

On De Giorgi's conjecture in dimension $N \geq 9$

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Abstract

A celebrated conjecture due to De Giorgi states that any bounded solution of the equation $\Delta u + (1 - u^2)u = 0$ in \mathbb{R}^N with $\partial_{y_N} u > 0$ must be such that its level sets $\{u = \lambda\}$ are all hyperplanes, *at least* for dimension $N \leq 8$. A counterexample for $N \geq 9$ has long been believed to exist. Starting from a minimal graph Γ which is not a hyperplane, found by Bombieri, De Giorgi and Giusti in \mathbb{R}^N , $N \geq 9$, we prove that for any small $\alpha > 0$ there is a bounded solution $u_\alpha(y)$ with $\partial_{y_N} u_\alpha > 0$, which resembles $\tanh\left(\frac{t}{\sqrt{2}}\right)$, where $t = t(y)$ denotes a choice of signed distance to the blown-up minimal graph $\Gamma_\alpha := \alpha^{-1}\Gamma$. This solution is a counterexample to De Giorgi's conjecture for $N \geq 9$.

Contents

1. Introduction	1486
2. The BDG minimal graph	1490
3. Local coordinates near Γ_α and the construction of a first approximation	1494
3.1. Coordinates in \mathbb{R}^9 near Γ and the Euclidean Laplacian	1497
3.2. Error of approximation	1499
3.3. Global first approximation	1500
4. The proof of Theorem 1	1501
4.1. Reduction by a gluing procedure	1502
4.2. An infinite dimensional Lyapunov-Schmidt reduction	1504
4.3. Solving the reduced problem	1504
4.4. Conclusion	1508
5. The proof of Lemma 4.1	1509
6. The proofs of Proposition 4.1 and Lemma 4.2	1511
6.1. Proof of Proposition 4.1	1512
6.2. Lipschitz dependence on h : The proof of Lemma 4.2	1513
6.3. Proof of Proposition 6.1	1514

6.4.	A priori estimates	1516
6.5.	Existence: Conclusion of the proof of Proposition 6.1	1519
6.6.	An equation on Γ_α	1521
7.	Solvability theory for the Jacobi operator:	
	Proof of Proposition 4.2	1521
7.1.	The approximate Jacobi operator	1522
7.2.	Supersolutions for the approximate Jacobi operator	1522
7.3.	Proof of Proposition 4.2(b)	1526
7.4.	Proof of Proposition 4.2(a)	1529
8.	Local coordinates on Γ :	
	The effect of curvature and closeness to Γ_0	1532
8.1.	The proof of Proposition 3.1	1532
8.2.	Comparing G and G_0	1536
8.3.	Approximating Γ by Γ_0 and their Jacobi operators:	
	Proof of Lemma 7.1	1541
8.4.	The proof of Lemma 7.8	1545
9.	Asymptotic behavior of the BDG graph:	
	Proofs of Lemma 2.1 and Theorem 2	1548
9.1.	Equation for g : Proof of Lemma 2.1	1548
9.2.	A new system of coordinates	1550
9.3.	Proof of Theorem 2	1555
9.4.	A refinement of the asymptotic behavior of F	1560
10.	Appendix: The proof of formula (7.4)	1564
	References	1566

1. Introduction

This paper deals with entire solutions of the Allen-Cahn equation

$$(1.1) \quad \Delta u + (1 - u^2)u = 0 \quad \text{in } \mathbb{R}^N.$$

Equation (1.1) arises in the gradient theory of phase transitions by Cahn-Hilliard and Allen-Cahn, in connection with the energy functional in bounded domains Ω

$$(1.2) \quad J_\varepsilon(u) = \frac{\varepsilon}{2} \int_\Omega |\nabla u|^2 + \frac{1}{4\varepsilon} \int_\Omega (1 - u^2)^2, \quad \varepsilon > 0,$$

whose Euler-Lagrange equation corresponds precisely to an ε -scaling of equation (1.1) in the expanding domain $\varepsilon^{-1}\Omega$. The theory of Γ -convergence developed in the 70's and 80's, showed a deep connection between this problem and the theory of minimal surfaces; see Modica, Mortola, Kohn, Sternberg, [21], [28], [29], [30], [36]. In fact, it is known that a family $\{u_\varepsilon\}_{\varepsilon>0}$ of local

minimizers of J_ε with uniformly bounded energy must converge as $\varepsilon \rightarrow 0$, up to subsequences, in L^1 -sense to a function of the form $\chi_E - \chi_{E^c}$ where χ denotes characteristic function of a set, and ∂E has minimal perimeter. Thus the interface between the stable *phases* $u = 1$ and $u = -1$, represented by the sets $\{u_\varepsilon = \lambda\}$ with $|\lambda| < 1$ approach a minimal hypersurface; see Caffarelli and Córdoba [7, 8] (also Röger and Tonegawa [32]) for stronger convergence and uniform regularity results on these level surfaces.

The above described connection led E. De Giorgi [9] to formulate in 1978 the following celebrated conjecture concerning entire solutions of equation (1.1).

DE GIORGI'S CONJECTURE. *Let u be a bounded solution of equation (1.1) such that $\partial_{x_N} u > 0$. Then the level sets $\{u = \lambda\}$ are all hyperplanes, at least for dimension $N \leq 8$.*

Equivalently, u depends on just one Euclidean variable so that it must have the form

$$(1.3) \quad u(x) = \tanh\left(\frac{x \cdot a - b}{\sqrt{2}}\right)$$

for some $b \in \mathbb{R}$ and some a with $|a| = 1$ and $a_N > 0$. We observe that the function

$$w(t) := \tanh\left(\frac{t}{\sqrt{2}}\right)$$

is the unique solution of the one-dimensional problem

$$w'' + (1 - w^2)w = 0, \quad w(0) = 0, \quad w(\pm\infty) = \pm 1.$$

The monotonicity of u implies that the scaled functions $u(x/\varepsilon)$ are, in a suitable sense, local minimizers of J_ε ; moreover, the level sets of u are all graphs. In this setting, De Giorgi's conjecture is a natural, parallel statement to *Bernstein's theorem* for minimal graphs, which in its most general form, due to Simons [35], states that any minimal hypersurface in \mathbb{R}^N , which is also a graph of a function of $N - 1$ variables, must be a hyperplane if $N \leq 8$. Strikingly, Bombieri, De Giorgi and Giusti [5] proved that this fact is false in dimension $N \geq 9$. This was most certainly the reason for the particle *at least* in De Giorgi's statement.

Great advance in De Giorgi's conjecture has been achieved in recent years, having been fully established in dimensions $N = 2$ by Ghoussoub and Gui [16] and for $N = 3$ by Ambrosio and Cabré [2]. Partial results in dimensions $N = 4, 5$ were obtained by Ghoussoub and Gui [17]. More recently Savin [33] established its validity for $4 \leq N \leq 8$ under the following additional assumption (see [1] for a discussion of this condition):

$$(1.4) \quad \lim_{x_N \rightarrow \pm\infty} u(x', x_N) = \pm 1.$$

Condition (1.4) is related to the so-called Gibbons' Conjecture.

GIBBONS' CONJECTURE. *Let u be a bounded solution of equation (1.1) satisfying*

$$(1.5) \quad \lim_{x_N \rightarrow \pm\infty} u(x', x_N) = \pm 1, \text{ uniformly in } x'.$$

Then the level sets $\{u = \lambda\}$ are all hyperplanes.

Gibbons' Conjecture has been proven in all dimensions with different methods by Barlow, Bass and Gui [3], Berestycki, Hamel, and Monneau [4], Caffarelli and Córdoba [8], and Farina [14]. In references [3], [8] it is proven that the conjecture is true for any solution that has one level set which is globally a Lipschitz graph. If the uniformity in (1.5) is dropped, then a counterexample can be built using the method by Pacard and the authors in [11], so that Savin's result is nearly optimal.

A counterexample to De Giorgi's conjecture in dimension $N \geq 9$ has been believed to exist for a long time, but the issue has remained elusive. Partial progress in this direction was made by Jerison and Monneau [19] and by Cabré and Terra [6]. See the survey article by Farina and Valdinoci [15].

In this paper we show that De Giorgi's conjecture is false in dimension $N \geq 9$ by constructing a bounded solution of equation (1.1) which is monotone in one direction and whose level sets are not hyperplanes. The basis of our construction is a minimal graph, different from a hyperplane, found by Bombieri, De Giorgi and Giusti [5]. In this work a solution of the zero mean curvature equation

$$(1.6) \quad \nabla \cdot \left(\frac{\nabla F}{\sqrt{1 + |\nabla F|^2}} \right) = 0 \quad \text{in } \mathbb{R}^{N-1},$$

different from a linear affine function was found, provided that $N \geq 9$. This solution is, in other words, a nontrivial minimal graph in \mathbb{R}^N . Let us observe that if $F(x')$ solves equation (1.6) then so does

$$F_\alpha(x') := \alpha^{-1}F(\alpha x'), \quad \alpha > 0,$$

and hence

$$(1.7) \quad \Gamma_\alpha = \{(x', x_N) \mid x' \in \mathbb{R}^{N-1}, x_N = F_\alpha(x')\}$$

is a minimal graph in \mathbb{R}^N . It turns out that the scaling parameter in (1.6) provides a natural bridge between (1.1) and (1.6).

Our main result states as follows:

THEOREM 1. *Let $N \geq 9$. There is a solution F to equation (1.6) which is not a linear affine function, such that for all $\alpha > 0$ sufficiently small, there*

exists a bounded solution $u_\alpha(y)$ of equation (1.1) such that $u_\alpha(0) = 0$,

$$\partial_{y_N} u_\alpha(y) > 0 \quad \text{for all } y \in \mathbb{R}^N,$$

and

$$(1.8) \quad |u_\alpha(y)| \rightarrow 1 \quad \text{as } \text{dist}(y, \Gamma_\alpha) \rightarrow +\infty,$$

uniformly for all small $\alpha > 0$, where Γ_α is given by (1.7).

Property (1.8) implies that the 0 level set of u_α lies inside the region $\text{dist}(y, \Gamma_\alpha) < R$ for some fixed $R > 0$ and all small α , and hence it cannot be a hyperplane. Much more accurate information about the solution will be drawn from the proof. The idea is simple. If $t(y)$ denotes a choice of signed distance to the graph Γ_α then, for a small fixed number $\delta > 0$, our solution looks like

$$u_\alpha(y) \sim \tanh\left(\frac{t}{\sqrt{2}}\right) \quad \text{if } |t| < \frac{\delta}{\alpha}.$$

As we have mentioned, a key ingredient of our proof is the existence of a nontrivial solution of equation (1.6) proven in [5]. We shall derive precise information about its asymptotic behavior, which in particular will help us to find global estimates for its derivatives. This is a crucial step since the mean curvature operator yields in general poor gradient estimates. In addition we shall derive a theory of existence and *a priori* estimates for the Jacobi operator of the minimal graph. Subsequently, a suitable first approximation for a solution of (1.1) is built. Next, we linearize our problem around the approximate solution in order to carry out an infinite-dimensional Lyapunov Schmidt reduction. This procedure eventually reduces the full problem (1.1) to one of solving a nonlinear, nonlocal equation which involves as a main term the Jacobi operator of the minimal graph. Schemes of this type have been successful in establishing existence of solutions to singular perturbation elliptic problems in various settings. For the Allen-Cahn equation in compact situations this has been done in the works del Pino, Kowalczyk and Wei [13], Kowalczyk [22], Pacard and Ritore [31]. In particular in [31] solutions concentrating on a minimal submanifold of a compact Riemannian manifold are found through an argument that shares some similarities with the one used here. In the non-compact setting, for both (1.1) and nonlinear Schrödinger equation, solutions have been constructed by del Pino, Kowalczyk and Wei [12], del Pino, Kowalczyk, Pacard and Wei [11], [10], and Malchiodi [24]. See also Malchiodi and Montenegro [25], [26]. We should emphasize here the importance of our earlier works [11], [10] in the context of the present paper, and especially the idea of constructing solutions concentrating on a family of unbounded sets, all coming from a suitably rescaled basic set. While in [11], [10] the concentration set was determined by solving a Toda system and the rescaling was the one appropriate

to this system, here the concentration set is the minimal graph and the rescaling is the one that leaves invariant the mean curvature operator. We mention that our work are partly motivated by earlier works of Kapouleas [20], Mazzeo and Pacard [27], and Mahmoudi, Mazzeo and Pacard [23] on construction of noncompact constant mean curvature surfaces in Euclidean three space.

Let us observe that a counterexample to De Giorgi's conjecture in $N = 9$ gives one in $\mathbb{R}^N = \mathbb{R}^9 \times \mathbb{R}^{N-9}$ for any $N > 9$, by extending the solution in \mathbb{R}^9 to the remaining variables in a constant manner. For this reason, in what follows we shall assume $N = 9$ in problem (1.1). We will also denote

$$f(u) := (1 - u^2)u.$$

We shall devote the rest of the paper to the proof of Theorem 1. The proof is rather long and technical, but has steps that are logically independent and can be divided into nearly independent blocks. The exposition is designed so that the proof is completed by page 1508, except for some steps which are isolated in the form of lemmas and propositions, and whose full proofs, postponed to the subsequent sections, are not necessary to follow the logical thread of the proof of Theorem 1. That is the purpose of the Sections 2–4.

In Section 2 the BDG graph and its asymptotic behavior are described. The proof of the main result there, Theorem 2, which involves a delicate improvement of the supersolution in [5], is carried out in Section 9. In Section 3 a first approximation, about which we linearize, is built and the error of approximation and its features are analyzed in detail. In Section 4 we present the full proof of the theorem in various steps, with several intermediate results stated, with proofs in turn are given in the proceeding Sections 5–9. Each of these last five sections is largely independent and can be read individually.

2. The BDG minimal graph

The minimal surface equation for a graph in \mathbb{R}^9 corresponds to the Euler-Lagrange equation for the functional

$$A(F) = \int \sqrt{1 + |\nabla F|^2} dx,$$

integrated over subsets of \mathbb{R}^8 . In other words, F represents a minimal graph if for any compactly supported test function ϕ

$$A'(F)[\phi] := \int \frac{\nabla F \cdot \nabla \phi}{\sqrt{1 + |\nabla F|^2}} dx = 0.$$

We observe that

$$A'(F)[\phi] = - \int H[F] \phi dx,$$

where

$$(2.1) \quad H[F] := \nabla \cdot \left(\frac{\nabla F}{\sqrt{1 + |\nabla F|^2}} \right) = 0 \quad \text{in } \mathbb{R}^8.$$

Quantity $H[F]$ corresponds to the mean curvature of the hypersurface in \mathbb{R}^9 ,

$$\Gamma := \{(x', F(x')) \mid x' \in \mathbb{R}^8\}.$$

The Bombieri-De Giorgi-Giusti minimal graph [5] is a nontrivial, entire smooth solution of equation (2.1) that enjoys some simple symmetries which we describe next.

Let us write $x' \in \mathbb{R}^8$ as $x' = (\mathbf{u}, \mathbf{v}) \in \mathbb{R}^4 \times \mathbb{R}^4$ and denote $u = |\mathbf{u}|$, $v = |\mathbf{v}|$. Let us consider the set

$$(2.2) \quad T := \{(\mathbf{u}, \mathbf{v}) \in \mathbb{R}^8 \mid v > u > 0\}.$$

We should remark here the set $\{u = v\} \in \mathbb{R}^8$ is the famous Simons minimal cone [35]. The solution found in [5] is radially symmetric in both variables, namely $F = F(u, v)$. In addition, F is positive in T and it vanishes along the Simons cone. Moreover, it satisfies

$$(2.3) \quad F(u, v) = -F(v, u) \quad \text{for all } u, v > 0.$$

Let us observe that for a function F that depends on (u, v) only, the area functional becomes, except for a multiplicative constant,

$$A(F) = \int \sqrt{1 + F_u^2 + F_v^2} \, u^3 v^3 \, du dv,$$

and hence equation (2.1) for such a function becomes

$$H[F] = \frac{1}{u^3 v^3} \partial_u \left(\frac{u^3 v^3 F_u}{\sqrt{1 + F_u^2 + F_v^2}} \right) + \frac{1}{u^3 v^3} \partial_v \left(\frac{u^3 v^3 F_v}{\sqrt{1 + F_u^2 + F_v^2}} \right) = 0.$$

It is useful to introduce in addition polar coordinates $(u, v) = (r \cos \theta, r \sin \theta)$ for which we get (up to a multiplicative constant)

$$A(F) = \int \sqrt{1 + F_r^2 + r^{-2} F_\theta^2} \, r^7 \sin^3 2\theta \, dr \, d\theta,$$

so that (2.1) reads

$$(2.4) \quad H[F] = \frac{1}{r^7 \sin^3 2\theta} \partial_r \left(\frac{F_r r^7 \sin^3 2\theta}{\sqrt{1 + F_r^2 + r^{-2} F_\theta^2}} \right) + \frac{1}{r^7 \sin^3 2\theta} \partial_\theta \left(\frac{F_\theta r^5 \sin^3 2\theta}{\sqrt{1 + F_r^2 + r^{-2} F_\theta^2}} \right) = 0.$$

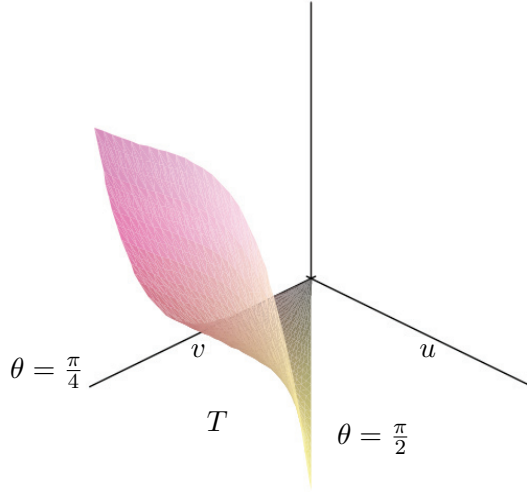


Figure 1. Schematic view of the function $F(u, v)$ representing Γ in the sector $T = \{0 < u < v\}$.

Set $F_0 = r^3g(\theta)$. Then we get

$$(2.5) \quad H[F_0] = \frac{1}{r^7 \sin^3 2\theta} \partial_r \left(\frac{3r^7 g \sin^3 2\theta}{\sqrt{r^{-4} + 9g^2 + g'^2}} \right) + \frac{1}{r \sin^3 2\theta} \partial_\theta \left(\frac{g' \sin^3 2\theta}{\sqrt{r^{-4} + 9g^2 + g'^2}} \right).$$

For F_0 to be a good approximation of a solution of the minimal surface equation $H[F] = 0$, we neglect terms of order r^{-4} in the denominators, and, additionally because of (2.3), we require that $g(\theta)$ solves the two-point boundary value problem

$$(2.6) \quad \frac{21g \sin^3 2\theta}{\sqrt{9g^2 + g'^2}} + \left(\frac{g' \sin^3 2\theta}{\sqrt{9g^2 + g'^2}} \right)' = 0 \quad \text{in } \left(\frac{\pi}{4}, \frac{\pi}{2} \right), \quad g\left(\frac{\pi}{4}\right) = 0 = g'\left(\frac{\pi}{2}\right).$$

Regarding (2.6), we have the following result.

LEMMA 2.1. *Problem (2.6) has a unique solution $g \in C^2([\frac{\pi}{4}, \frac{\pi}{2}])$ such that g and g' are positive in $(\frac{\pi}{4}, \frac{\pi}{2})$ and such that $g'(\frac{\pi}{4}) = 1$.*

We fix in what follows the function g as above and we set $F_0(x') = r^3g(\theta)$. Let us observe that

$$(2.7) \quad H[F_0] = O(r^{-5}) \quad \text{as } r = |x'| \rightarrow +\infty.$$

The next result, crucial in the arguments to follow, refines the existence result in [5] in what concerns the asymptotic behavior of the minimal graph, which turns out to be accurately described by F_0 ; see also Figure 1.

THEOREM 2. *There exists an entire solution $F = F(u, v)$ to equation (2.1) which satisfies (2.3) and such that*

$$(2.8) \quad F_0 \leq F \leq F_0 + \frac{\mathbf{C}}{r^\sigma} \min\{F_0, 1\} \quad \text{in } T, \quad r > R_0,$$

where $0 < \sigma < 1$, $\mathbf{C} \geq 1$ and R_0 , are positive constants.

We will carry out the proofs of Lemma 2.1 and Theorem 2 in Section 9. In what remains of this paper we will denote, for F and F_0 as in Theorem 2,

$$\Gamma = \{(x', F(x')) \mid x' \in \mathbb{R}^8\}, \quad \Gamma_0 = \{(x', F_0(x')) \mid x' \in \mathbb{R}^8\}.$$

By Γ_α we will denote the dilated surfaces $\Gamma_\alpha = \alpha^{-1}\Gamma$. Also, in the rest of this paper we shall use the notation

$$(2.9) \quad r(x) := |x'|, \quad r_\alpha(x) := r(\alpha x), \quad x = (x', x_9) \in \mathbb{R}^8 \times \mathbb{R} = \mathbb{R}^9.$$

We conclude this section by introducing the linearization of the mean curvature operator, corresponding to the second variation of the area functional, namely the linear operator $H'(F)$ defined by

$$H'(F)[\phi] := \frac{d}{dt} H(F + t\phi) |_{t=0} = \nabla \cdot \left(\frac{\nabla \phi}{\sqrt{1 + |\nabla F|^2}} - \frac{(\nabla F \cdot \nabla \phi)}{(1 + |\nabla F|^2)^{\frac{3}{2}}} \nabla F \right).$$

When the second variation is measured with respect to normal, rather than to vertical perturbations, we obtain the Jacobi operator of Γ , defined for smooth functions h on Γ as

$$\mathcal{J}_\Gamma[h] = \Delta_\Gamma h + |A_\Gamma(y)|^2 h,$$

where Δ_Γ is the Laplace-Beltrami operator on Γ and $|A_\Gamma|^2$ is the Euclidean norm of its second fundamental form, namely $|A_\Gamma|^2 = \sum_{i=1}^8 k_i^2$ where k_1, \dots, k_8 are the principal curvatures. See [35, Th. 3.2.2].

These two operators are linked through the simple relation

$$(2.10) \quad \mathcal{J}_\Gamma[h] = H'(F)[\phi], \quad \text{where } \phi(x') = \sqrt{1 + |\nabla F(x')|^2} h(x', F(x')).$$

Similarly, using formula (2.4), we compute for vertical perturbations $\phi = \phi(r, \theta)$ of Γ_0 ,

$$(2.11) \quad \begin{aligned} H'(F_0)[\phi] = & \frac{1}{r^7 \sin^3(2\theta)} \left\{ (9g^2 \tilde{w} r^3 \phi_\theta)_\theta + (r^5 g'^2 \tilde{w} \phi_r)_r \right. \\ & \left. - 3(gg' \tilde{w} r^4 \phi_r)_\theta - 3(gg' \tilde{w} r^4 \phi_\theta)_r \right\} \\ & + \frac{1}{r^7 \sin^3(2\theta)} \left\{ (r^{-1} \tilde{w} \phi_\theta)_\theta + (r \tilde{w} \phi_r)_r \right\}, \\ \tilde{w}(r, \theta) := & \frac{\sin^3 2\theta}{(r^{-4} + 9g^2 + g'^2)^{\frac{3}{2}}}. \end{aligned}$$

3. Local coordinates near Γ_α and the construction of a first approximation

We are studying the equation

$$(3.1) \quad \Delta U + f(U) = 0 \quad \text{in } \mathbb{R}^9, \quad f(U) = U(1 - U^2).$$

It is natural to look for a solution $U(x)$ that obeys the symmetries of Γ_α . Let us consider the linear isometry in \mathbb{R}^9 given by

$$(3.2) \quad \mathcal{Q}(\mathbf{u}, \mathbf{v}, x_9) = (P\mathbf{v}, Q\mathbf{u}, -x_9),$$

where P and Q are orthogonal transformations of \mathbb{R}^4 . We observe that this isometry leaves Γ_α invariant and that if $U(x)$ solves (3.1) then so does the function $-U(\mathcal{Q}x)$. We look for a solution with the property

$$(3.3) \quad U(\mathcal{Q}x) = -U(x)$$

for any \mathcal{Q} of the form (3.2). In other words, we look for $U = U(u, v, x_9)$ with

$$U(v, u, -x_9) = -U(u, v, x_9).$$

The proof of Theorem 1 relies on constructing a first, rather accurate approximation to a solution whose level sets are nearly parallel to Γ_α , and then linearize the equation around it to find an actual solution by fixed point arguments. A neighborhood of Γ_α can be parametrized as the set of all points of the form

$$(3.4) \quad x = X_\alpha(y, z) := y + z\nu(\alpha y), \quad y \in \Gamma_\alpha,$$

where $|z|$ is conveniently restricted for each y . We observe that $\nu(\alpha y)$ corresponds to the normal vector to Γ_α at the point y . It seems logical to consider $u_0(x) = w(z)$ as a first approximation to a solution near Γ_α . Rather than doing this, we consider a smooth small function h defined on Γ and set

$$u_0(x) := w(z - h(\alpha y)).$$

The function h is left as a parameter which will be later adjusted. Consistently, we ask that h obeys the symmetries of Γ requiring that for any \mathcal{Q} of the form (3.2) we have

$$(3.5) \quad h(y) = -h(\mathcal{Q}y) \quad \text{for all } y \in \Gamma.$$

We notice that this requirement implies that $h = 0$ on Simons cone $\{u = v\}$.

Suitably adapted to this initial guess is the change of variables

$$(3.6) \quad x = X_h(y, t) := y + (t + h(\alpha y))\nu(\alpha y), \quad y \in \Gamma_\alpha,$$

so that $u_0(x) = w(t)$.

Since $F(\mathbf{u}, \mathbf{v}) = F(u, v) = -F(v, u)$, we have that $\mathcal{Q}\nu(\alpha y) = -\nu(\alpha \mathcal{Q}y)$, and hence

$$(3.7) \quad X_h(\mathcal{Q}y, -t) = -\mathcal{Q}X_h(y, t).$$

Thus, if $V = V(x)$ and we set with some abuse of notation $V(y, t) := V(X_h(y, t))$, then

$$(3.8) \quad V(\mathcal{Q}x) = -V(x) \quad \text{if and only if} \quad V(y, t) = -V(\mathcal{Q}y, -t).$$

In particular, $u_0(x)$ satisfies the symmetry requirement (3.3) where it is defined, since the function w is odd.

To measure the accuracy of this approximation, and to set up the linearization scheme, we shall derive an expression for the Euclidean Laplacian Δ_x in terms of the coordinates (y, t) in a region where the map X_h defines a diffeomorphism onto an open neighborhood of Γ_α .

At this point we make explicit our assumptions on the parameter function h besides (3.5). We require that h is of size of order α and that it decays at infinity along Γ at a rate $O(r(y)^{-1})$, while its first and second derivatives decay at respective rates $O(r^{-2})$ and $O(r^{-3})$. Precisely, let us consider the norms

$$\|g\|_{\infty, \nu} := \|(1 + r^\nu)g\|_{L^\infty(\Gamma)}, \quad \|g\|_{p, \nu} := \sup_{y \in \Gamma} (1 + r(y)^\nu) \|g\|_{L^p(\Gamma \cap B(y, 1))}.$$

Let us fix numbers $M > 0$, $p > 9$ and assume that h satisfies

$$(3.9) \quad \|h\|_* := \|h\|_{\infty, 1} + \|D_\Gamma h\|_{\infty, 2} + \|D_\Gamma^2 h\|_{p, 3} \leq M\alpha.$$

In order to find the desired expression for the Laplacian in coordinates (3.6), we do so first in coordinates (3.4) for $\alpha = 1$. Let us consider the smooth map

$$(3.10) \quad (y, z) \in \Gamma \times \mathbb{R} \mapsto x = X(y, z) = y + z\nu(y) \in \mathbb{R}^9.$$

As we will justify below (Remark 8.1), there is a number $\delta > 0$ such that the map X is one-to-one inside the open set

$$(3.11) \quad \mathcal{O} = \{(y, z) \in \Gamma \times \mathbb{R} \mid |z| < \delta(r(y) + 1)\}.$$

It follows that X is actually a diffeomorphism onto its image, $\mathcal{N} = X(\mathcal{O})$.

The Euclidean Laplacian Δ_x can be computed by a well-known formula (see for instance [31]) in terms of the coordinates $(y, z) \in \mathcal{O}$ as

$$(3.12) \quad \Delta_x = \partial_{zz} + \Delta_{\Gamma_z} - H_{\Gamma_z}(y)\partial_z, \quad x = X(y, z), \quad (y, z) \in \mathcal{O},$$

where Γ_z is the manifold

$$\Gamma_z = \{y + z\nu(y) \mid y \in \Gamma\}.$$

By identification, the operator Δ_{Γ_z} is understood to act on functions of the variable y , and $H_{\Gamma_z}(y)$ is the mean curvature of Γ_z measured at $y + z\nu(y)$. To make expression (3.12) more explicit, we consider local coordinates around each point of Γ .

Let $p \in \Gamma$ be a point such that $r(p) = R$. Then a neighborhood of p in Γ can be locally represented in coordinates as the graph of a smooth function defined on its tangent space $T_p\Gamma$. Let us fix an orthonormal basis Π_1, \dots, Π_8 of $T_p\Gamma$. Then there is a neighborhood \mathcal{U} of 0 in \mathbb{R}^8 and a transformation of the form

$$(3.13) \quad \mathbf{y} \in \mathcal{U} \subset \mathbb{R}^8 \mapsto Y_p(\mathbf{y}) = p + \sum_{i=1}^8 \mathbf{y}_i \Pi_i + G_p(\mathbf{y})\nu(p)$$

onto a neighborhood of p in Γ . Here G_p is a smooth function with $D_{\mathbf{y}}G_p(0) = 0$. As we will prove in Section 8.1, the fact that curvatures at $y \in \Gamma$ are of order $O(r(y)^{-1})$ (as follows from a result by L. Simon [34]) yields:

PROPOSITION 3.1. *There exists a number $\theta_0 > 0$ independent of $p \in \Gamma$ such that \mathcal{U} can be taken to be the ball $B(0, \theta_0 R)$ whenever $R = r(p) > 1$. Moreover, the following estimates hold:*

$$|D_{\mathbf{y}}G_p(\mathbf{y})| \leq C \frac{|\mathbf{y}|}{R}, \quad |D_{\mathbf{y}}^m G_p(\mathbf{y})| \leq \frac{C}{R^{m-1}}, \quad m \geq 2 \quad \text{for all } |\mathbf{y}| \leq \theta_0 R.$$

The explicit dependence on p will be dropped below for notational simplicity. Let us denote by \mathbf{g}_{ij} the metric on Γ expressed in these local coordinates, namely

$$(3.14) \quad \mathbf{g}_{ij} = \langle \partial_i Y_p, \partial_j Y_p \rangle = \delta_{ij} + \partial_i G_p(\mathbf{y}) \partial_j G_p(\mathbf{y}).$$

Then, by Propoisition 3.1,

$$\mathbf{g}_{ij} = \delta_{ij} + O(|\mathbf{y}|^2 R^{-2}), \quad D_{\mathbf{y}}^m \mathbf{g}_{ij} = O(R^{-m-1}).$$

The Laplace-Beltrami operator on Γ is represented in coordinates $\mathbf{y} \in \mathcal{U}$ as

$$(3.15) \quad \Delta_{\Gamma} = \frac{1}{\sqrt{\det \mathbf{g}}} \partial_i (\sqrt{\det \mathbf{g}} \mathbf{g}^{ij} \partial_j) = a_{ij}^0(\mathbf{y}) \partial_{ij} + b_j^0(\mathbf{y}) \partial_j,$$

where

$$a_{ij}^0(\mathbf{y}) := g^{ij} = \delta_{ij} + O(|\mathbf{y}|^2 R^{-2}), \quad b_i^0(\mathbf{y}) := \frac{1}{\sqrt{\det g}} \partial_j (\sqrt{\det g} g^{ij}) = O(|\mathbf{y}| R^{-2}).$$

We should point out that here as well as throughout the remainder of this paper we use Einstein summation convention for repeated indices. Let us observe in addition that for $y = Y_p(\mathbf{y})$ we have that

$$\nu(y) = \frac{1}{\sqrt{1 + |D_{\mathbf{y}}G_p(\mathbf{y})|^2}} (\nu(p) - \partial_i G_p(\mathbf{y}) \Pi_i);$$

hence

$$(3.16) \quad D_{\mathbf{y}}\nu = O(R^{-1}), \quad D_{\mathbf{y}}^2\nu = O(R^{-2}).$$

3.1. *Coordinates in \mathbb{R}^9 near Γ and the Euclidean Laplacian.* From estimate (3.16) it can be proven that, normal rays emanating from two points y_1, y_2 , of Γ for which $r(y_1), r(y_2) > R$, cannot intersect before a distance of order R from Γ , which justifies the definiteness of the coordinates (y, z) in (3.11) (see Remark 8.1).

