to V_\lambda(L) \to \mathcal{A}^0_{\lambda,\chi_\lambda} \overset{d_0}{\to} \mathcal{A}^1_{\lambda,\chi_\lambda} \overset{d_1}{\to} \mathcal{A}^2_{\lambda,\chi_\lambda} \overset{d_2}{\to} \cdots \to \mathcal{A}^d_{\lambda,\chi_\lambda} \to 0. \tag{19}$$

Now we recall the following well-known fact; for example, see [BGG75]. Let $\xi \in X^*(T)$. Then $\xi \in S_i(\lambda)$ with $\chi_\xi = \chi_\lambda$ if and only if there exist $w \in W$ of length i such that $\xi = w * \lambda$. Moreover, this character appears with multiplicity one. Using the filtration of $\mathcal{A}^i_\lambda$ and the fact recalled above, we deduce that we have a filtration of I-modules on $\mathcal{A}^i_{\lambda,\chi_\lambda}$ such that the corresponding graded object is isomorphic to

$$(\mathcal{A}^i_{\lambda,\chi_\lambda})^{gr} \cong \bigoplus_{w|l(w)=i} \mathcal{A}_{w*\lambda}(L). \tag{20}$$

Now, we remark that since $\mathfrak{g}$ is reductive, we have an isomorphism of $\mathfrak{g}$-modules $\mathcal{A}^i_{\lambda,\chi_\lambda} \cong (\mathcal{A}^i_{\lambda,\chi_\lambda})^{gr}$. Since these spaces are locally analytic representation, this isomorphism is left equivariant for the action of a neighborhood of the identity in I. Since such a subgroup is a finite index in I and that L is characteristic zero, this isomorphism is moreover an isomorphism of I-modules.

We write $C^i_\lambda(L)$ for the continuous L-dual of $\mathcal{A}^i_{\lambda,\chi_\lambda}$. Then we have proved the following theorem.

THEOREM 3.3.10. *Let λ be an arithmetic weight of level p^m. There exists a long exact sequence of right I_m-modules*

$$\cdots \to C^{i+1}_\lambda(L) \xrightarrow{d_i} C^i_\lambda(L) \to \cdots \to C^0_\lambda(L) \to V^\vee_\lambda(L) \to 0, \tag{21}$$

where for each q we have

$$C^i_\lambda(L) = \bigoplus_{\substack{w\in\mathcal{W}_G,\\ l(w)=i}} \mathcal{D}_{w*\lambda}(L).$$

Proof. For algebraic dominant weight, this follows from dualizing the exact sequence (19) and the isomorphisms (20). In the general case, it suffices to remark that for any locally algebraic weight $\mu = \mu^{\mathrm{alg}}\varepsilon$, $f \mapsto f_\varepsilon$ (where f_ε is defined as in formula (14)) induces an isomorphism between $\mathcal{A}_{\mu^{\mathrm{alg}}}(L)$ and $\mathcal{A}_\mu(L)$ and between $W_{\mu^{\mathrm{alg}}}(L)$ and $W_\mu(L)$ when μ^{alg} is dominant. □

3.3.11. *Remark.* It could be proved that the maps d_q are defined as follows. Let $i \geq 1$ and $w \in \mathcal{W}_G$ of length i. Let α be a simple root. Assume that $l(s_\alpha w) = i+1$. Then $w^{-1}(\alpha) > 0$, and therefore $w(\lambda^{\mathrm{alg}})(H_\alpha) = \lambda^{\mathrm{alg}}(H_{w^{-1}(\alpha)}) \geq 0$, since λ^{alg} is dominant. We deduce that

$$w * \lambda^{\mathrm{alg}}(H_\alpha) = w(\lambda^{\mathrm{alg}})(H_\alpha) + (w(\rho)-\rho)(H_\alpha) \geq -1,$$

and we therefore have a map $\Theta_\alpha : \mathcal{A}_{w*\lambda}(L) \to \mathcal{A}_{s_\alpha w*\lambda}(L)$ defined by Proposition 3.2.11. Summing over the simple roots α satisfying $l(s_\alpha w) = i+1$ and then over the w of length i, we get after dualizing the map

$$d_i : \mathcal{C}^{i+1}_\lambda(L) \to \mathcal{C}^i_\lambda(L).$$

The proof of this description of these differential maps is left to the reader since it is not going to be used anywhere in this paper, although it would imply the following proposition for which we have a shorter and more conceptual proof. Let us introduce some notation first. For each q, we have a decomposition $d_q = \sum_{w,w'} d_{w,w'}$, where $d_{w,w'}$ is the map from $\mathcal{D}_{w*\lambda}(L)$ to $\mathcal{D}_{w'*\lambda}(L)$ induced by d_q where the sum is over w, w' with $l(w) = q+1$ and $l(w') = q$.

PROPOSITION 3.3.12. *Let $w, w' \in \mathcal{W}_G$ be such that $l(w) = l(w')+1$. Then we have*

$$d_{w,w'}(v * t) = (\xi(t)^{w'*\lambda^{\mathrm{alg}} - w*\lambda^{\mathrm{alg}}}).d_{w,w'}(v) * t$$

for each $t \in T^+$ and $v \in \mathcal{D}_{w*\lambda}(L)$.

Proof. For any arithmetic weight λ, let us denote by $\mathcal{D}^G_\lambda(L)$ the continuous dual of the locally analytic induction $\mathcal{A}^G_\lambda(L) := \left(\mathrm{ind}_{B^-(\overline{\mathbb{Q}}_p)}^{G(\mathbb{Q}_p)} L(\lambda)\right)^{\mathrm{an}}$ defined as the space of locally analytic L-valued function of $G(\mathbb{Q}_p)$ such that

$$f(n^-tg) = \lambda(t)f(g) \qquad \forall g \in G(\mathbb{Q}_p),\ t \in T(\mathbb{Q}_p),\ n^- \in N^-(\mathbb{Q}_p).$$

Here we have extended $\lambda = \lambda^{\mathrm{alg}}\varepsilon$ to a character of $T(\mathbb{Q}_p)$ by putting

$$\lambda(t) := \lambda^{\mathrm{alg}}(t)\varepsilon(t\xi(t)^{-1}) \qquad \forall t \in T(\mathbb{Q}_p).$$

Rewriting the construction of the locally analytic BGG complex with $B^-(\mathbb{Q}_p)\backslash G(\mathbb{Q}_p)$ in place of $B^-(\mathbb{Z}_p) \cap I\backslash I$ provides a resolution of $V_\lambda(L)^\vee$ as a $G(\mathbb{Q}_p)$-representation in which we replace the $\mathcal{D}_{w*\lambda}(L)$'s by the $\mathcal{D}^G_{w*\lambda}(L)$'s. The maps of the complex are then $G(\mathbb{Q}_p)$-equivariant for the action induced by the usual right translation in the argument of the locally analytic functions in $\mathcal{A}^G_{w*\lambda}(L)$. Now the restriction maps $\mathcal{A}^G_{w*\lambda}(L) \to \mathcal{A}_{w*\lambda}(L)$ induce a Δ^+-equivariant (for the $*$-action) inclusion of complexes

$$C^{G,q}_\lambda(L) = \bigoplus_{\substack{w\in\mathcal{W}_G,\\ lg(w)=q}} \mathcal{D}^G_{w*\lambda}(L) \hookrightarrow C^q_\lambda(L) = \bigoplus_{\substack{w\in\mathcal{W}_G,\\ lg(w)=q}} \mathcal{D}_{w*\lambda}(L),$$

which are compatible with the $G(\mathbb{Q}_p)$-equivariant maps $d_{w,w'}$. Since we have the following relation between the $*$-action and the usual right translation action on $\mathcal{A}^G_{w*\lambda}(L)$:

$$t * f = \xi(t)^{-w*\lambda} t \cdot f \qquad \forall t \in T^+ \text{ and } \forall f \in \mathcal{A}^G_{w*\lambda}(L),$$

we deduce that we have the relation of the proposition from the fact that the maps $d_{w,w'}$ are $G(\mathbb{Q}_p)$-equivariant for the usual right translation action. □

3.4. *Analytic variation and weight spaces.* We explain in this section how the spaces $\mathcal{A}_\lambda(L)$ can be interpolated when λ varies in the space of continuous $\overline{\mathbb{Q}}_p^\times$-valued characters of $T(\mathbb{Z}_p)$. We first recall the rigid analytic structure of this space.

3.4.1. *Open and closed disks.* Let $a \in \mathbb{Q}_p$ and let r be a rational power of p. We denote by $B^\circ_{a,r}$ (resp. $B_{a,r}$) the open unit disk (resp. the closed unit disk) of $\overline{\mathbb{Q}}_p$ of center a and radius r. These spaces are rigid analytic spaces defined over $\mathbb{Q}_p$ in the sense of Tate. These closed disks are in fact affinoid domains. The ring of analytic function on $B_{a,r}$ is given by the Tate algebra

$$\mathcal{O}(B_{a,r}) := \left\{ \sum_{n=0}^\infty a_n(z-a)^n \middle|\ \lim_{n\to\infty} |a_n|r^n = 0 \right\}.$$

The rigid analytic structure of $B^\circ_{a,r}$ is obtained by taking the following admissible covering of it by closed disks:

$$B_{a,r} = \bigcup_{r_n<r} B_{a,r_n},$$

where r_n is any sequence of rational powers of p converging to r.

3.4.2. *Weight spaces.* For any finitely generated $\mathbb{Z}_p$-module S, we can give a rigid analytic structure to $\mathrm{Hom}_{\mathrm{cont}}(S, \overline{\mathbb{Q}}_p^\times)$. For any algebraic extension L of $\mathbb{Q}_p$, let us write $\mathfrak{X}_S(L) := \mathrm{Hom}_{\mathrm{cont}}(S, L^\times)$. We write S_{tor} for the torsion part of S, and we let $S \cong S_{\mathrm{tor}} \times S_{\mathrm{free}}$ be a decomposition with S_{free} a free $\mathbb{Z}_p$-submodule of S. Let r be the rank of S_{free} over $\mathbb{Z}_p$. The choice of a $\mathbb{Z}_p$-basis of S_{free} therefore gives an isomorphism

$$\mathfrak{X}_S(\overline{\mathbb{Q}}_p) \cong S_{\mathrm{tor}}^* \times (B_{1,1}(\overline{\mathbb{Q}}_p)^\circ)^r \tag{22}$$

with $S_{\mathrm{tor}}^* := \mathrm{Hom}_{gp}(S_{\mathrm{tor}}, \overline{\mathbb{Q}}_p^*)$. We will fix such a basis once and for all. Let K be a finite extension of $\mathbb{Q}_p$ containing the values of all the characters in S_{tor}^*. Then for any finite extension L of K, we have $\mathfrak{X}_S(L) = S_{\mathrm{tor}}^* \times (B_{1,1}(L)^\circ)^r$. This gives $\mathfrak{X}_S(\overline{\mathbb{Q}}_p)$ a rigid analytic structure over $\mathbb{Q}_p$. Moreover, $\mathfrak{X}_S$ is isomorphic over K to a disjoint union of finitely many open unit disks of dimension r.

3.4.3. *Remark.* Notice that if S' is a $\mathbb{Z}_p$-submodule of S, then $\mathfrak{X}_{S/S'}$ can be identified to the Zariski closure of the characters of $\mathfrak{X}_S(\overline{\mathbb{Q}}_p)$ which are trivial on S'. This observation will be useful in the next section.

3.4.4. *Rigid analytic neighborhoods of S.* Let $x_1, \dots, x_r$ be the system of coordinates of S_{free} attached to the chosen basis giving the identification (22). For each integer n, consider the affinoid

$$S_n^{\mathrm{rig}} := S_{\mathrm{tor}} \times \sqcup_{(a_1,\dots,a_r)} B_{a_1,p^{-n}} \times \cdots \times B_{a_r,p^{-n}}, \tag{23}$$

where $(a_1, \dots, a_r)$ runs in a set of representatives of $(\mathbb{Z}/p^n\mathbb{Z})^r$. Using the system of coordinates $x_1, \dots, x_r$, we can embed S into $S_n^{\mathrm{rig}}(\overline{\mathbb{Q}}_p)$ via the identification $S = S_n^{\mathrm{rig}}(\mathbb{Q}_p)$ for all nonnegative integer n. So S_n can be seen as a rigid analytic neighborhood of S, and the ring of rigid analytic functions on S_n is isomorphic to the set of n-locally analytic functions on S. In other words, for any finite extension L of $\mathbb{Q}_p$, we have

$$\mathcal{A}_n(S, L) = \mathcal{O}(S_n^{\mathrm{rig}}{}_{/L}).$$

The construction above can be made for any p-adic space X in the sense of our definition of Section 3.2.1. We have considered here the compact case in which our rigid analytic neighborhoods are actually affinoid neighborhoods.

3.4.5. Let $\mathfrak{U} \subset \mathfrak{X}_S$ be a rigid analytic subspace. For any $s \in S$, we denote by $\langle s \rangle_{\mathfrak{U}}$ the function on $\mathfrak{U}(\overline{\mathbb{Q}}_p)$ defined by $\langle s \rangle_{\mathfrak{U}}(\lambda) = \lambda(s)$ for any $\lambda \in \mathfrak{U}(\overline{\mathbb{Q}}_p) \subset \mathrm{Hom}_{\mathrm{cont}}(S, \overline{\mathbb{Q}}_p^\times)$. Then $\langle s \rangle_{\mathfrak{U}}$ is an analytic function on S, and the map $s \mapsto \langle s \rangle_{\mathfrak{U}}$ defines continuous injective homomorphism from S into $\mathcal{O}(\mathfrak{U})^\times$.

The following lemma is the essential ingredient for the construction of analytic families of locally analytic induction spaces.

LEMMA 3.4.6. *For any affinoid subdomain $\mathfrak{U} \subset \mathfrak{X}_S$ defined over L, there exists a smallest integer $n(\mathfrak{U})$ such that any element $\lambda \in \mathfrak{U}(\mathbb{Q}_p)$ defined over a* *finite extension $L/\mathbb{Q}_p$ is $n(\mathfrak{U})$-locally L-analytic. Moreover, the map $(\lambda, s) \mapsto \lambda(s)$ induces a rigid analytic map $\mathfrak{U} \times S_{n(\mathfrak{U})} \to B_{1,1}$ defined over L.*

Proof. Since it is possible to cover $\mathfrak{U}$ with a finite number of closed disks, we may assume that $\mathfrak{U}$ is a closed disk. So let $\mathfrak{U} = B_{\lambda_0,R}$ for some $R < 1$. We can even assume that λ_0 is the trivial character. Let $u_1, \ldots, u_d$ be a $\mathbb{Z}_p$-basis of S_{free} defining the identification (22) and for any $\lambda \in \mathfrak{X}_S(\overline{\mathbb{Q}}_p)$. Let us put $\lambda_i = \lambda(u_i)$. Then we have $\lambda \in \mathfrak{U}(\overline{\mathbb{Q}}_p)$ if and only if $|\lambda_i - 1| \leq R < 1$. Now fix some integer $n = n_R$ depending only on R such that $|\lambda_i^{p^n} - 1| \leq p^{-1}$ for all $i = 1, \ldots, r$. Now if $s \in p^n S_{\text{free}}$, then we have $\lambda(s) = \prod_i \lambda_i^{s_i} = \prod_i (\lambda_i^{p^n})^{s_i/p^n}$, where the s_i's belong to $p^n\mathbb{Z}_p$ and are the coordinates of s with respect to the chosen basis $(u_1, \ldots, u_d)$. Therefore, for all $s \in p^n S_{\text{free}}$ and $\lambda \in \mathfrak{U}(\overline{\mathbb{Q}}_p)$, we have

$$\lambda(s) = \prod_i \left(\sum_{n_i=1}^{\infty} \binom{s_i/p^n}{n_i} \right) (\lambda_i^{p^n} - 1)^{n_i}, \tag{24}$$

where we have denoted $y \mapsto \binom{y}{n}$ for the well-known function on $\mathbb{Z}_p$ defined by

$$\frac{y(y-1)(y-2)\cdots(y-n+1)}{n!} \qquad \forall y \in \mathbb{Z}_p.$$

The series (24) is convergent for $(s_1, \ldots, s_r) \times (\lambda_1, \ldots, \lambda_r) \in B_{0,p^{-n}} \times B_{1,R}$ by our choice of n and thanks to the well-known Lemma 3.4.7 below. The claims of our lemma follow from this observation. $\square$

LEMMA 3.4.7. *The series $(1+z)^s = \sum_n \binom{s}{n} z^n$ converges for $z, s \in \mathbb{C}_p$ such that $|z| \leq p^{-1}$ and $|s|_p \leq 1$.*

Proof. This follows from an elementary evaluation of the p-adic valuation of the binomial coefficients and this is well known. $\square$

3.4.8. *Analytic families of analytic inductions.* We will consider now the case $S = T(\mathbb{Z}_p)$ and denote by $\mathfrak{X}_T$ the corresponding weight space.

Let $\mathfrak{U} \subset \mathfrak{X}_T$ be an affinoid subdomain defined over a finite extension $L/\mathbb{Q}_p$, and choose an integer $n \geq 0$. We define $\mathcal{A}_{\mathfrak{U},n}$ as the set of rigid analytic functions f on $\mathfrak{U} \times (\Omega_1)_n^{\text{rig}}{}_{/L}$ such that

$$f(\lambda, [tn]) = \lambda(t) f(\lambda, [n]) \tag{25}$$

for all $\lambda \in \mathfrak{U}(\overline{\mathbb{Q}}_p)$, $t \in T(\mathbb{Z}_p)_n^{\text{rig}}(\overline{\mathbb{Q}}_p)$ and $n \in N(\mathbb{Z}_p)_n^{\text{rig}}(\overline{\mathbb{Q}}_p)$. This space might be trivial. However, we have the following lemma.

LEMMA 3.4.9. *Let $\mathfrak{U} \subset \mathfrak{X}_T$ be an affinoid subdomain and n an integer such that $n \geq n(\mathfrak{U})$. Then we have a canonical bicontinuous isomorphism*

$$\mathcal{A}_{\mathfrak{U},n} \cong \mathcal{O}(\mathfrak{U}) \hat{\otimes}_L \mathcal{A}_n(N(\mathbb{Z}_p), L).$$

In particular, $\mathcal{A}_{\mathfrak{U},n}$ is a nontrivial $\mathcal{O}(\mathfrak{U})$-orthonormalizable Banach space.

Proof. The inclusion $\mathfrak{U} \times N(\mathbb{Z}_p)_n^{\mathrm{rig}} \hookrightarrow \mathfrak{U} \times (\Omega_1)_n^{\mathrm{rig}}$ induces a continuous map

$$\mathcal{A}_{\mathfrak{U},n} \to \mathcal{O}(\mathfrak{U} \times N(\mathbb{Z}_p)_n^{\mathrm{rig}}) \cong \mathcal{O}(\mathfrak{U}) \hat{\otimes}_K \mathcal{A}_n(N(\mathbb{Z}_p), K). \tag{26}$$

By relation (25), it is straightforward to see that this map is injective. To prove the surjectivity, it suffices to show that any pure tensor $\phi \otimes f$ is in the image. So consider the function

$$\psi(\lambda, [tn]) := \phi(\lambda)\lambda(t)f(n)$$

defined for any $(\lambda, t, n) \in (\mathfrak{U} \times T(\mathbb{Z}_p)_n^{\mathrm{rig}} \times N(\mathbb{Z}_p)_n^{\mathrm{rig}})(\overline{\mathbb{Q}}_p)$. By our assumption $n \geq n(\mathfrak{U})$ and Lemma 3.4.6, this function is clearly rigid analytic and its image, by the restriction map (26), is $\phi \otimes f$. Therefore (26) is an isomorphism, and since it is a surjective continuous between Banach spaces, its inverse is also continuous by the open mapping theorem. This proves our claim. □

COROLLARY 3.4.10. *For any $n \geq n(\mathfrak{U})$, the inclusion map $\mathcal{A}_{\mathfrak{U},n} \subset \mathcal{A}_{\mathfrak{U},n+1}$ is completely continuous.*

Proof. These inclusion maps are induced by the inclusions

$$\mathcal{A}_n(N(\mathbb{Z}_p), K) \subset \mathcal{A}_{n+1}(N(\mathbb{Z}_p), K)$$

which are completely continuous. Our claim follows from this observation and Lemma 3.2.2. □

We put now

$$\mathcal{A}_{\mathfrak{U}} := \bigcup_{n \geq n(\mathfrak{U})} \mathcal{A}_{\mathfrak{U},n}$$

and define the continuous $\mathcal{O}(\mathfrak{U})$-dual $\mathcal{D}'_{n,\mathfrak{U}} := \mathrm{Hom}_{\mathcal{O}(\mathfrak{U})}(\mathcal{A}_{n,\mathfrak{U}}, \mathcal{O}(\mathfrak{U}))$ from the previous lemma we have a canonical injective map

$$i_n : \mathcal{O}(\mathfrak{U}) \hat{\otimes}_L \mathcal{D}_n(N(\mathbb{Z}_p), L) \to \mathcal{D}'_{n,\mathfrak{U}},$$

and we write $\mathcal{D}_{n,\mathfrak{U}} \subset \mathcal{D}'_{\mathfrak{U}}$ for the image of i_n. Then we denote by $\mathcal{D}_{\mathfrak{U}}$ the projective limit over n of the $\mathcal{D}_{n,\mathfrak{U}}$'s. It follows easily from the definitions that $\mathcal{D}_{n,\mathfrak{U}}$ is an orthonormalizable $\mathcal{O}(\mathfrak{U})$-module and that $\mathcal{D}_{\mathfrak{U}}$ is a $\mathcal{O}(\mathfrak{U})$-projective compact Fréchet space.

3.4.11. *$*$-Action of* Δ^+. We can define the $*$-action on the spaces $\mathcal{A}_{\mathfrak{U},n}$, as in the case where $\mathfrak{U}$ is reduced to a singleton. For this, we remark that the right $*$-action of Δ^+ on Ω_1, being algebraic, extends into an action of Δ^+ on $(\Omega_1)_n^{\mathrm{rig}}$ for all $n \geq 0$. Similarly, we also remark that the left action of $T(\mathbb{Z}_p)$ on Ω_1, defined in Section 3.2.6, can be extended into a left action of $T(\mathbb{Z}_p)_n^{\mathrm{rig}}$ on $(\Omega_1)_n^{\mathrm{rig}}$. If $t \in T(\mathbb{Z}_p)_n^{\mathrm{rig}}$ and $f \in \mathcal{O}((\Omega_1)_n^{\mathrm{rig}})$, we write $t.f$ for the action of

f obtained by left translation on the argument of f. Now, as in Section 3.2.6, we have the identification

$$\mathcal{A}_{n,\mathfrak{U}} \cong \{f \in \mathcal{O}((\Omega_1)_n^{\mathrm{rig}}))\hat{\otimes}\mathcal{O}(\mathfrak{U})|\ t.f \otimes 1 = f \otimes \langle t\rangle_{\mathfrak{U}} \forall t \in T(\mathbb{Z}_p)_n^{\mathrm{rig}}\}.$$

Since the $*$-action of Δ^+ commutes with the left action of $T(\mathbb{Z}_p)_n^{\mathrm{rig}}$, it follows that we have a left action of Δ^+ on $\mathcal{A}_{n,\mathfrak{U}}$. We deduce that we have a left action of Δ^+ on $\mathcal{A}_{\mathfrak{U}}$ and a right action on $\mathcal{D}_{\mathfrak{U}}$.

3.4.12. *Remark.* Notice that if we define $\langle\cdot\rangle_{\mathfrak{U}}$ to be the $\mathcal{O}(\mathfrak{U})$-valued character of $T(\mathbb{Z}_p)$ given by $t \mapsto \langle t\rangle_{\mathfrak{U}}$, where $\langle t\rangle_{\mathfrak{U}} \in \mathcal{O}(\mathfrak{U})^\times$ is the analytic function on $\mathfrak{U}$ defined by $\lambda \mapsto \lambda(t)$, then $\mathcal{A}_{\mathfrak{U},n}$ can be seen as the n-locally $\mathcal{O}(\mathfrak{U})$-analytic induction of $\langle\cdot\rangle_{\mathfrak{U}}$. When $\mathfrak{U}$ is reduced to a single point $\mathfrak{U} = \{\lambda\}$, we recover the definition of $\mathcal{A}_{\lambda,n}(L)$ for a n-locally analytic character $\lambda \in \mathfrak{X}(L)$. In particular, we have the following lemma.

Lemma 3.4.13. *Let L be a finite extension of K and $\lambda \in \mathfrak{U}(L)$. Then we have the canonical isomorphism of Δ^+-modules*

$$\mathcal{A}_{\mathfrak{U}} \otimes_\lambda L \cong \mathcal{A}_\lambda(L) \text{ and } \mathcal{D}_{\mathfrak{U}} \otimes_\lambda L \cong \mathcal{D}_\lambda(L).$$

Proof. This follows from the definitions and Lemma 3.4.9. □

We also have the following very important lemma.

Lemma 3.4.14. *If $\delta \in \Delta^{++}$, then the $*$-action of δ defines a compact operator on the $\mathcal{O}(\mathfrak{U})$-projective compact Fréchet space $\mathcal{D}_{\mathfrak{U}}$.*

Proof. Assume that K is the field of definition of $\mathfrak{U}$. By Lemma 3.4.9, we have the bi-continuous isomorphism

$$\mathcal{D}_{\mathfrak{U}} \cong \mathcal{O}(\mathfrak{U})\hat{\otimes}_K \mathcal{D}(N(\mathbb{Z}_p), K).$$

From this remark, it is easy to see that the proof of our claim follows exactly the same lines as the proof of Lemma 3.2.8. The details are left to the reader. □

4. **Overconvergent finite slope cohomology**

In this section, $G_{/\mathbb{Q}}$ is a reductive group as in the first section. We fix a prime p and we assume that $G_{/\mathbb{Q}_p}$ is quasi-split as in the previous section.

4.1. *Hecke algebras and finite slope representations.* In this subsection, we define the notion of finite slope representation for the group G. We start by defining some Hecke algebras.

4.1.1. *The Hida-Hecke algebra.* We will freely use the notation of Section 3. For all positive integers m, let $\Delta_m^- := (\Delta_m^+)^{-1}$ and $\Delta_m^{--} := (\Delta_m^{++})^{-1}$. As usual we will drop m from the notation when it is equal to 1. We similarly define T^- and T^{--}. The spaces of distributions $\mathcal{D}_\lambda(L)$ and their quotients

$\mathbb{V}^{\vee}_{\lambda^{\rm alg}}(L)$ that we have defined in Section 3 are equipped with the right $*$-action of Δ^+. We will consider them now as left Δ^--modules since we have made the choice to consider the adelic action on the left to define Hecke operators.

Let m be a positive integer and let $C_c^\infty(\Delta_m^-//I_m,\mathbb{Z}_p)$ be the subspace of $\mathbb{Z}_p$-valued locally constant functions with compact support in Δ_m^- which are bi-invariant by I_m. This is an algebra for the convolution product defined with the Haar measure on $G(\mathbb{Q}_p)$ such that I_m is of measure 1. For $t\in T^-$, we write $u_t=u_{t,m}=I_mtI_m\in C_c^\infty(\Delta_m^-//I_m,\mathbb{Z}_p)$ for the characteristic function of the double class $I_mtI_m\subset\Delta_m^-$. Then it is well known, and it can be easily checked that

$$u_t.u_{t'}=u_{tt'}$$

for any $t,t'\in T^-$. Therefore, the map $t\mapsto u_t$ defines an algebra homomorphism $\mathbb{Z}_p[T^-/T(\mathbb{Z}_p)]\xrightarrow{\sim}C_c^\infty(\Delta_m^-//I_m,\mathbb{Z}_p)$. Then we put $\mathcal{U}_p=\mathcal{U}_p(G):=\mathbb{Z}_p[T^-/T(\mathbb{Z}_p)]$, and we will identify its elements as $\mathbb{Z}_p$-valued functions with compact support on Δ_m^- which are bi-invariant by I_m for m chosen according to the space on which we will let $\mathcal{U}_p$ act.

