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From real affine geometry to complex geometry

By Mark Gross and Bernd Siebert

Abstract

We construct from a real affine manifold with singularities (a tropical manifold) a degeneration of Calabi-Yau manifolds. This solves a fundamental problem in mirror symmetry. Furthermore, a striking feature of our approach is that it yields an explicit and canonical order-by-order description of the degeneration via families of tropical trees.

This gives complete control of the B-model side of mirror symmetry in terms of tropical geometry. For example, we expect that our deformation parameter is a canonical coordinate, and expect period calculations to be expressible in terms of tropical curves. We anticipate this will lead to a proof of mirror symmetry via tropical methods.

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Introduction

Toric geometry links the integral affine geometry of convex polytopes to complex geometry. On the complex side this correspondence works via equivariant partial completions of algebraic tori. It is thus genuinely linear in nature and deals exclusively with rational varieties. This paper provides a nonlinear extension of this correspondence producing (degenerations of) varieties with effective anti-canonical divisor. Among other things we obtain a new and rather surprising method for the construction of varieties with trivial canonical bundle by discrete methods. It generalizes the Batyrev-Borisov construction of Calabi-Yau varieties as complete intersections in toric varieties and a number of other, nontoric constructions. One may even hope to obtain all deformation classes of varieties with a trivial canonical bundle which contain maximally unipotent boundary points.

The data on the affine side consists of a topological manifold B built by gluing integral polytopes in an affine manner along their boundaries, along with compatible affine charts at the vertices of the resulting polyhedral decomposition $\mathscr{P}$ of B. This endows B with an integral affine structure on the complement of a codimension-two subset $\Delta \subseteq B$. The subset Δ is covered by the cells of the barycentric subdivision of $\mathscr{P}$ neither containing a vertex of $\mathscr{P}$ nor intersecting the interiors of top-dimensional cells of $\mathscr{P}$. This notion of affine manifolds with singularities allows for many more interesting closed examples than without singularities. For example, the two-torus is the only closed oriented surface with a nonsingular affine structure [Ben], [Mil58], while S^2 has many affine structures with singularities, for example as a base of an elliptically fibred K3 surface. Furthermore, such integral affine manifolds arise naturally from boundaries of reflexive polytopes [Gro05], [HZ05].

On the complex side we consider *toric degenerations* $\pi : \mathfrak{X} \to T$ of complex varieties as introduced in [GS06, Def. 4.1]; the central fibre X is a union of toric varieties glued torically by identifying pairs of toric prime divisors, and π is étale locally toric near the zero-dimensional toric strata of X. Beware that in general $\mathfrak{X}$ is not a toric variety. In the same paper it was shown how such a degeneration gives rise to an affine manifold with singularities and polyhedral decomposition $(B, \mathscr{P})$ as before ([GS06, Def. 4.13]). Combinatorially $\mathscr{P}$ is the dual intersection complex of the central fibre; that is, k-cells of $\mathscr{P}$ correspond to codimension-k intersections of irreducible components. The affine structure reflects the toric nature of the degeneration.

The main result, Theorem 1.30 of this paper, deals with the inverse problem: given $(B, \mathscr{P})$, find a toric degeneration $\pi : \mathfrak{X} \to T$ with dual intersection complex $(B, \mathscr{P})$.

In the case without singularities, Mumford already used toric methods to write down such degenerations, notably for the class of abelian varieties [Mum72].

To solve this problem in much greater generality, we need to make three assumptions. First, we need the existence of the central fibre of the toric degeneration as a *toric log Calabi-Yau space* as defined in [GS06, Def. 4.3]. Constructing such a space with dual intersection complex $(B, \mathscr{P})$ was the chief focus of [GS06]. This condition is necessary, as $\mathfrak{X} \to T$ induces such a log Calabi-Yau space structure on X. Also, Theorem 5.4 in [GS06] gives natural sufficient conditions (*positivity* and *simplicity*) in terms of the local affine monodromy around $\Delta \subseteq B$. Positivity is a kind of convexity property that again is necessary, while simplicity should be viewed as a maximal degeneration property that implies primitivity of the local monodromy. In instances where $(B, \mathscr{P})$ is nonsimple, the existence of a log Calabi-Yau structure can be explicitly checked by the results of Section 3.3 in [GS06], notably by Theorem 3.27.

Second, we assume the existence of a *polarization* for $(B, \mathscr{P})$. This is a multi-valued, convex, piecewise affine function on B. If $(B, \mathscr{P})$ is the dual intersection complex of a toric log Calabi-Yau space X, this condition is closely related to projectivity of X and is in fact equivalent to it provided $H^1(B, \mathbb{Q}) = 0$; see [GS06, Th. 2.34]. This condition is clearly not necessary as in some cases, for example in dimension two, log deformation theory gives the same result without any projectivity assumptions. While in our algorithm different polarizations lead to isomorphic families, the polarization is a basic ingredient that appears to be crucial for globalizing the local deformations consistently.

Third, we need a condition we term *locally rigid* (see Definition 1.26) on X. This is a somewhat technical condition, which essentially implies that at each step of our construction, the choices we need to make are unique. Without this condition, nonuniqueness can lead to obstructions to solving the problem. Simplicity implies local rigidity (Remark 1.29). As a result, the Main Theorem implies the following reconstruction theorem.

THEOREM 0.1. *Any polarized affine manifold with singularities with positive and simple monodromy arises as the dual intersection complex of a toric degeneration.*

In fact, our Main Theorem (Theorem 1.30) applies more broadly when the dual intersection complex $(B, \mathscr{P})$ is noncompact, corresponding to the case where a general fibre has only effective anti-canonical class. In principle, one should also be able to deal with the case when $(B, \mathscr{P})$ has boundary, where the corresponding complex manifold is not complete. However, in this situation we expect a Landau-Ginzburg potential to play an important role, and there are additional subtleties to the argument. We have chosen not to deal with these issues here and will consider this case elsewhere.

The proof of the Main Theorem gives far more than the existence of a toric degeneration. It gives a *canonical*, *explicit* k-th order deformation $X_k \to \operatorname{Spec} \Bbbk[t]/(t^{k+1})$ of X for any k. Furthermore, this degeneration is specified using data of a tropical nature.

Let us expand on this description. First, we explain the role the polarization plays. Given the polarization φ, a piecewise linear multi-valued function on B, one can construct the *discrete Legendre transform* of the triple $(B, \mathscr{P}, \varphi)$, which is another triple $(\check{B}, \check{\mathscr{P}}, \check{\varphi})$. If φ comes from an ample line bundle on a log Calabi-Yau space X with dual intersection complex $(B, \mathscr{P})$, then $(\check{B}, \check{\mathscr{P}})$ is the *intersection complex*, whose maximal cells are the Newton polytopes defining the polarized irreducible components of X (see [GS06, §§1.5 and 4.2]). The data governing the deformations of X then consist of what we call a *structure*, which is a collection of *slabs and walls*. These are codimension-one polyhedral subsets of B, contained locally in affine hyperplanes, along with some attached data of a ring automorphism which is used in our gluing construction. This structure has an important tropical interpretation. Morally, a structure can be viewed as a union of tropical trees in $\check{B}$ with leaves on Δ. We will not define the precise notion of tropical curves on affine manifolds with singularities, as this is not needed for the proof and the correct general definition is not yet entirely clear, but see [Gro09] for some further discussion of this. We inductively construct structures $\mathscr{S}_k$ for $k \geq 0$, with $\mathscr{S}_k$ providing sufficient data to construct $X_k \to \operatorname{Spec} \Bbbk[t]/(t^{k+1})$. Morally, $\mathscr{S}_k$ can be viewed as the union of "tropical trees of degree k".

The actual degenerations $X_k \to \operatorname{Spec} \Bbbk[t]/(t^{k+1})$ are constructed from the structure $\mathscr{S}_k$ by gluing together certain canonical thickenings of affine pieces of irreducible components of X, with the gluings specified by the automorphisms attached to the slabs and walls of $\mathscr{S}_k$. The main difficulty in the inductive construction of $\mathscr{S}_{k+1}$ from $\mathscr{S}_k$ is the need to maintain compatibility of this gluing. For this purpose, we adapt a key lemma of Kontsevich and Soibelman from [KS06], which essentially expresses commutators of automorphisms in a standard form as a product of automorphisms in this standard form.

We end this introduction with a number of remarks, historical and otherwise.

(1) Our construction can be viewed as a "nonlinear" generalization of Mumford's and Alexeev and Nakamura's construction of degenerations of abelian varieties [Mum72], [AN99], [Ale02]. If $B = \mathbb{R}^n/\Gamma$, where $\Gamma \subseteq \mathbb{Z}^n$ is a sublattice, then one obtains a degeneration of abelian varieties. Here Δ is empty, and the structures $\mathscr{S}_k$ can be taken to be empty too: there are no "corrections" to construct the deformation.

(2) Let $B_0 := B \setminus \Delta$. Then we can define $X(B_0) := T_{B_0}/\Lambda$, where Λ is the local system of integral flat vector fields. This is a torus bundle over B_0, and it inherits a natural complex structure from the tangent bundle T_{B_0}. One basic problem that arises in the Strominger-Yau-Zaslow approach to mirror symmetry [SYZ96] is that one would like to compactify $X(B_0)$ to a complex manifold $X(B)$. Because of the singularities, the complex structure is in fact not the correct one, and this compactification cannot be performed in the complex category, even when it can be performed in the topological category (see, e.g., [Gro09], [GS]). One needs to deform the complex structure before this compactification can be expected to exist. Typically, one considers an asymptotic version of this problem: consider $X_\epsilon(B_0) = T_{B_0}/\epsilon\Lambda$. Then for small $\epsilon > 0$, one expects that there is a small deformation of the complex structure on $X_\epsilon(B_0)$ which can be compactified. This problem has been discussed already in a number of places; see [Fuk05], [KS01], [Gro01]. The results of this paper, combined with the results of [GS] as described in [GS03], can be viewed as giving as complete a solution to this problem as one could hope for.

This problem was first attacked directly by Fukaya in [Fuk05] in the two-dimensional case. Fukaya gave heuristic arguments suggesting that the needed deformation should be concentrated along certain trees of gradient lines on B with leaves on Δ. This direct analytic approach seems to be very difficult; nevertheless, it gave a hint as to the relevant data for controlling the deformations.

(3) In [KS06], Kontsevich and Soibelman proposed an alternative approach to the reconstruction problem, suggesting one should construct a rigid analytic

space rather than a complex manifold from B. They showed how to do this in dimension two. Here, the same trees of gradient flow lines as in [Fuk05] emerge, this time with certain automorphisms attached to the edges of the trees. The proof relies on a group-theoretic lemma which we also use here. The advantage of using rigid analytic spaces is that most convergence issues become rather simple. However, there is one part of their argument which is rather technical: to prove convergence near the singularities, one has to control the gradient flow lines to avoid returning to some small neighbourhood of the singularities. This technical issue, along with some other points, appears to make this approach rather difficult to generalize directly to higher dimensions.

In a sense, we surmount these difficulties by discretizing the problem and passing to the discrete Legendre transform $(\check{B}, \check{\mathscr{P}})$. The advantage of working with the dual affine manifold $\check{B}$ is that the gradient flow lines on B become straight lines on $\check{B}$. These are obviously much easier to work with and control. On the other hand, this leaves us no ability to avoid a neighbourhood of the singularities. As a result, we have to deal with some compositions of automorphisms which involve terms of order 0; this introduces terms in our expressions with denominators, which have to be controlled. This is a significant technical problem, relatively easy in dimension two, but much harder in dimensions three and higher, and the solution to this problem occupies Section 3 of this paper. If one were to rewrite this paper in the dimension two case only, it would be considerably shorter. Given our current level of understanding, it seems that a price must be paid somewhere near the singularities, whether it be Kontsevich and Soibelman's genericity arguments or our algebraic arguments. It would be nice to find a simpler solution to these problems.

It is also worthwhile making a historical remark here. We had the original idea of constructing smoothings by gluing thickenings of affine pieces of irreducible components of X in 2003. It was also clear to us that the gluing maps should propagate along straight lines on $\check{B}$. However, we abandoned this approach for a while, attempting to find a Bogomolov-Tian-Todorov argument for smoothability. We returned to the question of explicit smoothings in 2005, and realized that the group-theoretic Kontsevich-Soibelman lemma applied in our situation, thus enabling us to complete the argument.

We also comment that if one is only interested in the two-dimensional case and one does not care about the explicit smoothings, but only the existence of a smoothing, then X can be smoothed using techniques of [Fri83] or [KN94] directly, as was known to us in 2001.

(4) We cannot overemphasize the importance of this result to understanding mirror symmetry. Our structures, in a sense, give a complete description of the B-model side of mirror symmetry, at a much deeper level than the usual description in terms of periods. Furthermore, our description of the B-model side

is tropical in nature. It is well known [Mik05], [NS06] that one should expect a correspondence between tropical curves on B and families of holomorphic curves on the corresponding degeneration $\mathfrak{X} \to T$. Thus it is an important point for understanding mirror symmetry that the construction of the corresponding complex manifold is controlled by tropical curves on $\check{B}$, hence by holomorphic curves on the mirror degeneration $\check{\mathfrak{X}} \to T$ corresponding to $\check{B}$. This is what one expects to see in mirror symmetry, and this gives *an explicit explanation for the connection between deformations and holomorphic curves in mirror symmetry.*

There remains the question of extracting explicit enumerative predictions from the structures we use to build our smoothings. We have performed some calculations in some three-dimensional examples, and from these, we feel highly confident in the following conjectures, stated with varying degrees of precision.

CONJECTURE 0.2. (1) *The coordinate t associated with the canonical k-th order deformations $X_k \to \operatorname{Spec} \Bbbk[t]/(t^{k+1})$ is a canonical coordinate in the usual sense in mirror symmetry.*

(2) *The enumerative predictions made by calculating periods of $X_k \to \operatorname{Spec} \Bbbk[t]/(t^{k+1})$ can be described explicitly in terms of contributions from each tropical rational curve on $\check{B}$ of "degree $\leq k$". The numerical contributions are determined by the automorphisms appearing in the structure $\mathscr{S}_k$.*

(3) *The automorphisms attached to walls of $\mathscr{S}_k$ can be interpreted as "raw enumerative data" which morally counts the number of holomorphic disks with boundary on Lagrangian tori of the mirror dual Strominger-Yau-Zaslow fibration. The tropical trees arising in the structures can be viewed as a tropical version of holomorphic disks.*

Ultimately, we believe it will be easier to read off enumerative information directly from the structures and that calculation of periods should be viewed as a crude way of extracting the much more detailed information present in the structures.

(5) Speculating further, we expect that our structures will yield a useful description of (higher) multiplication maps for homological mirror symmetry on the B-model side. Our smoothings $\mathfrak{X} \to T$ come along with canonical polarizations by an ample line bundle $\mathcal{L}$. A basis for the space of sections of $H^0(\mathfrak{X}, \mathcal{L}^{\otimes n})$ as an $\mathcal{O}_T$-algebra is given by the set of points $B\left(\frac{1}{n}\mathbb{Z}\right)$ of points on B whose coordinates are in $\frac{1}{n}\mathbb{Z}$. A structure should then allow us to give explicit descriptions of the multiplication maps $H^0(\mathfrak{X}, \mathcal{L}^{\otimes n_1}) \otimes H^0(\mathfrak{X}, \mathcal{L}^{\otimes n_2}) \to H^0(\mathfrak{X}, \mathcal{L}^{\otimes (n_1+n_2)})$, answering a question of Kontsevich. In discussions with Mohammed Abouzaid, it has become apparent that it seems likely that these multiplication maps and higher multiplication maps could be described in terms

of a "tropical Morse category" on B, once again making the B-model side look very much like the expected structure of the A-model (Fukaya category) side. Hopefully, this approach will ultimately lead to a proof of Homological Mirror Symmetry.

Further justification of these statements will have to wait for further work; however, [GS10] lays the groundwork for computation of periods.

Conventions. We work in the category $\underline{\mathrm{Sch}_{\Bbbk}}$ of separated schemes over an algebraically closed field $\Bbbk$ of characteristic 0. A *variety* is a scheme of finite type over $\Bbbk$. All our toric varieties are normal. A *toric monoid* is a finitely generated, saturated, integral monoid. These are precisely the monoids of the form $\sigma^\vee \cap \mathbb{Z}^n$ for $\sigma \subseteq \mathbb{R}^n$ a strictly convex, rational polyhedral cone.

1. **Fundamentals**

1.1. *Discrete data.* While our construction is part of the program laid out in [GS06], only a fraction of the techniques developed there is needed for it. To make this paper reasonably self-contained, Sections 1.2 and 1.3 therefore provide the relevant background. At the same time we discuss a generalization from the projective Calabi-Yau situation to semi-positive and noncomplete cases. For simplicity we restrict to the case without self-intersecting cells. The treatment of self-intersections is, however, straightforward; it is merely a matter of working with morphisms rather than inclusions and with algebraic spaces rather than schemes, as done consistently in [GS06].

To fix notation recall that a *convex polyhedron* is the intersection of finitely many closed affine half-spaces in $\mathbb{R}^n$. As all our polyhedra are convex, we usually drop the attribute "convex". A polyhedron is *rational* if the affine functions defining the half spaces can be taken with rational coefficients. The dimension of the smallest affine space containing a polyhedron Ξ is its *dimension.* Its *relative interior* $\operatorname{Int}\Xi$ is the interior inside this affine space, and the complement $\Xi\setminus\operatorname{Int}\Xi$ is called the *relative boundary* $\partial\Xi$. If $\dim\Xi = k$, then $\partial\Xi$ is itself a union of polyhedra of dimension at most $k-1$, called *faces*, obtained by intersection of Ξ with hyperplanes disjoint from $\operatorname{Int}\Xi$. Faces of dimensions $k-1$ and 0 are called *facets* and *vertices*, respectively. In contrast to [GS06], our polyhedra are not necessarily bounded, but we require the existence of at least one vertex (so half-spaces, for example, are not allowed). For $y \in \partial\Xi$, the *tangent cone* $K_y\Xi$ *of* Ξ *at* y is the cone generated by differences $z-y$ for $z \in \Xi$. If $\Xi' \subseteq \Xi$ is a face, we also define $K_{\Xi'}\Xi := K_y\Xi$ for any $y \in \operatorname{Int}\Xi'$. The closure of the cone $\mathbb{R}_{\geq 0}\cdot\big(\Xi\times\{1\}\big) \subseteq \mathbb{R}^n\times\mathbb{R} = \mathbb{R}^{n+1}$ is denoted $C(\Xi)$. Any polyhedron Ξ can be written as Minkowski sum $\Xi' + C$ of a bounded polyhedron Ξ' and a cone C. While Ξ' is not in general unique, C is determined as the Hausdorff limit $\lim_{\varepsilon\to 0}\varepsilon\Xi$ and is therefore called the *asymptotic cone of* Ξ.

Finally, if $C \subseteq \mathbb{R}^n$ is a cone, then $C^\vee$ denotes its dual as an additive monoid $\mathrm{Hom}(C, \mathbb{R}_{\geq 0})$, viewed as a cone in $\mathbb{R}^{\dim C} \simeq \mathrm{Hom}(C, \mathbb{R})$.

A rational polyhedron is *integral* or a *lattice polyhedron* if all its vertices are integral. The group of *integral affine transformations* $\mathrm{Aff}(\mathbb{Z}^n) = \mathbb{Z}^n \rtimes \mathrm{GL}(n, \mathbb{Z})$ acts on the set of integral polyhedra. Note that we require the translational part to also be integral. Finally, if Ξ is an integral polyhedron, $\Lambda_\Xi \simeq \mathbb{Z}^{\dim \Xi}$ denotes the free abelian group of integral tangent vector fields along Ξ. Note that for any $x \in \mathrm{Int}(\Xi)$, there is a canonical injection $\Lambda_\Xi \to T_{\Xi,x}$ inducing an isomorphism $T_{\Xi,x} \simeq \Lambda_{\Xi,\mathbb{R}} := \Lambda_\Xi \otimes_{\mathbb{Z}} \mathbb{R}$.

We consider topological manifolds with boundary built by gluing integral convex polyhedra along their faces in an integral affine manner. To this end consider the category $\underline{\mathrm{LPoly}}$ with integral, convex polyhedra as objects and integral affine isomorphisms onto faces and the identity as morphisms. An *integral polyhedral complex* is gluing data for a collection of such polyhedra given by a functor

$$F : \mathscr{P} \longrightarrow \underline{\mathrm{LPoly}}$$

for some category $\mathscr{P}$ such that if $\Xi \in F(\mathscr{P})$ and $\Xi' \subseteq \Xi$ is a face, then $\Xi' \in F(\mathscr{P})$. To avoid self-intersections we also require that for any $\tau, \sigma \in \mathscr{P}$, there is at most one morphism $e : \tau \to \sigma$. The associated topological space is the quotient

$$B = \coprod_{\sigma \in \mathscr{P}} F(\sigma) \Big/ \sim,$$

where two points $p \in F(\sigma)$, $p' \in F(\sigma')$ are equivalent if there exists $\tau \in \mathscr{P}$, $q \in F(\tau)$ and morphisms $e : \tau \to \sigma$, $e' : \tau \to \sigma'$ with $p = F(e)(q)$, $p' = F(e')(q)$. (Then B is the colimit of the composition of F with the forgetful functor from $\underline{\mathrm{LPoly}}$ to the category of topological spaces.) By abuse of notation we usually suppress F and consider the elements of $\mathscr{P}$ simply as subsets of B, called *cells*, with the structure of integral convex polyhedra understood. The set of k-dimensional cells is then denoted $\mathscr{P}^{[k]}$, the k-skeleton by $\mathscr{P}^{[\leq k]}$, and if $n = \sup\{\dim \sigma \,|\, \sigma \in \mathscr{P}\}$ is finite, we write $\mathscr{P}_{\max} := \mathscr{P}^{[n]}$. In this language morphisms are given by inclusions of subsets of B.

While B now has a well-defined affine structure on each cell, our construction also requires affine information in the normal directions. To add this information recall that the *open star* of $\tau \in \mathscr{P}$ is the following open neighbourhood of $\mathrm{Int}\,\tau$:

$$U_\tau = \bigcup_{\{\sigma \in \mathscr{P} \,|\, \mathrm{Hom}(\tau,\sigma) \neq \emptyset\}} \mathrm{Int}\,\sigma.$$

Definition 1.1. Let $\mathscr{P}$ be an integral polyhedral complex. A *fan structure* along $\tau \in \mathscr{P}$ is a continuous map $S_\tau : U_\tau \to \mathbb{R}^k$ with the following properties:

(i) $S_\tau^{-1}(0) = \mathrm{Int}\,\tau$.

(ii) If $e : \tau \to \sigma$ is a morphism, then $S_\tau|_{\operatorname{Int}\sigma}$ is an integral affine submersion onto its image; that is, it is induced by an epimorphism $\Lambda_\sigma \to W \cap \mathbb{Z}^k$ for some vector subspace $W \subseteq \mathbb{R}^k$.

(iii) The collection of cones $K_e := \mathbb{R}_{\geq 0} \cdot S_\tau(\sigma \cap U_\tau)$, $e : \tau \to \sigma$, defines a finite fan Σ_τ in $\mathbb{R}^k$.

Two fan structures $S_\tau, S'_\tau : U_\tau \to \mathbb{R}^k$ are considered *equivalent* if they differ only by an integral linear transformation of $\mathbb{R}^k$.

If $S_\tau : U_\tau \to \mathbb{R}^k$ is a fan structure along $\tau \in \mathscr{P}$ and $\sigma \supseteq \tau$, then $U_\sigma \subseteq U_\tau$. The *fan structure along σ induced by S_τ* is the composition

$$U_\sigma \longrightarrow U_\tau \xrightarrow{S_\tau} \mathbb{R}^k \longrightarrow \mathbb{R}^k / L_\sigma \simeq \mathbb{R}^l,$$

where $L_\sigma \subseteq \mathbb{R}^k$ is the linear span of $S_\tau(\operatorname{Int}\sigma)$. This is well defined up to equivalence.

Definition 1.2. An *integral tropical manifold* of dimension n is an integral polyhedral complex $\mathscr{P}$, which we assume countable, together with a fan structure $S_v : U_v \to \mathbb{R}^n$ at each vertex $v \in \mathscr{P}^{[0]}$ with the following properties:

(i) For any $v \in \mathscr{P}^{[0]}$, the support $|\Sigma_v| = \bigcup_{C \in \Sigma_v} C$ is convex with nonempty interior (hence is an n-dimensional topological manifold with boundary).

(ii) If v, w are vertices of $\tau \in \mathscr{P}$, then the fan structures along τ induced from S_v and S_w, respectively, are equivalent.

In the case with empty boundary and with all polyhedra bounded, this is what in [GS06] we called a "toric polyhedral decomposition of an integral affine manifold with singularities".

The underlying topological space $B = \bigcup_{\sigma \in \mathscr{P}} \sigma$ of an integral tropical manifold carries a well-defined integral affine structure outside of a closed subset of codimension two. This *discriminant locus* Δ can be taken as follows. For each bounded $\tau \in \mathscr{P}$ let $a_\tau \in \operatorname{Int}\tau$, and for each unbounded $\tau \in \mathscr{P}$ let $a_\tau \in \Lambda_{\tau,\mathbb{R}}$ be an element of the relative interior of the asymptotic cone of τ. The choice of a_τ in the unbounded case must be made subject to the constraint that if $\tau' \subseteq \tau$ is a face and τ' and τ have the same asymptotic cone, then $a_{\tau'} = a_\tau$. In this unbounded case, a_τ should be viewed as a point at infinity. Then for any chain $\tau_1 \subseteq \tau_2 \cdots \subseteq \tau_{n-1}$ with $\dim \tau_i = i$ and τ_i bounded if and only if $i \leq r$ $(r \geq 1)$,

$$\Delta_{\tau_1 \cdots \tau_{n-1}} := \operatorname{conv}\left\{ a_{\tau_i} \,\middle|\, 1 \leq i \leq r \right\} + \sum\nolimits_{i > r} \mathbb{R}_{\geq 0} \cdot a_{\tau_i} \subseteq \tau_{n-1}$$

is the Minkowski sum of an $(r-1)$-simplex with a simplicial cone of dimension at most $n - r - 1$. Define Δ as the union of all such polyhedra. In the unbounded case, if some unbounded edges are parallel, it might happen that these polyhedra are not all $(n-2)$-dimensional, but they are always contained in one of this type which is $(n-2)$-dimensional.

Now if $\rho \in \mathscr{P}^{[n-1]}$, then the connected components of $\rho \setminus \Delta$ are in one-to-one corespondence with the vertices of ρ. This is clear for bounded cells, while an unbounded cell together with its discriminant locus retracts onto a union of bounded cells. Thus the polyhedral structures on interiors of n-cells and the fan structures near the vertices define an atlas of integral affine charts on $B \setminus \Delta$. For $x \in \rho \setminus \Delta$, write $v[x]$ for the unique vertex in the same connected component of $\rho \setminus \Delta$ as x.

The fan structures and the polyhedral structure of the cells can be read off from this affine structure together with the decomposition into closed subsets provided by $\mathscr{P}$. This motivates the notation $(B, \mathscr{P})$ for an integral tropical manifold. In particular, we have a flat, torsion-free connection on $T_{B\setminus\Delta}$, which we use to parallel transport tangent vectors along homotopy classes of paths. By integrality there is also a locally constant sheaf Λ of integral tangent vectors on $B \setminus \Delta$. For $x \in (\sigma \cap B) \setminus \Delta$, we have a canonical isomorphism $\Lambda_\sigma \to \Lambda_x \cap T_{\sigma,x}$ that we will use liberally. If $\omega \subseteq \tau$, then this identification for $\sigma = \omega, \tau$ and $x \in \omega$ is compatible with the inclusion $\Lambda_\omega \to \Lambda_\tau$.

For bounded polyhedra a canonical choice of a_τ is the barycenter of τ, and this canonical choice, exhibiting Δ as a subcomplex of the barycentric subdivision, has been used in [GS06]. However, with this choice, Δ is not in sufficiently general position for our construction in this paper. We need the fact that the intersection of any proper *rational* affine subspace of an $(n-1)$-cell with Δ is transverse. The following lemma shows that in the bounded case, for a sufficiently general choice of the a_τ the corresponding discriminant locus $\Delta = \Delta(\{a_\tau\})$ does not contain any rational point. This implies that Δ intersects any proper rational affine subspace of an $(n-1)$-cell transversally, for otherwise it would contain a nonempty open subset of it, hence a rational point. We leave it to the reader to supply the more general unbounded case.

For the formulation of the lemma note that any $a \in B$ has a well-defined *field of definition* $\kappa(a) \subseteq \mathbb{R}$ generated over $\mathbb{Q}$ by its coordinate entries in any integral affine chart.

Lemma 1.3. *Given $(B, \mathscr{P})$ compact, assume that for all $1 \le r \le n-1$,*

$$\forall \tau_1 \subsetneq \cdots \subsetneq \tau_r : \ \operatorname{trdeg}_{\mathbb{Q}} \left(\kappa(a_{\tau_1}) \cdot \ldots \cdot \kappa(a_{\tau_r}) \right) = \sum_{i=1}^{r} \dim \tau_i .$$

Then $\Delta = \Delta(\{a_\tau\})$ contains no rational point.

Proof. It suffices to check the claim on one $(n-2)$-simplex $\Xi \subseteq \Delta$, say defined by $\tau_1 \subsetneq \cdots \subsetneq \tau_{n-1}$, $\dim \tau_i = i$. Let $e_1, \ldots, e_{n-1}$ be a $\mathbb{Q}$-basis for $\Lambda_{\tau_{n-1},\mathbb{Q}}$ adapted to the flag of $\mathbb{Q}$-vector subspaces $\Lambda_{\tau_1,\mathbb{Q}} \subsetneq \cdots \subsetneq \Lambda_{\tau_{n-1},\mathbb{Q}}$; that is, $\Lambda_{\tau_i,\mathbb{Q}} = \mathbb{Q}e_1 + \cdots + \mathbb{Q}e_i$. Then in the corresponding coordinate system,

$$a_{\tau_i} = \alpha_i^1 e_1 + \cdots + \alpha_i^i e_i, \quad i = 1, \ldots, n-1$$

for $\alpha_i^\mu \in \mathbb{R}$. Then $\kappa(a_{\tau_i}) = \mathbb{Q}(\alpha_i^1, \ldots, \alpha_i^i)$, and by assumption the α_i^μ are all algebraically independent over $\mathbb{Q}$.

Now assume that the convex hull of the a_{τ_i} contains a rational point. Then there exist $\lambda_i \in [0,1]$ with $\sum_i \lambda_i = 1$ such that

$$\sum_i \lambda_i a_{\tau_i} = \begin{pmatrix} \lambda_1\alpha_1^1 & +\cdots+ & \lambda_{n-1}\alpha_{n-1}^1 \\ & \ddots & \vdots \\ & & \lambda_{n-1}\alpha_{n-1}^{n-1} \end{pmatrix} \in \mathbb{Q}^{n-1}.$$

Solving inductively expresses each λ_i as a rational function in α_i^μ with coefficients in $\mathbb{Q}$, and $\sum_i \lambda_i = 1$ gives an algebraic relation between the α_i^μ. Now it is not hard to see that in solving inductively for $\lambda_{n-1}, \ldots, \lambda_1$, any two occurring monomials are different. Moreover, the coefficients can only all vanish if $\lambda_i = 0$ for all i, which is impossible since $\sum_i \lambda_i = 1$. We have thus found a nontrivial algebraic relation among the α_i^μ, contradicting algebraic independence. $\square$

To explain the meaning of Δ we now introduce the concept of *local monodromy*. Let $\omega \in \mathscr{P}^{[1]}$, $\rho \in \mathscr{P}^{[n-1]}$ with $\omega \subseteq \rho$, $\rho \not\subseteq \partial B$ and ω bounded. Then ρ is contained in two n-cells $\sigma^\pm$, and ω contains two vertices $v^\pm$. Following the change of affine charts given by (i) the fan structure at v^+, (ii) the polyhedral structure of σ^+, (iii) the fan structure at v^-, (iv) the polyhedral structure of σ^-, and back to (v) the fan structure at v^+ defines a transformation $T_{\omega\rho} \in \mathrm{SL}(\Lambda_{v^+})$. It is shown in [GS06, §1.5] that this transformation has the following form:

$$T_{\omega\rho}(m) = m + \kappa_{\omega\rho}\langle m, \check{d}_\rho\rangle d_\omega. \tag{1.1}$$

Here $d_\omega \in \Lambda_\omega \subseteq \Lambda_{v^+}$ and $\check{d}_\rho \in \Lambda_\rho^\perp \subseteq \Lambda_{v^+}^*$ are the primitive integral vectors pointing from v^+ to v^- and, in the chart at v^+, evaluating positively on σ^+, respectively. The constant $\kappa_{\omega\rho} \in \mathbb{Z}$ is independent of the choices of $v^\pm$ and $\sigma^\pm$. Geometrically meaningful integral tropical manifolds fulfill $\kappa_{\omega\rho} \geq 0$.

Definition 1.4 ([GS06, Def. 1.54]). An integral tropical manifold is *positive* if $\kappa_{\omega\rho} \geq 0$ for all $\omega \subseteq \rho$ with ω bounded and $\rho \not\subseteq \partial B$.

Slightly more generally one can consider an analogous sequence of changes of charts for two arbitrary vertices v, v' contained in an $(n-1)$-cell $\rho \not\subseteq \partial B$. Since v and v' can be connected by a sequence of 1-cells contained in ρ, the corresponding monodromy transformation takes the form

$$m \longmapsto m + \langle m, \check{d}_\rho\rangle m^\rho_{vv'} \tag{1.2}$$

for a well-defined $m^\rho_{vv'} \in \Lambda_\rho$. In particular, $m^\rho_{v^+v^-} = \kappa_{\omega\rho} d_\omega$. In the positive case this monodromy information can be conveniently gathered in the *monodromy polytope for* ρ:

$$\Delta(\rho) = \mathrm{conv}\{m^\rho_{vv'} \mid v' \in \rho\}. \tag{1.3}$$

Here $v \in \rho$ is a fixed vertex, and a different choice of v leads to a translation of $\Delta(\rho)$. Hence $\Delta(\rho)$ is a lattice polytope in $\Lambda_\rho \otimes_{\mathbb{Z}} \mathbb{R}$ that is well defined up to translation. Note that $\Delta(\rho)$ can have any dimension between 0 and $n-1$, and hence for fixed v, the map from vertices v' of ρ to vertices of $\Delta(\rho)$ needs not be injective.