Local coordinates $y = Y_p(\mathbf{y})$, $\mathbf{y} \in \mathcal{U} \subset \mathbb{R}^8$, as in (3.13) induce natural local coordinates in Γ_z , $Y_p(\mathbf{y}) + z\nu(\mathbf{y})$. The metric $\mathbf{g}_{ij}(z)$ on Γ_z can be computed:

$$(3.17) \quad \mathbf{g}_{ij}(z) = \langle \partial_i Y, \partial_j Y \rangle + z(\langle \partial_i Y, \partial_j \nu \rangle + \langle \partial_j Y, \partial_i \nu \rangle) + z^2 \langle \partial_i \nu, \partial_j \nu \rangle,$$

and hence for $r = r(y)$, and \mathbf{g}_{ij} as in (3.14),

$$\mathbf{g}_{ij}(z) = \mathbf{g}_{ij} + z O(|\mathbf{y}|r^{-2}) + z^2 O(r^{-2}), \quad D_{\mathbf{y}} g_{ij}(z) = D_{\mathbf{y}} g_{ij} + z O(r^{-2}) + z^2 O(r^{-3}).$$

Thus,

$$(3.18) \quad \Delta_{\Gamma_z} = \frac{1}{\sqrt{\det \mathbf{g}(z)}} \partial_i (\sqrt{\det \mathbf{g}(z)} \mathbf{g}^{ij}(z) \partial_j) = a_{ij}(\mathbf{y}, z) \partial_{ij} + b_i(\mathbf{y}, z) \partial_i,$$

where a_{ij}, b_i are smooth functions which can be expanded as

$$(3.19) \quad a_{ij}(\mathbf{y}, z) = a_{ij}^0(\mathbf{y}) + z a_{ij}^1(\mathbf{y}, z), \quad b_i(\mathbf{y}, z) = b_i^0(\mathbf{y}) + z \underbrace{(b_i^1(\mathbf{y}) + z b_i^2(\mathbf{y}, z))}_{\bar{b}_i^1(\mathbf{y}, z)}$$

with

$$a_{ij}^1(\mathbf{y}, z) = O(r^{-2}), \quad b_i^1(\mathbf{y}) = O(r^{-2}), \quad b_i^2(\mathbf{y}, z) = O(r^{-3}) \quad \text{for all } |\mathbf{y}| < 1.$$

Let us consider the remaining term in the expression (3.12). We have the validity of the formula

$$(3.20) \quad H(y, z) := H_{\Gamma_z}(y) = \sum_{i=1}^8 \frac{k_i}{1 - k_i z} = \sum_{j=1}^{\infty} z^{j-1} H_j(y), \quad H_j(y) := \sum_{i=1}^8 k_i^j,$$

where $k_i = k_i(y)$, $i = 1, \dots, 8$ are the principal curvatures of Γ at y , namely the eigenvalues of the second fundamental form $A_{\Gamma}(y)$, which correspond to the eigenvalues of $D_{\mathbf{y}}^2 G(0)$ for $y = p$ in the local coordinates (3.13). Since Γ is a minimal surface, we have that $H_1 = 0$. We will denote $|A_{\Gamma}(y)|^2 := H_2(y)$. We write, for later reference, for $m \geq 2$,

$$(3.21) \quad H(y, z) = z H_2(y) + z^2 H_3(y) + \dots + z^{m-2} H_{m-1}(y) + z^{m-1} \bar{H}_m(y, z),$$

where, since $k_i = O(r^{-1})$, we have

$$H_j(y) = O(r^{-j}), \quad \bar{H}_m(y, z) = O(r^{-m}).$$

Thus in local coordinates (y, z) , $y = Y_p(\mathbf{y})$, we have the validity of the expression

$$(3.22) \quad \Delta_x = \partial_{zz} + a_{ij}(\mathbf{y}, z) \partial_{ij} + b_i(\mathbf{y}, z) \partial_i - H(y, z) \partial_z,$$

with the coefficients described above.

We can use the above formula to derive an expression for the Laplacian near Γ_α by simple dilation as follows: We consider now the coordinates near Γ_α :
 (3.23)

$$x = X_\alpha(y, z) = y + z \nu(\alpha y), \quad (y, z) \in \mathcal{O}_\alpha = \left\{ y \in \Gamma_\alpha, |z| < \frac{\delta}{\alpha}(r(\alpha y) + 1) \right\}.$$

If $p \in \Gamma_\alpha$ and $p_\alpha := \alpha p \in \Gamma$, then the local coordinates $y = Y_{p_\alpha}(y)$ defined in (3.13) inherit corresponding coordinates in an α^{-1} -neighborhood of p by setting, with some $\theta > 0$ (depending in Γ),

$$(3.24) \quad y = Y_{p,\alpha}(y) := \alpha^{-1} Y_{p_\alpha}(\alpha y), \quad |y| < \frac{\theta}{\alpha}.$$

Let us consider a function $u(x)$ defined near Γ_α . Then letting $v(y, z) = u(X_\alpha(y, z))$, and defining $u(x) =: \tilde{u}(\alpha x)$ we find

$$\begin{aligned} \Delta_x u|_{x=X_\alpha(y,z)} &= \alpha^2 \Delta_{\tilde{x}} \tilde{u}(\tilde{x})|_{\tilde{x}=X(\alpha y, \alpha z)} \\ &= \alpha^2 (\partial_{\tilde{z}\tilde{z}} + a_{ij}(\tilde{y}, \tilde{z}) \partial_{ij} + b_i(\tilde{y}, \tilde{z}) \partial_i - H(\tilde{y}, \tilde{z}) \partial_{\tilde{z}}) \\ &\quad \times v(\alpha^{-1} \tilde{y}, \alpha^{-1} \tilde{z})|_{(\tilde{y}, \tilde{z})=(\alpha y, \alpha z)}, \end{aligned}$$

which means precisely that for the coordinates (3.23) we have

$$(3.25) \quad \Delta_x = \partial_{zz} + a_{ij}(\alpha y, \alpha z) \partial_{ij} + \alpha b_i(\alpha y, \alpha z) \partial_i - \alpha H(\alpha y, \alpha z) \partial_z.$$

Let us fix now a smooth, small function h defined on Γ as in (3.9) and consider coordinates (3.23) defined near Γ_α as

$$(3.26) \quad \begin{aligned} x &= X_h(y, t) = y + (t + h(\alpha y)) \nu(\alpha y), \\ (y, t) &\in \mathcal{O}_h = \left\{ y \in \Gamma_\alpha, |t + h(\alpha y)| < \frac{\delta}{\alpha}(r(\alpha y) + 1) \right\}. \end{aligned}$$

If $v(y, t) = u(X_h(y, t)) = \tilde{v}(y, t + h(\alpha y))$, then

$$\begin{aligned} \Delta_x u|_{x=X_h(y,t)} &= \Delta_x u|_{x=X_\alpha(y,t+h(\alpha y))} \\ &= (\partial_{zz} + a_{ij}(\alpha y, \alpha z) \partial_{ij} + \alpha b_i(\alpha y, \alpha z) \partial_i - \alpha H(\alpha y, \alpha z) \partial_z) \\ &\quad \times [\tilde{v}(y, z - h(\alpha y))]|_{(y,z)=(y,t+h(\alpha y))}, \end{aligned}$$

where by slight abuse of notation we are denoting by $h(\alpha y)$ the function $h \circ Y(\alpha y)$. Carrying out the differentiations and using the symmetry of a_{ij} , we arrive at the following expression for the Laplacian in coordinates (3.26)

$$(3.27) \quad \begin{aligned} \Delta_x &= \left(1 + \alpha^2 a_{ij} \partial_i h \partial_j h \right) \partial_{tt} + a_{ij} \partial_{ij} - 2\alpha a_{ij} \partial_i h \partial_{jt} + \alpha b_i \partial_i \\ &\quad - (\alpha^2 (a_{ij} \partial_{ij} h + b_i \partial_i h) + \alpha H) \partial_t, \end{aligned}$$

where all coefficients are evaluated at αy or $(\alpha y, \alpha(t + h(\alpha y)))$.

We observe that for $y = Y_{p,\alpha}(y)$, we have that (with some $\theta > 0$ small)

$$(3.28) \quad \Delta_{\Gamma_\alpha} = a_{ij}^0(\alpha y) \partial_{ij} + \alpha b_i^0(\alpha y) \partial_i, \quad |y| < \frac{\theta}{\alpha}.$$

Therefore if we write

$$(3.29) \quad \Delta_x = \partial_{tt} + \Delta_{\Gamma_\alpha} + B,$$

then, with the notation (3.19), the operator B acting on functions of $(y, t) \in \mathcal{O} \subset \Gamma_\alpha \times \mathbb{R}$ is given by

$$(3.30) \quad B = \alpha^2 a_{ij} \partial_i h \partial_j h \partial_{tt} + \alpha(t+h)(a_{ij}^1 \partial_{ij} + \alpha b_i^1 \partial_i) - 2\alpha a_{ij} \partial_i h \partial_j t - (\alpha^2(a_{ij} \partial_{ij} h + b_i \partial_i h) + \alpha H) \partial_t.$$

3.2. *Error of approximation.* Let us take as a first approximation to a solution of the Allen-Cahn equation simply the function $u_0(x) := w(t)$. We set

$$S(u) = \Delta u + f(u).$$

Since $w''(t) + f(w(t)) = 0$, we find that

$$S(u_0) = \alpha^2 a_{ij} \partial_i h \partial_j h w''(t) - (\alpha^2(a_{ij} \partial_{ij} h + b_i \partial_i h) + \alpha H) w'(t).$$

We expand $H(\alpha y, \alpha(t+h))$ according to (3.21) as

$$H = \alpha(t+h)|A_\Gamma(\alpha y)|^2 + \alpha^2(t+h)^2 H_3(\alpha y) + \alpha^3(t+h)^3 \bar{H}_4(\alpha y, \alpha(t+h)),$$

and we also expand

$$a_{ij} \partial_i h + b_i \partial_i h = \Delta_\Gamma h(\alpha y) + \alpha(t+h)(a_{ij}^1 \partial_{ij} h + b_i^1 \partial_i h).$$

Next we improve the approximation by eliminating the only term of size order α^2 in the error, namely $-\alpha^2|A_\Gamma(\alpha y)|^2 t w'(t)$. Let us consider the differential equation

$$\psi_0''(t) + f'(w(t))\psi_0(t) = t w'(t),$$

which has a unique bounded solution with $\psi_0(0) = 0$, given explicitly by the formula

$$\psi_0(t) = w'(t) \int_0^t w'(s)^{-2} \int_{-\infty}^s s w'(s)^2 ds.$$

Observe that this function is well defined and it is bounded since $\int_{-\infty}^\infty s w'(s)^2 ds = 0$ and $w'(t) \sim e^{-\sigma|t|}$ as $t \rightarrow \pm\infty$, any $\sigma < \sqrt{2}$. We consider as a second approximation

$$(3.31) \quad u_1 = u_0 + \phi_1, \quad \phi_1(y, t) := \alpha^2 |A_\Gamma(\alpha y)|^2 \psi_0(t).$$

Let us observe that

$$S(u_0 + \phi) = S(u_0) + \Delta_x \phi + f'(u_0)\phi + N_0(\phi), \quad N_0(\phi) = f(u_0 + \phi) - f(u_0) - f'(u_0)\phi.$$

We have that

$$\partial_{tt} \phi_1 + f'(u_0)\phi_1 = \alpha^2 |A(\alpha y)|^2 t w'.$$

Hence we get that the largest term in the error is cancelled. Indeed, we have

$$S(u_1) = S(u_0) + \alpha^2 |A_\Gamma(\alpha y)|^2 t w' + [\Delta_x - \partial_{tt}] \phi_1 + N_0(\phi_1).$$

Let us write $H_2(\alpha y) = |A_\Gamma(\alpha y)|^2$. We compute

$$\begin{aligned}
 (3.32) \quad S(u_1) = & -\alpha^2[\Delta_\Gamma h + |A_\Gamma|^2 h + \alpha(t+h)^2 H_3 + \alpha^2(t+h)^3 \bar{H}_4]w' \\
 & + \alpha^2 a_{ij} \partial_i h \partial_j h w'' + \alpha^3(t+h)(a_{ij}^1 \partial_{ij} h + b_i^1 \partial_i h)w' \\
 & - [\alpha^3 H + \alpha^4(a_{ij} \partial_{ij} h + b_i \partial_i h)]H_2 \psi'_0 \\
 & + \alpha^4((a_{ij} \partial_{ij} H_2 + b_i \partial_i H_2)\psi_0 - 2\alpha^4 a_{ij} \partial_i h \partial_j H_2 \psi'_0 \\
 & + \alpha^4 a_{ij} \partial_i h \partial_j h H_2 \psi''_0 + N_0(\alpha^2 H_2 \psi_0)),
 \end{aligned}$$

where all coefficients are evaluated at αy or $(\alpha y, \alpha(t+h(\alpha y)))$. Roughly speaking, the largest terms remaining in the above expression (recalling assumption (3.9)) are of size $O(\alpha^3 r_\alpha^{-3}(y)e^{-\sigma|t|})$. We introduce next a suitable norm to account for this type of decay. This norm will be used throughout the paper in the functional analytic set up.

For numbers $0 < \sigma < \sqrt{2}$, $p > 9$, $\nu > 0$, and a function g defined on $\Gamma_\alpha \times \mathbb{R}$, let us write

$$(3.33) \quad \|g\|_{p,\nu,\sigma} := \sup_{(y,t) \in \Gamma_\alpha \times \mathbb{R}} e^{\sigma|t|} r_\alpha^\nu(y) \|g\|_{L^p(B((y,t),1))}.$$

Then, for instance,

$$(3.34) \quad \|(\Delta_\Gamma h)(\alpha y)w'(t)\|_{p,3,\sigma} \leq C \sup_{y \in \Gamma} \|D_\Gamma^2 h\|_{L^p(B(y,\alpha) \cap \Gamma)} \alpha^{-\frac{8}{p}} \leq CM \alpha^{1-\frac{8}{p}}.$$

In all we get, assuming for instance that $S(u_1)$ is extended as zero outside \mathcal{O}_h ,

$$(3.35) \quad \|S(u_1)\|_{p,3,\sigma} \leq CM \alpha^{3-\frac{8}{p}}.$$

3.3. Global first approximation. The function u_1 built above is sufficient for our purposes as an approximation of the solution near Γ_α but it is only defined in a neighborhood of it. Let us consider the function \mathbb{H} defined in $\mathbb{R}^9 \setminus \Gamma_\alpha$ as

$$(3.36) \quad \mathbb{H}(x) := \begin{cases} 1, & \text{if } x_9 > F_\alpha(x'), \\ -1, & \text{if } x_9 < F_\alpha(x'). \end{cases}$$

The global approximation we will use consists simply of interpolating u_1 with \mathbb{H} outside of a large, expanding neighborhood of Γ_α using a cut-off function of $|t|$.

We recall that the set \mathcal{O}_h in $\Gamma_\alpha \times \mathbb{R}$ was defined as (see (3.26)):

$$(3.37) \quad \mathcal{O}_h = \left\{ (y, t) \in \Gamma_\alpha \times \mathbb{R}, \quad |t + h(\alpha y)| < \frac{\delta}{\alpha}(1 + r_\alpha(y)) \right\},$$

where δ is small positive number. We will denote $\mathcal{N}_\delta = X_h(\mathcal{O}_h)$. The fact that \mathcal{O}_h is actually expanding with r_α along Γ_α makes it possible to choose the cut-off in such a way that the error created has both smallness in α and fast decay in r_α .

Let $\eta(s)$ be a smooth cut-off function with $\eta(s) = 1$ for $s < 1$ and $\eta(s) = 0$ for $s > 2$. Let us introduce the cut-off functions $\zeta_m, m = 1, 2, \dots,$

$$(3.38) \quad \zeta_m(x) := \begin{cases} \eta(|t + h(\alpha y)| - \frac{\delta}{2\alpha}(1 + r_\alpha(y)) - m), & \text{if } x \in \mathcal{N}_\delta, \\ 0, & \text{if } x \notin \mathcal{N}_\delta. \end{cases}$$

Then we let our global approximation $\mathbf{w}(x)$ be simply defined as

$$(3.39) \quad \mathbf{w} := \zeta_5 u_1 + (1 - \zeta_5)\mathbb{H},$$

where \mathbb{H} is given by (3.36) and $u_1(x)$ is just understood to be $\mathbb{H}(x)$ outside \mathcal{N}_δ .

The global error of approximation becomes

$$(3.40) \quad S(\mathbf{w}) = \Delta \mathbf{w} + f(\mathbf{w}) = \zeta_5 S(u_1) + E,$$

where

$$E = 2\nabla \zeta_5 \nabla u_1 + \Delta \zeta_5 (u_1 - \mathbb{H}) + f(\zeta_5 u_1 + (1 - \zeta_5)\mathbb{H}) - \zeta_5 f(u_1).$$

The new error terms created are of exponentially small size and have fast decay with r_α . In fact, we have

$$|E| \leq C e^{-\frac{\delta}{\alpha}(1+r_\alpha)}.$$

Remark 3.1. Tracking back the way \mathbf{w} was built we see that it has the required symmetry near Γ_α , namely $\mathbf{w}(\mathcal{Q}y, -t) = -\mathbf{w}(y, t)$, which is as well respected by the cut-off functions. Using relation (3.8) we conclude that, globally in \mathbb{R}^9 , $\mathbf{w}(\mathcal{Q}x) = -\mathbf{w}(x)$. Since the orthogonal transformations P, Q in the definition of \mathcal{Q} in (3.2) are arbitrary, we get that $\mathbf{w} = \mathbf{w}(u, v, x_9)$ with $\mathbf{w}(v, u, -x_9) = -\mathbf{w}(u, v, x_9)$. It follows that exactly the same symmetry is obeyed by the error $S(\mathbf{w})$.

4. The proof of Theorem 1

We look for a solution u of the Allen-Cahn equation (3.1) in the form

$$U = \mathbf{w} + \varphi,$$

where \mathbf{w} is the global approximation defined in (3.39) and φ is in some suitable sense small, with the additional symmetry requirement

$$(4.1) \quad \varphi(\mathcal{Q}x) = -\varphi(x) \quad \text{for all } x \in \mathbb{R}^9,$$

so that (3.3) holds.

Thus we need to solve the following problem

$$(4.2) \quad \Delta \varphi + f'(\mathbf{w})\varphi = -S(\mathbf{w}) - N(\varphi),$$

where

$$N(\varphi) = f(\mathbf{w} + \varphi) - f(\mathbf{w}) - f'(\mathbf{w})\varphi.$$

The procedure of construction of a solution is made up of several steps which we explain next, postponing the proof of major facts for later sections.

4.1. *Reduction by a gluing procedure.* Here we perform a procedure that reduces (4.2) to a similar problem on entire $\Gamma_\alpha \times \mathbb{R}$, which in \mathcal{O}_h coincides with the expression of (4.2) in (y, t) coordinates, except for the addition of a very small nonlocal, nonlinear operator.

Let us consider the cut-off functions ζ_m introduced in (3.38). We look for a solution $\varphi(x)$ of problem (4.2) of the following form:

$$(4.3) \quad \varphi(x) = \zeta_2(x)\phi(y, t) + \psi(x),$$

where ϕ is defined in entire $\Gamma_\alpha \times \mathbb{R}$, $\psi(x)$ is defined in \mathbb{R}^9 and $\zeta_2(x)\phi(y, t)$ is understood to be zero outside \mathcal{N}_δ . We see that $\zeta_2(\mathcal{Q}x) = \zeta_2(x)$. Thanks to relation (3.8), φ will satisfy (4.1) if we require

$$(4.4) \quad \phi(\mathcal{Q}y, -t) = -\phi(y, t) \quad \text{for all } (y, t) \in \Gamma_\alpha \times \mathbb{R},$$

$$(4.5) \quad \psi(\mathcal{Q}x) = -\psi(x) \quad \text{for all } x \in \mathbb{R}^9.$$

We compute, using that $\zeta_2\zeta_1 = \zeta_1$,

$$(4.6) \quad \begin{aligned} S(\mathbf{w} + \varphi) &= \Delta\varphi + f'(\mathbf{w})\varphi + N(\varphi) + S(\mathbf{w}) \\ &= \zeta_2 [\Delta\phi + f'(u_1)\phi + \zeta_1(f'(u_1) - f'(1))\psi + \zeta_1N(\psi + \phi) + S(u_1)] \\ &\quad + \Delta\psi + [(1 - \zeta_1)f'(u_1) + \zeta_1f'(1)]\psi \\ &\quad + (1 - \zeta_2)S(\mathbf{w}) + (1 - \zeta_1)N(\psi + \zeta_2\phi) + 2\nabla\zeta_1\nabla\phi + \phi\Delta\zeta_1. \end{aligned}$$

We recall that $f'(\pm 1) = -2$.

Thus, we will construct a solution $\varphi = \zeta_2\phi + \psi$ to problem (4.2) if we require that the pair (ϕ, ψ) satisfies the following coupled system:

$$(4.7) \quad \begin{aligned} \Delta\phi + f'(u_1)\phi + \zeta_1(f'(u_1) + 2)\psi + \zeta_1N(\psi + \phi) + S(u_1) &= 0 \\ \text{for } |t| < \frac{\delta}{2\alpha}(1 + r_\alpha(y)) + 3, \end{aligned}$$

$$(4.8) \quad \begin{aligned} \Delta\psi + [(1 - \zeta_1)f'(u_1) - 2\zeta_1]\psi + (1 - \zeta_2)S(\mathbf{w}) + (1 - \zeta_1)N(\psi + \zeta_2\phi) \\ + 2\nabla\zeta_1\nabla\phi + \phi\Delta\zeta_1 = 0 \text{ in } \mathbb{R}^9. \end{aligned}$$

We will first extend equation (4.7) to entire $\Gamma_\alpha \times \mathbb{R}$ in the following manner. Let us set

$$(4.9) \quad \widetilde{B}(\phi) := \zeta_4[\Delta_x - \partial_{tt} - \Delta_{\Gamma_\alpha}]\phi = \zeta_4B(\phi),$$

where Δ_x is expressed in (y, t) coordinates using expression (3.27) with B described in (3.30), and $\widetilde{B}(\phi)$ is understood to be zero for (y, t) outside the support of ζ_4 . Similarly, we extend the local expression (3.32) for the error of approximation $S(u_1)$ in (y, t) coordinates, to entire $\Gamma_\alpha \times \mathbb{R}$ as

$$\widetilde{S}(u_1) = \zeta_4S(u_1),$$

with this expression understood to be zero outside the support of ζ_4 .

Thus we consider the extension of equation (4.7) given by

$$(4.10) \quad \begin{aligned} \partial_{tt}\phi + \Delta_{\Gamma_\alpha}\phi + \widetilde{B}(\phi) + f'(w(t))\phi \\ = -\widetilde{S}(u_1) - \{[f'(u_1) - f'(w)]\phi + \zeta_1(f'(u_1) + 2)\psi\} - \zeta_1 N(\psi + \phi) \end{aligned}$$

in $\Gamma_\alpha \times \mathbb{R}$.

Consistently with estimate (3.35) for the error, we consider the norm $\|\cdot\|_{p,\sigma,\nu}$ defined in (3.33) and consider for a function $\phi(y, t)$ the norm

$$(4.11) \quad \|\phi\|_{2,p,\sigma,\nu} := \|D^2\phi\|_{p,\sigma,\nu} + \|D\phi\|_{\infty,\sigma,\nu} + \|\phi\|_{\infty,\sigma,\nu}.$$

To solve the resulting system (4.7)–(4.8), we first solve equation (4.8) for ψ with a given ϕ , which is a small function in the above norm. For a function $\psi(x)$ defined in \mathbb{R}^9 , we define

$$(4.12) \quad \|\psi\|_{p,\nu,*} := \sup_{x \in \mathbb{R}^9} (1 + r(\alpha x))^\nu \|\psi\|_{L^p(B(x,1))}, \quad r(x', x_9) = |x'|.$$

Noting that the potential $[(1 - \zeta_1)f'(u_1) - 2\zeta_1]$ is strictly negative, so that the linear operator in (4.8) is qualitatively like $\Delta - 2$ and using contraction mapping principle, a solution $\psi = \Psi(\phi)$ is found according to the following lemma, whose detailed proof we carry out in Section 5.

LEMMA 4.1. *Let $\mu > 0$. Given ϕ satisfying the symmetry (4.4) and $\|\phi\|_{2,p,3,\sigma} \leq 1$, for all sufficiently small α , there exists a unique solution $\psi = \Psi(\phi)$ of problem (4.8) such that*

$$(4.13) \quad \|\psi\|_{2,p,3+\mu,*} := \|D^2\psi\|_{p,3+\mu,*} + \|\psi\|_{\infty,3+\mu,*} \leq C e^{-\frac{\sigma\delta}{\alpha}}.$$

Besides, Ψ satisfies the symmetry (4.5) and the Lipschitz condition

$$(4.14) \quad \|\Psi(\phi_1) - \Psi(\phi_2)\|_{2,p,3+\mu,*} \leq C e^{-\frac{\sigma\delta}{\alpha}} \|\phi_1 - \phi_2\|_{2,p,3,\sigma}.$$

Thus if we replace $\psi = \Psi(\phi)$ in the first equation (4.7) by setting

$$(4.15) \quad N(\phi) := \widetilde{B}(\phi) + [f'(u_1) - f'(w)]\phi + \zeta_1(f'(u_1) + 2)\Psi(\phi) + \zeta_1 N(\Psi(\phi) + \phi),$$

then our problem is reduced to finding a solution ϕ to the following nonlinear, nonlocal problem in $\Gamma_\alpha \times \mathbb{R}$:

$$(4.16) \quad \partial_{tt}\phi + \Delta_{\Gamma_\alpha}\phi + f'(w)\phi = -\widetilde{S}(u_1) - N(\phi) \quad \text{in } \Gamma_\alpha \times \mathbb{R}.$$

Examining the terms in (4.15), we notice that if ϕ satisfies the symmetry (4.4) then so do $N(\phi)$ and $\widetilde{S}(u_1)$. Thus we will solve the original problem (1.1) if we find a solution to problem (4.16). We will be able to do this for a certain specific choice of the parameter function h on which all elements in the right-hand side of (4.16) depend.

4.2. *An infinite dimensional Lyapunov-Schmidt reduction.* In order to find a solution to (4.16), we follow an infinite dimensional Lyapunov-Schmidt reduction procedure: we consider first the projected problem

$$(4.17) \quad \partial_{tt}\phi + \Delta_{\Gamma_\alpha}\phi + f'(w)\phi = -\tilde{S}(u_1) - \mathbf{N}(\phi) + c(y)w'(t) \quad \text{in } \Gamma_\alpha \times \mathbb{R},$$

$$\int_{\mathbb{R}} \phi(y, t) w'(t) dt = 0 \quad \text{for all } y \in \Gamma_\alpha,$$

where

$$(4.18) \quad c(y) := \frac{1}{\int_{\mathbb{R}} w'^2} \int_{\mathbb{R}} [\tilde{S}(u_1) + \mathbf{N}(\phi)] w'(t) dt.$$

The correction $c(y)w'(t)$ to the right-hand side provides unique solvability for any choice of the parameter h satisfying (3.9) in the sense of the following result, whose proof will be given in Section 6.1.

PROPOSITION 4.1. *Assume $p > 9$, $0 < \sigma < \sqrt{2}$. There exists a $K > 0$ such that for any sufficiently small α and any h satisfying (3.9), problem (4.17) has a unique solution $\phi = \Phi(h)$ that satisfies the symmetry (4.4) and such that*

$$(4.19) \quad \|\phi\|_{2,p,3,\sigma} \leq K\alpha^{3-\frac{8}{p}}, \quad \|\mathbf{N}(\phi)\|_{p,5,\sigma} \leq K\alpha^{5-\frac{8}{p}}.$$

Proposition 4.1 reduces the problem of finding a solution to problem (4.16) to that of finding a function h satisfying the constraint (3.9) such that $c(y) \equiv 0$ with c given by (4.18) for $\phi = \Phi(h)$, in other words such that

$$(4.20) \quad \int_{\mathbb{R}} [\tilde{S}(u_1) + \mathbf{N}(\Phi(h))](y, t) w'(t) dt = 0 \quad \text{for all } y \in \Gamma_\alpha.$$

4.3. *Solving the reduced problem.* We concentrate next in expressing the reduced problem (4.20) in a convenient form. We begin by computing an expansion of the quantity $\int_{\mathbb{R}} \tilde{S}(u_1) w'(t) dt$ making use of the expression (3.32) for $S(u_1)$. Let us decompose, using also expansion (3.21) for H ,

$$-\alpha^{-2}S(u_1) = [\Delta_\Gamma h + |A_\Gamma|^2 h + \alpha t^2 H_3]w' + E_1(y, t) + E_2(y, t),$$

where

$$E_1(y, t) = 2\alpha h H_3 t w' - \alpha b_i^1(\alpha y, 0) \partial_i h t w' - a_{ij}^0 \partial_i h \partial_j h w''$$

$$+ \alpha^2 [H_4 t^3 w' + H_2^2 t \psi_0' - H_2^2 f''(w) \psi_0^2 - (a_{ij}^0 \partial_{ij} H_2 + b_i^0 \partial_i H_2) \psi_0]$$

and

$$(4.21) \quad E_2(y, t) = [\alpha h^2 H_3 + \alpha^2((t+h)^3 - t^3)H_4 + \alpha^3(t+h)^4 \bar{H}_5]w'$$

$$- \alpha(t+h)[a_{ij}^1 \partial_i h \partial_j h w'' + a_{ij}^1 \partial_{ij} h w']$$

$$- \alpha \partial_i h [(t+h)b_i^1(\alpha y, \alpha(t+h)) - t b_i^1(\alpha y, 0)]w'$$

$$+ [\alpha^2 h H_2 + \alpha^3(t+h)^2 \bar{H}_3 + \alpha^2(a_{ij} \partial_{ij} h + b_i \partial_i h)]H_2 \psi_0'$$

$$\begin{aligned}
 &+ 2\alpha^2 a_{ij} \partial_i h \partial_j H_2 \psi'_0 - \alpha^3 (t + h) ((a_{ij}^1 \partial_{ij} H_2 + b_i^1 \partial_i H_2) \psi_0 \\
 &- \alpha^2 a_{ij} \partial_i h \partial_j h H_2 \psi''_0 - \alpha^{-2} [N_0(\alpha^2 H_2 \psi_0) - f''(w)(\alpha^2 H_2 \psi_0)^2].
 \end{aligned}$$

We recall that evaluation of the coefficients is made in local coordinates at y or $(\alpha y, \alpha(t + h(\alpha y)))$.

The logic of this decomposition is that terms in E_1 decay at most like $O(r_\alpha^{-4})$ but the functions of t involved in them are all odd, while those in E_2 decay like $O(r_\alpha^{-5})$, according to assumption (3.9) in h and the estimates we have obtained in the coefficients. We have

$$\int_{\mathbb{R}} E_1(y, t) w'(t) dt = 0,$$

while there is a constant C , possibly depending on the number M in constraint (3.9) such that for all h satisfying those relations, we have

$$(4.22) \quad |E_2(y, t)| \leq C(1 + r_\alpha^5)^{-1} [\alpha |(1 + r_\alpha^3) D_\Gamma^2 h(\alpha y)| + \alpha^2].$$

Thus, setting $c_1 = \int_{\mathbb{R}} w'^2 dt$, $c_2 = \int_{\mathbb{R}} t^2 w'^2 dt$, we find

$$(4.23) \quad -\alpha^{-2} \int_{\mathbb{R}} \tilde{S}(u_1)(y, t) w'(t) dt = c_1 [\Delta_\Gamma h + |A_\Gamma|^2 h](\alpha y) + c_2 \alpha H_3(\alpha y) - \mathcal{G}_1[h](\alpha y),$$

where, we recall, $H_3 = \sum_{i=1}^8 k_i^3$, and

$$\begin{aligned}
 (4.24) \quad \mathcal{G}_1[h](\alpha y) &:= \int_{\mathbb{R}} (\zeta_4 - 1) [(\Delta_\Gamma h + |A_\Gamma|^2 h + \alpha t^2 H_3) w' + E_1(y, t)] w' dt \\
 &+ \int_{\mathbb{R}} \zeta_4 E_2(y, t) w' dt.
 \end{aligned}$$

Let us observe that

$$(1 - \zeta_4)(|w'| + |w''|) \leq C e^{-\frac{\sigma}{\alpha} - \sigma r_\alpha};$$

hence the contribution of the first integral above is exponentially small in α and in r_α . Using (4.22) we get

$$(4.25) \quad \|\mathcal{G}_1[h]\|_{p,5} \leq C \alpha^2.$$

Now let us consider the operator

$$(4.26) \quad \mathcal{G}_2[h](\alpha y) = \alpha^{-2} \int_{\mathbb{R}} N(\Phi(h)) w' dt.$$

More generally, it will be convenient to consider a function $\psi(y, t)$ defined in $\Gamma_\alpha \times \mathbb{R}$ and the function g defined on Γ by the relation

$$g(y) = \int_{\mathbb{R}} \psi(\alpha^{-1} y, t) w' dt.$$

Then

$$\int_A |g(y)|^p dV_\Gamma(y) \leq C \sum_{|k| \geq 1} \alpha^8 \int_{\alpha^{-1} A} \int_{|t-k| < 1} |\psi(y, t)|^p dt dV_{\Gamma_\alpha}(y).$$

If $A = B(\bar{y}, 1) \cap \Gamma$, then $\alpha^{-1}A$ can be covered by $O(\alpha^{-8})$ balls of radius 1 in Γ_α . Thus

$$\int_{\alpha^{-1}A} \int_{|t-k|<1} |\psi(y, t)|^p dt dV_{\Gamma_\alpha}(y) \leq Cr(\bar{y})^{-\nu p} e^{-p\sigma|k|} \|\psi\|_{p,\nu,\sigma}^p,$$

and hence

$$(4.27) \quad \|g\|_{p,\nu} = \sup_{\bar{y} \in \Gamma} (1 + r(\bar{y})^\nu) \|g\|_{L^p(B(\bar{y},1) \cap \Gamma)} \leq C\alpha^{-\frac{8}{p}} \|\psi\|_{p,\nu,\sigma}.$$

Now, examining the expression (4.15) for the operator \mathbf{N} and using the bound (4.19) for $\Phi(h)$ we have that

$$\|\mathbf{N}(\Phi(h))\|_{p,5,\sigma} \leq C\alpha^4;$$

hence for \mathcal{G}_2 defined above, we get

$$\|\mathcal{G}_2(h)\|_{p,5,\sigma} \leq C\alpha^{2-\frac{8}{p}}$$

uniformly in h satisfying (3.9). In summary, the reduced equation (4.20) reads

$$(4.28) \quad \mathcal{J}_\Gamma[h](y) := \Delta_\Gamma h(y) + |A_\Gamma(y)|^2 h(y) = c\alpha H_3(y) + \mathcal{G}[h](y), \quad y \in \Gamma,$$

where

$$c = -c_2/c_1, \quad \mathcal{G}[h] := -c_1^{-1}(\mathcal{G}_1[h] + \mathcal{G}_2[h]).$$

The operator \mathcal{G} satisfies

$$(4.29) \quad \|\mathcal{G}[h]\|_{p,5} \leq C\alpha^{2-\frac{8}{p}}$$

for all h satisfying (3.9). Moreover, we observe that if $p(y, t)$ satisfies $p(\mathcal{Q}y, -t) = -p(y, t)$, then

$$\int_{\mathbb{R}} p(\mathcal{Q}y, t) w'(t) dt = - \int_{\mathbb{R}} p(y, t) w'(t) dt,$$

since w' is an even function. Since $p = \tilde{S}(u_1) + \mathbf{N}(\Phi(h))$ satisfies this requirement, we conclude that so does the operator $\mathcal{G}[h]$ and it is hence consistent to look for a solution h in this class of symmetries.