4.1.2. *Slope of a character of T^- or of $\mathcal{U}_p$.* The notion of slope that we introduce here is equivalent to a notion introduced by Emerton in [Eme06]. Recall that

$$X^*(T_{/F}):=\mathrm{Hom}_{\text{alg-gp}}(T_{/F},\mathbb{G}_{m/F})$$

and

$$X_*(T_{/F}):=\mathrm{Hom}_{\text{alg-gp}}(\mathbb{G}_{m/F},T),$$

where F is the smallest Galois extension of $\mathbb{Q}_p$ that splits G. Since T is defined over $\mathbb{Q}_p$, we have an action of the Galois group $\mathrm{Gal}(F/\mathbb{Q}_p)$ on $X^*(T_{/F})$ and $X_*(T_{/F})$. Recall also that we have a Galois equivariant duality pairing

$$(-,-):X^*(T_{/F})\otimes X_*(T_{/F})\to\mathbb{Z}$$

such that $\mu\circ\mu^\vee(t)=t^{(\mu,\mu^\vee)}$ for any $\mu\in X^*(T_{/F}),\mu^\vee\in X_*(T_{/F})$ and $t\in\mathbb{G}_m$. Let $X_*(T_{/F})^+$ be the dual cone of the cone generated by the positive roots of $X^*(T)$. Then, by definition, for any $\mu^\vee\in(X_*(T_{/F})^+)^{\mathrm{Gal}(F/\mathbb{Q}_p)}$, we have $\mu^\vee(p)\in T^{--}$.

Let θ now be a $\overline{\mathbb{Q}}_p$-valued character of $\mathcal{U}_p$. If $\theta(u_t)=0$ for some $t\in T^-$, then we say that the slope of θ is infinite and we write $\mu_\theta=\infty$. Otherwise θ induces a homomorphism of monoids from $T^-/T(\mathbb{Z}_p)$ into $\overline{\mathbb{Q}}_p^\times$ and can be easily extended to $T(\mathbb{Q}_p)/T(\mathbb{Z}_p)$. Such a character is said to be of finite slope. Equivalently θ is finite slope if $\theta(u_t)\neq 0$ for at least one $t\in T^{--}$. This can be easily checked since for any $t'\in T^{--}$, there exists a positive integer N such that $t^N=t't''$ with $t''\in T^{--}$.

When θ is of finite slope, the slope of θ is the element μ_θ of

$$X^*(T_{/F})^{\mathrm{Gal}(F/\mathbb{Q}_p)}_{\mathbb{Q}}:=X^*(T_{/F})^{\mathrm{Gal}(F/\mathbb{Q}_p)}\otimes\mathbb{Q}$$

defined by

$$(\mu_\theta, \mu^\vee) = v_p(\theta(u_{\mu^\vee(p)}))$$

for all $\mu^\vee \in (X_*(T_{/F})^+)^{\mathrm{Gal}(F/\mathbb{Q}_p)}$, where v_p denotes the p-adic valuation of $\overline{\mathbb{Q}}_p$ normalized by $v_p(p) = 1$. In particular, we have

$$|(a.\mu_\theta)(t)|_p = |\theta(u_{t^a})|_p$$

for any integer a such that $a.\mu_\theta \in X^*(T_{/F})$ and $t \in T^{--}$. Notice that if θ is integrally valued, then μ_θ belongs to the obtuse positive cone $X^*(T)_{+,\mathbb{Q}}$ generated over $\mathbb{Q}_+$ by the simple roots relative to the pair (B, T). Of course $X^*(T)_{+,\mathbb{Q}} \supset X^*(T)^+_{\mathbb{Q}}$, and the inclusion is strict in general. If $\mu, \mu' \in X^*(T)_{\mathbb{Q}}$, we write $\mu \geq \mu'$ if and only if $\mu - \mu' \in X^*(T)_{+,\mathbb{Q}}$.

Definition 4.1.3. Let λ^{alg} be an algebraic character of $T_{/F}$ and $\mu = \mu_\theta$. This slope will be said to be noncritical with respect to λ^{alg} if $\mu_\theta - \lambda^{\mathrm{alg}} + w * \lambda^{\mathrm{alg}} \notin X^*(T)_{+,\mathbb{Q}}$ for each $w \neq \mathrm{id}$. When λ^{alg} is implicit in the context, we will just say that θ or μ_θ is noncritical.

4.1.4. *Finite slope part of a $\mathcal{U}_p$-module.* Let $L \subset \overline{\mathbb{Q}}_p$ be a finite extension of $\mathbb{Q}_p$ and let V be a (possibly non-Hausdorff) quotient of Banach (or compact Fréchet), as in Section 2, L-vector topological spaces equipped with an action of $\mathcal{U}_p$ such that the action of u_t is completely continuous for any $t \in T^{--}$. For any character θ of T^- in $\overline{\mathbb{Q}}_p^\times$, we denote by $V_{\overline{\mathbb{Q}}_p}[\theta]$ the subspace of $V \otimes_L \overline{\mathbb{Q}}_p$ of vectors v such that for all $t \in T^-$, $(u_t - \theta(u_t))^q.v = 0$ for some integer q. Since the operators u_t commute, and their action is completely continuous on V, the $V_{\overline{\mathbb{Q}}_p}[\theta]$'s are finite-dimensional if θ is of finite slope.

Let $\mu \in X^*(T)_{\mathbb{Q}}^{\mathrm{Gal}(F/\mathbb{Q}_p)}$ and V as above. We put

$$V_{\overline{\mathbb{Q}}_p}^{\leq \mu} := \bigoplus_{\substack{\theta \mid \\ \mu_\theta \leq \mu}} V_{\overline{\mathbb{Q}}_p}[\theta] \subset V_{\overline{\mathbb{Q}}_p}.$$

Then this space is finite-dimensional over $\overline{\mathbb{Q}}_p$. We have $V_{\overline{\mathbb{Q}}_p}^{\leq \mu} = V^{\leq \mu} \otimes_L \overline{\mathbb{Q}}_p$ for $V^{\leq \mu} := V_{\overline{\mathbb{Q}}_p}^{\leq \mu} \cap V$. For any $t \in T^{--}$ and $h \in \mathbb{Q}$, we can define, as in Section 2, $V^{\leq h}$ and set V_{fs} as the inductive limit over h of the $V^{\leq h}$ when $h \to \infty$. Since $\mathcal{U}_p$ is commutative, this space is clearly stable by the action of $\mathcal{U}_p$, and we have

$$V^{\leq h} = \bigoplus_{\substack{\theta \\ v_p(\theta(u_t)) \leq h}} V_{\overline{\mathbb{Q}}_p}[\theta].$$

This implies that the inductive limit of the $V^{\leq \mu}$ for $\mu \in X^*(T)_{\mathbb{Q},+}$ is equal to V_{fs}.

4.1.5. *Global Hecke algebras.* We define the Hecke algebra $\mathcal{H}_p$ by

$$\mathcal{H}_p = \mathcal{H}_p(G) := C_c^\infty(G(\mathbb{A}_f^p), \mathbb{Q}_p) \otimes \mathcal{U}_p \subset C_c^\infty(G(\mathbb{A}_f), \mathbb{Q}_p).$$

To define the algebra structure on $\mathcal{H}_p$, we choose that the product is given by the convolution product for the Haar measure dg on $G(\mathbb{A}_f)$ such that $\mathrm{Meas}(K_{\max}^p.I, dg) = 1$ with $K_{\max}^p$ be the prime to p part of $K_{\max}$ which we defined in Section 1.1.2.

We denote by $\mathcal{H}_p'$ the ideal of $\mathcal{H}_p$ generated by $f = f^p \otimes u_t$ with $f^p \in C_c^\infty(G(\mathbb{A}_f^p), \mathbb{Q}_p)$ and $t \in T^{--}$. For any open compact subgroup $K^p \subset G(\mathbb{A}_f^p)$, we write

$$\mathcal{H}_p(K^p) := C_c^\infty(K^p \backslash G(\mathbb{A}_f^p)/K^p, \mathbb{Q}_p) \otimes \mathcal{U}_p$$

for the subalgebra of $\mathcal{H}_p$ of functions which are bi-invariant by K^p.

If S is a finite set of primes not containing p, then we also consider the Hecke algebra $R_{S,p}$ defined by

$$R_{S,p} := C_c^\infty(K_m^{S\cup\{p\}} \backslash G(\mathbb{A}_f^{S\cup\{p\}}, \mathbb{Z}_p)/K_m^{S\cup\{p\}}) \otimes \mathcal{U}_p,$$

where $K_m^{S\cup\{p\}}$ stands for a maximal compact of $G(\mathbb{A}_f^{S\cup\{p\}})$ which is hyperspecial at every prime $\ell \notin S \cup \{p\}$. It is well known that this algebra is commutative. Moreover, if $K^p \subset K_m$ is hyperspecial away from S, then $R_{S,p}$ can be seen as a subalgebra of the center of $\mathcal{H}_p(K^p)$ via the map $f \mapsto 1_{K_S} \otimes f$, where K_S stands for the open compact subgroup of $\prod_{l\in S} G(\mathbb{Q}_l)$ such that $K^p = K_S.K_m^{S\cup\{p\}}$. Here we have denoted 1_{K_S} as the characteristic function of K_S.

Definition 4.1.6. A $\overline{\mathbb{Q}}_p$-valued character θ of $R_{S,p}$ will be said to be of finite slope if its restriction to $\mathcal{U}_p$ is (i.e., $\theta(u_t) \neq 0$ for all $t \in T^-$).

We further generalize the above definition to admissible representations of $\mathcal{H}_p$.

4.1.7. *Finite slope admissible representations of* $\mathcal{H}_p$. Let (σ, V) be an admissible representation of $\mathcal{H}_p$ defined over a p-adic field E. Recall that this means that for any open compact subgroup $K^p \subset G(\mathbb{A}_f)$, the action of an element of $\mathcal{H}_p(K^p)$ on V defines an endomorphism of finite rank. Since $\mathcal{H}_p$ is the inductive limit of the $\mathcal{H}_p(K^p)$'s, the character map $f \mapsto \mathrm{tr}(\sigma(f))$ is well defined and will be denoted by J_σ. It is a classical fact that J_σ determines σ up to semi-simplification. Assume now that σ is absolutely irreducible. Since $\mathcal{U}_p$ is included in the center of $\mathcal{H}_p$, the action of $\mathcal{U}_p$ on V is then given by a character of degree one. We then say that σ is of finite slope if this character is and if σ^{K^p} contains an O_E-lattice stable by the action of $\mathbb{Z}_p$-valued Hecke operators in $\mathcal{H}_p(K^p)$.

We will say that this representation is of level K^p if the action of $\mathcal{H}_p(K^p)$ is nontrivial. In that case, we write V^{K^p} or σ^{K^p} for the image of $\sigma(1_{K^p})$, and we have an action of $\mathcal{H}_p(K^p)$ on this subspace. It is well known that this representation determines σ entirely. Let S be a finite set of primes such that K^p is maximal hyperspecial away from S. Since $R_{S,p}$ is included in the center of $\mathcal{H}_p(K^p)$, $R_{S,p}$ acts on σ^{K^p} by a character we denote as θ_σ. Then we say that σ is of finite slope if and only θ_σ is. A general admissible representation of $\mathcal{H}_p$ will be said to be of finite slope if all its absolutely irreducible sub-quotients are.

4.1.8. *p-regularized constant terms and parabolic induction.* Let $P \subset G$ be a standard parabolic subgroup (i.e., P contains the fixed minimal parabolic subgroup P_0). In particular, $P_{/\mathbb{Q}_p}$ contains the Borel subgroup B. We fix a Levi decomposition $P = MN$ such that $M_{/\mathbb{Q}_p}$ contains T. Such an M will be called a standard Levi subgroup of G. Then we can consider the Hecke algebras $\mathcal{H}_p(M)$ and $\mathcal{U}_p(M)$ and admissible finite slope representations or characters of them.

For any standard Levi subgroup $M \in \mathcal{L}_G$, recall that $\mathcal{W}^M$ are the elements w of the Weyl group of G such that $w^{-1}(\alpha) > 0$ for all positive roots α for the pair $(B \cap M, T)$. Let $w \in \mathcal{W}^M$. We are going to define a linear map from $\mathcal{H}_p(G)$ into $\mathcal{H}_p(M)$ $f \mapsto f^{\mathrm{reg}}_{M,w}$. To do so, we first define the image of an element $f \in \mathcal{H}'_p$ of the form $f = f^p \otimes u_t$. Notice first that for $w \in \mathcal{W}^M$ and $t \in T^-$, wtw^{-1} belongs to T^-_M, where T^-_M is defined as the set of $t \in T(\mathbb{Q}_p)$ such that $tN(\mathbb{Z}_p) \cap M(Z_p)t^{-1} \subset N(\mathbb{Z}_p)$. This follows from the very definition of $\mathcal{W}^M$ which tells us that $B \cap M = wBw^{-1} \cap M$ for all $w \in \mathcal{W}^M$.

For any $t \in T(\mathbb{Q}_p)$ and $w \in \mathcal{W}_{G_{/F},T_{/F}}$, we also write

$$\varepsilon_{\xi,w}(t) := \xi(t)^{w(\rho_P)+\rho_P}|t^{w^{-1}(\rho_P)+\rho_P}|_p.$$

It is easily checked that $\varepsilon_{\xi,w}$ induces a character of $T(\mathbb{Q}_p)/T(\mathbb{Z}_p)$ taking values in $O_F^\times$. If ξ satisfies the condition (11), then this character is trivial since $w^{-1}(\rho_P) + \rho_P = \sum_{\alpha \in R_P \cap w^{-1}(R_P)} \alpha$. The reason for introducing this normalization factor will become clear in the proof of Proposition 4.6.3.

We then define $f^{\mathrm{reg}}_{M,w}$ by

$$f^{\mathrm{reg}}_{M,w} = \varepsilon_{\xi,w}(t) f^p_M \otimes u_{wtw^{-1},M},$$

where f^p_M stands for the (nonunitary) constant term defined as in Section 1.4.1. By the remark above, $u_{wtw^{-1},M}$ is a well-defined element of $\mathcal{U}_p(M)$. For general f, we extend the definition by linearity. Now if σ^p_f is an irreducible admissible representation of $M(\mathbb{A}^p_f)$ and if $I^G_M(\sigma^p_f)$ is the (nonunitary) parabolic induction, then, as recalled in Section 1.4.1, we have that

$$\mathrm{tr}(f^p; I^G_M(\sigma^p_f)) = \mathrm{tr}(f^p_M; \sigma^p_f).$$

If σ is an irreducible finite slope representation of $\mathcal{H}_p(M)$, then we denote by $I^G_{M,w}$ the admissible finite slope representation defined by

$$I^G_{M,w}(\sigma) = I^G_M(\sigma^p_f) \otimes \theta_{\sigma,M,w},$$

where $\theta_{\sigma,M,w}$ is the character of $\mathcal{U}_p(G)$ defined by

$$u_t \mapsto \theta_\sigma(u_{wtw^{-1},M}).$$

It then follows from the definitions and what we have recalled that we have the character identity

$$J_{I^G_{M,w}(\sigma)}(f) = J_\sigma(f^{\mathrm{reg}}_{M,w}).$$

4.1.9. *p-stabilization of automorphic representations.* The main examples of finite slope representations are obtained as follows. Let $\pi = \pi_f \otimes \pi_\infty$ be an irreducible cohomological automorphic representation of $G(\mathbb{A})$ of weight λ^{alg}. It is defined over a number field, and we can therefore extend the scalar to a p-adic number field L. Then $\mathcal{H}_p$ has a nontrivial action on $\pi_f^{I'_m}$ for some integer m sufficiently large, and there exists some character ε of I_m/I'_m such that $\pi_f^{I'_m} \otimes L(\varepsilon^{-1})$ contains a nontrivial subspace invariant by I_m and over which we therefore have an action of $\mathcal{H}_p$. An irreducible constituent of this space for the action of $\mathcal{H}_p$ is called a p-stabilization of π. It is a standard fact that can be checked using the theory of Jacquet modules that a given representation π has only finitely many p-stabilizations. Notice also that this notion is purely local at p. If such a p-stabilization is of finite slope, it will be called a finite slope automorphic representation of G of weight $\lambda = \lambda^{\mathrm{alg}}\varepsilon$. It is then straightforward to see that it will also appear in the cohomology of $\mathbb{V}_{\lambda^{\mathrm{alg}}}(L)$ with $\lambda = \varepsilon.\lambda^{\mathrm{alg}}$.

4.1.10. *Finite slope character distributions.* Let L be a finite extension of $\mathbb{Q}_p$. We call an L-valued virtual finite slope character distribution J a $\mathbb{Q}_p$-linear map $J:\ \mathcal{H}'_p \to L$ such that there exists a collection of finite slope (absolutely) irreducible representations $\{\sigma_i; i \in \mathbb{Z}_{>0}\}$ and integers $m_i \in \mathbb{Z}$ such that

(i) For all $t \in T^{--}$, $h \in \mathbb{Q}$ and K^p, there are finitely many indexes i such that $m_i \neq 0$, $v_p(\theta_{\sigma_i}(u_t)) \leq h$ and $\sigma_i^{K^p} \neq 0$.

(ii) For all $f \in \mathcal{H}'_p$, we have

$$J(f) = \sum_{i=1}^{\infty} m_i J_{\sigma_i}(f).$$

Notice that the sum in (ii) is convergent thanks to the condition (i). If the m_i are nonnegative, then J is called a finite slope character distribution. We also say that J is an effective finite slope character distribution. In that case, for each open compact K^p, we can consider the space $V_J(K^p)$ as the completion of $\oplus_i (V_{\sigma_i}^{K^p})^{\oplus m_i}$ for the norm defined by $\|\sum_i v_i\| = \mathrm{Sup}_i \|v_i\|$. This defines a

p-adic Banach space over $\mathbb{C}_p$ over which $\mathcal{H}_p(K^p)$ acts continuously and such that the action of the elements of $\mathcal{H}'_p$ is completely continuous. Moreover, for each $f \in \mathcal{H}'_p(K^p)$, we have

$$\operatorname{tr}(f; V_J(K^p)) = J(f).$$

Let J be a virtual representation. Then for each isomorphism class of finite slope absolutely irreducible representation σ, we denote by $m_J(\sigma)$ the virtual multiplicity of σ in J. By definition, it means that we have the following equality for all $f \in \mathcal{H}'_p$:

$$J(f) = \sum_\sigma m_J(\sigma) J_\sigma(f).$$

Here σ runs in the set of isomorphism classes of finite slope absolutely irreducible representations. If $t \in T^{--}$, $h \in \mathbb{Q}$ and an open compact subgroup $K^p \subset G(\mathbb{A}_f^p)$, we denote by $\Sigma_J(K^p, t, h)$ the (finite) set of classes of irreducible σ such that $m_J(\sigma) \neq 0$, $v_p(\theta_\sigma(u_t)) \leq h$ and $\sigma^{K^p} \neq 0$.

4.1.11. *Fredholm determinants attached to a virtual finite slope character distribution.* Let J be a virtual finite slope character distribution and $f \in \mathcal{H}_p(K^p)$ of the form $f = f^p \otimes u_t$ with $t \in T^{--}$. Then we put

$$P_{J,f}(T) := \prod_{i=1}^{\infty} \det(1 - T\sigma_i(f))^{m_i}.$$

This is clearly a ratio of Fredholm series.

If $\alpha \in \overline{\mathbb{Q}}_p^\times$ and σ is an irreducible finite slope representation, we denote by $m_\sigma(f, \alpha)$ the multiplicity of the eigenvalue α for the Hecke operator f acting on V_σ. Then

$$m_J(f, \alpha) := \sum_\sigma m_J(\sigma).m_\sigma(f, \alpha)$$

is a well-defined finite sum by condition (i), and $m_J(f, \alpha)$ is the order of the zero $T = \alpha^{-1}$ for the meromorphic function $P_J(f, T)$ on $\mathbb{A}_1^{\mathrm{rig}}$. This integer is called the multiplicity of the eigenvalue α in J for the operator f.

Lemma 4.1.12. *If for all $f \in \mathcal{H}'_p$, $P_J(f, T)$ is an entire power series, then J is an effective finite slope character distribution.*

Proof. Let σ_0 such that $m_J(\sigma_0) \neq 0$. We want to prove that $m_J(\sigma_0) > 0$. Let K^p be such that $\sigma_0^{K^p} \neq 0$. Let $t \in T^{--}$ and let $h = v_p(\theta_{\sigma_0}(u_t))$. Also put

$$h' := \mathrm{M}in\{v_p(\theta_\sigma(u_t)),\ \forall \sigma \notin \Sigma_J(K^p, t, h) \text{ such that } m_J(\sigma) \neq 0 \text{ and } \sigma^{K^p} \neq 0\}.$$

From condition (i), it is easy to see that $h' > h$. Since $\Sigma_J(K^p, t, h)$ is finite, we know by Jacobson's lemma that there exists $f_1 \in \mathcal{H}_p(K^p)$ such that for all $\sigma \in \Sigma_J(K^p, t, h)$, we have $\sigma(f_1) = \mathrm{id}_{\sigma^{K^p}}$ if $\sigma \cong \sigma_0$ and $\sigma(f_1) = 0$ otherwise.

Now consider $f = (1_{K^p} \otimes u_{t^N})f_1$ with $N > v_1/(h' - h)$, with v_1 the valuation of the denominator of f_1. Then we have

$$P_J(f, T) = \det(1 - T.\sigma_0(1_{K^p} \otimes u_{t^N}))^{m_J(\sigma_0)} S(T),$$

where $S(T) = \prod_{\sigma | v_p(\theta_\sigma(u_t)) > h)} \det(1 - T\sigma((1_{K^p} \otimes u_{t^N}).f_1))^{m_J(\sigma)}$ is a meromorphic function with the set of zeroes and poles of p-adic norm greater or equal to $p^{h'N - v_1}$. Since by assumption $P_J(f, T)$ is an entire function of T and the set of zeroes of $\det(1 - T.\sigma_0(1_{K^p} \otimes u_{t^N}))$ are of p-adic norm smaller or equal to p^{hN}, this implies that $m_J(\sigma_0) > 0$. □

4.1.13. Assume now that J is effective; we have $P_{J,f}(T) = \det(1 - T.f; V_J(K^p))$. Let $t \in T^{--}$ and suppose that we have a factorization $P_{J,u_t}(T) = Q(T)S(T)$ with Q a polynomial such that $Q(0) = 1$ and Q and S relatively prime. Then we know, by Theorem 2.3.8, that we have a decomposition stable by u_t,

$$V_J(K^p) = N_J(Q) \oplus F_J(Q),$$

such that $Q^*(u_t)$ acts trivially on $N_J(Q)$ and is invertible on $F_J(Q)$. Moreover, there is a power series $R_{Q,S}$ such that $R_{Q,S}(u_t)$ is the projector of $V_J(K^p)$ onto $N_J(Q)$.

Since u_t is in the center of $\mathcal{H}_p(K^p)$, this decomposition is stable by the action of $\mathcal{H}_p(K^p)$, and for all $f \in \mathcal{H}_p(K^p)$, we have

$$J_{Q,t}(f) := J(f \circ R_{Q,S}(u_t)) = \mathrm{tr}(f \circ R_{Q,S}(u_t); V_J(K^p)) = \mathrm{tr}(f; N_J(Q)); \tag{27}$$

therefore $J_{Q,t}$ is a character of $\mathcal{H}_p(K^p)$ of degree *deg* Q.

4.2. *Automorphic finite slope representations.* We would like now to define the cohomology of arithmetic subgroups acting on the p-adic Δ^--modules that we defined in Section 3 and study the action of $\mathcal{H}_p$ on it. Because the standard resolution of group cohomology by inhomogeneous cochains is not of finite type, it is not suitable for topological properties. We will therefore use some projective resolutions of finite type. Their existence is due to the work of Borel and Serre. Ideally, we would like the cohomology of $\mathcal{D}_{\mathfrak{U}}$ to be *projective* over $\mathcal{O}(\mathfrak{U})$. In general, this is not quite true because of the possible presence of torsion in the cohomology and also because the cohomology might not be Hausdorff. To bypass these difficulties, we will work in the derived category of perfect complexes in the sense of Section 2. We will also construct finite slope character distributions for each p-adic weight and show that these are analytic functions of the weight. We will use some of the notations and definitions of Section 1.1.2.

4.2.1. *Resolution for arithmetic subgroups.* Let $\Gamma \subset G(\mathbb{Q})/Z_G(\mathbb{Q}) \subset G^{\mathrm{ad}}(\mathbb{Q})$ be a subgroup containing no nontrivial elements of finite order. Therefore, it acts freely and continuously from the left on the symmetric space $\mathcal{H}_G :=$

$G_\infty/Z_\infty K_\infty$ so that $\Gamma\backslash\mathcal{H}_G$ is a C^∞ manifold. Let d be its dimension. Unless G is anisotropic this quotient is not compact, and by the work of Borel-Serre, there exists a canonical compactification $\Gamma\backslash\bar{\mathcal{H}}_G$ where $\bar{\mathcal{H}}_G$ is a contractile real manifold with corners [BS73]. This fact will be useful to prove the following lemma.

Lemma 4.2.2. *Let Γ as above; then there exist length d finite free resolutions of the trivial Γ-module $\mathbb{Z}$. In other words, there exist exact sequences of Γ-modules of the form*

$$0 \to C_d(\Gamma) \to \cdots \to C_1(\Gamma) \to C_0(\Gamma) \to \mathbb{Z} \to 0,$$

where the $C_i(\Gamma)$'s are free $\mathbb{Z}[\Gamma]$-modules of finite rank.

Proof. Since $\Gamma\backslash\bar{\mathcal{H}}_G$ is compact, we may choose a finite triangulation of $\Gamma\backslash\bar{\mathcal{H}}_G$. We pull it back to $\bar{\mathcal{H}}_G$ by the canonical projection $\bar{\mathcal{H}}_G \to \Gamma\backslash\bar{\mathcal{H}}_G$ and denote by $C_q(\Gamma)$ the free $\mathbb{Z}$-module over the set of q-dimensional simplexes of the triangulation obtained by pull-back of $\bar{\mathcal{H}}_G$. This is the module of q-chains obtained from this triangulation. Since the action of Γ on $\bar{\mathcal{H}}_G$ is free, the $C_q(\Gamma)$ are free $\mathbb{Z}[\Gamma]$-modules and they are of finite type since the chosen triangulation of $\Gamma\backslash\bar{\mathcal{H}}_G$ is finite. We therefore obtain a complex of free $\mathbb{Z}[\Gamma]$-modules of finite rank

$$0 \to C_d(\Gamma) \overset{\partial_{d-1}}{\to} \cdots \overset{\partial_1}{\to} C_1(\Gamma) \overset{\partial_0}{\to} C_0(\Gamma) \to 0$$

whose homology is the singular homology of $\bar{\mathcal{H}}_G$. Since $\bar{\mathcal{H}}_G$ is contractile, this complex is exact except in degree zero. In degree zero, we have

$$C_0(\Gamma)/\partial_0(C_1(\Gamma)) = H_0(\bar{\mathcal{H}}_G, \mathbb{Z}) = \mathbb{Z}$$

which implies our claim. □

4.2.3. These resolutions are obviously not unique but two such resolutions are $\mathbb{Z}[\Gamma]$-homotopy equivalent. This is a standard fact that is true for any projective resolutions in abelian categories. We will use this fact in the following situation. Consider $\Gamma' \subset \Gamma$ to be a finite index subgroup; then the restriction to Γ' of any such resolution $C_\bullet(\Gamma)$ for Γ is $\mathbb{Z}[\Gamma']$-homotopy equivalent to any resolution $C_\bullet(\Gamma')$ for Γ'. We will use these resolutions to study the cohomology of these arithmetic groups. If M is a Γ-module, then we can compute $H^i(\Gamma, M) = \mathrm{E}xt^i_\Gamma(\mathbb{Z}, M)$ by taking the cohomology of $C^\bullet(\Gamma, M) := \mathrm{Hom}_\Gamma(C_\bullet(\Gamma), M)$. This complex is particularly nice because of the following corollary.

Corollary 4.2.4. *For any Γ and M as above, each term of the complex $C^\bullet(\Gamma, M)$ is isomorphic to finitely many copies of M.*

Proof. This follows from the fact that the $C_q(\Gamma)$'s are free of finite rank over $\mathbb{Z}[\Gamma]$. □

It is well known that we can define an action of Hecke operators on $H^\bullet(\Gamma, M)$. We will be explaining how we can directly define an action of them on $C^\bullet(\Gamma, M)$.