Remark 1.5. One can show that the affine structure extends to a neighbourhood of $\tau \in \mathscr{P}$ if and only if, for every $\omega \in \mathscr{P}^{[1]}$, $\rho \in \mathscr{P}^{[n-1]}$, with $\omega \subseteq \tau \subseteq \rho$, it holds that $\kappa_{\omega\rho} = 0$. This has been used in [GS06, Prop. 1.27] to find a smaller discriminant locus. In this paper we choose to work with the larger discriminant locus, as it slightly simplifies the presentation later on.

Integral tropical manifolds arise algebro-geometrically from certain degenerations of algebraic varieties whose central fibres are unions of toric varieties and which are toroidal ("log smooth") morphisms near the zero-dimensional toric strata of the central fibre. In the following we generalize the relevant definitions in [GS06] to pairs consisting of a variety and a divisor. Recall that an algebraic variety is called *algebraically convex* if there exists a proper map to an affine variety [GL73]. A toric variety is algebraically convex if and only if the defining fan has convex support.

Definition 1.6. A *totally degenerate* CY*-pair* is a reduced variety X together with a reduced divisor $D \subseteq X$ fulfilling the following conditions. Let $\nu : \widetilde{X} \to X$ be the normalization and $C \subseteq \widetilde{X}$ its conductor locus. Then $\widetilde{X}$ is a disjoint union of algebraically convex toric varieties, and C is a reduced divisor such that $[C] + \nu^*[D]$ is the sum of all toric prime divisors, $\nu|_C : C \to \nu(C)$ is unramified and generically two-to-one, and the square

$$\begin{tikzcd} C \arrow[r] \arrow[d] & \widetilde{X} \arrow[d, "\nu"] \\ \nu(C) \arrow[r] & X \end{tikzcd}$$

is cartesian and cocartesian.

In other words, if (X, D) is a totally degenerate CY-pair, then X is built from a collection of toric varieties by identifying pairs of toric prime divisors torically. The remaining toric prime divisors define D. Note that by the toric nature of the identification maps it makes sense to define a *toric stratum* of X as a toric stratum of any irreducible component.

Definition 1.7. Let T be the spectrum of a discrete valuation $\Bbbk$-algebra with a closed point O and a uniformizing parameter $t \in \mathcal{O}(T)$. Let $\mathfrak{X}$ be a $\Bbbk$-scheme and $\mathfrak{D}, X \subseteq \mathfrak{X}$ reduced divisors. A *log smooth morphism* $\pi : (\mathfrak{X}, X; \mathfrak{D}) \to (T, O)$ is a morphism $\pi : (\mathfrak{X}, X) \to (T, O)$ of pairs of $\Bbbk$-schemes with the following properties. For any $x \in \mathfrak{X}$ there exists an étale neighbourhood $U \to \mathfrak{X}$ of x such that $\pi|_U$ fits into a commutative diagram of the following form:

$$\begin{tikzcd} U \arrow[r, "\Phi"] \arrow[d, "\pi|_U"'] & \operatorname{Spec} \Bbbk[P] \arrow[d, "G"] \\ T \arrow[r, "\Psi"] & \operatorname{Spec} \Bbbk[\mathbb{N}]. \end{tikzcd}$$

Here P is a toric monoid, Ψ and G are defined respectively by mapping the generator $z^1 \in \Bbbk[\mathbb{N}]$ to t and to a nonconstant monomial $z^{m_0} \in \Bbbk[P]$, and Φ is étale with preimage of the toric boundary divisor equal to the pull-back to U of $X \cup \mathfrak{D}$.

This definition just rephrases that if we endow $\mathfrak{X}$ and T with the log structures $\mathcal{M}_{\mathfrak{X}}$ and $\mathcal{M}_T$ defined by $X \cup \mathfrak{D} \subseteq \mathfrak{X}$ and $O \subseteq T$, respectively, then the map of log spaces $(\mathfrak{X}, \mathcal{M}_X) \to (T, \mathcal{M}_T)$ is (log) smooth and integral. We refer to [GS06, §3.1] for a quick survey of the relevant log geometry. However, we will largely avoid the terminology of log structures here.

Definition 1.8. (cf. [GS06, Def. 4.1].) Let T be the spectrum of a discrete valuation $\Bbbk$-algebra and $O \in T$ its closed point. A *toric degeneration of* CY-*pairs over* T is a flat morphism $\pi : \mathfrak{X} \to T$ together with a reduced divisor $\mathfrak{D} \subseteq \mathfrak{X}$, with the following properties:

(i) $\mathfrak{X}$ is normal.
(ii) The central fibre $X := \pi^{-1}(O)$ together with $D = \mathfrak{D} \cap X$ is a totally degenerate CY-pair.
(iii) Away from a closed subset $\mathfrak{Z} \subseteq \mathfrak{X}$ of relative codimension two not containing any toric stratum of X, the map $\pi : (\mathfrak{X}, X; \mathfrak{D}) \to (T, O)$ is log smooth.

In this definition we dropped the requirement that π be proper from [GS06]. In the nonproper case the deformation theory of (X, D) does not appear to be very well behaved, but it still makes sense to talk about formal toric degenerations of CY-pairs as in the following definition.

Definition 1.9. Let T be the spectrum of a discrete valuation $\Bbbk$-algebra and $\widehat{O}$ the completion of T at its closed point. A *formal toric degeneration of* CY-*pairs over* $\widehat{O}$ is a flat morphism $\hat{\pi} : \widehat{X} \to \widehat{O}$ of formal schemes together with a reduced divisor $\widehat{D} \subseteq \widehat{X}$, with the following properties:

(i) $\widehat{X}$ is normal.

(ii) The central fibre $X = \hat{\pi}^{-1}(O) \subseteq \widehat{X}$ together with $D = \widehat{D} \cap X$ is a totally degenerate CY-pair.

(iii) Away from a closed subset $Z \subseteq X$ of relative codimension two and not containing any toric stratum of X, the map $\hat{\pi} : (\widehat{X}, X; \widehat{D}) \to (\widehat{O}, O)$ is étale locally on $\widehat{X}$ isomorphic to the completion of a log smooth morphism along the central fibre.

Clearly, a toric degeneration of CY-pairs induces a formal toric degeneration of CY-pairs by completion along the central fibre.

Remark 1.10. While this is not possible with our ad hoc definition of log (smooth) structures, it does make sense to talk about abstract log structures also on the codimension-two loci $\mathfrak{Z} \subseteq \mathfrak{X}$ and $Z \subseteq X$ in Definitions 1.8 and 1.9, respectively. Then $\mathfrak{Z}$ or Z contain the locus where this extension fails to be fine, that is, where the log structure fails to possess a chart locally. By abuse of notation, we therefore refer to $\mathfrak{Z}$ or Z as the *singular locus of the log structure.*

Before explaining how a toric degeneration defines an integral tropical manifold we would like to review the basic duality between convex piecewise linear functions and their Newton polyhedra, including the unbounded case; see [Roc70]. Let Σ be a not necessarily complete fan defined on $N_{\mathbb{R}}$, where as usual N is a finitely generated, free abelian group. Then, as a matter of convention, an (integral) *piecewise linear function* on Σ is a map

$$\varphi : N_{\mathbb{R}} \longrightarrow \mathbb{R} \cup \{\infty\}$$

that is an ordinary (integral) piecewise linear function on $|\Sigma|$ and that takes value ∞ everywhere else. The graph $\Gamma_\varphi \subseteq N_{\mathbb{R}} \times \mathbb{R}$ is the union of the ordinary graph of $\varphi|_{|\Sigma|}$ with

$$\Big\{(n,h) \in N_{\mathbb{R}} \oplus \mathbb{R} \,\Big|\, n \in \partial|\Sigma|, h \geq \varphi(n)\Big\}.$$

Now given an integral polyhedron $\Xi \subseteq M_{\mathbb{R}}$, $M = \mathrm{Hom}(N, \mathbb{Z})$, define

$$\varphi : N_{\mathbb{R}} \longrightarrow \mathbb{R} \cup \{\infty\}, \quad \varphi(n) = \sup(-n|_\Xi) = -\inf(n|_\Xi).$$

Then φ is a strictly convex piecewise linear function on the normal fan Σ of Ξ. For the normal fan we use the convention that its rays are generated by the inward normals to the facets of Ξ. The signs are chosen in such a way that if $C(\Xi) \subseteq M_{\mathbb{R}} \times \mathbb{R}$ denotes the closure of the cone generated by $\Xi \times \{1\}$, then $C(\Xi)^\vee \subseteq N_{\mathbb{R}} \times \mathbb{R}$ is the convex hull of Γ_φ. This description readily shows that $|\Sigma|$ is convex.

Conversely, if $\varphi : N_{\mathbb{R}} \to \mathbb{R} \cup \{\infty\}$ is a strictly convex piecewise linear function on a fan Σ on $N_{\mathbb{R}}$ with convex support, then its *Newton polyhedron*

$$\Xi = \{x \in M_{\mathbb{R}} \,|\, \varphi + x \geq 0\}$$

is an integral polyhedron. It is unbounded if and only if Σ is not complete. An alternative description is

$$\Xi = C_\varphi^\vee \cap \left(M_\mathbb{R} \times \{1\}\right)$$

for C_φ the convex hull of Γ_φ.

These two constructions set up a one-to-one correspondence between integral, strictly convex piecewise linear functions on fans in $N_\mathbb{R}$ with convex support on one side and integral polyhedra in $M_\mathbb{R}$ on the other side.

We are now ready to explain the first method of constructing an integral tropical manifold out of a toric degeneration of CY-pairs.

Example 1.11 (The fan picture [GS06, §4.1]). If $(\pi : \mathfrak{X} \to T, \mathfrak{D})$ is a toric degeneration of CY-pairs, we can define an integral tropical manifold as follows. For simplicity we assume that the irreducible components of $X = \pi^{-1}(O)$ do not self-intersect; that is, they are normal. Let $\underline{\mathrm{Strata}}(X)$ be the finite category consisting of toric strata of X, with inclusions defining the morphisms. For $S \in \underline{\mathrm{Strata}}(X)$, let $\eta_S \in \mathfrak{X}$ be the generic point and let $Y_1, \ldots, Y_r$ be the irreducible components of $X \cup \mathfrak{D}$ containing S. Choose the order in such a way that $Y_i \subseteq X$ if and only if $i \leq s$. Define the monoid

$$P_S := \left\{(m_1, \ldots, m_r) \in \mathbb{N}^r \,\middle|\, \textstyle\sum m_i [Y_i] \text{ is a Cartier divisor at } \eta_S \in \mathfrak{X}\right\}.$$

If $\Phi : U \to \operatorname{Spec} \Bbbk[P]$ is as in Definition 1.7 with η_S lifting to $\tilde{\eta}_S \in U$, then P_S is isomorphic to the monoid localization of P by $A := \{m \in P \,|\, \Phi^*(z^m) \in \mathcal{O}^\times_{U,\tilde{\eta}_S}\}$, that is, the quotient of the monoid $P - A \subseteq P^{\mathrm{gp}}$ by its invertible elements. This shows that P_S is a toric monoid. Hence P_S is the set of integral points of a rational polyhedral cone in $\mathbb{R}^r$. Define the polyhedron $F(S) \subseteq (\mathbb{R}^r)^*$ by intersecting the dual cone with the hyperplane $\langle\, . \,, \rho_S\rangle = 1$, where $\rho_S = (1, \ldots, 1, 0, \ldots, 0) \in P_S$ is the vector with entry 1 at the first s places:

$$F(S) := \left\{\lambda \in \operatorname{Hom}(P_S, \mathbb{R}_{\geq 0}) \,\middle|\, \lambda(\rho_S) = 1\right\}.$$

In this definition $\mathbb{R}_{\geq 0}$ is viewed as additive monoid. The fact that the central fibre X of the degeneration is reduced says that the integral distance of each facet of P_S to ρ_S equals 0 or 1. This implies that the vertices of $F(S)$ are integral. Note also that $F(S)$ is unbounded if and only if ρ_S lies in the boundary of the cone generated by P_S, which is the case if and only if $S \subseteq \mathfrak{D}$. Moreover, if $S_1 \subseteq S_2$, generization maps P_{S_1} surjectively to P_{S_2}, and this induces an inclusion of $F(S_2)$ as a face of $F(S_1)$. Thus

$$F : \underline{\mathrm{Strata}}(X)^{\mathrm{op}} \longrightarrow \underline{\mathrm{LPoly}}, \quad S \longmapsto F(S)$$

defines an integral polyhedral complex, the *dual intersection complex* $\check{\mathscr{P}}$ of X (denoted $\mathscr{P}$ in [GS06]). Finally, the toric irreducible components define compatible fan structures at the vertices, which by assumption on X have convex support. This makes $\check{B} = \coprod_{\sigma \in \check{\mathscr{P}}} F(\sigma)/\sim$ into an integral tropical manifold.

Because the irreducible components correspond to the fans at the vertices, we refer to this relation of integral tropical manifolds with toric degenerations as the *fan picture*. In this picture the maximal cells specify local models for $\mathfrak{X}$ at the zero-dimensional toric strata of X.

Note that this construction depends only on X and on the completion of $\mathcal{O}_{\mathfrak{X}}$ along $X \cup \mathfrak{D}$. Hence it works also for formal toric degenerations.

Example 1.12. [GS06], [Gro05] give many examples of affine manifolds obtained from toric degenerations of proper Calabi-Yau varieties. Here we give an example in the Fano case. Consider the equation $f_3(u_0, u_1, u_2, u_3) + tu_0u_1u_2 = 0$ defining $\mathfrak{X} \subseteq \mathbb{P}^3 \times \operatorname{Spec} \Bbbk[\![t]\!]$, with f_3 a general choice of cubic form. Then $\mathfrak{X} \to \operatorname{Spec} \Bbbk[\![t]\!]$ is a toric degeneration. The corresponding dual intersection complex $(B, \mathscr{P})$ looks like Figure 1.1. This picture is slightly misleading. The three unbounded rays are in fact parallel. The bounded two-cell is just a standard simplex. The discriminant locus Δ consists of the three points marked with crosses.

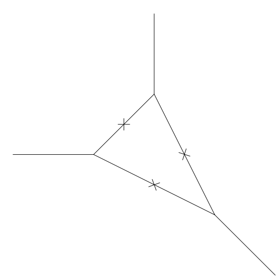


Figure 1.1.

The other method for producing an integral tropical manifold, which is even more relevant to this paper, requires a polarized central fibre.

Example 1.13 (The cone picture [GS06, §4.2]). Let $(\pi : \mathfrak{X} \to T, \mathfrak{D})$ be a toric degeneration of CY-pairs, and let $\mathcal{L}$ be an ample line bundle on $X = \pi^{-1}(O)$. Again we make the simplifying assumption that no components self-intersect. Then any $S \in \underline{\text{Strata}}(X)$ together with the restriction of $\mathcal{L}$ is a (not necessarily complete) polarized toric variety. The sections of powers of the line bundle define the integral points of the cone $C(\sigma) \subseteq \mathbb{R}^{n+1}$ over a (not necessarily bounded) integral polyhedron $\sigma \subseteq \mathbb{R}^n \times \{1\}$. Now σ is determined uniquely up to integral affine transformations, and inclusions of toric strata define integral affine inclusions of polyhedra as faces. Hence we have an integral polyhedral complex

$$F : \underline{\text{Strata}}(X) \longrightarrow \underline{\text{LPoly}},$$

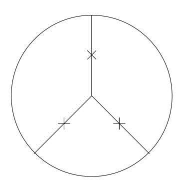
the *intersection complex* $\mathscr{P}$ of $(X, \mathcal{L})$. Note that the boundary of this polyhedral complex is covered by the cells corresponding to the toric strata of $\mathfrak{D} \cap X$.

The fan structures at the vertices this time come from log smoothness as follows. At a zero-dimensional toric stratum $\{x\} \subseteq X$ the degeneration is étale locally described by a toric morphism $\operatorname{Spec} \Bbbk[C \cap \mathbb{Z}^{n+1}] \to \operatorname{Spec} \Bbbk[\mathbb{N}]$ for some rational polyhedral cone $C \subseteq \mathbb{R}^{n+1}$. Denote by $\rho_S \in C \cap \mathbb{Z}^{n+1}$ the image of $1 \in \mathbb{N}$. Then the images of the faces of C not containing ρ_S under the projection $\mathbb{R}^{n+1} \to \mathbb{R}^{n+1}/\mathbb{R}\rho_S \simeq \mathbb{R}^n$ define an n-dimensional fan. Its support is convex because it is the image of a convex cone by a linear map. The cones in this fan are equal to tangent wedges of $F(\{x\}) \subseteq F(S)$ for $S \in \underline{\mathrm{Strata}}(X)$ containing x. This defines the fan structure at $F(\{x\})$. This construction again works also for formal toric degenerations of CY-pairs.

In this construction an irreducible component of X is defined by the cone over a maximal cell $\sigma \subseteq \mathbb{R}^n$ of $\mathscr{P}$ by $\operatorname{Proj}\big(\Bbbk[C(\sigma) \cap \mathbb{Z}^{n+1}]\big)$. This is why we call this correspondence the *cone picture*. In contrast to the fan picture, $(B, \mathscr{P})$ now carries information about the polarization, but we have lost some information about the local embedding into $\mathfrak{X}$ by projecting C down to $\mathbb{R}^{n+1}/\mathbb{R}\rho_S$. See Remark 1.15 for how to keep this information.

Note that in the construction we used a little less than an ample line bundle on X. It suffices to have an ample line bundle on each irreducible component with isomorphic restrictions on common toric prime divisors. We call such data a *pre-polarization* of X. If $H^2(B, \mathbb{Z}) \neq 0$, then a pre-polarization might not arise from a polarization; cf. [GS06, Th. 2.34].

Example 1.14. Returning to the degeneration of a cubic in Example 1.12, polarizing the degeneration with the restriction of $\mathcal{O}_{\mathbb{P}^3}(1)$, one obtains the intersection complex $(B, \mathscr{P})$ which looks like



Again, this figure is misleading. There are three standard simplices, and the boundary is in fact a straight line with respect to the affine structure.

Let $(B, \mathscr{P})$ be an integral tropical manifold. Define an (*integral*) *affine function* on an open set $U \subseteq B$ to be a continuous map $U \to \mathbb{R}$ that is (integral) affine on $U \setminus \Delta$. Similarly, an (integral) PL-*function* ("piecewise linear") on U is a continuous map $\varphi : U \to \mathbb{R}$ such that if $S_\tau : U_\tau \to \mathbb{R}^k$ is the fan structure along $\tau \in \mathscr{P}$, then $\varphi|_{U \cap U_\tau} = \lambda + S_\tau^*(\bar{\varphi})$ for an (integral) affine function $\lambda : U_\tau \to \mathbb{R}$ and a function $\bar{\varphi} : \mathbb{R}^k \to \mathbb{R}$ that is piecewise (integral) linear with respect to the fan Σ_τ [GS06, Def. 1.43]. The integral affine functions and integral PL-functions define sheaves $\mathcal{A}\!f\!f(B,\mathbb{Z})$ and $\mathcal{PL}_{\mathscr{P}}(B,\mathbb{Z})$ on B.

Remark 1.15. In the cone picture, the boundary of the cone C with $\operatorname{Spec}\Bbbk[C\cap\mathbb{Z}^{n+1}]\to\operatorname{Spec}\Bbbk[\mathbb{N}]$ describing $\pi:\mathfrak{X}\to T$ locally can be viewed as the graph of an integral PL-function on Σ_v, well defined up to integral affine functions. These glue to a section φ of the sheaf $\mathcal{PL}_{\mathscr{P}}(B,\mathbb{Z})/\mathcal{A}\!f\!f(B,\mathbb{Z})$ of multi-valued, integral PL-functions. This is additional information about étale models of the degeneration near the zero-dimensional strata, or in other words, about the log smooth morphism to (T,O).

Note that a local representative of φ as a PL-function is *strictly convex*, so it induces a strictly convex, integral affine function on each fan Σ_τ, $\tau\in\mathscr{P}$. We call such a φ a *polarization* of $(B,\mathscr{P})$ and $(B,\mathscr{P},\varphi)$ a *polarized*, *integral tropical manifold*.

Construction 1.16 (The discrete Legendre transform [GS06, §1.4]). There is a duality transformation on the set of polarized, integral tropical manifolds, the *discrete Legendre transformation* $(B,\mathscr{P},\varphi)\mapsto(\check{B},\check{\mathscr{P}},\check{\varphi})$, which is at the heart of our mirror symmetry construction. It works by defining $\check{\mathscr{P}}$ as the opposite category of $\mathscr{P}$. Thus any $\tau\in\mathscr{P}$ is also an object of $\check{\mathscr{P}}$, denoted $\check{\tau}$ for clarity. Then $\check{F}:\check{\mathscr{P}}\to\underline{\mathrm{LPoly}}$ maps $\check{\tau}$ to the Newton polyhedron of $\bar{\varphi}$, where $\varphi=\lambda+S_\tau^*(\bar{\varphi})$ is as in the definition of PL-functions above. If $\sigma\in\mathscr{P}$ is a maximal cell, then $\check{\sigma}\in\check{\mathscr{P}}$ is a vertex. In this case, the boundary of the dual of the cone over $\sigma\times\{1\}$ defines the graph of $\check{\varphi}$ at $\check{\sigma}$, hence also the fan structure at $\check{\sigma}$. Applying this transformation again retrieves the original polarized tropical manifold ([GS06, Prop. 1.51]). Moreover, the discrete Legendre transformation preserves positivity ([GS06, Prop. 1.55]). With the above correspondence between strictly convex piecewise linear functions on noncomplete fans and unbounded cells, the proofs of these facts for closed B in [GS06, §1.4] extend in a straighforward manner to the general case.

One important remark is that there is a homeomorphism of $B\setminus\partial B$ and $\check{B}\setminus\partial\check{B}$ mapping the discrimant loci onto each other. If B is closed, the homeomorphism can be constructed by piecewise affine identifications of the (perturbed) barycentric subdivisions. This is really just a homeomorphism and certainly does not in general preserve the affine structures. The general case is a little more subtle, and we leave the details to the interested reader because we do not need this result here. What we do use, however, is the consequence that we can view a local system on $B\setminus\Delta$ as a local system on $\check{B}\setminus\check{\Delta}$ and vice versa. In fact, a local system $\mathcal{F}$ on $B\setminus\check{\Delta}$ is nothing but a collection of groups $\mathcal{F}_v$, $\mathcal{F}_\sigma$, one for each vertex v and maximal cell σ of $\mathscr{P}$, together with a generization isomorphism

$$\psi_{\sigma v}:\mathcal{F}_v\longrightarrow\mathcal{F}_\sigma,$$

whenever $v \in \sigma$. The corresponding local system on $\check{B}$ then has the generization isomorphisms $\psi_{\check{v}\check{\sigma}}^{-1}$.

By construction it should also be clear that the discrete Legendre transform of the cone picture $(B, \mathscr{P}, \varphi)$ of a toric degeneration of CY-pairs with pre-polarized central fibre X leads to the fan picture $(\check{B}, \check{\mathscr{P}})$ of the same degeneration. The polarization $\check{\varphi}$ on $(\check{B}, \check{\mathscr{P}})$ thus obtained is directly related to the polarization of the irreducible components of X via the usual description of ample line bundles on a toric variety by strictly convex, integral PL-functions on the associated fan, well defined up to linear functions. This interpretation of a strictly convex, multi-valued, integral PL-function is what motivates us to call it a polarization of $(B, \mathscr{P})$.

1.2. *Algebraic data.* Our aim in this paper is to construct a toric degeneration of CY-pairs starting from a polarized, integral tropical manifold $(B, \mathscr{P}, \varphi)$, using the cone picture. This process requires additional, generally nondiscrete input that we now describe.

Let $v \in \mathscr{P}$ be a vertex. Choose a PL-function φ_v near v with $\varphi_v(v) = 0$ representing the polarization φ. The convex hull of the graph of φ_v defines a strictly convex, rational polyhedral cone $C_v \subseteq T_{B,v} \oplus \mathbb{R}$ and the associated toric monoid $P_v = C_v \cap (\Lambda_v \oplus \mathbb{Z})$. Then, according to Example 1.13 and Remark 1.15,

$$\Bbbk[t] \longrightarrow \Bbbk[P_v], \quad t \longmapsto z^{(0,1)}$$

is an étale local model for any pre-polarized toric degeneration of CY-pairs with cone picture $(B, \mathscr{P}, \varphi)$ near the zero-dimensional toric stratum $\{x\} \subseteq \mathfrak{X}$ corresponding to v. By integrality of φ it follows that the central fibre of this local model is the union of affine toric varieties

$$\bigcup_K \operatorname{Spec} \Bbbk[K \cap (\Lambda_v \oplus \mathbb{Z})],$$

where the union runs over all facets $K \subseteq C_v$ not containing $(0,1)$. These are indexed by maximal cells σ containing v. Now the projection $C_v \to T_{B,v}$ defines integral affine isomorphisms of these facets of C_v with the maximal cones in the fan Σ_v. Hence this union depends only on Σ_v, and we use the notation $\operatorname{Spec} \Bbbk[\Sigma_v]$ for it. The justification for this is that if Σ is a fan on $M_{\mathbb{R}}$ of convex, but not necessarily strictly convex cones, with $|\Sigma|$ convex, then

$$m + m' := \begin{cases} m + m', & \exists\, K \in \Sigma : m, m' \in K, \\ \infty, & \text{otherwise} \end{cases}$$

defines a monoid structure on $(M \cap |\Sigma|) \cup \{\infty\}$. By formally putting $z^\infty = 0$ this yields a monoid $\Bbbk$-algebra generated by monomials z^m, $m \in M \cap |\Sigma|$, which we denote as $\Bbbk[\Sigma]$. In the case of $\Sigma = \Sigma_v$ it is clearly isomorphic to

$\Bbbk[P_v]/(z^{(0,1)})$, and hence $\Bbbk[\Sigma_v]$ is indeed the coordinate ring of our local model of X.

Since the central fibre X of any toric degeneration is a union of toric varieties, open subspaces isomorphic to $\operatorname{Spec}\Bbbk[\Sigma_v]$ for $v \in \mathscr{P}^{[0]}$ cover X, and their mutual intersections have similar descriptions as unions of affine toric varieties. We thus arrive at the following gluing construction of X.

Construction 1.17 ([GS06, §2.2]). First define certain open subsets of $\operatorname{Spec}\Bbbk[\Sigma_v]$ as follows. If $\tau \in \mathscr{P}$ and $v \in \tau$ is a vertex, define the fan of convex, but not necessarily strictly convex, cones

$$\tau^{-1}\Sigma_v := \{K_e + \Lambda_{\tau,\mathbb{R}} \,|\, K_e \in \Sigma_v,\, e : v \to \sigma \text{ factors through } \tau\}.$$

Recall $\Lambda_\tau = \Lambda_v \cap T_{\tau,v}$. The quotient of $\tau^{-1}\Sigma_v$ by the linear space spanned by τ equals the fan Σ_τ defining the fan structure along τ. The fan $\tau^{-1}\Sigma_v$ depends only on τ. For a different choice of $v' \in \tau$, we can identify $\tau^{-1}\Sigma_v$ and $\tau^{-1}\Sigma_{v'}$ canonically. This is done via a piecewise linear identification of Λ_v and $\Lambda_{v'}$ which identifies the cones $K_e + \Lambda_{\tau,\mathbb{R}}$ and $K_{e'} + \Lambda_{\tau,\mathbb{R}}$ for $e : v \to \sigma$, $e' : v' \to \sigma$, via parallel transport between v and v' through $\sigma \in \mathscr{P}_{\max}$. While the induced bijection $\Lambda_v \to \Lambda_{v'}$ is not in general linear due to the effect of monodromy, the scheme

$$V(\tau) := \operatorname{Spec}\Bbbk[\tau^{-1}\Sigma_v]$$

is well defined up to unique isomorphism, independently of the choice of vertex $v \in \tau$; see [GS06, Construction 2.15]. Note that in [GS06], which mostly uses the fan picture, this space was denoted as $V(\check{\tau})$. The toric strata of $V(\tau)$ are in bijection with the cones in $\tau^{-1}\Sigma_v$, hence to morphisms $e : \tau \to \sigma$. The notation is $V_e \subseteq V(\tau)$.

Moreover, if $\omega \subseteq \tau$, there is a well-defined map of fans

$$\Big\{K_e + \Lambda_{\omega,\mathbb{R}} \in \omega^{-1}\Sigma_v \,\Big|\, e : v \to \sigma \text{ factors through } \omega \to \tau\Big\} \longrightarrow \tau^{-1}\Sigma_v,$$

which defines an open embedding

$$V(\tau) = \operatorname{Spec}\Bbbk[\tau^{-1}\Sigma_v] \longrightarrow \operatorname{Spec}\Bbbk[\omega^{-1}\Sigma_v] = V(\omega).$$

We can compose this embedding with any toric automorphism of $V(\tau)$. These are in bijection with maps

$$\mu : \Lambda_v \cap |\tau^{-1}\Sigma_v| \longrightarrow \Bbbk^\times \tag{1.4}$$

which are *piecewise multiplicative* with respect to $\tau^{-1}\Sigma_v$, meaning that the restriction to any cone in $\tau^{-1}\Sigma_v$ is a homomorphism of monoids. In other words, for each $\sigma \in \mathscr{P}_{\max}$ with $\tau \subseteq \sigma$, we have a homomorphism $\mu_\sigma : \Lambda_\sigma \to \Bbbk^\times$ such that for any σ, σ' containing τ, the restrictions $\mu_\sigma|_{\Lambda_{\sigma\cap\sigma'}}$ and $\mu_{\sigma'}|_{\Lambda_{\sigma\cap\sigma'}}$

coincide. On the irreducible component $V_e \subseteq V(\tau)$, $e : \tau \to \sigma$, the toric automorphism of $V(\tau)$ associated to μ is given by

$$z^m \longmapsto \mu_\sigma(m) \cdot z^m.$$

This description also shows that there is a space $\mathrm{PM}(\tau)$ of piecewise multiplicative functions along τ that depends only on the embedding of τ in B ($\mathrm{P\check{M}}(\check{\tau})$ in the notation of [GS06]). A choice of vertex $v \in \tau$ gives a representation of $\mathrm{PM}(\tau)$ by maps $\Lambda_v \cap |\tau^{-1}\Sigma_v| \to \Bbbk^\times$ that are piecewise multiplicative with respect to $\tau^{-1}\Sigma_v$.

To explain how piecewise multiplicative functions can be used to change the gluing of our affine pieces $V(v)$ we translate the definition of "open gluing data for the fan picture" ([GS06, Def. 2.25]) to the cone picture.

Definition 1.18. *Open gluing data* for $(B, \mathscr{P})$ are data $s = (s_e)_{e \in \mathrm{Hom}\, \mathscr{P}}$ with the following properties:

(1) $s_e \in \mathrm{PM}(\tau)$ for $e : \omega \to \tau$.
(2) $s_{\mathrm{id}_\tau} = 1$ for every $\tau \in \mathscr{P}$.
(3) If $e \in \mathrm{Hom}(\tau, \tau')$, $f \in \mathrm{Hom}(\tau', \tau'')$, then $s_{f\circ e} = s_f \cdot s_e$ wherever defined:

$$s_{f\circ e,\sigma} \;=\; s_{f,\sigma} \cdot s_{e,\sigma} \qquad \text{for all } \sigma \in \mathscr{P}_{\max} \text{ with } \sigma \supseteq \tau''.$$

Two open gluing data s, s' are *cohomologous* if there exist $t_\tau \in \mathrm{PM}(\tau)$, $\tau \in \mathscr{P}$ with $s'_e = t_\tau t_\omega^{-1} \cdot s_e$ for any $e : \omega \to \tau$.

"Open" refers to the fact that we glue the open sets $V(v) := \operatorname{Spec} \Bbbk[\Sigma_v]$ rather than their respective irreducible components. If s are open gluing data, then s_e for $e : \omega \to \tau$ defines an automorphism of $V(\tau) = \operatorname{Spec} \Bbbk[\tau^{-1}\Sigma_v]$ that we denote by the same symbol s_e. Thus, for any $e : \omega \to \tau$, we obtain an open embedding by composing $V(\tau) \to V(\omega)$ with s_e^{-1}. (For consistency we need to work with s_e^{-1} instead of s_e; see [GS06, proof of Lemma 2.29] for how this arises.) This yields a category of affine schemes and open embeddings, and saying that the open sets glue means that there is a scheme $X = X_0(\check{B}, \check{\mathscr{P}}, s)$, together with an open morphism

$$p : \coprod_{\omega \in \mathscr{P}} V(\omega) \longrightarrow X_0(\check{B}, \check{\mathscr{P}}, s)$$

that is a colimit for this category. The existence of $X_0(\check{B}, \check{\mathscr{P}}, s)$ is shown in [GS06, §2.2]. Moreover, two open gluing data give rise to isomorphic schemes if and only if they are cohomologous ([GS06, Prop. 2.32]).

Conversely, according to [GS06, Th. 4.14], any central fibre of a toric degeneration of CY-varieties with a pre-polarization arises in this way. All these results extend in a straightforward manner to CY-pairs. This ends Construction 1.17.

Remark 1.19. Our notation here differs somewhat from [GS06] since s are in fact open gluing data for the fan picture expressed in the cone picture, rather than gluing data for the cone picture as in [GS06, Def. 2.3]. In particular, according to [GS06, Th. 2.34], there is an obstruction to gluing the given ample line bundles on the irreducible components of $X_0(\check{B}, \check{\mathscr{P}}, s)$, which in turn needs not be projective.

The effect of monodromy on open gluing data $s = (s_e)$ (Definition 1.18) gives rise to a set of elements in $\Bbbk^\times$ that we now introduce for later use.

Definition 1.20 (Cf. [GS06, Def. 3.25]). Given $\mu \in \mathrm{PM}(\tau)$ for some $\tau \in \mathscr{P}$, for any $\rho \in \mathscr{P}^{[n-1]}$ containing τ and any vertex $v \in \tau$, we can measure the *change of μ along ρ with respect to v* as follows. Let σ, σ' be the unique maximal cells with $\rho = \sigma \cap \sigma'$. Let $m \in \Lambda_\sigma$ map to a generator of $\Lambda_\sigma/\Lambda_\rho \simeq \mathbb{Z}$, pointing from σ to σ'. Let $m' \in \Lambda_{\sigma'}$ be obtained by parallel transport of m through v. Then

$$D(\mu, \rho, v) := \frac{\mu_\sigma(m)}{\mu_{\sigma'}(m')} \in \Bbbk^\times \tag{1.5}$$

does not depend on the choice of m and is also invariant under changing μ by a homomorphism $\Lambda_v \to \Bbbk^\times$.

Remark 1.21. Formula 1.2 readily computes the dependence of $D(\mu, \rho, v)$ on v:

$$D(\mu, \rho, v') = \mu(m^\rho_{v'v})^{-1} \cdot D(\mu, \rho, v); \tag{1.6}$$

see [GS06, Rem. 3.26] for details.