It seems natural to attempt to solve problem (4.28) for functions h , with $\|h\|_* < M\alpha$ (see (3.9)) by a fixed point argument that involves an inverse for the Jacobi operator \mathcal{J}_Γ . Thus we consider the linear problem

$$(4.30) \quad \Delta_\Gamma h + |A_\Gamma(y)|^2 h = g, \quad y \in \Gamma.$$

We stress here the fact that functions h and g belong to the admissible class of symmetries. The solvability theory for (4.30) needs to consider separately the case $g = c\alpha H_3(y)$, which has a decay of order $O(r^{-3})$ and an additional vanishing property, and the case of a g with decay $O(r^{-5})$. We prove the following proposition in Section 7.

PROPOSITION 4.2. *The following statements hold:*

- (a) *If $g(y) = cH_3(y)$, then problem (4.30) has a solution h_0 with $\|h_0\|_* < +\infty$.*
- (b) *Given g with $\|g\|_{p,5} < +\infty$, there exists a unique solution $h := \mathcal{T}(g)$ to problem (4.30) with $\|h\|_* < +\infty$. Moreover, for a certain $C > 0$, $\|h\|_* \leq C\|g\|_{p,5}$.*

Writing $h := \alpha h_0 + h_1$, the equation becomes, in terms of h_1 ,

$$(4.31) \quad \Delta_\Gamma h_1 + |A_\Gamma(y)|^2 h_1 = \mathcal{G}[h_0 + h_1], \quad y \in \Gamma.$$

Finally, we solve problem (4.31) by an application of contraction mapping principle. We write it in the form

$$(4.32) \quad h_1 = \mathcal{T}(\mathcal{G}(h_0 + h_1)) =: \mathcal{M}(h_1), \quad \|h_1\|_* \leq \alpha^{2-\frac{8}{9}}.$$

Bound (4.29) and the proposition above implies that the \mathcal{M} applies the region $\|h_1\|_* \leq \alpha^{2-\frac{8}{9}}$ into itself if α is sufficiently small. Not only this, we will prove in Section 6.2:

LEMMA 4.2.

$$(4.33) \quad \|\mathcal{G}(h_1) - \mathcal{G}(h_2)\|_{p,5} \leq C\alpha^{1-\frac{16}{p}} \|h_1 - h_2\|_*$$

for all h_1, h_2 , satisfying (3.9).

Hence \mathcal{M} is also a contraction mapping. The existence of a unique solution of (4.32) follows. It is simply enough to choose the number M in (3.9) such that $M > \|h_0\|_*$.

Remark 4.1. We emphasize that, as we will see in Section 7, equation (4.30) can actually be solved with right-hand sides $g = O(r^{-4-\mu})$ for $h = O(r^{-2-\mu})$, whenever $0 < \mu < 1$, but we do not expect in general the existence of a solution $h = O(r^{-1})$ when $g = O(r^{-3})$. However assuming additionally that g has the form $g = g(\theta)^\tau r^{-3}$ where $\tau > \frac{1}{3}$ we can establish statement (a) of Proposition 4.2. We will prove that $H_3 = \sum_{i=1}^8 k_i^3$ is of the required form except for a term which decays fast in r . Individually, the principal curvatures k_i do not have this vanishing property but their mutual cancelations gives it for the average of their cubic powers. To track this property it is necessary to compare curvatures at a point of Γ with those at its closest neighbor in Γ_0 , and the suitably defined closeness for large r of the Jacobi operator on Γ to that on Γ_0 . We discuss these issues in Section 8.2 and Section 8.3, using as the basis the result of Theorem 2, whose self-contained proof we postpone to the last part of the paper.

4.4. *Conclusion.* Let us summarize the results of our considerations so far. Given the solution to the nonlinear projected problem ϕ and the corresponding solution h_α to the reduced problem found above we have found U_α such that

$$U_\alpha = \mathbf{w} + \zeta_2\phi + \psi(\phi)$$

and

$$\Delta U_\alpha + (1 - U_\alpha^2) U_\alpha = 0 \quad \text{in } \mathbb{R}^9.$$

The function U_α is a bounded function which obeys the symmetry of the minimal graph Γ_α :

$$(4.34) \quad U_\alpha(u, v, x_9) = -U_\alpha(v, u, -x_9),$$

from which it follows in particular $U_\alpha(0) = 0$. We show next that U_α is in fact monotone in the x_9 -direction. Let us observe that the function $\psi_\alpha := \partial_{x_9} U_\alpha$ is a solution of the linear equation

$$\Delta \psi_\alpha + f'(U_\alpha)\psi_\alpha = 0.$$

We claim that the construction yields the following: given $M > 0$, at points within distance at most M from Γ_α we have $\psi_\alpha > 0$ whenever α is sufficiently small. Indeed,

$$\partial_{x_9} U_\alpha(x) = \partial_{x_9} w(t) + O\left(\frac{\alpha^2}{1 + r_\alpha^2}\right) = w'(t)\partial_{x_9} t + O\left(\frac{\alpha^2}{1 + r_\alpha^2}\right).$$

The coordinates x and (y, t) are related by $x = y + (t + h(\alpha y))\nu(\alpha y)$; hence

$$e_9 = \partial_{x_9} y + \partial_{x_9} t\nu + \alpha[D_\Gamma h(\alpha y)\partial_{x_9} y]\nu + \alpha(t + h)[D_\Gamma \nu \partial_{x_9} y].$$

If $|t| \leq M$, then we deduce that $\partial_{x_9} y$ is uniformly bounded, and also

$$\partial_{x_9} t = \nu_9 + O\left(\frac{\alpha}{1 + r_\alpha^2}\right) = \frac{1}{\sqrt{1 + |\nabla F_\alpha|^2}} + O\left(\frac{\alpha}{1 + r_\alpha^2}\right) \geq \frac{c}{1 + r_\alpha^2}$$

by an estimate in [34]; see (8.33) below. This shows our claim.

Taking M sufficiently large (but independent of α) we can achieve $f'(U_\alpha) > -3/2$ outside of $\mathcal{N}_M := \{|t| \leq M\}$. We claim that we cannot have that $\psi_\alpha < 0$ in \mathcal{N}_M^c . Indeed, a nonpositive local minimum of ψ_α is discarded by maximum principle. If there was a sequence of points $x_n \in \mathbb{R}^9$, such that

$$\psi_\alpha(x_n) \rightarrow \inf_{\mathbb{R}^9} \psi_\alpha < 0,$$

$|x_n| \rightarrow \infty$, and at the same time $\text{dist}(x_n, \Gamma_\alpha) > M$, for a large M , the usual compactness argument applied to the sequence $\psi_n(x) = \psi_\alpha(x + x_n)$ would give us a nontrivial bounded solution of

$$\Delta \psi - c(x)\psi = 0 \quad \text{in } \mathbb{R}^9, \quad c(0) > 1,$$

with a negative minimum at the origin, hence a contradiction. We conclude that $\psi_\alpha > 0$ in entire \mathbb{R}^9 and the proof of the theorem is concluded, except for the steps postponed. We shall devote the rest of this paper to their proofs.

5. The proof of Lemma 4.1

Here we prove Lemma 4.1, which reduces the system (4.7)–(4.8) to solving the nonlocal equation (4.16). Let us consider equation (4.8):

$$(5.1) \quad \Delta\psi - W_\alpha(x)\psi + (1 - \zeta_2)S(\mathbf{w}) + (1 - \zeta_1)N(\psi + \zeta_2\phi) + 2\nabla\zeta_1\nabla\phi + \phi\Delta\zeta_1 = 0 \quad \text{in } \mathbb{R}^9,$$

where

$$W_\alpha(x) := [(1 - \zeta_1)(-f'(u_1)) + 2\zeta_1],$$

and we assume that ϕ satisfies the symmetry (4.4) with $\|\phi\|_{2,p,3} \leq 1$. Let us observe that $W_\alpha(\mathcal{Q}x) = W_\alpha(x)$ for all x and hence that the function $-\psi(\mathcal{Q}x)$ solves (5.1) if $\psi(x)$ does.

Let us consider first the linear problem

$$(5.2) \quad \Delta\psi - W_\alpha(x)\psi + g(x) = 0 \quad \text{in } \mathbb{R}^9.$$

We observe that globally we have $2 - \tau < W_\alpha(x) < 2 + \tau$ for arbitrarily small $\tau > 0$.

We recall that for $1 < p \leq +\infty$, we defined

$$\|g\|_{p,\nu,*} := \sup_{x \in \mathbb{R}^9} (1 + r(\alpha x))^\nu \|g\|_{L^p(B(x,1))}, \quad r(x', x_9) = |x'|.$$

LEMMA 5.1. *Given $p > 9$, $\nu \geq 0$, there is a $C > 0$ such that for all sufficiently small α and any g with $\|g\|_{p,\nu,*} < +\infty$, there exists a unique ψ solution to problem (5.2) with $\|\psi\|_{\infty,\nu,*} < +\infty$. This solution satisfies*

$$(5.3) \quad \|D^2\psi\|_{p,\nu,*} + \|\psi\|_{\infty,\nu,*} \leq C\|g\|_{p,\nu,*}.$$

Proof. We claim that the *a priori* estimate

$$(5.4) \quad \|\psi\|_{\infty,\nu,*} \leq C\|g\|_{p,\nu,*}$$

holds for solutions ψ with $\|\psi\|_{\infty,\nu,*} < +\infty$ to problem (5.2) with $\|g\|_{p,\nu,*} < +\infty$ provided that α is small enough. This and local elliptic estimates in turn imply the validity of (5.3). To prove the claim, let us assume the opposite, namely the existence $\alpha_n \rightarrow 0$, and solutions ψ_n to equation (5.2) with $\|\psi_n\|_{\infty,\nu,*} = 1$, $\|g_n\|_{p,\nu,*} \rightarrow 0$. Let us consider a point x_n with

$$(1 + r(\alpha_n x_n))^\nu \psi_n(x_n) \geq \frac{1}{2}$$

and define

$$\begin{aligned} \tilde{\psi}_n(x) &= \left(1 + r(\alpha_n(x_n + x))\right)^\nu \psi_n(x_n + x), \\ \tilde{g}_n(x) &= \left(1 + r(\alpha_n(x_n + x))\right)^\nu g_n(x_n + x), \\ \tilde{W}_n(x) &= W_{\alpha_n}(x_n + x). \end{aligned}$$

Then, we check that the equation satisfied by $\tilde{\psi}_n$ has the form

$$\Delta \tilde{\psi}_n - \tilde{W}_n(x) \tilde{\psi}_n + o(1) \nabla \tilde{\psi}_n + o(1) \tilde{\psi}_n = \tilde{g}_n.$$

Additionally, we know that $\tilde{\psi}_n$ is uniformly bounded; hence elliptic estimates imply L^∞ -bounds for the gradient and the existence of a subsequence uniformly convergent over compact subsets of \mathbb{R}^9 to a bounded solution $\tilde{\psi} \neq 0$ to an equation of the form

$$\Delta \tilde{\psi} - W_*(x) \tilde{\psi} = 0 \quad \text{in } \mathbb{R}^9,$$

where $0 < a \leq W_*(x) \leq b$. But maximum principle makes this situation impossible, hence estimate (5.4) holds.

Now, for existence, let us consider g with $\|g\|_{p,\nu,*} < +\infty$ and a collection of approximations g_n to g with $\|g_n\|_{\infty,\nu,*} < +\infty$, $g_n \rightarrow g$ in $L^p_{\text{loc}}(\mathbb{R}^9)$ and $\|g_n\|_{p,\nu,*} \leq C\|g\|_{p,\nu,*}$. The problem

$$\Delta \psi_n - W_n(x) \psi_n = g_n \quad \text{in } \mathbb{R}^9$$

can be solved since this equation has a positive supersolution of the form $C(1 + r(\alpha x))^{-\nu}$, provided that α is sufficiently small, independently of n . Let us call ψ_n the solution thus found, which satisfies $\|\psi_n\|_{\infty,\nu,*} < +\infty$. The *a priori* estimate shows that

$$\|D^2 \psi_n\|_{p,\nu,*} + \|\psi_n\|_{\infty,\nu,*} \leq C\|g\|_{p,\nu,*}.$$

Passing to the limit in the topology of uniform convergence over compacts, we find a subsequence which converges to a solution ψ to problem (5.2), with $\|\psi\|_{\infty,\nu,*} < +\infty$. The proof is complete. \square

Next, we conclude the proof of Lemma 4.1. Let us call $\psi := \Theta(g)$ the solution of problem (5.2) predicted by Lemma 5.1. Let us write problem (5.1) as a fixed point problem in the space X of $W^{2,p}_{\text{loc}}$ -functions ψ with $\|\psi\|_{2,p,3+\mu,*} < +\infty$,

$$(5.5) \quad \psi = \Theta(g_1 + K(\psi)),$$

where

$$g_1 = (1 - \zeta_2)S(\mathbf{w}) + 2\nabla\zeta_1\nabla\phi + \phi\Delta\zeta_1, \quad K(\psi) = (1 - \zeta_1)N(\psi + \zeta_2\phi).$$

Let us consider a function ϕ defined in $\Gamma_\alpha \times \mathbb{R}$ such that $\|\phi\|_{2,p,\nu,\sigma} \leq 1$. Let us observe that derivatives of the function ζ_1 are supported inside the set of

points x with

$$(5.6) \quad x = y + (t + h(\alpha y)) \nu(\alpha y), \quad \frac{\delta}{\alpha}(1 + r_\alpha(y)) - 5 < |t + h(\alpha y)| < \frac{\delta}{\alpha}(1 + r_\alpha(y)) + 5.$$

Note that if x satisfies (5.6), then

$$a r_\alpha(y) \leq r(\alpha x) \leq b r_\alpha(y), \quad e^{-\sigma|t|} \leq e^{-\frac{\sigma\delta}{2\alpha}} e^{-\sigma r_\alpha(x)}$$

for some positive numbers a, b . Setting $\delta' = \frac{\sigma}{2}\delta$, we have that for any $\mu > 0$,

$$|2\nabla\zeta_1\nabla\phi + \phi\Delta\zeta_1| \leq C e^{-\frac{\delta'}{\alpha}} (1 + r(\alpha x))^{-3-\mu} \|\phi\|_{2,p,\nu,\sigma}.$$

We also have that $\|S(\mathbf{w})\|_{p,3,\sigma} \leq C\alpha^{3-\frac{8}{p}}$; hence

$$\|(1 - \zeta_2)S(\mathbf{w})\|_{p,3,\sigma} \leq C e^{-\frac{\delta'}{\alpha}} (1 + r(\alpha x))^{-3-\mu}$$

and therefore

$$\|g_1\|_{p,3+\mu} \leq C e^{-\frac{\delta'}{\alpha}}.$$

Let us consider the set

$$\Lambda = \{\psi \in X \mid \|\psi\|_{2,p,3+\mu,*} \leq A e^{-\frac{\delta'}{\alpha}}\}$$

for a large number $A > 0$. Since

$$|K(\psi_1) - K(\psi_2)| \leq C(1 - \zeta_1) \sup_{t \in (0,1)} |t\psi_1 + (1-t)\psi_2 + \zeta_2\phi| |\psi_1 - \psi_2|,$$

we find that

$$\|K(\psi_1) - K(\psi_2)\|_{\infty,3+\mu} \leq C e^{-\frac{\delta'}{\alpha}} \|\psi_1 - \psi_2\|_{\infty,3+\mu}$$

while $\|K(0)\|_{\infty,\nu,*} \leq C e^{-\frac{\delta'}{\alpha}}$. It follows that the right-hand side of equation (5.5) defines a contraction mapping of Λ , and hence a unique solution $\psi = \Psi(\phi) \in \Lambda$ exists, provided that the number A in the definition of Λ is taken sufficiently large and $\|\phi\|_{2,p,3,\sigma} \leq 1$. In addition, it is direct to check the Lipschitz dependence of Ψ as stated in (4.14) on $\|\phi\|_{2,p,3,\sigma} \leq 1$. Since, as we have mentioned, $-\psi(Qx)$ satisfies the same equation, the symmetry assertion follows from uniqueness. The proof is concluded. \square

6. The proofs of Proposition 4.1 and Lemma 4.2

To solve problem (4.17), we derive first a solvability theory for the following linear problem:

$$(6.1) \quad \partial_{tt}\phi + \Delta_{\Gamma_\alpha}\phi + f'(w)\phi = g(y, t) + c(y)w'(t) \quad \text{in } \Gamma_\alpha \times \mathbb{R},$$

$$\int_{\mathbb{R}} \phi(y, t) w'(t) dt = 0 \quad \text{for all } y \in \Gamma_\alpha, \quad c(y) = -\frac{\int_{\mathbb{R}} g(y, t) w' dt}{\int_{\mathbb{R}} w'^2 dt}.$$

We have the following result.

PROPOSITION 6.1. *Given $p > 9$ and $0 < \sigma < \sqrt{2}$, there exists a constant $C > 0$ such that for all sufficiently small $\alpha > 0$, the following holds: given g with $\|g\|_{p,3,\sigma} < +\infty$, problem (6.1) has a unique solution ϕ with $\|\phi\|_{\infty,3,\sigma} < +\infty$, which in addition satisfies*

$$(6.2) \quad \|\phi\|_{2,p,3,\sigma} \leq C \|g\|_{p,3,\sigma}.$$

We will carry out the proof of Proposition 4.1 assuming for the moment the validity of the above result.

6.1. *Proof of Proposition 4.1.* Let $\phi = T(g)$ be the linear operator defined as the solution of (6.1) in Proposition 6.1. Then problem (4.17) can be reformulated as the fixed point problem

$$(6.3) \quad \phi = T(-\tilde{S}(u_1) - \mathbb{N}(\phi)), \quad \|\phi\|_{2,p,3,\sigma} \leq K\alpha^{3-\frac{8}{p}}.$$

We claim that there is a positive constant C , possibly dependent of M in (3.9), such that for all small α and any ϕ_1, ϕ_2 , with

$$\|\phi_l\|_{2,p,3,\sigma} \leq K\alpha^{3-\frac{8}{p}},$$

we have

$$(6.4) \quad \|\mathbb{N}(\phi_1) - \mathbb{N}(\phi_2)\|_{p,5,\sigma} \leq C\alpha \|\phi_1 - \phi_2\|_{2,p,3,\sigma}.$$

To prove this, we decompose the operator \mathbb{N} as

$$(6.5) \quad \mathbb{N}(\phi) := \underbrace{\tilde{B}(\phi)}_{\mathbb{N}_1(\phi)} + \underbrace{[f'(u_1) - f'(w)]\phi}_{\mathbb{N}_2(\phi)} + \underbrace{\zeta_1(f'(u_1) + 2)\Psi(\phi) + \zeta_1 N(\Psi(\phi) + \phi)}_{\mathbb{N}_3(\phi)}.$$

Let us start with \mathbb{N}_1 . This is a second order linear operator with coefficients of order α which decay at least like $O(r_\alpha^{-1})$. We recall that $\tilde{B} = \zeta_4 B$, where in local coordinates, B is given in (3.30). It is direct to see that

$$(6.6) \quad \|\mathbb{N}_1(\phi)\|_{p,5,\sigma} \leq C\alpha \|\phi\|_{2,p,3,\sigma}.$$

For instance, a computation similar to that in (3.34) yields that for $p \geq 9$, we have

$$\|\alpha^2(a_{ij}\partial_{ij}h)\partial_t\phi\|_{p,5,\sigma} \leq C\alpha^{2-\frac{8}{p}}\|D_\Gamma^2 h\|_{3,p}\|D\phi\|_{\infty,3,\sigma} \leq C\alpha^{3-\frac{8}{p}}\|\phi\|_{2,p,3,\sigma}.$$

Now, let us assume that $\|\phi_1\|_{2,p,3,\sigma}, \|\phi_2\|_{2,p,3,\sigma} \leq K\alpha^{3-\frac{8}{p}}$. Using Lemma 4.1, we immediately obtain

$$(6.7) \quad \|\mathbb{N}_2(\phi_1) - \mathbb{N}_2(\phi_2)\|_{p,3,\sigma} \leq C e^{-\sigma\frac{\delta}{\alpha}}\|\phi_1 - \phi_2\|_{p,3,\sigma}$$

and

$$(6.8) \quad \|\mathbb{N}_3(\phi_1) - \mathbb{N}_3(\phi_2)\|_{p,6,\sigma} \leq C(\|\phi_1\|_{\infty,3,\sigma} + \|\phi_2\|_{\infty,3,\sigma} + e^{-\sigma\frac{\delta}{\alpha}})\|\phi_1 - \phi_2\|_{\infty,3,\sigma}.$$

From (6.6), (6.7) and (6.8), inequality (6.4) follows. The proof of the claim is concluded.

To conclude the existence part of Proposition 4.1 we use the contraction mapping principle to deal with problem (6.3). First, using formula (3.32) we have that $\|\tilde{S}(u_1)\|_{p,3,\sigma} \leq C\alpha^{3-\frac{8}{p}}$. Let $B_\alpha = \{\phi \mid \|\phi\|_{2,p,3,\sigma} \leq K\alpha^{3-\frac{8}{p}}\}$ where K is a constant to be chosen. Second, we observe that for small α , and all $\phi \in B_\alpha$ we have $\|\mathbb{N}(\phi)\|_{p,4,\sigma} \leq C\alpha^{5-\frac{8}{p}}$. Then, from (6.4) we see that if K is fixed large enough independently of α , then the right-hand side of equation (5.5) defines a contraction mapping of B_α into itself. The contraction mapping principle implies the existence of a unique ϕ as stated. Finally, since the function $-\phi(\mathcal{Q}y, -t)$ satisfies the same equation, the symmetry assertion follows from uniqueness. \square

6.2. *Lipschitz dependence on h : The proof of Lemma 4.2.* We claim first that the solution $\phi = \Phi(h)$ in Proposition 4.1 has a Lipschitz dependence on h satisfying (3.9) in the sense that

$$(6.9) \quad \|\Phi(h_1) - \Phi(h_2)\|_{2,p,3,\sigma} \leq K\alpha^{2-\frac{8}{p}} \|h_1 - h_2\|_*$$

This is a consequence of various straightforward considerations of the Lipschitz character in h of the operator in the right-hand side of equation (4.17) for the norm $\|\cdot\|_*$ defined in (3.9). Let us recall expression (3.29) for the operator B , and consider as an example, two terms that depend linearly on h :

$$A(h_1, \phi) := \alpha a_{ij}^0 \partial_j h_1 \partial_{it} \phi.$$

Then

$$|A(h_1, \phi)| \leq C\alpha |\partial_j h_1| |\partial_{it} \phi|.$$

Hence

$$\|A(h_1, \phi)\|_{p,\nu+2,\sigma} \leq C\alpha \|(1 + r_\alpha^2) \partial_j h_1\|_\infty \|\partial_{it} \phi\|_{p,\nu,\sigma} \leq C\alpha^4 \|h_1\|_* \|\phi\|_{2,p,\nu,\sigma}.$$

Similarly, for $A(\phi, h_1) = \alpha^2 \Delta_\Gamma h_1 \partial_t \phi$, we have

$$|A(\phi, h_1)| \leq C\alpha^2 |\Delta_\Gamma h_1(\alpha y)| (1 + r_\alpha)^{-\nu} e^{-\sigma|t|} \|\phi\|_{2,p,\nu,\sigma}.$$

Hence

$$\|\alpha^2 \Delta_\Gamma h_1 \partial_t \phi\|_{p,\nu+2,\sigma} \leq C\alpha^{5-\frac{8}{p}} \|h_1\|_* \|\phi\|_{2,p,\nu,\sigma}.$$

We should take into account that some terms involve nonlinear, however mild dependence, in h . We recall for instance that $a_{ij}^1 = a_{ij}^1(\alpha y, \alpha(t + h_0 + h_1))$. Examining the rest of the terms involved we find that the whole operator \mathbb{N} produces a dependence on h_1 which is Lipschitz with small constant, and gaining decay in r_α ,

$$(6.10) \quad \|\mathbb{N}(h_1, \phi) - \mathbb{N}(h_2, \phi)\|_{p,\nu+1,\sigma} \leq C\alpha^2 \|h_1 - h_2\|_* \|\phi\|_{2,p,\nu,\sigma}.$$

Now, in the error term $\mathcal{R} = -\tilde{S}(u_1)$, we have that

$$(6.11) \quad \|\mathcal{R}(h_1) - \mathcal{R}(h_2)\|_{p,3,\sigma} \leq C\alpha^{2-\frac{8}{p}} \|h_1 - h_2\|_*.$$

To see this, again we check term by term expansion (4.21). For instance we have

$$|\alpha^2 a_{ij}^0 \partial_i h_0 \partial_j h_1| \leq C \alpha^2 (1 + r_\alpha)^{-3} e^{-\sigma|t|} \|h_1\|_*$$

so that

$$\|\alpha^2 a_{ij}^0 \partial_i h_0 \partial_j h_1\|_{p,3,\sigma} \leq C \alpha^2 \|h_1\|_*,$$

and the remaining terms are checked similarly. We observe that the factor $\alpha^{2-\frac{8}{p}}$ in (6.9) is due to the term $\alpha^2 \Delta_\Gamma h_1 w'$ in the expression for $S(u_1)$. Combining estimates (6.10), (6.11) and the fixed point characterization (5.5), we obtain the desired Lipschitz dependence (6.9) of Φ .

In particular, if we set $\phi_1 = \Phi(h_1)$, $\phi_2 = \Phi(h_2)$, we get, after invoking estimates (6.10) and (6.4),

$$\begin{aligned} (6.12) \quad & \|\mathbf{N}(h_1, \phi_1) - \mathbf{N}(h_2, \phi_2)\|_{p,5,\sigma} \\ & \leq \|\mathbf{N}(h_1, \phi_1) - \mathbf{N}(h_1, \phi_2)\|_{p,5,\sigma} + \|\mathbf{N}(h_1, \phi_2) - \mathbf{N}(h_2, \phi_2)\|_{p,5,\sigma} \\ & \leq C\alpha \|\phi_1 - \phi_2\|_{2,p,3,\sigma} + C\alpha^2 \|h_1 - h_2\|_* \|\phi_2\|_{2,p,3,\sigma} \\ & \leq C(\alpha^{3-\frac{8}{p}} + \alpha^{5-\frac{8}{p}}) \|h_1 - h_2\|_*. \end{aligned}$$

Now we recall that $\mathcal{G} = \mathcal{G}_1 + \mathcal{G}_2$, with the latter operators defined in (4.24) and (4.26). We have

$$\mathcal{G}_2(h) - \mathcal{G}_2(h_2) = \alpha^{-2} \int_{\mathbb{R}} (N(\Phi(h_1)) - N(\Phi(h_2))) (\alpha^{-1}y, t) w' dt$$

so that using relation (4.27) we get

$$\|\mathcal{G}_2(h_1) - \mathcal{G}_2(h_2)\|_{p,5} \leq C \alpha^{1-\frac{16}{p}} \|h_1 - h_2\|_*.$$

The operator \mathcal{G}_1 in (4.24) is analyzed in similar way, taking into account that the estimates in (6.11) involve terms carrying one more power of α and $O(r^{-5})$ as decay in r . We again get

$$\|\mathcal{G}_1(h_1) - \mathcal{G}_1(h_2)\|_{p,5} \leq C \alpha^{1-\frac{16}{p}} \|h_1 - h_2\|_*.$$

This concludes the proof. □

6.3. *Proof of Proposition 6.1.* At the core of the proof of the stated *a priori* estimates is the fact that the one-variable solution w of (1.1) is *nondegenerate* in $L^\infty(\mathbb{R}^9)$ in the sense that the linearized operator

$$L(\phi) = \Delta_y \phi + \partial_{tt} \phi + f'(w(t))\phi, \quad (y, t) \in \mathbb{R}^9 = \mathbb{R}^8 \times \mathbb{R}$$

satisfies the following:

LEMMA 6.1. *Let ϕ be a bounded, smooth solution of the problem*

$$(6.13) \quad L(\phi) = 0 \quad \text{in } \mathbb{R}^8 \times \mathbb{R}.$$

Then $\phi(y, t) = Cw'(t)$ for some $C \in \mathbb{R}$.

Proof. We begin by reviewing some known facts about the one-dimensional operator $L_0(\psi) = \psi'' + f'(w)\psi$. Assuming that $\psi(t)$ and its derivative decay sufficiently fast as $|t| \rightarrow +\infty$ and defining $\psi(t) = w'(t)\rho(t)$, we get that

$$\int_{\mathbb{R}} [|\psi'|^2 - f'(w)\psi^2] dt = \int_{\mathbb{R}} L_0(\psi)\psi dt = \int_{\mathbb{R}} w'^2 |\rho'|^2 dt;$$

therefore this quadratic form is positive unless ψ is a constant multiple of w' . Using this and a standard compactness argument, we get that there is a constant $\gamma > 0$ such that whenever $\int_{\mathbb{R}} \psi w' = 0$ with $\psi \in H^1(\mathbb{R})$, we have that

$$(6.14) \quad \int_{\mathbb{R}} (|\psi'|^2 - f'(w)\psi^2) dt \geq \gamma \int_{\mathbb{R}} (|\psi'|^2 + |\psi|^2) dt.$$

Now, let ϕ be a bounded solution of equation (7.3). We claim that ϕ has exponential decay in t , uniform in y . Let us consider a small number $\sigma > 0$ so that for a certain $t_0 > 0$ and all $|t| > t_0$, we have that

$$f'(w) < -2\sigma^2.$$

Let us consider for $\varepsilon > 0$ the function

$$g_\varepsilon(t, y) = e^{-\sigma(|t|-t_0)} + \varepsilon \sum_{i=1}^2 \cosh(\sigma y_i).$$

Then for $|t| > t_0$, we get that

$$L(g_\varepsilon) < 0 \quad \text{if } |t| > t_0.$$

As a conclusion, using maximum principle, we get

$$|\phi| \leq \|\phi\|_\infty g_\varepsilon \quad \text{if } |t| > t_0,$$

and letting $\varepsilon \rightarrow 0$, we then get

$$|\phi(y, t)| \leq C \|\phi\|_\infty e^{-\sigma|t|} \quad \text{if } |t| > t_0.$$

Let us observe the following fact. The function

$$\tilde{\phi}(y, t) = \phi(y, t) - \left(\int_{\mathbb{R}} w'(\zeta) \phi(y, \zeta) d\zeta \right) \frac{w'(t)}{\int_{\mathbb{R}} w'^2}$$

also satisfies $L(\tilde{\phi}) = 0$ and, in addition,

$$(6.15) \quad \int_{\mathbb{R}} w'(t) \tilde{\phi}(y, t) dt = 0 \quad \text{for all } y \in \mathbb{R}^8.$$

In view of the above discussion, it turns out that the function

$$\varphi(y) := \int_{\mathbb{R}} \tilde{\phi}^2(y, t) dt$$

is well defined. In fact, so are its first and second derivatives by elliptic regularity of ϕ , and differentiation under the integral sign is thus justified. Now let us observe that

$$\Delta_y \varphi(y) = 2 \int_{\mathbb{R}} \Delta_y \tilde{\phi} \cdot \tilde{\phi} dt + 2 \int_{\mathbb{R}} |\nabla_y \tilde{\phi}|^2,$$

and hence

$$\begin{aligned} (6.16) \quad 0 &= \int_{\mathbb{R}} (L(\tilde{\phi}) \cdot \tilde{\phi}) \\ &= \frac{1}{2} \Delta_y \varphi - \int_{\mathbb{R}} |\nabla_y \tilde{\phi}|^2 dz - \int_{\mathbb{R}} (|\tilde{\phi}_t|^2 - f'(w)\tilde{\phi}^2) dt. \end{aligned}$$

Let us observe that because of relations (6.15) and (6.14), we have

$$\int_{\mathbb{R}} (|\tilde{\phi}_t|^2 - f'(w)\tilde{\phi}^2) dt \geq \gamma \varphi.$$

It then follows that

$$\frac{1}{2} \Delta_y \varphi - \gamma \varphi \geq 0.$$

Since φ is bounded, from maximum principle we find that φ must be identically equal to zero. But this means

$$(6.17) \quad \phi(y, t) = \left(\int_{\mathbb{R}} w'(\zeta) \phi(y, \zeta) d\zeta \right) \frac{w'(t)}{\int_{\mathbb{R}} w'^2}.$$

Then the bounded function

$$g(y) = \int_{\mathbb{R}} w'(\zeta) \phi(y, \zeta) d\zeta$$

satisfies the equation

$$(6.18) \quad \Delta_y g = 0 \quad \text{in } \mathbb{R}^8.$$

Liouville’s theorem implies that $g \equiv \text{constant}$ and relation (6.17) yields $\phi(y, t) = Cw'(t)$ for some C . This concludes the proof. \square

6.4. *A priori estimates.* We shall consider problem (6.1) in a slightly more general form, also in a domain finite in y -direction. For a large number $R > 0$ let us set

$$\Gamma_\alpha^R := \{y \in \Gamma_\alpha \mid r(\alpha y) < R\}$$

and consider the variation of problem (6.1) given by

$$\begin{aligned} (6.19) \quad \partial_{tt}\phi + \Delta_{\Gamma_\alpha} \phi + f'(w(t))\phi &= g(y, t) + c(y)w'(t) \quad \text{in } \Gamma_\alpha^R \times \mathbb{R}, \\ \phi &= 0 \quad \text{on } \partial\Gamma_\alpha^R \times \mathbb{R}, \\ \int_{-\infty}^{\infty} \phi(y, t) w'(t) dt &= 0 \quad \text{for all } y \in \Gamma_\alpha^R, \end{aligned}$$

where we allow $R = +\infty$ and

$$c(y) \int_{\mathbb{R}} w'^2 dt = - \int_{\mathbb{R}} g(y, t) w' dt.$$

We begin by proving *a priori* estimates.