4.2.5. *Fonctoriality.* We first consider a somewhat general situation. Assume that we have two groups Γ and Γ' together with a group homomorphism $\phi : \Gamma \to \Gamma'$. Assume also that we have a right Γ-module N and a right Γ'-module M with a map of abelian groups $f : M \to N$, such that $f(m.\phi(\gamma)) = f(m).\gamma$ for all $m \in M$ and $\gamma \in \Gamma$. If we consider M as a Γ-module via ϕ, this f is Γ-equivariant. A pair (ϕ, f) like this is called compatible. It is trivial that f induces a map from $M^{\Gamma'}$ into N^Γ.

Let $C_\bullet(\Gamma)$ and $C_\bullet(\Gamma')$ be respectively two finite free resolutions of $\mathbb{Z}$ by $\mathbb{Z}[\Gamma]$ and $\mathbb{Z}[\Gamma']$-modules of finite rank. By considering $C_\bullet(\Gamma')$ as a Γ-module by ϕ, it is a Γ-resolution of $\mathbb{Z}$. Since $C_\bullet(\Gamma)$ is a Γ-projective resolution of $\mathbb{Z}$, it follows from the universal property of projective modules that we have a map, unique up to homotopy, $\phi_\bullet : C_\bullet(\Gamma) \to C_\bullet(\Gamma')$, which is compatible with ϕ.

We deduce that we have a map compatible with ϕ:

$$\mathrm{Hom}_{\mathbb{Z}}(C_\bullet(\Gamma'), M) \xrightarrow{(\phi_\bullet)^* \otimes f} \mathrm{Hom}_{\mathbb{Z}}(C_\bullet(\Gamma), N).$$

By taking respectively Γ and Γ' invariants, it induces a map $C^\bullet(\Gamma', M) \xrightarrow{f^\bullet} C^\bullet(\Gamma, N)$. Again this is uniquely defined up to homotopy.

If $\Gamma \subset \Gamma'$, $M = N$ and ϕ is the identity map, we obtain the restriction map

$$\mathrm{Res}_\Gamma^{\Gamma'} : C^\bullet(\Gamma, M) \to C^\bullet(\Gamma', M)$$

that induces the usual restriction map on the cohomology.

Assume now that Γ' is a subgroup of finite index of Γ. Let us fix a system of representatives $\{\gamma_i\}_i$ of $\Gamma' \backslash \Gamma$ (i.e., $\Gamma = \sqcup \Gamma' \gamma_i$). The corestriction map is obtained as follows. Again we choose free and finitely generated resolutions $C_\bullet(\Gamma')$ and $C_\bullet(\Gamma)$. Then $C_\bullet(\Gamma)$ is also a free and finitely generated resolution of the trivial $\mathbb{Z}[\Gamma']$-module $\mathbb{Z}$ since $\mathbb{Z}[\Gamma]$ is free of finite rank over $\mathbb{Z}[\Gamma']$. Therefore we have a Γ'-equivariant homotopy $\tau_{\Gamma,\Gamma'} : C_\bullet(\Gamma) \to C_\bullet(\Gamma')$. So we have the maps

$$\mathrm{Hom}_{\Gamma'}(C_\bullet(\Gamma'), M) \xrightarrow{\mathrm{Hom}(\tau_{\Gamma,\Gamma'}, \mathrm{id}_M)} \mathrm{Hom}_{\Gamma'}(C_\bullet(\Gamma), M) \xrightarrow{\sum_i \gamma_i} \mathrm{Hom}_\Gamma(C_\bullet(\Gamma), M),$$

where the second map is the usual average sum $m \mapsto \sum_i m.\gamma_i$ that sends Γ'-invariant to Γ-invariants. We call this composite the corestriction map

$$\mathrm{Cor}_{\Gamma'}^\Gamma : C^\bullet(\Gamma', M) \to C^\bullet(\Gamma, M).$$

It induces the usual corestriction map in cohomology. Again, this is uniquely defined up to homotopy.

4.2.6. *Hecke operators.* We refer to [Shi71] for the abstract definition and the properties of Hecke operators. Let Δ be the monoid containing Γ and Γ' and such that $\delta\Gamma\delta^{-1} \cap \Gamma'$ is of finite index in Γ' for all $\delta \in \Delta$. Then we can consider the abstract Hecke operators $\Gamma'\delta\Gamma \in \mathbb{Z}[\Gamma'\backslash\Delta/\Gamma]$. We say that Γ and Γ' are Δ-commensurable. Assume that M is a right Δ-module; then we can define the map $[\Gamma'\delta\Gamma] : C^\bullet(\Gamma', M) \to C^\bullet(\Gamma, M)$ as the composition

$$[\Gamma'\delta\Gamma] = \mathrm{Cor}^{\Gamma}_{\delta^{-1}\Gamma'\delta\cap\Gamma} \circ [\delta] \circ \mathrm{Res}^{\Gamma'}_{\Gamma'\cap\delta\Gamma\delta^{-1}},$$

where $[\delta]$ is defined by the pair of compatible maps $(x \mapsto \delta x \delta^{-1}, m \mapsto m.\delta)$ respectively from $\Gamma \cap \delta^{-1}\Gamma'\delta$ onto $\delta\Gamma\delta^{-1} \cap \Gamma'$ and from M into M. Again the action of $\Gamma'\delta\Gamma$ is well defined up to homotopy.

Consider now a third subgroup $\Gamma'' \subset \Delta$ which is Δ-commensurable with Γ' and Γ. Then we can compose the double classes $\Gamma'\delta\Gamma$ and $\Gamma''\delta'\Gamma'$ and get an element $\Gamma''\delta'\Gamma' \circ \Gamma'\delta\Gamma \in \mathbb{Z}[\Gamma''\backslash\Delta/\Gamma]$ (see [Shi71, Chap. 3], for example). Then we have

LEMMA 4.2.7. *The maps $[\Gamma''\delta'\Gamma'] \circ [\Gamma'\delta\Gamma]$ and $[\Gamma''\delta'\Gamma' \circ \Gamma'\delta\Gamma]$ are equivalent up to homotopy.*

Proof. This easily follows from the definitions and the fact that we have an equality when we define the maps on the level of the Φ-invariants for $\Phi = \Gamma, \Gamma', \Gamma''$. We leave the details to the reader. $\square$

4.2.8. *The adelic point of view.* Let K be a neat open compact subgroup of $G(\mathbb{A}_f)$ and let K_p be the image of K into $G(\mathbb{Q}_p)$. Let M be a left K-module such that K acts on M through its projection on K_p. We fix a decomposition as (1) so that the p-part of each g_i is trivial. Recall that $\Gamma(g_i, K)$ is defined as the image of $g_i.Kg_i^{-1} \cap G(\mathbb{Q})^+$ in $G(\mathbb{Q})/Z_G(\mathbb{Q})$. We put

$$R\Gamma^\bullet(K, M) := \oplus_i C^\bullet(\Gamma(g_i, K), M).$$

We can make another description. Consider the space $\overline{S}_G := G(\mathbb{Q})\backslash G(\mathbb{A}_f) \times \overline{\mathcal{H}}_G$. Then $\overline{S}_G(K) = \overline{S}_G/K$ is the Borel-Serre compactification of $S_G(K)$. Let π_K be the canonical projection $\overline{S}_G \to \overline{S}_G(K)$. Then choose a finite triangulation of $\overline{S}_G(K)$ and its pullback by π_K. Let us denote by $C_\bullet(K)$ the corresponding chain complex. It is equipped with a right action of K. It is an easy exercise to check that if we consider the decomposition of $\overline{S}_G(K)$ in the connected components determined by the g_i's and the triangulation on each connected component $\overline{\mathcal{H}}_G/\Gamma(g_i, K)$ associated to the chain complex $C_\bullet(\Gamma(g_i, K))$ (in the proof of Lemma 4.2.2), then we have the isomorphism

$$R\Gamma^\bullet(K, M) \cong \mathrm{Hom}_K(C_\bullet(K), M),$$

where the right action of K on M is given as usual by $m.k := k^{-1}.m$. In particular, this implies that if we had chosen another system of representatives

g_i', we would have obtained another complex homotopical to the first one. Therefore it defines an object in the homotopy category of abelian groups whose cohomology computes the cohomology of the local system $\widetilde{M}$ on $S_G(K)$.

Moreover, the map $M \mapsto R\Gamma^\bullet(K,M)$ is functorial with respect to left K-module and there is an isomorphism

$$H^\bullet(R\Gamma^\bullet(K,M)) \cong H^\bullet(S_G(K),M).$$

Suppose, more generally, that we have a pair (ϕ,f), where $\phi : K' \to K$ is a continuous and open group homomorphism and $f : M \to M'$ is a map of abelian groups where M (resp. M') is equipped with a left K-action (resp. with a left K'-action) such that $f(\phi(k').m) = k'.f(m)$ for all $k' \in K$ and $m \in M$. Using the description $R\Gamma^\bullet(K,M) = \mathrm{Hom}_K(C^\bullet(K),M)$ and the arguments of Section 4.2.5, we can define a map

$$R\Gamma^\bullet(K,M) \xrightarrow{(\phi^*,f)} R\Gamma^\bullet(K',M')$$

uniquely defined up to homotopy.

We now make a description using the decomposition in connected components associated to the g_i's and g_i''s respectively for K and K'. For simplicity, we assume that ϕ extends to a map from $G(\mathbb{A}_f)$ to itself since it will be the case for all the examples that will be considered. For each i, let j_i such that $\phi(g_i') = \gamma_i g_{j_i} k_i \in G(\mathbb{Q}) g_{j_i} K$. We can define maps $\phi_i : \Gamma(g_i',K') \to \Gamma(g_{j_i},K)$ and f_i by

$$\phi_i(x) := \gamma_i^{-1}\phi(x)\gamma_i, \qquad f_i(m) := f(\gamma_i.m).$$

Since the p-component of g_{j_i} is trivial, the p-component of γ_i belongs to the p-component of K and therefore γ_i acts on M. This justifies the definition of f_i. It is easy to check that the pairs (ϕ_i,f_i) satisfy the assumption of Section 4.2.5. We therefore have a map $C^\bullet(\Gamma(g_{j_i},K),M) \to C^\bullet(\Gamma(g_i',K'),M')$. Summing up over the i's, we get the map

$$R\Gamma^\bullet(K,M) \xrightarrow{(\phi^*,f)} R\Gamma^\bullet(K',M'),$$

which is uniquely defined up to a homotopy.

4.2.9. *Completely continuous action.* It is useful and important for our application to notice that if f satisfies certain properties, then so does the map (ϕ^*,f) between the complexes. For example, if M and M' are Banach spaces or compact Fréchet spaces equipped with continuous actions of K and K' and if f is completely continuous, then so is the map induced by (ϕ^*,f) at the level of the complexes. This follows from the very definition of our map, the fact that composition between continuous and compact maps is compact and that each term of the complex $R\Gamma^\bullet(K',M')$ is isomorphic to a finite number of copies of M'.

4.2.10. *Special cases.* Of course, if f is the identity map and K' is an open subgroup of K, then we get the restriction map that we denote by $\mathrm{Res}^K_{K'}$.

Another special case also arises if $\Delta_f \subset G(\mathbb{A}_f)$ is a monoid acting on M on the left via its projection into $G(\mathbb{Q}_p)$. For any $x \in \Delta_f$, we consider $\phi = \mathrm{Int}_x^{-1} : \ K' = xKx^{-1} \to K$ given by $\mathrm{Int}_x^{-1}(k) = x^{-1}kx$ and the map $M \to M$ given by $m \mapsto x.m$. We therefore get a map

$$R\Gamma^\bullet(K, M) \xrightarrow{R\Gamma(K,x)} R\Gamma^\bullet(xKx^{-1}, M),$$

which depends only on the coset $x.K$ modulo homotopy.

Consider now two open subgroups K, K' of Δ_f and $x \in \Delta_f$. Then we have the decompositions

$$K'xK = \sqcup_j x_j K \qquad \text{and} \qquad K' = \sqcup_j k_j.(K' \cap xKx^{-1})$$

with $x_j = k_j x$ for j running through a finite set of indices. Notice that $x_j K x_j^{-1} = xKx^{-1}$ is independent of j. Therefore it makes sense to define the action of the double coset $K'xK$ from $R\Gamma^\bullet(K, M)$ into $R\Gamma^\bullet(K', M)$ by

$$[K'xK] := \sum_j R\Gamma(K, x_j) = \sum_j R\Gamma(K' \cap xKx^{-1}, k_j) \circ \mathrm{Res}^{xKx^{-1}}_{K' \cap xKx^{-1}} \circ R\Gamma(x, K).$$

Again this action is defined up to homotopy and does not depend on the coset decomposition of $K'xK$ above. One can see also that it is homotopic to

$$\sum_j R\Gamma(K' \cap xKx^{-1}, k_j) \circ R\Gamma(x, x^{-1}K'x \cap K) \circ \mathrm{Res}^K_{x^{-1}K'x \cap K}.$$

When $K' \supset K$ and $x = 1$, we recover the corestriction map.

We now compare this action with the one defined by arithmetic subgroups. For this purpose, it is convenient to make M a right Δ_f^{-1}-module by the action $m.\delta^{-1} := \delta.m$ for all $\delta \in \Delta_f$ and $m \in M$. For each i, we write $g_i x = \gamma_{x,i} g_{j_i} h g_\infty$ with $h \in K$ as in Section 1.2.5. Then $\gamma_{x,i} \in \Delta_f \cap G(\mathbb{Q})$, and we have the equality in the homotopy category

$$[KxK] = \oplus_i [\Gamma_i \gamma_{x,i}^{-1} \Gamma_{j_i}],$$

where $[\Gamma_i \gamma_{x,i}^{-1} \Gamma_{j_i}]$ is the map from $C^\bullet(\Gamma_i, M)$ into $C^\bullet(\Gamma_{j_i}, M)$ defined previously. Of course, this equality is up to homotopy. The reader should compare this with relation (6) which was seen at the level of cohomology.

4.2.11. *Action of an automorphism on the resolutions.* We let ι be an automorphism of G such that $K^\iota = K$, and we fix an automorphism ι_M of M such that

$$g.\iota_M(m) = \alpha_M(g^\iota.m)$$

for all $g \in K$ and $m \in M$. Therefore, (ι, ι_M) induces an automorphism of $R\Gamma^\bullet(K, M)$ in the homotopy category. We can also describe this map using the decomposition in connected components. For any arithmetic subgroup Γ, we have a canonical isomorphism from $\mathrm{Hom}_\Gamma(C^\bullet(\Gamma), M)$ onto $\mathrm{Hom}_{\Gamma^\alpha}(C^\bullet(\Gamma^\iota), M)$ induced by $\phi \mapsto \phi^\iota$ for all $\phi \in \mathrm{Hom}_\Gamma(C^q(\Gamma), M)$ with $\phi^\iota(n) := \iota_M(\phi(n))$. Now note that since $K^\iota = K$, we should have $\Gamma(x, K)^\iota = \Gamma(x^\iota, K)$. Since for any representative systems $\{g_i\}$, $\{g_i^\iota\}$ is another system of representatives, we can see that the map induced by (ι, ι_M) is obtained from the maps above for $\Gamma = \Gamma(g_i, K)$ for each i up to homotopy.

4.3. *Finite slope cohomology.*

4.3.1. We will freely use the notations and assumptions of Section 3. Let A be a $\mathbb{Q}_p$-Banach algebra. Let K^p be a neat open compact subgroup of $G(\mathbb{A}_f^p)$. For such K^p, we choose representatives g_i for the cosets $G(\mathbb{Q})\backslash G(\mathbb{A}_f) \times \mathcal{H}_G)/K^p.I$ that are trivial at p. Then, from the previous discussion, it follows that the map $M \mapsto R\Gamma^\bullet(K^p.I_m, M)$ defines a functor from the category of left $A[\Delta_m^-/Z_p]$-Fréchet modules in the homotopy category of $\mathrm{Fre}(A)$. We let the algebra $\mathcal{U}_p$ act on the cohomology of $S_G(K^p.I_m)$ with coefficients $\widetilde{M}$ or on a complex $R\Gamma(K^p.I_m, M)$ through the projection on $C_c^\infty(\Delta_m^-//I_m, \mathbb{Z}_p)$ and the canonical action of the latter on the cohomology or the complex. Moreover, $R\Gamma^\bullet(K^p.I_m, M)$ is equipped with an action of the Hecke operators $\mathcal{H}_p(K^p)$ that defines an algebra homomorphism

$$\mathcal{H}_p(K^p) \to \mathrm{End}_{D^b_{pf}(\mathrm{Fre}(A))}(R\Gamma^\bullet(K^p.I_m, M)).$$

4.3.2. *Weight space revisited.* Let $Z(K^p) := Z_G(\mathbb{Q}) \cap K^p.I$. Then the natural map of $Z(K^p)$ inside $\Gamma(g_i, K^pI)$ is trivial for each i. Therefore $\mathcal{D}_\lambda$ and $V_\lambda^\vee(L)$ are $\Gamma(g_i, K^pI)$-modules only if λ is trivial on $Z(K^p)$. This is condition (2). We set $\mathfrak{X} = \mathfrak{X}_{K^p} \subset \mathfrak{X}_T$ to be the Zariski closure of the weights λ which are trivial on $Z(K^p)$. Let $Z_p(K^p)$ be the p-adic closure of $Z(K^p)$ inside $T(\mathbb{Z}_p)$. Then we have

$$\mathfrak{X}_{K^p}(L) := \mathrm{Hom}_{\mathrm{cont}}(T(\mathbb{Z}_p)/Z_p(K^p), L)$$

for any finite extension L of $\mathbb{Q}_p$. Moreover, $\mathfrak{X}_{K^p}$ is of dimension $rk_{\mathbb{Z}_p} T(\mathbb{Z}_p) - rk_{\mathbb{Z}_p} Z_p(K^p)$. Notice that if G is $\mathbb{Q}$-split or semi-simple, then $Z_p(K^p)$ is trivial since K^p is neat by assumption. Otherwise, its rank depends of the rank of some global units together with some Leopold defect. For instance, if $G = \mathrm{GSp}(2n, F)$ for a totally real field F, then $\mathfrak{X}_G$ is of dimension

$$(n+1)[F:\mathbb{Q}] - ([F:\mathbb{Q}] - 1 - \delta_{F,p}) = n[F:\mathbb{Q}] + 1 + \delta_{F,p},$$

where $\delta_{F,p}$ designs the defect of the Leopoldt conjecture for (F, p).

If S is a finite set of prime and K^p is maximal hyperspecial away from S, then $Z_p(K^p)$ does not depend of K^p if it is sufficiently small. In general, when K^p decreases, $\mathfrak{X}_{K^p}$ can get more connected components. However its dimension will stay the same.

4.3.3. *Finite slope cohomology.* Let M be any left $L[\Delta^-/Z_p(K^p)]$-module. We assume that M is a L-Banach or a compact L-Fréchet for which the elements in Δ^{--} act completely continuously. Let $t \in T^{--}$. We equip the complex $R\Gamma^\bullet(K^p I_m, M)$ with an action of the Hecke operators u_t (defined up to homotopy). By the definition of this action and the assumption on M, this operator is completely continuous on this complex and the latter has finite slope decomposition with respect to u_t. By the results of Chapter 2, this induces a slope decomposition for its cohomology. We then write $H^\bullet_{\mathrm{fs}}(S_G(K^p I_m), M) \subset H^\bullet(S_G(K^p I_m), M)$ for the finite slope part of its cohomology. Since for any $t, t' \in T^{--}$ there exits an N such that $t^N = t't''$ for $t'' \in T^-$ and u_t commutes with $u_{t'}$, it is easy to see that the finite slope part does not depend of the choice of t. We also put

$$H^q_{\mathrm{fs}}(\widetilde{S}_G, M) := \varinjlim_{K^p} H^q_{\mathrm{fs}}(S_G(K^p.I), M).$$

The spaces M that we will mainly consider are $\mathcal{D}_\lambda(L)$ and $V_\lambda^\vee(L)$ for weights λ trivial on $Z(K^p)$.

4.3.4. *Finite slope p-adic automorphic representations.* Let $\lambda \in \mathfrak{X}_{K^p}(\overline{\mathbb{Q}}_p)$. An irreducible finite slope representation σ of $\mathcal{H}_p$ will be called *p-adic automorphic of weight* λ if and only if it appears as a subquotient of the representation of $\mathcal{H}_p$ on $H^q_{\mathrm{fs}}(\widetilde{S}_G, \mathcal{D}_\lambda(L))$ for some integer q and some p-adic field L. It will be further called (M, w)-*ordinary Eisenstein* if there exist $M \in \mathcal{L}_G$, $w \in \mathcal{W}^M$ and a finite slope p-adic automorphic character σ_M of M of weight λ such that J_σ is a direct factor of the character $f \mapsto J_\tau(f^{\mathrm{reg}}_{M,w})$ for all $f \in \mathcal{H}_p$ and J_τ is the character of an automorphic finite slope representation for M. Moreover, M is supposed to be minimal for this property.

Proposition 4.3.5. *Let $\lambda \in \mathfrak{X}_{K^p}(L)$. For any irreducible finite slope representation σ of G, there is an integer $m^q(\sigma, \lambda) \in \mathbb{Z}$ such that for all $f \in \mathcal{H}'_p$, we have*

$$\mathrm{tr}(f; H^q_{\mathrm{fs}}(\widetilde{S}_G, \mathcal{D}_\lambda(L))) = \sum_\sigma m^q(\sigma, \lambda) J_\sigma(f).$$

In particular, σ is automorphic of weight λ if and only if $m^q(\sigma, \lambda) \neq 0$ for some q.

Proof. Let $t \in T^{--}$ and any $h \in \mathbb{Q}$. Then the $\leq h$-slope part for u_t of $H^q_{\mathrm{fs}}(\widetilde{S}_G, \mathcal{D}_\lambda(L))$ is $H^q_{\mathrm{fs}}(\widetilde{S}_G, \mathcal{D}_\lambda(L))^{\leq h} := \varinjlim_{K^p} H^q_{\mathrm{fs}}(S_G(K^p.I), \mathcal{D}_\lambda(L))^{\leq h}$. It is equipped with an action of $\mathcal{H}_p$ since u_t is in the center of $\mathcal{H}_p$, and this representation of $\mathcal{H}_p$ is clearly admissible. Therefore we have a decomposition

$$\mathrm{tr}(f; H^q_{\mathrm{fs}}(\widetilde{S}_G, \mathcal{D}_\lambda(L))^{\leq h}) = \sum_{\substack{\sigma \\ v_p(\theta_\sigma(u_t)) \leq h}} m^q(\sigma, \lambda) J_\sigma(f)$$

for all $f \in \mathcal{H}_p$. The proposition then easily follows by considering this equality with $f \in \mathcal{H}'_p$ and letting h go to infinity. $\square$

One defines the overconvergent Euler-Poincaré multiplicity by

$$m_G^\dagger(\sigma,\lambda) := \sum_q (-1)^q m^q(\sigma,\lambda). \tag{28}$$

In the sequel of this section, we want to relate the action of the Hecke operators on the finite slope cohomology of $\mathcal{D}_\lambda(L)$ to the one of $V_\lambda(L)$. We start by the following important lemma which goes back to Hida.

LEMMA 4.3.6. *Let M be a left Δ^-/Z_p-module. Let $t \in T^{--}$ and m be a positive integer. Then the following commutative diagram is commutative in the homotopy category of complexes of abelian groups*:

$$\begin{tikzcd} R\Gamma(K^p.I_m, M) \arrow[r, "u_t^m"] \arrow[dr, "{[I_{m+1}.t.I_m]}"] & R\Gamma(K^p.I_m, M) \\ R\Gamma(K^p.I_{m+1}, M) \arrow[u] \arrow[r, "u_t^{m+1}"] & R\Gamma(K^p.I_{m+1}, M). \arrow[u] \end{tikzcd}$$

In particular, if M is a Fréchet $L[\Delta^-/Z_p(K^p)$-vector space over which the elements of Δ^{--} act completely continuously, then the restriction map induces an isomorphism on the finite slope parts:

$$H^\bullet_{\mathrm{fs}}(S_G(K^pI), M) \xrightarrow{\sim} H^\bullet_{\mathrm{fs}}(S_G(K^pI_m), M).$$

Proof. The first part is a consequence of the definition of the action of double cosets and of the fact that the decomposition in right coset of the double cosets I_mtI_m and $I_{m+1}tI_m$ are the same. This last fact follows from $I_{m+1}\cap t^{-1}I_mt = I_m \cap t^{-1}I_mt$ for $t \in T^{--}$. To prove the second part of our lemma, we apply Lemma 2.3.4 to the following commutative diagram that follows from the first part of our lemma:

$$\begin{tikzcd} H^\bullet(K^p.I_m, M) \arrow[r, "u_t^m"] \arrow[dr, "{[I_{m+1}.t.I_m]}"] & H^\bullet(K^p.I_m, M) \\ H^\bullet(K^p.I_{m+1}, M) \arrow[u] \arrow[r, "u_t^{m+1}"] & H^\bullet(K^p.I_{m+1}, M). \arrow[u] \end{tikzcd}$$

□

4.3.7. Let λ be an algebraic dominant character of T. The next lemma shows that the finite slope part of the cohomology of $\mathbb{V}_\lambda^\vee(L)$ can be replaced by finite slope part of the cohomology of $V_\lambda^\vee(L)$.

LEMMA 4.3.8. *Let $\lambda = \lambda^{\mathrm{alg}}\varepsilon$ be an arithmetic weight of conductor p^{n_λ}. Then, for any $m \geq n_\lambda$, we have*

$$H^\bullet_{\mathrm{fs}}(S_G(K^p.I), V_\lambda^\vee(L)) \cong H^\bullet_{\mathrm{fs}}(S_G(K^p.I_m), \mathbb{V}_\lambda^\vee(\varepsilon, L)).$$

Proof. Let $V_{\lambda,n}(L)$ be the subspace of locally algebraic functions on I which are n-locally analytic on I'_{n_λ}. Then $V_\lambda(L) = \varinjlim_n V_{\lambda,n}(L)$. By Lemma 2.3.13, the canonical map

$$\widetilde{H}^\bullet(S_G(K^p.I_m), V_\lambda^\vee(L)) \to H^\bullet(S_G(K^p.I_m), V_{\lambda,n}^\vee(L))$$

induces an isomorphism on the finite slope part. The previous proposition applied to the $M = V_{\lambda,0}^\vee(L)$ together with the case $n = 0$ of the isomorphism above imply our claim since $\mathbb{V}_\lambda(\varepsilon, L) = V_{\lambda,0}(L)$. □

Definition 4.3.9. For any $t \in T^-$ and any $\lambda \in X^*(T)^+$, we put

$$N(\lambda, t) := \mathrm{Inf}_{w\neq \mathrm{id}} |t^{w*\lambda-\lambda}|_p.$$

We are now ready to compare the cohomology of $\mathcal{D}_\lambda(L)$ and of $\mathbb{V}_{\lambda^{\mathrm{alg}}}(\varepsilon, L)$.

Proposition 4.3.10. *Let $\lambda = \lambda^{\mathrm{alg}}\varepsilon \in \mathfrak{X}(L)$ be an arithmetic weight of conductor p_λ^n and μ be a noncritical slope with respect to λ^{alg}. Then for any positive integer $m \geq n_\lambda$, we have the canonical isomorphisms*

$$H^\bullet(S_G(K^p.I), \mathcal{D}_\lambda(L))^{\leq\mu} \cong H^\bullet(S_G(K^p.I_m), \mathbb{V}_\lambda^\vee(\varepsilon, L))^{\leq\mu}.$$

Similarly, for any rational $h < v_p(N(\lambda, t))$ and any Hecke operator $f = f^p \otimes u_t$ with $t \in T^{--}$, we have

$$H^\bullet(S_G(K^p.I), \mathcal{D}_\lambda(L))^{\leq h} \cong H^\bullet(S_G(K^p.I_m), \mathbb{V}_{\lambda^{\mathrm{alg}}}^\vee(\varepsilon, L))^{\leq h}$$

for the $\leq h$-slope decomposition with respect to the action of f.