Apart from open gluing data, which specify the central fibre as a scheme, we need some weak algebraic information about the embedding into $\mathfrak{X}$. This is what the log structure does, and it is more than just the discrete information retained by the polarization φ on $(B, \mathscr{P})$ in the cone picture. For the purposes of this paper it seems appropriate to explain this structure in an explicit, nonabstract form, following [GS06, p. 263f]. Let $X = X_0(\check{B}, \check{\mathscr{P}}, s)$ be a scheme obtained from open gluing data in the cone picture as just described. As seen in Construction 1.17, X has a covering by open sets isomorphic to $V(v) = \operatorname{Spec} \Bbbk[\Sigma_v]$, and $V(v)$ can be viewed as the toric Cartier divisor $z^{(0,1)} = 0$ in the affine toric variety $\operatorname{Spec} \Bbbk[P_v]$. For $m \in \Lambda_v \cap |\Sigma_v|$, the closure of the complement of the zero locus of z^m,

$$V_m(v) = \mathrm{cl}\{x \in V(v) \mid z^m \in \mathcal{O}^\times_{X,x}\},$$

is a union of irreducible components. Now a *chart* (*for a log smooth structure on $X = X_0(\check{B}, \check{\mathscr{P}}, s)$ of type given by φ*) is an open set $U \subseteq V(v)$ for some vertex v, together with $h_m \in \Gamma(U \cap V_m(v), \mathcal{O}^\times_{V(v)})$ for $m \in \Lambda_v \cap |\Sigma_v|$ that behaves piecewise multiplicatively with respect to Σ_v in the following sense:

$$m, m' \in \Lambda_v \cap |\Sigma_v| \implies h_m \cdot h_{m'} = h_{m+m'} \text{ on } V_{m+m'}(v). \tag{1.7}$$

The vertex v is part of the data defining a chart, and $p(U) \subseteq X$ is the *support* of the chart. Note that if there is no cone in Σ_v containing m, m', then $V_{m+m'}(v) = \emptyset$ and this condition is empty. Two charts (h_m), (h'_m) defined on the same open subset $U \subseteq V(v)$ are *equivalent* if there exists a homomorphism $\lambda : \Lambda_v \to \Gamma(U, \mathcal{O}_X^\times)$ with

$$h'_m = \lambda(m) \cdot h_m$$

for any $m \in \Lambda_v \cap |\Sigma_v|$.

In log geometry, a chart (for a fine, saturated log structure on X) is just a morphism from an (étale) open subset of X to an affine toric variety. This relates to our definition as follows: For any $m \in \Lambda_v \cap |\Sigma_v|$ consider $h_m \cdot z^m$ as function on U by continuation by zero. Then

$$z^m \longmapsto h_m \cdot z^m$$

defines an étale morphism $U \to \operatorname{Spec} \Bbbk[\Sigma_v]$; the composition with the closed embedding into $\operatorname{Spec} \Bbbk[P_v]$ is the associated chart in log geometry. Together with the distingushed monomial $z^{(0,1)} \in \operatorname{Spec} \Bbbk[P_v]$ this is a chart for a log smooth morphism to the standard log point. It provides our local model for a toric degeneration $(\mathfrak{X} \to T, \mathfrak{D})$ of CY-pairs with central fibre X as in Definition 1.8.

Remark 1.22. There is an important implicit dependence of our log-geometric chart on φ, which we suppress in our notion of charts. We can do this because φ fixes the *type of log structure* as explained in [GS06, Defs. 3.15 and 3.16].

A chart defined on $U \subseteq V(v)$ can be restricted to $U' \subseteq U$ simply by restricting the h_m. To compare arbitrary charts it remains to explain how to change the reference vertex. So let $(h_m)_{m \in \Lambda_v \cap |\Sigma_v|}$ be a chart defined on a nonempty $U \subseteq V(v)$, and assume $v' \in \mathscr{P}$ is another vertex with $p(U) \subseteq p\big(V(v')\big)$. Let σ be a maximal cell containing v and v'. If no such cell exists, $p\big(V(v)\big) \cap p\big(V(v')\big) = \emptyset$. Otherwise let $\Phi_{v'v}(s) : U \to U'$ be the gluing isomorphism, that is, the composition of $p|_U$ with the inverse of $p|_{V(v')}$. Write $e : v \to \sigma$, $e' : v' \to \sigma$, and denote by $K_e \in \Sigma_v$, $K_{e'} \in \Sigma_{v'}$ the tangent wedges of σ at v and v', respectively. Then for any $m \in K_e \cap \Lambda_v$, it holds that $V_e \subseteq V_m(v)$. Hence, by (1.7), the map

$$K_e \cap \Lambda_v \longrightarrow \Gamma(U \cap V_e, \mathcal{O}_{V_e}^\times), \quad m \longmapsto h_m|_{V_e \cap U}$$

is a homomorphism. Note that the collection of these homomorphisms for all σ determine the chart. Now let $\tau \subseteq \sigma$ denote the minimal cell containing v

and v'. Then parallel transport through σ gives the identification

$$K_{e'} + \Lambda_{\tau,\mathbb{R}} = K_e + \Lambda_{\tau,\mathbb{R}}.$$

Moreover, since the tangent wedge to v in τ is contained in K_e, the above homomorphism extends to $(K_e \cap \Lambda_v) + \Lambda_\tau$. Let h^σ_m denote the image of $m \in (K_e \cap \Lambda_v) + \Lambda_\tau$ under this extension. We are then able to define a chart on $U' \subseteq V(v')$ by pulling back the h^σ_m to U':

$$K_{e'} \cap \Lambda_v \longrightarrow \Gamma(U \cap V_{e'}, \mathcal{O}^\times_{V_{e'}}), \quad m \longmapsto \left(\Phi_{v'v}(s)^{-1}\right)^*(h^\sigma_m).$$

Finally define two charts, (h_m) on $U \subseteq V(v)$ and (h'_m) on $U' \subseteq V(v')$, to be *locally equivalent* if any $x \in U$ has an open neighbourhood W such that $(h_m|_W)$ is equivalent to the pull-back of $(h'_m|_{W'})$ to $V(v)$, where $W' = (p|_{V(v')})^{-1}(p(W))$.

Definition 1.23. An *atlas* (for a log smooth structure on $X = X_0(\check{B}, \check{\mathscr{P}}, s)$, of type defined by φ) is a system of locally equivalent charts. A *log smooth structure* on X (of type defined by φ) is a maximal atlas on the complement of a closed subset $Z \subseteq X$ of codimension two which does not contain any toric stratum.

A *pre-polarized toric log* CY-*pair* is a scheme of the form $X = X_0(\check{B}, \check{\mathscr{P}}, s)$ together with a polarization φ of $(B, \mathscr{P})$ and a log smooth structure.

A (formal) toric degeneration of CY-pairs $(\pi : \mathfrak{X} \to T, \mathfrak{D})$ induces a log smooth structure on the central fibre. Theorem 3.22 and Definition 4.17 in [GS06] describe the set of log smooth structures on X potentially arising in this way, as a quasi-affine subvariety of the space of sections of a coherent sheaf $\mathcal{LS}^+_{\mathrm{pre},X}$ supported on X_{sing}. On $V(v)$ this sheaf is isomorphic to $\bigoplus_e \mathcal{O}_{V_e}$, where the sum runs over all $e : v \to \rho$ with $\dim \rho = n-1$. A section $(f_e)_e$ of $\mathcal{LS}^+_{\mathrm{pre},V(v)} = \bigoplus_e \mathcal{O}_{V_e}$ over $U \subseteq V(v)\setminus Z$ defined by a log smooth structure obeys the following compatibility condition along toric strata of codimension two. Consider an $(n-2)$-cell $\tau \in \mathscr{P}$ with $v \in \tau$, let $\rho_1, \ldots, \rho_l$ be a cyclic ordering of the $(n-1)$-cells containing τ and write $h : v \to \tau$. Let $\check{d}_{\rho_i} \in \Lambda^\perp_{\rho_i} \subseteq \Lambda^*_v$ be generators compatible with this cyclic ordering. It turns out the following condition is satisfied by tuples (f_e) defining a log structure:

$$\prod_{i=1}^{l} \check{d}_{\rho_i} \otimes_{\mathbb{Z}} f_{e_i}|_{V_h} = 0 \otimes 1 \quad \text{in } \Lambda^*_v \otimes_{\mathbb{Z}} \Gamma(U, \mathcal{O}^\times_{V_h}). \tag{1.8}$$

Note that the product treats the first factor additively and the second factor multiplicatively. Conversely, any rational section of $\mathcal{LS}^+_{\mathrm{pre},V(v)}$ with zeros and poles not containing any toric stratum and fulfilling (1.8) defines a log smooth structure on $V(v)$. This is proved by showing that giving an equivalence class of charts on an open subset $U \subseteq V(v)$ is equivalent to giving sections $f_e \in \Gamma(U, \mathcal{O}^\times_{V_e})$ fulfilling (1.8).

Given a chart $(h_m)_{m\in\Lambda_v\cap|\Sigma_v|}$ on $U\subseteq V(v)$, the associated section f_e of $\mathcal{O}_{V_e}|_U$ for $e:v\to\rho$ is defined as follows. Since ρ is of codimension one, there exist two unique maximal cells σ^+, σ^-, with $\rho=\sigma^+\cap\sigma^-$. Write $g^\pm:v\to\sigma^\pm$. Working in an affine chart at v let $m^+\in\Lambda_v\cap K_{g^+}$ be a generator of Λ_v/Λ_ρ. For appropriate $m_0\in K_e$ it holds $m_0-m^+\in K_{g^-}$, and in any case $m_0+m_+\in K_{g^+}$. Then

$$f_e=\frac{h_{m_0}^2|_{V_e}}{h_{m_0-m^+}|_{V_e}\cdot h_{m_0+m^+}|_{V_e}}\in\mathcal{O}_{V_e}^\times(U).$$

This is independent of the choices of m^+ and m_0. The meaning of f_e is that at the generic point of V_e a local model for a toric degeneration with central fibre X is given by

$$V(zw-f_e\cdot t^l)\subseteq\operatorname{Spec}\Bbbk[z,w,t,x_1,\ldots,x_{n-1}]. \tag{1.9}$$

Explicitly, z and w may be taken as the continuations by zero of z^{m^+} and z^{-m^+}, and $l\in\mathbb{N}$ is the integral length of the one-cell $\check\rho\in\check{\mathscr{P}}$. From this description it should be plausible that the geometrically meaningful log smooth structures on $V(v)$ are defined by sections $(f_e)_e$ of $\bigoplus_e\mathcal{O}_{V_e}^\times$ over $V(v)\setminus Z$ that extend as sections of $\mathcal{LS}^+_{\mathrm{pre},V(v)}=\bigoplus_e\mathcal{O}_{V_e}$. Such log smooth structures are called *positive*. (This corresponds to positivity for integral tropical manifolds.) In fact, the log smooth structure associated to a toric degeneration of log CY-pairs is positive ([GS06, Prop. 4.20]).

The global structure of $\mathcal{LS}^+_{\mathrm{pre},X}$ follows from the formula describing the change of charts. Not surprisingly this depends on the choice of open gluing data s describing the patching of the open sets $V(v)$ to yield $X=X_0(\check B,\check{\mathscr{P}},s)$. Let $v,v'\in\mathscr{P}$ be vertices, $U\subseteq V(v)$ with $p(U)\subseteq p\big(V(v')\big)$ and $(f_e)_{e:v\to\rho}\in\Gamma(U,\mathcal{LS}^+_{\mathrm{pre},V(v)})$. Denote by $\Phi_{v'v}(s)$ the gluing isomorphism from $U\subseteq V(v)$ to an open subscheme $U'\subseteq V(v')$. Then the corresponding section of $\mathcal{LS}^+_{\mathrm{pre},V(v')}$ on U' is $(f_{e'})_{e':v'\to\rho}$ with

$$\Phi_{v'v}(s)^*(f_{e'})=\frac{D(s_{e'},\rho,v')}{D(s_e,\rho,v)}s_e(m^\rho_{v'v})z^{m^\rho_{v'v}}f_e; \tag{1.10}$$

see [GS06, Th. 3.27]. If $\tau\in\mathscr{P}$ is a cell containing v,v', then this equation can be written more symmetrically by viewing $f_{e'}$ and f_e as functions on an open subset of $V(\tau)$ via the canonical open embeddings $V(\tau)\to V(v')$, $V(\tau)\to V(v)$:

$$D(s_{g'},\rho,v')^{-1}s_{g'}^{-1}(f_{e'})=z^{m^\rho_{v'v}}D(s_g,\rho,v)^{-1}s_g^{-1}(f_e), \tag{1.11}$$

where $g:v\to\tau$, $g':v'\to\tau$. (Common factors arising from $\tau\to\rho$ cancel.)

These formulae provide an explicit description of $\mathcal{LS}^+_{\mathrm{pre},X}$ as an abstract sheaf. For each $\rho\in\mathscr{P}^{[n-1]}$ fix a vertex $v\in\rho$. Then the $-m^\rho_{vv'}\in\Lambda_\rho$ for $v'\in\rho$ define a PL-function on the normal fan of ρ ([GS06, Rem. 1.56]), hence an invertible sheaf $\mathcal{N}_\rho$ on the codimension-one stratum X_ρ corresponding to ρ. The isomorphism class of $\mathcal{N}_\rho$ is well defined since for a different v the PL-function only changes by a linear function. Now (1.10) shows

$$\mathcal{LS}^+_{\mathrm{pre},X} \simeq \bigoplus_{\rho\in\mathscr{P}^{n-1}} \mathcal{N}_\rho.$$

Summarizing, we now have a complete description of the space of positive log smooth structures on $X = X_0(\check{B},\check{\mathscr{P}},s)$ as sections of $\mathcal{LS}^+_{\mathrm{pre},X}$ with zeros not containing toric strata and fulfilling the compatibility condition (1.8) in codimension two. Sections of this sheaf are given explicitly via tuples $(f_e)_{e:v\to\rho}$ of regular functions on the codimension-one strata of each $V(v)\subseteq X$, obeying the compatibility condition (1.8) and the gluing conditions (1.10) or (1.11).

This description also defines uniquely, for each $\rho\in\mathscr{P}^{[n-1]}$, a codimension-two subscheme $Z_\rho\subseteq X$ with preimage in $V(v)$ the zero locus of $f_{v\to\rho}$ in $V_{v\to\rho}$. It is a Cartier divisor in the $(n-1)$-dimensional toric stratum $X_\rho\subseteq X$ associated to ρ. The canonical minimal choice of Z in Definition 1.23 is then

$$Z := \bigcup_{\rho\in\mathscr{P}^{[n-1]}} Z_\rho. \tag{1.12}$$

For given open gluing data, the space of sections of $\mathcal{LS}^+_{\mathrm{pre},X}$ giving rise to a positive log smooth structure on X can be empty or complicated. One main result in [GS06] is, however, that if $(B,\mathscr{P})$ is locally sufficiently rigid as an affine manifold ("*simple*") and positive, then the space of isomorphism classes of log CY-spaces with dual intersection complex $(B,\mathscr{P})$ equals $H^1(B,i_*\Lambda\otimes_{\mathbb{Z}}\Bbbk^\times)$, where $i: B\setminus\Delta\to B$ is the inclusion ([GS06, Th. 5.4]). This cohomology group is explicitly computable; it is a product of a finite group and $(\Bbbk^\times)^s$ with $s=\dim_{\mathbb{Q}} H^1(B,i_*\Lambda\otimes_{\mathbb{Z}}\mathbb{Q})$. Simplicity of $(B,\mathscr{P})$ requires certain polytopes associated to local monodromy to be elementary simplices ([GS06, Def. 1.60]). It implies primitivity of local monodromy in codimension two and can be explained by a complete list of local models in dimensions up to 3 ([GS06, Ex. 1.62]). In higher dimensions simplicity is a harder-to-understand maximal degeneracy condition. In the case of complete intersections in toric varieties it is related to Batyrev's MPCP resolutions [Gro05].

One final point in this section concerns the interaction of open gluing data with our chart description of log smooth structures. We already noted that cohomologous open gluing data lead to isomorphic X. But a change by a "coboundary" $(t_\sigma)_{\sigma\in\mathscr{P}}$ has the effect of composing our charts for the log smooth structure with the corresponding automorphisms of $V(\sigma)$. Because this changes the identification of $V(\sigma)$ with an open subset of X, this leads to a different section $(f_e)_{e:v\to\rho}$ of $\mathcal{LS}^+_{\mathrm{pre},X}|_{V(v)}=\mathcal{LS}^+_{\mathrm{pre},V(v)}$. We can, however, partly get rid of this nonuniqueness by requiring that for any $e: v\to\rho$ and

$x \in X$ the zero-dimensional stratum corresponding to v, we have

$$f_e(x) = 1. \tag{1.13}$$

In other words, the constant term of $f_e \in \Bbbk[K_e \cap \Lambda_v]$ equals 1. If this is the case, the log smooth structure and the corresponding section of $\mathcal{LS}^+_{\mathrm{pre},X}$ are called *normalized* for the given open gluing data ([GS06, Def. 4.23]).

If a section is not normalized, the constant terms of the f_e define the coboundary of a zero-cycle, whose application to the open gluing data leads to a normalized section ([GS06, p. 290]). Hence there is always some open gluing data for which the log smooth structure of a toric log CY-pair is normalized. The normalization condition may thus be interpreted as "gauge fixing", a process eliminating infinitesimal automorphisms. A generalization of this condition will turn up in our deformation process to get rid of pure t-terms (Definition 3.27).

1.3. *Statement of the Main Theorem.* The input to the Main Theorem is a pre-polarized, positive toric log CY-pair with intersection complex $(B, \mathscr{P}, \varphi)$. The log smooth structure needs to satisfy a certain local rigidity condition that we now explain. If $(B, \mathscr{P})$ is positive and simple, then for any open gluing data s the set of log smooth structures on $X = X_0(\check{B}, \check{\mathscr{P}}, s)$ is nonempty ([GS06, Th. 5.4]) and any choice fulfills the requested properties; see Remark 1.29 below. This implies the Reconstruction Theorem for integral tropical manifolds stated in the introduction.

The local rigidity condition involves the following notion of primitivity of Minkowski sums of polyhedra.

Definition 1.24. Let $\Xi_1, \ldots, \Xi_s \subseteq \mathbb{R}^n$ be polyhedra, $\Xi = \sum_i \Xi_i$ their Minkowski sum and $\Xi_i^{[0]}$, $\Xi^{[0]}$ the respective sets of vertices. For $v \in \Xi^{[0]}$, denote by $v(i) \in \Xi_i^{[0]}$ the unique solution to the equation $v = \sum_i v(i)$. Consider the following linear map:

$$F : \prod_i \mathrm{Map}\big(\Xi_i^{[0]}, \Bbbk\big) \longrightarrow \mathrm{Map}\big(\Xi^{[0]}, \Bbbk\big), \quad (a_i)_i \longmapsto \Big(v \mapsto \sum\nolimits_i a_i\big(v(i)\big)\Big).$$

Then $\Xi_1, \ldots, \Xi_s$ are called *Minkowski transverse* if $F(a_1, \ldots, a_s) = 0$ only has the trivial solutions

$$a_i(v) = \alpha_i \in \Bbbk, \quad \sum\nolimits_i \alpha_i = 0.$$

Remark 1.25. From $F(a_1, \ldots, a_s) = 0$, any edge of Ξ leads to a linear equation of the form

$$a_1(v_1) + \cdots + a_s(v_s) = a_1(v_1') + \cdots + a_s(v_s')$$

with $v_i, v_i' \in \Xi_i^{[0]}$, and the question is if these impose enough conditions to imply that a_i is constant. From this description it is not hard to see that $\Xi_1, \ldots, \Xi_s$

are Minkowski-transverse if they do not have parallel edges, for then only one v_i changes along any edge of Ξ, and there are enough edges to compare the values of any two vertices of any Ξ_i.

On the other hand, this is not a necessary condition. For example, the two polygons

$$\Xi_1 = \operatorname{conv}\Big\{(0,0),(1,0),(0,1)\Big\} \quad \text{and} \quad \Xi_2 = \operatorname{conv}\Big\{(0,0),(1,0),(1,1)\Big\}$$

are Minkowski transverse.

Suppose now we are given $X = X_0(\check{B}, \check{\mathscr{P}}, s)$ along with a positive log smooth structure. For the following definition, recall from (1.12) the components $Z_\rho \subseteq X$, $\rho \in \mathscr{P}^{[n-1]}$, of the singular locus of the log structure. For $\tau \subseteq \rho$ and $X_\tau \subseteq X$ the associated toric stratum, the Newton polytope of $Z_\rho \cap X_\tau$ is

$$\Delta_\tau(\rho) := \operatorname{conv}\{m^\rho_{vv'} \,|\, v' \in \tau\},$$

where $v \in \rho$ is a fixed choice of vertex. This is naturally a face of the monodromy polytope $\Delta(\rho)$ (1.3) parallel to Λ_τ and is well defined up to translation.

Definition 1.26. We call a positive, toric log Calabi-Yau space *locally rigid* if:

(i) For each $\rho \in \mathscr{P}^{[n-1]}$ and $\tau \in \mathscr{P}^{[n-2]}$, $\tau \subseteq \rho$, any integral point of $\Delta_\tau(\rho)$ is a vertex of $\Delta_\tau(\rho)$.

(ii) If $X_\tau \subseteq X$ denotes the toric stratum defined by $\tau \in \mathscr{P}^{[n-2]}$, then for any $\rho \in \mathscr{P}^{[n-1]}$ containing τ the intersection $Z_\rho \cap X_\tau$ is reduced and irreducible. Moreover, no more than three yield the same subset of X_τ.

(iii) For any $\tau \in \mathscr{P}^{[n-2]}$ let Ξ_i, $i = 1, \ldots, s$, be an enumeration of $\Delta_\tau(\rho)$ for $\rho \in \mathscr{P}^{[n-1]}$, $\tau \subseteq \rho$, modulo translation. Then $\Xi_1, \ldots, \Xi_s$ are Minkowski transverse.

Remark 1.27. Let $\tau \in \mathscr{P}^{[n-2]}$. By (ii) the polynomials $f_{v\to\rho}|_{V_{v\to\tau}}$ defining $Z_\rho \cap X_\tau$ locally are irreducible. The compatibility condition (1.8) then implies that for any ρ with $Z_\rho \cap X_\tau \neq \emptyset$,

$$\Big\{\check{d}_{\rho'} \,\Big|\, Z_{\rho'} \cap X_\tau = Z_\rho \cap X_\tau\Big\}$$

are the edge vectors of a polygon with edges of unit integral length in the two-dimensional affine space $\Lambda_\tau^\perp \subseteq \Lambda^*_{v,\mathbb{R}}$, well defined up to translation. By the second requirement in (ii) it has 2 or 3 edges. We thus obtain a set $\{\check{\Xi}_i\}$ of line segments and triangles and a corresponding set of convex PL-functions φ_i on Σ_τ.

Example 1.28. To illustrate the concept of local rigidity we give three local examples in dimension 4. Figure 1.2 shows the respective fans Σ_τ together with the values of a PL-function φ_τ pulling back to φ at the generators of the rays (in square brackets) and the functions $f_{v\to\rho}$. The five rays in the right-most figure

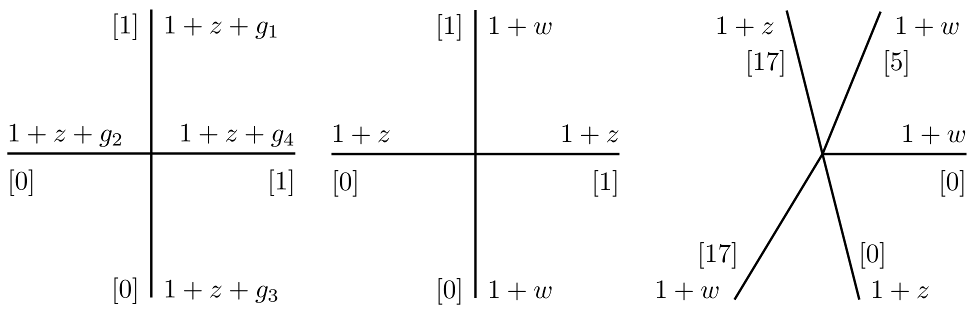


Figure 1.2.

are generated by $(1,0)$, $(2,5)$, $(-1,4)$, $(-3,-5)$ and $(1,-4)$. In either case $z = z^{(0,0,1,0)}, w = z^{(0,0,0,1)}$, denote monomials generating the coordinate ring of the maximal torus of X_τ, while $g_1,\ldots,g_4$ are arbitrary functions vanishing on X_τ. Note that all examples are normalized and fulfill (1.8). In the left figure all four functions $f_{v\to\rho}$ have the same restriction to X_τ, thus violating condition (ii). On the other hand, the middle and right examples are locally rigid. The polytopes according to Remark 1.27 are $\check\Xi_1 = [0,1]\times\{0\}$, $\check\Xi_2 = \{0\}\times[0,1]$, and $\check\Xi_1 = \operatorname{conv}\{0,(4,1)\}$, $\check\Xi_2 = \operatorname{conv}\{(0,0),(0,1),(-5,3)\}$, respectively.

Remark 1.29. If $(B,\mathscr{P})$ is simple ([GS06, Def. 1.60]), then $X_0(\check B,\check{\mathscr{P}},s)$ is locally rigid for any choice of open gluing data s. In fact, (i) follows readily from the fact that in this case $\Delta(\rho)$ is an elementary simplex for any $\rho\in\mathscr{P}^{[n-1]}$. This also implies that $Z_\rho\cap X_\tau$ is reduced and irreducible, as $\Delta_\tau(\rho)$ is the Newton polytope of $Z_\rho\cap X_\tau$. As for the second condition in (ii), let $v\in\tau$ be a vertex. Simplicity implies the existence of $p\le\operatorname{codim}\tau = 2$ polytopes $\check\Delta_i\subseteq\Lambda^\perp_{\tau,\mathbb{R}}\subseteq\Lambda^*_v$ with the following properties: (1) Each ray of Σ_τ labelled by a codimension-one ρ with $Z_\rho\cap X_\tau\neq\emptyset$ is generated by the inward normal of some $\check\Delta_i$. (2) The convex hull of $\bigcup_i\check\Delta_i\times\{e_i\}\subseteq\Lambda^\perp_{\tau,\mathbb{R}}\times\mathbb{R}^p$ is an elementary simplex. By (2) the tangent spaces of $\check\Delta_i$ are transverse. Thus either $p=1$ and $\check\Delta_1$ is a triangle or a line segment, or $p=2$ and $\check\Delta_1$, $\check\Delta_2$, are two nonparallel line segments (cf. [GS06, p. 217] for the latter case). This implies (ii). Condition (iii) follows from Remark 1.25 since Ξ_i are elementary simplices with $T_{\Xi_1}\oplus\cdots\oplus T_{\Xi_s}$ an internal direct sum.

We are now in position to state the main result of this paper. The notions of pre-polarized toric log CY-pair, formal toric degeneration of CY-pairs, local rigidity and positivity have been introduced in Definitions 1.23, 1.9, 1.26 and 1.4, respectively.

THEOREM 1.30. *Any locally rigid, positive, pre-polarized toric log* CY*-pair with proper irreducible components arises from a formal toric degeneration of* CY*-pairs.*

Note that the hypothesis of properness is equivalent to the boundedness of every cell of $\mathscr{P}$ in the intersection complex $(B, \mathscr{P})$. We are confident that this hypothesis is not necessary, but the unbounded case seems to raise a number of interesting points involving Landau-Ginzburg potentials, which are better dealt with elsewhere. However, most of the arguments we give will work in general, and we will, in the course of the proof, remark when we are using this boundedness hypothesis.

If we want actual families, we need to restrict to the projective or compact analytic setting.

COROLLARY 1.31. *Any projective, locally rigid, positive toric log* CY*-pair* (X, D) *with* $H^1(X, \mathcal{O}_X) = H^2(X, \mathcal{O}_X) = 0$ *arises from a projective toric degeneration of* CY*-pairs over* $\Bbbk[\![t]\!]$.

Proof. The cohomological assumptions imply that an ample line bundle on X extends to the formal degeneration. The result then follows from Grothendieck's Existence Theorem in formal geometry; see [Gro63, 5.4.5]. □

Remark 1.32. The assumption on cohomology is indeed superfluous. In fact, one can show that any ample line bundle L on X extends to the formal degeneration by applying our construction to the total space of L. Details of this observation will appear elsewhere.

Note that if $H^2(B, \Bbbk^\times) = 0$ for B the dual intersection complex associated to the toric log-CY pair, then projectivity of $X_0(\check{B}, \check{\mathscr{P}}, s)$ follows from the existence of the pre-polarization ([GS06, Th. 2.34]).

For the analytic formulation we just remark that all the notions we have introduced so far have straightforward analogues in the complex-analytic world. In view of the existence of versal deformations of (pairs of) compact complex spaces [Dou74], [Gra74], we obtain the following result.

COROLLARY 1.33. *Any compact, locally rigid, positive toric log* CY*-pair arises from a toric degeneration of analytic* CY*-pairs over* $\Bbbk[\![t]\!]$.

2. **Main objects of the construction**

The rest of the paper is devoted to the proof of Theorem 1.30. We thus fix, once and for all, a polarized, integral tropical manifold $(B, \mathscr{P}, \varphi)$, open gluing data s for $(B, \mathscr{P})$, and a positive log smooth structure on $X = X_0(\check{B}, \check{\mathscr{P}}, s)$

given by a compatible set of sections $(f_e)_e$ of $\mathcal{LS}^+_{\mathrm{pre},V(v)}$ fulfilling the multiplicative condition (1.11). We also assume that the discriminant locus $\Delta = \Delta(\{a_\tau\})$ does not contain any rational points, as discussed before Lemma 1.3.

In this entire section, we do not assume the cells of $\mathscr{P}$ to be bounded.

2.1. *Exponents*, *orders*, *rings*. We construct the deformation of (X, D) order-by-order. In each step the deformation is a colimit, in the category of separated schemes, of a system of affine schemes. This system is obtained by chopping B into polyhedral pieces, called *chambers*, of growing number for higher order; then $\mathscr{P}$ induces a stratification of each chamber, and there will be one ring for each inclusion of such strata. Homomorphisms are obtained either by changing strata within one chamber or by passing from one chamber to a neighbouring one. We start by explaining how to define the rings.

Recall that for a vertex $v \in B$ we have the local model $V(v) \subseteq \operatorname{Spec} \Bbbk[P_v]$ for $X \subseteq \mathfrak{X}$. Here P_v are the integral points over the graph in $T_{B,v} \oplus \mathbb{R}$ of a local representative φ_v of φ with $\varphi_v(v) = 0$. The disadvantage of this description is that it depends on the choice of representative of φ. To derive a more invariant point of view recall that the Legendre dual to v is a maximal cell $\check{v} \subseteq \Lambda^*_{v,\mathbb{R}}$ with vertices $\check{\sigma} = -\lambda_\sigma \in \Lambda^*_v$, where λ_σ are the linear functions defined by φ_v for the maximal cells σ containing v. We may then view $m = (\overline{m}, h) \in \Lambda_v \oplus \mathbb{Z}$ as an affine function on $\check{v}$ via the sequence of identifications

$$\Lambda_v \oplus \mathbb{Z} = (\Lambda^*_v)^* \oplus \mathbb{Z} = \Gamma(\operatorname{Int} \check{v}, \mathcal{A}\!f\!f(\check{B}, \mathbb{Z})).$$

The value of this affine function, denoted by m also, at $\check{\sigma}$ is

$$m(\check{\sigma}) = \langle \overline{m}, -\lambda_\sigma \rangle + h. \tag{2.1}$$

This gives the following description of P_v in terms of affine functions on $\check{v}$.

LEMMA 2.1. $P_v = \left\{ m = (\overline{m}, h) \in \Lambda_v \oplus \mathbb{Z} \,\middle|\, m|_{\check{v}} \geq 0 \right\}$.

Proof. The condition $m|_{\check{v}} \geq 0$ is equivalent to requiring that $m = (\overline{m}, h)$ lies in the dual of the cone generated by $\check{v} \times \{1\} \subseteq \Lambda^*_{v,\mathbb{R}} \oplus \mathbb{R}$. By our definition of Newton polyhedra this agrees with the convex hull of the graph of φ_v, whose integral points are P_v. $\square$

The preceding discussion motivates the following definition.

Definition 2.2. An *exponent* at a point $x \in B \setminus \Delta$ is an element of the stalk of $\mathcal{A}\!f\!f(\check{B}, \mathbb{Z})$ at x. An exponent on $\sigma \in \mathscr{P}_{\max}$ is an exponent at any $x \in \operatorname{Int} \sigma$, that is, an element of $\mathcal{A}\!f\!f(\check{B}, \mathbb{Z})_{\check{\sigma}}$. An exponent m on σ defines exponents at any $x \in \sigma \setminus \Delta$ that we denote by the same symbol.

The image of an exponent m at x (on σ) under the projection $\mathcal{A}\!f\!f(\check{B}, \mathbb{Z})_x \to \Lambda_x$ ($\mathcal{A}\!f\!f(\check{B}, \mathbb{Z})_{\check{\sigma}} \to \Lambda_\sigma$) is denoted $\overline{m}$.

Here we view $\mathcal{A}\!f\!f(\check{B},\mathbb{Z})$ as a locally constant sheaf on B as explained in Construction 1.16. Note that there is an exact sequence

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathcal{A}\!f\!f(\check{B},\mathbb{Z}) \stackrel{m\mapsto\overline{m}}{\longrightarrow} i_*\Lambda \longrightarrow 0;$$

see [GS06, Def. 1.11]. An exponent m at $x \in B \setminus \Delta$ extends uniquely to a section of $\mathcal{A}\!f\!f(\check{B},\mathbb{Z})$ on the interior of each $\sigma \in \mathscr{P}_{\max}$ containing x. This defines an element of $\mathcal{A}\!f\!f(\check{B},\mathbb{Z})_{\check{\sigma}}$ that we denote m_σ. Note that if $x \in \sigma \cap \sigma'$ for another maximal cell σ' containing x, then parallel transport in $\check{v}$ for a vertex v in the same connected component of $(\sigma\cap\sigma')\setminus\Delta$ as x maps $m_\sigma \in \mathcal{A}\!f\!f(\check{B},\mathbb{Z})_{\check{\sigma}}$ to $m_{\sigma'} \in \mathcal{A}\!f\!f(\check{B},\mathbb{Z})_{\check{\sigma}'}$

To define our rings we need various order functions on exponents. For $m \in P_v$ and $\sigma \in \mathscr{P}_{\max}$, $v \in \sigma$, the monomial z^m does not vanish on the irreducible component $V_{v\to\sigma}$ of $\operatorname{Spec} \Bbbk[\Sigma_v] \subseteq \operatorname{Spec} \Bbbk[P_v]$ if and only if $h = \lambda_\sigma(\overline{m})$. Thus by (2.1), $m_\sigma(\check{\sigma})$ equals the order of vanishing of z^m along $V_{v\to\sigma}$.