LEMMA 6.2. *Let us assume that $0 < \sigma < \sqrt{2}$ and $\nu \geq 0$. Then there exists a constant $C > 0$ such that for all small α and all large R , and every solution ϕ to problem (6.19) with $\|\phi\|_{\infty, \nu, \sigma} < +\infty$ and right-hand side g satisfying $\|g\|_{p, \nu, \sigma} < +\infty$, we have*

$$(6.20) \quad \|D^2\phi\|_{p, \nu, \sigma} + \|D\phi\|_{\infty, \nu, \sigma} + \|\phi\|_{\infty, \nu, \sigma} \leq C\|g\|_{p, \nu, \sigma}.$$

Proof. For the purpose of establishing the *a priori* estimate (6.19), it clearly suffices to consider the case $c(y) \equiv 0$. By local elliptic estimates, it is enough to show that

$$(6.21) \quad \|\phi\|_{\infty, \nu, \sigma} \leq C\|g\|_{p, \nu, \sigma}.$$

Let us assume by contradiction that (6.21) does not hold. Then we have the existence of sequences $\alpha = \alpha_n \rightarrow 0$, $R = R_n \rightarrow \infty$, g_n with $\|g_n\|_{p, \nu, \sigma} \rightarrow 0$, ϕ_n with $\|\phi_n\|_{\infty, \nu, \sigma} = 1$ such that

$$(6.22) \quad \begin{aligned} \partial_{tt}\phi_n + \Delta_{\Gamma_\alpha} \phi_n + f'(w(t))\phi_n &= g_n && \text{in } \Gamma_\alpha^R \times \mathbb{R}, \\ \phi_n &= 0 && \text{on } \partial\Gamma_\alpha^R \times \mathbb{R}, \\ \int_{-\infty}^{\infty} \phi_n(y, t) w'(t) dt &= 0 && \text{for all } y \in \Gamma_\alpha^R. \end{aligned}$$

Then we can find points $(p_n, t_n) \in \Gamma_\alpha^R \times \mathbb{R}$ such that

$$(6.23) \quad e^{-\sigma|t_n|} (1 + r(\alpha_n p_n))^\nu |\phi_n(p_n, t_n)| \geq \frac{1}{2}.$$

Let us consider the local coordinates for Γ_{α_n} around p_n , defined by (3.24):

$$Y_{p_n, \alpha_n}(\mathbf{y}) = \alpha_n^{-1} Y_{\alpha_n p_n}(\alpha_n \mathbf{y}), \quad |\mathbf{y}| < \frac{\theta}{\alpha_n},$$

where $Y_p(\mathbf{y})$ is given by (3.13). We observe that, read in these coordinates, $\phi_n(\mathbf{y}, t)$ satisfies $|\phi_n(0, t_n)| \geq \gamma > 0$.

We consider different possibilities. Let us assume first that

$$r_\alpha(p_n) + |t_n| = O(1) \quad \text{as } n \rightarrow \infty.$$

We recall that the Laplace-Beltrami operator of Γ_{α_n} written in local coordinates has the form

$$\Delta_{\Gamma_{\alpha_n}} = a_{ij}^0(\alpha_n \mathbf{y}) \partial_{ij} + \alpha_n b_j^0(\alpha_n \mathbf{y}) \partial_j,$$

where, uniformly on $|\mathbf{y}| < \theta \alpha^{-1}$, we have

$$a_{ij}^0(\alpha_n \mathbf{y}) = \delta_{ij} + o(1), \quad b_i^0 = O(1) \quad \text{as } \alpha \rightarrow 0.$$

Then

$$a_{ij}^0(\alpha_n \mathbf{y}) \partial_{ij} \phi_n + \alpha_n b_j^0(\alpha_n \mathbf{y}) \partial_j \phi_n + \partial_{tt} \phi_n + f'(w(t)) \phi_n = g_n(\mathbf{y}, t), \quad |\mathbf{y}| < \frac{\theta}{\alpha}.$$

Since ϕ_n is bounded, and $g_n \rightarrow 0$ in $L^p_{loc}(\mathbb{R}^9)$, we obtain local uniform $W^{2,p}$ -bound. Hence we may assume, passing to a subsequence, that ϕ_n converges uniformly in compact subsets of \mathbb{R}^9 to a function $\phi(\mathbf{y}, t)$ that satisfies

$$\Delta_{\mathbb{R}^8} \phi + \partial_{tt} \phi + f'(w(t)) \phi = 0.$$

Thus ϕ is nonzero and bounded. But Lemma 6.1 implies that, necessarily, $\tilde{\phi}(\mathbf{y}, t) = Cw'(t)$. On the other hand, we have

$$0 = \int_{\mathbb{R}} \phi_n(\mathbf{y}, t) w'(t) dt \longrightarrow \int_{\mathbb{R}} \phi(\mathbf{y}, t) w'(t) dt \quad \text{as } n \rightarrow \infty.$$

Hence, necessarily $\phi \equiv 0$. But we have $|\phi_n(0, t_n)| \geq \gamma > 0$, and since t_n and $r(\alpha_n y_n)$ were bounded, the local uniform convergence implies $\phi \neq 0$. We have reached a contradiction.

If $r_\alpha(p_n) = O(1)$ but t_n is unbounded, say, $t_n \rightarrow +\infty$, the situation is similar. The difference is that we now define

$$\tilde{\phi}_n(\mathbf{y}, t) = e^{\sigma(t_n+t)} \phi_n(\mathbf{y}, t_n + t), \quad \tilde{g}_n(\mathbf{y}, t) = e^{\sigma(t_n+t)} g_n(\mathbf{y}, t_n + t).$$

Then $\tilde{\phi}_n$ is uniformly bounded, and $\tilde{g}_n \rightarrow 0$ in $L^p_{loc}(\mathbb{R}^9)$. Now $\tilde{\phi}_n$ satisfies

$$a_{ij}^0(\alpha_n \mathbf{y}) \partial_{ij} \tilde{\phi}_n + \partial_{tt} \tilde{\phi}_n + \alpha_n b_j(\alpha_n \mathbf{y}) \partial_j \tilde{\phi}_n - 2\sigma \partial_t \tilde{\phi}_n + (f'(w(t+t_n)) + \sigma^2) \tilde{\phi}_n = \tilde{g}_n.$$

Passing to the limit we obtain

$$(6.24) \quad \Delta_{\mathbb{R}^8} \tilde{\phi} + \partial_{tt} \tilde{\phi} - 2\sigma \partial_t \tilde{\phi} - (2 - \sigma^2) \tilde{\phi} = 0 \quad \text{in } \mathbb{R}^9,$$

where $\tilde{\phi} \neq 0$. But since by assumption $2 - \sigma^2 > 0$, the maximum principle implies that $\tilde{\phi} \equiv 0$. We obtain a contradiction.

Let us consider the case $r(\alpha_n p_n) \rightarrow +\infty$ but $r(\alpha_n p_n) \ll R_n$. Assume first that the sequence t_n is bounded and set

$$\tilde{\phi}_n(\mathbf{y}, t) := (1 + r(\alpha_n \mathbf{y}))^\nu \phi_n(\mathbf{y}, t).$$

Direct differentiation yields

$$\begin{aligned} \partial_j (r_\alpha^{-\nu} \tilde{\phi}_n) &= r_\alpha^{-\nu} [\partial_j \tilde{\phi} + O(\alpha r_\alpha^{-1}) \tilde{\phi}], \\ \partial_{ij} (r_\alpha^{-\nu} \tilde{\phi}_n) &= r_\alpha^{-\nu} [\partial_{ij} \tilde{\phi} + O(\alpha r_\alpha^{-1}) \partial_i \tilde{\phi} + O(\alpha^2 r_\alpha^{-2}) \tilde{\phi}], \end{aligned}$$

and the equation satisfied by $\tilde{\phi}_n$ therefore has the form

$$\Delta_{\mathbf{y}} \tilde{\phi}_n + \partial_{tt} \tilde{\phi}_n + o(1) \partial_{ij} \tilde{\phi}_n + o(1) \partial_j \tilde{\phi}_n + o(1) \tilde{\phi}_n + f'(w(t)) \tilde{\phi}_n = \tilde{g}_n,$$

where $\tilde{\phi}_n$ is bounded, $\tilde{g}_n \rightarrow 0$ in $L^p_{loc}(\mathbb{R}^9)$. From elliptic estimates, we also get uniform bounds for $\|\partial_j \tilde{\phi}_n\|_\infty$ and $\|\partial_{ij} \tilde{\phi}_n\|_{p,0,0}$. In the limit, we obtain a $\tilde{\phi} \neq 0$ bounded solution of

$$(6.25) \quad \Delta_{\mathbf{y}} \tilde{\phi} + \partial_{tt} \tilde{\phi} + f'(w(t)) \tilde{\phi} = 0, \quad \int_{\mathbb{R}} \tilde{\phi}(\mathbf{y}, t) w'(t) dt = 0,$$

a situation which is discarded in the same way as before if $\tilde{\phi}$ is defined in \mathbb{R}^9 .

Now, if t_n is still bounded but $r(\alpha_n y_n) - R_n = O(1)$, then passing to the limit we find the limit equation (6.25) satisfied in a half-space, which after a rotation in the \mathbf{y} -plane can be assumed to be

$$H = \{(\mathbf{y}, t) \in \mathbb{R}^8 \times \mathbb{R} \mid y_8 < 0\},$$

$$\text{with } \phi(\tilde{\mathbf{y}}, 0, t) = 0 \text{ for all } \tilde{\mathbf{y}} = (\mathbf{y}, \dots, y_7) \in \mathbb{R}^7, t \in \mathbb{R}.$$

By Schwarz's reflection, the odd extension of $\tilde{\phi}$, which is defined for $y_8 > 0$, by $\tilde{\phi}(\tilde{\mathbf{y}}, y_8, t) = -\tilde{\phi}(\tilde{\mathbf{y}}, -y_8, t)$, satisfies the same equation, and thus the problem reduces to one of the previous cases again yielding a contradiction.

Let us now assume that $r(\alpha_n p_n) \rightarrow +\infty$ and $|t_n| \rightarrow +\infty$. If $t_n \rightarrow +\infty$, we define

$$\tilde{\phi}_n(\mathbf{y}, t) = (1 + r(\alpha_n \mathbf{y}))^\nu e^{t_n+t} \phi_n(\mathbf{y}, t_n + t).$$

In this case we end up in the limit with a $\tilde{\phi} \neq 0$ bounded and satisfying the equation

$$\Delta_{\mathbf{y}} \tilde{\phi} + \partial_{tt} \tilde{\phi} - 2\sigma \partial_t \tilde{\phi} - (2 - \sigma^2) \tilde{\phi} = 0,$$

either in the entire space or in a half-space under zero boundary condition. This implies again $\tilde{\phi} = 0$, and a contradiction has been reached. All cases have been discarded, and the proof is concluded. \square

6.5. *Existence: Conclusion of the proof of Proposition 6.1.* Let us now prove existence. We assume first that g has compact support in $\Gamma_\alpha \times \mathbb{R}$:

$$(6.26) \quad \begin{aligned} \partial_{tt} \phi + \Delta_{\Gamma_\alpha} \phi + f'(w(t)) \phi &= g(y, t) + c(y) w'(t) \quad \text{in } \Gamma_\alpha^R \times \mathbb{R}, \\ \phi &= 0 \quad \text{on } \partial \Gamma_\alpha^R \times \mathbb{R}, \\ \int_{-\infty}^{\infty} \phi(y, t) w'(t) dt &= 0 \quad \text{for all } y \in \Gamma_\alpha^R, \end{aligned}$$

where we allow $R = +\infty$ and

$$c(y) \int_{\mathbb{R}} w'^2 dt = - \int_{\mathbb{R}} g(y, t) w' dt.$$

Problem (6.26) has a weak formulation which is as follows. Let

$$H = \{ \phi \in H^1_0(\Gamma_\alpha^R \times \mathbb{R}) \mid \int_{\mathbb{R}} \phi(y, t) w'(t) dt = 0 \text{ for all } y \in \Gamma_\alpha^R \}.$$

H is a closed subspace of $H_0^1(\Gamma_\alpha^R \times \mathbb{R})$, hence a Hilbert space when endowed with its natural norm:

$$\|\phi\|_H^2 = \int_{\Gamma_\alpha^R} \int_{\mathbb{R}} (|\partial_t \phi|^2 + |\nabla_{\Gamma_\alpha} \phi|^2 - f'(w(t)) \phi^2) dV_{\Gamma_\alpha} dt.$$

Function ϕ is then a weak solution of problem (6.26) if $\phi \in H$ and satisfies

$$\begin{aligned} a(\phi, \psi) &:= \int_{\Gamma_\alpha^R \times \mathbb{R}} (\partial_t \phi \partial_t \psi + \nabla_{\Gamma_\alpha} \phi \cdot \nabla_{\Gamma_\alpha} \psi - f'(w(t)) \phi \psi) dV_{\Gamma_\alpha} dt \\ &= - \int_{\Gamma_\alpha^R \times \mathbb{R}} g \psi dV_{\Gamma_\alpha} dt \quad \text{for all } \psi \in H. \end{aligned}$$

Indeed, decomposing a general smooth compactly supported test function in the form

$$\psi(y, t) = a(y)w'(t) + \tilde{\psi}(y, t), \quad \tilde{\psi} \in H,$$

we obtain, after an integration by parts and using the orthogonality constraint in ϕ , that equation (6.26) is satisfied in the usual weak sense. Moreover, standard elliptic estimates yield that a weak solution of problem (6.26) is also classical provided that g is regular enough.

Let us observe that because of the orthogonality condition defining H , we have

$$\gamma \int_{\Gamma_\alpha^R \times \mathbb{R}} \psi^2 dV_{\Gamma_\alpha} dt \leq a(\psi, \psi) \quad \text{for all } \psi \in H.$$

Hence the bilinear form a is coercive in H , and existence of a unique weak solution follows from Riesz’s theorem. If g is regular and compactly supported, ϕ is also regular. Local elliptic regularity implies in particular that ϕ is bounded. Indeed for some $t_0 > 0$, the equation satisfied by ϕ is

$$(6.27) \quad \Delta \phi + f'(w(t)) \phi = c(y)w'(t), \quad |t| > t_0, \quad y \in \Gamma_\alpha^R,$$

and $c(y)$ is bounded. Then, enlarging t_0 if necessary, we see that for $\sigma < \sqrt{2}$, the function $v(y, t) := Ce^{-\sigma|t|} + \varepsilon e^{\sigma|t|}$ is a positive supersolution of equation (6.27) for a large enough choice of C and arbitrary $\varepsilon > 0$. Hence $|\phi| \leq Ce^{-\sigma|t|}$, from maximum principle. Since Γ_α^R is bounded, we conclude that $\|\phi\|_{p,\nu,\sigma} < +\infty$. From Lemma 6.2, we obtain that if R is large enough, then

$$(6.28) \quad \|D^2 \phi\|_{p,\nu,\sigma} + \|D\phi\|_{\infty,\nu,\sigma} + \|\phi\|_{\infty,\nu,\sigma} \leq C\|g\|_{p,\nu,\sigma}.$$

Now let us consider problem (6.26) for $R = +\infty$, allowed above, and for $\|g\|_{p,\nu,\sigma} < +\infty$. Then solving the equation for finite R and suitable compactly supported g_R , we generate a sequence of approximations ϕ_R which is uniformly controlled in R by the above estimate. If g_R is chosen so that $g_R \rightarrow g$ in $L_{\text{loc}}^p(\Gamma_\alpha \times \mathbb{R})$ and $\|g_R\|_{p,\nu,\sigma} \leq C\|g\|_{p,\nu,\sigma}$, we obtain that ϕ_R is locally uniformly bounded, and by extracting a subsequence, it converges uniformly locally over compacts to a solution ϕ to the full problem which respects the estimate (6.2). This concludes the proof of existence, and hence that of the proposition. \square

6.6. *An equation on Γ_α .* With arguments similar to those above, we analyze the following equation that will be relevant in the study of the Jacobi operator in Section 7:

$$(6.29) \quad \Delta_{\Gamma_\alpha} h - h = g \quad \text{in } \Gamma_\alpha.$$

We prove:

COROLLARY 6.1. *Let $p > 8$, $\nu \geq 0$. Then there exists $C > 0$ such that for all sufficiently small α and any $g \in L^p_{\text{loc}}(\Gamma_\alpha)$ with*

$$\sup_{y \in \Gamma_\alpha} (1 + r_\alpha^\nu(y)) \|g\|_{L^p(B(y,1) \cap \Gamma_\alpha)} < +\infty,$$

there exists a unique solution h of problem (6.29) with $\|(1 + r_\alpha^\nu)h\|_\infty < +\infty$. This solution satisfies

$$\|D_{\Gamma_\alpha}^2 h\|_{p,\nu} + \|D_{\Gamma_\alpha} h\|_{\infty,\nu} + \|h\|_{\infty,\nu} \leq \|g\|_{p,\nu}.$$

Proof. With the notation used above, we consider the approximate problem

$$(6.30) \quad \Delta_{\Gamma_\alpha} h - h = g \quad \text{in } \Gamma_\alpha^R, \quad h = 0 \quad \text{on } \partial\Gamma_\alpha^R,$$

where we allow $R = +\infty$. Exactly the same arguments used in the proof of Lemma 6.2 lead to the existence of a constant $C > 0$ such that for all small α and all large R , such that for any solution h with $\|(1 + r_\alpha^\nu)h\|_\infty < +\infty$, we have the *a priori* estimate

$$\begin{aligned} \sup_{y \in \Gamma_\alpha^R} (1 + r_\alpha^\nu(y)) \|D_{\Gamma_\alpha}^2 h\|_{L^p(B(y,1) \cap \Gamma_\alpha^R)} + \|(1 + r_\alpha^\nu)D_{\Gamma_\alpha} h\|_\infty + \|(1 + r_\alpha^\nu)h\|_\infty \\ \leq C \sup_{y \in \Gamma_\alpha^R} (1 + r_\alpha^\nu(y)) \|g\|_{L^p(B(y,1) \cap \Gamma_\alpha^R)}. \end{aligned}$$

This estimate and the Fredholm alternative yields the existence of a unique solution h_R of (6.30). Letting $R \rightarrow +\infty$ possibly passing to a subsequence, we obtain the existence of a solution as predicted. \square

7. Solvability theory for the Jacobi operator: Proof of Proposition 4.2

In this section we consider the linear problem

$$(7.1) \quad \mathcal{J}_\Gamma[h] = \Delta_\Gamma h + |A_\Gamma(y)|^2 h = \mathbf{g}(y) \quad \text{in } \Gamma$$

and derive estimates and existence results that lead to the proof of Proposition 4.2. For this, the main tool we use is the method of barriers. This is suitable for the operator \mathcal{J}_Γ since it has a positive, bounded element in its kernel. In fact $Z = \frac{1}{\sqrt{1+|\nabla F|^2}}$ satisfies $\mathcal{J}_\Gamma[Z] = 0$.

7.1. *The approximate Jacobi operator.* The surfaces Γ and Γ_0 are uniformly close for r large. Let $p \in \Gamma$ with $r(p) \gg 1$ and let $\nu(p)$ be the unit normal to Γ at p . Let $\pi(p) \in \Gamma_0$ be a point such that for some $t_p \in \mathbb{R}$, we have

$$(7.2) \quad \pi(p) = p + t_p \nu(p).$$

As we will see below, the point $\pi(p)$ exists and is unique when $r(p) \gg 1$, and the map $p \mapsto \pi(p)$ is smooth.

Computations on Γ_0 can be made in very explicit terms since F_0 is explicit. Hence it is important to relate them with analogous computations carried out on Γ , at least for r large. This leads us to considering the *approximate Jacobi operator* \mathcal{J}_{Γ_0} , corresponding to first variation of mean curvature (or second variation of area) at Γ_0 , measured along normal perturbations. This corresponds to the operator acting on functions $h : \Gamma_0 \rightarrow \mathbb{R}$ given by

$$(7.3) \quad \mathcal{J}_{\Gamma_0}[h](y) := H'(F_0)[\phi](x'), \quad \phi(x') = \sqrt{1 + |\nabla F_0(x')|^2} h(y), \quad y = (x', F_0(x')).$$

The expression for \mathcal{J}_{Γ_0} is similar to that in (4.28) for \mathcal{J}_{Γ} , but it involves a correction that gives account of the fact that Γ_0 is not a minimal surface, while very close to being so. In fact we have

$$(7.4) \quad \mathcal{J}_{\Gamma_0}[h] := \Delta_{\Gamma_0} h + |A_{\Gamma_0}(y)|^2 h + O(r^{-4}) D_{\Gamma_0}^2 h + O(r^{-5}) D_{\Gamma_0} h + O(r^{-6}) h.$$

This expression follows from a standard calculation which we carry out in coordinates adapted to the graph in the appendix.

For large r , \mathcal{J}_{Γ} is “close to” the *approximate Jacobi operator* \mathcal{J}_{Γ_0} in the sense of the following result, whose proof we carry out in Section 8.3.

LEMMA 7.1. *Assume that h and h_0 are smooth functions defined respectively on Γ and Γ_0 for r large, and related through the formula*

$$h_0(\pi(y)) = h(y), \quad y \in \Gamma, \quad r(y) > r_0.$$

There exists a $\sigma > 0$ such that

$$(7.5) \quad \mathcal{J}_{\Gamma}[h](y) = [\mathcal{J}_{\Gamma_0}[h_0] + O(r^{-2-\sigma}) D_{\Gamma_0}^2 h_0 + O(r^{-3-\sigma}) D_{\Gamma_0} h_0 + O(r^{-4-\sigma}) h_0](\pi(y)).$$

7.2. *Supersolutions for the approximate Jacobi operator.* We look for positive supersolutions of \mathcal{J}_{Γ_0} far away from the origin, or in other words for positive functions h which satisfy a differential inequality of the form

$$(7.6) \quad -\mathcal{J}_{\Gamma_0}[h] \geq \mathbf{g}(y) \quad \text{in } \Gamma, \quad r(y) > r_0,$$

for a class of right-hand sides that are decaying in $r = r(y)$ and additionally satisfy either

$$(7.7) \quad \mathbf{g}(y) = \frac{1}{r^{4+\mu}},$$

or

$$(7.8) \quad \mathbf{g}(y) = \frac{g(\theta)^\tau}{r^3},$$

where (r, θ) are the polar coordinates in \mathbb{R}^8 introduced in Section 2 and function g satisfies Lemma 2.1, and $\mu \in (0, 1)$, $\tau \in (\frac{1}{3}, \frac{2}{3})$.

We want to establish the following key result.

LEMMA 7.2. *For a function \mathbf{g} as in (7.7) with $0 < \mu < 1$, there exists a positive supersolution h of (7.2) such that*

$$cr^{-2-\mu} \leq h(y) \leq Cr^{-2-\mu}, \quad r > r_0.$$

Proof. We recall that $\mathcal{J}_{\Gamma_0}[h] = H'(F_0)[\sqrt{1 + |\nabla F_0|^2}h]$ and that in polar coordinates we can write (see (2.7))

$$(7.9) \quad H'(F_0)[\phi] := \tilde{L} := \tilde{L}_0 + \tilde{L}_1,$$

with

$$(7.10) \quad \tilde{L}_0(\phi) = \frac{1}{r^7 \sin^3(2\theta)} \left\{ (9g^2 \tilde{w}r^3 \phi_\theta)_\theta + (r^5 g'^2 \tilde{w} \phi_r)_r - 3(gg' \tilde{w}r^4 \phi_r)_\theta - 3(gg' \tilde{w}r^4 \phi_\theta)_r \right\},$$

and

$$(7.11) \quad \tilde{L}_1(\phi) = \frac{1}{r^7 \sin^3(2\theta)} \left\{ (r^{-1} \tilde{w} \phi_\theta)_\theta + (r \tilde{w} \phi_r)_r \right\},$$

$$(7.12) \quad \tilde{w}(r, \theta) := \frac{\sin^3 2\theta}{(r^{-4} + 9g^2 + g'^2)^{\frac{3}{2}}}.$$

We can expand

$$\tilde{w}(\theta, r) = \tilde{w}_0(\theta) + r^{-4} w_1(r, \theta),$$

where

$$\tilde{w}_0(\theta) := \frac{\sin^3(2\theta)}{(9g^2 + g'^2)^{\frac{3}{2}}}, \quad w_1(r, \theta) = -\frac{3}{2} \frac{\sin^3(2\theta)}{(9g^2 + g'^2)^{\frac{5}{2}}} + O(r^{-4} \sin^3(2\theta)).$$

We set

$$L_0(\phi) = \frac{1}{r^7 \sin^3(2\theta)} \left\{ (9g^2 \tilde{w}_0 r^3 \phi_\theta)_\theta + (r^5 g'^2 \tilde{w}_0 \phi_r)_r - 3(gg' \tilde{w}_0 r^4 \phi_r)_\theta - 3(gg' \tilde{w}_0 r^4 \phi_\theta)_r \right\}.$$

Let us compute this last operator for a function of the form

$$\phi(r, \theta) = r^\beta q(\theta).$$

We obtain

$$r^7 \sin^3(2\theta) L_0(r^\beta q(\theta)) = r^{3+\beta} \left[9(g^2 \tilde{w}_0 q')' - 3\beta(gg'q \tilde{w}_0)' + \tilde{w}_0(\beta + 4)(\beta g'^2 q - 3gg'q') \right].$$

It is clear, by direct substitution, that $L_0(F_0^{\frac{\beta}{3}}) = L_0(r^\beta g^{\frac{\beta}{3}}) = 0$. Hence $q = g^{\frac{\beta}{3}}$ annihilates the operator on the right-hand side. As a consequence, the operator takes a divergence form with $h \equiv g^{-\frac{\beta}{3}}q$, namely

$$r^7 \sin^3(2\theta) L_0(r^\beta q(\theta)) = 9r^{3+\beta} g^{\frac{\beta+4}{3}} \left[\tilde{w}_0 g^{\frac{2}{3}} (g^{-\frac{\beta}{3}}q)' \right]'$$

We want to find a positive function q such that the following equation holds:

$$-L_0(r^\beta q(\theta)) = 9 \frac{g(\theta)^\tau}{r^{4-\beta}}, \quad \theta \in \left(\frac{\pi}{4}, \frac{\pi}{2} \right),$$

or equivalently,

$$-\left[\tilde{w}_0 g^{\frac{2}{3}} (g^{-\frac{\beta}{3}}q)' \right]' = g^{\tau - \frac{\beta+4}{3}} \sin^3(2\theta).$$

Then we can solve explicitly for q by direct integration, getting

$$q(\theta) = g^{\frac{\beta}{3}}(\theta) \left(\int_{\frac{\pi}{4}}^\theta \frac{ds}{\tilde{w}_0(s)g^{\frac{2}{3}}(s)} \int_s^{\frac{\pi}{2}} g^{\tau - \frac{\beta+4}{3}}(s') \sin^3(2s') ds' \right),$$

or equivalently

$$(7.13) \quad q(\theta) = g^{\frac{\beta}{3}}(\theta) \int_{\frac{\pi}{4}}^\theta g^{-\frac{2}{3}}(9g^2 + g'^2)^{\frac{3}{2}} \frac{ds}{\sin^3(2s)} \times \int_s^{\frac{\pi}{2}} g^{\tau - \frac{\beta+4}{3}}(\tau') \sin^3(2\tau') d\tau', \quad \theta \in \left(\frac{\pi}{4}, \frac{\pi}{2} \right),$$

provided of course that the choices of τ and β make this formula well defined.

We will analyze this formula in the two cases of interest.

Let us consider the case $\tau = 0$, $\beta = -\mu$, $0 < \mu < 1$, corresponding to the right-hand side of (7.7). Then

$$q(\theta) = g^{-\frac{\mu}{3}}(\theta) \int_{\frac{\pi}{4}}^\theta g^{-\frac{2}{3}}(9g^2 + g'^2)^{\frac{3}{2}} \frac{ds}{\sin^3(2s)} \times \int_s^{\frac{\pi}{2}} g^{\frac{\mu}{3} - \frac{4}{3}}(s') \sin^3(2s') ds', \quad \theta \in \left(\frac{\pi}{4}, \frac{\pi}{2} \right].$$

Since $g'(\frac{\pi}{4}) > 0$, q is well defined, positive and smooth in $(\frac{\pi}{4}, \frac{\pi}{2}]$. More than this: for instance expanding $g(\theta) = g_1x + g_3x^3 + \dots$ for $x = \theta - \frac{\pi}{4}$, and similarly with the other functions involved in the formula, we realize that q in reality extends smoothly up to $\theta = \frac{\pi}{4}$ in the form

$$q(\theta) = q_0 + q_2x^2 + q_4x^4 + \dots,$$

and we have $q_0 = q(\frac{\pi}{4}) > 0$, $q'(\frac{\pi}{4}) = 0$. Hence if we extend q by even reflection around $\frac{\pi}{4}$: $q(\theta) = q(\frac{\pi}{2} - \theta)$ if $\theta \in (0, \frac{\pi}{4}]$, then the symmetric, positive function $\phi_0 := q(\theta)r^{-\mu}$ satisfies $-L_0(\phi_0) = 9r^{-4-\mu}$ in \mathbb{R}^8 . Since q is smooth, we also find that the remaining terms in the expansion of $H'(F_0)[\phi_0]$ contribute quantities of size $O(r^{-8-\mu})$. Thus

$$-H'(F_0)[\phi_0] \geq \frac{1}{r^{4+\mu}},$$

or equivalently

$$-\mathcal{J}_{\Gamma_0}[h_0] \geq \frac{1}{r^{4+\mu}} \quad \text{in } \Gamma_0, \quad r > r_0, \quad \phi_0 =: \sqrt{1 + |\nabla F_0|^2}h_0,$$

which is what we were looking for since $h_0 = O(r^{-2-\mu})$. □

In the case of g given by (7.8), we consider the problem in the sector

$$(7.14) \quad \Gamma_{0+} = \left\{ y \in \Gamma \mid \theta \in \left(\frac{\pi}{4}, \frac{\pi}{2} \right) \right\},$$

$$(7.15) \quad -\mathcal{J}_{\Gamma_0}[h] \geq \frac{g(\theta)^\tau}{r^3} \quad \text{in } \Gamma_{0+}, \quad r(y) > r_0.$$

We prove:

LEMMA 7.3. *If $\frac{1}{3} < \tau < \frac{2}{3}$, then there exists a supersolution h of (7.15), smooth and positive in Γ_{0+} with $h = 0$ on $\partial\Gamma_{0+}$ and*

$$h(y) \leq Cr^{-1}, \quad y \in \Gamma_{0+}, \quad r > r_0.$$

Proof. We consider now the case $\beta = 1$, $\frac{1}{3} < \tau < \frac{2}{3}$, in formula (7.13), corresponding to the case (7.8). Now we get

$$(7.16) \quad q(\theta) = g^{\frac{1}{3}}(\theta), \int_{\frac{\pi}{4}}^{\theta} g^{-\frac{2}{3}}(9g^2 + g'^2)^{\frac{3}{2}} \frac{ds}{\sin^3(2s)} \\ \times \int_s^{\frac{\pi}{2}} g^{\tau-\frac{5}{3}}(s') \sin^3(2s') ds', \quad \theta \in \left(\frac{\pi}{4}, \frac{\pi}{2} \right).$$

Here q is smooth up to $\theta = \frac{\pi}{2}$ with $q'(\frac{\pi}{2}) = 0$ and it extends continuously to $\theta = \frac{\pi}{4}$ with $q(\frac{\pi}{4}) = 0$. Again setting $x = \theta - \frac{\pi}{4}$, we see that now h becomes expanded near $\frac{\pi}{4}$ as

$$q(\theta) = x^\sigma (q_0 + q_2x^2 + q_4x^4 + \dots), \quad q_0 > 0.$$

Here we have used that fact that $\frac{1}{3} < \tau < \frac{2}{3}$. In particular,

$$(7.17) \quad q''(\theta) = -\tau(1-\tau)q_0x^{\tau-2} + O(x^\tau) = -cg(\theta)^{\tau-2} + O(g(\theta)^\tau), \quad c > 0.$$

By direct substitution, we see that for large r ,

$$-\tilde{L}_0(rq(\theta)) = 9\frac{g(\theta)^\tau}{r^3} + O(g(\theta)^\tau r^{-7}),$$

while for $\theta - \frac{\pi}{4} \ll 1$, we have, using (7.11) and (7.17),

$$-\tilde{L}_1(rq(\theta)) = cg^{\tau-2}r^{-7} + O(g^{\tau-2}r^{-11}) + O(g^{\tau}r^{-7}), \quad c > 0,$$

and in general $-\tilde{L}_1(rq(\theta)) = O(g^{\tau-2}r^{-7})$. Combining the above estimates, we see that for all sufficiently large r ,

$$-\tilde{L}(rq(\theta)) > \frac{g(\theta)^\tau}{r^3},$$

and the desired conclusion follows with $h = \frac{rq(\theta)}{\sqrt{1+|\nabla F_0|^2}} = O(r^{-1})$. □

Remark 7.1. The result of Lemma 7.3 is of course true if $\tau \geq \frac{2}{3}$. The supersolution found will then be near $\theta = \frac{\pi}{4}$ of the order $O(g(\theta)^\tau r^{-1})$ for any $\tau < \frac{2}{3}$. On the other hand, if we choose directly $\tau \geq \frac{2}{3}$ in formula (7.16), this boundary behavior gets refined to $O(g(\theta)^{\frac{2}{3}} \log g(\theta) r^{-1})$ if $\tau = \frac{2}{3}$ and to $O(g(\theta)^{\frac{2}{3}} r^{-1})$ if $\tau > \frac{2}{3}$. In all cases these supersolutions are not smooth up $\theta = \frac{\pi}{4}$.