Proof. We just prove the first part. The second part can be similarly proven. By the previous lemmas, it suffices to show that

$$H^\bullet(S_G(K^p.I), \mathcal{D}_\lambda(L))^{\leq\mu} \cong H^\bullet(S_G(K^p.I), V_\lambda^\vee(L))^{\leq\mu}.$$

For any simple root α, recall that we have defined in Proposition 3.2.11 a homomorphism of left I-module

$$\Theta_\alpha : \mathcal{A}_\lambda(L) \to \mathcal{A}_{s_\alpha*\lambda}(L).$$

Let us write Θ_α^* for the dual homomorphism. Then by Proposition 3.2.12, we have a canonical exact sequence

$$\bigoplus_{\alpha\in\Delta} \mathcal{D}_{s_\alpha*\lambda}(L) \to \mathcal{D}_\lambda(L) \to V_\lambda^\vee(L) \to 0.$$

Let us fix an ordering $\alpha_1, \ldots, \alpha_r$ of the simple roots in Δ such that for each integer i between 1 and r, $\Theta_i^* = \Theta_{\alpha_1}^* + \cdots + \Theta_{\alpha_i}^*$. For $i = 0$, we define $\Theta_0^* = 0$. Then $\mathrm{Coker}\,\Theta_0^* = \mathcal{D}_\lambda(L)$, $\mathrm{Coker}\,\Theta_r^* \cong V_\lambda^\vee(L)$, and for each integer $i \in \{1, \ldots, r\}$, we have an exact sequence

$$0 \to Q_i \xrightarrow{\Theta_{\alpha_i}^*} \mathrm{Coker}\ \Theta_{i-1}^* \to \mathrm{Coker}\ \Theta_i^* \to 0,$$

where Q_i is the exact quotient of $\mathcal{D}_{s_{\alpha_i}*\lambda}(L)$ making the short sequence exact. This induces the long exact sequence

$$H^q(S_G(K^p.I),Q_i)^{\leq \mu-(\lambda^{\mathrm{alg}}(H_{\alpha_i})+1)\alpha_i} \to H^q(S_G(K^p.I),\mathrm{Coker}\ \Theta^*_{i-1})^{\leq\mu}$$
$$\to H^q(S_G(K^p.I),\mathrm{Coker}\ \Theta^*_{i})^{\leq\mu} \to H^{q+1}(S_G(K^p.I),Q_i)^{\leq \mu-(\lambda^{\mathrm{alg}}(H_{\alpha_i})+1)\alpha_i}.$$

There is a shift in the slope truncation because the operator $\Theta^*_{\alpha_i}$ is not exactly equivariant for the action of $\mathcal{U}_p$. In fact, we have the following formula for any eigenvector v in $H^\bullet(S_G(K^p.I_m),Q_i)[\theta]$:

$$u_t(\Theta^*_{\alpha_i}(v)) = \alpha_i(t)^{\lambda^{\mathrm{alg}}(H_{\alpha_i})+1}\Theta_{\alpha_i}(u_t.v) = \theta(t)\alpha_i(t)^{\lambda^{\mathrm{alg}}(H_{\alpha_i})+1}.\Theta^*_{\alpha_i}(v),$$

which implies our claim since the character of $\Theta^*_{\alpha_i}(v)$ is therefore of slope $\mu_\theta-(\lambda^{\mathrm{alg}}(H_{\alpha_i})+1)\alpha_i$.

Now since Q_i contains a stable O_L-lattice under the left action of Δ^-, the slope of any character occurring in $H^\bullet(S_G(K^p.I),Q_i)$ must belong to $X^*(T)_{\mathbb{Q},+}$. Since μ is not critical with respect to λ, $\mu-(\lambda^{\mathrm{alg}}(H_{\alpha_i})+1)\alpha_i\notin X^*(T)_{\mathbb{Q},+}$, which implies $H^\bullet(S_G(K^p.I),Q_i)^{<\mu_\theta-(\lambda^{\mathrm{alg}}(H_{\alpha_i})+1)\alpha_i}=0$. Therefore $H^q(K^p.I,\mathrm{Coker}\,\Theta^*_{i-1})^{\leq\mu}$ is independent of i. This fact for $i=r$ and $i=0$ means that

$$H^\bullet(S_G(K^p.I),\mathcal{D}_\lambda(L))^{\leq\mu}\cong H^\bullet(S_G(K^p.I),V_\lambda^\vee(L))^{\leq\mu}. \qquad \square$$

4.3.11. *More multiplicities.* Let θ be a finite slope L-valued character of the Hecke algebra $R_{S,p}$. For any $\lambda\in\mathfrak{X}(L)$ and any K^p which is maximal outside S, let us consider

$$m^\dagger(\theta,\lambda,K^p):=\sum_q(-1)^q\dim_L H^q(S_G(K^p.I),\mathcal{D}_\lambda(L))[\theta].$$

When λ is arithmetic, we also define

$$m^{\mathrm{cl}}(\theta,\lambda,K^p):=\sum_q(-1)^q\dim_L H^q(S_G(K^p.I),V_\lambda^\vee(L))[\theta].$$

An immediate consequence of the previous proposition is the following classicity result on multiplicities.

COROLLARY 4.3.12. *Let λ be an arithmetic weight. Then for any θ such that μ_θ is noncritical with respect to λ^{alg}, we have*

$$m^{\mathrm{cl}}(\theta,\lambda,K^p)=m^\dagger(\theta,\lambda,K^p).$$

4.4. *A spectral sequence.* Let λ be an arithmetic weight of level p^m. We can refine the classicity result explained above by using the BGG complex. For this purpose, we consider the following double complex:

$$C_\lambda^{i,j}:=R\Gamma^i(K^pI_m,C^j_\lambda(L))=\bigoplus_{w|l(w)=j}R\Gamma^i(K^pI_m,\mathcal{D}_{w*\lambda}(L)).$$

Since the BGG complex is exact except in degree 0, where its cohomolgy is isomorphic to $V_\lambda^\vee(L)$, the spectral sequence that one obtains, by taking cohomology with respect to j first degenerates and converges to $H^{i+j}(S_G(K^pI), V_\lambda^\vee(L))$. On the other hand, the spectral sequence obtained by taking cohomology with respect to i first has a $E_1^{i,j}$ term given by $\oplus_{w|l(w)=j} H^i(S_G(K^pI), \mathcal{D}_{w*\lambda}(L))$. Then Proposition 4.3.10 is a corollary of the following theorem.

THEOREM 4.4.1. *Let $\lambda = \lambda^{\mathrm{alg}}\varepsilon$ be an arithmetic weight of level p^m and a slope $\mu \in X^*(T)_{\mathbb{Q}}$. Then we have the following spectral sequence*:

$$\bigoplus_{w|l(w)=j} H^i_{\mathrm{fs}}(K^pI_m, \mathcal{D}_{w*\lambda}(L))^{\leq \mu + w*\lambda - \lambda} \Rightarrow H^{i+j}_{\mathrm{fs}}(S_G(K^pI_m), \mathbb{V}^\vee_{\lambda^{\mathrm{alg}}}(\varepsilon, L))^{\leq \mu}.$$

Proof. This follows from the fact that the two spectral sequences attached to the double complex $(C_\lambda^{i,j})_{\mathrm{fs}}$ converge to the same limit. The slope truncation follows from the fact (due to Proposition 3.3.12) that the differential map $E_1^{i,j} \to E_1^{i,j+1}$ is equivariant with respect to the action of $\mathcal{U}_p$ if one renormalizes the $*$-action of u_t on $H^i_{\mathrm{fs}}(K^pI_m, \mathcal{D}_{w*\lambda}(L))$ by multiplying by the factor $\xi(t)^{w*\lambda^{\mathrm{alg}}}$. □

4.5. *The p-adic automorphic character distributions.* In the beginning of this chapter, we have defined finite slope p-adic character distributions as certain p-adic linear functionals on $\mathcal{H}'_p$. In this section, we define the finite slope p-adic (virtual) character distributions of the p-adic automorphic spectrum that we will decompose as alternating sums of cuspidal and Eisenstein parts.

4.5.1. *Definition of $I_G^\dagger$ and I_G^{cl}.* Let L be a finite extension of $\mathbb{Q}_p$ and fix $\lambda \in \mathfrak{X}(L)$. For $f \in \mathcal{H}'_p$, we put

$$I_G^\dagger(f, \lambda) := \mathrm{tr}(f; H^\bullet_{\mathrm{fs}}(\widetilde{S}_G, \mathcal{D}_\lambda(L))).$$

If $f \in \mathcal{H}'_p(K^p)$, we easily see that

$$\begin{aligned} I_G^\dagger(f, \lambda) &= \mathrm{Meas}(K^p, dg) \times \mathrm{tr}(f; H^\bullet_{\mathrm{fs}}(S_G(K^p.I), \mathcal{D}_\lambda(L))) \\ &= \mathrm{Meas}(K^p, dg) \times \mathrm{tr}(f; R\Gamma^\bullet(K^p.I, \mathcal{D}_\lambda(L))). \end{aligned}$$

This second equality comes from Corollary 2.3.11. If $\lambda \in \mathfrak{X}(L)$ is algebraic dominant, one also defines $I_G^{\mathrm{cl}}(f, \lambda)$ by

$$I_G^{\mathrm{cl}}(f, \lambda) := \mathrm{tr}^*(f; H^\bullet(\widetilde{S}_G, \mathbb{V}^\vee_{\lambda^{\mathrm{alg}}}(L)),$$

where the superscript "$*$" is here to remind the reader that $\mathbb{V}^\vee_{\lambda^{\mathrm{alg}}}(L)$ is considered as a left Δ^--module for the $*$-action. If $f = f^p \otimes u_t$ with $t \in T^{--}$, it follows from the comments of Section 1.2.5, relation (15) and Lemma 4.3.8 that we have the following formula:

$$I_G^{\mathrm{cl}}(f, \lambda) = \xi(t)^\lambda.\mathrm{tr}^{\mathrm{st}}(f; H^\bullet(\widetilde{S}_G, \mathbb{V}^\vee_\lambda(\mathbb{C})), \tag{29}$$

where the superscript "st" means that we have considered the standard action of the Hecke operators on $H^\bullet(\widetilde{S}_G, \mathbb{V}_\lambda^\vee(\mathbb{C}))$ as defined in Section 1.2.5.

LEMMA 4.5.2. *Let λ be an algebraic dominant weight and let $f = f^p \otimes u_t \in \mathcal{H}'_p(K^p)$. Then the following congruence holds*:

$$I_G^\dagger(f,\lambda) \equiv I_G^{\mathrm{cl}}(f,\lambda) \mod N(\lambda,t)\mathrm{Meas}(K^p,dg),$$

with $N(\lambda,t)$ the power of p defined in Definition 4.3.9.

Proof. Let K^p be such that f^p is bi-K^p-invariant. Let h be the largest slope (strictly) less than $v_p(N(\lambda,t))$ and ocurring in the cohomology of $\mathcal{D}_\lambda(L)$ or $\mathbb{V}_\lambda(L)$. Then one has

$$\mathrm{tr}(f, H^\bullet_{\mathrm{fs}}(S_G(K^p.I),\mathcal{D}_\lambda(L))^{\leq h} \equiv I_G^\dagger(f,\lambda) \mod N(\lambda,t)\mathrm{Meas}(K^p,dg),$$
$$\mathrm{tr}(f, H^\bullet(S_G(K^p.I_m),V_\lambda^\vee(L))^{\leq h} \equiv I_G^{\mathrm{cl}}(f,\lambda) \mod N(\lambda,t)\mathrm{Meas}(K^p,dg).$$

By Proposition 4.3.10, the left-hand side of both congruences are equal and the lemma is proved. □

4.5.3. *Twist with respect to a pair (w,λ)*. For any pair (w,λ) with $w \in W$ and λ a locally algebraic weight, one defines a $\mathbb{Q}_p$-linear automorphism of the Hecke algebra $\mathcal{H}_p$

$$f \mapsto f^{w,\lambda}$$

defined by

$$f^{w,\lambda} := \xi(t)^{w*\lambda^{\mathrm{alg}}-\lambda^{\mathrm{alg}}} f^p \otimes u_t$$

if $f = f^p \otimes u_t$ with $t \in T^-$ and $f^p \in C_c^\infty(G(\mathbb{A}_f^p),\mathbb{Q}_p)$ and extended to $\mathcal{H}_p$ by linearity. For any character θ of $R_{S,p}$, we then consider the twisted character $\theta^{w,\lambda}$ defined by

$$\theta^{w,\lambda}(f) := \theta(f^{w,\lambda})$$

for all $f \in R_{S,p}$. Similarly, for any irreducible finite slope representation σ, we denote by $\sigma^{w,\lambda}$ the twisted finite slope representation defined by

$$\sigma^{w,\lambda}(f) := \sigma(f^{w,\lambda}).$$

It is straightforward to verify that we have

$$\mu_{\theta^{w,\lambda}} = \mu_\theta + \lambda^{\mathrm{alg}} - w*\lambda^{\mathrm{alg}}. \tag{30}$$

In particular, when μ_θ is not critical with respect to λ^{alg}, $\mu_{\theta^{w,\lambda}} \notin X^*(T)_{\mathbb{Q},+}$, and therefore $m^\dagger(\theta^{w,\lambda}w,\lambda,K^p) = 0$ as long as $w \neq \mathrm{id}$ since the cohomology as an integral structure (if the multiplicity is not zero, it means that $\theta^{w,\lambda}$ must be integrally valued and its slope must belong to $X^*(T)_{\mathbb{Q},+}$). In view of Corollary 4.5.5 below, this gives another proof of Corollary 4.3.12.

THEOREM 4.5.4. *Let $f \in \mathcal{H}'_p$; then for any locally algebraic character λ, we have*

$$I_G^{\mathrm{cl}}(f,\lambda) = \sum_w (-1)^{l(w)} I_G^{\dagger}(f^{w,\lambda}, w * \lambda).$$

Proof. One applies the finite slope spectral sequence of Theorem 4.4.1. One again needs to pay attention to the fact that the action on the BGG resolution is the standard action (i.e., the action of t on $\mathcal{D}_{w*\lambda}(L)$ is the $*$-action multiplied by $\xi(t)^{w*\lambda^{\mathrm{alg}}}$). The reason why the twists of the Hecke operators appear here is that the distribution $I_G^{\dagger}$ is the trace with respect to the $*$-action. The details are left to the reader. □

Recall that for any irreducible finite slope representation σ, we have defined in (28) the Euler-Poincaré multiplicity $m_G^{\dagger}(\sigma,\lambda)$. It satisfies

$$I_G^{\dagger}(f,\lambda) = \sum_{\sigma} m_G^{\dagger}(\sigma,\lambda) J_{\sigma}(f),$$

where the sum runs over (absolutely) irreducible finite slope representations. If, moreover, λ is locally algebraic, then we can also define $m^{\mathrm{cl}}(\sigma,\lambda)$ in a similar way by replacing $I_G^{\dagger}(f,\lambda)$ by $I_G^{\mathrm{cl}}(f,\lambda)$. Then we have the following straightforward corollary.

COROLLARY 4.5.5. *Let λ be an arithmetic weight. Then for any finite slope irreducible representation σ, we have*

$$m_G^{\mathrm{cl}}(\sigma,\lambda) = \sum_{w \in W} (-1)^{l(w)} m_G^{\dagger}(\sigma^{w,\lambda}, w * \lambda).$$

There is a similar formula for the multiplicities $m^{\dagger}(\theta,\lambda,K^p)$ and $m^{\mathrm{cl}}(\theta,\lambda,K^p)$.

4.6. *The Eisenstein and cuspidal finite slope p-adic character distributions.* As in the classical case, the p-adic automorphic distribution (which is not in general effective) can be decomposed as a sum indexed on Levi M and elements of $\mathcal{W}^M$ of Eisenstein and a cuspidal p-adic character distributions.

4.6.1. *Definitions of $I_{G,M,w}^{\dagger}$ and $I_{G,0}^{\dagger}$.* For any standard Levi $M \in \mathcal{L}_G$ and $w \in \mathcal{W}^M$, recall that we have defined linear maps from $\mathcal{H}_p(G)$ into $\mathcal{H}_p(M)$ $f \mapsto f_{M,w}^{\mathrm{reg}}$ to relate the character of the parabolic induction of an admissible finite slope to the character of the representation which is induced. We define character distributions $I_{G,0}^{\dagger}$ and $I_{G,M,w}^{\dagger}$ for any $M \in \mathcal{L}_G$ and $w \in \mathcal{W}_{\mathrm{Eis}}^M$, by induction on the $rk(G)$. If $rk(G) = 0$, we put

$$I_{G,0}^{\dagger}(f,\lambda) = I_{G,G}^{\dagger}(f,\lambda) := I_G^{\dagger}(f,\lambda).$$

Assume now that $rk(G) = r$ and that these distributions are defined for groups of rank less than r. Then, for any proper Levi $M \in \mathcal{L}_G$ and $f = f^p \otimes u_t$, we put

$$I^\dagger_{G,M,w}(f,\lambda) := I^\dagger_{M,0}(f^{\mathrm{reg}}_{M,w}, w*\lambda + 2\rho_P)$$

where $2\rho_P$ stands for the sum of positive roots of the unipotent radical of the standard parabolic subgroup of Levi M and

$$I^\dagger_{G,M}(f,\lambda) := \sum_{w\in\mathcal{W}^M_{\mathrm{Eis}}} (-1)^{l(w)+\dim\mathfrak{n}_M} I^\dagger_{G,M,w}(f,\lambda),$$

where $\mathfrak{n}_M$ stands for the Lie algebra of the standard parabolic of Levi M. We define

$$I^\dagger_{G,0}(f,\lambda) := I^\dagger_G(f,\lambda) - \sum_{\substack{M\in\mathcal{L}_G\\ M\neq G}} I^\dagger_{G,M}(f,\lambda).$$

Let $\lambda = \lambda^{\mathrm{alg}}\varepsilon$ be an arithmetic weight of level p^m. In view of the formula (29), for $f\in\mathcal{H}_p(K^p)$ we define

$$I^{\mathrm{cl}}_{G,0}(f,\lambda) := \mathrm{Meas}(K^p).\xi(t)^\lambda.\mathrm{tr}^{\mathrm{st}}(f; H^\bullet_{\mathrm{cusp}}(S_G(K^p.I_m), \mathbb{V}^\vee_\lambda(\mathbb{C})(\varepsilon))).$$

LEMMA 4.6.2. *For any $f = f^p\otimes u_t\in\mathcal{H}_p$ and regular arithmetic weight λ, we have*

$$I^{\mathrm{cl}}_G(f,\lambda) = \sum_{M\in\mathcal{L}^c_G}\sum_{w_0\in\mathcal{W}^M_{\mathrm{Eis}}}\sum_{w\in\mathcal{W}^M} \cdot (-1)^{l(w)+\dim\mathfrak{n}_M}\xi(t)^{\lambda-w^{-1}w_0*\lambda}.I^{\mathrm{cl}}_{M,0}(f^{\mathrm{reg}}_{M,w}, w_0*\lambda+2\rho_{P_M}).$$

Proof. This is a consequence of the trace formula of Franke and a standard computation that we now explain. In order to have lighter notations, we will write the proof only when λ is algebraic (i.e., $\varepsilon = 1$) and $f = 1_{K^p}\otimes u_t$ since the proof is strictly the same in the general case. We first fix a standard parabolic P of Levi subgroup M. For any algebraic dominant character μ , let us write $\sigma_\mu := H^\bullet_{\mathrm{cusp}}(\widetilde{S}_M, V^\vee_\mu(\mathbb{C}))^{K^p}$, which is viewed as a representation of $M(\mathbb{Q}_p)$. Let P be the standard parabolic subgroup with Levi M and let N be its unipotent radical. Then by relation (8), we have

$$\mathrm{tr}^{\mathrm{st}}(f_M,\sigma_\mu) = \mathrm{tr}^{\mathrm{st}}(u_t : \mathrm{Ind}^{G(\mathbb{Q}_p)}_{P(\mathbb{Q}_p)}\sigma_\mu).$$

Here again, the subscript "st" stands as usual for "standard action" and the parabolic induction is the smooth nonunitary parabolic induction. If $\phi\in(\mathrm{Ind}^{G(\mathbb{Q}_p)}_{P(\mathbb{Q}_p)}\sigma_\mu)^I$ and $w\in\mathcal{W}^M$, then $\phi(w)$ is invariant by $wIw^{-1}\cap M(\mathbb{Q}_p) = I\cap M(\mathbb{Q}_p) = I_M$ since $w\in\mathcal{W}^M$. Therefore, from the decomposition

$$G(\mathbb{Q}_p) = \bigsqcup_{w\in\mathcal{W}^M} P(\mathbb{Q}_p)wI,$$

we see that the map $\phi\mapsto(\phi(w))_{w\in\mathcal{W}^M}$ defines an isomorphism

$$\left(\mathrm{Ind}^{G(\mathbb{Q}_p)}_{P(\mathbb{Q}_p)}\sigma_\mu\right)^I \cong (\sigma_\mu^{I_M})^{\mathcal{W}^M}.$$

Then a classical computation gives

$$\begin{aligned}
&\mathrm{tr}^{\mathrm{st}}(u_t : \left(\mathrm{Ind}_{P(\mathbb{Q}_p)}^{G(\mathbb{Q}_p)}\sigma_\mu\right)^I) \\
&\quad = \sum_{w\in\mathcal{W}^M}[N_w(\mathbb{Z}_p) : tN_w(\mathbb{Z}_p)t^{-1}].\mathrm{tr}^{\mathrm{st}}(I_M wtw^{-1} I_M : \sigma_\mu^{I_M}) \\
&\quad = \sum_{w\in\mathcal{W}^M}\frac{[N_w(\mathbb{Z}_p) : tN_w(\mathbb{Z}_p)t^{-1}]}{\mathrm{Meas}(K^p)}\xi(t)^{-w^{-1}\mu} I_{M,0}^{\mathrm{cl}}(1_{K^p}\otimes u_{wtw^{-1},M},\mu).
\end{aligned}$$

Here we have considered $I_M wtw^{-1} I_M$ as the element $u_{wtw^{-1},M}\in\mathcal{U}_p(M)$ thanks to the remarks of Section 4.1.8 and $N_w := N\cap w^{-1}Nw$. Notice that

$$[N_w(\mathbb{Z}_p) : tN_w(\mathbb{Z}_p)t^{-1}] = \prod_{\alpha\in R_P\cap w^{-1}(R_P)}|\alpha(t)|_p^{-1} = |t^{w^{-1}(\rho_P)+\rho_P}|_p^{-1}.$$

If $\mu = w_0 * \lambda + 2\rho_P = w_0(\lambda+\rho_P)+\rho_P$ for some $w_0\in\mathcal{W}^M_{\mathrm{Eis}}$, we therefore have

$$\begin{aligned}
&\xi(t)^\lambda[N_w(\mathbb{Z}_p) : tN_w(\mathbb{Z}_p)t^{-1}]\xi(t)^{-\mu} \\
&\quad = \xi(t)^{\lambda-w^{-1}(w_0(\lambda+\rho_P)+\rho_P)}|t^{w^{-1}(\rho_P)+\rho_P}|_p^{-1} \\
&\quad = \xi(t)^{\lambda-w^{-1}(w_0(\lambda+\rho_P)+\rho_P)+w^{-1}(\rho_P)+\rho_P}\varepsilon_{\xi,w}(t) = \xi(t)^{\lambda-w^{-1}w_0*\lambda}\varepsilon_{\xi,w}(t).
\end{aligned}$$

Recall also that we have

$$\varepsilon_{\xi,w}(t)^{-1}(1_{K^p})_M\otimes u_{wtw^{-1},M} = \mathrm{Meas}(K^p)^{-1} f_{M,w}^{\mathrm{reg}}.$$

Combining all the previous identities, we get

$$\begin{aligned}
(31)\qquad &\xi(t)^\lambda.\mathrm{tr}^{\mathrm{st}}(f_M, H^\bullet_{\mathrm{cusp}}(\widetilde{S},\mathbb{V}^M_{w_0*\lambda+2\rho_P}(\mathbb{C})^\vee) \\
&\qquad = \sum_{w\in\mathcal{W}^M}\xi(t)^{\lambda-w^{-1}w_0*\lambda} I^{\mathrm{cl}}_{M,0}(f^{\mathrm{reg}}_{M,w}, w_0*\lambda+2\rho_P).
\end{aligned}$$

Recall that by Theorem 1.4.2 due to J. Franke, since λ is regular, we have

$$\begin{aligned}
&\mathrm{tr}^{\mathrm{st}}(f : H^\bullet(\widetilde{S}_G,\mathbb{V}^\vee_\lambda(\mathbb{C}))) \\
&\qquad = \sum_{M\in\mathcal{L}_G}\sum_{w_0\in\mathcal{W}^M_{\mathrm{Eis}}}(-1)^{l(w_0)+\dim\mathfrak{n}_M}\mathrm{tr}^{\mathrm{st}}(f_M : H^\bullet_{\mathrm{cusp}}(\widetilde{S}_M,\mathbb{V}^M_{w_0*\lambda+2\rho_{P_M}})).
\end{aligned}$$

Therefore, after multiplying by $\xi(t)^\lambda$, we get

$$\begin{aligned}
I^{\mathrm{cl}}_G(f,\lambda) = &\sum_{M\in\mathcal{L}_G}\sum_{w_0\in\mathcal{W}^M_{\mathrm{Eis}}} \\
&\cdot(-1)^{l(w_0)+\dim\mathfrak{n}_M}\xi(t)^\lambda\mathrm{tr}^{\mathrm{st}}(f_M : H^\bullet_{\mathrm{cusp}}(\widetilde{S}_M,\mathbb{V}^M_{w_0*\lambda+2\rho_{P_M}})).
\end{aligned}$$

The statement that we have claimed now results from the combination of this formula and (31). ∎

Corollary 4.6.3. *Let $f = f^p \otimes u_t \in \mathcal{H}_p(K^p)$ be $\mathbb{Z}_p$-valued and let λ be a regular arithmetic weight. Then we have the congruence*

$$I^{\dagger}_{G,0}(f,\lambda) \equiv I^{\mathrm{cl}}_{G,0}(f,\lambda) \pmod{\mathrm{Meas}(K^p, dg).N(\lambda,t)}.$$

Proof. Notice first that since $f = f^p \otimes u_t \in \mathcal{H}_p(K^p)$ is $\mathbb{Z}_p$-valued, the images of f by all the character distributions we have defined are $\mathrm{Meas}(K^p).\mathbb{Z}_p$-valued. We prove this proposition by induction on the rank of G. The case of rank 0 follows from the previous paragraph (see Lemma 4.5.2). We now assume that the proposition is satisfied for all proper Levi subgroups M of $\mathcal{L}_G$. If $w_0 \neq w$, then $N(\lambda,t)$ divides $\xi(t)^{\lambda - w^{-1}w_0*\lambda}$. It therefore follows from the previous lemma that we have the following congruence modulo $\mathrm{Meas}(K^p)N(\lambda,t)$:

$$I^{\mathrm{cl}}_{G,0} \equiv I^{\mathrm{cl}}_{G}(f,\lambda) - \sum_{w \in \mathcal{W}^M_{\mathrm{Eis}}} (-1)^{l(w)+\dim \mathfrak{n}_M} I^{\mathrm{cl}}_{M,0}(f^{\mathrm{reg}}_{M,w}, w*\lambda + 2\rho_{P_M}).$$

By Lemma 4.5.2, we also have

$$I^{\mathrm{cl}}_{G}(f,\lambda) \equiv I^{\dagger}_{G}(f,\lambda) \pmod{\mathrm{Meas}(K^p, dg).N(\lambda,t)}.$$

We can now conclude the proof, using the induction hypotheses and the definition of $I^{\dagger}_{G,0}$. □

4.6.4. For any $\lambda \in \mathfrak{X}(\overline{\mathbb{Q}}_p)$, by Proposition 4.3.5, we have

$$I^{\dagger}_{G}(f,\lambda) = \sum_{\sigma} m^{\dagger}(\sigma,\lambda) J_{\sigma}(f),$$

with $m^{\dagger}(\sigma,\lambda) = \sum_g (-1)^q m^q(\sigma,\lambda) \in \mathbb{Z}$. By induction on the rank of G, we easily see that $I^{\dagger}_{G,0}(f,\lambda)$ and $I^{\dagger}_{G,M,w}(f,\lambda)$ are finite slope character distributions, and we have spectral decompositions

$$I^{\dagger}_{G,0}(f,\lambda) = \sum_{\sigma} m^{\dagger}_0(\sigma,\lambda) J_{\sigma}(f)$$

and

$$I^{\dagger}_{G,M,w}(f,\lambda) = \sum_{\sigma} m^{\dagger}_{G,M,w}(\sigma,\lambda) J_{\sigma}(f),$$

where $m^{\dagger}_{G,M,w}$ and $m^{\dagger}_{G,0}(\sigma,\lambda) \in \mathbb{Z}$ denote the corresponding multiplicities. If λ is a dominant algebraic weight, then we also denote $m^{\mathrm{cl}}_{G,0}(\sigma,\lambda)$ as the multiplicity of σ with respect to the finite slope character distributions $f \mapsto I^{\mathrm{cl}}_{G,0}(f,\lambda)$.