Definition 2.3. 1) Let m be an exponent at $x \in B\setminus\Delta$. Then the *order* of m on $\sigma \in \mathscr{P}_{\max}$, $x \in \sigma$ is

$$\operatorname{ord}_\sigma(m) := m_\sigma(\check{\sigma}).$$

Denote by $\mathscr{P}^\partial_{\max}$ the set of codimension-one cells of $\mathscr{P}$ contained in ∂B. Then for $x \in \rho$, the *order* of m on $\rho \in \mathscr{P}^\partial_{\max}$ is

$$\operatorname{ord}^\partial_\rho(m) := \langle \overline{m}, n_\rho\rangle,$$

where $n_\rho \in \Lambda^*_x$ is an inward pointing primitive normal to ρ.

For $A \subseteq B$ a subset contained in a cell of $\mathscr{P}$ and with $x \in A$, define

$$\operatorname{ord}_A(m) := \max\Big(\Big\{ \operatorname{ord}_\sigma(m) \,\Big|\, \sigma \in \mathscr{P}_{\max},\ A \subseteq \sigma\Big\} \cup \Big\{ \operatorname{ord}^\partial_\rho(m) \,\Big|\, \rho \in \mathscr{P}^\partial_{\max},\ A \subseteq \rho\Big\}\Big).$$

Note that for $A = \sigma$ a maximal cell this agrees with the previous definition.

2) Let $\omega \in \mathscr{P}$ be the minimal cell containing x. Define

$$P_x := \left\{ m \in \mathcal{A}\!f\!f(\check{B},\mathbb{Z})_x \,\middle|\, \begin{array}{l} \forall \sigma \in \mathscr{P}_{\max}, x \in \sigma : \operatorname{ord}_\sigma(m) \geq 0 \\ \exists \sigma' \in \mathscr{P}_{\max}, \omega \subseteq \sigma' : \overline{m} \in K_\omega \sigma' \end{array} \right\}.$$

The notion of order is compatible with local monodromy.

Lemma 2.4. *Let $\sigma, \sigma' \in \mathscr{P}_{\max}$ and let m be an exponent on σ. If m' is the result of parallel transport of m along a closed loop inside $(\sigma \cup \sigma')\setminus\Delta$, then*

$$\operatorname{ord}_\sigma(m) = \operatorname{ord}_\sigma(m').$$

If $\rho \in \mathscr{P}^\partial_{\max}$, ρ is a face of σ and m' is the result of parallel transport of m along a closed loop inside $\operatorname{Int}(\sigma) \cup \operatorname{Int}(\sigma') \cup ((\rho \cap \sigma') \setminus \Delta)$, *then*

$$\operatorname{ord}^\partial_\rho(m) = \operatorname{ord}^\partial_\rho(m').$$

Proof. On $(\sigma\cup\sigma')\setminus\Delta$, the locally constant sheaf $\mathcal{A}\!f\!f(\check{B},\mathbb{Z})$ splits non-canonically as $\Lambda\oplus\mathbb{Z}$. This follows from [GS06, Prop. 1.12] in connection with [GS06, Prop. 1.29] applied to $\tau=\sigma\cap\sigma'$. Moreover, by [GS06, Prop. 1.29] again, the monodromy for paths in $(\sigma\cup\sigma')\setminus\Delta$ acts trivially on $\Lambda_\tau^\perp\subseteq\Lambda_x^*$, which in an affine chart at $x\in\tau$ contains $\check\sigma$. Hence $m_\sigma(\check\sigma)$ remains unchanged under monodromy.

Similarly, for the second statement, let $\tau=\rho\cap\sigma'$. Then monodromy of loops in $\mathrm{Int}(\sigma)\cup\mathrm{Int}(\sigma')\cup(\tau\setminus\Delta)$ preserves $\Lambda_\tau^\perp\subseteq\Lambda_x^*$. But $\Lambda_\tau^\perp$ contains the normal to ρ, hence the result. □

In view of the lemma it makes sense to define, for $m\in\mathcal{A}\!f\!f(\check{B},\mathbb{Z})_{\check\sigma}$, the order on neighbouring maximal cells.

Definition 2.5. Let $\sigma,\sigma'\in\mathscr{P}_{\max}$ with $\sigma\cap\sigma'\neq\emptyset$ and $m\in\mathcal{A}\!f\!f(\check{B},\mathbb{Z})_{\check\sigma}$ an exponent on σ. Define the *order of m on σ'* as follows. Let $m'\in\mathcal{A}\!f\!f(\check{B},\mathbb{Z})_{\check\sigma'}$ be the result of parallel transport of m inside $\check v$ for any vertex $v\in\sigma\cap\sigma'$. Then

$$\mathrm{ord}_{\sigma'}(m):=\mathrm{ord}_{\sigma'}(m').$$

If, in addition, $\rho'\in\mathscr{P}^\partial_{\max}$ and ρ' is a face of σ', let $m'\in\mathcal{A}\!f\!f(\check{B},\mathbb{Z})_{\check\sigma'}$ be the result of parallel transport of m inside $\check v$ for any vertex $v\in\sigma\cap\rho'$. Then

$$\mathrm{ord}^\partial_{\rho'}(m):=\mathrm{ord}^\partial_{\rho'}(m').$$

With these definitions it now also makes sense, for m an exponent on $\sigma\in\mathscr{P}_{\max}$ and $A\subseteq\sigma$, to define $\mathrm{ord}_A(m)$ just as in Definition 2.3 above for exponents at a point.

Much of our strategy depends on the idea that if an exponent m is propagated in the direction $-\overline{m}$, the order of m increases. This is analogous to the behaviour of the order function ord_l in [KS06, §10.3].

Proposition 2.6. *Let m be an exponent at $x\in B\setminus(\Delta\cup\partial B)$, and let $\tau\in\mathscr{P}$ be the minimal cell containing x. If $\sigma^+,\sigma^-\in\mathscr{P}$ are maximal cells containing τ such that the corresponding maximal cones in Σ_τ contain $\overline{m}$ and $-\overline{m}$, respectively, then*

$$\mathrm{ord}_{\sigma^-}(m)=\max\Big\{\mathrm{ord}_\sigma(m)\,\Big|\,\sigma\in\mathscr{P}_{\max},\tau\subseteq\sigma\Big\},$$
$$\mathrm{ord}_{\sigma^+}(m)=\min\Big\{\mathrm{ord}_\sigma(m)\,\Big|\,\sigma\in\mathscr{P}_{\max},\tau\subseteq\sigma\Big\}.$$

Proof. Σ_τ is the normal fan of $\check\tau$; given $e^\pm:\tau\to\sigma^\pm$, the cones $K_{e^\pm}$ of Σ_τ are the normal cones to the vertices $\check\sigma^\pm$ of $\check\tau$ (see [GS06, Def. 1.38]). In particular, on $\check\tau$ an element of $K_{e^\pm}$ achieves its minimal value at $\check\sigma^\pm$, from which the result follows. □

Next we construct standard thickenings of the rings describing the toric strata locally. These will be our basic building blocks.

Construction 2.7 (The rings). For $\omega \in \mathscr{P}$ and $\sigma \in \mathscr{P}_{\max}$ with $\omega \subseteq \sigma$, define the monoid

$$P_{\omega,\sigma} := \left\{ m \in \mathcal{A}\!f\!f(\check{B},\mathbb{Z})_{\check{\sigma}} \;\middle|\; \begin{array}{l} \forall\, \sigma' \in \mathscr{P}_{\max},\, \omega \subseteq \sigma' : \operatorname{ord}_{\sigma'}(m) \geq 0 \\ \exists\, \sigma' \in \mathscr{P}_{\max},\, \omega \subseteq \sigma' : \overline{m} \in K_\omega \sigma' \end{array} \right\}.$$

The condition $\overline{m} \in K_\omega\sigma'$ is only relevant if $\omega \subseteq \partial B$. Note that if v is a vertex, then by Lemma 2.1, for any choice of representative φ_v of φ at v and any $\sigma \in \mathscr{P}_{\max}$, it holds that $P_{v,\sigma} = P_v$ canonically.

For any $\sigma' \in \mathscr{P}_{\max}$ containing ω, parallel transport through a vertex $v \in \sigma \cap \sigma'$ induces an isomorphism $P_{\omega,\sigma} \simeq P_{\omega,\sigma'}$. This isomorphism, however, generally depends on the choice of v. Thus σ serves as a reference cell.

Another manifestation of this phenomenon is as follows. If $x \in \operatorname{Int}(\omega) \setminus \Delta$, there is a canonical isomorphism $P_{\omega,\sigma} \simeq P_x$. Thus for any $x, x' \in \operatorname{Int}(\omega) \setminus \Delta$, any choice of maximal cell σ containing ω induces an isomorphism $P_x \simeq P_{x'}$, but this isomorphism generally depends on the choice of σ.

If $g : \omega \to \tau \in \operatorname{Hom}(\mathscr{P})$ and $\sigma \in \mathscr{P}_{\max}$ with $\tau \subseteq \sigma$, then for each $k \in \mathbb{N}$ we have a monoid ideal

$$P^{>k}_{g,\sigma} := \left\{ m \in P_{\omega,\sigma} \;\middle|\; \operatorname{ord}_\tau(m) > k \right\} \subseteq P_{\omega,\sigma}.$$

Let $I^{>k}_{g,\sigma}$ denote the ideal in $\Bbbk[P_{\omega,\sigma}]$ generated by $P^{>k}_{g,\sigma}$ and define

$$R^k_{g,\sigma} := \left(\Bbbk[P_{\omega,\sigma}] / I^{>k}_{g,\sigma} \right)_{f_{g,\sigma}}.$$

The function $f_{g,\sigma}$ at which we localize is constructed from the given section $(f_e)_e$ of $\mathcal{LS}^+_{\mathrm{pre},X}$ as follows. Choose a vertex $v \in \omega$ and write $e : v \to \omega$. Recall that $s_e \in \operatorname{PM}(\omega)$ is a map $\Lambda_v \cap |\omega^{-1}\Sigma_v| \to \Bbbk^\times$ that is piecewise multiplicative with respect to $\omega^{-1}\Sigma_v$. Thus the restriction of s_e to the maximal cone in $\omega^{-1}\Sigma_v$ given by σ defines a homomorphism $\zeta : \Lambda_\sigma \to \Bbbk^\times$. Composing this homomorphism with $\mathcal{A}\!f\!f(\check{B},\mathbb{Z})_{\check{\sigma}} \to \Lambda_\sigma$ leads to the following ring automorphism of $\Bbbk[P_{\omega,\sigma}]$:

$$s_{e,\sigma} : \Bbbk[P_{\omega,\sigma}] \longrightarrow \Bbbk[P_{\omega,\sigma}], \quad s_{e,\sigma}(z^m) = \zeta(\overline{m}) \cdot z^m.$$

Now for any $\rho \in \mathscr{P}^{[n-1]}$ containing τ, denote by e_ρ the composition $v \to \omega \to \tau \to \rho$ and let $K_{e_\rho} \in \Sigma_v$ be the corresponding cone of codimension one. Since $V_{v\to\rho} \subseteq V(v)$ equals $\operatorname{Spec} \Bbbk[K_{e_\rho} \cap \Lambda_v]$ we have the expansion $f_{e_\rho} = \sum_{\overline{m} \in K_{e_\rho} \cap \Lambda_v} f_{e_\rho,\overline{m}} z^{\overline{m}}$. The restriction of this function to $V_{v\to\tau}$ lifts canonically to $\Bbbk[P_{v,\sigma}]$ as $\sum_{m \in P_{v,\sigma}} f_{e_\rho,m} z^m$ with

$$f_{e_\rho,m} := \begin{cases} f_{e_\rho,\overline{m}}, & \overline{m} \in K_{e_\rho},\ \operatorname{ord}_\tau(m) = 0, \\ 0, & \text{otherwise.} \end{cases}$$

With the inclusion $P_{v,\sigma} \subseteq P_{\omega,\sigma}$ via parallel transport inside σ understood, we now define

$$f_{\rho,e,\sigma} := s_{e,\sigma}^{-1}\Bigg(\sum_{m\in P_{v,\sigma}} f_{e_\rho,m} z^m\Bigg) \in \Bbbk[P_{\omega,\sigma}] \tag{2.2}$$

and the localizing element as

$$f_{g,\sigma}^v := \prod_{\rho\supseteq\tau} f_{\rho,e,\sigma}.$$

If $\tau \in \mathscr{P}_{\max}$, then this product is empty, and we take $f_{g,\sigma}^v = 1$. Note that by the normalization condition, $f_{g,\sigma}^v$ has constant term 1.

For a different choice of vertex $e' : v' \to \omega$, equation (1.11) implies

$$s_{e',\sigma}^{-1}(f_{e'_\rho}) = C \cdot z^{m^\rho_{v'v}} s_{e,\sigma}^{-1}(f_{e_\rho})$$

for some $C \in \Bbbk^\times$. (Here we view $m^\rho_{v'v} \in \Lambda_\rho \subseteq \Lambda_\sigma$ as an element of $P_{\omega,\sigma}$ by taking the unique lift m under $P_{\omega,\sigma} \to \Lambda_\sigma$ with $\mathrm{ord}_\rho(m) = 0$. Similar identifications will occur throughout the text without further notice.) Thus

$$\begin{aligned} f_{g,\sigma}^{v'} &= \prod_{\rho\supseteq\tau} s_{e',\sigma}^{-1}\Bigg(\sum_{m\in P_{v',\sigma}} f_{e'_\rho,m} z^m\Bigg) \\ &= C' \prod_{\rho\supseteq\tau} z^{m^\rho_{v'v}} s_{e,\sigma}^{-1}\Bigg(\sum_{m\in P_{v,\sigma}} f_{e_\rho,m} z^m\Bigg) \;=\; C' z^{\ell m^\rho_{v'v}} f_{g,\sigma}^v, \end{aligned}$$

where $e'_\rho : v' \to \rho$ and $C' \in \Bbbk^\times$ is another constant. Now the monomials $z^{m^\rho_{v'v}}$ are invertible in $\Bbbk[P_{\omega,\sigma}]$ since $m^\rho_{v'v} \in \Lambda_\omega$. Hence the localization of $\Bbbk[P_{\omega,\sigma}]/I_{g,\sigma}^{>k}$ at $f_{g,\sigma}^v$ does not depend on the choice of $v \in \omega$. We set $f_{g,\sigma} := f_{g,\sigma}^v$ for any $v \in \omega$, viewed as well defined only up to invertible functions in $\Bbbk[P_{\omega,\sigma}]$.

More generally, if $I \subseteq \Bbbk[P_{\omega,\sigma}]$ is any monomial ideal with radical $I_{g,\sigma}^{>0}$, set

$$R_{g,\sigma}^I := \big(\Bbbk[P_{\omega,\sigma}]/I\big)_{f_{g,\sigma}}.$$

Any of these rings contains the distinguished monomial $z^{\mathbb{1}}$, where $\mathbb{1} \in \mathcal{A}\!f\!f(\check{B},\mathbb{Z})_{\check{\sigma}}$ is the constant 1 function. These monomials correspond to the deformation parameter t and are preserved by all our constructions. We therefore write $t = z^{\mathbb{1}}$ and keep in mind that we really work with $\Bbbk[t]$-algebras.

Remark 2.8. The meaning of the rings $R^k_{\omega\to\tau,\sigma}$ is as follows. The choice of $e : v \to \omega$ determines the local model $V(v) = \operatorname{Spec}\Bbbk[\Sigma_v] \subseteq \operatorname{Spec}\Bbbk[P_{v,\sigma}]$ for $X \subseteq \mathfrak{X}$, and the open subsets $\operatorname{Spec}\Bbbk[\omega^{-1}\Sigma_v] \subseteq \operatorname{Spec}\Bbbk[\Sigma_v]$ and $\operatorname{Spec}\Bbbk[P_{\omega,\sigma}] \subseteq \operatorname{Spec}\Bbbk[P_{v,\sigma}]$. The open gluing construction of X yields the open embedding

$$\Phi_{v,\omega}(s) : V(\omega) \longrightarrow V(v)$$

by twisting $\operatorname{Spec}\Bbbk[\omega^{-1}\Sigma_v] \to \operatorname{Spec}\Bbbk[\Sigma_v]$ by s_e^{-1}. Moreover, $g:\omega\to\tau$ determines the toric stratum

$$V_g = \operatorname{Spec}\left(\Bbbk[P_{\omega,\sigma}]/I_{g,\sigma}^{>0}\right) \subseteq V(\omega).$$

Thus $\operatorname{Spec}\left(\Bbbk[P_{\omega,\sigma}]/I_{g,\sigma}^{>k}\right)$ is the k-th order thickening of V_g inside $\operatorname{Spec}\Bbbk[P_{\omega,\sigma}]$.

As for the localization recall that the singular locus Z of the log structure on $V(v)$ is the union of the zero loci Z_e of f_e for $e: v\to\rho$, $\rho\in\mathscr{P}^{[n-1]}$. Thus the zero locus of $f_{g,\sigma}^v$ equals the $\Phi_{v,\omega}(s)$-preimage of $p^{-1}(Z_\tau)$ for $Z_\tau := \bigcup_{e:\tau\to\rho} Z_e$. In summary, $\operatorname{Spec} R_{g,\sigma}^k$ is isomorphic to the k-th order thickening of $V_g\setminus \Phi_{v,\omega}(s)^{-1}(p^{-1}(Z_\tau))$ inside $V(\omega)$.

Remark 2.9. Let $g:\omega\to\tau$, $g':\omega'\to\tau'$ and assume that $\omega\subseteq\omega'$ and $\tau\supseteq\tau'$. We also fix a reference cell $\sigma\in\mathscr{P}_{\max}$ containing τ. Then $P_{\omega',\sigma}$ differs from $P_{\omega,\sigma}$ by making invertible those $m\in P_{\omega,\sigma}$ with $\operatorname{ord}_{\omega'}(m)=0$. Moreover, for any $m\in P_{\omega,\sigma}$ it holds $\operatorname{ord}_{\tau'}(m)\ge\operatorname{ord}_\tau(m)$ since $\tau'\subseteq\tau$. Hence $I_{g,\sigma}^{>k}\subseteq I_{g',\sigma}^{>k}$, and we obtain the canonical homomorphism

$$\psi_0 : \Bbbk[P_{\omega,\sigma}]/I_{g,\sigma}^{>k} \longrightarrow \Bbbk[P_{\omega',\sigma}]/I_{g',\sigma}^{>k}.$$

If $\omega=\omega'$, then $\psi_0(f_{g,\sigma})$ divides $f_{g',\sigma}$, and hence ψ_0 induces a map $R_{g,\sigma}^k\to R_{g',\sigma}^k$.

In the general case we need to take into account the twisting by the open gluing data as follows. The piecewise multiplicative function s_a, $a:\omega\to\omega'$, coming from the open gluing data, is given on σ by a homomorphism $s_{a,\sigma}:\Lambda_\sigma\to\Bbbk^\times$. This defines a ring automorphism of $\Bbbk[P_{\omega',\sigma}]$ respecting orders. Hence it induces an automorphism of $\Bbbk[P_{\omega',\sigma}]/I_{g',\sigma}^{>k}$ that we also denote by $s_{a,\sigma}$. The special case where ω is a vertex is the case used in Construction 2.7. Now if $e:v\to\omega$, then $s_{a\circ e,\sigma}=s_{a,\sigma}\cdot s_{e,\sigma}$, which implies

$$f_{\rho,a\circ e,\sigma} = (s_{a,\sigma}^{-1}\circ\psi_0)\big(f_{\rho,e,\sigma}\big).$$

Thus $s_{a,\sigma}^{-1}\circ\psi_0$ defines a well-defined map

$$\psi_0(s) : R_{g,\sigma}^k \longrightarrow R_{g',\sigma}^k.$$

Example 2.10. To illustrate the use of the rings $R_{g,\sigma}^k$ in our construction let us look at a simple example that captures the situation in codimension one. Assume that ρ is a one-dimensional cell in a two-dimensional B, with vertices v_1, v_2, and monodromy constant $\kappa:=\kappa_{\rho\rho}\ge 0$; see (1.1). For simplicity we assume the open gluing data to be trivial ($s_e=1$ for all e). Let the dual cell $\check\rho$ have integral length l. Let σ_1, σ_2 be the two maximal cells with $\rho=\sigma_1\cap\sigma_2$. Then $\operatorname{Spec}\Bbbk[P_{\rho,\sigma_i}]$ is isomorphic to $\mathbb{A}^1\setminus\{0\}$ times the two-dimensional A_{l-1}-singularity. With $g_i:\rho\to\sigma_i$ and $\operatorname{id}_\rho:\rho\to\rho$, we have the two maps $R_{g_i,\sigma_i}^k\to R_{\operatorname{id}_\rho,\sigma_i}^k$, which in appropriate coordinates are just canonical quotient homomorphisms composed with a localization:

$$\begin{aligned}
&\Bbbk[w,w^{-1},x_1,y_1,t]/(x_1y_1-t^l,y_1^{\bar k+1})\\
&\qquad\qquad\longrightarrow \Bbbk[w,w^{-1},x_1,y_1,t]_{f_{\mathrm{id}_\rho,\sigma_1}}/(x_1y_1-t^l,x_1^{\bar k+1},y_1^{\bar k+1}),\\
&\Bbbk[w,w^{-1},x_2,y_2,t]/(x_2y_2-t^l,x_2^{\bar k+1})\\
&\qquad\qquad\longrightarrow \Bbbk[w,w^{-1},x_2,y_2,t]_{f_{\mathrm{id}_\rho,\sigma_2}}/(x_2y_2-t^l,x_2^{\bar k+1},y_2^{\bar k+1}).
\end{aligned}$$

Here we assumed for simplicity that $k+1=(\bar k+1)\cdot l$ for $\bar k\in\mathbb{N}$. Now by parallel transport through v_i we obtain two isomorphisms

$$\psi_i : R^k_{\mathrm{id}_\rho,\sigma_1}\to R^k_{\mathrm{id}_\rho,\sigma_2}.$$

This gives two fibre products $R^k_{g_1,\sigma_1}\times_{R^k_{\mathrm{id}_\rho,\sigma_2}} R^k_{g_2,\sigma_2}$, and we will prove in a more general context in Lemma 2.34 that each is isomorphic to

$$\operatorname{Spec}\Bbbk[w,w^{-1},x,y,t]/(xy-t^l,t^{k+1}).$$

However, for $k+1\ge l$ there exists no isomorphism between these two fibre products inducing the identity on $R^k_{g_i,\sigma_i}$ unless $\kappa=0$. In fact, if $w=z^m$ with $\overline{m}\in\Lambda_\rho$ the generator pointing from v_1 to v_2, then ψ_1 and ψ_2 are related by the automorphism

$$\psi_2^{-1}\circ\psi_1 : w\mapsto w,\ x_1\mapsto w^{-\kappa}x_1,\ y_1\mapsto w^{\kappa}y_1,\ t\mapsto t$$

of $R^k_{\mathrm{id}_\rho,\sigma_1}$, which is not the identity unless $\kappa=0$ or $k<l$. For a continuation of this discussion see Example 2.19.

The example illustrates that monodromy yields an obstruction to gluing the standard k-th order deformations of the local models of X consistently. To remedy this we need to compose the maps between rings by automorphisms. These automorphisms are the subject of the next subsection, in a log setting for our rings that we now discuss.

By construction $R^I_{\omega\to\tau,\sigma}$ comes with a homomorphism of monoids

$$(P_{\omega,\sigma},+)\to(R^I_{\omega\to\tau,\sigma},\cdot).$$

This yields a chart for a log structure on $\operatorname{Spec} R^I_{\omega\to\tau,\sigma}$, and it will be very important in the algorithm to trace this information. We therefore now introduce a category of rings with charts.

Definition 2.11. A *log ring* is a ring R together with a monoid homomorphism $\alpha : P\to(R,\cdot)$. A *morphism* of log rings (or *log morphism*) $(\alpha : P\to R)\to(\alpha' : P'\to R')$ consists of a ring homomorphism $\psi : R\to R'$ and monoid homomorphisms

$$\beta : P\to P',\quad \theta : P\to(R')^\times$$

such that

$$\psi \circ \alpha = \theta \cdot (\alpha' \circ \beta). \tag{2.3}$$

If $(\beta, \theta, \psi) : (P \to R) \to (P' \to R')$ and $(\beta', \theta', \psi') : (P' \to R') \to (P'' \to R'')$ are log morphisms, their composition is defined as

$$\big(\beta' \circ \beta, (\psi' \circ \theta) \cdot (\theta' \circ \beta), \psi' \circ \psi\big).$$

This is indeed a log morphism from $(P \to R)$ to $(P'' \to R'')$ as one easily checks. Two log morphisms $(\beta_1, \theta_1, \psi_1)$, $(\beta_2, \theta_2, \psi_2)$ from $\alpha : P \to R$ to $\alpha' : P' \to R'$ are *equivalent* if there exists a homomorphism $\eta : P \to (P')^\times$ such that

$$\beta_2 = \beta_1 + \eta, \quad \theta_2 = \theta_1 \cdot (\alpha' \circ \eta)^{-1}, \quad \psi_2 = \psi_1.$$

Log rings and log morphisms modulo equivalence define the category $\underline{\text{LogRings}}$.

Remark 2.12. 1) This definition just rephrases the basic notions of log geometry [Kat89] on the level of rings. In particular, a log ring $\alpha : P \to R$ is the same as an affine scheme $X = \operatorname{Spec} R$ with a chart for a log structure $\gamma : P \to \Gamma(X, \mathcal{M}_X)$ in the Zariski topology. Note that such a chart induces a canonical isomorphism

$$\big(\mathcal{O}_X^\times \oplus P_X\big) \big/ \big\{(h, m) \,\big|\, h \cdot \alpha(m) = 1\big\} \longrightarrow \mathcal{M}_X,$$

so we can represent elements of $\mathcal{M}_X$ as pairs (h, m), $h \in \mathcal{O}_X^\times$, $m \in P$.

Similarly, a log morphism (β, θ, ψ) between the log rings $\alpha : P \to R$ and $\alpha' : P' \to R'$ gives rise to a morphism of the associated affine log schemes as follows. The map $\underline{f} : X' := \operatorname{Spec} R' \to X := \operatorname{Spec} R$ of the underlying schemes is defined by ψ. Then by (2.3),

$$\underline{f}^{-1}(\mathcal{O}_X^\times \oplus P_X) \longrightarrow \mathcal{O}_{X'}^\times \oplus P'_{X'}, \quad (h, m) \longmapsto \big(\psi(h) \cdot \theta(m), \beta(m)\big)$$

descends to a morphism $\underline{f}^{-1}\mathcal{M}_X \to \mathcal{M}_{X'}$. Indeed, $h \cdot \alpha(m) = 1$ implies

$$1 = \psi\big(h \cdot \alpha(m)\big) = \psi(h) \cdot (\psi \circ \alpha)(m) \overset{(2.3)}{=} \psi(h) \cdot \theta(m) \cdot \alpha'\big(\beta(m)\big).$$

Conversely, under the assumption that for the chart $\gamma' : P' \to \Gamma(X', \mathcal{M}_{X'})$ no nonzero element of P' maps to an invertible element, any morphism of log schemes $(\underline{f}, f^\flat) : (X', \mathcal{M}_{X'}) \to (X, \mathcal{M}_X)$ arises in this fashion. In fact, under the stated condition, the composition

$$P \xrightarrow{\gamma} \Gamma(X, \mathcal{M}_X) \xrightarrow{f^\flat} \Gamma(X', \mathcal{M}_{X'}) \xrightarrow{\kappa} \Gamma(X', \mathcal{M}_{X'}/\mathcal{O}_{X'}^\times)$$

with κ the quotient homomorphism, factors canonically over $\kappa \circ \gamma'$, thus defining $\beta : P \to P'$. Comparison of $\gamma' \circ \beta$ with $f^\flat \circ \gamma$ then defines θ. Note that on the side of log rings the stated condition translates into the requirement ${\alpha'}^{-1}({R'}^\times) = \{0\}$.

Thus, at least for log rings $\alpha : P \to R$ fulfilling $\alpha^{-1}(R^\times) = \{0\}$, our discussion also shows that the notion of equivalence is compatible with compositions of morphisms.

2) In our case, $R = R^k_{g,\sigma}$ is a localization of a quotient of $\Bbbk[P_{\omega,\sigma}]$ and hence carries canonically the structure of a log ring via $\alpha : P_{\omega,\sigma} \to R^k_{g,\sigma}$, $\alpha(m) = z^m$. Because $P_{\omega,\sigma}$ generates $R^k_{g,\sigma}$ up to localization, (2.3) determines the underlying ring homomorphism ψ of a log morphism from β and θ. Moreover, in the cases we are interested in, β is either canonically given or is fixed in the discussion and θ factors through the projection $P_{\omega,\sigma} \to \Lambda_\sigma$. By abuse of notation we then talk of a group homomorphism $\theta : \Lambda_\sigma \to (R_{g',\sigma})^\times$ as being a log morphism. We write $\overline{\theta}$ for the associated ring homomorphism, and use $\theta(m)$ and $\theta(\overline{m})$ interchangeably. Explicitly, we have

$$\overline{\theta}(z^m) := \theta(\overline{m}) \cdot z^{\beta(m)}$$

for the underlying ring homomorphism, and the composition of two log morphisms θ_1, θ_2 reads

$$(\theta_1 \circ \theta_2)(m) = \theta_1(m) \cdot \overline{\theta_1}\big(\theta_2(m)\big). \tag{2.4}$$

2.2. *Automorphism groups.* We will now discuss various groups of log automorphisms of the rings which appear in our construction. For this subsection, fix $g : \omega \to \tau$, $\sigma \in \mathscr{P}_{\max}$ with $\tau \subseteq \sigma$, and a monomial ideal $I \subseteq \Bbbk[P_{\omega,\sigma}]$ with radical $I_0 := I^{>0}_{g,\sigma}$. Let $f := f^v_{g,\sigma} \in \Bbbk[P_{\omega,\sigma}]$, $v \in \omega$ a vertex, be a localizing element as in Construction 2.7. Write $P := P_{\omega,\sigma}$ and $R^I := R^I_{g,\sigma} = (\Bbbk[P]/I)_f$. Recall also the projection $P \to \Lambda_\sigma$, $m \mapsto \overline{m}$, and the conventions of Remark 2.12. We are interested in log automorphisms of $P \to R^I$.

Remark 2.13. 1) The inverse of a log automorphism θ is

$$\theta^{-1}(m) = \overline{\theta}^{-1}\left(\frac{1}{\theta(m)}\right).$$

2) The formula for multiple compositions is

$$\theta_1 \circ \theta_2 \circ \cdots \circ \theta_r = \theta_1 \cdot (\overline{\theta}_1 \circ \theta_2) \cdot \ldots \cdot (\overline{\theta}_1 \circ \overline{\theta}_2 \circ \cdots \circ \overline{\theta}_{r-1} \circ \theta_r). \tag{2.5}$$

On the right-hand side the composition symbol denotes ordinary composition of maps and the centered dots denote multiplication of maps with target R^I.

We will now describe the group of all log automorphisms $\theta : \Lambda_\sigma \to (R^I)^\times$ with the property that $\theta(m) = 1 \mod I_0$, by describing the Lie algebra of this group. We first consider the *module of log derivations* of R^I, defined by

$$\Theta(R^I) := R^I \otimes_{\mathbb{Z}} \Lambda^*_\sigma = \mathrm{Hom}(\Lambda_\sigma, R^I).$$

We view an element $\xi \in \Theta(R^I)$ as an additive map $\xi : P \to R^I$ factoring through $P \to \Lambda_\sigma$. In particular, $a \otimes n$ defines the map

$$P \ni m \longmapsto a\langle \overline{m}, n\rangle.$$

Note that $\xi \in \Theta(R^I)$ also induces an ordinary $\Bbbk$-derivation of R^I via

$$\overline{\xi}(z^m) := \xi(m) z^m.$$

It is then suggestive to write $a\partial_n$ for $a \otimes n \in \Theta(R^I)$ or its associated ordinary derivation:

$$(a\partial_n)(z^m) = a\langle \overline{m}, n\rangle z^m.$$

The adjoint action of the group of automorphisms on derivations lifts to the log setting by defining, for θ a log automorphism and ξ a log derivation,

$$\mathrm{Ad}_\theta\, \xi := (\overline{\theta} \circ \overline{\xi} \circ \theta^{-1}) \cdot \theta + \overline{\theta} \circ \xi. \tag{2.6}$$

Given $\xi_1, \ldots, \xi_n \in \Theta(R^I)$, we can define a higher order log differential operator, a map $\xi_1 \circ \cdots \circ \xi_n : \Lambda_\sigma \to R^I$, inductively by the formula

$$(\xi_1 \circ \cdots \circ \xi_n)(m) = \xi_1(m) \cdot (\xi_2 \circ \cdots \circ \xi_n)(m) + \overline{\xi}_1(\xi_2 \circ \cdots \circ \xi_n(m)) \tag{2.7}$$

so that

$$z^m(\xi_1 \circ \cdots \circ \xi_n)(m) = (\overline{\xi}_1 \circ \cdots \circ \overline{\xi}_n)(z^m),$$

where the composition on the right-hand side is just the composition of ordinary $\Bbbk$-endomorphisms of R^I. The powers of $\xi \in \Theta(R^I)$ fulfill a higher order Leibniz rule:

$$\xi^n(m_1 + m_2) = \sum_{i=0}^{n} \binom{n}{i} \xi^i(m_1)\xi^{n-i}(m_2), \quad m_1, m_2 \in \Lambda_\sigma. \tag{2.8}$$

Proposition 2.14. *The group*

$$G^I := \Big\{\theta : \Lambda_\sigma \to (R^I)^\times \,\Big|\, \theta \text{ is a log automorphism}, \forall m \in \Lambda_\sigma : \theta(m) = 1 \mod I_0\Big\}$$

is an algebraic group with Lie algebra $\mathfrak{g}^I := I_0 \cdot \Theta(R^I)$ endowed with the Lie bracket

$$[\xi_1, \xi_2] := \xi_1 \circ \xi_2 - \xi_2 \circ \xi_1.$$

Proof. From (2.8) it follows that if $\xi \in I_0 \cdot \Theta(R^I)$, the formula

$$\exp(\xi)(m) := 1 + \sum_{i=1}^{\infty} \frac{\xi^i(m)}{i!} \tag{2.9}$$

defines an element $\exp(\xi) \in G^I$ since

$$\exp(\xi)(m_1 + m_2) = \exp(\xi)(m_1) \cdot \exp(\xi)(m_2).$$

Note that the sum is finite because $\sqrt{I} = I_0$.