7.3. *Proof of Proposition 4.2(b).* This result is just a special case of the following:

PROPOSITION 7.1. *Let $4 < \nu < 5$. There exists a positive constant $C > 0$ such that if g satisfies $\|g\|_{p,\nu} < +\infty$, then there is a unique solution of the equation*

$$(7.18) \quad \mathcal{J}_\Gamma[h] = \mathbf{g} \quad \text{in } \Gamma$$

such that $\|h\|_{\infty,\nu-2} < +\infty$. This solution satisfies

$$\|D_\Gamma^2 h\|_{p,\nu-\frac{8}{p}} + \|D_\Gamma h\|_{\infty,\nu-1} + \|h\|_{\infty,\nu-2} \leq C \|g\|_{p,\nu}.$$

For the proof, we first show the existence of the supersolution in Lemma 7.2 for Γ_0 replaced with Γ .

LEMMA 7.4. *For $0 < \mu < 1$, there exists a positive supersolution h of*

$$(7.19) \quad -\mathcal{J}_\Gamma[h] \geq \frac{1}{r^{4+\mu}} \quad \text{in } \Gamma, \quad r(y) > r_0,$$

such that

$$h(y) \leq Cr^{-2-\mu}, \quad r > r_0.$$

Proof. Let h_0 be the supersolution built in Lemma 7.2 for

$$(7.20) \quad -\mathcal{J}_{\Gamma_0}[h_0] \geq \frac{1}{r^{4+\mu}} \quad \text{in } \Gamma_0, \quad r(y) > r_0,$$

and consider the function h defined on Γ as $h(y) = 2h_0(\pi(y))$. Then according to Lemma 7.1, we have that

$$\mathcal{J}_\Gamma[h](y) \geq \frac{2}{r(\pi(y))^{4+\mu}} + \Theta(\pi(y)),$$

where

$$\Theta(y) = O(r^{-2-\sigma})D_{\Gamma_0}^2 h_0 + O(r^{-3-\sigma})D_{\Gamma_0} h_0 + O(r^{-4-\sigma})h_0.$$

Using the explicit form of h_0 in the proof of the previous lemma, we compute directly that

$$\Theta(y) = O(r^{-6-\sigma-\mu}).$$

Finally, since $\pi(p)$ is uniformly close to p , we have that $r(\pi(y)) = r(y) + O(1)$, and thus we find that for all large r_0 ,

$$-\mathcal{J}_\Gamma[h] \geq \frac{1}{r^{4+\mu}} \quad \text{in } \Gamma, \quad r(y) > r_0.$$

The proof is concluded. □

A second element needed is a regularity estimate for equation (7.30).

LEMMA 7.5. *Let $p > 8$, $\nu \geq 2$. Then there exists a $C > 0$ such that if $\|\mathbf{g}\|_{\infty,\nu} + \|h\|_{\infty,\nu-2} < +\infty$ and h solves (7.30), then*

$$(7.21) \quad \|D_\Gamma^2 h\|_{p,\nu-\frac{8}{p}} + \|D_\Gamma h\|_{\infty,\nu-1} \leq C (\|h\|_{\infty,\nu-2} + \|\mathbf{g}\|_{\infty,\nu}).$$

Proof. Without loss of generality, we may assume that $\|h\|_{\infty,\nu-2} + \|\mathbf{g}\|_{\infty,\nu} \leq 1$. We use the local coordinates (3.13). Then, around a point p with $r(p) = R$, for any sufficiently large R , the equation reads on $B(0, 2\theta R)$ for a small, fixed $\theta > 0$ as

$$a_{ij}^0(\mathbf{y})\partial_{ij}h + b_i^0(\mathbf{y})\partial_i h = -|A_\Gamma(\mathbf{y})|^2 h + \mathbf{g}(\mathbf{y}) \quad \text{in } B(0, 2\theta R).$$

Consider the scalings

$$\tilde{h}(\mathbf{y}) = R^{\nu-2}h(R\mathbf{y}), \quad \tilde{\mathbf{g}}(\mathbf{y}) = R^\nu \mathbf{g}(R\mathbf{y}).$$

Then $|A_\Gamma(R\mathbf{y})|^2|\tilde{h}| + |\tilde{\mathbf{g}}| \leq C$ in $B(0, 2\theta)$, and

$$a_{ij}^0(R\mathbf{y})\partial_{ij}\tilde{h} + b_i^0(R\mathbf{y})\partial_i \tilde{h} = \tilde{g} \quad \text{in } B(0, 2\theta), \quad \tilde{g} := |A_\Gamma(R\mathbf{y})|^2\tilde{h} + \tilde{\mathbf{g}},$$

where $a_{ij} = \delta_{ij} + O(\theta)$. By interior elliptic regularity we find that

$$\|\partial_i \tilde{h}\|_{L^\infty(B(0,\theta))} + \|\partial_{ij} \tilde{h}\|_{L^p(B(0,\theta))} \leq C,$$

and, in particular, $|\partial_i \tilde{h}(0)| = R^{\nu-1}|\partial_i h(p)| \leq C$ so that

$$|D_\Gamma h(p)| \leq CR^{1-\nu},$$

$$\int_{B(0,\theta)} R^{\nu p-8} |\partial_{ij} h|^p(R\mathbf{y}) R^8 \, d\mathbf{y} = R^{\nu p-8} \int_{B(0,\theta R)} |\partial_{ij} h|^p(\mathbf{y}) \, d\mathbf{y} \leq C.$$

Hence

$$r(p)^{\nu-1} |D_\Gamma h(p)| + r(p)^{\nu-\frac{8}{p}} \|D_\Gamma^2 h\|_{L^p(B(p,1)\cap\Gamma)} \leq C$$

provided that $r(p)$ is large enough. On a bounded region, the corresponding estimate follows from interior elliptic estimates, and hence estimate (7.21) follows. \square

Proof of Proposition 7.1. We begin by proving existence assuming that $\|\mathbf{g}\|_{\infty,\nu} < +\infty$. Let us consider the approximate problems

$$(7.22) \quad \mathcal{J}_\Gamma[h] = \mathbf{g}(y) \quad \text{in } \Gamma \cap B(0, R), \quad h = 0 \quad \text{on } \partial(\Gamma \cap B(0, R))$$

where we allow $R = +\infty$.

We claim the existence of a $C > 0$ uniform in R and \mathbf{g} such that the *a priori* estimate

$$(7.23) \quad \|h\|_{\infty,\nu-2} \leq C \|\mathbf{g}\|_{\infty,\nu}$$

holds. Let us assume the opposite, namely the existence of sequences $R = R_n \rightarrow +\infty$, $h = h_n$ and $\mathbf{g} = \mathbf{g}_n$ such that (7.30) holds, but $\|h_n\|_{\infty,\nu-2} = 1$, $\|\mathbf{g}_n\|_{\infty,\nu} \rightarrow 0$.

Passing to a subsequence, we may assume that $h_n \rightarrow h$ locally uniformly in Γ , where h satisfies the homogeneous equation $\mathcal{J}_\Gamma[h] = 0$ and $\|h\|_{\infty,\nu-2} \leq 1$. We claim that $h = 0$. To prove this, we let $Z = \frac{1}{\sqrt{1+|\nabla F|^2}}$ and observe that $\mathcal{J}_\Gamma[Z] = 0$. Since $h = o(r^{-2})$ as $r \rightarrow +\infty$, it follows that given $\varepsilon > 0$, we have that $|h(y)| \leq \varepsilon Z$ whenever $r(y)$ is large enough. It follows from the maximum principle that

$$|h(y)| \leq \frac{\varepsilon}{\sqrt{1+|\nabla F|^2}} \quad \text{in } \Gamma$$

and hence that $h = 0$, as claimed.

Now, from Lemma 7.4, we know that there is a positive supersolution \bar{h} of $-\mathcal{J}_\Gamma[\bar{h}] \geq r^{-\nu}$ for $r > r_0$ such that $\bar{h} \geq Cr^{2-\nu}$. We also have that $|\mathbf{g}_n| \leq \rho_n r^{2-\nu}$ with $\rho_n \rightarrow 0$. Furthermore,

$$-\mathcal{J}_\Gamma[\pm h_n - o(1)\bar{h}] \leq 0 \quad \text{in } \{r_0 < r < R_n\} \cap \Gamma,$$

and $\pm h_n - o(1)\bar{h} \leq 0$ on the boundary of this set, where we are using that $h_n \rightarrow 0$ locally uniformly. From maximum principle, we conclude that for all large n , $|h_n| \leq o(1)\bar{h}$ and thus $\|h_n\|_{\infty,\nu-2} \rightarrow 0$, a contradiction that proves the validity of the *a priori* estimate (6.2).

Now, as for existence of a solution to (7.18) for a given g , we use the *a priori* estimate found. The approximate problem is indeed uniquely solvable when $R < +\infty$, thanks to the *a priori* estimate and Fredholm alternative. Possibly passing to a subsequence, we get that h_R converges locally uniformly

to a solution h of the equation. The limiting function clearly satisfies the estimate (6.2). Now, Lemma 7.5 yields the stronger estimate

$$(7.24) \quad \|D_\Gamma^2 h\|_{p, \nu - \frac{8}{p}} + \|D_\Gamma h\|_{\infty, \nu - 1} + \|h\|_{\infty, \nu - 2} \leq C \|\mathbf{g}\|_{\infty, \nu}$$

for any $p > 8$

Let us assume now that we only have $\|\mathbf{g}\|_{p, \nu} < +\infty$. We find a solution to equation (7.18) by reducing the problem to one in which \mathbf{g} is replaced by a $\bar{\mathbf{g}}$ with $\|\bar{\mathbf{g}}\|_{\infty, \nu} < +\infty$. We do this using the result of Corollary 6.1. Let us consider the equation

$$-\Delta_\Gamma \psi + \lambda^{-2} \psi = \mathbf{g} \quad \text{in } \Gamma,$$

where $\lambda > 0$ is a small number, to be chosen. The transformation $\bar{\psi}(y) := \psi(\lambda y)$ makes this equation is equivalent to

$$-\Delta_{\Gamma_\lambda} \bar{\psi} + \bar{\psi} = \lambda^2 \mathbf{g}(\lambda y) \quad \text{in } \Gamma_\lambda.$$

From the result of Corollary 6.1 with λ replacing α , we find a sufficiently small λ for which this problem has a unique solution respecting the corresponding decay estimate for the right-hand side. In terms of ψ the estimate achieved reads

$$\|D_\Gamma^2 \psi\|_{p, \nu} + \|D_\Gamma \psi\|_{\infty, \nu} + \|\psi\|_{\infty, \nu} \leq C_\lambda \|\mathbf{g}\|_{p, \nu}.$$

We denote $\psi := \psi(\mathbf{g})$. Then writing in equation (4.24) $h = \psi(g) + h_1$ we obtain the following equation for h_1 :

$$(7.25) \quad \Delta_\Gamma h_1 + |A_\Gamma(y)|^2 h_1 = \bar{\mathbf{g}}(y) \quad \text{in } \Gamma,$$

where

$$\bar{\mathbf{g}} = \lambda^2 \psi(\mathbf{g}) - |A_\Gamma(y)|^2 \psi(\mathbf{g}).$$

Clearly $\|\bar{\mathbf{g}}\|_{\infty, \nu} \leq C \|\mathbf{g}\|_{p, \nu}$. But we know by the previous step that there exists a unique solution h_1 to (7.25), which satisfies

$$\|D_\Gamma^2 h_1\|_{p, \nu - \frac{8}{p}} + \|D_\Gamma h_1\|_{\infty, \nu - 1} + \|h_1\|_{\infty, \nu - 2} \leq C \|\bar{\mathbf{g}}\|_{\infty, \nu},$$

and the result follows. □

7.4. *Proof of Proposition 4.2(a).*

LEMMA 7.6. *The results of Lemma 7.5 and Proposition 7.1 remain unchanged when Γ is replaced by Γ_0 for the problem*

$$\mathcal{J}_{\Gamma_0}[h] = \mathbf{g} \quad \text{in } \Gamma_0.$$

Proof. The proof of the analog of Lemma 7.5 is identical, taking into account suitable local coordinates $y = Y_0(\mathbf{y})$ for Γ_0 , for instance for large r one can use those introduced in (8.28) below that lead to exactly the same asymptotic properties for the Laplace Beltrami operator. The proof of the

corresponding result to Proposition 7.1 is also the same, on the basis of the supersolution found on Γ_0 and the fact that $\mathcal{J}_{\Gamma_0}[\frac{1}{\sqrt{1+|\nabla F_0|^2}}] = 0$. \square

Our next task is to solve the problem

$$(7.26) \quad \mathcal{J}_{\Gamma_0}[h] = \mathbf{g} \quad \text{in } \Gamma_0,$$

where we assume now that \mathbf{g} decays only at rate $O(r^{-3})$ but it is symmetric in the sense that

$$\mathbf{g}(Qy) = -\mathbf{g}(Qy) \quad \text{for all } y \in \Gamma_0$$

and for all Q of the form (3.2). In particular, $\mathbf{g} = \mathbf{g}(r, \theta)$.

We look for a solution to (7.26) that shares the same symmetries. Thus it suffices to solve the problem in Γ_{0+} with h symmetric and vanishing at its boundary, namely

$$(7.27) \quad \mathcal{J}_{\Gamma_0}[h] = \mathbf{g} \quad \text{in } \Gamma_{0+}, \quad h = 0 \quad \text{on } \partial\Gamma_{0+},$$

since then the odd extension of $h = h(r, \theta)$ through $\theta = \frac{\pi}{4}$ will satisfy (7.26). We require in addition that in polar coordinates, the function \mathbf{g} is dominated in the following way:

$$(7.28) \quad |\mathbf{g}(y)| \leq \frac{Cg(\theta)}{r^3 + 1} \quad \text{in } \Gamma_{0+}.$$

We prove:

LEMMA 7.7. *Let $p > 8$ and assume that \mathbf{g} satisfies (7.28). Then there exists a solution h to problem (7.27) such that*

$$(7.29) \quad \|D_{\Gamma_0}^2 h\|_{p, 3-\frac{8}{p}} + \|D_{\Gamma_0} h\|_{\infty, 2} + \|h\|_{\infty, 1} < +\infty.$$

Proof. Let us consider the supersolution h_0 for (7.27) defined by $r > r_0$ given by Lemma 7.3. (We fix an arbitrary exponent $\tau \in (\frac{1}{3}, \frac{2}{3})$). Let $\eta(r)$ be a smooth cut-off function such that $\eta(r) = 1$ for $r < r_0$ and $\eta(r) = 0$ for $r > r_0 + 1$. We consider the function, defined in entire Γ_{0+} as

$$h_1 = \eta + (1 - \eta)h_0.$$

Then

$$\begin{aligned} -\mathcal{J}_{\Gamma_0}[h_1] &= -(1 - \eta)\mathcal{J}_{\Gamma_0}[h_0] + \bar{\mathbf{g}}_0 \geq (1 - \eta)g^\tau(\theta)r^{-3} + \bar{\mathbf{g}}_0 \\ &\geq \varsigma(1 - \eta)g^\tau(\theta)(1 + r)^{-3} + \bar{\mathbf{g}}_0, \end{aligned}$$

where $\bar{\mathbf{g}}_0$ is compactly supported and $\varsigma > 0$ is a constant depending in r_0 . Let h_2 be the unique solution of

$$-\mathcal{J}_{\Gamma_0}[h_2] = |\bar{\mathbf{g}}_0| + \varsigma\eta g(\theta)^\tau(1 + r)^{-3},$$

given by Lemma 7.6, which is positive in Γ_{0+} and symmetric. Then if $h_3 := h_1 + h_2$, we get

$$-\mathcal{J}_{\Gamma_0}[h_3] \geq \varsigma g(\theta)^\tau (1+r)^{-3} \geq C\varsigma g(\theta)(1+r)^{-3},$$

and hence h_3 is a positive supersolution of the problem (7.27)–(7.28).

Since \mathcal{J}_{Γ_0} satisfies maximum principle, we have that the approximation scheme

$$(7.30) \quad \mathcal{J}_{\Gamma_0}[h_R] = \mathbf{g}(y) \quad \text{in } \Gamma_{0+} \cap B(0, R), \quad h = 0 \quad \text{on } \partial(\Gamma_{0+} \cap B(0, R))$$

is such that its unique solution satisfies $|h_R| \leq Ch_3$. Standard diagonal argument gives a subsequence of h_R which converges locally uniformly to a smooth solution h of

$$(7.31) \quad \mathcal{J}_{\Gamma_0}[h] = \mathbf{g} \quad \text{in } \Gamma_{0+}, \quad h = 0 \quad \text{on } \partial\Gamma_{0+},$$

with the property that $\|h\|_{\infty,1} \leq C$. Observe that we also have $\|\mathbf{g}\|_{\infty,3} \leq C$. From Lemma 7.6, we then get that for any $p > 8$,

$$(7.32) \quad \|D_{\Gamma_0}^2 h\|_{p,3-\frac{8}{p}} + \|D_{\Gamma_0} h\|_{\infty,2} + \|h\|_{\infty,1} \leq C,$$

as desired. □

To conclude with the proof of the proposition, we need to consider the equation

$$(7.33) \quad \mathcal{J}_{\Gamma}[h] = H_3(y) := \sum_{i=1}^8 k_i^3(y) \quad \text{in } \Gamma.$$

A main fact we need is the following lemma, whose proof is postponed to Section 8.4.

LEMMA 7.8. *Let $k_i^0(y)$ denote the principal curvatures at a point $y \in \Gamma_{0+}$ (see (7.15) for the definition of Γ_{0+}). Then we have that for all large enough $r(y)$,*

$$(7.34) \quad \left| \sum_{i=1}^8 k_{i0}^3(y) \right| \leq C \frac{g(\theta)}{r^3} + O(r^{-5}) \quad \text{on } \Gamma_{0+},$$

$$(7.35) \quad \sum_{i=1}^8 k_i^3(y) = \sum_{i=1}^8 k_{i0}^3(\pi(y)) + O(r^{-5}).$$

Let us conclude the proof of the proposition. From Lemma 7.7 and using an odd extension by reflection, we see that there exists a solution h_0 of

$$\mathcal{J}_{\Gamma_0}[h_0] = \sum_{i=1}^8 k_{i0}^3 \quad \text{in } \Gamma,$$

satisfying the appropriate estimates. Let $h_1(y) = h_0(\pi(y))$ for $r(y) > r_0$, and extended smoothly in an arbitrary way to all of Γ . Then according to Lemma 7.1, we find that for large r ,

$$(7.36) \quad \mathcal{J}_\Gamma[h_1](y) = \sum_{i=1}^8 k_{i0}^3(\pi(y)) + \left[O(r^{-2-\sigma})D_{\Gamma_0}^2 h_0 + O(r^{-3-\sigma})D_{\Gamma_0} h_0 + O(r^{-4-\sigma})h_0 \right](\pi(y)) \quad \text{in } \Gamma.$$

To solve problem (7.33) we set $h = h_1 + h_2$ and then get the equation for h_2 ,

$$\mathcal{J}_\Gamma[h_2] = \Theta(y) \quad \text{in } \Gamma,$$

where, using relation (7.35) and Lemma 7.6, we get

$$\|\Theta\|_{p,5} < +\infty.$$

Then we choose h_2 to be unique solution to that problem given by Proposition 7.1. The function h built this way satisfies the requirements of the proposition. \square

**8. Local coordinates on Γ :
The effect of curvature and closeness to Γ_0**

8.1. *The proof of Proposition 3.1.* Let $p_0 = (x_0, F(x_0))$ with $|x_0| = R$. Then there is a function $G(y)$ such that, for some $\rho, a > 0$,

$$\Gamma \cap B_\rho(p_0) = p_0 + \{(y, G(y)) \mid |y| < a\},$$

where $y = (y_1, \dots, y_8)$ are the Euclidean coordinates on $T_{p_0}\Gamma$. More precisely, $F(x)$ and $G(y)$ are linked through the following relation:

$$(8.1) \quad \begin{bmatrix} x \\ F(x) \end{bmatrix} = \begin{bmatrix} x_0 \\ F(x_0) \end{bmatrix} + \Pi y + G(y)\nu(p_0).$$

Here

$$\Pi y = \sum_{j=1}^8 y_j \Pi_j, \quad y \in \mathbb{R}^8,$$

where $\{\Pi_1, \Pi_2, \dots, \Pi_8\}$ is a choice of an orthonormal basis for the tangent space to the minimal graph at the point $p_0 = (x_0, F(x_0))$, and

$$\nu(p_0) = \frac{1}{\sqrt{1 + |\nabla F(x_0)|^2}} \begin{bmatrix} \nabla F(x_0) \\ -1 \end{bmatrix},$$

so that

$$G(y) = \frac{1}{\sqrt{1 + |\nabla F(x_0)|^2}} (F(x) - F(x_0) - \nabla F(x_0) \cdot (x - x_0)).$$

The implicit function theorem implies that G and x , given in equation (8.1), are smooth functions of y , at least while $|y| < a$ for a sufficiently small number

$a > 0$. Clearly when p_0 is restricted to some fixed compact set, then there exists a $\theta > 0$ such that

$$a = \theta(1 + R), \quad R = |x_0|.$$

To show a similar bound for all $p_0 \in \Gamma$, we will assume $|x_0| = R > 1$. The bound we are seeking amounts to estimating (from below) the largest a so that

$$\sup_{|y| < a} |D_y G(y)| < +\infty.$$

Here and below, by D_y, D_y^2 , etc. we will denote the derivatives with respect to the local variable y . Let $\nu(z)$ denote unit normal at the point $z = (y, G(y))$ (with some abuse of notation $\nu(p_0) \equiv \nu(0)$). Let us set

$$\hat{y} = \frac{y}{|y|}$$

and consider the following curve on the minimal surface:

$$r \mapsto \gamma(r) := (r\hat{y}, G(r\hat{y})), \quad 0 < r \leq |y|.$$

Then

$$\partial_r \nu(\gamma(r)) = A_\Gamma(\gamma(r))[(\hat{y}, D_y G(r\hat{y}) \cdot \hat{y})],$$

where A_Γ is the second fundamental form on Γ and $D_y G(r\hat{y}) = D_y G(y)|_{y=r\hat{y}}$. Thus

$$|\nu(\gamma(r)) - \nu(0)| \leq \sup_{0 < s < r} |A_\Gamma(\gamma(s))| \int_0^r (1 + |D_y G(s\hat{y})|) ds.$$

We will now make use of Simon's estimate ([34, Th. 4, p. 673 and Rem. 2, p. 674]) which yields

$$\sup_{0 < s < r} |A_\Gamma(\gamma(s))| < \frac{c}{R},$$

since we can assume that $|y| < \theta R$, with some small $\theta > 0$. In addition, we have that

$$|\nu(\gamma(r)) - \nu(0)| \geq \frac{|D_y G(r\hat{y})|}{1 + |D_y G(r\hat{y})|};$$

hence

$$\frac{|D_y G(r\hat{y})|}{1 + |D_y G(r\hat{y})|} \leq \frac{c}{R} \int_0^r (1 + |D_y G(s\hat{y})|) ds.$$

Let us write $\varepsilon = \frac{c}{R}$ and

$$\psi(r) := \int_0^r (1 + |D_y G(s\hat{y})|) ds.$$

The above inequality reads

$$1 - \frac{1}{\psi'(r)} \leq \varepsilon \psi(r),$$

or

$$(1 - \varepsilon \psi(r)) \psi'(r) \leq 1,$$

so that for all sufficiently small (relative to the size of ε) $r > 0$, we have that

$$1 - (1 - \varepsilon\psi(r))^2 \leq 2\varepsilon r.$$

Since $\psi(0) = 0$, it follows that

$$(1 - 2\varepsilon r)^{\frac{1}{2}} \leq (1 - \varepsilon\psi(r));$$

hence

$$1 - \frac{1}{1 + |D_y G(r\hat{y})|} \leq \varepsilon\psi(r) \leq 1 - (1 - 2\varepsilon r)^{\frac{1}{2}},$$

which implies

$$|D_y G(y)| \leq (1 - 2\varepsilon|y|)^{-\frac{1}{2}} - 1 \leq 8\varepsilon|y|,$$

provided that $\varepsilon|y| < \frac{1}{4}$. Hence we have established that there are positive numbers θ, c , independent of R such that

$$(8.2) \quad |D_y G(y)| \leq \frac{c}{R}|y| \quad \text{for all } |y| < \theta R.$$

In particular, we obtain a uniform bound on $D_y G(y)$ for $|y| \leq \theta R$, while at the same time

$$(8.3) \quad |\nu(y, G(y)) - \nu(0)| \leq \frac{c}{R}|y| \quad \text{for all } |y| < \theta R.$$

This guarantees the fact that our minimal surface indeed defines a graph over the tangent plane at p_0 , at least for $|y| \leq \theta R$. The quantities $x(y)$ and $G(y)$ linked by equation (8.1) are thus well defined, provided that $|y| < \theta R$. The implicit function theorem yields, in addition, their differentiability. We have

$$(8.4) \quad \begin{bmatrix} D_y x(y) \\ D_x F(x) D_y x(y) \end{bmatrix} = \Pi + D_y G(y)\nu(p_0),$$

and, in particular, $|D_y x(y)|$ is uniformly bounded in $|y| < \theta R$. The above relation also tells us that

$$(8.5) \quad |D_y^m x(y)| \leq |D_y^m G(y)|, \quad m \geq 2, \quad |y| < \theta R.$$

Let us estimate now the derivatives of G . Since $G(y)$ represents a minimal graph, we have that

$$(8.6) \quad H[G] = \nabla_y \cdot \left(\frac{\nabla_y G}{\sqrt{1 + |\nabla_y G|^2}} \right) = 0 \quad \text{in } B(0, \theta R) \subset \mathbb{R}^8.$$

Let us consider now the change of variable

$$\tilde{G}(y) = \frac{1}{R}G(Ry)$$

and observe that \tilde{G} is bounded and satisfies

$$(8.7) \quad H[\tilde{G}] = \nabla_y \cdot \left(\frac{\nabla_y \tilde{G}}{\sqrt{1 + |\nabla_y \tilde{G}|^2}} \right) = 0 \quad \text{in } B(0, \theta).$$

In fact from (8.2), we have

$$|\tilde{G}(y)| \leq C \quad \text{for all } |y| \leq \theta;$$

hence, potentially reducing θ , from standard estimates for the minimal surface equation (see for instance [18]) we find

$$(8.8) \quad |D_y \tilde{G}(y)| \leq C \quad \text{for all } |y| \leq \theta,$$

with a similar estimate for $D_y^2 \tilde{G}$, and in general the same bound for $D_y^m \tilde{G}$, $m \geq 2$ in this region. As a conclusion, using also (8.5) we obtain

$$(8.9) \quad |D_y^m x(y)| + |D_y^m G(y)| \leq \frac{C}{R^{m-1}} \quad \text{for all } |y| \leq \theta R$$

for $m = 2, 3, \dots$. This estimate and (8.2) provide in particular the result of the lemma. \square

Remark 8.1. From the above considerations it follows that the local coordinates near Γ in (3.11) are well defined. Indeed this is the case as long as the function $x \mapsto (y, z)$ is invertible. We claim that this holds, and consequently that the Fermi coordinates are well defined if

$$(8.10) \quad |z| \leq \theta |A_\Gamma(y)|^{-1},$$

whenever $r(y)$, the distance from the origin of the projection of $y \in \Gamma$ onto \mathbb{R}^8 , is large enough, and θ is chosen to be a small number. We argue by contradiction; i.e., we assume that $x \mapsto (y, z)$ is not one-to-one. Because of the symmetry of the surface Γ , it is enough to consider the situation in which, for certain $x = (x', x_9)$ such that $x' \in T$, we have the existence of two different points $y_1, y_2 \in \Gamma \cap T$ such that

$$(8.11) \quad x = y_i + z\nu(y_i), \quad i = 1, 2,$$

with z satisfying (8.10). We may assume that $|r(y_1)| = R_1$ is large. Then it follows that

$$(8.12) \quad |y_1 - y_2| \leq |z| |\nu(y_1) - \nu(y_2)| \leq \theta |A_\Gamma(y_1)|^{-1}.$$

In the portion of Γ where (8.12) holds, we in fact have

$$(8.13) \quad \begin{aligned} |y_1 - y_2| &\leq |z| |\nu(y_1) - \nu(y_2)| \\ &\leq \theta |A_\Gamma(y_1)|^{-1} \sup_{|y_1 - y| \leq \theta |A_\Gamma(y_1)|} |A_\Gamma(y)| |y_1 - y_2|. \\ &\leq C\theta \frac{R_1 + 1}{R_1} |y_1 - y_2|. \end{aligned}$$

We get a contradiction if we take $\theta > 0$ to be sufficiently small, and thus the claim follows.

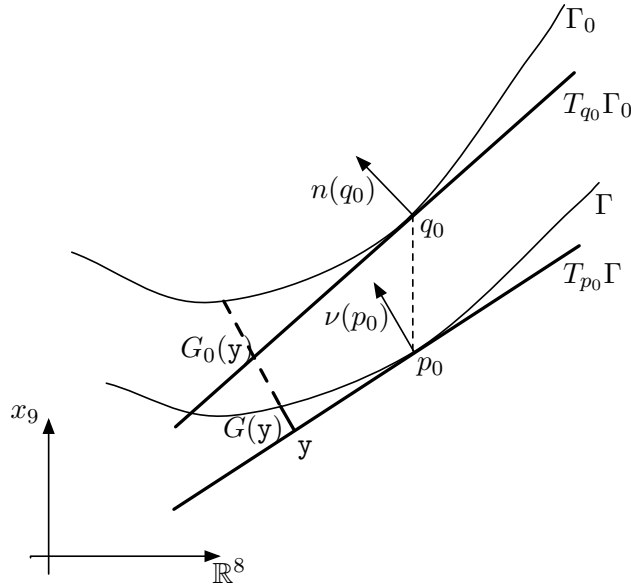


Figure 2. Local configuration of the two surfaces Γ and Γ_0 .

8.2. *Comparing G and G_0 .* We want to estimate with higher accuracy derivatives of G , in their relation with the approximate minimal graph Γ_0 , $x_9 = F_0(x)$. We shall establish next that in the situation considered above we also have that Γ_0 can be represented as the graph of a function $G_0(y)$ over the tangent plane to Γ at the point p_0 , at least in a ball on that plane of radius θR for a sufficiently small, fixed $\theta > 0$ and for all large R . Below we let n and ν denote respective normal vectors to Γ_0 and Γ , with the convention $n \cdot \nu \geq 0$. For convenience the situation is presented schematically in Figure 2.