Corollary 4.6.5. *Let λ be a regular arithmetic weight. If σ is not critical with respect to λ^{alg}, then we have*

$$m^{\mathrm{cl}}_{G,0}(\sigma,\lambda) = m^{\dagger}_{G,0}(\sigma,\lambda).$$

Proof. Using an appropriate $t \in T^{--}$, this is an easy consequence of Corollary 4.6.3. The details are left to the reader. □

Definition 4.6.6. Let $(\lambda_n)_n$ be a sequence of algebraic dominant weight in $\mathfrak{X}(\mathbb{Q}_p)$ such that $(\lambda_n)_n$ is converging p-adically to a weight λ in $\mathfrak{X}(\mathbb{Q}_p)$. We say that this sequence is highly regular if, for all positive simple root α, we have

$$\lim_{n\to\infty} \lambda_n(H_\alpha) = +\infty.$$

This notion is used in the following situation. If $t \in T^{--}$, then

$$\lim_{n\to\infty} N(\lambda_n, t) = 0, \tag{32}$$

where the limit is understood for the p-adic topology.

Corollary 4.6.8. *Let $(\lambda_n)_n$ be a highly regular sequence of dominant weight converging p-adically to a weight $\lambda \in \mathfrak{X}(L)$. Then for any Hecke operator $f = f^p \otimes u_t \in \mathcal{H}'_p$, we have*

$$\lim_{n\to\infty} I^{\mathrm{cl}}_{G,?}(f, \lambda_n) = I^{\dagger}_{G,?}(f, \lambda)$$

for $? = \emptyset$, 0.

Proof. This is a direct consequence of the congruences of Lemma 4.5.2 and Corollary 4.6.3. □

4.7. *Automorphic Fredholm series.*

4.7.1. *Definition.* We consider automorphic Fredholm series only when $G(\mathbb{R})$ has discrete series. Under this hypothesis, d_G stands for half the dimension of the corresponding locally symmetric space. For any $f \in \mathcal{H}'_p$, $\lambda \in \mathfrak{X}(\overline{\mathbb{Q}}_p)$ and $? = \emptyset, 0$, let us denote by $P^{\dagger}_{G,?}(f, \lambda, X)$ the Fredholm power series associated to the finite slope character distribution:

$$h \mapsto (-1)^{d_G}\mathrm{Meas}(K^p)^{-1}.I^{\dagger}_{G,?}(h, \lambda)$$

for K^p the maximal open compact subgroup of $G(\mathbb{A}_f^p)$ such that f is K^p bi-invariant. If, moreover, λ is an arithmetic weight, then we set

$$P^{\mathrm{cl}}_G(f, \lambda, X) := \det(1 - X.f; H^\bullet(S_G(K^p, I), \mathbb{V}^\vee_{\lambda^{\mathrm{alg}}}(L)))^{(-1)^{d_G}}.$$

Lemma 4.7.2. *Assume that λ is a regular arithmetic weight and $f = f^p \otimes u_t$ with $t \in T^{--}$. Then the power series*

$$P^{n-cl}_{G,0}(f, \lambda, X) := \frac{P^{\dagger}_{G,0}(f, \lambda, X)}{P^{\mathrm{cl}}_{G,0}(f, \lambda, X)}$$

is a meromorphic function of X on $\mathbb{C}_p$ (i.e., the ratio of two Fredholm series in X) with coefficients in O_L, and its set of zeroes and poles lies in

$$\{x \in \mathbb{C}_p^\times,\ \textit{such that } |x|_p \geq N(\lambda, t)\}.$$

Proof. This follows from the definition of the Fredholm power series and Corollary 4.6.5. □

Let $\Lambda_{\mathfrak{X}} := \mathbb{Z}_p[[T(\mathbb{Z}_p)/Z_p]] \subset \mathcal{O}(\mathfrak{X})$ and $\Lambda_{\mathfrak{X},\mathbb{Q}_p} = \Lambda_{\mathfrak{X}} \otimes \mathbb{Q}_p$. We have the following theorem.

THEOREM 4.7.3. *Let $f \in \mathcal{H}'_p(K^p)$ and $\mathfrak{X} = \mathfrak{X}_{K^p}$; then the following properties hold*:

(i) *The functions of λ defined by $I^\dagger_G(f,\lambda)$, $I^\dagger_{G,M,w}(f,\lambda)$ and $I^\dagger_{G,0}(f,\lambda)$ belong to $\Lambda_{\mathfrak{X},\mathbb{Q}_p}$. In particular, they are analytic on $\mathfrak{X}$.*

(ii) *If G_∞ has no discrete series, then $I^\dagger_{G,0}(f,\lambda) \equiv 0$.*

(ii)′ *If M_∞ has no discrete series or if $\dim(\mathfrak{X}_{K^p\cap M}) < \dim(\mathfrak{X}_{K^p})$, then $I^\dagger_{G,M,w}(f,\lambda) \equiv 0$.*

(iii) *Assume that $f = f^p \otimes u_t$ with $t \in T^{--}$; then we have $P^\dagger_{G,0}(f,\lambda,X) \in \Lambda_{\mathfrak{X}}\{\{X\}\}$. In other words, it defines an analytic function on $\mathfrak{X} \times \mathbb{A}^1_{\mathrm{rig}}$.*

Proof. Again from the definitions, it suffices to prove (i) for $I^\dagger_G(f,\lambda)$ since the other cases will follow from an induction argument on the rank of G. Let $\mathcal{U} \subset \mathfrak{X}$ be a an open affinoid subdomain and let $n \geq n_{\mathfrak{U}}$. Then we have

$$R\Gamma^\bullet(K^p.I,\mathcal{D}_{\mathfrak{U},n}) \otimes_\lambda L \cong R\Gamma^\bullet(K^p.I,\mathcal{D}_{\lambda,n}(L))$$

for any $\lambda \in \mathfrak{U}(L)$. Therefore $F_{\mathfrak{U}} := \mathrm{Meas}(K^p)\mathrm{tr}(f, R\Gamma^\bullet(K^p.I,\mathcal{D}_{\mathfrak{U},n}))$ is a function inside $\mathcal{O}(\mathfrak{U})$ satisfying $F_{\mathfrak{U}}(\lambda) = I^\dagger_G(f,\lambda)$ for any $\lambda \in \mathfrak{U}(\overline{\mathbb{Q}}_p)$. Let $\mathcal{O}^0(\mathfrak{U})$ be the ring of analytic functions on $\mathfrak{U}$ which are bounded by 1. If f is $\mathrm{Meas}(K^p)^{-1}.\mathbb{Z}_p$-valued, then $F_{\mathfrak{U}} \in \mathcal{O}^0(\mathfrak{U})$ since f preserves the $\mathcal{O}^0(\mathfrak{U})$-lattice $R\Gamma^\bullet(K^p.I,(\mathcal{D}_{\mathfrak{U},n})^0)$. Here $(\mathcal{D}_{\mathfrak{U},n})^0$ is the intersection of $\mathcal{D}_{\mathfrak{U},n}$ with the $\mathcal{O}^0(\mathfrak{U})$-dual of the lattice of functions f of $\mathcal{A}_{\mathfrak{U},n}$ bounded by 1 on $\mathfrak{U} \times I$. Since this can be done for any such $\mathfrak{U} \subset \mathfrak{X}$, we deduce that $I^\dagger_G(f,\lambda) \in \varprojlim_{\mathfrak{U}} \mathcal{O}^0(\mathfrak{U}) = \Lambda_{\mathfrak{X}}$. Therefore (i) follows.

For the proof of (ii), we see from (i) that it suffices to show the vanishing of $I^\dagger_{G,0}(f,\lambda)$ for all algebraic dominant weights λ because those weights are Zariski dense in $\mathfrak{X}$. Let λ be such a weight and let $(\lambda_n)_n$ be a highly regular sequence converging p-adically to λ. For each n, $I^{\mathrm{cl}}_{G,0}(f,\lambda_n) = 0$ since G_∞ has no discrete series and λ_n is dominant regular. By Corollary 4.6.8, this implies that

$$I^\dagger_{G,0}(f,\lambda) = \lim_{n\to\infty} I^{\mathrm{cl}}_{G,0}(f,\lambda_n) = 0,$$

which concludes the proof of (ii). The first part of assertion (ii)′ follows from assertion (ii) for the group M. The second assertion follows from the fact that for any algebraic dominant $\lambda \in \mathfrak{X}_{K^p}(L)$ such that $w * \lambda + 2\rho_P$ is nontrivial on $Z_M(\mathbb{Q}) \cap K^p.I$, we have

$$I^{\mathrm{cl}}_{M,0}(f^{\mathrm{reg}}_{M,w}, w*\lambda + 2\rho_P) \equiv 0 \pmod{\mathrm{Meas}(K^p)N_M(wtw^{-1}, w*\lambda+2\rho_P)}.$$

Since those λ's are Zariski dense by our hypothesis on the dimensions of the weight spaces for G and M, this implies, using a highly regular sequence, that $I^{\dagger}_{M,0}(f^{\mathrm{reg}}_{M,w}, w * \lambda + 2\rho_P) = 0$. Using again the Zariski density of those λ's, one can conclude our proof since this is an analytic function of λ by (i).

We now prove point (iii) for which we may assume that G_∞ has discrete series by point (ii). Let $\mathfrak{U} \subset \mathfrak{X}$ as above in the proof of (i). We furthermore assume that it contains algebraic weights. This implies that algebraic weights are dense in $\mathfrak{U}$. We will also assume that $\mathcal{O}(\mathfrak{U})$ is factorial which is the case for instance if $\mathfrak{U}$ is a closed disc. Since $\mathfrak{X}$ can be covered by a union of such discs, we may assume that $\mathcal{O}(\mathfrak{U})$ is a factorial ring. We need to prove that $P^{\dagger}_{G,0}(f, \lambda, X) \in \mathcal{O}^0(\mathfrak{U})\{\{X\}\}$. From the construction and the description above in the proof of (i), this series is the ratio of two series in $\mathcal{O}^0(\mathfrak{U})\{\{X\}\}$. Since $\mathcal{O}(\mathfrak{U})$ is factorial, by [CM98, Th. 1.3.11] we have a prime factorization of $P^{\dagger}_{G,0}(f, \lambda, X)$ as

$$P^{\dagger}_{G,0}(f, \lambda, X) = \prod_{i \in I_T} P_i^{m_i}(X)$$

with $m_i \in \mathbb{Z}\backslash\{0\}$ and $\{P_i(X)\}_i$ a set of distinct prime Fredholm series in $\mathcal{O}^0(\mathfrak{U})\{\{X\}\}$. We therefore can write $P^{\dagger}_{G,0}(f, \lambda, X) = \frac{N(\lambda,X)}{D(\lambda,X)}$ with $N(\lambda, X)$ and $D(\lambda, X)$ relatively prime Fredholm series.

Before continuing the proof, we refer the reader to Section 5.1.4 for the definition and basic properties of the hypersurface $Z(F) \subset \mathfrak{U} \times \mathbb{A}^1_{\mathrm{rig}}$ defined for any Fredholm series $F(\lambda, X) \in \mathcal{O}(\mathfrak{U})\{\{X\}\}$.

Assume now that $D(\lambda, X) \neq 1$. Since $(N, D) = 1$, it follows that $Y := Z(D) - Z(D) \cap Z(N)$ is a nonempty open rigid subvariety of the hypersurface $Z(D)$. Since the projection of $\pi :\ Z(D) \to \mathfrak{U}$ is flat, for any open affinoid subdomain $\mathfrak{W} \subset Y$, $\pi(\mathfrak{W})$ is an open affinoid subdomain of $\mathfrak{U}$, and it therefore contains a Zariski dense set of algebraic weights.

Let us fix such a $\mathfrak{W}$ and let $w = (\lambda, x) \in \mathfrak{W}(\overline{\mathbb{Q}}_p)$ such that $\lambda \in \mathfrak{U}(\mathbb{Q}_p)$ is an algebraic weight. Since $x \neq 0$, we can easily choose an element $w' = (\lambda', x')$, p-adically close to w inside $\mathfrak{W}(\overline{\mathbb{Q}}_p)$ so that λ' is regular dominant and $|x'|_p < N(\lambda', t)$. Since (λ', x') is a pole of $P^{\dagger}_{G,0}(f, \lambda', X)$, it therefore follows, from Lemma 4.7.2, that x' is a pole of the rational fraction $P^{\mathrm{cl}}_{G,0}(f, \lambda', X)$. But, since λ' is regular dominant,

$$P^{\mathrm{cl}}_{G,0}(f, \lambda', X) = \det(1 - X.f|H^{d(G)}_!(S_G(K^p.I, \mathbb{V}^\vee_{\lambda'}(L))$$

it is a polynomial and has no pole. This contradiction implies $D(\lambda, X) = 1$, and therefore $P^{\dagger}_{G,0}(f, \lambda, X) \in \mathcal{O}^0(\mathfrak{U})\{\{X\}\}$ as claimed. $\square$

COROLLARY 4.7.4. *For any finite slope representation σ, $w \in \mathcal{W}^M_{\mathrm{Eis}}$ and $\lambda \in \mathfrak{X}(\overline{\mathbb{Q}}_p)$, we have*

$$(-1)^{d_M} m^{\dagger}_{G,M,w}(\sigma, \lambda) \geq 0,$$

and $m^{\dagger}_{G,M,w}(\sigma,\lambda)$ *is always* 0 *unless* $M \in \mathcal{L}^c_G$. *In particular, for each* $\lambda \in \mathfrak{X}(\overline{\mathbb{Q}}_p)$, *the map* $f \mapsto (-1)^{d_m} I^{\dagger}_{G,M,w}(f,\lambda)$ *is an effective finite slope character distribution.*

Proof. Since $I^{\dagger}_{G,M,w}(f,\lambda) = I^{\dagger}_{M,0}(f^{\mathrm{reg}}_{M,w})$, it is sufficient to prove the result for $G = M$. This is a consequence of Lemma 4.1.12 together with the parts (iii) and (ii) of Theorem 4.7.3. □

4.7.5. *A twisted version.* Let ι be a finite order automorphism of G preserving the pair $(B_{/F}, T_{/F})$ for F the finite extension of $\mathbb{Q}_p$ that splits G and the center of G. Especially it preserves the Iwahori subgroups I_m and the subgroup $T(\mathbb{Z}_p)$. It therefore acts on $\mathfrak{X}$ by $\lambda^{\iota}(t) := \lambda(t^{\iota^{-1}})$ and this action preserves the cone of dominant weights. We denote by $\mathfrak{X}^{\iota} \subset \mathfrak{X}$ the subvariety of weights λ fixed by ι. Since ι preserves I, it acts on $f \in \mathcal{A}(I,L)$ by $(\iota.f)(g) := f(g^{\iota^{-1}})$. Moreover, if $\lambda \in \mathfrak{X}^{\iota}(L)$, then this action leaves $\mathcal{A}_{\lambda}$ stable. The Fréchet space $\mathcal{D}_{\lambda}$ inherits an action of ι compatible with the action of ι on the groups I in the sense of Section 4.2.11. For any $f \in \mathcal{H}_p$, we can therefore study the traces of $\iota \times f$ (a notation for ι composed with f) on the Fréchet spaces or Fréchet complexes we have defined. We can especially define the distributions $I^{*}_{G}(\iota \times f,\lambda)$, $I^{*}_{G,0}(\iota \times f,\lambda)$ and of $I^{*}_{G,M}(\iota \times f,\lambda)$ as well as a twisted multiplicity $m^{*}(\sigma \times \iota,\lambda)$ with $* = \dagger$ or cl. We can also define the corresponding power series $P^{*}_{G}(\iota \times f,\lambda,X)$, $P^{*}_{G,0}(\iota \times f,\lambda,X)$ and of $P^{*}_{G,M}(\iota \times f,\lambda,X)$. The following definition will be relevant in the theorem below which is the twisted version of Theorem 4.7.3.

Definition 4.7.6. We say that ι is of Cartan type if there exists $g_{\infty} \in G_{\infty}$ such that $\mathrm{Int}(g_{\infty}) \circ \iota$ is a Cartan involution of G_{∞}. For instance, if G_{∞} has discrete series, one can show that $\iota = \mathrm{id}$ is of Cartan type.

Let $\Lambda_{\mathfrak{X}^{\iota}} := \mathbb{Z}_p[[T^{\iota}/Z_p]] \subset \mathcal{O}(\mathfrak{X}^{\iota})$ and $\Lambda_{\mathfrak{X}^{\iota},\mathbb{Q}_p} = \Lambda_{\mathfrak{X}^{\iota}} \otimes \mathbb{Q}_p$. In the twisted situation, a variant of Theorem 4.7.3 is the following.

THEOREM 4.7.7. *For any* $f \in \mathcal{H}'_p$, *the following properties hold*:

(i) *The functions of* λ *defined by* $I^{\dagger}_{G}(\iota \times f,\lambda)$, $I^{\dagger}_{G,M}(\iota \times f,\lambda)$ *and* $I^{\dagger}_{G,0}(\iota \times f,\lambda)$ *belong to* $\Lambda_{\mathfrak{X}^{\iota},\mathbb{Q}_p}$. *In particular, they are analytic on* $\mathfrak{X}^{\iota}$.

(ii) *If* ι *is not of Cartan type, then* $I^{\dagger}_{G,0}(\iota \times f,\lambda) = 0$ *for all* $\lambda \in \mathfrak{X}^{\iota}(\overline{\mathbb{Q}}_p)$.

(iii) *Assume that* $f = f^p \otimes u_t$ *with* $t \in T^{--}$; *then we have* $P^{\dagger}_{G,0}(\iota \times f,\lambda,X) \in \Lambda_{\mathfrak{X}}\{\{X\}\}$. *In particular, it defines an analytic function on* $\mathfrak{X}^{\iota} \times \mathbb{A}^1_{\mathrm{rig}}$.

Proof. It is similar to the proof of Theorem 4.7.3. A detailed construction and proof will appear in Zhengyu Xiang's thesis. In particular, one needs to write the decomposition of the twisted finite slope character distribution attached to rational parabolic subgroups which are stable by the involution ι.

It can be done exactly in the same way. One has to replace the trace formula of Franke by the associated twisted trace formula which can be in turn obtained from Franke's spectral sequence expressing the cohomology of $V_\lambda^\vee(\mathbb{C})$ as a limit of the spectral sequence constructed out of the cuspidal cohomology of the standard Levi subgroups when λ is regular. $\square$

4.7.8. The next two sections will be devoted to some applications of the important analyticity property of the distributions $I^\dagger_{G,?}(f,\lambda)$ we have defined in the previous paragraphs. The first application is the construction of eigenvarieties *à la Coleman-Mazur*. The second is the proof of a formula for these distribution in geometric terms à la Arthur-Selberg.

5. **Construction of eigenvarieties**

5.1. *Spectral varieties.*

5.1.1. *Analytic families of finite slope character distributions.* Let $\mathfrak{X}$ be a rigid analytic space defined over an extension of $\mathbb{Q}_p$. A $\mathbb{Q}_p$-linear map

$$J = J_{\mathfrak{X}} :\ \mathcal{H}'_p \to \Lambda_{\mathfrak{X},\mathbb{Q}_p} \subset \mathcal{O}(\mathfrak{X})$$

is called a $\mathfrak{X}$-family of character distribution if, for all $\lambda \in \mathfrak{X}(\overline{\mathbb{Q}}_p)$, the composite J_λ of this map with the evaluation map at λ is an effective finite slope character distribution. For any irreducible finite slope representation σ, we write $m_J(\sigma,\lambda) \in \mathbb{Z}_{\geq 0}$ for the multiplicity of J_σ in J_λ. Let K^p be an open compact subgroup of $G(\mathbb{A}_f^p)$. The goal of this section is to attach to the pair $(J_{\mathfrak{X}}, K^p)$ an eigenvariety over $\mathfrak{X}$ parametrizing the spherical Hecke eigensystems of the irreducible finite slope representations σ for which $m_J(\sigma,\lambda) > 0$ and $\sigma^{K^p} \neq 0$. We will then apply our construction to the analytic families of finite slope distributions we have studied in the previous chapter. Let S be the smallest finite set of primes such that K^p is hyperspecial away from S. In our main application, $\mathfrak{X}$ will be the weight space $\mathfrak{X}_{K^p}$ (which is actually only depending upon S) introduced in the previous chapter and J will be $I^\dagger_{G,0}$. One can also construct Eisenstein components attached to the distributions $I^\dagger_{G,M,w}$'s.

5.1.2. *Remark.* Before starting up the task that we propose to perform in this chapter, we would like to mention that our construction extends a construction of K. Buzzard in [Buz07] who did such a construction with the weaker hypothesis that for each affinoid $\mathfrak{U} \subset \mathfrak{X}$, there is a $\mathfrak{U}$-family of orthonormalizable $\mathcal{O}(\mathfrak{U})$-Banach spaces equipped with an action of the Hecke algebra and such that certain Hecke operators at p are completely continuous on them.

5.1.3. Let $f \in \mathcal{H}'_p$. Then for each $\lambda \in \mathfrak{X}(\overline{\mathbb{Q}}_p)$, let us write $P_J(f,\lambda,X)$ for the Fredholm power series (in X) attached to f and J_λ. Then we can write

$P_J(f,\lambda,X)=1-J_\lambda(f)X+\cdots$. Therefore, we see that the first term of the X-expansion for $P_J(f,\lambda,X)$ is an analytic function of λ. A similar statement is also true for all the coefficients of $P_J(f,\lambda,X)$ since the coefficient of X^n can be expressed as a polynomial of degree n in $J_\lambda(f),J_\lambda(f^2),\dots,J_\lambda(f^n)$. Thus, there exists $P_J(f,X)\in\Lambda_{\mathfrak{X},\mathbb{Q}_p}\{\{T\}\}$ such that $P_J(f,X)(\lambda)=P_J(f,\lambda,X)$ for all $\lambda\in\mathfrak{X}(\overline{\mathbb{Q}}_p)$.

5.1.4. *Fredholm hypersurfaces.* We recall some of the definitions due to Coleman and Mazur of Fredholm hypersurfaces. We refer the reader to [CM98] and [Buz07] for the notions and properties recalled here. One says that an element $P\in\Lambda_{\mathfrak{W}}\{\{X\}\}$ is a Fredholm series if $P(0)=1$. For such a P, we denote by $Z(P)$ the rigid subvariety of $\mathfrak{X}\times\mathbb{A}^1_{\mathrm{rig}}$ cut out by P. It is called a Fredholm hypersurface, and its projection onto $\mathfrak{X}$ is flat. $Z(P)(\overline{\mathbb{Q}}_p)$ is equipped with the natural topology such that the inclusion $Z(P)(\overline{\mathbb{Q}}_p)\subset\overline{\mathbb{Q}}_p\times\mathfrak{X}(\overline{\mathbb{Q}}_p)$ is continuous. An admissible affinoid subdomain of $Z(P)$ can be obtained as follows. Let $\mathfrak{U}\subset\mathfrak{X}$ be an affinoid subdomain and assume that we have a factorization $P|_{\mathfrak{U}}=Q\cdot R$ with $Q,R\in\mathcal{O}(\mathfrak{U})\{\{X\}\}$, with Q a polynomial of degree d relatively prime to R such that $Q(0)=1$. Then $\mathfrak{W}_{Q,\mathfrak{U}}:=\mathrm{Sp}(\mathcal{O}(\mathfrak{U})[X]/(X^dQ(X^{-1}))(\overline{\mathbb{Q}}_p)$ imbeds naturally in $Z(P)(\overline{\mathbb{Q}}_p)$ where, for any Tate algebra A, we denote by $\mathrm{Sp}(A)$ the corresponding rigid affinoid variety as in [BGR84]. Moreover, $\mathfrak{W}_{Q,\mathfrak{U}}$ is open if and only if $\mathfrak{U}$ is. Any such subset will be called admissible. It is not difficult to check that it defines a Grothendieck topology on the ring space $Z(P)$ by taking finite covering by admissible open subsets. This gives $Z(P)$ the structure of a rigid analytic variety. Of course, this ringed space is not necessarily reduced. Its reduction $Z(P)_{\mathrm{red}}$ is a union of irreducible components that are themselves of the form $Z(P)$ for irreducible Fredholm series P.

5.1.5. *Spectral varieties attached to J.* For any $f=f^p\otimes u_t\in\mathcal{H}'_p(K^p)$ with $t\in T^{--}$, we denote by $\mathfrak{Z}_J(f):=Z(P_J(f))\subset\mathfrak{X}\times\mathbb{A}^1_{\mathrm{rig}}$ the Fredholm hypersurface cut out by the Fredholm series $P_J(f,\lambda,X)$.

PROPOSITION 5.1.6. *Let $x=(\lambda_x,\alpha_x)\in\mathfrak{X}(\overline{\mathbb{Q}}_p)\times\overline{\mathbb{Q}}_p^\times$. Then $x\in\mathfrak{Z}_J(f)(\overline{\mathbb{Q}}_p)$ if and only if α_x^{-1} appears as an eigenvalue of f acting on $V_{J_{\lambda_x}}(K^p)$ with a nontrivial multiplicity.*

Proof. This is obvious from the definition. □

5.2. *First construction of the eigenvarieties.*

5.2.1. We fix $K^p\subset G(\mathbb{A}_f^p)$. Let S be the smallest finite set of primes away from which K^p is hyperspecial. Let $\widetilde{R}_{S,p}$ be the p-adic completion of $R_{S,p}[u_t^{-1},\ t\in T^-]$. For any subfield $L\subset\overline{\mathbb{Q}}_p$, we define

$$\mathfrak{R}_{S,p}(L)=\mathrm{Hom}_{\mathrm{ct.alg.}}(\widetilde{R}_{S,p},L).$$

By construction, the characters of $R_{S,p}$ contained in $\mathfrak{R}_{S,p}(L)$ are those of finite slope. The canonical p-adic topology of $\mathfrak{R}_{S,p}(L)$ is the topology induced by the metric $|\theta - \theta'| = \mathrm{Sup}_{f \in R_{S,p}} |\theta(f) - \theta'(f)|_p$. In particular, for all $f \in \widetilde{R}_{S,p}$, the map from $\mathfrak{R}_{S,p}(L)$ into L defined by $\theta \mapsto \theta(f)$ is continuous. We consider $\mathfrak{Y} = \mathfrak{Y}_{S,p} := \mathfrak{X} \times \mathfrak{R}_{S,p}$. A point y of $\mathfrak{Y}(\overline{\mathbb{Q}}_p)$ is a pair (λ_y, θ_y), where λ_y is a weight and θ_y is a homomorphism $R_{S,p} \to \overline{\mathbb{Q}}_p$ of finite slope.

5.2.2. *Construction of* $\mathfrak{E}_{K^p,J}$. Let $\hat{R}_S$ be the p-adic completion of the $\mathbb{Z}_p$-valued smooth function on $G(\mathbb{A}_f^{S \cup \{p\}})$ which are bi-invariant by $K_m^{S \cup \{p\}}$. A Hecke operator $f \in \mathcal{H}_p'$ will be said to be K^p-admissible if it is of the form $f = 1_{K_S} \otimes f' \otimes u_t$ with $f' \in \hat{R}_S^\times$ and $t \in T^{--}$. For any such f, one defines a map of ringed spaces R_f from $\mathfrak{Y}_{S,p}$ into $\mathfrak{X} \times \mathbb{A}^1_{\mathrm{rig}}$ by $y = (\lambda_y, \theta_y) \mapsto (\lambda_y, \theta_y(f)^{-1})$ on the set of L-points and on the ring of functions $R_f^* : \mathcal{O}(\mathfrak{X})\{\{X\}\} \to \mathcal{O}(\mathfrak{X}) \hat{\otimes} \widetilde{R}_{S,p}$ defined by

$$\sum_{n=0}^{\infty} a_n \cdot X^n \mapsto \sum_{n=0}^{\infty} a_n \cdot (f)^{-n}.$$

We define the eigenvariety $\mathfrak{E}_{K^p,J}$ as the following infinite fiber product over $\mathfrak{Y}_{S,p}$:

$$\mathfrak{E}_{K^p,J} := \prod_f R_f^{-1}(\mathfrak{Z}_J(f)),$$

where the fiber product is indexed on the set of K^p-admissible Hecke operators f. For each admissible f, we will denote by r_f the restriction of R_f to the eigenvariety.