Conversely, let $\theta \in G^I$. Define inductively $N_i : \Lambda_\sigma \to R$ by $N_0 := 1$ and

$$N_i := \theta \cdot \left(\overline{\theta} \circ N_{i-1}\right) - N_{i-1}.$$

The induced map $\overline{N}_i : z^m \mapsto N_i(m)z^m$ equals $(\overline{\theta} - \mathrm{id})^i$. Note that N_i takes values in I_0^i. Thus we can define $\log(\theta) : \Lambda_\sigma \to R^I$ by

$$\log(\theta) := \sum_{i=1}^{\infty} \frac{(-1)^{i+1}}{i} N_i.$$

Again, this is a finite sum. Noting inductively that

$$N_n(m_1 + m_2) = \sum_{i+j+k=n, i,j,k\geq 0} \frac{n!}{i!j!k!} N_{i+j}(m_1) N_{i+k}(m_2),$$

it follows by direct computation that $\log(\theta)$ is additive. Hence $\log(\theta) \in I_0 \cdot \Theta(R^I)$, and then the usual power series identity implies $\theta = \exp(\log(\theta))$. $\square$

On the $\Bbbk$-basis $z^m \partial_n$ of $\mathfrak{g}^I$, the formula for the Lie bracket is

(2.10)
$$\begin{aligned}[] [z^m\partial_n, z^{m'}\partial_{n'}] &= (z^m \partial_n(z^{m'}))\partial_{n'} - (z^{m'}\partial_{n'}(z^m))\partial_n \\ &= z^{m+m'}(\langle \overline{m'}, n\rangle \partial_{n'} - \langle \overline{m}, n'\rangle \partial_n) = z^{m+m'} \partial_{\langle \overline{m'}, n\rangle n' - \langle \overline{m}, n'\rangle n}. \end{aligned}$$

In particular, $\mathfrak{g}^I$ is a nilpotent Lie algebra.

Later on we will often need to control how the basic elements $\exp(z^m\partial_n)$ commute with certain more general log automorphisms. For this we record the following lemma.

LEMMA 2.15. *For $h \in (R^I)^\times$ consider the log automorphism*

$$\theta : m \longmapsto h^{-\langle \overline{m}, n_0\rangle},$$

of R^I, where $n_0 \in \Lambda_\sigma^*$ annihilates any exponent occurring in h. Then for $m \in \Lambda_\sigma$, $n \in \Lambda_\sigma^*$,

$$\mathrm{Ad}_\theta(z^m\partial_n) = z^m\Big(h^{-\langle \overline{m}, n_0\rangle}\partial_n + h^{-\langle \overline{m}, n_0\rangle - 1}(\partial_n h)\partial_{n_0}\Big).$$

Proof. Using the fact that every monomial in h is left-invariant by $\overline{\theta}$ we get $\theta^{-1}(m) = h^{\langle \overline{m}, n_0\rangle}$ and, with $\xi = z^m\partial_n$,

$$\begin{aligned} \mathrm{Ad}_\theta(\xi)(m') &\overset{(2.6)}{=} (\overline{\theta}\circ\overline{\xi}\circ\theta^{-1})(m')\cdot\theta(m') + (\overline{\theta}\circ\xi)(m') \\ &= (\overline{\theta}\circ\overline{\xi})\Big(h^{\langle \overline{m'}, n_0\rangle}\Big)\cdot h^{-\langle \overline{m'}, n_0\rangle} + \overline{\theta}\Big(\langle \overline{m'}, n\rangle z^m\Big) \\ &= \overline{\theta}\Big(\langle \overline{m'}, n_0\rangle h^{\langle \overline{m'}, n_0\rangle - 1}(\partial_n h) z^m\Big) h^{-\langle \overline{m'}, n_0\rangle} + \langle \overline{m'}, n\rangle h^{-\langle \overline{m}, n_0\rangle} z^m \\ &= \langle \overline{m'}, n_0\rangle h^{-\langle \overline{m}, n_0\rangle - 1}(\partial_n h) z^m + \langle \overline{m'}, n\rangle h^{-\langle \overline{m}, n_0\rangle} z^m. \end{aligned}$$ $\square$

For any sub-Lie algebra $\mathfrak{h} \subseteq \mathfrak{g}^I$, we obtain a subgroup $H = \exp(\mathfrak{h})$ of G^I consisting of exponentials of elements of $\mathfrak{h}$. We shall consider a number of such subgroups.

In what follows, fix a codimension-two subspace $T_\mathfrak{j} \subseteq \Lambda_{\sigma,\mathbb{R}}$ defined over $\mathbb{Q}$, and write $\Lambda_\mathfrak{j} = T_\mathfrak{j} \cap \Lambda_\sigma$. Later on $T_\mathfrak{j}$ will be the tangent space to a polyhedral subset of σ of codimension two. Write $P^{>0} = P^{>0}_{g,\sigma} \subseteq P_{\omega,\sigma}$ and P^I for the monoid ideals generating I_0 and I. Then each of the following subspaces of $\mathfrak{g}^I$ are Lie subalgebras, as is easily checked using (2.10):

$$\begin{aligned}
\mathfrak{g}^I_{\mathrm{j}} &:= \bigoplus_{m\in P^{>0}\setminus P^I} z^m\big(\Bbbk\otimes\Lambda_{\mathrm{j}}^\perp\big),\\
\tilde{\mathfrak{h}}^I_{\mathrm{j}} &:= \bigoplus_{m\in P^{>0}\setminus P^I} z^m\big(\Bbbk\otimes(\overline{m}^\perp\cap\Lambda_{\mathrm{j}}^\perp)\big),\\
\mathfrak{h}^I_{\mathrm{j}} &:= \bigoplus_{\substack{m\in P^{>0}\setminus P^I\\ \overline{m}\neq 0}} z^m\big(\Bbbk\otimes(\overline{m}^\perp\cap\Lambda_{\mathrm{j}}^\perp)\big),\\
{}^\perp\mathfrak{h}^I_{\mathrm{j}} &:= \bigoplus_{\substack{m\in P^{>0}\setminus P^I\\ \overline{m}\notin\Lambda_{\mathrm{j}}}} z^m\big(\Bbbk\otimes(\overline{m}^\perp\cap\Lambda_{\mathrm{j}}^\perp)\big),\\
{}^\parallel\mathfrak{h}^I_{\mathrm{j}} &:= \bigoplus_{\substack{m\in P^{>0}\setminus P^I\\ \overline{m}\in\Lambda_{\mathrm{j}}\setminus\{0\}}} z^m\big(\Bbbk\otimes\Lambda_{\mathrm{j}}^\perp\big).
\end{aligned}$$

The corresponding subgroups of G^I are denoted G^I_{j}, $\widetilde{H}^I_{\mathrm{j}}$, H^I_{j}, ${}^\perp H^I_{\mathrm{j}}$, and ${}^\parallel H^I_{\mathrm{j}}$, respectively. Of these the most essential one for our construction is H^I_{j} with Lie algebra generated by derivations $z^m\partial_n$, where ∂_n acts trivially on z^m and m points in a specific direction ($\overline{m}\neq 0$).

Remarks 2.16. (1) All $\theta\in G^I_{\mathrm{j}}$ satisfy $\theta(m)=1$ whenever $\overline{m}\in\Lambda_{\mathrm{j}}$.

(2) The log automorphism associated to an element of $\tilde{\mathfrak{h}}^I_{\mathrm{j}}$ of the form $a\partial_n$ is easy to write down explicitly:

$$\exp(a\partial_n)(m) = \exp(\langle\overline{m},n\rangle a) = \exp(a)^{\langle\overline{m},n\rangle}.$$

Here $\exp(a)$ is the usual exponential of a function, which is a polynomial in a because $a\in I_0$. Indeed, a involves only monomials z^m with $\langle\overline{m},n\rangle=0$, and hence the composition formula (2.7) inductively shows

$$(a\partial_n)^i(m) = \big(\langle\overline{m},n\rangle a\big)^i.$$

(3) Denote by

$$\Omega^p(R^I) = R^I\otimes\bigwedge\nolimits^p\Lambda_\sigma$$

the module of logarithmic p-forms on R^I. We write $f\otimes m = f\,\mathrm{dlog}\,m$ for $f\in R^I$, $m\in\bigwedge^p\Lambda_\sigma$. We have the (ordinary) exterior derivative

$$d: R^I\longrightarrow\Omega^1(R^I)=\mathrm{Hom}(\Lambda_\sigma^*,R^I),\quad f\longmapsto(n\mapsto\partial_n f)\,.$$

This gives

$$d:\Omega^p(R^I)\longrightarrow\Omega^{p+1}(R^I)$$

in the usual way, and $\mathfrak{g}^I$ acts on $\Omega^p(R^I)$ by Lie derivative. In particular, $\mathfrak{g}^I$ acts on $\Omega^{\dim B}(R^I)$ by $\xi(\Omega) = \mathcal{L}_\xi(\Omega) = d(\iota(\xi)\Omega)$ for $\xi \in \mathfrak{g}^I$, $\Omega \in \Omega^{\dim B}(R^I)$. It is then not difficult to see that $\tilde{\mathfrak{h}}^I_{\mathrm{j}}$ consists of those elements of $\mathfrak{g}^I_{\mathrm{j}}$ which preserve

$$\Omega_{\mathrm{std}} = \mathrm{dlog}(m_1 \wedge \cdots \wedge m_n) = \mathrm{dlog}(m_1) \wedge \cdots \wedge \mathrm{dlog}(m_n),$$

where $m_1 \wedge \cdots \wedge m_n$ is a primitive generator of $\bigwedge^{\dim B} \Lambda_\sigma$. In fact,

$$\mathcal{L}_{z^m \partial_n} \Omega_{\mathrm{std}} = d\big(z^m \iota_{\partial_n} \Omega_{\mathrm{std}}\big) = \langle m, n\rangle z^m \Omega_{\mathrm{std}}.$$

Note also that any log automorphism of R^I acts on $\Omega^p(R^I)$ by

$$\theta\big(a\,\mathrm{dlog}(m_1) \wedge \cdots \wedge \mathrm{dlog}(m_p)\big) := \overline{\theta}(a)\Big(\mathrm{dlog}\, m_1 + \frac{d\theta(m_1)}{\theta(m_1)}\Big) \wedge \cdots \wedge \Big(\mathrm{dlog}\, m_p + \frac{d\theta(m_p)}{\theta(m_p)}\Big).$$

One can check that whenever $\theta \in G^I$ this agrees with the exponential of the action of $\mathfrak{g}^I$ on $\Omega^p(R^I)$; that is, for $\xi \in \mathfrak{g}^I$ and $\alpha \in \Omega^p(R^I)$ it holds that

$$\big(\exp(\xi)\big)(\alpha) = \sum_{i=0}^{\infty} \frac{1}{i!} \mathcal{L}^i_\xi(\alpha).$$

Thus $\widetilde{H}^I_{\mathrm{j}}$ consists of those log automorphisms in G^I_{j} preserving Ω_{std}.

(4) Note that ${}^{\|}\mathfrak{h}^I_{\mathrm{j}}$ is abelian and $[{}^{\|}\mathfrak{h}^I_{\mathrm{j}}, {}^{\perp}\mathfrak{h}^I_{\mathrm{j}}] \subseteq {}^{\perp}\mathfrak{h}^I_{\mathrm{j}}$, so we get an exact sequence of Lie algebras

$$0 \longrightarrow {}^{\perp}\mathfrak{h}^I_{\mathrm{j}} \longrightarrow \mathfrak{h}^I_{\mathrm{j}} \longrightarrow {}^{\|}\mathfrak{h}^I_{\mathrm{j}} \longrightarrow 0$$

and hence an exact sequence of groups

$$1 \longrightarrow {}^{\perp}H^I_{\mathrm{j}} \longrightarrow H^I_{\mathrm{j}} \longrightarrow {}^{\|}H^I_{\mathrm{j}} \longrightarrow 1.$$

2.3. *Slabs, walls and structures.* Our construction involves splitting B into smaller and smaller pieces which are separated by *slabs* and *walls*. We begin with the subdivision of B given by $\mathscr{P}$; the codimension-one elements of $\mathscr{P}$ define slabs. We then proceed to subdivide B through a scattering process by adding walls, which are codimension-one polyhedra contained in maximal elements of $\mathscr{P}$. These walls split these maximal cells into chambers. The choice of words "wall" and "slab" is inspired by the first author's house, which is built on a slab. Just as with this house, over time, the slabs develop cracks, and here are subdivided, while once a wall is introduced, it remains unmodified during the process of further subdivisions of $\mathscr{P}$. A slab also carries additional data, namely the starting data determined by the log structure and some higher order corrections, while a wall only carries higher order data. A further difference is that walls, unlike slabs, have a built-in directionality. Both slabs and walls lead to log automorphisms of rings $R^k_{g,\sigma}$, which will be used to glue together these rings to create k-th order deformations.

For the following definition, recall the open star $U_\tau = \bigcup_{\sigma\in\mathscr{P},\sigma\supseteq\tau}\operatorname{Int}\sigma$ of a cell τ and the notation $v[x]\in\rho$ for the unique vertex in the same connected component of $\rho\setminus\Delta$ of some $x\in\rho\setminus\Delta$, $\rho\in\mathscr{P}^{[n-1]}$.

Definition 2.17. A *slab* is a convex, rational, $(n-1)$-dimensional polyhedral subset $\mathfrak{b}$ of a cell $\rho_{\mathfrak{b}}\in\mathscr{P}^{[n-1]}$ together with elements

$$f_{\mathfrak{b},x} = \sum_{m\in P_x,\,\overline{m}\in\Lambda_{\rho_{\mathfrak{b}}}} c_m z^m \in \Bbbk[P_x],$$

one for each $x\in\mathfrak{b}\setminus\Delta$, satisfying the following properties:

(i) If $x,x'\in\mathfrak{b}\setminus\Delta$, $\Pi:\Bbbk[P_x^{\mathrm{gp}}]\to\Bbbk[P_{x'}^{\mathrm{gp}}]$, is defined by parallel transport along a path inside $\mathrm{cl}(U_{\rho_{\mathfrak{b}}})\setminus\Delta$ and $v=v[x], v'=v[x']$, then

$$D(s_{e'},\rho_{\mathfrak{b}},v')^{-1}s_{e'}^{-1}(f_{\mathfrak{b},x'}) = z^{m^{\rho_{\mathfrak{b}}}_{v'v}}\Pi\Big(D(s_e,\rho_{\mathfrak{b}},v)^{-1}s_e^{-1}(f_{\mathfrak{b},x})\Big), \tag{2.11}$$

where $e:v\to\rho_{\mathfrak{b}}$, $e':v'\to\rho_{\mathfrak{b}}$.

(ii) If $e:v\to\rho_{\mathfrak{b}}$ with $v=v[x]$, and $\Pi:\Bbbk[P_x^{\mathrm{gp}}]\to\Bbbk[P_v^{\mathrm{gp}}]$ is defined by parallel transport from x to v along a path inside $\rho_{\mathfrak{b}}\setminus\Delta$, then

$$f_e = \Pi\Big(\sum_{m\in P_x,\,\mathrm{ord}_{\mathfrak{b}}(m)=0} c_m z^m\Big),$$

where (f_e) is the section of $\mathcal{LS}^+_{\mathrm{pre},X}$ defining the log smooth structure on X.

Remarks 2.18. 1) By condition (i) the functions $f_{\mathfrak{b},x}$ determine each other by parallel transport inside $\mathrm{cl}(U_{\rho_{\mathfrak{b}}})\setminus\Delta$. In particular, a slab carries only finitely many nonzero coefficients as information.

2) Condition (ii) says that (f_e) determines the part of $f_{\mathfrak{b},x}$ of order 0 for every $x\in\mathfrak{b}\setminus\Delta$. Note that by (1.11) this is compatible with condition (i).

Example 2.19. Continuing on Example 2.10 let us show how slabs resolve the problem of incompatible gluings due to monodromy. In this example, we take an additional slab $\mathfrak{b}$ with support the one-dimensional cell $\rho=\sigma_1\cap\sigma_2$. We view the functions $f_{\mathfrak{b},v_i}$ as elements of $R^k_{\mathrm{id}_\rho,\sigma_2}$ via parallel transport from v_i into σ_2. Let $\pi_i:\Lambda_{\sigma_i}\to\mathbb{Z}$ be the projection with kernel Λ_ρ and which is positive on vectors pointing from σ_1 to σ_2. In going from σ_1 to σ_2, compose the isomorphism $\psi_i:R^k_{\mathrm{id}_\rho,\sigma_1}\to R^k_{\mathrm{id}_\rho,\sigma_2}$, obtained via parallel transport through the vertex $v_i\in\rho$, with

$$z^m\longmapsto (f_{\mathfrak{b},v_i})^{-\pi_1(\overline{m})}\cdot z^m.$$

In appropriate coordinates these are the homomorphisms of $\Bbbk[w,w^{-1}]$-algebras sending x_1,y_1 to $f_{\mathfrak{b},v_1}\cdot x_2$, $f_{\mathfrak{b},v_1}^{-1}\cdot y_2$ $(i=1)$, and to $f_{\mathfrak{b},v_2}\cdot w^\kappa x_2$, $f_{\mathfrak{b},v_2}^{-1}\cdot w^{-\kappa}y_2$ $(i=2)$, respectively. Now (2.11) requires $f_{\mathfrak{b},v_2}=w^{-\kappa}f_{\mathfrak{b},v_1}$, and this is exactly what is needed to make the two homomorphisms agree. Thus $R^k_{g_1,\sigma_1} \times_{R^k_{\mathrm{id}_\rho,\sigma_2}} R^k_{g,\sigma_2}$ is well defined.

Explicitly, computing in the chart at v_1, the fibre product is generated as $\Bbbk[w, w^{-1}, t]$-algebra by $X := (x_1, f_{\mathfrak{b},v_1} x_2)$, $Y := (f_{\mathfrak{b},v_1} y_1, y_2)$, with single relation $XY - F_{\mathfrak{b},v_1} t^l$, where $F_{\mathfrak{b},v_1} = (f_{\mathfrak{b},v_1}, f_{\mathfrak{b},v_1})$. Note how this fits with the interpretation of the section $f_{v_1 \to \rho}$ of $\mathcal{LS}^+_{\mathrm{pre},V(v_1)}|_{V(\rho)}$ in (1.9).

Definition 2.20. A *wall* is a convex, rational, $(n-1)$-dimensional polyhedral subset $\mathfrak{p}$ of a maximal cell $\sigma_{\mathfrak{p}} \in \mathscr{P}^{[n]}$ with $\mathfrak{p} \cap \operatorname{Int} \sigma_{\mathfrak{p}} \neq \emptyset$ together with

(i) an $(n-2)$-face $\mathfrak{q} \subseteq \mathfrak{p}$, $\mathfrak{q} \not\subseteq \partial B$, the *base* of $\mathfrak{p}$;
(ii) an exponent $m_{\mathfrak{p}}$ on $\sigma_{\mathfrak{p}}$ with $\operatorname{ord}_{\sigma_{\mathfrak{p}}}(m_{\mathfrak{p}}) > 0$ and $m_{\mathfrak{p},x} \in P_x$ for every $x \in \mathfrak{p} \setminus \Delta$; and
(iii) $c_{\mathfrak{p}} \in \Bbbk$,

such that

$$\mathfrak{p} = (\mathfrak{q} - \mathbb{R}_{\geq 0} \overline{m}_{\mathfrak{p}}) \cap \sigma_{\mathfrak{p}}.$$

Here we view $\sigma_{\mathfrak{p}}$ as a polyhedron in $\Lambda_{\sigma_{\mathfrak{p}},\mathbb{R}}$ and $m_{\mathfrak{p}}$ as an element of $\mathcal{Aff}(\check{B}, \mathbb{Z})_{\check{\sigma}_{\mathfrak{p}}}$. The notation is $(\mathfrak{p}, m_{\mathfrak{p}}, c_{\mathfrak{p}})$, or simply $\mathfrak{p}$ if $m_{\mathfrak{p}}$ and $c_{\mathfrak{p}}$ are understood.

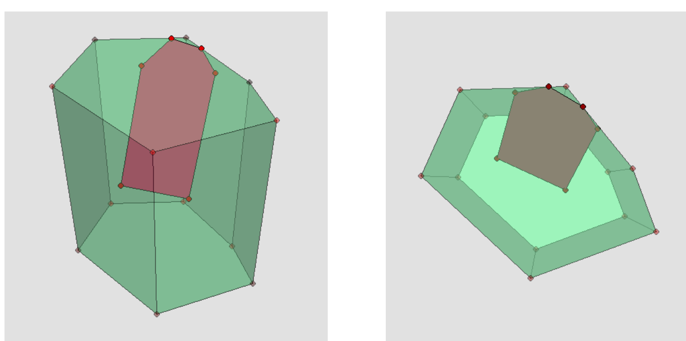


Figure 2.1. Perspective and top views of a wall in a maximal cell ($n = 3$).

Remarks 2.21. 1) Analogous to slabs we have a function for each $x \in \mathfrak{p} \setminus \Delta$, defined by

$$f_{\mathfrak{p},x} := 1 + c_{\mathfrak{p}} z^{m_{\mathfrak{p},x}}.$$

The various $f_{\mathfrak{p},x}$ are transformed to one another via parallel transport inside $\sigma_{\mathfrak{p}} \setminus \Delta$. Note that the $f_{\mathfrak{p},x}$ are honest functions while the $f_{\mathfrak{b},x}$ of a slab are really sections of the line bundle with transition functions $z^{m^{\rho_{\mathfrak{b}}}_{v'v}}$.

2) We have $\partial\mathfrak{p} = \mathrm{Base}(\mathfrak{p}) \cup \mathrm{Sides}(\mathfrak{p}) \cup \mathrm{Top}(\mathfrak{p})$ with

$$\begin{aligned}\mathrm{Base}(\mathfrak{p}) &:= \mathfrak{q},\\ \mathrm{Sides}(\mathfrak{p}) &:= (\partial\mathfrak{q} - \mathbb{R}_{\geq 0}\overline{m_{\mathfrak{p}}}) \cap \sigma_{\mathfrak{p}},\\ \mathrm{Top}(\mathfrak{p}) &:= \mathrm{cl}\left(\partial\mathfrak{p} \setminus (\mathfrak{q} \cup \mathrm{Sides}(\mathfrak{p}))\right).\end{aligned}$$

In the following we will consider systems of slabs and walls fulfilling certain additional conditions.

Definition 2.22. Let $\mathscr{S} = \mathscr{S}^b \cup \mathscr{S}^p$ with $\mathscr{S}^b$ and $\mathscr{S}^p$ locally finite sets of slabs and walls, respectively. Define the *support* of $\mathscr{S}$ as

$$|\mathscr{S}| := \bigcup_{\mathfrak{b}\in\mathscr{S}} \mathfrak{b}.$$

A *chamber of* $\mathscr{S}$ is the closure of a connected component of $B \setminus |\mathscr{S}|$. The set of chambers of $\mathscr{S}$ is denoted as Chambers($\mathscr{S}$). Two chambers $\mathfrak{u}$, $\mathfrak{u}'$, are *adjacent* if $\dim \mathfrak{u} \cap \mathfrak{u}' = n - 1$.

A *structure* is a locally finite set of slabs and walls $\mathscr{S}$ along with a polyhedral decomposition $\mathscr{P}_{\mathscr{S}}$ of $|\mathscr{S}|$, fulfilling the following conditions:

(i) The map associating to a slab $\mathfrak{b} \in \mathscr{S}$ its underlying polyhedral subset of B defines an injection from $\mathscr{S}^b$ to $\mathscr{P}_{\mathscr{S}}^{[n-1]}$, and any $\rho \in \mathscr{P}^{[n-1]}$ is contained in $|\mathscr{S}^b|$.
(ii) Each chamber of $\mathscr{S}$ is convex and its interior is disjoint from any wall.
(iii) Any wall in $\mathscr{S}$ is a union of elements of $\mathscr{P}_{\mathscr{S}}$.
(iv) Any $\sigma \in \mathscr{P}_{\max}$ contains only finitely many slabs or walls in $\mathscr{S}$.

2.4. *The gluing morphisms.* We assume a structure $\mathscr{S}$ to be given. Then for each chamber $\mathfrak{u} \in \mathrm{Chambers}(\mathscr{S})$ there exists a unique $\sigma_{\mathfrak{u}} \in \mathscr{P}_{\max}$ with $\mathfrak{u} \subseteq \sigma_{\mathfrak{u}}$. Thus for each pair $(g, \mathfrak{u})$ with $(g : \omega \to \tau) \in \mathrm{Hom}(\mathscr{P})$ and $\tau \subseteq \sigma_{\mathfrak{u}}$, we have the rings $R^k_{g,\sigma_{\mathfrak{u}}}$, $k \in \mathbb{N}$. These are the rings whose spectra we wish to glue. Technically this is done by a functor from a "gluing category" to the category of log rings.

Definition 2.23. For a structure $\mathscr{S}$ define $\underline{\mathrm{Glue}}(\mathscr{S})$ as the category with objects $(g, \mathfrak{u})$ with $(g : \omega \to \tau) \in \mathrm{Hom}(\mathscr{P})$, $\mathfrak{u} \in \mathrm{Chambers}(\mathscr{S})$ and $\omega \cap \mathfrak{u} \neq \emptyset$, $\tau \subseteq \sigma_{\mathfrak{u}}$. (Then also $\tau \cap \mathfrak{u} \neq \emptyset$, $\omega \subseteq \sigma_{\mathfrak{u}}$.) We call ω and τ the *domain* and *target* of $(g, \mathfrak{u})$, respectively. There is a (unique) morphism

$$(g : \omega \to \tau, \mathfrak{u}) \longrightarrow (g' : \omega' \to \tau', \mathfrak{u}')$$

if and only if $\omega \subseteq \omega'$ and $\tau \supseteq \tau'$.

Note that each morphism $\mathfrak{e} : (g, \mathfrak{u}) \to (g', \mathfrak{u}')$ in this category decomposes into a sequence of morphisms of the following two basic types:

(I) $\omega \subseteq \omega'$, $\tau \supseteq \tau'$, $\mathfrak{u} = \mathfrak{u}'$ (*change of strata*).

(II) $\omega = \omega'$, $\tau = \tau'$, $\dim \mathfrak{u} \cap \mathfrak{u}' = n-1$, $\omega \cap \mathfrak{u} \cap \mathfrak{u}' \neq \emptyset$ (*change of chamber*).

For these two types of morphisms we now define a morphism of log rings $R^k_{g:\omega\to\tau,\sigma_\mathfrak{u}} \to R^k_{g':\omega'\to\tau',\sigma_{\mathfrak{u}'}}$ by specifying homomorphisms of monoids $\beta : P_{\omega,\sigma_\mathfrak{u}} \to P_{\omega',\sigma_{\mathfrak{u}'}}$ and

$$\theta : \Lambda_{\sigma_\mathfrak{u}} \longrightarrow \left(R^k_{g',\sigma_{\mathfrak{u}'}}\right)^\times,$$

following the conventions of Remark 2.12(2). At this point our definition will still depend on choices, but this dependence will disappear after imposing the condition of consistency on $\mathscr{S}$ below (Definition 2.28). To put these log morphisms into context recall from Remark 2.8 that if $(g : \omega \to \tau, \mathfrak{u}) \in \underline{\mathrm{Glue}}(\mathscr{S})$, then $\operatorname{Spec} R^k_{g,\sigma_\mathfrak{u}}$ is a k-th order thickening of an open subset of V_g. Hence the target τ of $(g : \omega \to \tau, \mathfrak{u})$ selects the toric stratum while its domain ω selects the affine open subset to consider. Thus a morphism of Type I in $\underline{\mathrm{Glue}}(\mathscr{S})$ should map to a composition of the closed embedding associated to $\tau' \to \tau$ composed with the open embedding associated to $\omega \to \omega'$. Changing chambers (II) leads to the application of log isomorphisms.

Construction 2.24 (The basic gluing morphisms).

(I) (Change of strata). Let

$$\mathfrak{e} : (g : \omega \to \tau, \mathfrak{u}) \to (g' : \omega' \to \tau', \mathfrak{u})$$

be a morphism in $\underline{\mathrm{Glue}}(\mathscr{S})$ of Type I and let $a : \omega \to \omega'$. Denote by $s_{a,\sigma_\mathfrak{u}} : \Lambda_{\sigma_\mathfrak{u}} \to \Bbbk^\times$ the homomorphism defined by $s_a \in \mathrm{PM}(\omega)$ for $\sigma_\mathfrak{u}$. Take $\beta : P_{\omega,\sigma_\mathfrak{u}} \to P_{\omega',\sigma_\mathfrak{u}}$ to be the canonical map and

$$\theta : \Lambda_{\sigma_\mathfrak{u}} \longrightarrow (R^k_{g',\sigma_\mathfrak{u}})^\times, \quad m \longmapsto s^{-1}_{a,\sigma_\mathfrak{u}}(m).$$

Note that $\overline{\theta}$ is the canonical ring homomorphism from Remark 2.9.

(II) (Change of chambers). Let

$$\mathfrak{e} : (g : \omega \to \tau, \mathfrak{u}) \to (g : \omega \to \tau, \mathfrak{u}')$$

be a morphism in $\underline{\mathrm{Glue}}(\mathscr{S})$ of Type II. Then either $\mathfrak{u} \cap \mathfrak{u}'$ intersects the interior of a maximal cell (that is, $\sigma_\mathfrak{u} = \sigma_{\mathfrak{u}'}$) or not. This leads to the following two cases:

(1) $\underline{\sigma_\mathfrak{u} = \sigma_{\mathfrak{u}'}}$. Write $\sigma := \sigma_\mathfrak{u} = \sigma_{\mathfrak{u}'}$ and $h : \omega \to \sigma$. The intersection $\mathfrak{u} \cap \mathfrak{u}'$ is an $(n-1)$-dimensional convex polyhedron not contained in the $(n-1)$-skeleton of $\mathscr{P}$. Since $\omega \cap \mathfrak{u} \cap \mathfrak{u}' \neq \emptyset$, there exists $\mathfrak{v} \in \mathscr{P}^{[n-1]}_\mathscr{S}$ with $\mathfrak{v} \subseteq \mathfrak{u} \cap \mathfrak{u}'$, $\omega \cap \mathfrak{v} \neq \emptyset$. Then $\operatorname{Int} \mathfrak{v} \subseteq \operatorname{Int} \sigma$, and any wall $\mathfrak{p}$ with $\mathfrak{v} \subseteq \mathfrak{p}$ has the property $\omega \cap \mathfrak{p} \neq \emptyset$. Because Δ does not contain rational points then even $\omega \cap (\mathfrak{p} \setminus \Delta) \neq \emptyset$. This is the first place where we need the perturbation of Δ.

Let $\mathfrak{p}_1, \ldots, \mathfrak{p}_r \in \mathscr{S}^p$ be the walls containing $\mathfrak{v}$. Choose $x \in \mathrm{Int}(\mathfrak{v})$, and let $f_{\mathfrak{p}_i,x} = 1 + c_{\mathfrak{p}_i} z^{m_{\mathfrak{p}_i,x}}$ denote the function associated to $\mathfrak{p}_i$ at x according to

Remark 2.21(1). Note that any nonzero exponent m of this function fulfills $\mathrm{ord}_\tau(m) \geq \mathrm{ord}_\sigma(m) > 0$. Denote by f_i the image of $f_{\mathfrak{p}_i,x}$ in $R^k_{g,\sigma}$.

The tangent space of $\mathfrak{u}\cap\mathfrak{u}'$ defines an $(n-1)$-dimensional rational subspace $T_{\mathfrak{u}\cap\mathfrak{u}'} \subseteq \Lambda_{\sigma,\mathbb{R}}$. Let $\pi : \Lambda_\sigma \to \mathbb{Z}$ be the epimorphism which contracts $T_{\mathfrak{u}\cap\mathfrak{u}'} \cap \Lambda_\sigma$ and which is positive on vectors pointing from $\mathfrak{u}$ to $\mathfrak{u}'$.

Now define $\beta = \mathrm{id} : P_{\omega,\sigma} \to P_{\omega,\sigma}$ and

$$\theta = \theta(\mathfrak{v}) : \Lambda_\sigma \longrightarrow (R^k_{g,\sigma})^\times, \quad m \longmapsto s_{h,\sigma}\Big(\prod_{i=1}^r f_i\Big)^{-\pi(m)}.$$

This yields a log automorphism of $R^k_{g,\sigma}$ because $f_i = 1 \mod I^{>0}_{g,\sigma}$, and hence the associated automorphism of $\Bbbk[P_{\omega,\sigma}]/I^{>k}_{g,\sigma}$ changes the localizing element only by an invertible function.

Without further assumptions our definition of θ depends on the choice of $\mathfrak{v}$. We keep this dependence in mind for the time being by adding $\mathfrak{v}$ to the notation at appropriate places.

(2) $\underline{\sigma_\mathfrak{u} \neq \sigma_{\mathfrak{u}'}}$. In this case $\mathfrak{u}\cap\mathfrak{u}'$ is contained in an $(n-1)$-cell $\rho \in \mathscr{P}$. Let $\mathfrak{v} \in \mathscr{P}^{[n-1]}_{\mathscr{S}}$ be such that $\mathfrak{v} \subseteq \mathfrak{u}\cap\mathfrak{u}'$ and $\omega\cap\mathfrak{v} \neq \emptyset$. Again we will emphasize the dependence on $\mathfrak{v}$ in the notation. Since all polyhedra are rational and Δ does not contain rational points, it holds that $(\omega\cap\mathfrak{v})\setminus\Delta \neq \emptyset$. Let $x \in (\omega\cap\mathfrak{v})\setminus\Delta$, and write $e : v \to \omega$, $v := v[x]$. Denote by $\mathfrak{b}$ the unique slab with underlying polyhedral set $\mathfrak{v}$.

Now define β by parallel transport through v:

$$\mathcal{A}\!f\!f(\check{B},\mathbb{Z})_{\check{\sigma}_\mathfrak{u}} \longrightarrow \mathcal{A}\!f\!f(\check{B},\mathbb{Z})_v \longrightarrow \mathcal{A}\!f\!f(\check{B},\mathbb{Z})_{\check{\sigma}_{\mathfrak{u}'}},$$

and

$$\theta = \theta(\mathfrak{v}) : \Lambda_{\sigma_\mathfrak{u}} \longrightarrow (R^k_{g,\sigma_{\mathfrak{u}'}})^\times, \quad m \longmapsto \big(D(s_e,\rho,v)^{-1} s_e^{-1}(f_{\mathfrak{b},x})\big)^{-\pi(m)}.$$

Here $\pi : \Lambda_{\sigma_u} \to \mathbb{Z}$ is the epimorphism with kernel Λ_ρ which is positive on vectors pointing from $\mathfrak{u}$ to $\mathfrak{u}'$, and $f_{\mathfrak{b},x}$ is considered as an element of $\Bbbk[P_{\omega,\sigma_{\mathfrak{u}'}}]$ via a chart at v. Since β respects orders (Lemma 2.4) it identifies $P_{\omega,\sigma_\mathfrak{u}} \subseteq \mathcal{A}\!f\!f(\check{B},\mathbb{Z})_{\check{\sigma}_\mathfrak{u}}$ with $P_{\omega,\sigma_{\mathfrak{u}'}}$ and $I^{>k}_{g,\sigma_\mathfrak{u}}$ with $I^{>k}_{g,\sigma_{\mathfrak{u}'}}$. Hence β and θ define a ring isomorphism $\Bbbk[P_{\omega,\sigma_\mathfrak{u}}]/I^{>k}_{g,\sigma_\mathfrak{u}} \to \Bbbk[P_{\omega,\sigma_{\mathfrak{u}'}}]/I^{>k}_{g,\sigma_{\mathfrak{u}'}}$. This isomorphism respects the localizing elements as these only involve monomials of order zero with respect to ρ and hence are tangent to ρ. This shows that θ indeed defines a log isomorphism.