To prove the above claim we will show that for fixed, sufficiently small θ , we have the estimate

$$(8.14) \quad |n(q) - \nu(p_0)| < C\theta \quad \text{for all } q \in \Gamma_0 \cap B(p_0, \theta R).$$

Since by Theorem 2

$$F(x) - F_0(x) = O(|x|^{-\sigma}), \quad \text{some } \sigma \in (0, 1),$$

we have that the points $p_0 = (x_0, F(x_0))$ and $q_0 = (x_0, F_0(x_0))$ satisfy

$$(8.15) \quad |p_0 - q_0| \leq \frac{C}{R^\sigma}.$$

Let $T_{p_0}\Gamma, T_{q_0}\Gamma_0$, be the corresponding tangent hyperplanes, namely

$$\begin{aligned} T_{p_0}\Gamma &= \{z \in \mathbb{R}^9 \mid (z - p_0) \cdot \nu(p_0) = 0\}, \\ T_{q_0}\Gamma_0 &= \{z \in \mathbb{R}^9 \mid (z - q_0) \cdot n(q_0) = 0\}. \end{aligned}$$

We assume that $\nu(p_0) \cdot n(q_0) \geq 0$. We claim that there is a number $M > 0$ such that for all large R ,

$$(8.16) \quad |\nu(p_0) - n(q_0)| \leq \frac{5M}{R}.$$

Let us assume the opposite and let us consider a point $z \in T_{q_0}\Gamma_0$ with

$$\theta R > |z - q_0| > \frac{\theta}{2}R,$$

with $\theta > 0$ as in (8.2). Let us write $\cos \alpha = \nu(p_0) \cdot n(q_0)$ with $0 \leq \alpha \leq \frac{\pi}{2}$. Then, using (8.15) we get

$$(8.17) \quad \text{dist}(z, T_{p_0}\Gamma) \geq |z - p_0| \sin \alpha \geq \left(\frac{\theta}{2}R - R^{-\sigma}\right) |\nu(p_0) - n(q_0)| \geq M\theta.$$

Now let $\tilde{q} \in \Gamma_0$ be the point whose projection onto $T_{q_0}\Gamma_0$ is z . Point \tilde{q} is unique by the analog of (8.3) for the surface Γ_0 . Let us write $\tilde{q} = (\tilde{x}, F_0(\tilde{x}))$. Notice that $|\tilde{x}| \sim R$. We will also set $\tilde{p} = (\tilde{x}, F(\tilde{x})) \in \Gamma$. Since the second fundamental form of the surface Γ_0 satisfies an estimate similar to the one for Γ , we may assume, reducing θ if necessary, that

$$\text{dist}(\tilde{q}, T_{q_0}\Gamma_0) \leq c\theta.$$

Now, estimate (8.2) implies that

$$\text{dist}(\tilde{p}, T_{p_0}\Gamma) \leq c\theta.$$

If M is fixed so that $M\theta$ is sufficiently large, the above two relations and (8.15) are not compatible with (8.17). Indeed, we get

$$\begin{aligned} M\theta &\leq \text{dist}(z, T_{p_0}\Gamma) \leq \text{dist}(\tilde{p}, \tilde{q}) + \text{dist}(\tilde{p}, T_{p_0}\Gamma_0) + \text{dist}(\tilde{q}, T_{q_0}\Gamma_0) \\ &\leq c\theta + \text{dist}(\tilde{p}, \tilde{q}) \\ &\leq \frac{C}{R^\sigma} + c\theta; \end{aligned}$$

hence (8.16) holds. Moreover, using estimate (8.3) and the analogous estimate for the variation of n , we have the validity of the estimate

$$|n(q) - n(q_0)| + |\nu(p) - \nu(p_0)| < C\theta \quad \text{for all } p \in \Gamma \cap B(p_0, \theta R), q \in \Gamma_0 \cap B(q_0, \theta R).$$

Furthermore, we observe that the analog of estimate (8.3) implies that in the set $\Gamma_0 \cap B(q_0, \theta R)$, the distance between Γ_0 and its tangent plane at q_0 varies by no more than $c\theta$. From this and (8.15), (8.16), the desired conclusion (8.14) immediately follows (taking θ smaller if necessary). Hence the function $G_0(\mathbf{y})$ is well defined for $|\mathbf{y}| < \theta R$.

Let us observe that F_0 and G_0 are linked through the following relation:

$$(8.18) \quad \begin{bmatrix} \tilde{x} \\ F_0(\tilde{x}) \end{bmatrix} = \begin{bmatrix} x_0 \\ F(x_0) \end{bmatrix} + \Pi \mathbf{y} + G_0(\mathbf{y})\nu(p_0).$$

By the implicit function theorem, \tilde{x} and $G_0(\mathbf{y})$ define differentiable functions of \mathbf{y} for $|\mathbf{y}| \leq \theta R$. We shall establish derivative estimates for G_0 similar to those found for G . We claim that

$$(8.19) \quad |D_{\mathbf{y}}^m \tilde{x}(\mathbf{y})| + |D_{\mathbf{y}}^m G_0(\mathbf{y})| \leq \frac{C}{R^{m-1}} \quad \text{for all } |\mathbf{y}| \leq \theta R,$$

for $m = 1, 2, \dots$. Differentiation of relation (8.18) yields

$$(8.20) \quad \begin{bmatrix} \partial_j \tilde{x} \\ DF_0(\tilde{x}) \partial_j \tilde{x} \end{bmatrix} = \Pi_j + \partial_j G_0 \nu(p_0).$$

Let $q = (\tilde{x}, F_0(\tilde{x}))$ and

$$n(q) = \frac{1}{\sqrt{1 + |\nabla F_0(\tilde{x})|^2}} \begin{bmatrix} DF_0(\tilde{x}) \\ -1 \end{bmatrix}.$$

From (8.20) and the fact that $n(q) \cdot \nu(p_0) \geq c > 0$, we then get

$$|\partial_j G_0(\mathbf{y})| \leq C |\Pi_j \cdot n(q)| \leq C.$$

Using again relation (8.20), we also get

$$|\partial_j \tilde{x}(\mathbf{y})| \leq C.$$

Let us differentiate again. Now we get

$$(8.21) \quad \begin{bmatrix} \partial_{jk} \tilde{x} \\ DF_0(\tilde{x}) \partial_{jk} \tilde{x} \end{bmatrix} + \begin{bmatrix} 0 \\ D^2 F_0(\tilde{x}) [\partial_j \tilde{x}, \partial_j \tilde{x}] \end{bmatrix} = \partial_{jk} G_0 \nu(p_0).$$

Again, taking the dot product against $\nu(p_0)$, we get

$$|\partial_{jk} G_0(\mathbf{y})| \leq C \frac{|D^2 F_0(\tilde{x})|}{\sqrt{1 + |\nabla F_0(\tilde{x})|^2}} \leq \frac{C}{R},$$

and thus

$$|\partial_{jk} \tilde{x}(\mathbf{y})| \leq \frac{C}{R}.$$

Iterating this argument, using that

$$|D^m F_0(\tilde{x})| \leq CR^{3-m}, \quad m = 1, 2, \dots$$

the desired result (8.19) follows.

Let us write

$$G(\mathbf{y}) = G_0(\mathbf{y}) + \mathbf{h}(\mathbf{y}).$$

We will estimate first the size of $\mathbf{h}(\mathbf{y})$ in the ball $|\mathbf{y}| \leq \theta R$. We claim that we have

$$(8.22) \quad |\mathbf{h}(\mathbf{y})| \leq CR^{-1-\sigma} \quad \text{for all } |\mathbf{y}| \leq \theta R.$$

The first observation we make is that when $\mathbf{y} = 0$, we have

$$(8.23) \quad |\mathbf{h}(0)| = |G_0(0)| \leq \frac{C}{R^{1+\sigma}}.$$

To show this let \tilde{x} be such that

$$\begin{bmatrix} \tilde{x} \\ F_0(\tilde{x}) \end{bmatrix} = \begin{bmatrix} x_0 \\ F(x_0) \end{bmatrix} + G_0(0)\nu(p_0),$$

and let \tilde{y} be such that

$$\begin{bmatrix} \tilde{x} \\ F(\tilde{x}) \end{bmatrix} = \begin{bmatrix} x_0 \\ F(x_0) \end{bmatrix} + \Pi\tilde{y} + G(\tilde{y})\nu(p_0).$$

Comparing these two expressions and using $|F(\tilde{x}) - F_0(\tilde{x})| \sim R^{-\sigma}$, we see that $|\tilde{y}| \sim R^{-\sigma}$; hence, by (8.2), we get that $|G(\tilde{y})| \sim R^{-1-2\sigma}$. Now multiplying the above relations by $\nu(p_0)$ and subtracting them, we infer (8.23) since by [34, Th. 4, p. 673 and Th. 5, p. 680], we have that

$$|\nu_9(p_0)| = \frac{1}{\sqrt{1 + |DF(p_0)|^2}} \leq \frac{C}{R}.$$

To prove (8.22), now we let $p_1 = (x_1, F(x_1)) \in \Gamma \cap B(p_0, \theta R)$ so that

$$p_1 = p_0 + \Pi y + G(y)\nu(p_0), \quad |y| \leq \theta R.$$

Then $|G(y) - G_0(y)|$ corresponds to the length of the segment in the direction $\nu(p_0)$ starting at p_1 , which ends on the surface Γ_0 . Let $p_2 = (x_1, F_0(x_1))$. Then

$$|p_1 - p_2| \leq CR^{-\sigma}.$$

Let us consider the tangent plane $T_{p_2}\Gamma_0$ to Γ_0 at p_2 , with normal $\nu(p_2)$. Then, $\Gamma_0 \cap B(p_2, CR^{-\sigma})$ lies within a distance $O(R^{-1-\sigma})$ from $T_{p_2}\Gamma_0$; more precisely,

$$\Gamma_0 \cap B(p_2, CR^{-\sigma}) \subset \mathcal{C}_R,$$

where \mathcal{C}_R is the cylinder

$$\mathcal{C}_R = \{ \bar{z} + s\nu(p_2) \mid \bar{z} \in T_{p_2}\Gamma_0, |\bar{z} - p_2| \leq CR^{-\sigma}, |s| \leq CR^{-1-\sigma} \}.$$

Using (8.23) we may assume that $p_1 \in \mathcal{C}_R$. In particular, the line starting from p_1 with direction $\nu(p_1)$ intersects Γ_0 inside this cylinder. Since $\nu(p_1) \cdot \nu(p_2) \geq c > 0$, the length of this segment is of the same order as the height of the cylinder, and we then get

$$|G(y) - G_0(y)| \leq CR^{-1-\sigma};$$

hence (8.22) holds.

Next we shall improve the previous estimate. We claim that we have

$$(8.24) \quad |D_y^m \mathbf{h}(y)| \leq \frac{c}{R^{m+1+\sigma}} \quad \text{in } |y| < \theta R$$

for $m = 0, 1, 2, \dots$. Let us set

$$\tilde{G}(y) = \frac{1}{R}G(Ry), \quad \tilde{G}_0(y) = \frac{1}{R}G_0(Ry), \quad \tilde{\mathbf{h}}(y) = \frac{1}{R}\mathbf{h}(Ry).$$

We compute (for brevity dropping the subscript in the derivatives)

$$\sqrt{1 + |\nabla\tilde{G}|^2}H[\tilde{G}] = \Delta\tilde{G} - \frac{D^2\tilde{G}[\nabla\tilde{G}, \nabla\tilde{G}]}{1 + |\nabla\tilde{G}|^2} = 0.$$

Now,

$$\frac{D^2\tilde{G}[\nabla\tilde{G}, \nabla\tilde{G}]}{1 + |\nabla\tilde{G}|^2} = \frac{D^2\tilde{\mathbf{h}}[\nabla\tilde{G}, \nabla\tilde{G}]}{1 + |\nabla\tilde{G}|^2} + \frac{D^2\tilde{G}_0[\nabla\tilde{G}, \nabla\tilde{G}]}{1 + |\nabla\tilde{G}|^2},$$

and

$$\frac{D^2\tilde{G}_0[\nabla\tilde{G}, \nabla\tilde{G}]}{1 + |\nabla\tilde{G}|^2} = \frac{D^2\tilde{G}_0[\nabla\tilde{G}_0, \nabla\tilde{G}_0]}{1 + |\nabla\tilde{G}_0|^2} + \frac{D^2\tilde{G}_0[2\nabla\tilde{G}_0 + \nabla\mathbf{h}, \nabla\mathbf{h}]}{1 + |\nabla\tilde{G}|^2}.$$

Furthermore,

$$\begin{aligned} \frac{D^2\tilde{G}_0[\nabla\tilde{G}_0, \nabla\tilde{G}_0]}{1 + |\nabla\tilde{G}|^2} &= \frac{D^2\tilde{G}_0[\nabla\tilde{G}_0, \nabla\tilde{G}_0]}{1 + |\nabla\tilde{G}_0|^2} \\ &\quad - \frac{D^2\tilde{G}_0[\nabla\tilde{G}_0, \nabla\tilde{G}_0](2\nabla\tilde{G}_0 + \nabla\tilde{\mathbf{h}}) \cdot \nabla\tilde{\mathbf{h}}}{(1 + |\nabla\tilde{G}_0|^2)(1 + |\nabla\tilde{G}|^2)}. \end{aligned}$$

Collecting terms we see that $\tilde{\mathbf{h}}$ satisfies the equation

$$\Delta\tilde{\mathbf{h}} - \frac{D^2\tilde{\mathbf{h}}[\nabla\tilde{G}, \nabla\tilde{G}]}{1 + |\nabla\tilde{G}|^2} + b \cdot \nabla\tilde{\mathbf{h}} + E = 0 \quad \text{in } B(0, \theta),$$

where

$$E = \Delta\tilde{G}_0 - \frac{D^2\tilde{G}_0[\nabla\tilde{G}_0, \nabla\tilde{G}_0]}{1 + |\nabla\tilde{G}_0|^2} = \sqrt{1 + |\nabla\tilde{G}_0|^2}H(\tilde{G}_0),$$

and

$$b = -\frac{D^2\tilde{G}_0[\nabla\tilde{G}_0, \nabla\tilde{G}_0](2\nabla\tilde{G}_0 + \nabla h)}{(1 + |\nabla\tilde{G}_0|^2)(1 + |\nabla\tilde{G}|^2)} + \frac{D^2\tilde{G}_0[2\nabla\tilde{G}_0 + \nabla h]}{1 + |\nabla\tilde{G}|^2}.$$

Notice that

$$|\nabla\tilde{G}(\mathbf{y})| \leq C, \quad |\tilde{\mathbf{h}}(\mathbf{y})| \leq CR^{-2-\sigma} \quad \text{in } |\mathbf{y}| < \theta.$$

Also by (9.36) it follows that the mean curvature of Γ_0 decays like R^{-5} . From

$$\begin{aligned} |E(\mathbf{y})| &= R \left| \left(\Delta G_0 - \frac{D^2G_0[\nabla G_0, \nabla G_0]}{1 + |\nabla G_0|^2} \right) (R\mathbf{y}) \right| \\ &= R\sqrt{1 + |\nabla G_0|^2}H[G_0](R\mathbf{y}) \\ &= R\sqrt{1 + |\nabla G_0(R\mathbf{y})|^2}H[F_0](\tilde{x}(R\mathbf{y})) \end{aligned}$$

(in the notation of (8.18)), we then find

$$|E(\mathbf{y})| = O(R^{-4}),$$

and, as a conclusion, reducing θ if needed,

$$|D_{\mathbf{y}}\tilde{\mathbf{h}}(\mathbf{y})| \leq \frac{c}{R^{2+\sigma}} \quad \text{in } |\mathbf{y}| < \theta,$$

so that for \mathbf{h} , accordingly we get

$$|D_{\mathbf{y}}\mathbf{h}(\mathbf{y})| \leq \frac{c}{R^{2+\sigma}} \quad \text{in } |\mathbf{y}| < \theta R.$$

On the other hand, using (8.19) we have for instance that

$$D_{\mathbf{y}}H[G_0](\mathbf{y}) = D_x H[F_0](\tilde{x}(\mathbf{y}))D_{\mathbf{y}}\tilde{x}(\mathbf{y}) = O(R^{-6});$$

hence

$$|D_{\mathbf{y}}E(\mathbf{y})| = O(R^{-4}).$$

More generally, since

$$D_x^m H[F_0](x) = O(|x|^{-5-m}),$$

we get

$$D_{\mathbf{y}}^m E(\mathbf{y}) = O(R^{-4}).$$

Thus, estimates (8.19), (8.9) and standard higher regularity elliptic estimates yield

$$|D_{\mathbf{y}}^m \tilde{\mathbf{h}}(\mathbf{y})| \leq \frac{c}{R^{2+\sigma}} \quad \text{in } |\mathbf{y}| < \theta R.$$

Hence

$$|D_{\mathbf{y}}^m \mathbf{h}(\mathbf{y})| \leq \frac{c}{R^{m+1+\sigma}} \quad \text{in } |\mathbf{y}| < \theta R$$

for $m \geq 1$.

8.3. *Approximating Γ by Γ_0 and their Jacobi operators:*

Proof of Lemma 7.1. The surfaces Γ and Γ_0 are uniformly close for r large. Let $p \in \Gamma$ with $r(p) \gg 1$. Let us consider the point $\pi(p) \in \Gamma_0$ defined in (7.2). Using local coordinates (8.18) around p , we have

$$\pi(p) = p + G_0(0)\nu(p).$$

Here of course the function G_0 depends on p . From this it follows that $\pi(p)$ exists and is unique when $r(p) \gg 1$. As we will see below, the map $p \mapsto \pi(p)$ is smooth.

We recall that the Jacobi operators associated to Γ and Γ_0 , respectively, are

$$\mathcal{J}_{\Gamma}[h] = \Delta_{\Gamma}h + |A_{\Gamma}|^2h, \quad \mathcal{J}_{\Gamma_0}[h] = H'(F_0)[\sqrt{1 + |\nabla F_0|^2}],$$

where we recall that from (7.4), \mathcal{J}_{Γ_0} is the sum of $\Delta_{\Gamma_0} + |A_{\Gamma_0}|^2$ perturbed by a second order operator with very rapidly decaying coefficients.

Let us consider two smooth functions h and h_0 defined on Γ and Γ_0 for r large, and related through the formula

$$h_0(\pi(y)) = h(y), \quad y \in \Gamma, \quad r(y) > r_0.$$

Then, to prove Lemma 7.1 we have to establish the relation

(8.25)

$$\mathcal{J}_{\Gamma}[h](y) = [\mathcal{J}_{\Gamma_0}[h_0] + O(r^{-2-\sigma})D_{\Gamma_0}^2 h_0 + O(r^{-3-\sigma})D_{\Gamma_0} h_0 + O(r^{-4-\sigma})h_0](\pi(y)).$$

8.3.1. *Projection map $\pi(p)$ and its derivatives.* We show next that this map is smooth and estimate its derivatives. In local coordinates \mathbf{y} we have that in a neighborhood of $\mathbf{y} = 0$,

$$(8.26) \quad \pi(\mathbf{y}) = p + \sum_{i=1}^8 \mathbf{y}_i \Pi_i + G(\mathbf{y})\nu(0) + (G_0(0) + t)\nu(\mathbf{y}) = p + \sum_{i=1}^8 \tilde{\mathbf{y}}_i \Pi_i + G_0(\tilde{\mathbf{y}})\nu(0)$$

for certain scalar function $t(\mathbf{y})$ and vector function $\tilde{\mathbf{y}}(\mathbf{y})$. Here and in what follows, with some abuse of notation, we write $f(\mathbf{y})$ to mean $f(Y(\mathbf{y}))$. Thus we should have $t(0) = 0$, $\tilde{\mathbf{y}}(0) = 0$. Local existence and smoothness of these functions can be found by the implicit function theorem. Indeed (8.26) is equivalent to the system

$$A(\mathbf{y}, \tilde{\mathbf{y}}, t) := \begin{bmatrix} \mathbf{y}_1 - \tilde{\mathbf{y}}_1 + (G_0(0) + t)\nu(\mathbf{y}) \cdot \Pi_1 \\ \vdots \\ \mathbf{y}_8 - \tilde{\mathbf{y}}_8 + (G_0(0) + t)\nu(\mathbf{y}) \cdot \Pi_8 \\ G(\mathbf{y}) + (G_0(0) + t)\nu(\mathbf{y}) \cdot \nu(0) - G_0(\tilde{\mathbf{y}}) \end{bmatrix} = 0.$$

Note that $A(0, 0, 0) = 0$ and that

$$D_{\mathbf{y},t}A(0, 0, 0) = \begin{bmatrix} \text{Id}_{\mathbb{R}^8} + G_0(0)D_{\mathbf{y}}^2G(0) & 0 \\ D_{\mathbf{y}}G_0(0) & 1 \end{bmatrix} = \text{Id}_{\mathbb{R}^9} + O(r^{-2-\sigma})$$

is invertible; hence the existence of the smooth functions $\tilde{\mathbf{y}}(\mathbf{y})$ and $t(\mathbf{y})$ as required follows. Moreover, implicit differentiation yields

$$D_{\mathbf{y}}t(0) = [D_{\mathbf{y},t}A(0, 0, 0)]^{-1}G'_0(0) = O(r^{-2-\sigma}),$$

while

$$D_{\mathbf{y}}\tilde{\mathbf{y}}(0) = \text{Id}_{\mathbb{R}^8} + O(r^{-2-\sigma}).$$

Iterating the implicit differentiation, using that one negative power of r is gained in successive differentiations of the coefficients $G(\mathbf{y})$ and $\nu(\mathbf{y})$, we find that

$$D_{\mathbf{y}}^m \tilde{\mathbf{y}}(0), D_{\mathbf{y}}^m t(0) = O(r^{-m-1-\sigma}), \quad m \geq 2.$$

8.3.2. *Comparing Δ_Γ and Δ_{Γ_0} .* Given a smooth function $f(y)$ defined on Γ for all large r , it is natural to associate to it the function f_0 defined on Γ_0 for large r by the formula

$$(8.27) \quad f_0(\pi(y)) := f(y).$$

The question is now how to compare the quantities $[\Delta_\Gamma f](y)$ and $[\Delta_{\Gamma_0} f_0](\pi(y))$. Given a point p on Γ , the corresponding local coordinates \mathbf{y} are good, both for parametrizing locally Γ near p and Γ_0 near $\pi(p)$ respectively by

$$(8.28) \quad Y(\mathbf{y}) = p + \mathbf{y}_i \Pi_i + G(\mathbf{y})\nu(p) \quad \text{and} \quad Y_0(\mathbf{y}) = p + \mathbf{y}_i \Pi_i + G_0(\mathbf{y})\nu(p).$$

The observation is that, by definition, $\pi(Y(\mathbf{y})) = Y_0(\tilde{\mathbf{y}}(\mathbf{y}))$ and thus the relation $(f_0 \circ \pi)(Y(\mathbf{y})) = f(Y(\mathbf{y}))$ means $f_0(Y_0(\tilde{\mathbf{y}}(\mathbf{y}))) = f(Y(\mathbf{y}))$. In other words, with the usual abuse of notation,

$$f_0(\tilde{\mathbf{y}}(\mathbf{y})) = f(\mathbf{y}),$$

and the question is to compare $\Delta_\Gamma f(\mathbf{y})$ and $(\Delta_{\Gamma_0} f_0)(\tilde{\mathbf{y}}(\mathbf{y}))$ where these two operators are expressed in the local coordinates \mathbf{y} .

Let us recall that the metric tensor \mathbf{g} on Γ near p satisfies the estimate

$$(8.29) \quad \mathbf{g}_{ij} = \delta_{ij} + \partial_i G(\mathbf{y}) \partial_j G(\mathbf{y}) = \delta_{ij} + O(|\mathbf{y}|^2 r^{-2}), \quad |\mathbf{y}| \leq \theta r, \quad r = r(p),$$

where $\partial_i = \partial_{y_i}$. Similar estimates hold for the metric tensor \mathbf{g}_0 on the surface Γ_0 expressed in the same local coordinates. In fact we have

$$(8.30) \quad \begin{aligned} \mathbf{g}_{0,ij} &= \langle \partial_i Y_0, \partial_j Y_0 \rangle = \delta_{ij} + \partial_i G_0(\mathbf{y}) \partial_j G_0(\mathbf{y}) \\ &= \mathbf{g}_{ij} - \partial_i G(\mathbf{y}) \partial_j \mathbf{h}(\mathbf{y}) - \partial_j G(\mathbf{y}) \partial_i \mathbf{h}(\mathbf{y}) + \partial_i \mathbf{h}(\mathbf{y}) \partial_j \mathbf{h}(\mathbf{y}) \\ &= \mathbf{g}_{ij} + |\mathbf{y}| O(r^{-3-\sigma}). \end{aligned}$$

Hence if we write

$$\Delta_\Gamma = a_{ij}^0(\mathbf{y}) \partial_{ij} + b_i^0(\mathbf{y}) \partial_i, \quad \Delta_{\Gamma_0} = \bar{a}_{ij}^0(\mathbf{y}) \partial_{ij} + \bar{b}_i^0(\mathbf{y}) \partial_i,$$

then we now find for $|\mathbf{y}| < 1$,

$$\Delta_\Gamma = [\bar{a}_{ij}^0(\mathbf{y}) + O(r^{-3-\sigma})] \partial_{ij} + [\bar{b}_i^0(\mathbf{y}) + O(r^{-3-\sigma})] \partial_i.$$

We compute

$$\partial_i(f_0 \circ \tilde{\mathbf{y}}) = (\partial_k f_0 \circ \tilde{\mathbf{y}}) \partial_i \tilde{\mathbf{y}}_k, \quad \partial_{ij}(f_0 \circ \tilde{\mathbf{y}}) = (\partial_{kl} f_0 \circ \tilde{\mathbf{y}}) \partial_i \tilde{\mathbf{y}}_k \partial_j \tilde{\mathbf{y}}_l + (\partial_k f_0 \circ \tilde{\mathbf{y}}) \partial_{ij} \tilde{\mathbf{y}}_k.$$

We recall that we found at $\mathbf{y} = 0$

$$\partial_i \tilde{\mathbf{y}}_k = \delta_{ik} + O(r^{-2-\sigma}), \quad \partial_{ij} \tilde{\mathbf{y}}_k = O(r^{-3-\sigma}),$$

and hence

$$\Delta_\Gamma(f_0 \circ \tilde{\mathbf{y}})(0) = \Delta_{\Gamma_0} f_0(0) + O(r^{-2-\sigma})(\partial_{ij} f_0)(0) + O(r^{-3-\sigma})(\partial_i f_0)(0),$$

so that

$$(8.31) \quad \Delta_\Gamma f(p) = \Delta_{\Gamma_0} f_0(\pi(p)) + O(r^{-2-\sigma}) [D_{\Gamma_0}^2 f_0](\pi(p)) + O(r^{-3-\sigma}) [D_{\Gamma_0} f_0](\pi(p)).$$

8.3.3. *Comparing curvatures: Conclusion of proof of Lemma 7.1.* Let us consider the second fundamental form on Γ , A_Γ , and the second fundamental form on Γ_0 , A_{Γ_0} . We observe that for a given point $p \in \Gamma$, we get that in the local coordinates \mathbf{y} (3.13), the matrix representing $A_\Gamma(p)$ in the basis Π_1, \dots, Π_8 of $T_p\Gamma$ is $A = -D_{\mathbf{y}}^2G(0)$ since $D_{\mathbf{y}}G(0) = 0$.

We consider next Γ_0 described by the coordinates $Y_0(\mathbf{y})$ near the point $q = \pi(p)$. The tangent space $T_{\pi(p)}\Gamma_0$ is spanned by the vectors

$$\tilde{\Pi}_j := \Pi_j + \partial_j G_0(0)\nu_0 = \Pi_j + O(r^{-2-\sigma}),$$

and the the normal vector to Γ_0 at the point $p + \Pi\mathbf{y} + G(\mathbf{y})\nu_0$ is given by

$$n(\mathbf{y}) = \frac{1}{\sqrt{1 + |\nabla G_0(\mathbf{y})|^2}} (-\partial_j G_0(\mathbf{y})\Pi_j + \nu_0).$$

We have that

$$\partial_j n(0) = \sum_{i=1}^8 a_{ij} \tilde{\Pi}_i$$

for certain numbers a_{ij} . By definition, the matrix of the second fundamental form of $A_{\Gamma_0}(p)$ with respect to the basis $\tilde{\Pi}_i$ corresponds to the 8×8 matrix $A_0 = [a_{ij}]$. Now,

$$\partial_j n(0) = -\frac{1}{\sqrt{1 + |\nabla G_0(0)|^2}} \partial_{ij} G_0(0)\Pi_j - \frac{\partial_{ij} G_0(0)\partial_j G_0(0)}{(1 + |\nabla G_0(0)|^2)^{\frac{3}{2}}} (-\partial_{ij} G_0(0)\Pi_j + \nu_0);$$

hence

$$\partial_j n(0) = -\partial_{ij} G_0(0)\Pi_j + O(r^{-4-\sigma}) = -\partial_{ij} G_0(0)\tilde{\Pi}_j + O(r^{-3-\sigma}),$$

and therefore

$$a_{ij} = -\partial_{ij} G_0(0) + O(r^{-3-\sigma}) = -\partial_{ij} G(0) + O(r^{-3-\sigma}).$$

In summary, the matrix representing $A_{\Gamma_0}(\pi(p))$ is

$$A_0 = -D_{\mathbf{y}}^2G(0) + O(r^{-3-\sigma}).$$

The eigenvalues of this symmetric matrix, which are of order $O(r^{-1})$, differ at most $O(r^{-3-\sigma})$ from those of $A = -D_{\mathbf{y}}^2G(0)$. As a conclusion, we get in particular that

$$(8.32) \quad |A_\Gamma(p)|^2 = |A_{\Gamma_0}(\pi(p))|^2 + O(r^{-4-\sigma}).$$

Let us consider now the operators \mathcal{J}_Γ and \mathcal{J}_{Γ_0} . According to relations (8.31) and (3.33), and using formula (7.4), we find that if $h_0(\pi(y)) = h(y)$, $y \in \Gamma$, then

$$\mathcal{J}_\Gamma[h](y) = [\mathcal{J}_{\Gamma_0}[h_0] + O(r^{-2-\sigma})D_{\Gamma_0}^2 h_0 + O(r^{-3-\sigma})D_{\Gamma_0} h_0 + O(r^{-4-\sigma})h_0](\pi(y)),$$

and the proof of the Lemma 7.1 is thus concluded. \square

Remark 8.2. The estimates obtained for the second fundamental form of Γ in comparison to that in Γ_0 makes it simple to see that for some $a > 0$

$$(8.33) \quad \frac{a}{r^2} \leq \frac{1}{\sqrt{1 + |\nabla F(p)|^2}} \leq \frac{a^{-1}}{r^2}$$

for all $r(p)$ sufficiently large, which is a special case of the estimate in [34, Th. 5, p. 679].

8.4. *The proof of Lemma 7.8.*

8.4.1. *Proof of estimate (7.35).* Denoting by k_i and k_{i0} the principal curvatures respectively on Γ and Γ_0 we get, according to the considerations above on the second fundamental forms,

$$\sum_{i=1}^8 k_i^3(p) = \sum_{i=1}^8 (k_{i0} + O(r^{-3-\sigma}))^3(\pi(p)) = \sum_{i=1}^8 k_{i0}^3(p) + O(r^{-5-\sigma}),$$

and thus estimate (7.35) in Lemma 7.8 holds.