From the definition, $\mathfrak{E}_{K^p,J}$ is clearly a ringed space whose underlying topological space is the set of $\overline{\mathbb{Q}}_p$-points $\mathfrak{E}_{K^p,J}(\overline{\mathbb{Q}}_p)$ with the topology induced by the canonical p-adic topology of $\mathfrak{X} \times \mathfrak{R}_{S,p}(\overline{\mathbb{Q}}_p)$. Of course, we have

$$\mathfrak{E}_{K^p,J}(\overline{\mathbb{Q}}_p) = \bigcap_f R_f^{-1}(\mathfrak{Z}_{G,J}(f)(\overline{\mathbb{Q}}_p)).$$

By definition of the canonical topology of $\mathfrak{R}_{S,p}(\overline{\mathbb{Q}}_p)$, the maps r_f are therefore continuous. We also need to define a G-topology of $\mathfrak{E}_{K^p,J}$. We say that an open subset of $\mathfrak{E}_{K^p,J}$ is an admissible open subset if it is the union of open subsets of the form $(r_{f_1} \times \cdots \times r_{f_r})^{-1}(\mathfrak{W})$. Here $f_1, \ldots f_r$ stand for K^p-admissible operators and $\mathfrak{W}$ is an open admissible affinoid subdomain of $\mathfrak{Z}_J(f_1) \times \cdots \times \mathfrak{Z}_J(f_r)$. Similarly, we define the admissible coverings as the inverse images by the projections r_f's of the admissible coverings of the corresponding spectral varieties. Naturally $\mathfrak{E}_{K^p,J}$ is also a ringed space for its G-topology. Notice also that by construction, for any K^p-admissible f, we have a map of ringed spaces $r_f : \mathfrak{E}_{K^p,J} \to \mathfrak{Z}_J(f)$ fitting in a canonical diagram:

$$\begin{tikzcd} \mathfrak{E}_{K^p,J} \arrow[d, "\mathbf{w}"] \arrow[dr, "r_f"] & \\ \mathfrak{X} & \mathfrak{Z}_{G,J}(f). \arrow[l] \end{tikzcd}$$

In the next subsections, we will prove the desired expected properties of the eigenvarities we have defined from those of the spectral varieties. We first give a description of the points of the eigenvariety $\mathfrak{E}_{K^p,J}$. Let us denote by $m_J(\lambda,\theta,K^p)$ the multiplicity of θ in $V_{J_\lambda}(K^p)$. Then we have the following

Proposition 5.2.3. *Let K^p be an open compact subgroup of $G(\mathbb{A}_f^p)$ and let $y = (\lambda_y, \theta_y) \in (\mathfrak{X} \times \mathfrak{R}_{S,p})(\overline{\mathbb{Q}}_p)$. Then $m_J(\lambda_y,\theta_y,K^p) > 0$ if and only if $y \in \mathfrak{E}_{K^p,J}(\overline{\mathbb{Q}}_p)$. Moreover, if $y = (\lambda_y,\theta_y) \in \mathfrak{E}_{K^p,J}(\overline{\mathbb{Q}}_p)$, then there exists a K^p-admissible f such that*

$$r_f^{-1}(r_f(y)) = \{y\}.$$

Proof. The argument of this lemma is essentially due to Coleman and Mazur. For any $f \in \mathcal{H}'_p(K^p)$ and $\beta \in \overline{\mathbb{Q}}_p$, let us denote by $m_J(\lambda_y,f,\beta,K^p)$ the multiplicity of the eigenvalue β for f acting on $V_{J_{\lambda_y}}(K^p)$. Then we have

$$m_J(\lambda_y, f, \theta_y(f), K^p) = \sum_{\substack{\theta \\ \theta(f)=\theta_y(f)}} m_J(\lambda_y,\theta,K^p).$$

Assume now that $m_J(\lambda_y,\theta_y,K^p) > 0$. Then for any K^p-admissible f, we deduce from the formula above that $m_J(\lambda_y,f,\theta_y(f),K^p) > 0$, and therefore $r_f(y) = (\lambda_y,\theta_y(f)^{-1}) \in \mathfrak{Z}_J(f)(\overline{\mathbb{Q}}_p)$ by Proposition 5.1.6. Since this is true for all K^p-admissible f, we deduce that $y \in \mathfrak{E}_{K^p,J}(\overline{\mathbb{Q}}_p)$.

Let now L be the finite extension of $\mathbb{Q}_p$ such that $y \in \mathfrak{E}_{K^p,J}(L)$. Let $t \in T^{--}$ and $h = v_p(\theta_y(u_t))$. Let us consider the action of $R_{S,p}$ on the L-Banach space $V_{J_{\lambda_y}}(K^p)$ and let $V := V_{J_{\lambda_y}}(K^p)^{\leq h}$ be its $\leq h$-slope part of $V_{J_{\lambda_y}}(K^p)$ for the action of the operator $1_{K^p} \otimes u_t$. Let A be the image of $R_{S,p}$ inside $\mathrm{End}_L(V)$ and consider $f_1, f_2, \dots, f_r$ to be a subset of $R_{S,p}$ whose images in A form a system of generators of A over L. Let $R \in \mathbb{Z}_p$ be of positive p-adic valuation such that two distinct eigenvalues α and α' of the operators $1_{K^p} \otimes u_t, f_1, \dots, f_r$ acting on V must satisfy $v_p(\alpha - \alpha') < v_p(R)$. Then we consider the operators $h_1, \dots, h_r$ defined by $h_1 = f_1$ and the induction formula $h_{i+1} = f_{i+1} \cdot (1 + R.h_i)$, and we take $f = (1_{K^p} \otimes u_t)(1 + R.h_q)$. Let now θ be a character of $R_{S,p}$ occurring in the representation $V_{\lambda_y}(K^p)$ and such that $\theta(f) = \theta_y(f)$. In particular, this implies that $v_p(\theta(u_t)) = v_p(\theta_y(u_t))$, and therefore θ occurs in V. Moreover, this implies that $v_p(\theta(u_t) - \theta_y(u_t)) \geq v_p(R)$, and therefore $\theta(u_u) = \theta_y(u_t)$ by our assumption on R. We deduce that $\theta(h_q) = \theta_y(h_q)$. Repeating the previous argument, we deduce that $\theta(f_q) = \theta_y(f_q)$ and then by a descending induction that $\theta(f_i) = \theta_y(f_i)$ for $i = q, q-1, \ldots, 1$, and therefore $\theta = \theta_y$ since θ and θ_y agree on a system of generators of the Hecke algebra acting on V.

For this f, we therefore have $m_J(\lambda_y, f, \theta_y(f), K^p) = m_J(\lambda_y, \theta_y, K^p)$. Since $r_f(y) \in \mathfrak{Z}_J(f)(\overline{\mathbb{Q}}_p)$, this implies that $m_J(\lambda_y, \theta_y, K^p) > 0$ by Proposition 5.1.6, and we have $r_f^{-1}(r_f(y)) \cap \mathfrak{E}_{K^p,J}(\overline{\mathbb{Q}}_p) = \{y\}$. $\square$

5.3. *Second construction.* We now give a construction of a rigid analytic variety whose set of points is in bijection with $\mathfrak{E}_{J,K^p}(\overline{\mathbb{Q}}_p)$. We first construct the local pieces and we show how we can glue them together to construct a rigid analytic space over $\mathfrak{X}$.

5.3.1. *Construction of local pieces.* We fix $t \in T^{--}$ and write f_0 for the K^p-admissible Hecke operator $f_0 := 1_{K^p} \otimes u_t$. Let $\mathfrak{U} \subset \mathfrak{X}$ be an affinoid subset of $\mathfrak{X}$ and let $\mathfrak{W}_{Q,\mathfrak{U}}$ be the admissible affinoid subset of $\mathfrak{Z}_J(f_0)$ over $\mathfrak{U}$ attached to an admissible factorization $P_J(f, X)|_{\mathfrak{U}} = Q(X)S(X) \in \mathcal{O}(\mathfrak{U})\{\{X\}\}$. For any $\lambda \in \mathfrak{U}(\overline{\mathbb{Q}}_p)$, let us write $Q_\lambda(X)$ for the evaluation of $Q(X)$ at λ. Then recall that there is a unique $\mathcal{H}_p(K^p)$-stable decomposition (see §4.1.13)

$$V_{J_\lambda}(K^p) = N_{J_\lambda}(Q_\lambda) \oplus F_{J_\lambda}(Q_\lambda)$$

such that $N_{J_\lambda}(Q_\lambda)$ is finite-dimensional of dimension $\deg(Q)$, $Q_\lambda(X)$ is the characteristic polynomial of f acting on $N_{J_\lambda}(Q_\lambda)$ and $Q_\lambda^*(f)$ is invertible on $F_{J_\lambda}(Q_\lambda)$. Let $R_{Q,S}(X) \in X\mathcal{O}(\mathfrak{U})\{\{X\}\}$ be the entire power series attached to Q and S by Theorem 2.3.8. Then for any $\lambda \in \mathfrak{U}(\overline{\mathbb{Q}}_p)$, $R_{Q,S}(f_0)(\lambda)$ acts on $V_{J_\lambda}(K^p)$ as the projector on $N_{J_\lambda}(Q_\lambda)$ with respect to the above decomposition. In particular, for any $f \in \mathcal{H}_p(K^p)$ the trace $J_{Q_\lambda,t}(f)$ of f acting on $N_{J_\lambda}(Q_\lambda)$ is equal to $J(f.R_Q(f_0)(\lambda))$ (see §4.1.13). This implies that the map $T_{Q,\mathfrak{U}} : f \mapsto J(f.R_Q(f_0)) \in \mathcal{O}(\mathfrak{U})$ is a pseudo-representation of $\mathcal{H}_p(K^p)$ of dimension the degree of Q (see, for instance, [Tay91] for the definitions and basic properties of pseudo-representations). Then we put $R_{\mathfrak{U}} := R_{S,p} \otimes \mathcal{O}(\mathfrak{U})$ and $h_{J,Q,\mathfrak{U}} := R_{\mathfrak{U}}/\operatorname{Ker}(T_{Q,\mathfrak{U}}) \cap R_{\mathfrak{U}}$ with

$$\operatorname{Ker}(T_{Q,\mathfrak{U}}) := \{f \in \mathcal{H}_p(K^p) \otimes \mathcal{O}(\mathfrak{U}) | T_{Q,\mathfrak{U}}(ff') = 0,\ \forall f' \in \mathcal{H}_p(K^p)\}.$$

By the basic properties of pseudo-representations, we see that $h_{J,Q,\mathfrak{U}}$ is a finite algebra over $\mathcal{O}(\mathfrak{U})$ and is therefore an affinoid algebra. Let us assume that $\mathfrak{U}$ is reduced. By the theory of pseudo-representations, a theorem of Taylor [Tay91] implies that $h_{J,Q,\mathfrak{U}}$ is the image of $R_{\mathfrak{U}}$ by a semi-simple representation $\rho_{J,Q,\mathfrak{U}}$ of $\mathcal{H}_p(K^p)$ of dimension $\deg(Q)$ defined over a finite extension of the total ring of fractions of $\mathcal{O}(\mathfrak{U})$. Moreover, for all $f \in \mathcal{H}_p(K^p)$, $\operatorname{Ch}_{J,Q,\mathfrak{U}}(f, X) := \det(1 - X\rho_{J,Q,\mathfrak{U}}(f))$ has coefficients in $\mathcal{O}(\mathfrak{U})$ since $\operatorname{tr}(\rho_{J,Q,\mathfrak{U}}(f)) = T_{Q,\mathfrak{U}}(f) \in \mathcal{O}(\mathfrak{U})$ for all $f \in \mathcal{H}_p(K^p)$.

We then write $Y_{J,Q,\mathfrak{U}} := \mathrm{Sp}(h_{J,Q,\mathfrak{U}})$ for the corresponding affinoid variety. There is a canonical map $Y_{J,Q,\mathfrak{U}} \to \mathfrak{U}$ which fits in a canonical diagram

$$\begin{array}{ccc} Y_{J,Q,\mathfrak{U}} & \subset & \mathfrak{Y}_{S,p} = \mathfrak{X} \times \mathfrak{R}_{S,p} \\ \downarrow & & \downarrow \\ \mathfrak{U} & \subset & \mathfrak{X}. \end{array}$$

Moreover, the map R_{f_0} induces a canonical surjective map

$$r_{f_0}:\ Y_{J,Q,\mathfrak{U}} \to \mathfrak{W}_{Q,\mathfrak{U}} \subset \mathfrak{Z}_J(f_0)$$

above $\mathfrak{U}$ which is finite since both the source and the target are finite over $\mathfrak{U}$.

In this construction, $\mathfrak{U}$ is not necessarily supposed to be open. In the particular case where $\mathfrak{U}$ is reduced to a point $\mathfrak{U} = \{\lambda\}$, then $Y_{J,Q,\{\lambda\}} = \mathrm{Sp}(h_{J,Q,\{\lambda\}})$ is the set of finite slope characters θ of $R_{S,p}$ such that $Q_\lambda(\theta(u_t)^{-1}) = 0$ and $m_J(\lambda, \theta, K^p) > 0$. Moreover, $h_{J,Q,\{\lambda\}}$ is the unique quotient of $R_{S,p}\otimes\mathbb{Q}_p$ having these characters.

LEMMA 5.3.2. *Let $\mathfrak{U}' \subset \mathfrak{U}$ be an affinoid subdomain* (*not necessarily open*); *then the kernel of the canonical surjective map $h_{J,Q,\mathfrak{U}} \otimes_{\mathcal{O}(\mathfrak{U})} \mathcal{O}(\mathfrak{U}') \to h_{J,Q,\mathfrak{U}'}$ is contained in the nilradical of $h_{J,Q,\mathfrak{U}} \otimes_{\mathcal{O}(\mathfrak{U})} \mathcal{O}(\mathfrak{U}')$. In particular,*

$$Y_{J,Q,\mathfrak{U}}(\overline{\mathbb{Q}}_p) = r_{f_0}^{-1}(\mathfrak{W}_{Q,\mathfrak{U}}(\overline{\mathbb{Q}}_p)) = \{(\lambda,\theta) \in (\mathfrak{E}_{J,K^p} \times_{\mathfrak{X}} \mathfrak{U})(\overline{\mathbb{Q}}_p) | Q_\lambda(\theta(f_0)^{-1}) = 0\}.$$

Proof. We may clearly assume that $\mathfrak{U}$ is reduced as this case will trivially imply the general case. Let f be in the kernel of this map. Then the coefficients of the characteristic polynomial $\mathrm{Ch}_{J,Q,\mathfrak{U}}(f,X)$ must belong to the kernel of $\mathfrak{b} := \ker(\mathcal{O}(\mathfrak{U}) \to \mathcal{O}(\mathfrak{U}'))$. By Cayley-Hamilton's theorem, $\mathrm{Ch}_{J,Q,\mathfrak{U}}(f, \rho_{J,Q,\mathfrak{U}}(f)) = 0$, and therefore $f^{\deg(Q)}$ can be expressed as a polynomial in f with coefficients in $\mathfrak{b}$. This implies that f is nilpotent in $h_{J,Q,\mathfrak{U}}\otimes_{\mathcal{O}(\mathfrak{U})}\mathcal{O}(\mathfrak{U})/\mathfrak{b} = h_{J,Q,\mathfrak{U}}\otimes_{\mathcal{O}(\mathfrak{U})}\mathcal{O}(\mathfrak{U}')$ which proves the first part of the proposition. The second part follows from the case $\mathfrak{U}' = \{\lambda\}$ for any $\lambda \in \mathfrak{U}(\overline{\mathbb{Q}}_p)$ and the description of $Y_{J,Q,\{\lambda\}}$ which was done before. □

5.3.3. *Gluing of the local pieces.* We need to show that the pieces $Y_{J,Q,\mathfrak{U}}$ glue together when the $\mathfrak{W}_{Q,\mathfrak{U}}$ do in the spectral variety $\mathfrak{Z}_J(f_0)$.

LEMMA 5.3.4. *Let $\mathfrak{U}' \subset \mathfrak{U}$ be an inclusion of open affinoid subdomains of $\mathfrak{X}$; then the canonical surjective map $h_{J,Q,\mathfrak{U}} \otimes_{\mathcal{O}(\mathfrak{U})} \mathcal{O}(\mathfrak{U}') \to h_{J,Q,\mathfrak{U}'}$ is an isomorphism.*

Proof. Since $\mathfrak{U}' \subset \mathfrak{U}$ is an inclusion of open affinoid subdomains of $\mathfrak{X}$, the map $\mathcal{O}(\mathfrak{U}) \mapsto \mathcal{O}(\mathfrak{U}')$ is flat. Let $R_{\mathfrak{U}} := R_{S,p}\otimes\mathcal{O}(\mathfrak{U})$; then $h_{J,Q,\mathfrak{U}} = R_{\mathfrak{U}}/\ker(T_{Q,\mathfrak{U}})$. Now since $\mathcal{O}(\mathfrak{U}) \to \mathcal{O}(\mathfrak{U}')$ is flat, we have $\ker(T_{Q,\mathfrak{U}'}) \cong \ker(T_{Q,\mathfrak{U}}) \otimes_{\mathcal{O}(\mathfrak{U})} \mathcal{O}(\mathfrak{U}')$.

Therefore,

$$\begin{aligned}
h_{J,Q,\mathfrak{U}'} &= R_{\mathfrak{U}'}/\ker(T_{Q,\mathfrak{U}'})\\
&= (R_{\mathfrak{U}} \otimes_{\mathcal{O}(\mathfrak{U})} \mathcal{O}(\mathfrak{U}'))/(\ker(T_{Q,\mathfrak{U}}) \otimes_{\mathcal{O}(\mathfrak{U})} \mathcal{O}(\mathfrak{U}'))\\
&= (R_{\mathfrak{U}}/\ker(T_{Q,\mathfrak{U}})) \otimes_{\mathcal{O}(\mathfrak{U})} \mathcal{O}(\mathfrak{U}')\\
&= h_{J,Q,\mathfrak{U}} \otimes_{\mathcal{O}(\mathfrak{U})} \mathcal{O}(\mathfrak{U}'). \qquad \square
\end{aligned}$$

PROPOSITION 5.3.5. *Assume that we have a factorization $Q = Q'Q''$ with $(Q', Q'') = 1$ in $\mathcal{O}(\mathfrak{U})[X]$. Then the canonical inclusion map $\mathfrak{W}_{Q',\mathfrak{U}} \hookrightarrow \mathfrak{W}_{Q,\mathfrak{U}}$ induces the following canonical isomorphism*:

$$\begin{tikzcd}
Y_{J,Q',\mathfrak{U}} \arrow[rr, phantom, "\cong"] \arrow[drr, "r_{f_0}"'] & & Y_{J,Q,\mathfrak{U}} \times_{\mathfrak{W}_{Q,\mathfrak{U}}} \mathfrak{W}_{Q',\mathfrak{U}} \arrow[r] \arrow[d] & Y_{J,Q,\mathfrak{U}} \arrow[d, "r_{f_0}"'] \\
& & \mathfrak{W}_{Q',\mathfrak{U}} \arrow[r] & \mathfrak{W}_{Q,\mathfrak{U}}.
\end{tikzcd}$$

Proof. Since $(Q', Q'') = 1$, we have $N_{J_\lambda}(Q) = N_{J_\lambda}(Q') \oplus N_{J_\lambda}(Q'')$ for all $\lambda \in \mathfrak{U}(\overline{\mathbb{Q}}_p)$. Moreover, if we write $1 = Q'(X)R'(X)+Q''(X)R''(X)$ in $\mathcal{O}(\mathfrak{U})[X]$, then the evaluation at λ of $e_0' := Q''(f_0)R''(f_0)$ (resp. $e_0'' := Q'(f_0)R'(f_0)$) acting on $N_{J_\lambda}(Q)$ is the projector onto $N_{J_\lambda}(Q')$ (resp. onto $N_{J_\lambda}(Q'')$). We deduce that

$$\ker(T_{Q,\mathfrak{U}}) = \ker(T_{Q',\mathfrak{U}}) \cap \ker(T_{Q'',\mathfrak{U}}). \tag{33}$$

Indeed notice first that the splitting above shows that $T_{Q,\mathfrak{U}} = T_{Q',\mathfrak{U}} + T_{Q'',\mathfrak{U}}$, and therefore the intersection is included inside the left-hand side of equality (33). Now if $g \in \ker(T_{Q,\mathfrak{U}})$, then $T_{Q',\mathfrak{U}}(gf) = T_{Q,\mathfrak{U}}(gfe_0') = 0$ for all $f \in R_{\mathfrak{U}}$, which implies that $\ker(T_{Q,\mathfrak{U}}) \subset \ker(T_{Q',\mathfrak{U}})$. Similarly, $\ker(T_{Q,\mathfrak{U}}) \subset \ker(T_{Q'',\mathfrak{U}})$ which finishes the proof of (33). On the other hand, $R_{\mathfrak{U}} = \ker(T_{Q',\mathfrak{U}}) + \ker(T_{Q'',\mathfrak{U}})$ since any $f \in R_{\mathfrak{U}}$ can be written as $fe_0'' + fe_0'$. By the Chinese Remainder Theorem, this implies we have a canonical isomorphism

$$h_{J,Q,\mathfrak{U}} \cong h_{J,Q',\mathfrak{U}} \times h_{J,Q'',\mathfrak{U}}$$

compatible with the canonical maps from $R_{S,p}$ in the algebras $h_{J,Q,\mathfrak{U}}$, $h_{J,Q',\mathfrak{U}}$ and $h_{J,Q'',\mathfrak{U}}$, respectively. This easily implies the claim of the proposition. $\square$

5.3.6. From the previous proposition and lemma, we can deduce, as was done by other authors (for example, see [Buz07, §5]) from Propositions 9.3.2/1 and 9.3.3/1 of [BGR84], that the $Y_{J,Q,\mathfrak{U}}$ glue together into a reduced rigid analytic variety $\mathfrak{E}'_{J,K^p}$ with a finite map r_{f_0} over $\mathfrak{Z}_J(f_0)$ such that for each pair $(Q, \mathfrak{U})$, we have

$$\mathfrak{E}'_{J,K^p} \times_{\mathfrak{Z}_J(f_0)} \mathfrak{W}_{Q,\mathfrak{U}} = Y_{J,Q,\mathfrak{U}}.$$

The cocycle conditions defining the descent data are satisfied since they are satisfied for the spectral variety $\mathfrak{Z}_J(f_0)$. Moreover, by Lemma 5.3.2, $\mathfrak{E}'_{J,K^p}(\overline{\mathbb{Q}}_p) =$

$\mathfrak{E}_{J,K^p}(\overline{\mathbb{Q}}_p)$. This implies that $\mathfrak{E}_{J,K^p}$ is a rigid analytic space whose corresponding reduced closed subspace is $\mathfrak{E}'_{J,K^p}$. We will therefore denote the later by $\mathfrak{E}^{\mathrm{red}}_{J,K^p}$. We may summarize our results by the following theorem.

THEOREM 5.3.7. *Let J and K^p as before. Then $\mathfrak{E}_{J,K^p} \subset \mathfrak{Y}_{S,p}$ is a rigid analytic equidimensional space over $\mathbb{Q}_p$ satisfying the following properties*:

(i) *For any $y = (\lambda_y, \theta_y) \in \mathfrak{Y}_{S,p}(\overline{\mathbb{Q}}_p)$, we have $y \in \mathfrak{E}_{J,K^p}(\overline{\mathbb{Q}}_p)$ if and only if $m_J(\lambda, \theta, K^p) > 0$.*

(ii) *For any K^p-admissible Hecke operator f, the projection map*

$$r_f : \ \mathfrak{E}_{J,K^p} \to \mathfrak{Z}_J(f)$$

is a finite surjective morphism.

(iii) *$\mathfrak{E}_{J,K^p}$ is equi-dimensional of dimension* $\dim \mathfrak{X}$.

Proof. Point (i) is Proposition 5.2.3. Point (iii) follows from the fact that $\mathfrak{E}_{J,K^p} = \cup_{Q,\mathfrak{U}} \mathrm{Sp}(h_{J,Q,\mathfrak{U}})$, where $(Q, \mathfrak{U})$ runs over the open affinoid subdomain of $\mathfrak{U}$ and Q over the polynomial of $\mathcal{O}(\mathfrak{U})[X]$ inducing a prime factorization of $P_J(f_0, X) \in \mathcal{O}(\mathfrak{U})\{\{X\}\}$ and the fact that $h_{J,Q,\mathfrak{U}}$ is a finite torsion free $\mathcal{O}(\mathfrak{U})$-algebra. We are left with point (ii). The surjectivity follows from the fact that for any K^p-admissible f, we have

$$m_J(\lambda_y, f, \theta_y(f), K^p) = \sum_{\substack{\theta \\ \theta(f)=\theta_y(f)}} m_J(\lambda_y, \theta, K^p)$$

and the caracterization of the points of $\mathfrak{E}_{J,K^p}$ and $\mathfrak{Z}_J(f)$ given by Propositions 5.1.6 and 5.2.3. Now let $(Q, \mathfrak{U})$ as before and let $Q_f^*(X) := \mathrm{Ch}_{J,Q,\mathfrak{U}}(f, X)$. Then the map r_f^* is induced by the $\mathcal{O}(\mathfrak{U})$-algebra homomorphism

$$\mathcal{O}(\mathfrak{U})[X]/(Q_f^*(X)) \to h_{J,Q,\mathfrak{U}}$$

induced by $X \mapsto f$. Since $h_{J,Q,\mathfrak{U}}$ is finite over $\mathcal{O}(\mathfrak{U})$ and the image of $Y_{J,Q,\mathfrak{U}}$ by r_f is clearly $\mathrm{Sp}(\mathcal{O}(\mathfrak{U})[X]/(Q_f^*(X)))$, we deduce that r_f is finite on $\mathfrak{E}^{\mathrm{red}}_{J,K^p}$. Since it factorizes through $\mathfrak{E}_{J,K^p}$, it is also finite on the latter. This finishes the proof of our theorem. □

COROLLARY 5.3.8. *Every irreducible component of $\mathfrak{E}_{J,K^p}$ projects surjectively onto a Zariski dense subset of $\mathfrak{X}$.*

Proof. Let $\mathfrak{V}$ be an irreducible component of $\mathfrak{E}_{J,K^p}$ and let us choose a point $y \in \mathfrak{V}(\overline{\mathbb{Q}}_p)$ belonging to only one irreducible component of $\mathfrak{E}_{K^p}$. By Proposition 5.2.3, we can choose a K^p-admissible f such that $r_f^{-1}(r_f(y))$ is reduced to $\{y\}$. Let $\mathfrak{W}$ be an irreducible component of $\mathfrak{Z}_J(f)$ containing $r_f(y)$. Because r_f is finite and surjective, there is one irreducible component $\mathfrak{V}'$ of $r_f^{-1}(\mathfrak{W})$ such that $r_f(\mathfrak{V}') = \mathfrak{W}$. In particular, $\mathfrak{V}'(\overline{\mathbb{Q}}_p) \cap r_f^{-1}(r_f(y)) \neq \emptyset$, and therefore $y \in \mathfrak{V}'(\overline{\mathbb{Q}}_p)$. Since $\mathfrak{V}$ is the irreducible component of $\mathfrak{E}_{J,K^p}$ containing

y, we must have $\mathfrak{V}' = \mathfrak{V}$. Note that we also have $r_f(\mathfrak{V}) = \mathfrak{W}$ which implies that there is also only one irreducible component of $\mathfrak{Z}_J(f)$ containing $r_f(y)$. Since the statement of our corollary is true for the projection $\mathfrak{Z}_J(f) \to \mathfrak{X}$, we deduce that the projection of $\mathfrak{V}$ onto $\mathfrak{X}$ has a Zariski dense image. $\square$

5.3.9. *Families of irreducible finite slope representations of $\mathcal{H}_p(K^p)$.* An irreducible component of $\mathfrak{E}_{K^p,J}$ can be seen as a family of finite slope characters of $R_{S,p}$. We now want to generalize this to representations of $\mathcal{H}_p(K^p)$ having positive multiplicity with respect to J. We have the following proposition.