Our construction also seems to depend on $x \in \mathfrak{v}\cap(\omega\setminus\Delta)$. We now show that this is not the case. In fact, a different choice x' gives a vertex $v' \in \omega$ leading to $e' : v' \to \omega$, $\theta' : \Lambda_{\sigma_\mathfrak{u}} \longrightarrow (R^k_{g,\sigma_{\mathfrak{u}'}})^\times$ instead of $e : v \to \omega$, $\theta : \Lambda_{\sigma_\mathfrak{u}} \longrightarrow (R^k_{g,\sigma_{\mathfrak{u}'}})^\times$. Parallel transport through v' instead of v yields

$$\beta'(m) = \beta(m) + \pi(\overline{m}) \cdot m^\rho_{v'v}.$$

Then $(\beta,\theta,\overline{\theta})$ and $(\beta',\theta',\overline{\theta'})$ are equivalent via $\eta:\Lambda_{\sigma_{\mathfrak{u}}}\to(P_{\omega,\sigma_{\mathfrak{u}'}})^\times$, $\eta(m)=\pi(\overline{m})\cdot m^\rho_{v'v}$:

$$\begin{aligned}\theta'(m)&=\Big(D(s_{e'},\rho,v')^{-1}s_{e'}^{-1}(f_{\mathfrak{b},x'})\Big)^{-\pi(m)}\\&=\Big(D(s_e,\rho,v)^{-1}s_e^{-1}(f_{\mathfrak{b},x})z^{m^\rho_{v'v}}\Big)^{-\pi(m)}\\&=\theta(m)\cdot z^{-\pi(m)\cdot m^\rho_{v'v}}\;=\;\theta(m)\cdot(\alpha'\circ\eta(m))^{-1}.\end{aligned}\tag{2.12}$$

Here we used the change of coordinates formula (2.11) for $f_{\mathfrak{b},x}$. This proves independence of the choice of x, up to equivalence.

Remark 2.25. 1) The consistency check (2.12) in Construction 2.24(II.2) forces the change of coordinates formula (2.11) in the definition of slabs.

2) While the log isomorphism in Construction 2.24(II.2) is only defined up to equivalence, a choice of vertex $v\in\omega$ distinguishes a representative of the equivalence class. In fact, quite generally for log isomorphisms, equivalent log isomorphisms can be distinguished by the underlying homomorphism of monoids, which in the case at hand is given by parallel transport through v.

Changing chambers commutes with changing strata.

LEMMA 2.26. *Assume that $g:\omega\to\tau$, $g':\omega'\to\tau'$, and $\mathfrak{u},\mathfrak{u}'\in\mathrm{Chambers}(\mathscr{S})$, fulfill $\omega\subseteq\omega'$, $\tau\supseteq\tau'$, $\omega\cap\mathfrak{u}\cap\mathfrak{u}'\neq\emptyset$, $\dim\mathfrak{u}\cap\mathfrak{u}'=n-1$ and $\tau\subseteq\sigma_{\mathfrak{u}}\cap\sigma_{\mathfrak{u}'}$. For the morphisms*

$$\begin{aligned}\mathfrak{e}_1&:=\big((g,\mathfrak{u})\longrightarrow(g,\mathfrak{u}')\big),&\quad \mathfrak{e}_2&:=\big((g,\mathfrak{u}')\longrightarrow(g',\mathfrak{u}')\big),\\ \mathfrak{f}_1&:=\big((g,\mathfrak{u})\longrightarrow(g',\mathfrak{u})\big),&\quad \mathfrak{f}_2&:=\big((g',\mathfrak{u})\longrightarrow(g',\mathfrak{u}')\big)\end{aligned}$$

in $\underline{\mathrm{Glue}}(\mathscr{S})$ let $\theta(\mathfrak{e}_i)$, $\theta(\mathfrak{f}_i)$ be the basic gluing morphisms from Construction 2.24, where $\theta(\mathfrak{e}_1)$ and $\theta(\mathfrak{f}_2)$ are computed using the same $\mathfrak{v}\in\mathscr{P}^{[n-1]}_{\mathscr{S}}$. Then

$$\theta(\mathfrak{e}_2)\circ\theta(\mathfrak{e}_1)=\theta(\mathfrak{f}_2)\circ\theta(\mathfrak{f}_1).$$

Proof. Denote $a:\omega\to\omega'$. Let us first assume that $\mathfrak{u},\mathfrak{u}'$, are contained in the same maximal cell σ, that is, $\sigma_{\mathfrak{u}}=\sigma_{\mathfrak{u}'}$. With $h:\omega\to\sigma$, $h':\omega'\to\sigma$ it holds that $h=h'\circ a$ and hence $s_{h,\sigma}=s_{h',\sigma}\cdot s_{a,\sigma}$. Then from Construction 2.24(I) and (II.1) we obtain

$$\begin{aligned}\theta(\mathfrak{e}_2)\circ\theta(\mathfrak{e}_1)(m)&=\overline{\theta(\mathfrak{e}_2)}\big(\theta(\mathfrak{e}_1)(m)\big)\cdot\theta(\mathfrak{e}_2)(m)\\&=\overline{\theta(\mathfrak{e}_2)}\Big(s_{h,\sigma}\big(\textstyle\prod_i f_i\big)^{-\pi(m)}\Big)\cdot s_{a,\sigma}^{-1}(m)\\&=s_{h',\sigma}\big(\textstyle\prod_i f_i\big)^{-\pi(m)}\cdot s_{a,\sigma}^{-1}(m)=\theta(\mathfrak{f}_2)(m)\cdot\overline{\theta(\mathfrak{f}_2)}\big(s_{a,\sigma}^{-1}(m)\big)\\&=\theta(\mathfrak{f}_2)(m)\cdot\overline{\theta(\mathfrak{f}_2)}\big(\theta(\mathfrak{f}_1)(m)\big)=\theta(\mathfrak{f}_2)\circ\theta(\mathfrak{f}_1)(m).\end{aligned}$$

Otherwise there exists $\rho \in \mathscr{P}^{[n-1]}$ with $\mathfrak{u} \cap \mathfrak{u}' \subseteq \rho$. Writing $e : v \to \omega$, $e' : v \to \omega'$ for $v = v[x]$ as in Construction 2.24(II.2), by the definitions of $D(s_e, \rho, v)$ and $D(s_{e'}, \rho, v)$ (Definition 1.20), we have

$$\left(\frac{D(s_{e'}, \rho, v)}{D(s_e, \rho, v)}\right)^{\pi(m)} = \frac{s_{e',\sigma_{\mathfrak{u}}}(m)}{s_{e',\sigma_{\mathfrak{u}'}}(m)} \cdot \frac{s_{e,\sigma_{\mathfrak{u}'}}(m)}{s_{e,\sigma_{\mathfrak{u}}}(m)} = \frac{s_{a,\sigma_{\mathfrak{u}}}(m)}{s_{a,\sigma_{\mathfrak{u}'}}(m)}.$$

Hence

$$\begin{aligned}\theta(\mathfrak{e}_2) \circ \theta(\mathfrak{e}_1)(m) &= \overline{\theta(\mathfrak{e}_2)}\big(D(s_e, \rho, v)^{-1} s_e^{-1}(f_{\mathfrak{b},x})\big)^{-\pi(m)} \cdot \theta(\mathfrak{e}_2)(m) \\ &= s_{a,\sigma_{\mathfrak{u}'}}^{-1}\big(D(s_e, \rho, v)^{-1} s_e^{-1}(f_{\mathfrak{b},x})\big)^{-\pi(m)} \cdot s_{a,\sigma_{\mathfrak{u}'}}^{-1}(m) \\ &= \big(D(s_{e'}, \rho, v)^{-1} s_{e'}^{-1}(f_{\mathfrak{b},x})\big)^{-\pi(m)} \cdot s_{a,\sigma_{\mathfrak{u}}}^{-1}(m) = \theta(\mathfrak{f}_2) \circ \theta(\mathfrak{f}_1)(m),\end{aligned}$$

as desired. □

2.5. *Loops around joints and consistency.* To use compositions of basic gluing morphisms to define a functor from $\underline{\mathrm{Glue}}(\mathscr{S})$ to $\underline{\mathrm{LogRings}}$ requires a compatibility condition that we now discuss.

Definition 2.27. A *joint* $\mathfrak{j}$ of $\mathscr{S}$ is an $(n-2)$-cell of $\mathscr{P}_{\mathscr{S}}$ with $\mathfrak{j} \not\subseteq \partial B$. The set of joints of $\mathscr{S}$ is denoted $\mathrm{Joints}(\mathscr{S})$.

The minimal cell $\sigma_{\mathfrak{j}} \in \mathscr{P}$ containing a joint $\mathfrak{j}$ has codimension at most two, and we speak of codimension-zero, one and two joints, respectively. For $v \in \sigma_{\mathfrak{j}}$ a vertex we define the normal space of $\mathfrak{j}$ as

$$\mathcal{Q}^v_{\mathfrak{j},\mathbb{R}} := \Lambda_{v,\mathbb{R}}/\Lambda_{\mathfrak{j},\mathbb{R}}.$$

To denote the image of an object in $\mathcal{Q}^v_{\mathfrak{j},\mathbb{R}}$ we use a double bar. For example, if $\sigma \in \mathscr{P}_{\max}$ contains $\mathfrak{j}$, we have the canonical map

$$\mathcal{A}\!f\!f(\check{B}, \mathbb{Z})_{\check{\sigma}} \longrightarrow \Lambda_v \longrightarrow \mathcal{Q}^v_{\mathfrak{j},\mathbb{R}}, \quad m \longmapsto \overline{\overline{m}}.$$

Moreover, for a cell inside B containing $\mathfrak{j}$, say $\tau \in \mathscr{P}$, denote by $\overline{\overline{\tau}} \subseteq \mathcal{Q}^v_{\mathfrak{j},\mathbb{R}}$ the image of the tangent wedge to τ along $\mathfrak{j}$. This is a convex cone, which is strictly convex if and only if $\mathfrak{j}$ is contained in an $(n-2)$-dimensional face of τ.

Note that for any $\sigma \in \mathscr{P}_{\max}$ containing v, parallel transport defines a canonical isomorphism

$$\mathcal{Q}^v_{\mathfrak{j},\mathbb{R}} = \Lambda_{\sigma,\mathbb{R}}/\Lambda_{\mathfrak{j},\mathbb{R}},$$

and local monodromy acts trivially on the right-hand side if $\operatorname{codim} \sigma_{\mathfrak{j}} \in \{0, 2\}$. Thus, in this case, $\mathcal{Q}^v_{\mathfrak{j},\mathbb{R}}$ can be defined independently of v, while if $\sigma_{\mathfrak{j}} = \rho \in \mathscr{P}^{[n-1]}$, we can only define the two half-planes separated by the line $\overline{\overline{\rho}}$ invariantly.

Now the $(n-1)$-cells of $\mathscr{P}_{\mathscr{S}}$ containing $\mathfrak{j}$ define a set of distinct half-lines in $\mathcal{Q}^v_{\mathfrak{j},\mathbb{R}}$. Let $\mathfrak{v}_1, \ldots, \mathfrak{v}_l, \mathfrak{v}_{l+1} = \mathfrak{v}_1$, be a cyclic numbering of these cells induced by an orientation on $\mathcal{Q}^v_{\mathfrak{j},\mathbb{R}}$. Then by Definition 2.22(ii) for any i there exists a unique $\mathfrak{u}_i \in \mathrm{Chambers}(\mathscr{S})$ with $\partial\overline{\overline{\mathfrak{u}}}_i = \overline{\overline{\mathfrak{v}}}_i \cup \overline{\overline{\mathfrak{v}}}_{i+1}$.

Now let $(g : \omega \to \tau) \in \mathrm{Hom}(\mathscr{P})$ and assume $\mathfrak{j} \cap \omega \neq \emptyset$ and $\tau \subseteq \sigma_{\mathfrak{j}}$. Then for each i we have a morphism

$$\mathfrak{e} : (g : \omega \to \tau, \mathfrak{u}_i) \to (g : \omega \to \tau, \mathfrak{u}_{i+1})$$

in $\underline{\mathrm{Glue}}(\mathscr{S})$ changing chambers. By Construction 2.24 we thus obtain the sequence of log isomorphisms

$$\theta_i = \theta(\mathfrak{v}_i) : \Lambda_{\sigma_{\mathfrak{u}_i}} \to (R^k_{g,\sigma_{\mathfrak{u}_{i+1}}})^\times$$

from $R^k_{g,\sigma_{\mathfrak{u}_i}}$ to $R^k_{g,\sigma_{\mathfrak{u}_{i+1}}}$. Note that the equivalence class of θ_i only depends on g, $\mathfrak{v}_i$, $\mathfrak{j}$ and an orientation on $\mathcal{Q}^v_{\mathfrak{j},\mathbb{R}}$.

Definition 2.28. The structure $\mathscr{S}$ is *consistent at the joint* $\mathfrak{j}$ *to order* k if for any $(g : \omega \to \tau) \in \mathrm{Hom}(\mathscr{P})$ with $\mathfrak{j} \cap \omega \neq \emptyset$ and $\tau \subseteq \sigma_{\mathfrak{j}}$, the composition

$$\theta^k_{\mathfrak{j}} := \theta_l \circ \cdots \circ \theta_1 : \Lambda_{\sigma_{\mathfrak{u}_1}} \longrightarrow (R^k_{g,\sigma_{\mathfrak{u}_1}})^\times \tag{2.13}$$

equals 1. A structure is *consistent to order* k if it is consistent to order k at every joint.

Remark 2.29. 1) Note that relabelling $\mathfrak{u}_1, \ldots, \mathfrak{u}_l$ to $\mathfrak{u}_i, \ldots, \mathfrak{u}_l, \mathfrak{u}_1, \ldots, \mathfrak{u}_{i-1}$, only leads to conjugation of $\theta^k_{\mathfrak{j}}$ with an isomorphism $R^k_{g,\sigma_{\mathfrak{u}_1}} \to R^k_{g,\sigma_{\mathfrak{u}_i}}$, and reversing the cyclic order produces the inverse. Thus consistency around a joint really only depends on $\mathscr{S}$ and $\mathfrak{j}$. Moreover, $\theta^k_{\mathfrak{j}}$ is well defined once a reference chamber $\mathfrak{u}_1$ and orientation of $\mathcal{Q}^v_{\mathfrak{j}}$ are chosen for a vertex $v \in \sigma_{\mathfrak{j}}$.

2) The notion of consistency implicitly depends on the choice of open gluing data s.

3) Consistency of a structure $\mathscr{S}$ does not depend on the choice of polyhedral decomposition $\mathscr{P}_{\mathscr{S}}$. In fact, if $\mathscr{S}$ is consistent at every joint of $\mathscr{P}_{\mathscr{S}}$ it is also consistent at every joint of any refinement of $\mathscr{P}_{\mathscr{S}}$.

4) Assume $\omega \subseteq \omega' \subseteq \tau' \subseteq \tau$ and we compute (2.13) both for $g : \omega \to \tau$ and $g' : \omega' \to \tau'$, with the same sequence of chambers, resulting in log automorphisms $\theta^k_{\mathfrak{j}}$ and $\theta^{k\prime}_{\mathfrak{j}}$. It then follows from Lemma 2.26 that $\theta^{k\prime}_{\mathfrak{j}}$ equals the composition of $\theta^k_{\mathfrak{j}}$ with the canonical homomorphism

$$(R^k_{g,\sigma_{\mathfrak{u}_1}})^\times \longrightarrow (R^k_{g',\sigma_{\mathfrak{u}_1}})^\times.$$

5) By 4), it suffices to consider the case $\omega = \tau = \sigma_{\mathfrak{j}}$. In fact, $R^k_{\omega\to\sigma_{\mathfrak{j}},\sigma_{\mathfrak{u}_1}} \to R^k_{\mathrm{id}_{\sigma_{\mathfrak{j}}},\sigma_{\mathfrak{u}_1}}$ is the localization at nonzero divisors, while $R^k_{\omega\to\sigma_{\mathfrak{j}},\sigma_{\mathfrak{u}_1}} \to R^k_{\omega\to\tau,\sigma_{\mathfrak{u}_1}}$ is a surjection followed by a localization. Hence a log automorphism of $R^k_{\omega\to\tau,\sigma_{\mathfrak{u}_1}}$ compatible with the identity on $R^k_{\mathrm{id}_{\sigma_{\mathfrak{j}}},\sigma_{\mathfrak{u}_1}}$ is the identity.

For consistent structures we can define a *gluing functor*

$$F_s^k : \underline{\mathrm{Glue}}(\mathscr{S}) \longrightarrow \underline{\mathrm{LogRings}},$$

mapping $(g, \mathfrak{u})$ to $R^k_{g,\sigma_\mathfrak{u}}$ and morphisms of types I or II to the basic gluing morphisms of the respective types defined in Construction 2.24. In fact, the following is true.

LEMMA 2.30. *Assume that the structure $\mathscr{S}$ is consistent to order k. Let $\mathfrak{e} = \mathfrak{e}_r \circ \cdots \circ \mathfrak{e}_1 = \mathfrak{e}'_{r'} \circ \cdots \circ \mathfrak{e}'_1$ be two decompositions of $\mathfrak{e} \in \mathrm{Hom}(\underline{\mathrm{Glue}}(\mathscr{S}))$ into basic morphisms. Then if $\theta_r, \ldots, \theta_1$ and $\theta'_{r'}, \ldots, \theta'_1$ are the associated basic log morphisms, it holds that*

$$\theta_r \circ \cdots \circ \theta_1 = \theta'_{r'} \circ \cdots \circ \theta'_1.$$

Proof. The proof proceeds in three steps.

Step 1. *Independence of choices.* The construction of the morphism associated to a change of chambers $(g : \omega \to \tau, \mathfrak{u}) \to (g : \omega \to \tau, \mathfrak{u}')$ (Construction 2.24(II) required a choice of $\mathfrak{v} \in \mathscr{P}^{[n-1]}_{\mathscr{S}}$. We claim independence of this choice for consistent structures. We use the notation from Construction 2.24. If $\mathfrak{v}' \in \mathscr{P}^{[n-1]}_{\mathscr{S}}$ is another $(n-1)$-cell with $\mathfrak{v}' \subseteq \mathfrak{u} \cap \mathfrak{u}'$, $\omega \cap \mathfrak{v}' \neq \emptyset$ and adjacent to $\mathfrak{v}$ (i.e. $\dim \mathfrak{v} \cap \mathfrak{v}' = n-2$), then $\mathfrak{j} := \mathfrak{v} \cap \mathfrak{v}'$ is a joint and the log isomorphisms θ and θ' constructed via $\mathfrak{v}$ and $\mathfrak{v}'$, respectively, differ by a composition $\theta_l \circ \cdots \circ \theta_1$ associated to a loop around $\mathfrak{j}$ (Definition 2.28). Hence they are equal. The general case follows since any two $(n-1)$-cells of $\mathscr{P}_{\mathscr{S}}$ contained in $\mathfrak{u} \cap \mathfrak{u}'$ and intersecting ω can be connected by a sequence of adjacent cells with the same properties.

Step 2. *Reduction to the case $g = g'$.* By construction the basic gluing morphisms changing strata are compatible with compositions in $\mathrm{Hom}(\mathscr{P})$: If $\mathfrak{e}_i$ and $\mathfrak{e}_{i+1}$ both change strata for a fixed chamber $\mathfrak{u} = \mathfrak{u}_i = \mathfrak{u}_{i+1} = \mathfrak{u}_{i+2}$, then

$$F_s^k(\mathfrak{e}_i) \circ F_s^k(\mathfrak{e}_{i+1}) = F_s^k(\mathfrak{f}),$$

where $\mathfrak{f} : (g_i, \mathfrak{u}) \to (g_{i+2}, \mathfrak{u})$. Since, by Lemma 2.26, changing chambers commutes with changing strata, we can thus assume that only $\mathfrak{e}_1$ changes strata. Since the analogous factorization for $\mathfrak{e}'$ leads to the same change of strata this reduces the claim to a sequence of changes of chambers.

Step 3. *Spaces of chambers.* Given $g : \omega \to \tau$ look at all chambers $\mathfrak{u}$ such that $(g, \mathfrak{u}) \in \underline{\mathrm{Glue}}(\mathscr{S})$:

$$A := \left\{ \mathfrak{u} \in \mathrm{Chambers}(\mathscr{S}) \,\middle|\, \omega \cap \mathfrak{u} \neq \emptyset,\ \tau \subseteq \sigma_\mathfrak{u} \right\}.$$

Define Σ to be the abstract two-dimensional cell complex with A as set of vertices, edges connecting adjacent $\mathfrak{u}, \mathfrak{u}' \in A$ and a disk glued into any 1-cycle of chambers $\mathfrak{u}_1, \ldots, \mathfrak{u}_l$ forming a loop around a joint. An edge with vertices $\mathfrak{u}, \mathfrak{u}'$, defines a change of chamber isomorphism (of either type) $R^k_{g,\sigma_\mathfrak{u}} \to R^k_{g,\sigma_{\mathfrak{u}'}}$.

Consistency says that the composition of these isomorphisms following the boundary of a 2-cell is the identity. Thus we obtain the desired independence of the sequence of adjacent chambers connecting $\mathfrak{u}_1$ with $\mathfrak{u}_r$ once we know $H^1(\Sigma, \mathbb{Z}) = 0$. This follows from the following lemma.

LEMMA 2.31. $\pi_1(\Sigma) = 0$.

Proof. Denote by

$$V := \bigcup_{\sigma \in \mathscr{P}_{\max}, \tau \subseteq \sigma} \sigma$$

the closed star of τ with respect to $\mathscr{P}$. Then for $\mathfrak{u} \in \mathrm{Chambers}(\mathscr{S})$ the condition $\tau \subseteq \sigma_{\mathfrak{u}}$ is equivalent to $\mathfrak{u} \subseteq V$, and such chambers define a decomposition of V into closed polyhedra. This is generally not a proper polyhedral decomposition because the intersection of two chambers needs not be a face of either chamber. We can however refine the decomposition of V into chambers into an honest polyhedral decomposition $\mathscr{P}_V$, for example by replacing each slab or wall by the hyperplane containing it. Since ω is topologically a ball, the cells of $\mathscr{P}_V$ intersecting ω form a polyhedral decomposition $\mathscr{P}'$ of an n-cell, and so does the combinatorial dual decomposition $\check{\mathscr{P}}'$. Thus the two-skeleton of $\check{\mathscr{P}}'$ is simply connected. Now Σ is obtained from $\check{\mathscr{P}}'$ by contracting all edges corresponding to adjacent n-cells of $\mathscr{P}$ lying in the same chamber. Thus also Σ is simply connected. □

This finishes the proof of Lemma 2.30. □

2.6. *Construction of finite order deformation.* Given a structure $\mathscr{S}$ that is consistent to order k we are now able to construct the desired deformation (X_k, D_k) of (X, D) over $\mathrm{Spec}\left(\Bbbk[t]/(t^{k+1})\right)$, by taking, in a certain sense, the colimit of $\mathrm{Spec}\, R^k_{g,\sigma_{\mathfrak{u}}}$ over $(g, \mathfrak{u}) \in \underline{\mathrm{Glue}}(\mathscr{S})$. Taken literally this would lead to a nonseparated scheme because we have removed closed subsets (the zero loci of the localizing elements $f_{g,\sigma_{\mathfrak{u}}}$) from lower-dimensional strata. Instead we proceed as follows. Denote by

$$\overline{F^k_s} : \underline{\mathrm{Glue}}(\mathscr{S}) \longrightarrow \underline{\mathrm{Rings}}$$

the composition of F^k_s with the forgetful functor $\underline{\mathrm{LogRings}} \to \underline{\mathrm{Rings}}$. For $(g : \omega \to \tau, \mathfrak{u}) \in \underline{\mathrm{Glue}}(\mathscr{S})$, the underlying topological space of $\mathrm{Spec}\, \overline{F^k_s}(g, \mathfrak{u}) = \mathrm{Spec}\, R^k_{g,\sigma_{\mathfrak{u}}}$ is, according to Remark 2.8, canonically an open subset of $V_g \subseteq V(\omega)$. Denote by

$$i(g, \mathfrak{u}) : \left|\mathrm{Spec}\, R^k_{g,\sigma_{\mathfrak{u}}}\right| \longrightarrow \left|V(\omega)\right|$$

the inclusion of the underlying topological spaces. Then

$$\mathcal{O}_k(g, \mathfrak{u}) := i(g, \mathfrak{u})_* \mathcal{O}_{\mathrm{Spec}\, R^k_{g,\sigma_{\mathfrak{u}}}}$$

defines a sheaf of $\Bbbk[t]$-algebras on $V(\omega)$, and if $\mathfrak{e} : (g : \omega \to \tau, \mathfrak{u}) \to (g' : \omega' \to \tau', \mathfrak{u}')$ is a morphism in $\underline{\mathrm{Glue}}(\mathscr{S})$ and

$$\Phi_{\omega\omega'}(s) : V(\omega') \to V(\omega)$$

is the open embedding defined from the composition of $p|_{V(\omega')}$ with the inverse of $p|_{V(\omega)}$, then $F_s^k(\mathfrak{e})$ defines a homomorphism of sheaves of $\Bbbk[t]$-algebras

$$\mathcal{O}_k(g, \mathfrak{u}) \longrightarrow \big(\Phi_{\omega\omega'}(s)\big)_* \mathcal{O}_k(g', \mathfrak{u}').$$

In fact, recall from the discussion following Definition 1.18 that $\Phi_{\omega\omega'}(s)$ is given by the canonical embedding $\operatorname{Spec} \Bbbk[\omega'^{-1}\Sigma_v] \to \operatorname{Spec} \Bbbk[\omega^{-1}\Sigma_v]$ for some $v \in \omega$, composed with the automorphism s_h^{-1} of $\operatorname{Spec} \Bbbk[\omega'^{-1}\Sigma]$ coming from open gluing data, where $h : \omega \to \omega'$. By Remark 2.9 this is compatible with the reduction modulo t of $\operatorname{Spec} \overline{F_s^k}(\mathfrak{e})$. In particular, we have

$$|\Phi_{\omega\omega'}(s)| \circ i(g', \mathfrak{u}') = i(g, \mathfrak{u}) \circ |\operatorname{Spec} \overline{F_s^k}(\mathfrak{e})|.$$

Thus

$$(g : \omega \to \tau, \mathfrak{u}) \longmapsto \big(|V(\omega)|, \mathcal{O}_k(g, \mathfrak{u})\big) \tag{2.14}$$

defines a contravariant functor from $\underline{\mathrm{Glue}}(\mathscr{S})$ to the category of ringed spaces. The aim of this subsection is to show that the colimit of this functor defines a deformation of X over $\Bbbk[t]/(t^{k+1})$ of the desired form. This will be achieved in Proposition 2.39.

We first construct deformations $V^k(\omega)$ of the standard affine sets $V(\omega)$ that cover X and then glue by open embeddings. Thus for the time being keep ω fixed and consider

$$V^k(\omega) := \Big(\Big|V(\omega)\Big|, \varprojlim \mathcal{O}_k(g, \mathfrak{u})\Big),$$

where the inverse limit runs over all $(g, \mathfrak{u}) \in \underline{\mathrm{Glue}}(\mathscr{S})$ with domain ω.

Let $x \in \mathrm{Int}(\omega) \setminus \Delta$. For each $g : \omega \to \tau$ we can choose $\mathfrak{u}_g \in \mathrm{Chambers}(\mathscr{S})$ with $x \in \mathfrak{u}_g$ and such that $(g, \mathfrak{u}_g) \in \underline{\mathrm{Glue}}(\mathscr{S})$. Then by Lemma 2.30, if $(g, \mathfrak{u}) \in \underline{\mathrm{Glue}}(\mathscr{S})$ with domain ω, there is a unique isomorphism $R^k_{g,\sigma_{\mathfrak{u}_g}} \to R^k_{g,\sigma_{\mathfrak{u}}}$ by a composition of changes of chambers. This shows that we may replace the system of rings $R^k_{g,\sigma_{\mathfrak{u}}}$ and sheaves $\mathcal{O}_k(g, \mathfrak{u})$, for $(g, \mathfrak{u}) \in \underline{\mathrm{Glue}}(\mathscr{S})$ with domain ω, with the system

$$R_\tau := R^k_{g:\omega\to\tau,\sigma_{\mathfrak{u}_g}} \quad \text{and} \quad \mathcal{O}_k(\tau) := \mathcal{O}_k(g : \omega \to \tau, \mathfrak{u}_\tau),$$

respectively, where now τ runs over all cells containing ω and $\mathfrak{u}_\tau := \mathfrak{u}_{\omega\to\tau}$. In doing this keep in mind that the homomorphisms

$$\psi_{\tau'\tau} : R_\tau \to R_{\tau'}$$

thus obtained now involve compositions of changes of chambers.

Eventually we will argue inductively, noting that reduction modulo t^{l+1} defines rings R^l_τ, sheaves $\mathcal{O}_l(\tau)$ and ringed spaces $V^l(\omega)$ for $l < k$. These are related by the complex

$$0 \longrightarrow \mathcal{O}_{V(\omega)} \xrightarrow{\cdot t^l} \mathcal{O}_{V^l(\omega)} \longrightarrow \mathcal{O}_{V^{l-1}(\omega)} \longrightarrow 0 \tag{2.15}$$

of sheaves of $\Bbbk[t]$-algebras on $V(\omega)$, and we have to show this complex is exact. Then $V^l(\omega)$ is a flat lifting of $V^{l-1}(\omega)$ from $\Bbbk[t]/(t^l)$ to $\Bbbk[t]/(t^{l+1})$. It then follows inductively that $V^k(\omega)$ is affine, of finite type over $\Bbbk$ and flat over $\Bbbk[t]/(t^{k+1})$. The overall strategy for showing exactness is to write down an isomorphism with standard local models outside a codimension-three locus contained in Z_ω, the preimage of Z under $p : V(\omega) \to X$, and then to extend. This will also show log smoothness away from Z in the limit $k \to \infty$.

The first step is the construction of a chart for $V^k(\omega)$ away from Z_ω. We write $P = P_{\omega,\sigma_{\mathfrak{u}_{\mathrm{id}_\omega}}} = P_x$. Recall that for any $\tau \supseteq \omega$, we have the homomorphism

$$\alpha_\tau : P \longrightarrow R_\tau$$

endowing R_τ with the structure of a log ring. Whenever $\tau \supseteq \tau' \supseteq \omega$ there is a log morphism

$$\theta_{\tau'\tau} : \Lambda_x \simeq \Lambda_{\sigma_{\mathfrak{u}_\tau}} \longrightarrow (R_{\tau'})^\times$$

such that $\overline{\theta}_{\tau'\tau} = \psi_{\tau'\tau}$.

Proposition 2.32. *Let $p \in |V(\omega)| \setminus Z_\omega$. Since $|V(\omega)|$ coincides with the topological space underlying* $\operatorname{Spec} \Bbbk[P]/(t^{k+1})$, *there is a prime ideal $\mathfrak{p} \subseteq \Bbbk[P]/(t^{k+1})$ corresponding to p. Furthermore, for $g : \omega \to \tau$, $|\operatorname{Spec} R_\tau|$ is identified with $|V_g| \setminus Z_\omega$. Thus if $p \in |V_g|$, there is a prime ideal $\mathfrak{p}_\tau \subseteq R_\tau$ corresponding to p. Then there is an isomorphism*

$$\psi : \left(\Bbbk[P]/(t^{k+1})\right)_{\mathfrak{p}} \longrightarrow \varprojlim (R_\tau)_{\mathfrak{p}_\tau},$$

where the inverse limit is over all $g : \omega \to \tau$ with $p \in |V_g|$, and the maps of the inverse system are the localizations of the maps $\psi_{\tau'\tau}$.

Proof. For any $\tau' \subseteq \tau$ with $\omega \subseteq \tau'$ it holds that $(\omega \to \tau', \mathfrak{u}_\tau) \in \underline{\mathrm{Glue}}(\mathscr{S})$. Note that $\psi_{\tau'\tau}$ is the composition of the canonical homomorphism

$$R_\tau = R^k_{\omega\to\tau,\sigma_{\mathfrak{u}_\tau}} \longrightarrow R^k_{\omega\to\tau',\sigma_{\mathfrak{u}_\tau}},$$

changing strata, with a change of chambers isomorphism

$$R^k_{\omega\to\tau',\sigma_{\mathfrak{u}_\tau}} \longrightarrow R^k_{\omega\to\tau',\sigma_{\mathfrak{u}_{\tau'}}} = R_{\tau'}.$$

In particular, $\psi_{\tau'\tau}$ needs not be surjective because the change of strata homomorphism may involve a localization. However, after localizing at the ideals $\mathfrak{p}_{\tau'}$ and $\mathfrak{p}_\tau$ respectively, this map becomes surjective.