8.4.2. *Proof of estimate (7.34).* To prove (7.34) on Γ_0 , we compute explicitly its second fundamental form. The surface Γ_0 given by the graph of $F_0 = F_0(u, v)$ can be parametrized by the map

$$(u, v, \hat{\mathbf{u}}, \hat{\mathbf{v}}) \in \mathbb{R}_+ \times \mathbb{R}_+ \times S^3 \times S^3 \mapsto (u\hat{\mathbf{u}}, v\hat{\mathbf{v}}, F_0(u, v)).$$

Let us consider an arbitrary point $p \in \Gamma_0$, $p = (u\hat{\mathbf{u}}, v\hat{\mathbf{v}}, F_0(u, v))$ and local parametrizations of S^3 given by $\mathbf{u} = \mathbf{u}(t)$, $\mathbf{v} = \mathbf{v}(s)$, $t, s \in \mathbb{R}^3$, with

$$\mathbf{u}(0) = \hat{\mathbf{u}}, \quad \mathbf{v}(0) = \hat{\mathbf{v}}, \quad \partial_{t_i}\mathbf{u}(0) = \tau_i, \quad \partial_{s_i}\mathbf{v}(0) = \sigma_i,$$

where τ_i, σ_i $i = 1, 2, 3$ are the vectors of an orthonormal basis, respectively of $T_{\hat{\mathbf{u}}}S^3$ and $T_{\hat{\mathbf{v}}}S^3$. Then we have

$$\begin{aligned} T_p\Gamma &= \text{span} \{ (\hat{\mathbf{u}}, 0, F_{0u}), (0, \hat{\mathbf{v}}, F_{0v}), (u\tau_i, 0, 0), (0, v\sigma_i, 0), i = 1, 2, 3 \} \\ &:= \text{span} \{ e_1, e_2, f_i, g_i, i = 1, 2, 3 \}, \end{aligned}$$

and

$$n(p) = \frac{(F_{0u}\hat{\mathbf{u}}, F_{0v}\hat{\mathbf{v}}, -1)}{\sqrt{1 + |\nabla F_0|^2}}.$$

A direct computation yields

$$\begin{aligned} n_u \cdot e_1 &= \frac{F_{0uu}}{\sqrt{1 + |\nabla F_0|^2}}, & n_u \cdot e_2 &= \frac{F_{0vu}}{\sqrt{1 + |\nabla F_0|^2}}, \\ n_v \cdot e_1 &= \frac{F_{0uv}}{\sqrt{1 + |\nabla F_0|^2}}, & n_v \cdot e_2 &= \frac{F_{0vv}}{\sqrt{1 + |\nabla F_0|^2}}, \\ n_u \cdot f_i &= 0 = n_u \cdot g_i, & n_v \cdot f_i &= 0 = n_v \cdot g_i. \end{aligned}$$

Likewise, we get

$$n_{t_i} = \frac{(F_{0u}\tau_i, 0, 0)}{\sqrt{1 + |\nabla F_0|^2}}, \quad n_{s_i} = \frac{(0, F_{0u}\sigma_i, 0)}{\sqrt{1 + |\nabla F_0|^2}};$$

hence

$$\begin{aligned} n_{t_i} \cdot f_i &= \frac{uF_{0u}}{\sqrt{1 + |\nabla F_0|^2}}, \quad n_{s_i} \cdot e_1 = n_{s_i} \cdot e_2 = n_{s_i} \cdot g_j = n_{s_i} \cdot f_k = 0, \\ n_{s_i} \cdot g_i &= \frac{vF_{0v}}{\sqrt{1 + |\nabla F_0|^2}}, \quad n_{t_i} \cdot e_1 = n_{t_i} \cdot e_2 = n_{t_i} \cdot f_j = n_{t_i} \cdot g_k = 0, \\ & \quad j = 1, 2, 3, \quad k \neq i. \end{aligned}$$

The matrix of the second fundamental form $A_{\Gamma_0}(p)$ relative to the basis of $T_p\Gamma_0$,

$$T_p\Gamma_0 = \text{span} \{e_1, e_2, f_1, f_2, f_3, g_1, g_2, g_3\}$$

is by definition the 8×8 matrix $A = (a_{ij})$ such that

$$\begin{aligned} n_u &= a_{11}e_1 + a_{12}e_2 + \sum_{j=3}^5 a_{1j}f_j + \sum_{j=6}^8 a_{1j}g_{j-5}, \\ n_v &= a_{21}e_1 + a_{22}e_2 + \sum_{j=3}^5 a_{2j}f_j + \sum_{j=6}^8 a_{2j}g_{j-5}, \\ n_{s_i} &= a_{2+i1}e_1 + a_{2+i2}e_2 + \sum_{j=3}^5 a_{2+ij}f_j + \sum_{j=6}^8 a_{2+ij}g_{j-5}, \quad i = 1, 2, 3, \\ n_{t_i} &= a_{5+i1}e_1 + a_{5+i2}e_2 + \sum_{j=3}^5 a_{5+ij}f_j + \sum_{j=6}^8 a_{5+ij}g_{j-5}, \quad i = 1, 2, 3. \end{aligned}$$

Using the above computations, we readily get that A is a block matrix of the form

$$A = \begin{bmatrix} A_1 & 0 & 0 \\ 0 & A_2 & 0 \\ 0 & 0 & A_3 \end{bmatrix},$$

where

$$A_1 = \frac{1}{\sqrt{1 + |\nabla F_0|^2}} \begin{bmatrix} F_{0uu} & F_{0uv} \\ F_{0vu} & F_{0vv} \end{bmatrix} \begin{bmatrix} 1 + F_{0u}^2 & 1 + F_{0u}F_{0v} \\ 1 + F_{0v}F_{0u} & 1 + F_{0v}^2 \end{bmatrix}^{-1},$$

and

$$A_2 = \frac{F_{0u}}{u\sqrt{1 + |\nabla F_0|^2}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad A_3 = \frac{F_{0v}}{v\sqrt{1 + |\nabla F_0|^2}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

The principal curvatures are the eigenvalues k_{i0} of the matrix A . Thus we find

$$\begin{aligned} k_{10} &= \lambda_1, & k_{20} &= \lambda_2, \\ k_{30} &= k_{40} = k_{50} = \frac{F_{0u}}{u\sqrt{1 + |\nabla F_0|^2}} = \mu_1, \\ k_{60} &= k_{70} = k_{80} = \frac{F_{0v}}{v\sqrt{1 + |\nabla F_0|^2}} = \mu_2, \end{aligned}$$

where $\lambda_i, i = 1, 2$ are the eigenvalues of the 2×2 block A_1 . Expressing μ_1, μ_2 , in polar coordinates,

$$\begin{aligned} \mu_1 &:= \frac{F_{0u}}{u\sqrt{1 + |\nabla F_0|^2}} = \frac{1}{r\sqrt{9g^2 + g'^2}}(3g(\theta)\cos\theta - g'(\theta)\sin\theta), \\ \mu_2 &:= \frac{F_{0v}}{v\sqrt{1 + |\nabla F_0|^2}} = \frac{1}{r\sqrt{9g^2 + g'^2}}(3g(\theta)\sin\theta + g'(\theta)\cos\theta); \end{aligned}$$

hence

$$\begin{aligned} \mu_1^3 + \mu_2^3 &= \frac{\mathcal{R}}{r^3(9g^2 + g'^2)^{\frac{3}{2}}}, \\ \mathcal{R} &:= [(3g(\theta)\cos\theta - g'(\theta)\sin\theta)^3 + (3g(\theta)\sin\theta + g'(\theta)\cos\theta)^3]. \end{aligned}$$

Now, since $g(\theta)$ vanishes at $\frac{\pi}{4}$ with $g'(\frac{\pi}{4}) > 0$, we get

$$\mathcal{R} = O(g(\theta)) + g'(\theta)^3(\cos^3\theta - \sin^3\theta) = O(g(\theta)),$$

and therefore

$$\sum_{i=3}^8 k_{i0}^3 = O(g(\theta)r^{-3}).$$

It remains to estimate $k_{10}^3 + k_{20}^3$.

We know that, globally, all principal curvatures are $O(r^{-1})$. Let us consider the case $\theta \in (\frac{\pi}{4}, \frac{3\pi}{5})$. Since second derivatives of F in (u, v) are of order $O(r)$, we then get that

$$\begin{aligned} A_1 &= O(r^{-5}) \begin{bmatrix} r^{-4} + 9g^2\cos^2\theta + g'^2\sin^2\theta & r^{-4} - 3g'g\sin\theta\cos\theta \\ r^{-4} - 3g'g\sin\theta\cos\theta & r^{-4} + 9g^2\sin^2\theta + g'^2\cos^2\theta \end{bmatrix}^{-1} \\ &= O(r^{-5}) \begin{bmatrix} 9g^2\cos^2\theta + g'^2\sin^2\theta & -3g'g\sin\theta\cos\theta \\ -3g'g\sin\theta\cos\theta & 9g^2\sin^2\theta + g'^2\cos^2\theta \end{bmatrix}^{-1} + O(r^{-9}). \end{aligned}$$

The latter inverse is uniformly bounded in the region considered. As a conclusion we get that the eigenvalues of this matrix are of size at most $O(r^{-5})$ near $\theta = \frac{\pi}{4}$, while for θ away from $\frac{\pi}{4}$ the eigenvalues are of the size $O(r^{-1})$. Globally we then get

$$\sum_{i=1}^8 k_{i0}^3 = O(g(\theta)r^{-3}) + O(r^{-5}),$$

and the proof is concluded. □

**9. Asymptotic behavior of the BDG graph:
Proofs of Lemma 2.1 and Theorem 2**

9.1. *Equation for g: Proof of Lemma 2.1.* We want to solve the problem

$$(9.1) \quad \frac{21 \sin^3(2\theta)g}{\sqrt{9g^2 + g_\theta^2}} + \left(\frac{\sin^3(2\theta)g_\theta}{\sqrt{9g^2 + g_\theta^2}} \right)_\theta = 0, \quad \theta \in \left(\frac{\pi}{4}, \frac{\pi}{2} \right),$$

with the boundary conditions

$$(9.2) \quad g\left(\frac{\pi}{4}\right) = 0, \quad g_\theta\left(\frac{\pi}{2}\right) = 0.$$

Let us observe that if $g(\theta)$ is a solution of (9.1), then so is $Cg(\theta)$ for any constant C . The following lemma proves the existence of solutions to (9.1).

LEMMA 9.1. *Problem (9.1) has a solution such that*

$$(9.3) \quad g(\theta) \geq 0, \quad g_{\theta\theta}(\theta) \leq 0, \quad g_\theta(\theta) \geq 0,$$

and the last inequality is strict for $\theta \in [\frac{\pi}{4}, \frac{\pi}{2})$.

Proof. If g is a solution to (9.1), then the function

$$\psi(\theta) = \frac{g_\theta(\theta)}{g(\theta)}, \quad g(\theta) \neq 0$$

satisfies the following equation:

$$(9.4) \quad 9\psi' + (9 + \psi^2)[21 + 6 \cot(2\theta)\psi] = 0.$$

Our strategy is to solve (9.4) first and then find the function g . To this end we will look for a solution of (9.4) in the interval $I = (\pi/4, \pi/2)$ with

$$(9.5) \quad \psi(\pi/2) = 0.$$

In order to define the function g we also need ψ to be defined and positive in the whole interval $(\frac{\pi}{4}, \frac{\pi}{2}]$ and $\lim_{\theta \rightarrow \frac{\pi}{4}^+} \psi(\theta) = +\infty$. Let $(\theta^*, \frac{\pi}{2}]$, $\frac{\pi}{4} \leq \theta^*$, be the maximal interval for which the solution of (9.4) exists.

We set $\psi_+(\theta) = -11 \tan(2\theta)$. Then we have

$$9\psi'_+ + (9 + \psi_+^2)[21 + 6 \cot(2\theta)\psi_+] < 0, \quad \theta \in \left(\frac{\pi}{4}, \frac{\pi}{2} \right],$$

$$\psi_+\left(\frac{\pi}{2}\right) = 0 = \psi\left(\frac{\pi}{2}\right), \quad \psi'_+\left(\frac{\pi}{2}\right) = -22 < -21 = \psi'\left(\frac{\pi}{2}\right).$$

Substituting $\psi_-(\theta) = -2 \tan(2\theta)$ for ψ in (9.4), we get

$$(9.6) \quad 9\psi'_- + (9 + \psi_-^2)[21 + 6 \cot(2\theta)\psi_-] > 0.$$

We have $\psi(\pi/2) = \psi_-(\pi/2) = 0$ and, from (9.4),

$$\psi'(\pi/2) = -21 < -4 = \psi'_-(\pi/2).$$

From this we get that the maximal solution of (9.4) satisfies

$$(9.7) \quad \psi_+(\theta) = -11 \tan(2\theta) > \psi(\theta) \geq \psi_-(\theta) = -2 \tan(2\theta) \geq 0, \quad \theta \in (\theta^*, \pi/2)$$

and that $\theta^* = \frac{\pi}{4}$. Let us now define

$$(9.8) \quad g(\theta) = \exp \left\{ - \int_{\theta}^{\pi/2} \psi(t) dt \right\},$$

where ψ is the unique solution of (9.4)–(9.5). Clearly we have $g_{\theta}(\pi/2) = 0$ and from (9.7) it follows that $g(\pi/4) = 0$. Thus g defined in (9.8) is a solution of (9.1)–(9.2).

We have $g_{\theta} > 0$ in $(\frac{\pi}{4}, \frac{\pi}{2})$, since $g_{\theta} = g\psi$. To show that $g_{\theta}(\frac{\pi}{4}) > 0$ we will improve the upper bound on ψ . Let us define

$$\psi_1 = -2 \tan(2\theta) + \tilde{\psi}, \quad \text{where } \tilde{\psi} = A(-\tan(2\theta))^{\eta},$$

and $\frac{2}{3} < \eta < 1, A > 1$, are to be chosen. Direct calculation gives

$$\begin{aligned} 9\psi_1' + (9 + \psi_1^2)[21 + 6\psi_1 \cot(2\theta)] &= 9\tilde{\psi}' \cos^2(2\theta) + 45 \cos^2(2\theta) \\ &+ 6\tilde{\psi} \cot(2\theta)[4 + 5 \cos^2(2\theta)] + 36\tilde{\psi} \sin(2\theta)(-\cos(2\theta)) \\ &+ 9\tilde{\psi}^2 \cos^2(2\theta) + 6\tilde{\psi}^2 \cot(2\theta)[4 \sin(2\theta)(-\cos(2\theta)) + \tilde{\psi} \cos^2(2\theta)]. \end{aligned}$$

Using the definition of $\tilde{\psi}$, after some calculation, we find that the last expression is negative for $\theta \in (\frac{\pi}{4}, \frac{\pi}{2})$ when

$$\begin{aligned} 0 > -18A\eta + 45(-\tan(2\theta))^{1-\eta} \cos^2(2\theta) - 6A[4 + 5 \cos^2(2\theta)] + 36A \sin^2(2\theta) \\ - 15A^2(-\tan(2\theta))^{1+\eta} \cos^2(2\theta) - 6A^3(-\tan(2\theta))^{1-2\eta} \sin(2\theta)(-\cos(2\theta)), \end{aligned}$$

which can be achieved if $\frac{2}{3} < \eta < 1$ and A is chosen sufficiently large. Since $\eta < 1$, it follows that

$$\psi(\theta) \leq \psi_1(\theta), \quad \theta \in \left(\frac{\pi}{4}, \frac{\pi}{2} \right);$$

hence, for certain constant $C > 0$,

$$(9.9) \quad -C \cos(2\theta) \leq g(\theta) \leq -\cos(2\theta), \quad \theta \in \left[\frac{\pi}{4}, \frac{\pi}{2} \right].$$

In fact the inequalities in (9.3) are strict for $\theta \in (\frac{\pi}{4}, \frac{\pi}{2})$. It follows in addition that

$$g_{\theta}(\theta) \geq C \sin(2\theta), \quad \theta \in \left[\frac{\pi}{4}, \frac{\pi}{2} \right].$$

This shows, in particular, that $g_{\theta} > 0$ in $[\frac{\pi}{4}, \frac{\pi}{2})$. The remaining estimate for $g_{\theta\theta}$ follows from the second order equation for g . □

Given function g as above let us define

$$(9.10) \quad \cos \phi = \frac{3g}{\sqrt{9g^2 + g_\theta^2}}, \quad \sin \phi = \frac{g\theta}{\sqrt{9g^2 + g_\theta^2}}.$$

We see from Lemma 9.1 that ϕ satisfies

$$(9.11) \quad \phi' + 7 + 6 \cot(2\theta) \tan \phi = 0, \quad \phi\left(\frac{\pi}{4}\right) = \frac{\pi}{2}, \quad \phi\left(\frac{\pi}{2}\right) = 0.$$

We need the following lemma.

LEMMA 9.2. *It holds that*

$$(9.12) \quad \phi'\left(\frac{\pi}{4}\right) = -3, \quad \phi'\left(\frac{\pi}{2}\right) = -\frac{7}{4}, \quad \phi'(\theta) > -3 \text{ for } \theta \in \left(\frac{\pi}{4}, \frac{\pi}{2}\right).$$

Proof. To prove the first identity we observe that $\tan \phi = \frac{1}{3}\psi$, which after differentiation yields

$$(9.13) \quad \phi' = \frac{1}{3}\psi' \cos^2 \phi = -\frac{1}{3}[21 + 6 \cot(2\theta)\psi] \geq -3$$

since $\psi(\theta) \geq -2 \tan(2\theta)$. Now considering (9.11) we see that when $\theta \rightarrow \pi/4^+$, we can have $\phi'(\pi/4^+) = -3$ or $\phi'(\pi/4^+) = -4$. From (9.13) we get the required formula.

The second identity follows from simple analysis near $\theta = \frac{\pi}{2}$.

To prove the last estimate, we suppose that there exists a point $\theta_1 \in (\frac{\pi}{4}, \frac{\pi}{2})$ such that $\phi'(\theta_1) = -3$. We claim that $\phi''(\theta_1) < 0$. This gives a contradiction. (We may take θ_1 to be the point closest to $\frac{\pi}{2}$. Then necessarily $\phi''(\theta_1) \geq 0$.) In fact, from (9.11), we deduce that

$$2 \sin(2\theta_1) \cos \phi + 3 \cos(2\theta_1) \sin \phi = 0,$$

which is equivalent to

$$(9.14) \quad 5 \sin(2\theta_1 + \phi) = \sin(2\theta_1 - \phi).$$

Note that $2\theta - \phi \in (0, \pi)$ and hence $0 < 2\theta - \phi < 2\theta + \phi < \pi$. Now we compute

$$\begin{aligned} \phi''(\theta_1) &= \frac{6}{\sin^2 \theta_1 \cos^2 \phi} \left(\sin 2\phi - \frac{1}{2} \sin 4\theta_1 \phi' \right) \\ &= \frac{6}{\sin^2 \theta_1 \cos^2 \phi} \sin(2\theta_1 - \phi) \cos(2\theta_1) \cos \phi < 0, \end{aligned}$$

which completes the proof. □

9.2. *A new system of coordinates.* One of the key results of our paper is a refinement of the results in [5] which amounts to finding more precise information about the asymptotic behavior of the minimal graph of Bombieri, De Giorgi and Giusti. This is the purpose of introducing function F_0 . It is easy to see that far enough from the origin F_0 is a subsolution of the mean curvature equation and therefore, at least away from the origin, the BDG

minimal should lie above the graph of F_0 . Finding a supersolution which asymptotically behaves like F_0 is, however, a different story. We observe that the supersolution found in [5] asymptotically resembles something like $\sim Mr^3$ with $M \gg 1$ and therefore lies above a multiple of F_0 . On the other hand our approach requires a more accurate estimate $F \sim F_0$ away from the origin.

For this reason we next introduce new coordinates (s, t) in the sector T which depend on the function $F_0 = r^3g(\theta)$. These coordinates, which are given explicitly in (9.17)–(9.18), correspond to “geographical” orthogonal coordinates for the graph of F_0 . The coordinate t is simply its height and s measures a weighted length along the level sets. The weight takes into account the actual higher dimensional character of the coordinate s (its two-dimensional analog would simply be arclength on the level curves of F_0). Expressing the mean curvature operator in these coordinates leads to formula (9.24). Its main feature is that the degeneracy of the mean curvature operator for a function close to F_0 is removed. This expression is a useful tool for separating terms of the mean curvature operator with distinct features when we examine suitable candidates for a supersolution of the minimal surface equation.

LEMMA 9.3. *There exists a diffeomorphism $\Phi : Q \rightarrow T$, where $Q = \{(t, s) \mid t > 0, s > 0\}$ such that $\Phi(t, s) = \mathbf{u}(t, s) = (u(t, s), v(t, s))$ and \mathbf{u} satisfies the coupled system of differential equations*

$$(9.15) \quad \frac{\partial \mathbf{u}}{\partial t} = \frac{\nabla F_0}{|\nabla F_0|^2}, \quad \frac{\partial \mathbf{u}}{\partial s} = \frac{1}{(uv)^3} \frac{\nabla F_0^\perp}{|\nabla F_0|},$$

where we denote

$$\nabla F = (F_u, F_v), \quad \nabla F^\perp = (F_v, -F_u).$$

Moreover, Φ maps $(t = 0, s)$ onto the line $u = v$ and $(t, s = 0)$ onto $(u = 0, v)$.

Proof. Introducing polar coordinates

$$u = r \cos \theta, \quad v = r \sin \theta,$$

and using (9.15), we find

$$(9.16) \quad \begin{cases} \frac{\partial r}{\partial t} = \frac{F_{0,r}}{|\nabla F_0|^2} = \frac{3g}{r^2(9g^2+g_\theta^2)} \\ \frac{\partial \theta}{\partial t} = \frac{F_{0,\theta}}{|\nabla F_0|^2} = \frac{g_\theta}{r^3(9g^2+g_\theta^2)} \end{cases}, \quad \begin{cases} \frac{\partial r}{\partial s} = \frac{8F_{0\theta}}{r^7 \sin^3(2\theta)|\nabla F_0|} = \frac{8g_\theta}{r^6 \sin^3(2\theta)\sqrt{9g^2+g_\theta^2}} \\ \frac{\partial \theta}{\partial s} = \frac{-8F_{0r}}{r^7 \sin^3(2\theta)|\nabla F_0|} = -\frac{24g}{r^7 \sin^3(2\theta)\sqrt{9g^2+g_\theta^2}} \end{cases}.$$

Using the formal relations

$$\begin{bmatrix} t_r & t_\theta \\ s_r & s_\theta \end{bmatrix} \begin{bmatrix} r_t & r_s \\ \theta_t & \theta_s \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

we arrive in particular at the equations for s :

$$3gs_r + \frac{g\theta}{r}s_\theta = 0,$$

$$\frac{8g_\theta s_r}{r^6 \sin^3 2\theta \sqrt{9g^2 + g_\theta^2}} - \frac{24gs_\theta}{r^7 \sin^3 2\theta \sqrt{9g^2 + g_\theta^2}} = 1,$$

or

$$\begin{cases} \frac{\partial s}{\partial \theta} = -\frac{3r^7 \sin^3 2\theta g}{8\sqrt{9g^2 + g_\theta^2}} \\ \frac{\partial s}{\partial r} = \frac{r^6 \sin^3 2\theta g_\theta}{8\sqrt{9g^2 + g_\theta^2}} \end{cases},$$

which are satisfied by the function

$$(9.17) \quad s = \frac{r^7 \sin^3(2\theta)g_\theta}{56\sqrt{9g^2 + g_\theta^2}}$$

because of the equation satisfied by g . Similarly, for t , we obtain the solution

$$(9.18) \quad t = r^3 g(\theta).$$

Using the properties of the function g , we can directly check that function given by formulas (9.17)–(9.18) is a diffeomorphism with the required properties. \square

For future reference let us keep in mind that setting $\sin \phi$, $\cos \phi$ as in formula 9.10, we simply find

$$(9.19) \quad \frac{\partial r}{\partial s} = \frac{r}{7s} \sin^2 \phi, \quad \frac{\partial \theta}{\partial s} = -\frac{1}{14s} \sin(2\phi),$$

and

$$(9.20) \quad \frac{\partial r}{\partial t} = \frac{r}{3t} \cos^2 \phi, \quad \frac{\partial \theta}{\partial t} = \frac{1}{6t} \sin(2\phi).$$

Our next goal is to express the mean curvature operator in terms of the variables (t, s) . Denoting by \mathbf{u}' the matrix $(\mathbf{u}_t, \mathbf{u}_s)$, the minimal surface equation is transformed to

$$(9.21) \quad (uv)^{-3} \frac{1}{\sqrt{\det \mathbf{u}'\mathbf{u}'^T}} \nabla_{t,s} \cdot \left(\frac{(uv)^3 \sqrt{\det \mathbf{u}'\mathbf{u}'^T}}{\sqrt{1 + |\nabla F|^2}} (\mathbf{u}'\mathbf{u}'^T)^{-1} \nabla_{t,s} F \right) = 0.$$

From Lemma 9.3 we find

$$(9.22) \quad \langle \mathbf{u}_t, \mathbf{u}_t \rangle = \frac{1}{|\nabla F_0|^2}, \quad \langle \mathbf{u}_t, \mathbf{u}_s \rangle = 0, \quad \langle \mathbf{u}_s, \mathbf{u}_s \rangle = \frac{1}{(uv)^6} := \rho^2;$$

hence we compute

$$(9.23) \quad \det \mathbf{u}' = \frac{-\rho}{|\nabla F_0|}, \quad (\mathbf{u}'\mathbf{u}'^T)^{-1} = \begin{pmatrix} |\nabla F_0|^2 & 0 \\ 0 & \rho^{-2} \end{pmatrix}.$$

Then equation (9.21) becomes

$$(9.24) \quad |\nabla F_0| \partial_t \left(\frac{|\nabla F_0| \partial_t F}{\sqrt{1 + |\nabla F|^2}} \right) + |\nabla F_0| \partial_s \left(\frac{\rho^{-2} \partial_s F}{|\nabla F_0| \sqrt{1 + |\nabla F|^2}} \right) = 0.$$

Let us observe that

$$\begin{aligned} \nabla F &= \left\langle \nabla F, \frac{\nabla F_0}{|\nabla F_0|} \right\rangle \frac{\nabla F_0}{|\nabla F_0|} + \left\langle \nabla F, \frac{\nabla F_0^\perp}{|\nabla F_0|} \right\rangle \frac{\nabla F_0^\perp}{|\nabla F_0|} \\ &= F_t \nabla F_0 + \rho^{-1} F_s \frac{\nabla F_0^\perp}{|\nabla F_0|}. \end{aligned}$$

From this we have

$$\begin{aligned} 1 + |\nabla F|^2 &= 1 + |\nabla F_0|^2 \left(F_t^2 + \frac{\rho^{-2} F_s^2}{|\nabla F_0|^2} \right) \\ &= |\nabla F_0|^2 \left(\frac{1}{|\nabla F_0|^2} + F_t^2 + \frac{\rho^{-2} F_s^2}{|\nabla F_0|^2} \right). \end{aligned}$$

Denoting by $Q(\nabla_{t,s} F)$ the function

$$Q(\nabla_{t,s} F) = \frac{1}{|\nabla F_0|^2} + F_t^2 + \frac{\rho^{-2} F_s^2}{|\nabla F_0|^2},$$

we see the mean curvature equation is equivalent to

$$H[F] = \frac{|\nabla F_0|}{Q^{3/2}(\nabla_{t,s} F)} G[F] = 0,$$

where

$$(9.25) \quad \begin{aligned} G[F] &= Q(\nabla_{t,s} F) F_{tt} - \frac{1}{2} \partial_t Q(\nabla_{t,s} F) F_t \\ &\quad + Q(\nabla_{t,s} F) \partial_s \left(\frac{\rho^{-2} F_s}{|\nabla F_0|^2} \right) - \frac{1}{2} \partial_s Q(\nabla_{t,s} F) \frac{\rho^{-2} F_s}{|\nabla F_0|^2}. \end{aligned}$$

Now we derive the mean curvature operator for functions of the form

$$F = F_0 + A\varphi(t, s) = t + A\varphi(t, s),$$

where A is a real number. Our goal is to write the resulting equation in the form of a polynomial in A . In general we assume that for $r \gg 1$,

$$(9.26) \quad |\varphi_t| + \frac{|\varphi_s \rho^{-1}|}{|\nabla F_0|} = o(1).$$

We compute

$$\begin{aligned} \nabla F &= \nabla F_0 + \left\langle \nabla \varphi, \frac{\nabla F_0}{|\nabla F_0|} \right\rangle \frac{\nabla F_0}{|\nabla F_0|} + \left\langle \nabla \varphi, \frac{\nabla F_0^\perp}{|\nabla F_0|} \right\rangle \frac{\nabla F_0^\perp}{|\nabla F_0|} \\ &= \nabla F_0 + \varphi_t \nabla F_0 + \rho^{-1} \varphi_s \frac{\nabla F_0^\perp}{|\nabla F_0|}. \end{aligned}$$

Then we have

$$\begin{aligned} 1 + |\nabla F|^2 &= 1 + |\nabla F_0|^2 \left[(1 + A\varphi_t)^2 + A^2 \frac{\rho^{-2}\phi_s^2}{|\nabla F_0|^2} \right] \\ &= |\nabla F_0|^2 \left(1 + \frac{1}{|\nabla F_0|^2} + 2A\varphi_t + A^2 R_1 \right), \end{aligned}$$

where we denote

$$R_1 = \varphi_t^2 + \frac{\rho^{-2}\varphi_s^2}{|\nabla F_0|^2}.$$

It is convenient to introduce

$$R = \left(1 + \frac{1}{|\nabla F_0|^2} + 2A\varphi_t + A^2 R_1 \right).$$

With this notation, we have

$$\begin{aligned} (9.27) \quad & |\nabla F_0|^{-1} R^{3/2} H[F_0 + A\varphi] \\ &= \left[AR\partial_t^2\varphi - \frac{1}{2}(1 + A\partial_t\varphi)\partial_t R + AR\partial_s \left(\frac{\rho^{-2}\partial_s\varphi}{|\nabla F_0|^2} \right) - \frac{1}{2}A \frac{\rho^{-2}\partial_s\varphi}{|\nabla F_0|^2} \partial_s R \right] \\ &= -\frac{1}{2}\partial_t|\nabla F_0|^{-2} + A \left[|\nabla F_0|^{-2}\partial_t^2\varphi - \frac{1}{2}\partial_t|\nabla F_0|^{-2}\partial_t\varphi \right. \\ &\quad \left. + \partial_s \left(\frac{\rho^{-2}\partial_s\varphi}{|\nabla F_0|^2} \right) (1 + |\nabla F_0|^{-2}) - \frac{1}{2} \left(\frac{\rho^{-2}\partial_s\varphi}{|\nabla F_0|^2} \right) \partial_s |\nabla F_0|^{-2} \right] \\ &\quad + A^2 \left[\partial_t\varphi\partial_t^2\varphi - \frac{1}{2}\partial_t R_1 + 2\partial_t\varphi\partial_s \left(\frac{\rho^{-2}\partial_s\varphi}{|\nabla F_0|^2} \right) - \left(\frac{\rho^{-2}\partial_s\varphi}{|\nabla F_0|^2} \right) \partial_{ts}^2\varphi \right] \\ &\quad + A^3 \left[R_1\partial_t^2\varphi - \frac{1}{2}\partial_t\varphi\partial_t R_1 + R_1\partial_s \left(\frac{\rho^{-2}\partial_s\varphi}{|\nabla F_0|^2} \right) - \frac{1}{2} \left(\frac{\rho^{-2}\partial_s\varphi}{|\nabla F_0|^2} \right) \partial_s R_1 \right]. \end{aligned}$$

In the sequel we will refer to the consecutive term in (9.27) as the A^0 , A^1 , A^2 and A^3 terms respectively. For future reference we observe that the A^0 term can be written as

$$(9.28) \quad -\frac{1}{2}\partial_t|\nabla F_0|^{-2} = |\nabla F_0|^{-1}(1 + |\nabla F_0|^{-2})^{3/2}H[F_0]$$

and the A^1 term can be written as

$$\begin{aligned} (9.29) \quad & \left[\cdot \right] = |\nabla F_0|^{-1}\tilde{L}_0[\varphi] \\ & - \frac{3}{2}\partial_t|\nabla F_0|^{-2}\partial_t\varphi + |\nabla F_0|^{-2}\partial_s \left(\frac{\rho^{-2}\partial_s\varphi}{|\nabla F_0|^2} \right) - \frac{1}{2} \left(\frac{\rho^{-2}\partial_s\varphi}{|\nabla F_0|^2} \right) \partial_s |\nabla F_0|^{-2}, \end{aligned}$$

where

$$(9.30) \quad \tilde{L}_0[\varphi] = |\nabla F_0| \left[\partial_t \left(\frac{\partial_t\varphi}{|\nabla F_0|^2} \right) + \partial_s \left(\frac{\rho^{-2}\partial_s\varphi}{|\nabla F_0|^2} \right) \right].$$

9.3. *Proof of Theorem 2.* Taking the existence result in [5] as the point of departure, we find the asymptotic behavior of the minimal graph by proving Theorem 2. Our approach, which is based on a comparison principle, relies on a refinement of the supersolution/subsolution in [5]. We need the following comparison principle.

LEMMA 9.4. *Let Ω be a smooth and open bounded domain. If F_1 and F_2 satisfies*

$$(9.31) \quad H[F_1] \leq H[F_2] \text{ in } \Omega, \quad F_1 \geq F_2 \text{ on } \partial\Omega,$$

then

$$(9.32) \quad F_1 \geq F_2 \text{ in } \Omega.$$

Proof. The proof is simple since

$$H[F_1] - H[F_2] = \sum_{i,j} a_{ij} \frac{\partial^2}{\partial x_i \partial x_j} (F_1 - F_2)$$

where the matrix (a_{ij}) is uniformly elliptic in Ω . By the usual maximum principle, we obtain the desired result. \square

Let us observe that from (9.9), we have

$$(9.33) \quad \min \left(\frac{-\cos(2\theta)}{g(\theta)} \right) \geq 1, \quad \theta \in \left(\frac{\pi}{4}, \frac{\pi}{2} \right).$$

Thus for $F_0 = r^3 g(\theta)$, it holds that

$$(9.34) \quad F_0 = r^3 g(\theta) \leq (v^2 - u^2)(v^2 + u^2)^{\frac{1}{2}}.$$

We will now construct a subsolution to the mean curvature equation.

LEMMA 9.5. *Let $H[F]$ denote the mean curvature operator. We have*

$$(9.35) \quad H[F_0] \geq 0.$$

It holds as well that

$$(9.36) \quad H[F_0] = O(r^{-5}).$$

Proof. Since $H[F]$ and $G[F]$ (defined in (9.25)) differ only by a nonnegative factor, it suffices to show that

$$(9.37) \quad G[F_0] \geq 0.$$

In fact, let $F = F_0 = t$. We then have

$$\begin{aligned} G[F_0] &= -\frac{1}{2} \partial_t Q(\nabla_{t,s} F_0) \\ &= -\frac{1}{2} \partial_t \left(\frac{1}{|\nabla F_0|^2} \right), \end{aligned}$$

where

$$\frac{1}{|\nabla F_0|^2} = \frac{1}{r^4(9g^2 + g_\theta^2)} = \frac{r^2 \cos^2 \phi}{9t^2}.$$

By formula (9.20), we have

$$\begin{aligned} (9.38) \quad -\partial_t \left(\frac{r^2 \cos^2 \phi}{9t^2} \right) &= \frac{r^2}{9t^3} \left[2 \cos^2 \phi - \frac{2tr_t \cos^2 \phi}{r} + t\phi'\theta_t \sin(2\phi) \right] \\ &= \frac{2r^2 \cos^2 \phi}{9t^3} \left[\frac{2}{3} \cos^2 \phi + \frac{1}{3} \sin^2 \phi(\phi' + 3) \right] \\ &\geq 0, \end{aligned}$$

where we have used the fact that $\phi'(\theta) \geq -3$. Estimate (9.36) follows easily from the expansions; see also (2.7). This ends the proof. \square

By the standard theory of the mean curvature equation for each fixed $R > 0$, there exists a unique solution to the following problem:

$$(9.39) \quad \frac{1}{(uv)^3} \nabla \cdot \left(\frac{(uv)^3 \nabla F}{\sqrt{1 + |\nabla F|^2}} \right) = 0 \text{ in } \Gamma_R, \quad F = F_0 \text{ on } \partial\Gamma_R,$$

where $\Gamma_R = B_R \cap T$, $T = \{u, v > 0, u < v\}$. Let us denote the solution to (9.39) by F_R .