PROPOSITION 5.3.10. *Let $\lambda_0 \in \mathfrak{X}(\overline{\mathbb{Q}}_p)$ and σ_0 be an irreducible finite slope representation such that $m_J(\lambda_0, \sigma_0) > 0$. Let $y_0 = (\lambda_0, \theta_{\sigma_0})$ be the corresponding point of the eigenvariety $\mathfrak{E}_{J,K^p}$. Then there exist*

(i) *a finite flat covering $\mathfrak{V}$ over an affinoid open subdomain $\mathfrak{W}$ of $\mathfrak{E}_{J,K^p}$ containing y_0 which is finite and generically flat over its projection $\mathfrak{U} \subset \mathfrak{X}$ in to weight space,*
(ii) *a point $x_0 \in \mathfrak{V}(\overline{\mathbb{Q}}_p)$ above y_0,*
(iii) *for all $x \in \mathfrak{V}(\overline{\mathbb{Q}}_p)$, a (nonempty) finite set Π_x of irreducible finite slope representations σ of $\mathcal{H}_p(K^p)$ such that $\theta_\sigma = \theta_x$ is the character of $R_{S,p}$ attached to the projection of x into $\mathfrak{W}(\overline{\mathbb{Q}}_p) \subset \mathfrak{E}_{J,K^p}(\overline{\mathbb{Q}}_p)$,*
(iv) *a nontrivial linear map $I_{\mathfrak{V}} : \mathcal{H}_p(K^p) \to \mathcal{O}(\mathfrak{V})$,*

such that if for any $x \in \mathfrak{V}(\overline{\mathbb{Q}}_p)$, we write I_x for the composite of $I_{\mathfrak{V}}$ with the evaluation map at x from $\mathcal{O}(\mathfrak{V})$ into $\overline{\mathbb{Q}}_p$ and λ_x for the image of x in $\mathfrak{X}(\overline{\mathbb{Q}}_p)$. Then

(a) *For all $x \in \mathfrak{V}(\overline{\mathbb{Q}}_p)$, $I_x = \sum_{\sigma\in\Pi_x} m_x(\sigma) J_\sigma$ with $m_x(\sigma) > 0$ only if $m_J(\lambda_x, \sigma) > 0$.*
(b) *There exists a Zariski dense subset $\mathfrak{V}(\overline{\mathbb{Q}}_p)^{\mathrm{generic}} \in \mathfrak{V}(\overline{\mathbb{Q}}_p)$ such that Π_x is a singleton $\{\sigma_x\}$ and $m_x(\sigma) = m_J(\lambda_x, \sigma_x)$ is constant for all $x \in \mathfrak{V}(\overline{\mathbb{Q}}_p)^{\mathrm{generic}}$;*
(c) *$\sigma_0 \in \Pi_{x_0}$;*
(d) *Let $\theta_{\mathfrak{W}}$ be the canonical character of $R_{S,p}$ corresponding to $\mathfrak{W}$; then $I_{\mathfrak{V}}(ff') = \theta_{\mathfrak{W}}(f) I_{\mathfrak{W}}(f')$ for all $f \in R_{S,p}$ and $f' \in \mathcal{H}_p(K^p)$.*

Proof. Let $y_0 = (\lambda_0, \theta_0)$ with $\theta_0 = \theta_{\sigma_0}$. Then $y_0 \in \mathfrak{E}_{J,K^p}(\overline{\mathbb{Q}}_p)$. Let $\mathfrak{W}$ be an affinoid neighborhood of y_0 inside $\mathfrak{E}^{\mathrm{red}}_{J,K^p}$. We may assume that $Y = Y_{J,Q,\mathfrak{U}}$ for some admissible pair $(Q, \mathfrak{U})$ with $\mathfrak{U}$ an open affinoid neighborhood of λ_0 inside $\mathfrak{X}$. Then we consider the character $T_{J,Q,\mathfrak{U}}$ representation $\rho_{J,Q,\mathfrak{U}}$. Because the image of $\mathcal{H}_p(K^p) \otimes \mathcal{O}(\mathfrak{U})$ by $\rho_{J,Q,\mathfrak{U}}$ is finite over $\mathcal{O}(\mathfrak{U})$, after extending the scalar to a finite extension of $\mathcal{O}(Y)$ or equivalently after replacing Y by a finite cover Y', it decomposes as a sum of isotypical components $T_{J,Q,\mathfrak{U}} = T_1 + \cdots + T_m$, where the T_i's are characters of isotypical representations of $\mathcal{H}_p(K^p)$ defined on the fraction ring of $\mathcal{O}(Y')$. Since these are isotypical, the action of $R_{S,p}$ on the semi-simple representation of each T_i is given by characters taking values in the ring of analytic functions of some irreducible components of Y. Moreover, there exists a point x_0 of $Y'(\overline{\mathbb{Q}}_p)$ above $y_0 = (\lambda_0, \theta_0) \in Y(\overline{\mathbb{Q}}_p)$ such that the specialization of $T_{J,Q,\mathfrak{U}}$ at x_0 contains J_{σ_0} as a summand. *A fortiori* there exists i_0 such that the specialization at x_0 at T_{i_0} contains J_{σ_0} as a summand. Let $\mathfrak{V}$ be the irreducible component of Y' containing x_0 such that the character T_{i_0} is defined over the fraction field of $\mathcal{O}(\mathfrak{V})$ and let $\mathfrak{W}$ be the image of $\mathfrak{V}$ via the finite projection $Y' \to Y$. Let us call $I_{\mathfrak{V}}$ the composite of T_{i_0} with the restriction map $\mathcal{O}(Y') \to \mathcal{O}(\mathfrak{V})$. Then we must have $I_{\mathfrak{V}}(ff') = \theta_{\mathfrak{W}}(f)I_{\mathfrak{V}}(f')$. From the definitions and construction, it is clear that (a), (c) and (d) are satisfied. Point (c) is a direct consequence of the following easy fact: If a character $I_{\mathfrak{V}}$ from an algebra A into an affinoid algebra $\mathcal{O}(\mathfrak{V})$ is generically irreducible, then the set of $y \in \mathfrak{V}(\overline{\mathbb{Q}}_p)$, such that the specialization at y of $I_{\mathfrak{V}}$ is reducible, is a proper closed affinoid subspace of $\mathfrak{V}$. □

5.4. *Application to finite slope automorphic character distributions.* We can apply the formal result of the previous section to the families of finite slope character distributions we have constructed in the previous chapter. Let $\mathcal{L}_G^c \subset \mathcal{L}_G$ be the subset of standard Levi subgroups of G having discrete series. For each $J = (-1)^{d_M} I^\dagger_{G,M,w}$, when $M \in \mathcal{L}_G^c$ and $w \in \mathcal{W}^M_{\mathrm{Eis}}$, we obtain an eigenvariety $\mathfrak{E}_{K^p,M,w}$. If $G(\mathbb{R})$ has discrete series, then we denote by $\mathfrak{E}_{K^p,0}$ the previous eigenvariety when $M = G$. We also write

$$\mathfrak{E}_{K^p} := \bigcup_{M\in\mathcal{L}_G^c} \bigcup_{w\in\mathcal{W}^M_{\mathrm{Eis}}} \mathfrak{E}_{K^p,M,w},$$

$$\mathfrak{E}_{K^p,M} := \bigcup_{w\in\mathcal{W}^M_{\mathrm{Eis}}} \mathfrak{E}_{K^p,M,w} \quad \mathfrak{E}_{K^p,Eis} := \bigcup_{\substack{M\in\mathcal{L}_G^c\\ M\neq G}} \mathfrak{E}_{K^p,M}.$$

From the previous section and the construction and properties of the automorphic p-adic finite slope distributions of the previous chapter, these eigenvarieties are equidimensional. It is also useful to notice the following proposition.

Proposition 5.4.1. *Let $M, M' \in \mathcal{L}_G^c$ with $M \neq M'$; then the intersection of the subvarieties $\mathfrak{E}_{K^p,M}$ and $\mathfrak{E}_{K^p,M'}$ is of dimension smaller than the dimension of $\mathfrak{X}_{K^p}$.*

Proof. By an easy reduction step, we can reduce to the case $M' = G$ and M any proper Levi of G in $\mathcal{L}_G^c$. Assume that the statement of this proposition is wrong. That means that we have an irreducible component in the intersection of dimension $\dim \mathfrak{X}$. By the corollary above, we can find a noncritical point y such that λ_y is a very regular point (as regular as we want in fact) in this intersection. This point y would be classical and associated to cuspidal representation and an Eisenstein series at the same time. This cuspidal representation would be CAP in the sense of Piatetski-Shapiro. By a result of M. Harris [Har84], this is not possible if λ_y is chosen sufficiently regular. $\square$

5.4.2. *Noncritical and classical points.* A point $y = (\lambda_y, \theta_y) \in \mathfrak{E}_{K^p}(\overline{\mathbb{Q}}_p)$ is called classical if λ_y is arithmetic and if θ_y is attached to an automorphic classical finite slope representation of weight λ_y^{alg}. A point $y = (\lambda_y, \theta_y)$ is said to be noncritical if λ_y is an arithmetic weight and if θ_y is noncritical with respect to λ_y^{alg}. Finally y is said to be regular if λ_y is arithmetic regular. By the discussion of the previous chapter (in particular Corollary 4.6.5), we know that any noncritical regular point of the eigenvariety is classical. In particular, these points in any open affinoid subdomain of the eigenvariety are Zariski dense.

5.4.3. *Families of finite slope automorphic representations.* Let π be an automorphic representation of $G(\mathbb{A})$ occurring in the cohomology with the system of coefficient $\mathbb{V}_{\lambda_0}^\vee(\mathbb{C})$. It is defined over a $\overline{\mathbb{Q}}_p$ after we have fixed embeddings of $\overline{\mathbb{Q}}$ in $\mathbb{C}$ and $\overline{\mathbb{Q}}_p$. We assume that $\pi_f^{I'_m.K^p}$ is nonzero for some $m > 0$ and some open compact subgroup K^p. We further fix a $\overline{\mathbb{Q}}_p$-valued finite order character ε of $T(\mathbb{Z}/p^m\mathbb{Z})$ and let θ_0 be a finite slope character of $R_{S,p}$ occurring in $\pi_f^{I'_m.K^p} \otimes \overline{\mathbb{Q}}_p(\varepsilon^{-1})$. This determines what we have called a p-stabilization of π, and we suppose that it is of finite slope. Let σ_0 be the irreducible constituent of the restriction of $\pi_f^{I'_m.K^p} \otimes \overline{\mathbb{Q}}_p(\varepsilon^{-1})$ to $\mathcal{H}_p(K^p)$ such that $J_{\sigma_0}(fg) = \theta_0(f) J_{\sigma_0}(g)$ for any $f \in R_{S,p}$ and any $g \in \mathcal{H}_p(K^p)$. Then σ_0 is a finite slope automorphic representation, and the following theorem is perhaps the most striking result of this paper.

THEOREM 5.4.4. *Assume that $m_0^\dagger(\sigma_0, \lambda_0) \neq 0$. Then, there exists*

(1) *an affinoid open neighborhood $\mathfrak{U} \subset \mathfrak{X}$ of λ_0;*
(2) *a finite cover $\mathfrak{V}$ of $\mathfrak{U}$ with structural morphism $\mathbf{w}$;*
(3) *a homomorphism $\theta_{\mathfrak{V}} : R_{S,p} \to \mathcal{O}(\mathfrak{V})$;*
(4) *a character distribution $I_{\mathfrak{V}} : \mathcal{H}_p(K^p) \to \mathcal{O}(\mathfrak{V})$;*
(5) *a point $y_0 \in \mathfrak{V}(\overline{\mathbb{Q}}_p)$ above λ_0;*
(6) *a Zariski dense subset $\Sigma \subset \mathfrak{V}(\overline{\mathbb{Q}}_p)$ such that $\lambda_y = \mathbf{w}(y)$ is an arithmetic weight for all $y \in \Sigma$;*
(7) *for each $y \in \Sigma$, a finite set Π_y of irreducible finite slope cohomological cuspidal representations of weight $\lambda_y = \mathbf{w}(y)$;*

satisfying the following:

(i) *The specialization of $\theta_{\mathfrak{V}}$ at y_0 is equal to θ_0.*
(ii) *The character distribution I_{σ_0} is an irreducible component of the specialization I_{y_0} of $I_{\mathfrak{V}}$ at y_0.*

(iii) *For any* $y \in \Sigma$, *the specialization* θ_y *of* $\theta_{\mathfrak{Y}}$ *at* y *is a character occurring in the representation of* $R_{S,p}$ *in* π^{K^p} *for all* $\pi \in \Pi_y$.

(iv) *For each* $y \in \Sigma$ *the specialization* I_y *of* $I_{\mathfrak{Y}}$ *at* y *satisfies*

$$I_y(f) = \sum_{\sigma \in \Pi_y} m^{\mathrm{cl}}(\sigma, \lambda_y)\mathrm{tr}(\pi_y(f)),$$

where $m^{\mathrm{cl}}(\sigma, \lambda_y)$ *is the Euler-Poincaré characteristic of* σ *defined as*

$$m^{\mathrm{cl}}(\sigma, \lambda_y) := \sum_i (-1)^i \dim_{\mathbb{C}} \mathrm{Hom}_{\mathcal{H}_p}(\sigma, \varinjlim_{K^p} H^i(S_G(K^p.I_m), \mathbb{V}^\vee_{\lambda_y^{\mathrm{alg}}}(\varepsilon_y, \mathbb{C})))$$

with $\lambda_y = \lambda_y^{\mathrm{alg}} \varepsilon_y$.

Moreover, Π_y *contains only one representation for* y *in a Zariski dense subset of* Σ.

Proof. We just apply Proposition 5.3.10 to $J = (-1)^{d_G} I^\dagger_{G,0}$. Then we consider Σ as the subset of points $y = (\lambda, \theta)$ of $\mathfrak{Y}(\overline{\mathbb{Q}}_p)$ such that λ is arithmetic regular and θ is of noncritical slope with respect to λ. This set is easily seen to be Zariski dense in $\mathfrak{Y}(\overline{\mathbb{Q}}_p)$ since the projection of $\mathfrak{Y}$ onto $\mathfrak{X}$ contains the arithmetic point λ_0. Moreover, these points are classical and correspond to cuspidal representations by Corollary 4.6.5. More precisely, for $y \in \Sigma$ and $\sigma \in \Pi_y$, we have $m^\dagger_{G,0}(\sigma, \lambda_y) = m^{\mathrm{cl}}(\sigma, \lambda_y)$. This concludes the proof of Theorem 5.4.4. $\square$

5.5. *Examples.*

5.5.1. *About the hypothesis.* We now explain that the hypothesis of the previous theorem is satisfied for a very large class of automorphic representations. We assume that $G(\mathbb{R})$ has discrete series. For π_f an irreducible representation of G_f and λ a dominant algebraic weight, let us define the Euler-Poincaré multiplicity of π_f with respect to λ by

$$m_{\mathrm{EP}}(\pi_f, \lambda) = \sum_i (-1)^i \dim_{\mathbb{C}} \mathrm{Hom}_{G_f}(\pi_f, H^i(\widetilde{S}_G, \mathbb{V}^\vee_\lambda(\mathbb{C}))).$$

When the weight λ is regular let us also define

$$m(\pi_f, \lambda) := (-1)^{d_G} \sum_{\pi_\infty \in \Pi_\lambda} m(\pi_f \otimes \pi_\infty).$$

By the results recalled in the first chapter, we then have

$$m(\pi_f, \lambda) = m_{\mathrm{EP}}(\pi_f, \lambda).$$

If σ is a p-stabilization of π_f, we therefore have

$$m^{\mathrm{cl}}(\sigma, \lambda) = m(\pi_f, \lambda) \times \dim \mathrm{Hom}_{\mathcal{H}_p}(\sigma, (\pi_f|_{\mathcal{H}_p})^{\mathrm{ss}}),$$

where $(\pi_f|_{\mathcal{H}_p})^{\mathrm{ss}}$ stands for the semi-simplification of the restriction of π_f to $\mathcal{H}_p$. So if this p-stabilization is noncritical with respect to λ, then $m^\dagger(\sigma, \lambda)$ is nonzero if and only if $m(\pi_f, \lambda) \neq 0$, which is the case, for example, when π_f is attached to a cuspidal representation of weight λ. This applies in particular to all cuspidal forms of regular weight for symplectic or unitary groups over totally real fields. In those cases, the previous theorem applies and there exist families passing through σ.

However, if $m(\pi_f, \lambda) \neq 0$ but σ is critical, then it is not clear in general that $m^\dagger(\sigma, \lambda) \neq 0$. In what follows, we look at some examples of this situation.

5.5.2. *An example in the* GL(2)*-case.* Let $G = \mathrm{GL}(2)_{/\mathbb{Q}}$. Assume that $\pi_f = \mathrm{triv}$ is the trivial one-dimensional representation and $\lambda_0 = 1$ is the trivial weight. There is only one p-stabilization σ since π_f is one-dimensional, and this is the trivial representation itself. The U_p operator attached to the double class of $\mathrm{diag}(1, p^{-1})$ has eigenvalue p, and the eigenvalue of T_ℓ for $\ell \neq p$ is $1 + \ell$. It is easy to check that π_f shows up only in degree 0 since the Eisenstein series E_2 is not holomorphic. Therefore in that case $m^{\mathrm{cl}}(\sigma, \lambda) = 1$. However, $m_0^\dagger(\sigma, \lambda)$ must vanish since otherwise one would have a p-adic family of slope 1 passing through the p-adic form $E_2(q) - E_2(q^p)$, and we know that this is impossible by a theorem of Coleman-Gouvea-Jochnowitz (see also [SU02]). We could in fact show that $m_0^\dagger(\sigma, \lambda) = 0$ in that case using the multiplicity formula of Corollary 4.5.5. Notice that $m_{\mathrm{Eis}}^\dagger(\sigma, \lambda) = 0$ since σ is not ordinary.

5.5.3. *An example in the* GSp(4)*-case.* In this example, we will use the standard notations without defining all of them in detail. In particular, we refer the reader to [TU99] or [SU06a] for the definitions on the group $G = \mathrm{GSp}(4)$ and its corresponding automorphic forms. Let (B, T) be the Borel pair with T the diagonal torus and B the Borel subgroup stabilizing the standard flag of the symplectic space attached to G. Let s_1 and s_2 be the symmetries of $X^*(T)$ attached, respectively, to the small and long simple roots attached to the pair (B, T). An algebraic weight λ is a triple $(a, b; c)$ with $a, b, c \in \mathbb{Z}$ and $a + b \equiv c \pmod 2$, and it maps T to the multiplicative group by the rule $\mathrm{diag}(t_1, t_2, t_1^{-1}\nu, t_2^{-1}\nu) \mapsto t_1^a t_2^b \nu^{(c-a-b)/2}$. A weight is dominant if $a \geq b \geq 0$. The simple short and long roots are respectively $(1, -1; 0)$ and $(0, 2; 0)$ and the corresponding simple reflexion s_1 and s_2 act on $X^*(T)$ by $s_1(a, b; c) = (b, a; c)$ and $s_2(a, b; c) = (a, -b; c)$. Since $\rho = (2, 1; 0)$, we have $s_1 * (a, b; c) = (b - 1, a + 1; c)$ and $s_2 * (a, b; c) = (a, -b - 2; c)$.

We denote respectively by $M_S \cong \mathrm{GL}_2 \times \mathbb{G}_m$ and $M_K \cong \mathrm{GL}_2 \times \mathbb{G}_m$ the standard Levi of the Siegel and Klingen parabolic subgroup of G. The corresponding isomorphisms are given in the Siegel case by $(g, \nu) \mapsto \mathrm{diag}(g, {}^t g^{-1}\nu)$ and in the Klingen case by

$$\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, x\right) \mapsto \begin{pmatrix} x & 0 & 0 & 0 \\ 0 & a & 0 & b \\ 0 & 0 & (ad-bc)x^{-1} & \\ 0 & c & 0 & d \end{pmatrix}.$$

One can easily check that $W_{\mathrm{Eis}}^{M_S} = \{\mathrm{id}, s_2\}$ and $W_{\mathrm{Eis}}^{M_K} = \{\mathrm{id}, s_1\}$.

Let π be a unitary cuspidal representation of $\mathrm{PGL}(2)_{/\mathbb{Q}}$ whose corresponding classical Hecke newform f is of weight $2k-2$ with k an integer greater than 2. We make the following assumption:

$$L(f, k-1) = L(\pi, 1/2) = 0.$$

Under this hypothesis, there exists a unique cuspidal nontempered representation $\mathrm{SK}(\pi)$ on $\mathrm{GSp}(4)_{/\mathbb{Q}}$ such that its degree 4 L-function is given by

$$L(\mathrm{SK}(\pi), s) = L(\pi, s)\zeta(s - 1/2)\zeta(s + 1/2),$$

and each local component of $\mathrm{SK}(\pi)$ at a finite place is the nontempered Langlands quotient of the unitary parabolic induction from the Siegel parabolic $\mathrm{Ind}_{M_S}^{G}(\pi_f \otimes \|\cdot\|^{1/2} \times \|\cdot\|^{-1/2}$, (i.e., $\pi_f \otimes \|\cdot\|^{1/2} \rtimes \|\cdot\|^{-1/2}$ with the notation of Sally-Tadič [ST93]).

The corresponding p-adic Galois representation is, up to twist, given by

$$\rho_{\mathrm{SK}(\pi)} = \rho_f \oplus \mathbb{Q}_p(1-k) \oplus \mathbb{Q}_p(2-k),$$

where we have denoted by ρ_f the p-adic Galois representation attached to f which has a determinant given by the cyclotomic character raised to the $(3-2k)$-th power. This automorphic representation is called a Saito-Kurokawa lifting of f. When $k > 2$, $\mathrm{SK}(\pi) \otimes |\nu|^{3-k}$ is cohomological of weight $\lambda_k = (k-3, k-3; 2k-6)$. Here we have denoted by $|\nu|$ the adelic norm of the multiplier of G.

Let $\varepsilon = \varepsilon(\pi, 1/2) = \pm 1$. This sign determines the nature of the archimedean component of $\mathrm{SK}(\pi)$. When $\varepsilon = 1$, then $\mathrm{SK}(\pi)_\infty$ and its contragredient are nontempered and cohomological in degree-2 and 4. Therefore $\mathrm{SK}(\pi)_f$ shows up in the cuspidal cohomology in degree-2 and 4 with multiplicity one. If $\varepsilon = -1$, then $\mathrm{SK}(\pi)_\infty$ and its contragredient are the holomorphic and antiholomorphic discrete series. Therefore $\mathrm{SK}(\pi_f)$ appears in the cuspidal cohomology of degree-3 with multiplicity two. As remarked by Harder, in both cases there are also Eisenstein classes attached to $\mathrm{SK}(\pi_f)$ providing a multiplicity one subspace in the Eisenstein cohomology in degree 2 and 3 isomorphic to $\mathrm{SK}(\pi)_f$. All together, we deduce that $m_{\mathrm{EP}}(\mathrm{SK}(\pi)_f) = -2$ if $\varepsilon = -1$ and $m_{\mathrm{EP}}(\mathrm{SK}(\pi)_f) = 2$ if $\varepsilon = +1$. In other words, $m_{\mathrm{EP}}(\mathrm{SK}(\pi)_f) = 2\varepsilon(\pi, 1/2)$. For an account of these facts recalled here and their credits, the reader can consult [Har93], [Sch05], [SU06a], [Wal80], [Wal91].

Let us assume that π is unramified at p and that we can fix a root α of the Hecke polynomial of f at p such that $0 < v = v_p(\alpha) < k-2$. (The case $v = 0$ can be done but it requires a bit more work, and the corresponding result is already known by the work [SU06a].) An element $t = \mathrm{diag}(t_1, t_2, t_1^{-1}\nu, t_2^{-1}\nu)$ belongs to T^- if $2v_p(t_1) \geq 2v_p(t_2) \geq v_p(\nu)$. The slope of a p-stabilization is therefore determined by the eigenvalues of the two Hecke operators $U_{1,p} = I\,\mathrm{diag}(1,1,p,p)^{-1}I$ and $U_{2,p} = I\,\mathrm{diag}(1,p,p^2,p)^{-1}I$. In [SU06a], it is explained that there exists a p-stabilization σ_α of $\mathrm{SK}(\pi)\otimes|\nu|^{3-k}$ such that the eigenvalue of $U_{1,p}$ is α, and the one of $U_{2,p}$ is αp^{k-2}. After renormalization, the eigenvalues are therefore respectively α and αp.

This implies that the slope of this p-stabilization is given by $\mu_{\sigma_\alpha}=(v+1,v-1;0)$ with $v=v_p(\alpha_p)$. We want to compare μ_{σ_α} to $w*\lambda_k-\lambda_k$ for $w=s_1,s_2$. We have $s_1*\lambda_k-\lambda_k=(-1,1;0)$ and $s_2*\lambda_k-\lambda_k=(0,-2k+4;0)$. Therefore $\mu_{\sigma_\alpha}+s_1*\lambda_k-\lambda_k=(v,v;0)$ belongs to the boundary of the obtuse cone. Thus μ_σ is critical. Notice also that $\mu_\sigma+s_2*\lambda_k-\lambda_k=(1+v,v-2k+3;0)$ does not belong to the obtuse cone by our assumption on v. From these remarks we can deduce that

$$m^{\mathrm{cl}}(\sigma_\alpha,\lambda_k)=m^\dagger(\sigma_\alpha,\lambda_k)-m^\dagger(\sigma_\alpha^{s_1,\lambda_k},s_1*\lambda_k). \tag{34}$$

We will show the following

Proposition 5.5.4. *With the above hypothesis and notation, we have*

$$m_0^\dagger(\sigma_\alpha,\lambda_k)=2(\varepsilon(\pi,1/2)-1).$$

In particular, it is nonzero if and only if $\varepsilon=\varepsilon(\pi,1/2)=-1$.

Proof. First we notice that $m^{\mathrm{cl}}(\sigma,\lambda_k)=m_{\mathrm{EP}}(\mathrm{SK}(\pi)_f)$. To show the proposition, we now need to make use of formula (34). To relate $m^\dagger(\sigma_\alpha,\lambda_k)$ to $m_0^\dagger(\sigma_\alpha,\lambda_k)$ and also to compute $m^\dagger(\sigma_\alpha^{s_1,\lambda_k},s_1*\lambda_k)$, we now have to study the Eisenstein multiplicities.