Note also that if $\omega \subseteq \tau'' \subseteq \tau' \subseteq \tau$, then by consistency,

$$\theta_{\tau''\tau} = \theta_{\tau''\tau'} \circ \theta_{\tau'\tau} = \theta_{\tau''\tau'} \cdot (\psi_{\tau''\tau'} \circ \theta_{\tau'\tau}).$$

It follows that $(\theta_{\tau'\tau})_{\tau' \subsetneq \tau}$ is a barycentric 1-cocycle for the system of groups $\mathrm{Hom}(\Lambda_x, (R_\tau)^\times_{\mathfrak{p}_\tau})$, as considered in [GS06, A.1]. Since the homomorphisms $(R_\tau)^\times_{\mathfrak{p}_\tau} \to (R_{\tau'})^\times_{\mathfrak{p}_{\tau'}}$ are surjective it is straightforward to check the exactness criterion $(*)$ in [GS06, Prop. A.1]. Hence there exist $\theta_\tau : \Lambda_x \to (R_\tau)^\times_{\mathfrak{p}_\tau}$ such that for any $\tau' \subsetneq \tau$,

$$\theta_{\tau'\tau} = \theta_{\tau'}/(\psi_{\tau'\tau} \circ \theta_\tau).$$

Then $\theta_\tau \cdot \alpha_\tau$ defines a compatible system of homomorphisms

$$\psi_\tau : \left(\Bbbk[P]/(t^{k+1})\right)_{\mathfrak{p}} \to (R_\tau)_{\mathfrak{p}_\tau},$$

hence the desired map ψ. Note that ψ_τ is the composition of the canonical quotient

$$\left(\Bbbk[P]/(t^{k+1})\right)_{\mathfrak{p}} \longrightarrow \left(\Bbbk[P]/I^{>k}_{\omega\to\tau}\right)_{\mathfrak{p}}$$

and an isomorphism of $\left(\Bbbk[P]/I^{>k}_{\omega\to\tau}\right)_{\mathfrak{p}}$ with $(R_\tau)_{\mathfrak{p}_\tau}$. Now it is easy to check that $\Bbbk[P]/(t^{k+1})$ is the inverse limit of the rings $\Bbbk[P]/I^{>k}_{\omega\to\tau}$ with the canonical quotient homomorphisms between them, and an analogous statement holds for the localizations. The ψ_τ induce an isomorphism between this inverse system and $(R_\tau)_{\mathfrak{p}_\tau}$. This shows that ψ is indeed an isomorphism. □

COROLLARY 2.33. *For any $l \le k$, Sequence* (2.15) *is exact on $V(\omega) \setminus Z_\omega$.*

Following Example 2.19 we can also check directly exactness of (2.15) at general points of Z_ω.

LEMMA 2.34. *For any $\rho \in \mathscr{P}^{[n-1]}$ containing ω, Sequence* (2.15) *is also exact at all points of the maximal torus* $\mathrm{Spec}\,\Bbbk[\Lambda_\rho] \subseteq V_{\omega\to\rho}$.

Proof. Let σ^+, σ^-, be the maximal cells with $\rho = \sigma^+ \cap \sigma^-$. Only three of the sheaves $\mathcal{O}_k(\tau)$ have support on $\mathrm{Spec}\,\Bbbk[\Lambda_\rho]$, namely $\mathcal{O}_k(\rho)$ and $\mathcal{O}_k(\sigma^\pm)$. Moreover, by choosing $\mathfrak{u}_{\sigma^\pm}$ as adjacent chambers and $\mathfrak{u}_\rho = \mathfrak{u}_{\sigma^+}$, we can assume that $R_{\sigma^+} \to R_\rho$ is the canonical homomorphism while $R_{\sigma^-} \to R_\rho$ is the canonical homomorphism composed with a change of chamber automorphism

$$\psi : R_\rho \longrightarrow R_\rho, \quad z^m \longmapsto f_\rho^{-\pi(\overline{m})} \cdot z^m$$

as defined in Construction 2.24(II.2). Recall that $\pi : \Lambda_{\sigma^-} \to \mathbb{Z}$ maps to 1 a generator of $\Lambda_{\sigma^-}/\Lambda_\rho$ pointing from σ^- to σ^+, and f_ρ has zero locus $Z_\omega \cap |\mathrm{Spec}\,\Bbbk[\Lambda_\rho]| \subseteq V(\omega)$. Denote by $F_\rho \subseteq P$ the face corresponding to ρ. Since ρ is codimension one, $P + F_\rho^{\mathrm{gp}} \simeq \Lambda_\rho \oplus S_e$ for some $e \ge 1$ with $S_e \subseteq \mathbb{Z}^2$ the monoid generated by $(1,0)$, $(e,-1)$, $(0,1)$. Denote by R_+, R_- and $R_\cap$

the localizations at the multiplicative system $\{z^m\}_{m\in F_\rho}$ of R_{σ^+}, R_{σ^-} and R_ρ, respectively. Explicitly, we may write

$$\begin{aligned}
R_- &= \Bbbk[\Lambda_\rho][x,y,t]/\Big(xy-t^e, y^\beta t^\gamma \,\Big|\, \beta e+\gamma\ge k+1\Big),\\
R_+ &= \Bbbk[\Lambda_\rho][x,y,t]/\Big(xy-t^e, x^\alpha t^\gamma \,\Big|\, \alpha e+\gamma\ge k+1\Big),\\
R_\cap &= \Big(\Bbbk[\Lambda_\rho][x,y,t]/\Big(xy-t^e, x^\alpha y^\beta t^\gamma \,\Big|\, \max\{\alpha,\beta\}e+\gamma\ge k+1\Big)\Big)_{f_\rho},
\end{aligned}$$

and $R_+\to R_\cap$ is the canonical quotient followed by localization at f_ρ, while $R_-\to R_\cap$ is the homomorphism of $\Bbbk[\Lambda_\rho][t]/(t^{k+1})$-algebras with

$$x\longmapsto f_\rho x,\quad y\longmapsto f_\rho^{-1}y.$$

We claim that

$$\begin{gathered}
R_\cup := \Bbbk[\Lambda_\rho][X,Y,t]/(XY-f_\rho t^e, t^{k+1}) \longrightarrow R_-\times_{R_\cap} R_+,\\
X\longmapsto (x, f_\rho x),\ Y\longmapsto (f_\rho y, y)
\end{gathered}$$

is an isomorphism of $\Bbbk[\Lambda_\rho][t]$-algebras. In fact, the rings R_-, R_+, are generated by 1, x^i, y^j, $i,j>0$, as $\Bbbk[\Lambda_\rho][t]/(t^{k+1})$-modules, and the same monomials generate $R_\cap$ as $\Bbbk[\Lambda_\rho]_{f_\rho}[t]/(t^{k+1})$-module. Moreover, the $\Bbbk[\Lambda_\rho][t]$-submodules of R_- (R_+) generated by x^i, $i\ge 0$, (y^j, $j\ge 0$), is a free direct summand. Thus if $g_\pm\in R_\pm$, we may write uniquely $g_-=\sum_{i\ge 0}a_ix^i+h_-(y,t)$, $g_+=\sum_{j\ge 0}b_jy^j+h_+(x,t)$ with $h_\pm(0,t)=0$. Thus $(g_-,g_+)\in R_-\times_{R_\cap}R_+$ if and only if

$$a_0=b_0,\quad h_-(y,t)=\sum_{j>0}b_jf_\rho^jy^j,\quad h_+(x,t)=\sum_{i>0}a_if_\rho^ix^i$$

as elements of $R_\cap$. If this is the case, then (g_-,g_+) is the image of $\sum_{i\ge 0}a_iX^i+\sum_{j>0}b_jY^j\in R_\cup$. This shows surjectivity. Injectivity follows along the same lines by noting that $R_\cup$ is a free $\Bbbk[\Lambda_\rho][t]/(t^{k+1})$-module with basis X^i, Y^j, $i\ge 0$, $j>0$.

To complete the proof it remains to observe that

$$(t^l)\subseteq \Bbbk[\Lambda_\rho][X,Y,t]/(XY-f_\rho t^e, t^{l+1})$$

is a free $\Bbbk[\Lambda_\rho][X,Y]/(XY)$-module for $0<l\le k$. In fact, by the same argument as before, each element of (t^l) can be uniquely written as

$$t^l\Big(\sum_{i\ge 0}a_iX^i+\sum_{j>0}b_jY^j\Big).\qquad\square$$

We now know that Sequence (2.15) is exact on $V(\omega)\setminus Z'_\omega$, where Z'_ω is the intersection of Z_ω with the union of the codimension-two strata of $V(\omega)$. To extend across Z'_ω the crucial technical result is the following.

LEMMA 2.35. *For $\omega \subseteq \tau \subseteq \sigma$, $\sigma \in \mathscr{P}_{\max}$, let $Y = \operatorname{Spec}(\Bbbk[P]/I^{>k}_{\omega\to\tau,\sigma})$, and let $p \in Y$ be a scheme-theoretic point contained in a proper toric stratum of Y, but p not the generic point of a toric stratum. Then* $\operatorname{depth}\mathcal{O}_{Y,p} \geq 2$.

Proof. There is a τ' with $\omega \subseteq \tau' \subsetneq \tau$ such that $V_{\omega\to\tau'} \subseteq Y^{\mathrm{red}}$ is the smallest toric stratum containing p. Then $p \in V_{\tau'\to\tau'}$, the open torus orbit of $V_{\omega\to\tau'}$. So p is a point in the open subscheme $U = \operatorname{Spec}(\Bbbk[P_{\tau',\sigma}]/I^{>k}_{\tau'\to\tau,\sigma})$ of Y. Note that $P_{\tau',\sigma}$ splits noncanonically as

$$P_{\tau',\sigma} = P' \times \mathbb{Z}^r,$$

where $r = \dim\tau'$ and P' is a sharp monoid (i.e., containing no invertible elements other than 0). In particular, there is a monomial ideal $I' \subseteq \Bbbk[P']$ such that

$$U \simeq \operatorname{Spec}(\Bbbk[P']/I' \otimes_{\Bbbk} \Bbbk[\mathbb{Z}^r]).$$

Let $\mathfrak{p} \subseteq \Bbbk[P']/I' \otimes_{\Bbbk} \Bbbk[\mathbb{Z}^r]$ be the prime ideal corresponding to p. We need to find a regular sequence $a_1, a_2 \in \mathfrak{p}$ of length two. Take $a_1 = 1 \otimes f$, where $f \in \mathfrak{p} \cap \Bbbk[\mathbb{Z}^r]$ is a nonzero element, which exists since p is not the generic point of $V_{\tau'\to\tau}$. Take $a_2 = z^m \otimes 1$, where $m \in P'$ is an element in the interior of the face of P' corresponding to τ. It is then easy to see that a_1, a_2, form a regular sequence. Indeed, view $\Bbbk[P']/I'$ and $\Bbbk[\mathbb{Z}^r]/(f)$ as $\Bbbk$-vector spaces. Then tensoring the injective map

$$\Bbbk[\mathbb{Z}^r] \xrightarrow{\cdot f} \Bbbk[\mathbb{Z}^r]$$

with $\Bbbk[P']/I'$ shows that a_1 is not a zero-divisor, and tensoring the injective map

$$\Bbbk[P']/I' \xrightarrow{\cdot z^m} \Bbbk[P']/I'$$

with $\Bbbk[\mathbb{Z}^r]/(f)$ shows that a_2 is not a zero-divisor in $(\Bbbk[P']/I' \otimes_{\Bbbk} \Bbbk[\mathbb{Z}^r])/(a_1)$. □

Remark 2.36. The assumption in Lemma 2.35 that p is not the generic point of a toric stratum is necessary. In particular, unlike V_g the thickening $\operatorname{Spec} R^k_{g,\sigma}$ needs not fulfill Serre's condition S_2. As an example, take ω a point in a two-dimensional B, a polarization φ_ω with Newton polyhedron the unit square, $\tau \supseteq \omega$ a maximal cell and p the zero-dimensional toric stratum $V_{\omega\to\omega} \subseteq \operatorname{Spec} R_\tau$. Then in appropriate coordinates

$$\Bbbk[P] = \Bbbk[x,y,z,w]/(xy - zw), \quad I^{>k}_{\omega\to\tau,\tau} = (x,z)^{k+1},$$

and $y^{-1}z = w^{-1}x$ is a regular function on $\operatorname{Spec}(R_\tau) \setminus \{p\}$ that does not extend. Such an extension would be possible by the depth argument if $\operatorname{depth}\mathcal{O}_{\operatorname{Spec} R_\tau, p} \geq 2$.

LEMMA 2.37. $j_*\mathcal{O}_{V^k(\omega)\setminus Z_\omega} = \mathcal{O}_{V^k(\omega)}$.

Proof. If $U \subseteq V(\omega)$ is an open set, then by definition,

$$\mathcal{O}_{V^k(\omega)}(U) = \varprojlim \mathcal{O}_{\operatorname{Spec} R_\tau}(U),$$
$$j_*\mathcal{O}_{V^k(\omega)\setminus Z_\omega}(U) = \varprojlim \mathcal{O}_{\operatorname{Spec} R_\tau}(U \setminus Z_\omega).$$

Now for any maximal cell $\sigma \supseteq \omega$, Lemma 2.35 shows with the usual depth argument that the restriction map $\mathcal{O}_{\operatorname{Spec} R_\sigma}(U) \to \mathcal{O}_{\operatorname{Spec} R_\sigma}(U \setminus Z_\omega)$ is a bijection. Hence the canonical map $\mathcal{O}_{V^k(\omega)} \to j_*\mathcal{O}_{V^k(\omega)\setminus Z_\omega}$ is an isomorphism since membership of a tuple (f_σ), $f_\sigma \in \mathcal{O}_{\operatorname{Spec} R_\sigma}(U)$, in $\varprojlim \mathcal{O}_{\operatorname{Spec} R_\tau}(U)$ can be checked on $U \setminus Z_\omega$. □

We are now in position to conclude exactness of (2.15) at all points.

PROPOSITION 2.38. *For any $l \le k$, Sequence* (2.15) *is exact.*

Proof. We know exactness of (2.15) on $V(\omega) \setminus Z'_\omega$, where Z'_ω is the intersection of Z_ω with the union of the codimension-two strata of $V(\omega)$. Pushing forward by $j' : V(\omega) \setminus Z'_\omega \to V(\omega)$ yields the exact sequence

$$0 \longrightarrow j'_*\mathcal{O}_{V(\omega)\setminus Z'_\omega} \xrightarrow{\cdot t^l} j'_*\mathcal{O}_{V^l(\omega)\setminus Z'_\omega} \longrightarrow j'_*\mathcal{O}_{V^{l-1}(\omega)\setminus Z'_\omega} \longrightarrow R^1 j'_*\mathcal{O}_{V(\omega)\setminus Z'_\omega}.$$

The term on the far right vanishes since $V(\omega)$ is Cohen-Macaulay and $\operatorname{codim} Z'_\omega \ge 3$. □

We have now established that $V^k(\omega)$ is a flat deformation of $V(\omega)$ over $\operatorname{Spec} \Bbbk[t]/(t^{k+1})$. In particular, $V^k(\omega)$ is an affine scheme of finite type over $\Bbbk$. Moreover, whenever $\omega \subseteq \omega'$, the functor (2.14) induces a map of schemes

$$\Phi^k_{\omega\omega'}(s) : V^k(\omega') \longrightarrow V^k(\omega).$$

These morphisms are compatible with sequences $\omega \subseteq \omega' \subseteq \omega''$:

$$\Phi^k_{\omega\omega''}(s) = \Phi^k_{\omega\omega'}(s) \circ \Phi^k_{\omega'\omega''}(s);$$

hence they define a functor from $\mathscr{P}$ to the category of schemes. Define X_k as the colimit of this functor.

PROPOSITION 2.39. *The maps $\Phi^k_{\omega\omega'}(s)$ are open embeddings. In particular, X_k is a scheme locally of finite type and flat over $\Bbbk[t]/(t^{k+1})$.*

Proof. Recall that in studying $\mathcal{O}_{V^k(\omega)}$ we reduced the inverse limit over $(\omega \to \tau, \mathfrak{u}) \in \underline{\mathrm{Glue}}(\mathscr{S})$ to an inverse limit over cells τ containing ω by choosing one chamber $\mathfrak{u}_\tau$ for each τ. Now using the same choice of $\mathfrak{u}_\tau$ for ω and ω', for $\tau \supseteq \omega'$, we see that $V^k(\omega') \to V^k(\omega)$ is defined by

$$\varprojlim_{\tau\supseteq\omega'} \Phi_{\omega\omega'}(s)^{-1}\mathcal{O}^k(\omega \to \tau, \mathfrak{u}_\tau) \longrightarrow \varprojlim_{\tau\supseteq\omega'} \mathcal{O}^k(\omega' \to \tau, \mathfrak{u}_\tau). \tag{2.16}$$

Note that on the left-hand side we dropped the sheaves $\mathcal{O}^k(\omega \to \tau, \mathfrak{u}_\tau)$ for cells τ containing ω but not containing ω' because they are supported away from the image of $\Phi_{\omega\omega'}(s)$. Now on the ring level, the τ-component of (2.16) is the identity of $R^k_{\omega'\to\tau,\sigma_{\mathfrak{u}_\tau}}$. Thus (2.16) is an isomorphism.

It remains to remark that for vertices $v, v' \in \mathscr{P}$ the open sets $p\big(V(v)\big)$ and $p\big(V(v')\big)$ intersect in $p\big(V(\omega)\big)$ for ω the minimal cell containing v, v'. Thus $\mathcal{O}_{X_k}$, as a sheaf on $|X|$, is isomorphic on $p(V(\omega))$ to $p_*\mathcal{O}_{V^k(\omega)}$. Hence X_k is a scheme with the claimed properties. At this point we use crucially that the cells of $\mathscr{P}$ do not self-intersect; otherwise we would end up with an algebraic space here. $\square$

Remark 2.40. Proposition 2.32 also endows X_k with an abstract log structure, together with a log smooth morphism to $\operatorname{Spec} \Bbbk[t]/(t^{k+1})$ with the log structure generated by $\mathbb{N} \to \Bbbk[t]/(t^{k+1})$, $1 \mapsto t$. While this is not relevant to this paper it is important in order-by-order computations involving the log structure, such as in the study of variations of Hodge structures.

2.7. *The limit $k \to \infty$.* So far we have dealt with a fixed structure $\mathscr{S}$ that was consistent to order k. We now wish to take the limit $k \to \infty$ by considering a sequence $\mathscr{S}_k$ of structures that are compatible in the following way.

Definition 2.41. Two structures $\mathscr{S}$, $\mathscr{S}'$, are *compatible to order k* if the following conditions hold:

(1) If $\mathfrak{p} = (\mathfrak{p}, m, c) \in \mathscr{S}$ is a wall with $c \neq 0$ and $\operatorname{ord}_{\sigma_\mathfrak{p}}(m) \leq k$, then $\mathfrak{p} \in \mathscr{S}'$, and the analogous statement holds for $\mathscr{S}$ and $\mathscr{S}'$ interchanged.
(2) If $x \in \big(\operatorname{Int}(\mathfrak{b}) \cap \operatorname{Int}(\mathfrak{b}')\big) \setminus \Delta$ for slabs $\mathfrak{b} \in \mathscr{S}$, $\mathfrak{b}' \in \mathscr{S}'$, then $f_{\mathfrak{b},x}, f_{\mathfrak{b}',x} \in \Bbbk[P_x]$ agree modulo t^{k+1}.

If $\mathscr{S}$, $\mathscr{S}'$, are compatible to order k and $\mathscr{S}$ is consistent to order k, then $\mathscr{S}'$ is also consistent to order k and the two deformations X_k and X'_k constructed from $\mathscr{S}$ and $\mathscr{S}'$, respectively, are canonically isomorphic.

We are now in a position to reduce the Main Theorem to the construction of a sequence of compatible structures.

PROPOSITION 2.42. *Assume there is a sequence $(\mathscr{S}_k)_{k\geq 0}$ of structures on $(B, \mathscr{P}, \varphi)$ such that for any k,*

(1) *$\mathscr{S}_k$ is consistent to order k;*
(2) *$\mathscr{S}_k$ and $\mathscr{S}_{k+1}$ are compatible to order k.*

Then there exists a formal toric degeneration of CY*-pairs $(\hat{\pi} : \widehat{X} \to \widehat{O}, \widehat{D})$ with central fibre (X, D) and intersection complex $(B, \mathscr{P}, \varphi)$ for the given prepolarization on X.*

Proof. By compatibility of $\mathscr{S}_k$ and $\mathscr{S}_{k+1}$, we have a closed embedding $X_k \to X_{k+1}$ exhibiting X_{k+1} as flat deformation of X_k for any k. Thus $\widehat{X} :=$

$\varinjlim_k X_k$ is a formal scheme, flat over $\Bbbk[\![t]\!]$. Moreover, the charts

$$\psi^k : \left(\Bbbk[P]/(t^{k+1})\right)_{\mathfrak{p}} \longrightarrow \varprojlim (R_\tau)_{\mathfrak{p}_\tau}$$

constructed in Proposition 2.32 are also compatible for various k and compatible with the open embeddings and with other choices of f. Hence for any $p \in X \setminus Z$ we obtain an isomorphism of $\mathcal{O}_{\widehat{X},p} = \varprojlim \mathcal{O}_{X_k,p}$ with a localization of $\varprojlim_k \Bbbk[P]/(t^{k+1})$. Define the deformation $\widehat{D} \subseteq \widehat{X}$ of D by interpreting ψ^k as the chart for a log structure. Explicitly, since D is a toric Cartier divisor, there exists $m \in P$, unique up to an invertible element, such that $(z^m, t) \subseteq \varprojlim (R_\tau)_{\mathfrak{p}_\tau}$ is the ideal of D. Define $\widehat{D}$ as the closure of the divisor defined by z^m. Note that $\widehat{D}$ fulfills (iii) in Definition 1.9 by construction. Since $\operatorname{codim} Z \geq 2$, this also shows regularity of $\widehat{X}$ in codimension one. Furthermore, since $\varprojlim$ commutes with push-forward by $j : X \setminus Z \to X$, Lemma 2.37 implies $\mathcal{O}_{\widehat{X}} = j_* \mathcal{O}_{\widehat{X}\setminus Z}$. This shows that $\widehat{X}$ is S_2, and hence $\widehat{X}$ is normal as required in Definition 1.9(ii). Finally, the central fibre is isomorphic to (X, D) by construction. $\square$

The rest of the paper is devoted to the construction of a sequence of structures $\mathscr{S}_k$ as demanded in Proposition 2.42.

3. **The algorithm**

This section is devoted to the core construction of this paper, the inductive generation of structures $(\mathscr{S}_k)_{k\geq 0}$ as required in Theorem 2.42. We continue with the polarized tropical manifold $(B, \mathscr{P}, \varphi)$, open gluing data for the cone picture s and data $(f_e)_e$ defining a positive log smooth structure on $X = X_0(\check{B}, \check{\mathscr{P}}, s)$, as fixed at the beginning of Section 2. We now also assume local rigidity (Definition 1.26).

THEOREM 3.1. *If all cells of B are bounded, there exists a sequence $(\mathscr{S}_k)_{k\geq 0}$ of structures on $(B, \mathscr{P}, \varphi)$ such that for any k,*

(1) *$\mathscr{S}_k$ is consistent to order k;*
(2) *$\mathscr{S}_k$ and $\mathscr{S}_{k+1}$ are compatible to order k.*

The proof of this theorem occupies the whole section, with the proof of one technical result deferred to Section 4. As remarked earlier, most of the arguments do not require bounded cells, so we shall work with the general case, making it clear where we require the boundedness hypothesis.

3.1. *The initial structure.* Take $\mathscr{S}_0$ to consist only of slabs $\mathfrak{b}$, where $\mathfrak{b} = \rho_{\mathfrak{b}}$ is a codimension-one cell of $\mathscr{P}$ and $f_{\mathfrak{b},x} = \Pi^{-1}(f_e)$, where $e : v = v[x] \to \rho_{\mathfrak{b}}$ is the vertex in the connected component of $\rho_{\mathfrak{b}} \setminus \Delta$ containing x and $\Pi : \Bbbk[P_x^{\mathrm{gp}}] \to \Bbbk[P_v^{\mathrm{gp}}]$ is given by parallel transport from x to v along a path inside $\rho_{\mathfrak{b}} \setminus \Delta$. Then $\mathrm{Chambers}(\mathscr{S}_0) = \mathscr{P}^{[n]}$, and we can take $\mathscr{P}_{\mathscr{S}} = \mathscr{P}^{[\leq n-1]}$.

PROPOSITION 3.2. *$\mathscr{S}_0$ is consistent to order* 0.

Proof. The only joints $\mathfrak{j}$ of $\mathscr{S}_0$ are the codimension-two cells of $\mathscr{P}$, so take $\mathfrak{j} = \tau \in \mathscr{P}^{[n-2]}$, $\tau \not\subseteq \partial B$. Let $\sigma_1, \ldots, \sigma_l$ be the maximal cells of $\mathscr{P}$ containing τ, ordered cyclically (that is, $\sigma_i \cap \sigma_{i+1} = \rho_i \in \mathscr{P}^{[n-1]}$ with $\sigma_{l+1} := \sigma_1$), and take $\check{d}_{\rho_i}$ (as in (1.1)) to be negative on σ_i. To check consistency for $g : \omega \to \tau'$ with $\tau' \subseteq \sigma_i$, for all i it suffices to consider $\omega = \tau' = \tau$, by Remark 2.29(5). Then by Construction 2.24(II.2), letting $x \in \mathrm{Int}(\tau) \setminus \Delta$ and $e : v = v[x] \to \tau$,

$$\theta_i := F_s^0(\mathrm{id}_\tau, \sigma_i) : m \longmapsto \big(D(s_e, \rho_i, v)^{-1} s_e^{-1}(f_{\mathfrak{b}_{\rho_i},x})\big)^{-\langle m, \check{d}_{\rho_i}\rangle}.$$

Thus in $R^0_{\mathrm{id}_\tau,\sigma_1}$, letting $e_{\rho_i} : v \to \rho_i$,

$$\begin{aligned}(\theta_l \circ \cdots \circ \theta_1)(m) &= \Big(\prod_{i=1}^{l} D(s_e, \rho_i, v)^{\langle m, \check{d}_{\rho_i}\rangle}\Big) s_e^{-1}\Big(\prod_{i=1}^{l} \big(f_{e_{\rho_i}}|_{V_{\mathrm{id}_\tau}}\big)^{-\langle m, \check{d}_{\rho_i}\rangle}\Big)\\ &= \Big(\prod_{i=1}^{l} \frac{s_{e,\sigma_i}(m)}{s_{e,\sigma_{i+1}}(m)}\Big) s_e^{-1}\Big(\prod_{i=1}^{l} \big(f_{e_{\rho_i}}|_{V_{\mathrm{id}_\tau}}\big)^{-\langle m, \check{d}_{\rho_i}\rangle}\Big) \;=\; 1,\end{aligned}$$

the last equality by (1.8). This is the desired consistency. □

Note that according to Remark 2.40 the structure $\mathscr{S}_0$ defines an abstract log structure on X. Checking consistency to order 0 means verifying the multiplicative condition (1.8) for the associated section of $\mathcal{LS}^+_{\mathrm{pre},X}$. By construction this is indeed just the log structure we started with.

3.2. *Scattering diagrams.* Given $\mathscr{S}_{k-1}$, the construction of $\mathscr{S}_k$ proceeds in three steps. The first of these introduces new walls, of order k, by a procedure that is strictly local around a joint and is the subject of this subsection. The second step performs various semi-global adjustments involving several joints. The remaining trouble terms are removed in the last step by a normalization procedure applied to each slab.

Recall from Section 2.5 the space $\mathcal{Q}^v_{\mathfrak{j},\mathbb{R}} = \Lambda_{v,\mathbb{R}}/\Lambda_{\mathfrak{j},\mathbb{R}}$ for a joint $\mathfrak{j}$ and the notation $\overline{\overline{m}}$, $\overline{\overline{\tau}}$ etc. We think of $\mathcal{Q}^v_{\mathfrak{j},\mathbb{R}}$ as being divided by those half-lines $\mathfrak{c}$ emanating from the origin that are contained in $\overline{\overline{\rho}}$ for some $\rho = \rho_{\mathfrak{c}} \in \mathscr{P}^{[n-1]}$, $\rho \supseteq \mathfrak{j}$. We refer to these half-lines as *cuts*. Observe that if $\operatorname{codim} \sigma_{\mathfrak{j}} = 1$, there are two cuts separating $\mathcal{Q}^v_{\mathfrak{j},\mathbb{R}}$ into two half-planes, while in the codimension-two case the cuts subdivide $\mathcal{Q}^v_{\mathfrak{j},\mathbb{R}}$ into a number of strictly convex cones. In the codimension-zero case there are no cuts at all.

Once an orientation on $\mathcal{Q}^v_{\mathfrak{j},\mathbb{R}}$ is chosen and $\mathfrak{c} = \mathbb{R}_{\geq 0} \cdot \overline{\overline{m}} \subseteq \mathcal{Q}^v_{\mathfrak{j},\mathbb{R}}$, $m \in \Lambda_\sigma \setminus \{0\}$, is a (rational) half-line emanating from the origin, the unique generator $n_{\mathfrak{c}}$ of $m^\perp \cap \Lambda_{\mathfrak{j}}^\perp \simeq \mathbb{Z}$ with the property that $\langle m', n_{\mathfrak{c}}\rangle > 0$ for m, m', mapping to an oriented basis of $\mathcal{Q}^v_{\mathfrak{j},\mathbb{R}}$, is called the *normal vector to* $\mathfrak{c}$.

Consistency at $\mathfrak{j}$ depends only on the local properties around $\mathfrak{j}$ of slabs and walls containing $\mathfrak{j}$ and hence can be studied on $\mathcal{Q}^v_{\mathfrak{j},\mathbb{R}}$. The following definition is an abstraction of the situation.

Definition 3.3. A *ray* in $\mathcal{Q}^v_{\mathfrak{j},\mathbb{R}}$ is a triple $(\mathfrak{r}, m_\mathfrak{r}, c_\mathfrak{r})$, where $\mathfrak{r}$ is a one-dimensional, rational cone $\mathbb{R}_{\geq 0}\cdot\overline{\overline{q}}$, $q \in \Lambda_v \setminus \Lambda_\mathfrak{j}$; $m_\mathfrak{r}$ is a nonzero exponent on a maximal cell σ with $\pm\overline{\overline{m}}_\mathfrak{r} \in \mathfrak{r}\cap\overline{\overline{\sigma}}$ and such that $m \in P_x$ for all $x \in \mathfrak{j}\setminus\Delta$; $c_\mathfrak{r}$ is a constant in $\Bbbk$. By abuse of notation we often just write $\mathfrak{r}$ to refer to $(\mathfrak{r}, m_\mathfrak{r}, c_\mathfrak{r})$. A ray is called *incoming*, *outgoing* and *undirectional* in the respective cases $\overline{\overline{m}}_\mathfrak{r} \in \mathfrak{r}\setminus\{0\}$, $-\overline{\overline{m}}_\mathfrak{r} \in \mathfrak{r}\setminus\{0\}$ and $\overline{\overline{m}}_\mathfrak{r} = 0$. The *order* of a ray $\mathfrak{r}$ is defined as $\operatorname{ord}_\mathfrak{j}(m_\mathfrak{r})$.

A *scattering diagram for* $\mathfrak{j}$ at a vertex $v \in \sigma_\mathfrak{j}$ consists of

(1) a choice of $\omega \in \mathscr{P}$ with $\mathfrak{j}\cap\operatorname{Int}\omega \neq \emptyset$, $\omega \subseteq \sigma_\mathfrak{j}$ and $v \in \omega$;
(2) a finite set of rays $\mathfrak{r} = (\mathfrak{r}, m_\mathfrak{r}, c_\mathfrak{r})$;
(3) for each cut $\mathfrak{c} \subseteq \mathcal{Q}^v_{\mathfrak{j},\mathbb{R}}$ and any $x \in (\mathfrak{j}\cap\operatorname{Int}\omega)\setminus\Delta$ a function $f_{\mathfrak{c},x} \in \Bbbk[P_x]$ with the same properties as the functions $f_{\mathfrak{b},x}$ in Definition 2.17;
(4) an orientation of $\mathcal{Q}^v_{\mathfrak{j},\mathbb{R}}$.

The notation is $\mathfrak{D} = \{\mathfrak{r}, f_\mathfrak{c}\}$ with ω and the orientation of $\mathcal{Q}^v_{\mathfrak{j},\mathbb{R}}$ understood. For rays $\mathfrak{r}$ and cuts $\mathfrak{c}$ of a scattering diagram we write $n_\mathfrak{r}$ and $n_\mathfrak{c}$, respectively, for the now well-defined normal vectors.

Two scattering diagrams $\mathfrak{D} = \{\mathfrak{r}, f_\mathfrak{c}\}$, $\mathfrak{D}' = \{\mathfrak{r}', f'_\mathfrak{c}\}$ for $\mathfrak{j}$ at v defined with the same ω are *equivalent modulo a monomial ideal* $J \subseteq \Bbbk[P_{\omega,\sigma}]$, where $\sigma \supseteq \mathfrak{j}$, if (1) for any $m \in P_{\omega,\sigma}$ with $z^m \notin J$, and $\varepsilon \in \{-1, 1\}$, it holds

$$\prod_{\{\mathfrak{r}\in\mathfrak{D}\,|\,m_\mathfrak{r}=m,\,\varepsilon\cdot\overline{\overline{m}}\in\mathfrak{r}\}} (1 + c_\mathfrak{r} z^{m_\mathfrak{r}}) = \prod_{\{\mathfrak{r}'\in\mathfrak{D}'\,|\,m_{\mathfrak{r}'}=m,\,\varepsilon\cdot\overline{\overline{m}}\in\mathfrak{r}'\}} (1 + c_{\mathfrak{r}'} z^{m_{\mathfrak{r}'}}) \mod J,$$

where we use parallel transport through v to interpret m as an exponent on other maximal cells containing $\mathfrak{j}$; (2) for any cut $\mathfrak{c}$ and any $x \in (\mathfrak{j}\cap\operatorname{Int}\omega)\setminus\Delta$ the functions $f_{\mathfrak{c},x}, f'_{\mathfrak{c},x} \in \Bbbk[P_x]$ agree modulo terms of $\operatorname{ord}_\mathfrak{j}$ at least $k+1$.