Using (9.34), the comparison principle and the supersolution found in [5],

$$(9.40) \quad F_0 \leq F_R \leq \mathcal{H} \left((v^2 - u^2) + (v^2 - u^2)(u^2 + v^2)^{1/2} (1 + \mathbf{A}(|\cos(2\theta)|)^{\lambda-1}) \right),$$

where

$$\mathcal{H}(t) := \int_0^t \exp \left(\mathbf{B} \int_{|w|}^\infty \frac{dt}{t^{2-\lambda}(1 + t^{2\alpha\lambda-2\alpha})} \right) dw,$$

$\lambda > 1$ is a positive fixed number, $\alpha = \frac{3}{2}$, and \mathbf{A}, \mathbf{B} are sufficiently large positive constants. This inequality, combined with standard elliptic estimates, implies that as $R \rightarrow +\infty$, $F_R \rightarrow F$ which is a solution to the mean curvature equation $H[F] = 0$ with

$$(9.41) \quad F_0 \leq F \leq \mathcal{H} \left((v^2 - u^2) + (v^2 - u^2)(u^2 + v^2)^{1/2} (1 + \mathbf{A}(|\cos(2\theta)|)^{\lambda-1}) \right).$$

Next we need the following key lemma.

LEMMA 9.6. *There exists $\sigma_0 \in (0, 1)$ such that for each $\sigma \in (0, \sigma_0)$, there exists $a_0 > 1$ such that for each sufficiently large $\tilde{A} \geq 1$, we have*

$$(9.42) \quad H \left[F_0 + \frac{\tilde{A}F_0}{r^\sigma} \right] \leq 0 \text{ for } r > a_0.$$

Moreover, under the same assumptions for each sufficiently large $A \geq 1$, we have

$$(9.43) \quad H \left[F_0 + \frac{A}{r^\sigma} \right] \leq 0 \text{ for } r > a_0 A^{\frac{1}{3+\sigma}}.$$

Proof. We will consider (9.42) first. We will use formula (9.27) to write $H[F_0 + \frac{\tilde{A}F_0}{r^\sigma}]$ multiplied by a nonnegative factor as a polynomial in \tilde{A} . Explicit computation (9.27) yields

$$|\nabla F_0|^{-1}R^{3/2}H\left[F_0 + \frac{\tilde{A}F_0}{r^\sigma}\right] = H_0 + \tilde{A}H_1 + \tilde{A}^2H_2 + \tilde{A}^3H_3,$$

where

(9.44)

$$H_0 = |\nabla F_0|^{-1}(1 + |\nabla F_0|^{-2})^{3/2}H[F_0] = \frac{r^2 \cos^2 \phi}{9t^3} \left[\frac{2}{3} \cos^2 \phi + \frac{1}{3} \sin^2 \phi(\phi' + 3) \right],$$

$$H_1 = \frac{-7\sigma \cos^2 \phi}{9tr^\sigma} (7 + (2\phi' - \sigma) \sin^2 \phi) + \frac{\cos^2 \phi}{tr^\sigma} O(r^{-4}).$$

Below we will show in addition that

(9.45)

$$H_2 = \frac{\cos^2 \phi}{tr^\sigma} O(r^{-\sigma}) \leq 0,$$

$$H_3 = \frac{\cos^2 \phi}{tr^\sigma} O(r^{-2\sigma}) \leq 0.$$

We assume for the moment the validity of these estimates. Let us observe that the first term in (9.44) is bounded by

(9.46)

$$H_0 \leq c_1 \frac{r^2 \cos^4 \phi}{t^3} \leq c_1 \frac{\cos^2 \phi}{tr^4}.$$

Estimate (9.46) follows from (9.44) and the fact that $\phi(\pi/4) = \pi/2$, $\phi'((\pi/4)^+) = -3$, $\phi''((\pi/4)^+) = 0$. Summarizing, we have

(9.47)

$$H[F_0 + \frac{\tilde{A}F_0}{r^\sigma}] \leq H_0 + \tilde{A}H_1$$

$$\leq \frac{-7\tilde{A}\sigma \cos^2 \phi}{9tr^\sigma} (7 + (2\phi' - \sigma) \sin^2 \phi) + \frac{\cos^2 \phi}{tr^\sigma} O(r^{-4+\sigma})$$

$$\leq 0.$$

To prove (9.43) we use a similar argument. Writing $H[F_0 + \frac{A}{r^\sigma}]$ as a polynomial in A , we get that the A^0 term is equal to H_0 in (9.44) and

(9.48)

$$H_1 = \frac{-7\sigma \cos^2 \phi}{9g^2(\theta)r^{6+\sigma}} (7 + (2\phi' - \sigma) \sin^2 \phi) + \frac{1}{r^{6+\sigma}} O(r^{-1}).$$

The other terms satisfy

$$H_2 = \frac{1}{r^{6+\sigma}} O(r^{-3-\sigma}), \quad (A^2 \text{ term}),$$

$$H_3 = \frac{1}{r^{6+\sigma}} O(r^{-6-2\sigma}), \quad (A^3 \text{ term}).$$

Since $H_0 = O(r^{-7})$, the lemma follows by combing the above estimates.

It remains to establish inequalities (9.45). We will collect first some terms appearing in the expansion formula (9.27). We have $\varphi(t, s) = tr^{-\sigma}$ and

$$(9.49) \quad \begin{aligned} \partial_t \varphi &= \frac{1}{r^\sigma} \left(1 - \frac{\sigma}{3} \cos^2 \phi\right), \quad \partial_s \varphi = -\frac{\sigma t \sin^2 \phi}{7r^\sigma s}, \\ \partial_t^2 \varphi &= \frac{C\sigma}{9r^\sigma t} \cos^2 \phi [\sigma \cos^2 \phi - 3 + 2\phi' \sin^2 \phi], \\ \partial_{ts} \varphi &= -\frac{\sigma \sin^2 \phi}{7r^\sigma s} \left(1 + \frac{2\phi' \cos^2 \phi}{3} - \frac{\sigma \cos^2 \phi}{3}\right). \end{aligned}$$

We also have

$$(9.50) \quad \begin{aligned} \frac{\rho^{-2} \varphi_s}{|\nabla F_0|^2} &= -\frac{7s\sigma \cos^2 \phi}{9tr^\sigma}, \\ \partial_s \left(\frac{\rho^{-2} \varphi_s}{|\nabla F_0|^2} \right) &= -\frac{7\sigma \cos^2 \phi}{9tr^\sigma} \left[1 + \frac{\sin^2 \phi}{7} (2\phi' - \sigma)\right], \\ \frac{\rho^{-2} \varphi_s^2}{|\nabla F_0|^2} &= \frac{\sigma^2 \sin^2 \phi \cos^2 \phi}{9r^{2\sigma}}, \\ \partial_s \left(\frac{\rho^{-2} \varphi_s^2}{|\nabla F_0|^2} \right) &= -\frac{2\sigma^2 \sin^2 \phi \cos^2 \phi}{63r^{2\sigma} s} (\sigma \sin^2 \phi + \phi' \cos(2\phi)), \\ \partial_t \left(\frac{\rho^{-2} \varphi_s^2}{|\nabla F_0|^2} \right) &= \frac{2\sigma^2 \sin^2 \phi \cos^2 \phi}{27tr^{2\sigma}} [-\sigma \cos^2 \phi + \phi' \cos(2\phi)]. \end{aligned}$$

Using formula (9.27), we get

$$(9.51) \quad \begin{aligned} H_2 &= -\frac{1}{2} \partial_t \left(\frac{\rho^{-2} \varphi_s^2}{|\nabla F_0|^2} \right) + 2\partial_t \varphi \partial_s \left(\frac{\rho^{-2} \varphi_s}{|\nabla F_0|^2} \right) - \left(\frac{\rho^{-2} \varphi_s}{|\nabla F_0|^2} \right) \partial_{ts} \varphi, \\ H_3 &= \left(\frac{\rho^{-2} \varphi_s^2}{|\nabla F_0|^2} \right) \partial_t^2 \varphi - \frac{1}{2} \partial_t \varphi \partial_t \left(\frac{\rho^{-2} \varphi_s^2}{|\nabla F_0|^2} \right) + \left[(\partial_t \varphi)^2 + \left(\frac{\rho^{-2} \varphi_s^2}{|\nabla F_0|^2} \right) \right] \partial_s \left(\frac{\rho^{-2} \varphi_s}{|\nabla F_0|^2} \right) \\ &\quad - \partial_t \varphi \partial_{ts} \varphi \left(\frac{\rho^{-2} \varphi_s}{|\nabla F_0|^2} \right) - \frac{1}{2} \left(\frac{\rho^{-2} \varphi_s}{|\nabla F_0|^2} \right) \partial_s \left(\frac{\rho^{-2} \varphi_s^2}{|\nabla F_0|^2} \right). \end{aligned}$$

From (9.49)–(9.51), by direct calculation we get

$$(9.52) \quad \begin{aligned} H_2 &= \frac{\sigma^2 \sin^2 \phi \cos^2 \phi}{27tr^{2\sigma}} [\sigma \cos^2 \phi - \cos(2\phi)\phi'] \\ &\quad - \frac{2\sigma \cos^2 \phi}{27tr^{2\sigma}} (3 - \sigma \cos^2 \phi) [7 + (2\phi' - \sigma) \sin^2 \phi] \\ &\quad + \frac{\sigma \cos^2 \phi \sin^2 \phi}{27tr^{2\sigma}} (3 - \sigma \cos^2 \phi + 2 \cos^2 \phi \phi') \\ &= \frac{\sigma \cos^2 \phi}{27tr^{2\sigma}} [-6(7 + 2\phi' \sin^2 \phi) + (3 + 2 \cos^2 \phi \phi') \sin^2 \phi + O(\sigma)] \\ &= \frac{\sigma \cos^2 \phi}{27tr^{2\sigma}} [-42 + \sin^2 \phi (-12\phi' + 3 + 2\phi' \cos^2 \phi) + O(\sigma)] < 0 \end{aligned}$$

and

$$\begin{aligned}
 (9.53) \quad H_3 &= \frac{\sigma^2 \sin^2 \phi \cos^2 \phi}{81tr^{3\sigma}} [\sigma \cos^2 \phi - 3 \cos 2\phi] \phi' \\
 &\quad - \frac{\sigma \cos^2 \phi}{81tr^{3\sigma}} (9 - 6\sigma \cos^2 \phi + \sigma^2 \cos^2 \phi) (7 + (2\phi' - \sigma) \sin^2 \phi) \\
 &\quad + \frac{\sigma^3 \sin^2 \phi \cos^4 \phi}{81tr^{3\sigma}} (\sigma \sin^2 \phi + \cos(2\phi) \phi') \\
 &\quad + \frac{\sigma \sin^2 \phi \cos^2 \phi}{81tr^{3\sigma}} (3 - \sigma \cos^2 \phi) (3 - \sigma \cos^2 \phi + 2 \cos^2 \phi \phi') \\
 &= \frac{\sigma \cos^2 \phi}{27tr^{3\sigma}} \left[\sin^2 \phi (3 + 2 \cos^2 \phi \phi') - 3(7 + (2\phi' - \sigma) \sin^2 \phi) \right. \\
 &\quad \left. - \sigma \sin^2 \cos(2\phi) \phi' + O(\sigma^2 \cos^2 \phi) \right] \\
 &= \frac{\sigma \cos^2 \phi}{27tr^{3\sigma}} \left[-21 \cos^2 \phi - (6 - \sigma) \sin^2 \phi (\phi' + 3) \right. \\
 &\quad \left. + (2 - 2\sigma) \cos^2 \phi \sin^2 \phi \phi' + O(\sigma^2 \cos^2 \phi) \right] \leq 0
 \end{aligned}$$

when $\sigma > 0$ is sufficiently small. From this we get (9.42). The proof of (9.43) is similar. \square

Now we can prove Theorem 2. In fact, from (9.40), we have

$$(9.54) \quad F_0 \leq F_R \leq F_0 + \frac{\tilde{A}F_0}{r^\sigma} \text{ for } r = a_0$$

if we choose $\tilde{A} \geq 1$ such that

$$(9.55) \quad \max_{\theta} \frac{\mathcal{H} \left(a_0 (-\cos(2\theta)) + a_0^{3/2} (-\cos(2\theta)) (1 + \mathbf{A}(|\cos(2\theta)|)^{\lambda-1}) \right)}{(a_0^3 + \tilde{A}a_0^{3-\sigma})g(\theta)} \leq 1,$$

which is possible since $\sup_{\theta} \frac{|\cos(2\theta)|}{g(\theta)} < +\infty$ (this follows from (9.9) and the fact that $g_{\theta}(\frac{\pi}{4}) > 0$). Note that (9.55) holds for any \tilde{A} large.

By the comparison principle in the domain $\Gamma_R \setminus B_{a_0}$ (noting that the function $F_0 + \frac{\tilde{A}F_0}{r^\sigma}$ is a super-solution for $r > a_0$ by Lemma 9.6 and the function F_0 is a sub-solution by Lemma 9.5), we deduce that

$$(9.56) \quad F_0 \leq F_R \leq F_0 + \frac{\tilde{A}F_0}{r^\sigma} \text{ in } \Gamma_R \setminus B_{a_0},$$

and hence

$$(9.57) \quad F_0 \leq F_R \leq F_0 + \tilde{A}r^{3-\sigma} \text{ in } \Gamma_R \setminus B_{a_0}$$

for \tilde{A} large.

Let $A \geq 1$ be a constant to be chosen later and let us consider the region $\Gamma_R \cap \{r > R_0\}$, where $R_0 = a_0 A^{\frac{1}{3+\sigma}}$. From (9.57), we then have

$$(9.58) \quad F_0 \leq F_R \leq F_0 + \tilde{A} R_0^{3-\sigma} \leq F_0 + \frac{A}{R_0^\sigma} \quad \text{for } r = R_0$$

if we choose

$$(9.59) \quad \tilde{A} \leq \frac{A}{R_0^3} = \frac{A}{a_0^3 A^{\frac{3}{3+\sigma}}} = a_0^{-3} A^{\frac{\sigma}{3+\sigma}}.$$

By the comparison principle applied now in $\Gamma_R \cap \{r > R_0\}$, using Lemma 9.6, we then obtain

$$(9.60) \quad F_0 \leq F_R \leq F_0 + \frac{A}{r^\sigma}, \quad \text{for } r \geq R_0 = a_0 A^{\frac{1}{3+\sigma}}.$$

The assertion of the theorem follows now by combing (9.56) and (9.60) and letting $R \rightarrow \infty$. □

9.4. *A refinement of the asymptotic behavior of F .* While Theorem 2 is enough for our purposes, we establish next a result that estimates accurately the BDG graph near ∂T , which is interesting in its own right.

THEOREM 3. *There exists $\sigma_0 \in (0, 1)$ such that for each $\sigma \in (0, \sigma_0)$, there exists $a_0 > 1$ such that for each sufficiently large $A \geq 1$, we have*

$$(9.61) \quad H \left[F_0 + \frac{A \tanh(F_0 r^{-1})}{r^\sigma} \right] \leq 0 \quad \text{for } r > a_0 A^{\frac{1}{1+\sigma}}.$$

As a consequence there are constants C, R_0 , such that the solution to the mean curvature equation described in Theorem 2 satisfies

$$(9.62) \quad F_0 \leq F \leq F_0 + \frac{C \tanh(F_0 r^{-1})}{r^\sigma} \quad \text{for } r > R_0.$$

Proof. Let us prove (9.61) first. We will denote

$$F = F_0 + \frac{A}{r^\sigma} \varphi(t, s), \quad \varphi(t, s) = \tanh(t/r).$$

Note that the A^0 and A^1 terms in (9.27) are

$$(9.63) \quad \begin{aligned} & -\frac{1}{2} \partial_t \left(\frac{1}{|\nabla F_0|^2} \right) + \frac{A}{|\nabla F_0|^2} \partial_t^2 \varphi - \frac{A}{2} \partial_t \left(\frac{1}{|\nabla F_0|^2} \right) \partial_t \varphi \\ & + A \left(1 + \frac{1}{|\nabla F_0|^2} \right) \partial_s \left(\frac{\rho^{-2} \varphi_s}{|\nabla F_0|^2} \right) - \frac{A}{2} \partial_s \left(\frac{1}{|\nabla F_0|^2} \right) \frac{\rho^{-2} \varphi_s}{|\nabla F_0|^2} \\ & = |\nabla F_0|^{-1} H[F_0] + A |\nabla F_0|^{-1} \left[|\nabla F_0| \partial_t \left(\frac{\varphi_t}{|\nabla F_0|^2} \right) + |\nabla F_0| \partial_s \left(\frac{\rho^{-2} \varphi_s}{|\nabla F_0|^2} \right) \right] \\ & \quad - A \left[\frac{3}{2} \partial_t \left(\frac{1}{|\nabla F_0|^2} \right) \partial_t \varphi - \frac{1}{|\nabla F_0|^2} \partial_s \left(\frac{\rho^{-2} \varphi_s}{|\nabla F_0|^2} \right) + \frac{1}{2} \partial_s \left(\frac{1}{|\nabla F_0|^2} \right) \frac{\rho^{-2} \varphi_s}{|\nabla F_0|^2} \right]. \end{aligned}$$

By (9.46), we have

$$(9.64) \quad H_0 = |\nabla F_0|^{-1} H[F_0] \leq c_1 \frac{\cos^2 \phi}{tr^4} \leq c_1 \frac{\cos \phi}{r^7}.$$

Now we will deal with the first A^1 term in (9.63). This term is given explicitly in (9.29). We recall here that in (9.30) we have defined the following operator:

$$(9.65) \quad \tilde{L}_0[\varphi] := |\nabla F_0| \partial_t \left(\frac{\varphi_t}{|\nabla F_0|^2} \right) + |\nabla F_0| \partial_s \left(\frac{\rho^{-2} \varphi_s}{|\nabla F_0|^2} \right).$$

We will prove the following lemma.

LEMMA 9.7. *There exists $\sigma_0 > 0$ such that for each $\sigma \in (0, \sigma_0)$, there exist $a_0 > 0$ and $c_0 > 0$ such that*

$$(9.66) \quad \tilde{L}_0[r^{-\sigma} \tanh(t/r)] \leq -\frac{c_0}{r^{4+\sigma}} \min\{1, t/r\}, \quad r > a_0.$$

Proof. Let us denote

$$\beta(\eta) = \tanh(\eta), \quad \eta = \frac{t}{r}, \quad \beta_1(\eta) = \beta(\eta) - \frac{1}{\sigma} \beta' \eta$$

and

$$(9.67) \quad \varphi = \beta(\eta)r^{-\sigma}, \quad \sigma > 0.$$

Then we compute

$$\partial_s \varphi = -\frac{\sigma r^{-\sigma} \sin^2 \phi}{7s} \beta_1;$$

hence

$$\partial_s \left(\frac{\rho^{-2} \partial_s \varphi}{|\nabla F_0|^2} \right) = -c_1 \sigma \partial_s \left(\frac{r^{-\sigma}}{t^2} \cos^2 \phi \beta_1 \right),$$

where $c_1 > 0$. From now on, by $c_i > 0$ we will denote generic positive constants.

We obtain

$$(9.68) \quad \begin{aligned} & \partial_s \left(\frac{\rho^{-2}}{|\nabla F_0|^2} \partial_s \varphi \right) \\ &= -\frac{c_1 \sigma r^{-6-\sigma}}{9g^2 + g_\theta^2} \left\{ \beta_1 \left[1 + \frac{2 \sin^2 \phi}{7} \left(\frac{-\sigma}{2} + \phi' \right) \right] - \frac{\eta \sin^2 \phi}{7} \beta_1' \right\}. \end{aligned}$$

On the other hand, we have

$$\partial_t \varphi = -\frac{\sigma r^{-\sigma}}{3t} \beta \cos^2 \phi + \beta' \left(1 - \frac{\cos^2 \phi}{3} \right) r^{-\sigma-1}$$

and

$$\frac{\partial_t \varphi}{|\nabla F_0|^2} = \frac{r^{1-\sigma} \cos^2 \phi}{9t^2} \left[\frac{-\sigma}{3} \left(\frac{\beta}{\eta} \right) \cos^2 \phi + \left(1 - \frac{\cos^2 \phi}{3} \right) \beta' \right];$$

hence

(9.69)

$$\begin{aligned} \partial_t \left(\frac{1}{|\nabla F_0|^2} \partial_t \varphi \right) &= \left[-\frac{2r^{1-\sigma} \cos^2 \phi}{9t^3} + \frac{r^{1-\sigma} \sin^2(2\phi) \phi'}{63t^3} \right] \left(1 - \frac{\cos^2 \phi}{3} \right) \beta' \\ &\quad + \frac{r^{-\sigma} \cos^2 \phi}{9t^2} \left[\frac{-\sigma}{3} \left(\frac{\beta}{\eta} \right)' \cos^2 \phi + \left(1 - \frac{\cos^2 \phi}{3} \right) \beta'' \right] \left(1 - \frac{\cos^2 \phi}{3} \right) \\ &\quad + O\left(\frac{\cos \phi}{r^{8+\sigma}} \right). \end{aligned}$$

The first term in (9.69) is negative. The second term can be estimated as follows:

$$\begin{aligned} (9.70) \quad \frac{r^{-\sigma} \cos^2 \phi}{9t^2} \left[\frac{-\sigma}{3} \left(\frac{\beta}{\eta} \right)' \cos^2 \phi + \left(1 - \frac{\cos^2 \phi}{3} \right) \beta'' \right] \left(1 - \frac{\cos^2 \phi}{3} \right) \\ \leq \frac{c_2}{r^{6+\sigma}} \left[\frac{-\sigma}{3} \left(\frac{\beta}{\eta} \right)' \cos^2 \phi + \frac{2}{3} \beta'' \right]. \end{aligned}$$

Combining (9.68) and (9.70), we have

$$\begin{aligned} (9.71) \quad \tilde{L}_0[\varphi] &\leq \frac{c_3}{r^{4+\sigma}} \left\{ -\sigma \beta_1 \left[1 + \frac{2 \sin^2 \phi}{7} \left(\frac{-\sigma}{2} + \phi' \right) \right] + \frac{\eta \sigma \sin^2 \phi}{7} \beta'_1 \right. \\ &\quad \left. + \left[\frac{-\sigma}{3} \left(\frac{\beta}{\eta} \right)' \cos^2 \phi + \frac{2}{3} \beta'' \right] \right\} + O\left(\frac{\cos \phi}{r^{6+\sigma}} \right). \end{aligned}$$

Denoting the term in brackets above by \tilde{a} we can estimate as follows:

$$\tilde{a} \leq \beta'' \left(c_4 \eta^2 \sin^2 \phi + \frac{2}{3} \right) - c_5 \sigma \left[\beta - c_6 |\beta' \eta| - c_7 \left| \left(\frac{\beta}{\eta} \right)' \right| \right].$$

Given small $\varepsilon_0 > 0$, let $\eta_0 > 0$ be such that

$$\beta - c_6 |\beta' \eta| - c_7 \left| \left(\frac{\beta}{\eta} \right)' \right| \geq \varepsilon_0, \quad \eta \geq \eta_0;$$

hence for $\eta > \eta_0$, we have

$$(9.72) \quad \tilde{a} \leq -c_8 \varepsilon_0 \sigma \quad \text{for } \sigma \in (0, 1/2).$$

On the other hand, when $0 \leq \eta \leq \eta_0$, then

$$(9.73) \quad \tilde{a} \leq -c_9 \eta \left(\frac{1}{7} \eta^2 + \frac{2}{3} \right) - c_{10} \sigma \eta \leq -c_{11} \eta,$$

where $\sigma \in (0, \sigma_0)$ with $\sigma_0 > 0$ small. Finally let us consider the last term in (9.71). When $\eta \leq 1$, then

$$\frac{\cos \phi}{r^{6+\sigma}} \leq \frac{c_{12} \eta}{r^{8+\sigma}},$$

while when $1 \leq \eta$, then

$$\frac{\cos \phi}{r^{6+\sigma}} \leq \frac{1}{r^{6+\sigma}}.$$

Summarizing the above and (9.71)–(9.73), we have that for each $\sigma \in (0, \sigma_0)$, where σ_0 is small, there exists $r_0 > 0, c_0$, such that

$$(9.74) \quad \begin{aligned} \tilde{L}_0[\varphi] &\leq -\left(\frac{c_{13}}{r^{4+\sigma}} - \frac{c_{14}}{r^{6+\sigma}}\right) \min\{1, \eta\} \\ &\leq -\frac{c_0}{r^{4+\sigma}} \min\{1, \eta\}, \quad r > r_0. \end{aligned} \quad \square$$

Continuing the proof of Theorem 3 we notice that

$$(9.75) \quad -\frac{3}{2} \partial_t \left(\frac{1}{|\nabla F_0|^2} \right) \partial_t \varphi \leq \frac{c_{15} \cos \phi}{r^{8+\sigma}} \leq \frac{c_{15} \min\{\eta, 1\}}{r^{8+\sigma}}$$

since $\cos \phi \leq \frac{\eta}{r^2}$, and

$$(9.76) \quad \frac{1}{|\nabla F_0|^2} \partial_s \left(\frac{\rho^{-2} \varphi_s}{|\nabla F_0|^2} \right) \leq \frac{c_{16}}{r^{8+\sigma}} \min\{\eta, 1\}$$

$$(9.77) \quad -\frac{1}{2} \partial_s \left(\frac{1}{|\nabla F_0|^2} \right) \frac{\rho^{-2} \varphi_s}{|\nabla F_0|^2} \leq c_{17} \frac{|\beta_1(\eta)|}{r^{10+\sigma}} \leq c_{17} \frac{\min\{\eta, 1\}}{r^{10+\sigma}}.$$

We analyze the A^2 term and A^3 terms in the expansion of

$$H[F_0 + Ar^{-\sigma} \tanh(t/r)].$$

A typical term in (9.27) is

$$(9.78) \quad \begin{aligned} -\frac{1}{2} \partial_t \left(\frac{\rho^{-2} F_s^2}{|\nabla F_0|^2} \right) &= -\frac{\sigma^2 \sin^2 \phi \cos^2 \phi}{27t^3 r^{2\sigma}} [-\sigma \cos^2 \phi - 3 + \cos(2\phi) \phi'] \beta_1^2 \\ &\quad - \frac{\sigma^2 \sin^2 \phi \cos^2 \phi}{9t^3 r^{2\sigma}} 2\beta_1 \beta_1' \eta \left(1 - \frac{\cos^2 \phi}{3} \right) \\ &= \sin^2 \phi \min\{\eta, 1\} O(r^{-7-2\sigma}). \end{aligned}$$

Other A^2 terms are estimated in a similar way. Direct calculation shows that the A^3 term satisfies

$$(9.79) \quad H_3 = \sin^2 \phi \min\{\eta, 1\} O(r^{-8-3\sigma}).$$

In conclusion, we have

$$(9.80) \quad \begin{aligned} H[F_0 + Ar^{-\sigma} \tanh(F_0/r)] &\leq \left(\frac{c_1}{r^7} - \frac{c_0 A}{r^{6+\sigma}} + \frac{c_{18} A^2}{r^{7+2\sigma}} + \frac{c_{19} A^3}{r^{7+3\sigma}} \right) \min\{1, \eta\} \\ &\leq 0, \end{aligned}$$

if we choose a_0 large and $r \geq a_0 A^{\frac{1}{1+\sigma}}$. This proves (9.61).

Now we will show (9.62). From (9.57), we have

$$(9.81) \quad F_0 \leq F_R \leq F_0 + \tilde{A} F_0 r^{-\sigma} \quad \text{for } r \geq a_0$$

for some $\tilde{A} \geq 1$.

Let us consider the region

$$\Sigma := B_R \cap \{v > u\} \cap \{r > R_0\} \cap \left\{0 \leq \frac{F_0}{r} < 1\right\},$$

where $R_0 = a_0 A^{\frac{1}{1+\sigma}}$, and A is to be chosen. From (9.57), in Σ we have

$$(9.82) \quad F_0 \leq F_R \leq F_0 + \tilde{A} F_0 R_0^{-\sigma} \leq F_0 + \frac{A \tanh(F_0 R_0^{-1})}{R_0^\sigma} \quad \text{for } r = R_0$$

if we choose

$$(9.83) \quad \tilde{A} \leq \frac{A \tanh(F_0 R_0^{-1})}{R_0 F_0 R_0^{-1}} = A^{\sigma/(1+\sigma)} a_0^{-1} \sup_{|\eta| < 1} \frac{\tanh \eta}{\eta}.$$

Consider now the boundary $\{\frac{F_0}{r} = 1\}$. By (9.60),

$$(9.84) \quad \begin{aligned} F_0 \leq F_R &\leq F_0 + \frac{A \tanh(1)}{r^\sigma} \\ &\leq F_0 + \frac{A \tanh(F_0/r)}{r^\sigma} \quad \text{for } r \geq R_0 \geq a_0 (\tanh(1) A)^{\frac{1}{3+\sigma}} \text{ and } F_0/r = 1 \end{aligned}$$

if we choose (cf. (9.59))

$$(9.85) \quad \tilde{A} \leq a_0^{-3} (\tanh(1) A)^{\frac{\sigma}{3+\sigma}}.$$

Choosing A larger if necessary, we can assume that in addition to (9.83), also (9.85) is satisfied. By the comparison principle applied to Σ , we then obtain

$$(9.86) \quad F_0 \leq F_R \leq F_0 + \frac{A \tanh(F_0/r)}{r^\sigma} \quad \text{for } r \geq R_0.$$

Passing to the limit $R \rightarrow \infty$ we then get

$$(9.87) \quad F_0 \leq F \leq F_0 + \frac{A \tanh(F_0/r)}{r^\sigma} \quad \text{for } r \geq R_0$$

in Σ . Combining this with the statement of Theorem 2 to estimate F for $r > R_0$ in the complement of Σ , we complete the proof. \square

10. Appendix: The proof of formula (7.4)

In this appendix we carry out the main computation leading to formula (7.4) for the approximate Jacobi operator

$$\mathcal{J}_{\Gamma_0}[h] := H'(F_0)[\sqrt{1 + |\nabla F_0|^2} h].$$

Following the notation in Section 9.2, the minimal surface equation $H[F] = 0$ becomes

$$(10.1) \quad H[F] := |\nabla F_0| \partial_t \left(\frac{|\nabla F_0|}{\sqrt{1 + |\nabla F|^2}} \partial_t F \right) + |\nabla F_0| \partial_s \left(\frac{\rho^{-2}}{|\nabla F_0| \sqrt{1 + |\nabla F|^2}} \partial_s F \right) = 0.$$

It is easy to see that

$$(10.2) \quad H'[F_0](\phi) = |\nabla F_0| \partial_t \left(\frac{|\nabla F_0|}{(1 + |\nabla F_0|^2)^{3/2}} \partial_t \phi \right) + |\nabla F_0| \partial_s \left(\frac{\rho^{-2}}{|\nabla F_0| \sqrt{1 + |\nabla F_0|^2}} \partial_s \phi \right).$$

Let us now set

$$\phi = \sqrt{1 + |\nabla F_0|^2} h.$$

Then after some simple computations, we obtain

$$\begin{aligned} |\nabla F_0| \partial_t \left(\frac{|\nabla F_0|}{(1 + |\nabla F_0|^2)^{3/2}} \partial_t \phi \right) &= \frac{|\nabla F_0|^2}{1 + |\nabla F_0|^2} \partial_t^2 h - \frac{|\nabla F_0|}{(1 + |\nabla F_0|^2)^2} \partial_t |\nabla F_0| \partial_t h \\ &\quad + |\nabla F_0| \partial_t \left(\frac{|\nabla F_0|}{(1 + |\nabla F_0|^2)^2} \partial_t |\nabla F_0| \right) h \end{aligned}$$

and

$$\begin{aligned} |\nabla F_0| \partial_s \left(\frac{\rho^{-2}}{|\nabla F_0| \sqrt{1 + |\nabla F_0|^2}} \partial_s \phi \right) &= \partial_s (\rho^{-2} \partial_s h) \\ &\quad - \frac{1}{|\nabla F_0| (1 + |\nabla F_0|^2)} \partial_s |\nabla F_0| \rho^{-2} \partial_s h \\ &\quad + |\nabla F_0| \partial_s \left(\frac{\rho^{-2}}{1 + |\nabla F_0|^2} \partial_s |\nabla F_0| \right) h. \end{aligned}$$

Note that

$$(10.3) \quad \frac{|\nabla F_0|}{(1 + |\nabla F_0|^2)^2} \partial_t |\nabla F_0| = O(r^{-7}), \quad \frac{1}{|\nabla F_0| (1 + |\nabla F_0|^2)} \rho^{-1} \partial_s |\nabla F_0| = O(r^{-5}).$$

The operator in terms of h then becomes

$$\begin{aligned} J_{\Gamma_0}[h] &:= H'[F_0](\sqrt{1 + |\nabla F_0|^2} h) = \partial_t^2 h + \partial_s (\rho^{-2} \partial_s h) \\ &\quad + h \left(|\nabla F_0| \partial_t^2 \left(-\frac{1}{|\nabla F_0|} \right) + |\nabla F_0| \partial_s (\rho^{-2} \partial_s \left(-\frac{1}{|\nabla F_0|} \right)) \right) \\ &\quad + \mathcal{O}(r^{-4} |\partial_t^2 h| + r^{-7} |\partial_t h| + r^{-5} |\rho^{-1} h_s| + r^{-6} |h|). \end{aligned}$$

The desired expression (7.4) is then deduced from the following two identities:

$$\Delta_{\Gamma_0} h = \partial_t^2 h + \partial_s (\rho^{-2} \partial_s h)$$

and

$$|A_{\Gamma_0}|^2 = \left(|\nabla F_0| \partial_t^2 \left(-\frac{1}{|\nabla F_0|} \right) + |\nabla F_0| \partial_s (\rho^{-2} \partial_s \left(-\frac{1}{|\nabla F_0|} \right)) \right),$$

which follow from standard computations. We omit the details.

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