If $m^\dagger_{G,M_K,w}(\sigma_\alpha,\lambda_k)\neq 0$, then there exists σ_w for $w=\mathrm{id}$ or s_1 a finite slope representation of $\mathrm{GL}(2)_{/\mathbb{Q}}$ and a Dirichlet character χ_w such that if $w=\mathrm{id}$, then σ_w is of weight $(k-3;6-2k)$ with $\theta_{\sigma_w}(U_p)=\alpha$ and $\chi_w(p)\theta_{\sigma_{\mathrm{id}}}(p.\mathrm{id})=\theta_\sigma(U_{2,p})=p\alpha$, or, if $w=s_1$, then σ_w is of weight $(k-2,6-2k)$ with $\theta_{\sigma_w}(U_p)=a_p$ and $\chi_{\mathrm{id}}(p)\theta_{\sigma_w}(p.id)=\theta_\sigma(U_{2,p})=p\alpha$. Since we know that α is a Weil number of weight $2k-3$, we see that these situations cannot occur. Therefore $m^\dagger_{G,M_K,id}(\sigma_\alpha,\lambda_k)=m^\dagger_{G,M_K,s_1}(\sigma_\alpha,\lambda_k)=0$. Similarly, we could show that $m^\dagger_{G,M_K,w}(\sigma_\alpha^{s_1,\lambda_k},s_1*\lambda_k)=0$ for $w\in\{\mathrm{id},s_1\}$. We also have the same vanishing results if we replace M_K by T by similar but simpler arguments. We therefore deduce that

$$m^\dagger(\sigma_\alpha,\lambda_k)=m_0^\dagger(\sigma_\alpha,\lambda_k)-m^\dagger_{G,M_S,id}(\sigma_\alpha,\lambda_k)+m^\dagger_{G,M_S,s_2}(\sigma_\alpha,s_2*\lambda_k).$$

If $m^\dagger_{G,M_S,id}(\sigma_\alpha,\lambda_k)\neq 0$, then we have $\alpha=\theta_{\sigma_\alpha}(U_{1,p})=\chi(p)$ for some Dirichlet character χ. This is impossible since α is a Weil number of weight $2k-3$. If $m^\dagger_{G,M_S,s_2}(\sigma_\alpha,\lambda_k)\neq 0$, then there exists a finite slope (cuspidal) representation σ_2 of $\mathrm{GL}(2)$ of weight $2k-2$ and Dirichlet characters χ_2 and χ_2' such that $\alpha=\theta_{\sigma_\alpha}(U_{1,p})=\theta_{\sigma_2}(U_p)\chi_2(p)$ and $p.\alpha=\theta_{\sigma_\alpha}(U_{2,p})=\theta_{\sigma_2}(U_p)\chi_2'(p)$. This is impossible; we therefore have

$$m^\dagger(\sigma_\alpha,\lambda_k)=m_0^\dagger(\sigma_\alpha,\lambda_k).$$

We now need to compute $m^\dagger_{G,M_S,w}(\sigma_\alpha^{s_1,\lambda_k}, s_1 * \lambda_k)$ for $w \in \{\mathrm{id}, s_2\}$. Write $\sigma'_\alpha = \sigma_\alpha^{s_1,\lambda_k}$ and $\lambda'_k = s_1 * \lambda_k = (k-4, k-2; 6-2k)$. We have $\theta_{\sigma'_\alpha}(U_{1,p}) = \alpha$ and $\theta_{\sigma'_\alpha}(U_{2,p}) = p^{-1}\theta_{\sigma_\alpha}(U_{2,p}) = \alpha$. If $m^\dagger_{G,M_S,id}(\sigma_\alpha^{s_1,\lambda_k}, s_1 * \lambda_k) \neq 0$, then $\alpha = \theta_{\sigma'_\alpha}(U_{1,p}) = \chi(p)$ for some Dirichlet character χ which is again impossible since α is a Weil number. Now from the formula defining the distribution I_{G,M_S,s_2}, we see that $m^\dagger_{G,M_S,s_2}(\sigma_\alpha^{s_1,\lambda_k}, s_1*\lambda_k) = m^\dagger_{\mathrm{GL}(2),0}(\pi_\alpha, (0; 2k-4)) = -2$, where π is the p-stabilization of π such that $\theta_{\pi_\alpha}(U_p) = \alpha$. Therefore we deduce that

$$m^\dagger(\sigma'_\alpha, \lambda'_k) = m^\dagger_0(\sigma'_\alpha, \lambda'_k) - 2.$$

Combining all the previous considerations, we therefore get

$$m^\dagger_0(\sigma_\alpha, \lambda_k) = 2\varepsilon - 2 + m^\dagger_0(\sigma'_\alpha, \lambda'_k).$$

Since $m^\dagger_0(\sigma'_\alpha, \lambda'_k) \leq 0$, this, in particular, implies that $m^\dagger_0(\sigma_\alpha, \lambda_k) < 0$ if $\varepsilon = -1$. In fact, one can show that $m^\dagger_0(\sigma'_\alpha, \lambda'_k) = 0$. Otherwise one would be able to construct a very peculiar 3-dimensional family of generically large Galois representations which would have a stable line by the inertia subgroup at p. This family would have a specialization whose semi-simplification would be isomorphic to $\rho_{SK(\pi)}$. Moreover, at this specialization the generic stable line by the inertia subgroup at p would be the line $\mathbb{Q}_p(2-k)$. Then by arguments similar to those of [SU06a], [SU06b], one would get infinitely many elements in the Selmer groups corresponding to the Galois representations $\mathbb{Q}_p(-1)$ or $\rho_f(k-2)$. But the former cannot exist by Class Field Theory, and the latter is known to be impossible by a theorem of Kato [Kat04]. This finishes the proof of our proposition. It shows that the use of nontempered cuspidal representation does not lead to results like the one of [SU06a] when the sign of the functional equation is $+1$. This is why it is better to use Eisenstein series as it is explained in [SU06b]. This result also implies that the main theorem of [SU06a] is also true in the nonordinary case. We leave the verification of details of this fact to the conscientious reader. □

5.6. *The twisted eigenvarieties for* GL_n. In this section, we assume (for simplicity)[22] that $G = \mathrm{GL}_{n/F}$ over a totally real number field or a CM fields F. We denote by F^+ its maximal totally real subfield and by c the complex conjugation automorphism of F.

Let T and B be respectively the diagonal torus of G and the group of upper triangular matrices in G. Let J be the anti-diagonal matrix defined by

$$J = (\delta_{i,n-j})_{1 \leq i,j \leq n}.$$

[22] One could extend this to a more general situation.

We consider the involution ι defined by $\iota(g) = g^\iota := J^t g^{-1} J^{-1}$ if F is totally real and by $\iota(g) = g^\iota := J^t g^{-c} J^{-1}$ if F is CM. This is clearly an involution of Cartan type.

It is easy to check that T and B are stable under the action of ι. A weight $\lambda \in \mathfrak{X}^\iota(\overline{\mathbb{Q}}_p)$ is such that

$$\lambda(\mathrm{diag}(t_1, \ldots, t_n)) = \chi_1(t_1) \ldots \chi_n(t_n)$$

with $\chi_j = \chi_{n-j}$ if F is totally real and $\chi_j = \chi_{n-j} \circ c$ if F is CM and where the χ_i's are characters of $(O_F \otimes \mathbb{Z}_p)^\times$. So $\dim \mathfrak{X}^\iota = ([n/2]-1).[F^+ : \mathbb{Q}] + 1 + \delta_{F^+,p}$. This is the subvariety of essentially self-dual weights.

For any $f \in C_v^\infty(G(\mathbb{A}_f), \mathbb{Q}_p)$, we set f^ι defined by

$$f^\iota(g) := f(g^\iota),$$

and for any character θ of $R_{S,p}$, we denote by θ^ι the character defined by

$$\theta^\iota(f) = \theta(f^\iota).$$

Then it is possible to construct rigid analytic spaces $\mathfrak{E}^\iota_{K^p,0}$ by applying the construction we have made in the beginning of this section for the finite slope character distributions $I^\dagger_{\mathrm{GL}_n,0}(f \times \iota, f, \lambda)$, and we have the following theorem. A detailed proof of the following theorem will appear in the forthcoming thesis of Z. Xiang [Xia12].

THEOREM 5.6.1. *Let K^p be an open compact subgroup of* $\mathrm{GL}_n(\mathbb{A}_f^p \otimes F)$. *The ringed spaces $\mathfrak{E}^\iota_{K^p,0}$ are rigid analytic varieties and the following properties hold*:

(i) $\theta_y = \theta_y^\iota$ *and* $\lambda_y = \lambda_y^\iota$ *for any* $y = (\lambda_y, \theta_y) \in \mathfrak{E}^\iota_{K^p}(\overline{\mathbb{Q}}_p)$. *In particular,* $\mathfrak{E}^\iota_{K^p}$ *sits over* $\mathfrak{X}^\iota$.

(ii) *Let* $y = (\lambda_y, \theta_y) \in (\mathfrak{X} \times \mathfrak{Y}_S^*)(\overline{\mathbb{Q}}_p)$. *Then* $y \in \mathfrak{E}^\iota_{K^p,0}(\overline{\mathbb{Q}}_p)$ *if and only if* $m^\dagger(\lambda, \theta \times \iota, K^p) \neq 0$.

(iii) *For any K^p-admissible Hecke operator f, the projection map* $\mathfrak{E}^\iota_{K^p,0}(\overline{\mathbb{Q}}_p) \to \mathfrak{Z}_G(f)$ *is a finite surjective morphism.*

(iv) *The restriction of the map* $\mathfrak{E}^\iota_{K^p} \to \mathfrak{X}^\iota$ *to any irreducible component is generically flat. In particular,* $\mathfrak{E}^\iota_{K^p}$ *is equidimensional and has the same dimension as* $\dim \mathfrak{X}^\iota$.

5.7. *Some more eigenvarieties.* We would like to end this section by discussing some conjectures on p-adic families. What we have constructed here is the Zariski closure of all the points $y = (\theta, \lambda)$ for which $m^\dagger(\theta, \lambda, K^p)$ does not vanish. We have proved that they form a nice rigid analytic variety $\mathfrak{E}_{K^p}$ which is generically flat over weight space $\mathfrak{X}$. But what about the other cohomological systems of Hecke eigenvalues for which the Euler-Poincaré multiplicity vanishes? Can we still construct an eigenvariety containing all the points (θ, λ) such that $\bar{H}^*(S_G(K^p I_m), \mathcal{D}_\lambda)[\theta] \neq 0$? What is the dimension of the irreducible components passing through such a point when $m^\dagger(\theta,\lambda,K^p)=0$? We would like to give some speculative answers to these questions.

5.7.1. *The full eigenvariety.* A point $y=(\theta,\lambda)\in\mathfrak{Y}_{S,p}(\overline{\mathbb{Q}}_p)$ is said to be cohomological of level K^p if $\bar{H}^*(S_G(K^pI_m),\mathcal{D}_\lambda)[\theta]\neq 0$.

We want to give a construction of a variety that contains all the cohomological points. This construction is mainly due to Ash-Stevens[23] but our construction is a variant. We will only sketch it. We fix $f=u_t$ with $t\in T^{--}$. We consider the action of f on the Banach spaces $R\Gamma^q(K^p.I_m,\mathcal{D}_{\mathfrak{U},n})$'s. For each degree q, let $R^q_{\mathfrak{U}}(f,\lambda,X)\in\mathcal{O}^0(\mathfrak{U})\{\{X\}\}$ the Fredholm determinant of f acting on $R\Gamma^q(K^p.I_m,\mathcal{D}_{\mathfrak{U},n})$ and let $R_{\mathfrak{U}}(f,\lambda,X):=\prod_q R^q_{\mathfrak{U}}(\lambda,X)$. (We do not take alternatings product here.) Since we can make this construction for any $\mathfrak{U}$, we can easily see that there exists $R_{\mathfrak{U}}(f,\lambda,X)\in\Lambda_{\mathfrak{X}}\{\{X\}\}$ that specializes to $R_{\mathfrak{U}}(f,\lambda,X)$ by the canonical map $\Lambda_{\mathfrak{X}}\to\mathcal{O}(\mathfrak{U})$. We then denote by $\mathfrak{Z}'(f)$ the spectral variety[24] associated to $R_{\mathfrak{X}}(f,\lambda,X)$. We now choose $\mathfrak{W}\subset\mathfrak{Z}(f)$ an admissible affinoid subdomain of $\mathfrak{Z}(f)$. We denote by $\mathfrak{U}$ its image by the projection onto weight space $\mathfrak{X}$. Let $e_{\mathfrak{W}}$ be the idempotent attached to $\mathfrak{W}$ by the associated factorization of the Fredholm series of f. Consider the complex $e_{\mathfrak{W}}.R\Gamma(K^pI_m,\mathcal{D}_{\mathfrak{U}})$. Each term is of finite rank over $\mathcal{O}(\mathfrak{U})$, and we consider the action of $R_{S,p}$ on it modulo homotopy. Let $B_{\mathfrak{W}}$ be the $\mathcal{O}(\mathfrak{U})$ algebra generated by the image of $R_{S,p}$ in $\mathrm{End}_{D^b}(e_{\mathfrak{W}}.R\Gamma(K^pI_m,\mathcal{D}_{\mathfrak{U}})$ which is of finite type over $\mathcal{O}(\mathfrak{U})$. Then put $\mathbb{E}(\mathfrak{W}):=\mathrm{Sp}(B_{\mathfrak{W}})$. It is a finite affinoid domain over $\mathfrak{U}$. We define the eigenvariety $\mathbb{E}_{K^p}$ by gluing the $\mathbb{E}(\mathfrak{W})$. A detail of the gluing of similar pieces has been written up by Z. Xiang [Xia12].

5.7.2. *Conjectures.* It would not be difficult to check that the eigenvariety $\mathfrak{E}_{K^p}$ we have constructed in this paper is the union of the components of $\mathbb{E}_{K^p}$ of dimension d. We would like to formulate two conjectures on the dimension of the other components.

CONJECTURE 5.7.3. *Let $x=(\lambda,\theta)$ be a point of the eigenvariety contained in only one irreducible component C of $\mathbb{E}_{K^p}$. Then the projection of C onto $\mathfrak{X}$ is codimension d in weight space if and only if there exist two nonnegative integers p,q, and a positive integer $m(\theta,\lambda,K^p)$ such that*

(a) *The θ-generalized eigenspace of $H^r(S_G(K^pI_m),\mathcal{D}_\lambda(L))$ is nonzero only if $p\le r\le q$ and its dimension is $m(\theta,\lambda,K^p)\times\binom{q-p}{r-p}$.*
(b) *$d=q-p$.*

We now give a few examples to support this conjecture.

[23] They basically proved that the eigenvariety is locally (for its canonical p-adic topology) rigid analytic which is actually all we need for the conjectures that we state here.

[24] It is not hard to see that $\mathfrak{Z}(f)\subset\mathfrak{Z}'(f)$ but the latter might be much bigger as it depends of the resolution we have chosen unlike $\mathfrak{Z}(f)$.

Example 1: *$G = \mathrm{SL}(2, F)$ with F a totally real field of degree d over $\mathbb{Q}$.* In this case, weight space is dimension d. Consider $x = (\theta, k)$ with θ the system of Hecke eigenvalue associated to an Eisenstein series of weight k. It is determined by a Hecke character ψ of the idèle class group of F. If the p-stabilization is chosen ordinary, then it is known that there is a p-adic family of Eisenstein series of dimension $1 + \delta$, where δ is the defect of the Leopold conjecture for F and p. The study of the Eisenstein cohomology (see [Har87]) shows that the Eisenstein classes occur in degree q only if $d \leq q \leq 2d - 1$ with multiplicity $\binom{d-1}{q-d}$ since the rank of the group of units is $d - 1$. We see that in that case, our conjecture is satisfied if and only if Leopold conjecture is true for (F, p).

Example 2: *$G = D^\times$ with D a quaternion algebra over a number field having exactly one complex place and which is ramified at all the real places.* In this situation, the cuspidal cohomology is nontrivial only in degree 1 and 2 so in this situation $q = 2$ and $p = 1$, and it is expected by the conjecture that the projection onto weight space of the irreducible components are codimension 1. In fact, in the ordinary case this is a theorem of Hida [Hid94].

Example 3: $G = \mathrm{GL}(n, \mathbb{Q})$. . We write $n = 2m$ or $n = 2m+1$ according to the parity of n. In this situation, the cuspidal cohomology $H^*_{\mathrm{cusp}}(S_G(K), \mathbb{V}_\lambda^\vee)$ with regular λ vanishes except in degree i with $m^2 \leq i \leq m^2+m-1$ if n is even and $m(m+1) \leq i \leq m(m+1)+m$ if n is odd. Notice also that the cohomology vanishes if λ is not essentially self-dual. (See [Clo90] for these assertions.) The prediction of our conjectures then says that the dimension of the eigenvariety should be $2m - (m-1) = m + 1$ in the even case and $2m + 1 - m = m + 1$ in the odd case (compare to a conjecture of Hida in [Hid98]). One can remark that $m + 1$ is actually the dimension of the subvariety of essentially self-dual weights denoted $\mathfrak{X}^\iota$ in the previous section. For $n = 3$, the arguments of Hida in [Hid94] implies that our conjecture is true in the $\mathrm{GL}(3)_{/\mathbb{Q}}$-case.

The following proposition, proved independently by G. Stevens and by the author, gives some more evidence for the conjecture above. Its proof will be published in another paper in which we hope to give more evidence for Conjecture 5.7.3.

Proposition 5.7.4. *Let $x = (\lambda, \theta)$ be a point of the eigenvariety $\mathbb{E}_{K^p}$. Assume that $H^r(S_G(K^pI_m), \mathcal{D}_\lambda(L))[\theta] \neq 0$ for exactly q consecutive degrees r. Then there is a component of the eigenvariety containing x of dimension at least* $\dim \mathfrak{X} - q$.

This especially means that our conjecture states the opposite inequalities. It therefore should be seen as a non-abelian generalization of Leopoldt conjecture. We end this section by a giving a refined conjecture when there are several irreducible components (of possibly different dimensions).

CONJECTURE 5.7.5. *Let $x=(\lambda,\theta)$ be a point of the eigenvariety $\mathbb{E}_{K^p}$ and let $C_1,\dots,C_s$ the irreducible components containing x.*

For each $i=1,\dots,s$, there exists p_i,q_i,m_i such that

(a) *The θ-generalized eigenspace of $H^r(S_G(K^pI_m),\mathcal{D}_\lambda(L))$ has rank $\sum_{i=1}^s m_i \times \binom{q_i-p_i}{r-p_i}$.*

(b) *The component C_i has dimension* $\dim\mathfrak{X}-d_i$ *with $d_i=q_i-p_i$.*

6. A p-adic trace formula

6.1. *Spectral side of the p-adic trace formula.* Let $\lambda\in\mathfrak{X}(\overline{\mathbb{Q}}_p)$. By Section 4, we know that we can write the p-adic finite slope character distribution $f\mapsto I_G^\dagger(f,\lambda)$ defined over $\mathcal{H}'_p$ as a sum

$$I_G^\dagger(f,\lambda)=\sum_\sigma m_G^\dagger(\lambda,\sigma).I_\sigma(f), \tag{35}$$

where σ runs in the set of irreducible finite slope representations of $\mathcal{H}_p$. This sum is infinite but we know it is p-adically convergent.

The aim of this section is to apply the p-adic analyticity with respect to the weight of the map $\lambda\mapsto I_G^\dagger(f,\lambda)$ to establish a formula for (35) in geometric terms similar to Arthur-Selber type trace formulas. Inspired by Franke's trace formula for the Lefschetz numbers, we introduce an overconvergent version of it and show that it equals the corresponding p-adic automorphic distribution $I_G^\dagger(f,\lambda)$. It is possible to obtain a similar result for the distribution $I_{G,0}^\dagger$, but we have decided to do this in a future paper, so as to keep this article a reasonable length.

6.2. *Franke's trace formula for Lefschetz numbers.* The purpose of this paragraph is to recall Franke's formula. We start by recalling the main terms involved in it.

6.2.1. *Tamagawa numbers.* Let $H_{/\mathbb{Q}}$ be a connected reductive group. Assume that $H(\mathbb{R})$ contains a compact Cartan subgroup and let $\bar{H}(\mathbb{R})$ be the compact modulo center inner form of $H(\mathbb{R})$. We denote by A_H the maximal split-torus of the center of H and denote by $K_H(\mathbb{R})$ one of its maximal compact subgroup. Let dh be a Haar measure on H. Then the Tamagawa number $\chi(H)=\chi(H,dh)$ associated to H is defined by

$$\chi(H,dh):=(-1)^{d_H}\frac{\mathrm{vol}(H(\mathbb{Q})\backslash H(A)/A_H(\mathbb{R})^0)}{\mathrm{vol}(\bar{H}(\mathbb{R})/A_H(\mathbb{R})^0)}\cdot\frac{w_{H(\mathbb{R})}}{w_{K_H(\mathbb{R})}},$$

where dh denotes an Haar measure on $H(\mathbb{A}_f)$ and where the Haar measures used for $\bar{H}(\mathbb{R})$ and $H(\mathbb{R})$ correspond by the (inner) isomorphism between $H_{/\mathbb{C}}$ and $\bar{H}_{/\mathbb{C}}$. The cardinality of the Weyl group of $H(\mathbb{R})$ and $K_H(\mathbb{R})$ are respectively denoted by $w_{H(\mathbb{R})}$ and $w_{K_H(\mathbb{R})}$. The factor $\frac{w_{H(\mathbb{R})}}{w_{K_H(\mathbb{R})}}$ is then equal to the number of representations in a discrete series L-packet of $H(\mathbb{R})$.

6.2.2. *Semi-simple elements.* We write $G(\mathbb{Q})^{\mathrm{ss}}$ for the set of semi-simple elements of $G(\mathbb{Q})$. For all $\gamma \in G(\mathbb{Q})^{\mathrm{ss}}$, we write G_γ for the centralizor of γ in G, G_γ^0 for its connected component, and we put $i(\gamma) := [G_\gamma : G_\gamma^0]$. To define the integral orbitals intervening in the trace formula, we use the fixed Haar measure on G we have taken at the beginning of this paper. Let $h \in C_c^\infty(G(\mathbb{A}_f), \mathbb{Q})$; then we define the orbital integral

$$O_\gamma(h, dg_\gamma) := \int_{G(\mathbb{A}_f)/G_\gamma(\mathbb{A}_f)} h(g\gamma g^{-1}) d\bar{g},$$

where $d\bar{g}$ is the quotient of the Haar measure dg by the Haar measure dg_γ on G_γ. If $\gamma \in G(\mathbb{Q})^{\mathrm{ss}}$, following Franke, we define $\varepsilon(\gamma) \in \{-1, 0, 1\}$ by putting $\varepsilon(\gamma) = 0$ if $G_\gamma(\mathbb{R})$ does not contain a Cartan subgroup which is compact modulo $A_{G_\gamma}^0(\mathbb{R})$. In other words, $\varepsilon(\gamma) = 0$ when γ is not elliptic. Otherwise we put $\varepsilon(\gamma) = (-1)^{\dim A_G(\mathbb{R})/A_{G_\gamma}(\mathbb{R})}$.

Theorem 6.2.3 (Franke). *Let f be any Hecke operators and let λ be an algebraic dominant weight; then we have*

$$\begin{aligned}\mathrm{tr}(f; H^\bullet(\tilde{S}_G, \mathbb{V}_\lambda^\vee(\mathbb{C}))&\\ &= (-1)^{d_G} \sum_{\gamma \in G(\mathbb{Q})^{ss}/\sim} \epsilon(\gamma) \frac{\chi(G_\gamma^0, dg_\gamma)}{i(\gamma)} O_\gamma(f, dg_\gamma) tr(\gamma, \mathbb{V}_\lambda^\vee(\mathbb{C}))\end{aligned}$$

Proof. This is formula (24) of [Fra98] built on the formula of Theorem 1.4.2 and Arthur's trace formula on L_2-Lefschetz numbers. □

6.3. *A formula for $I_G^\dagger(f, \lambda)$.* Before establishing our p-adic trace formula, we introduce a certain p-adic function on Δ^{--}.

Lemma 6.3.1. *For any $g \in \Delta^{--}$ as above, the map $\lambda \mapsto \Phi_G^\dagger(g, \lambda) := \mathrm{tr}(g, \mathcal{D}_\lambda(L))$ is analytic on $\mathfrak{X}_T(\overline{\mathbb{Q}}_p)$.*

Proof. For any affinoid subdomain $\mathfrak{U} \subset \mathfrak{X}_T$ and $\lambda \in \mathfrak{U}(L)$, we have

$$\mathcal{D}_{\mathfrak{U},n} \otimes_\lambda L \cong \mathcal{D}_{\lambda,n}(L) \tag{36}$$

for any $n \geq n_{\mathfrak{U}}$. Now $\mathcal{D}_{\mathfrak{U},n}$ is $\mathcal{O}(\mathfrak{U})$-orthonormalizable, and therefore $\phi_{\mathfrak{U}} := \mathrm{tr}(\gamma, \mathcal{D}_{\mathfrak{U},n}) \in \mathcal{O}(\mathfrak{U})$ is analytic on $\mathfrak{U}$ and by (36) satisfies $\phi_{\mathfrak{U}}(\lambda) = \mathrm{tr}(g; \mathcal{D}_{\lambda,n}) = \Phi_G^\dagger(g, \lambda)$ for all $\lambda \in \mathfrak{U}(\overline{\mathbb{Q}}_p)$. □

The following lemma is analogue to Corollary 4.6.8.

Lemma 6.3.2. *Let λ be an algebraic dominant weight and $(\lambda_n)_n$ be a very regular sequence converging to λ. Then for any $g \in \Delta^{--}$, we have*

$$\lim_{n \to \infty} \lambda_n(\xi(t_g)) \mathrm{tr}(g, \mathbb{V}_{\lambda_n}^\vee(L)) = \Phi_G^\dagger(g, \lambda).$$

Proof. Let $g \in \Delta^{--}$. From Theorem 3.3.10 and Proposition 3.3.12 it then follows that we have the congruence

$$\mathrm{tr}(g; \mathcal{D}_\lambda(L)) \equiv \lambda(\xi(t_g))\mathrm{tr}(g; V_\lambda^\vee(L)) = \lambda(\xi(t_g))\mathrm{tr}(g; \mathbb{V}_\lambda^\vee(L)) \mod N(\lambda, t_g).$$

We now deduce the result from (32) and the analyticity of the map $\lambda \mapsto \Phi_G^\dagger(\gamma, \lambda)$. □

We are now ready to state and prove a formula for the distribution $I_G^\dagger$. It is given by the following theorem. We consider a Haar measure dg of $G(\mathbb{A}_f)$ such that the p-component gives the measure 1 to the Iwahori subgroup I. Then we have the following

THEOREM 6.3.3. *Let $f = f^p \otimes f_p \in \mathcal{H}'_p$ and $\lambda \in \mathfrak{X}(\overline{\mathbb{Q}}_p)$; then we have*

$$I_G^\dagger(f, \lambda) = (-1)^{d_G} \sum_{\gamma \in G(\mathbb{Q})^{ss}/\simeq} \varepsilon(\gamma) \frac{\chi(G_\gamma^0, dg_\gamma)}{i_G(\gamma)} O_\gamma(f^p, dg_\gamma) I_{G,\gamma}^\dagger(f, \lambda),$$

where the sum is taken over the conjugacy classes of $\mathbb{R}$-elliptic semi-simple elements of $G(\mathbb{Q})$, with

$$I_{G,\gamma}^\dagger(f, \lambda) = \int_{G(\mathbb{Q}_p)/G_\gamma(\mathbb{Q}_p)} f_p(x\gamma x^{-1}) \Phi_G^\dagger(x\gamma x^{-1}, \lambda) d\bar{x}.$$

Proof. It is sufficient to prove the formula for $f = f^p \otimes u_t$. It is clear that both the right- and left-hand side of the formula are analytic functions of λ. To prove the equality, it is therefore sufficient to prove it for λ algebraic and dominant. For such a λ, consider a highly regular sequence $(\lambda_n)_n$ converging p-adically to λ. We multiply both sides of the trace formula of Theorem 6.2.3 for each λ_n by $\xi(t)^{\lambda_n}$ and consider the term corresponding to some semi-simple element γ:

$$\begin{aligned}
&\xi(t)^{\lambda_n} \frac{\chi(G_\gamma^0, dg_\gamma)}{i_G(\gamma)} O_\gamma(f, dg_\gamma) \mathrm{tr}(\gamma, \mathbb{V}_{\lambda_n}^\vee) \\
&\quad = \frac{\chi(G_\gamma^0, dg_\gamma)}{i_G(\gamma)} \xi(t)^{\lambda_n} O_\gamma(f^p, dg_\gamma) \int_{G(\mathbb{Q}_p)/G_\gamma(\mathbb{Q}_p)} f_p(x\gamma x^{-1}) \mathrm{tr}(\gamma, \mathbb{V}_{\lambda_n}^\vee(L)) dx \\
&\quad = \frac{\chi(G_\gamma^0, dg_\gamma)}{i_G(\gamma)} O_\gamma(f^p, dg_\gamma) \int_{G(\mathbb{Q}_p)/G_\gamma(\mathbb{Q}_p)} f_p(x\gamma x^{-1}) \xi(t)^{\lambda_n} \mathrm{tr}(x^{-1}\gamma x, \mathbb{V}_{\lambda_n}^\vee(L)) dx.
\end{aligned}$$

Then pass to the limit when n goes to infinity. From Lemma 6.3.2 we then get

$$\frac{\chi(G_\gamma^0, dg_\gamma)}{i_G(\gamma)} O_\gamma(f^p, dg_\gamma) \int_{G(\mathbb{Q}_p)/G_\gamma(\mathbb{Q}_p)} f_p(x\gamma x^{-1}) \Phi_G^\dagger(x^{-1}\gamma x, \lambda) dx.$$

Therefore the limit of the right-hand side of the trace formula of Franke for λ_n multiplied by $\xi(t)^{\lambda_n}$ has the limit of the right-hand side of the formula stated in the theorem. The fact that the limit of the left-hand side is $I_G^\dagger(f, \lambda)$ was known by Corollary 4.6.8. □

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Columbia University, New York, NY
Current address: Institut de Mathématiques de Jussieu, Paris Cedex, France
E-mail: urban@math.jussieu.fr