Given a scattering diagram $\mathfrak{D} = \{\mathfrak{r}_i, f_\mathfrak{c}\}$ and $g: \omega \to \tau$ with $\mathfrak{j}\cap\operatorname{Int}\omega \neq \emptyset$, $\tau \subseteq \sigma_\mathfrak{j}$, and $\sigma \in \mathscr{P}_{\max}$ containing $\mathfrak{j}$, we obtain a log isomorphism of $R^k_{g,\sigma}$ just as from a loop around a joint. Specifically, let $\sigma_1, \ldots, \sigma_r = \sigma_0$ be a cyclic ordering of the maximal cells containing $\mathfrak{j}$ compatible with the orientation of $\mathcal{Q}^v_{\mathfrak{j},\mathbb{R}}$, and let $\rho_j = \sigma_{j-1}\cap\sigma_j$. This induces a cyclic ordering of the cuts $\mathfrak{c}_j \subseteq \overline{\overline{\sigma}}_{j-1}\cap\overline{\overline{\sigma}}_j$. In the codimension-two case this inclusion defines $\mathfrak{c}_j$ uniquely, while there are two choices in the case of codimension one. Assume that the rays are labelled cyclically as well and in such a way that $\mathfrak{r}_i \subseteq \overline{\overline{\sigma}}_j$ if and only if $i_{j-1} < i \leq i_j$. Then for $\mathfrak{r}_i \subseteq \overline{\overline{\sigma}}_j$ and for any k we have the log automorphism

$$\theta_i : \Lambda_{\sigma_j} \longrightarrow (R^k_{g,\sigma_j})^\times, \quad m \longmapsto \left(s_{\omega\to\sigma_j}(1 + c_{\mathfrak{r}_i} z^{m_{\mathfrak{r}_i}})\right)^{-\langle\overline{m}, n_{\mathfrak{r}_i}\rangle}$$

of R^k_{g,σ_j}, as in Construction 2.24(II.1), where we think of passing through $\mathfrak{r}_i$ in the sense of the cyclic ordering of the σ_i. Note that θ_i can also be written as

$$\theta_i = \exp\Big(-\log\big(s_{\omega\to\sigma_j}(1+c_{\mathfrak{r}_i}z^{m_{\mathfrak{r}_i}})\big)\partial_{n_{\mathfrak{r}_i}}\Big);$$

hence θ_i is an element of the group $H_{\mathfrak{j}}^{I^{>k}_{g,\sigma}}$ acting on R^k_{g,σ_j}. Similarly, following Construction 2.24(II.2) with $x \in \mathfrak{j}\setminus\Delta$ such that $v[x]=v$, the functions $f_{\mathfrak{c}_j,x}$ define the log isomorphism

$$\theta_{\mathfrak{c}_j} : P_{\omega,\sigma_{j-1}} \longrightarrow (R^k_{g,\sigma_j})^\times, \quad m \longmapsto \Big(D(s_{v\to\omega},\rho_j,v)^{-1}s^{-1}_{v\to\omega}(f_{\mathfrak{c}_j,x})\Big)^{-\langle\overline{m},n_{\mathfrak{c}_i}\rangle}$$

from $R^k_{g,\sigma_{j-1}}$ to R^k_{g,σ_j}, with monoid homomorphism defined by parallel transport through x. Note that if we chose an x with $v[x]\neq v$, we would still obtain an equivalent log isomorphism as verified in (2.12); see also Remark 2.25(2). Define

$$\begin{aligned}(3.1)\quad \theta^k_{\mathfrak{D},g} := {}&\big(\theta_{i_r}\circ\cdots\circ\theta_{i_{r-1}+1}\big)\\ &\circ\theta_{\mathfrak{c}_r}\circ\big(\theta_{i_{r-1}}\circ\cdots\circ\theta_{i_{r-2}+1}\big)\circ\theta_{\mathfrak{c}_{r-1}}\circ\cdots\circ\big(\theta_{i_1}\circ\cdots\circ\theta_1\big)\circ\theta_{\mathfrak{c}_1}.\end{aligned}$$

After distinguishing σ_1 this is a well-defined representative of a log automorphism of R^k_{g,σ_1}. In fact, any two log automorphisms associated to rays or slabs in the same direction commute, so this composition is independent of the chosen indexing.

By definition, $\theta^k_{\mathfrak{D},g}$ depends only on the equivalence class of $\mathfrak{D}$ to order k. Note also that reversing orientations leads to $(\theta^k_{\mathfrak{D},g})^{-1}$, while a different choice of σ_1 leads to conjugation of $\theta^k_{\mathfrak{D},g}$ by a log isomorphism $R^k_{g,\sigma_j}\to R^k_{g,\sigma_1}$ for some j. We are often only interested in properties invariant under these changes and hence suppress them in the notation for $\theta^k_{\mathfrak{D},g}$.

Construction 3.4. A structure $\mathscr{S}$ induces a scattering diagram $\mathfrak{D}_{\mathfrak{j}} = \mathfrak{D}_{\mathfrak{j}}(\mathscr{S},\omega,v)$ for each joint $\mathfrak{j}$, $\omega\in\mathscr{P}$ with $\mathfrak{j}\cap\operatorname{Int}\omega\neq\emptyset$, $\omega\subseteq\sigma_{\mathfrak{j}}$ and vertex $v\in\omega$ as follows. The slabs containing $\mathfrak{j}$ readily define the functions $f_{\mathfrak{c},x}$. For a wall $\mathfrak{p}$ containing $\mathfrak{j}$ there are the following possibilities:

(1) $\mathfrak{j}\subseteq\partial\mathfrak{p}$. Then add the ray $(\overline{\overline{\mathfrak{p}}}, m_{\mathfrak{p}}, c_{\mathfrak{p}})$ to $\mathfrak{D}_{\mathfrak{j}}$. This ray is incoming, outgoing or unoriented if $\mathfrak{j}\subseteq\operatorname{Top}(\mathfrak{p})$, $\mathfrak{j}\subseteq\operatorname{Base}(\mathfrak{p})$ or $\mathfrak{j}\subseteq\operatorname{Sides}(\mathfrak{p})$, respectively.
(2) $\mathfrak{j}\cap\operatorname{Int}\mathfrak{p}\neq\emptyset$. Then $\overline{\overline{\mathfrak{p}}}$ is a line through the origin, defining two one-dimensional half lines $\mathfrak{r}$, $\mathfrak{r}'=-\mathfrak{r}\subseteq\mathcal{Q}_{\mathfrak{j},\mathbb{R}}$. Then add the pair of rays $(\mathfrak{r},m_{\mathfrak{p}},c_{\mathfrak{p}})$, $(\mathfrak{r}',m_{\mathfrak{p}},c_{\mathfrak{p}})$. These are either both undirectional or a pair of an incoming and an outgoing ray.

Note that consistency of $\mathscr{S}$ around $\mathfrak{j}$ to order k can be expressed by $\theta^k_{\mathfrak{D}_{\mathfrak{j}},\mathrm{id}_{\sigma_{\mathfrak{j}}}}=1$.

Remark 3.5. For a different choice of vertex $v'\in\omega$ there is a piecewise linear identification of $\mathcal{Q}^v_{\mathfrak{j},\mathbb{R}}$ with $\mathcal{Q}^{v'}_{\mathfrak{j},\mathbb{R}}$ defined on $\overline{\overline{\sigma}}$ by parallel transport from v

to v' inside $\sigma \in \mathscr{P}_{\max}$. This identifies the scattering diagrams $\mathfrak{D}_{\mathfrak{j}} = \mathfrak{D}_{\mathfrak{j}}(\mathscr{S}, \omega, v)$ and $\mathfrak{D}'_{\mathfrak{j}} = \mathfrak{D}_{\mathfrak{j}}(\mathscr{S}, \omega, v')$. Note that the respective computations in (3.1), for the same maximal cell $\sigma_1 \supseteq \mathfrak{j}$, then only differ by changing the underlying monoid homomorphisms by parallel transport from v to v' in σ_1. In particular, we have the equality of representatives of log automorphisms of $R^k_{\omega\to\sigma_{\mathfrak{j}},\sigma_1}$

$$\theta^k_{\mathfrak{D}_{\mathfrak{j}}} = \theta^k_{\mathfrak{D}'_{\mathfrak{j}}}.$$

Similarly, any of the considerations with scattering diagrams below are independent of the choice of v.

Assuming $\theta^{k-1}_{\mathfrak{D},\mathrm{id}_{\sigma_{\mathfrak{j}}}} = 1$, we now use the structure of the group ${}^{\perp}H_{\mathfrak{j}}^{I^{>k}_{g,\sigma}}$ to try to achieve $\theta^k_{\mathfrak{D},\mathrm{id}_{\sigma_{\mathfrak{j}}}} = 1$ by adding some rays and, in the codimension-two case, changing the functions $f_{\mathfrak{c}}$ to order k. The key idea is captured in the following lemma of Kontsevich and Soibelman ([KS06, Th. 6]), adapted to our setting. For the rest of this subsection fix the joint $\mathfrak{j}$, $\sigma \in \mathscr{P}_{\max}$ containing $\mathfrak{j}$, and $g : \omega \to \sigma_{\mathfrak{j}}$ with $\mathfrak{j} \cap \operatorname{Int}\omega \neq \emptyset$ and $v \in \omega$. Write $I_k = I^{>k}_{g,\sigma} \subseteq \Bbbk[P_{\omega,\sigma}]$.

Definition 3.6. For $K \subseteq \mathcal{Q}^v_{\mathfrak{j},\mathbb{R}}$ a strictly convex cone ($K \cap -K = \{0\}$), not necessarily closed and $I \subseteq \Bbbk[P_{\omega,\sigma}]$ a monomial ideal with radical I_0, we define the following Lie subalgebras of $\mathfrak{g}^I_{\mathfrak{j}}$:

$$\mathfrak{g}^I_{\mathfrak{j},K} := \bigoplus_{\substack{z^m \in I_0\setminus I\\ -\overline{\overline{m}}\in K\setminus\{0\}}} z^m(\Bbbk \otimes \Lambda^{\perp}_{\mathfrak{j}}),$$

$$\mathfrak{h}^I_{\mathfrak{j},K} := \bigoplus_{\substack{z^m \in I_0\setminus I\\ -\overline{\overline{m}}\in K\setminus\{0\}}} z^m\big(\Bbbk \otimes (\overline{m}^{\perp} \cap \Lambda^{\perp}_{\mathfrak{j}})\big) = \mathfrak{g}^I_{\mathfrak{j},K} \cap \mathfrak{h}^I_{\mathfrak{j}}.$$

The corresponding Lie groups are denoted as $G^I_{\mathfrak{j},K}$ and $H^I_{\mathfrak{j},K}$.

Note that $\mathfrak{h}^I_{\mathfrak{j},K} \subseteq {}^{\perp}\mathfrak{h}^I_{\mathfrak{j}}$.

Lemma 3.7. *Let $\mathfrak{j}$ be a joint with $\sigma_{\mathfrak{j}} \in \mathscr{P}_{\max}$ and $K \subseteq \mathcal{Q}^v_{\mathfrak{j},\mathbb{R}}$ a strictly convex cone. Then for any $\theta \in H^{I_k}_{\mathfrak{j},K}$, there exists a scattering diagram $\mathfrak{D}$ for $\mathfrak{j}$ consisting entirely of outgoing rays $\mathfrak{r}$ with $\operatorname{Int}\mathfrak{r} \subseteq K$ such that $\theta = \theta^k_{\mathfrak{D},g}$. Moreover, $\mathfrak{D}$ is unique up to equivalence to order k.*

Proof. For $k = 0$ we may take $\mathfrak{D} = \emptyset$. By induction on k we may thus assume there exists a unique scattering diagram $\mathfrak{D}'$ with $\theta^{k-1}_{\mathfrak{D}',g} = \theta \mod I_{k-1}$. Then by the definition of $\mathfrak{h}^{I_k}_{\mathfrak{j},K}$ we can write uniquely

$$\theta^k_{\mathfrak{D}',g} \circ \theta^{-1} = \exp\Big(\sum_i c_i z^{m_i}\partial_{n_i}\Big), \tag{3.2}$$

where $z^{m_i} \in I_{k-1} \setminus I_k$, $-\overline{\overline{m}}_i \in K \setminus \{0\}$, $n_i \in \overline{m_i}^{\perp} \cap \Lambda^{\perp}_{\mathfrak{j}}$ and $c_i \in \Bbbk \setminus \{0\}$. By changing c_i we may assume $n_i = n_{-\mathbb{R}_{\geq 0}\overline{\overline{m}}_i}$. Define $\mathfrak{D}$ by adding the outgoing rays $(-\mathbb{R}_{\geq 0}\overline{\overline{m}}_i, m_i, c_i)$ to $\mathfrak{D}'$. Noting that $[z^{m_i}\partial_{n_i}, \mathfrak{h}_{\mathfrak{j}}^{I_k}] = 0$, we see that

$$\theta^k_{\mathfrak{D},g} \circ \theta^{-1} = \theta^k_{\mathfrak{D}',g} \circ \theta^{-1} \circ \prod_i \exp\big(-c_i z^{m_i}\partial_{n_i}\big) = \mathrm{id}\,.$$

Uniqueness follows from the uniqueness of the expansion in (3.2). □

As we will see, the same idea as in the proof of Lemma 3.7 can be used to add rays to a codimension-zero scattering diagram $\mathfrak{D}'$ with $\theta^{k-1}_{\mathfrak{D}',g} = 1$ to construct a scattering diagram $\mathfrak{D}$ with $\theta^k_{\mathfrak{D},g} \in \ker\big({}^{\parallel}H^k_{\mathfrak{j}} \to {}^{\parallel}H^{k-1}_{\mathfrak{j}}\big)$, uniquely up to equivalence to order k. Note that the remaining exponents m with $\overline{m} \in \Lambda_{\mathfrak{j}}$ have to be dealt with by other arguments since outgoing rays always lead to elements in the subgroup ${}^{\perp}H^k_{\mathfrak{j}} \subseteq H^k_{\mathfrak{j}}$.

In higher codimension, under the presence of slabs, this is much more subtle because we have to convert between computations in the various groups R^k_{g,σ_i} using the log isomorphisms associated to slabs. In particular, it is not clear that the commutation does not introduce poles in directions different from the cuts $\mathfrak{c}$ corresponding to slabs. We will also have weaker uniqueness properties because one can always replace a ray in the direction of a cut $\mathfrak{c}$ with a change of $f_{\mathfrak{c}}$. The detailed study of this situation is the subject of the technical last section. Here we content ourselves with a statement of the results needed for the construction of $\mathscr{S}_k$.

For enhanced readability we introduce the following notation. Recall we have fixed $\mathfrak{j}$, $\sigma \in \mathscr{P}_{\max}$, with $\mathfrak{j} \subseteq \sigma$, and $g : \omega \to \sigma_{\mathfrak{j}}$ with $\mathfrak{j} \cap \mathrm{Int}\,\omega \neq \emptyset$. We work with various log automorphisms of $R^k_{g,\sigma} = \big(\Bbbk[P_{\omega,\sigma}]/I_k\big)_{f_{g,\sigma_{\mathfrak{j}}}}$, $I_k = I^{>k}_{g,\sigma}$.

Convention 3.8. 1) For a set $V_\mu \subseteq \Lambda_\sigma$, a subspace $W_\mu \subseteq \Lambda^*_\sigma$ and elements $f_\mu \in (R^k_{g,\sigma})^\times$, we write

$$O^k\Big(\sum_\mu \frac{V_\mu}{f_\mu} \otimes W_\mu\Big)$$

for the set of log automorphisms of $R^k_{g,\sigma}$ of the form $\theta = \exp\Big(\sum_{\mu,i} c_{\mu,i}\frac{z^{m_{\mu,i}}}{f_\mu}\partial_{n_{\mu,i}}\Big)$ with $\overline{m_{\mu,i}} \in V_\mu$, $n_{\mu,i} \in W_\mu$, $\mathrm{ord}_{\sigma_{\mathfrak{j}}}(m_{\mu,i}) \geq k$.

2) Let $v \in \omega$ be a vertex, $e : v \to \omega$. Then for $\rho \in \mathscr{P}^{[n-1]}$ containing $\mathfrak{j}$, define

$$f_\rho = f_{\rho,v} := D(s_e, \rho, v)^{-1} f_{\rho,e,\sigma} \in \Bbbk[P_{\omega,\sigma}]$$

with $f_{\rho,e,\sigma}$ defined in (2.2). Note that according to (1.11) a different choice $e' : v' \to \omega$ leads to

$$f_{\rho,v'} = z^{m^\rho_{v'v}} f_{\rho,v}. \tag{3.3}$$

In particular, f_ρ is well defined up to multiplication by z^m with $\overline{m} \in \Lambda_\omega$, $\mathrm{ord}_\omega(m) = 0$.

For example, with this notation

$$\ker\left({}^{\|}H_{\mathfrak{j}}^{I_k} \to {}^{\|}H_{\mathfrak{j}}^{I_{k-1}}\right) = O^k\big((\Lambda_{\mathfrak{j}} \setminus \{0\}) \otimes \Lambda_{\mathfrak{j}}^{\perp}\big).$$

We are now ready to state the main result of Section 4.

Proposition 3.9. *Let $\mathfrak{D}'$ be a scattering diagram for $\mathfrak{j} \in \mathrm{Joints}(\mathscr{S}_{k-1})$ with $\theta^{k-1}_{\mathfrak{D}',g} = 1$, $g: \omega \to \sigma_{\mathfrak{j}}$. Then there exists a scattering diagram $\mathfrak{D}$, equivalent to order $k-1$ to $\mathfrak{D}'$ and with the sets of rays differing only by outgoing rays $\mathfrak{r}$ with $\mathfrak{r} \not\subseteq \overline{\overline{\rho}}$ for any $\rho \in \mathscr{P}^{[n-1]}$ containing $\mathfrak{j}$, such that*

$$\theta^k_{\mathfrak{D},g} \in \begin{cases} O^k\big((\Lambda_{\mathfrak{j}} \setminus \{0\}) \otimes \Lambda_{\mathfrak{j}}^{\perp}\big), & \operatorname{codim}\sigma_{\mathfrak{j}} = 0,\\ O^k\Big(\Lambda_{\mathfrak{j}} \otimes \Lambda_{\mathfrak{j}}^{\perp} + \dfrac{\Lambda_\rho}{f_\rho} \otimes \Lambda_\rho^{\perp}\Big), & \operatorname{codim}\sigma_{\mathfrak{j}} = 1\ (\rho = \sigma_{\mathfrak{j}}),\\ O^k\Big(\Lambda_{\mathfrak{j}} \otimes \Lambda_{\mathfrak{j}}^{\perp} + \sum_{\rho \supseteq \mathfrak{j}} \dfrac{\Lambda_{\mathfrak{j}}}{f_\rho} \otimes \Lambda_\rho^{\perp}\Big), & \operatorname{codim}\sigma_{\mathfrak{j}} = 2. \end{cases} \tag{3.4}$$

If $\operatorname{codim}\sigma_{\mathfrak{j}} < 2$, *the functions $f_{\mathfrak{c},x}$ of $\mathfrak{D}$ and $\mathfrak{D}'$ coincide, while if* $\operatorname{codim}\sigma_{\mathfrak{j}} = 2$ *they may be changed by adding multiples of z^m with $-\overline{\overline{m}} \in \mathfrak{c} \setminus \{0\}$.*

Moreover, up to equivalence, $\mathfrak{D}$ is the unique scattering diagram with these properties.

Proof for $\operatorname{codim}\sigma_{\mathfrak{j}} = 0$. Arguing similarly to Lemma 3.7 we have the unique decomposition

$$\theta^k_{\mathfrak{D},g} = \exp\Big(\sum_i c_i z^{m_i} \partial_{n_i}\Big), \tag{3.5}$$

but this time only $z^{m_i} \in I_{k-1} \setminus I_k$, $\overline{m_i} \neq 0$, $n_i \in \overline{m_i}^{\perp} \cap \Lambda_{\mathfrak{j}}^{\perp}$. Define $\mathfrak{D}$ by adding to $\mathfrak{D}'$ for each i with $\overline{\overline{m}}_i \neq 0$ the ray $(-\mathbb{R}_{\geq 0}\overline{\overline{m}}_i, m_i, c_i)$, assuming without loss of generality that $n_i = n_{-\mathbb{R}_{\geq 0} m_i}$. Since the log automorphisms of these rays are in the center of $H_{\mathfrak{j}}^{I_k}$ it holds that

$$\theta^k_{\mathfrak{D},g} = \theta^k_{\mathfrak{D}',g} \circ \prod_{\{i \,|\, \overline{\overline{m_i}} \neq 0\}} \exp\big(-c_i z^{m_i} \partial_{n_i}\big) = \exp\Big(\sum_{\{i \,|\, \overline{\overline{m_i}} = 0\}} c_i z^{m_i} \partial_{n_i}\Big).$$

This is of the desired form. Finally, uniqueness follows from the uniqueness statement for (3.5).

The proof for $\operatorname{codim}\sigma_{\mathfrak{j}} > 0$ occupies Section 4. □

It is also important for this section to record the effect on $\theta^k_{\mathfrak{D},g}$ of certain simple changes to $\mathfrak{D}$.

Proposition 3.10. *Let $\mathfrak{D}$ be a scattering diagram for $\mathfrak{j}$ and assume $\theta^k_{\mathfrak{D},g}$ fulfills* (3.4) *of Proposition* 3.9.

(1) *If $\widehat{\mathfrak{D}}$ is obtained from $\mathfrak{D}$ by adding the term cz^m to some $f_{\mathfrak{c}}$ with* $\operatorname{ord}_{\mathfrak{j}} m = k$ *and* $\overline{m} \in \Lambda_{\sigma_{\mathfrak{j}}}$, *then*

$$\theta^k_{\hat{\mathfrak{D}},g} = \theta^k_{\mathfrak{D},g} \circ \exp\Big(-c' \frac{z^m}{f_{\rho_{\mathfrak{c}}}} \partial_{n_{\mathfrak{c}}} \Big)$$

with $c' = D(s_e, \rho_{\mathfrak{c}}, v)^{-1} \cdot s_e^{-1}(\overline{m}) \cdot c$, $e : v \to \omega$.

(2) *If $\widehat{\mathfrak{D}}$ is obtained from $\mathfrak{D}$ by adding an undirectional ray $(\mathfrak{r}, m, c)$ with* $\operatorname{ord}_{\mathfrak{j}} m = k$, *then*

$$\theta^k_{\hat{\mathfrak{D}},g} = \begin{cases} \theta^k_{\mathfrak{D},g} \circ \exp\Big(-c' z^m \partial_{n_{\mathfrak{r}}} \Big), & \operatorname{codim} \sigma_{\mathfrak{j}} \neq 1, \\ \theta^k_{\mathfrak{D},g} \circ \exp\Big(-c' z^m \partial_{n_{\mathfrak{r}}} \Big) \circ O^k\Big(\dfrac{\Lambda_\rho}{f_\rho} \otimes \Lambda_\rho^\perp \Big), & \operatorname{codim} \sigma_{\mathfrak{j}} = 1 \ (\rho = \sigma_{\mathfrak{j}}) \end{cases}$$

with $c' = s_h(m) \cdot c$, $h : \omega \to \sigma_j$.

Proof. First we observe that the change in (1) has the same effect as composing the log isomorphism associated to $\mathfrak{c}$ by $\exp(-c' z^m / f_{\rho_{\mathfrak{c}}} \partial_{n_{\mathfrak{c}}})$. Note that since $\overline{m} \in \Lambda_{\sigma_{\mathfrak{j}}}$, it holds that $\operatorname{ord}_{\sigma_{\mathfrak{j}}}(m + m') > k$ whenever $\operatorname{ord}_{\sigma_{\mathfrak{j}}}(m') > 0$. Thus by Lemma 2.15 this log isomorphism commutes with any of the other log isomorphisms, hence the result. Adding an undirectional ray is similar, but in the codimension-one case $f_{\mathfrak{c}}$ involves monomials $z^{m'}$ with $\overline{m'} \in \Lambda_{\rho_{\mathfrak{c}}} \setminus \Lambda_{\mathfrak{j}}$ since $\rho_{\mathfrak{c}} = \sigma_{\mathfrak{j}}$. But again, by Lemma 2.15, we obtain

$$\operatorname{Ad}_{\theta_{\mathfrak{c}}}(-cz^m \partial_{n_{\mathfrak{r}}}) = -cz^m \Big(\partial_{n_{\mathfrak{r}}} + f_{\rho_{\mathfrak{c}}}^{-1} (\partial_{n_{\mathfrak{r}}} f_{\rho_{\mathfrak{c}}}) \partial_{n_{\mathfrak{c}}} \Big)$$

since $\langle \overline{m}, n_{\mathfrak{c}} \rangle = 0$. The exponential of this expression is of the form $\exp(-cz^m \partial_{n_{\mathfrak{r}}}) \circ O^k\big((\Lambda_{\rho_{\mathfrak{c}}} / f_{\rho_{\mathfrak{c}}}) \otimes \Lambda_{\rho_{\mathfrak{c}}}^\perp \big)$. □

3.3. *Step* I: *Scattering at joints.* We now begin the algorithm providing the induction step, the construction of $\mathscr{S}_k$ from $\mathscr{S}_{k-1}$. Since the various parts of it are scattered throughout four subsections, text providing instructions for this process is shaded.

I.1. *Refinement of slabs.* The notion of compatibility of structures (Definition 2.41) allows arbitrary refinements of slabs. To be able to use local methods we now impose the following conditions on $\mathscr{S}_{k-1}$, which can be achieved by subdivision of slabs:

$$\text{If } \mathfrak{b} \in \mathscr{S}_{k-1} \text{ is a slab and } \mathfrak{b} \cap \partial\rho_{\mathfrak{b}} \neq \emptyset, \text{ there exists } \tau \subseteq \partial\rho_{\mathfrak{b}} \text{ with} \\ \mathfrak{b} \cap \operatorname{Int} \tau \neq \emptyset \quad \text{and} \quad \tau' \in \mathscr{P}^{[\leq n-2]}, \operatorname{Int} \tau' \cap \mathfrak{b} \neq \emptyset \implies \tau' \subseteq \tau. \tag{3.6}$$

Of course, this also means refining the polyhedral decomposition $\mathscr{P}_{\mathscr{S}_{k-1}}$.

Note that (3.6) implies that if $\dim \mathfrak{b} \cap \partial\rho_{\mathfrak{b}} = n-2$ for a slab $\mathfrak{b}$, then $\mathfrak{b} \cap \partial\rho_{\mathfrak{b}}$ is a joint.

For each joint $\mathfrak{j}$ of $\mathscr{S}_{k-1}$ we now obtain a scattering diagram $\mathfrak{D}'_{\mathfrak{j}}$ (Construction 3.4) to which we can apply Proposition 3.9 with $g = \mathrm{id}_{\sigma_{\mathfrak{j}}}$. Each ray of the scattering diagram thus obtained defines a new wall with base $\mathfrak{j}$. However, if $\operatorname{codim}\sigma_{\mathfrak{j}} = 2$, this also involves a change of slabs by terms of order k and thus influences the computation at other joints. We therefore deal with joints of codimension two first.

I.2. *Adjustments of slabs from joints of codimension two.* For each codimension-two joint $\mathfrak{j}$ and any slab $\mathfrak{b}$ containing $\mathfrak{j}$, Proposition 3.9, applied to any $g : \omega \to \sigma_{\mathfrak{j}}$, defines a change $\tilde{f}_{\mathfrak{b},x}$ of $f_{\mathfrak{b},x}$, for all $x \in \mathfrak{b} \cap \omega$, by terms of order k along $\sigma_{\mathfrak{j}}$. Use (2.11) to extend this modification of slab function $f_{\mathfrak{b},x}$ to all $x \in \mathfrak{b}$. In view of uniqueness and Remark 2.29(4) and (5), the results for different choices of ω containing x coincide.

Moreover, by (3.6) any slab contains at most one joint $\mathfrak{j}$ with $\operatorname{codim}\sigma_{\mathfrak{j}} = 2$. Hence the corrections from different joints are independent of each other. After applying these changes to $\mathscr{S}_{k-1}$ simultaneously, we may therefore assume Proposition 3.9 applies without any change of slabs. Note that this replacement does not affect the equivalence class of $\mathscr{S}_{k-1}$ to order $k-1$.

The next step produces the new walls. We require the following lemma.

LEMMA 3.11. *Let $\mathfrak{j} \subseteq B$, $\mathfrak{j} \not\subseteq \partial B$, be an $(n-2)$-dimensional polyhedral subset of some $\sigma \in \mathscr{P}_{\max}$ and m a monomial on σ with $\operatorname{ord}_\sigma(m) = k \geq 0$. Assume furthermore $m \in P_x$, for all $x \in \mathfrak{j}$ and $-\overline{m} \notin \Lambda_{\mathfrak{j}}$, to be contained in the tangent wedge to σ along $\mathfrak{j}$. Then $m \in P_x$ for any $x \in \mathfrak{p} \setminus \Delta$ with*

$$\mathfrak{p} := (\mathfrak{j} - \mathbb{R}_{\geq 0} \cdot \overline{m}) \cap \sigma,$$

and for any $x \in \operatorname{Top}(\mathfrak{p}) := \operatorname{cl}\left(\partial\mathfrak{p} \setminus \left(\mathfrak{j} \cup (\partial\mathfrak{j} - \mathbb{R}_{\geq 0}\overline{m})\right)\right)$, $x \notin \partial B$, there exists $\sigma' \in \mathscr{P}_{\max}$ with $x \in \sigma'$ and $\operatorname{ord}_{\sigma'}(m) > k$.

Proof. For any $\sigma' \in \mathscr{P}_{\max}$ with $\sigma' \cap (\mathfrak{p} \setminus \mathfrak{j}) \neq \emptyset$, Proposition 2.6 with $\sigma^+ = \sigma$ shows

$$\operatorname{ord}_{\sigma'}(m) \geq \operatorname{ord}_\sigma(m) = k \geq 0.$$

Thus $m \in P_x$ for any $x \in \mathfrak{p}$. On the other hand, if $x \in \operatorname{Top}(\mathfrak{p})$, then $\overline{m}$ is not tangent to the minimal cell τ containing x. Thus m, as an affine function on $\check{\tau}$, is nonconstant. In particular, as it takes its minimal value on $\check{\sigma}$, there exists a vertex $\check{\sigma}'$ of $\check{\tau}$ such that $\operatorname{ord}_{\sigma'}(m) = m(\check{\sigma}') > m(\check{\sigma}) = \operatorname{ord}_\sigma(m) = k$. □

I.3. *Scattering at joints.* For a joint $\mathfrak{j}$ of $\mathscr{S}_{k-1}$ let $\mathfrak{D}_{\mathfrak{j}}$ be the scattering diagram obtained from $\mathfrak{D}'_{\mathfrak{j}} := \mathfrak{D}_{\mathfrak{j}}(\mathscr{S}_{k-1}, \sigma_{\mathfrak{j}}, v)$, for some choice of vertex $v \in \sigma_{\mathfrak{j}}$, by the application of Proposition 3.9 with $g = \operatorname{id}_{\sigma_{\mathfrak{j}}}$. By I.2 the functions $f_{\mathfrak{c},x}$ remain unchanged, so $\mathfrak{D}_{\mathfrak{j}}$ differs from $\mathfrak{D}'_{\mathfrak{j}}$ only by outgoing rays in directions different from directions of slabs. Moreover, Proposition 3.9 applied with various $g : \omega \to \sigma_{\mathfrak{j}}$, for ω with $\mathfrak{j} \cap \operatorname{Int}\omega \neq \emptyset$, implies $m_{\mathfrak{r}} \in P_x$ for any $\mathfrak{r} \in \mathfrak{D}_{\mathfrak{j}} \setminus \mathfrak{D}'_{\mathfrak{j}}$ and

$x \in \mathfrak{j} \setminus \Delta$. In fact, by Remark 3.5 we may assume that the vertex $v \in \sigma_\mathfrak{j}$ lies in ω. Hence v can also be used for the scattering diagram $\mathfrak{D}_\mathfrak{j}(\omega)$ obtained for $\omega \to \sigma_\mathfrak{j}$. Then as in Remark 2.29 one sees that uniqueness implies equivalence of $\mathfrak{D}_\mathfrak{j}$ and $\mathfrak{D}_\mathfrak{j}(\omega)$ to order k. This shows that $m_\mathfrak{r} \in P_x$ for $x \in (\mathfrak{j} \cap \operatorname{Int}\omega) \setminus \Delta$.

Define $\mathscr{S}_k^{\mathrm{I}}$ by adding to $\mathscr{S}_{k-1}$, for any joint $\mathfrak{j}$ and any ray $\mathfrak{r} \in \mathfrak{D}_\mathfrak{j} \setminus \mathfrak{D}'_\mathfrak{j}$, the wall $(\mathfrak{p}_\mathfrak{r}, m_\mathfrak{r}, c_\mathfrak{r})$ with

$$\mathfrak{p}_\mathfrak{r} := \big(\mathfrak{j} - \mathbb{R}_{\geq 0} \cdot \overline{m_\mathfrak{r}}\big) \cap \sigma,$$

where σ is the unique maximal cell with $\mathfrak{r} \subseteq \overline{\overline{\sigma}}$. This is indeed a wall by Lemma 3.11. Add some more walls $\mathfrak{p}$ with $c_\mathfrak{p} = 0$ to achieve the requirement of Definition 2.22(ii), for example by covering $H_\mathfrak{p} \cap \sigma$ by such walls, for each added wall $\mathfrak{p} \subseteq \sigma$, with $H_\mathfrak{p} \subseteq \Lambda_{\sigma,\mathbb{R}}$ the affine hyperplane containing $\mathfrak{p}$. (This step is indeed not necessary as follows by the arguments in Section 3.4, but we will not prove this.) Choose also a polyhedral decomposition $\mathscr{P}_{\mathscr{S}_k^{\mathrm{I}}}$ which on $|\mathscr{S}_{k-1}|$ refines $\mathscr{P}_{\mathscr{S}_{k-1}}$ and such that each slab or wall of $\mathscr{S}_k^{\mathrm{I}}$ is a union of $(n-1)$-cells of $\mathscr{P}_{\mathscr{S}_k^{\mathrm{I}}}$.

We now have produced a new structure $\mathscr{S}_k^{\mathrm{I}}$, with the superscript "I" indicating that it is the result of Step I of the algorithm. By subdividing slabs we may assume (i) in the definition of structures (Definition 2.22) to continue to hold, while (3.6) from I.1 is true in any case.

We now check that the corrections at the joints of $\mathscr{S}_{k-1}$ have the desired effect. In particular, we have to verify that new walls do not influence the computations at joints different from their bases.

PROPOSITION 3.12. *For any* $\mathfrak{j} \in \operatorname{Joints}(\mathscr{S}_k^{\mathrm{I}})$, *the scattering diagram* $\mathfrak{D}_\mathfrak{j} = \mathfrak{D}_\mathfrak{j}(\mathscr{S}_k^{\mathrm{I}}, \sigma_\mathfrak{j}, v)$, $v \in \sigma_\mathfrak{j}$, *fulfills*

$$\theta^k_{\mathfrak{D}_\mathfrak{j},g} \in \begin{cases} O^k\big((\Lambda_\mathfrak{j} \setminus \{0\}) \otimes \Lambda_\mathfrak{j}^\perp\big), & \operatorname{codim}\sigma_\mathfrak{j} = 0,\\ O^k\Big(\Lambda_\mathfrak{j} \otimes \Lambda_\mathfrak{j}^\perp + \dfrac{\Lambda_\rho}{f_\rho} \otimes \Lambda_\rho^\perp\Big), & \operatorname{codim}\sigma_\mathfrak{j} = 1 \ (\rho = \sigma_\mathfrak{j}),\\ O^k\Big(\Lambda_\mathfrak{j} \otimes \Lambda_\mathfrak{j}^\perp + \sum_{\rho \supseteq \mathfrak{j}} \dfrac{\Lambda_\mathfrak{j}}{f_\rho} \otimes \Lambda_\rho^\perp\Big), & \operatorname{codim}\sigma_\mathfrak{j} = 2. \end{cases} \tag{3.7}$$

Proof. $\mathscr{S}_k^{\mathrm{I}}$ differs from $\mathscr{S}_{k-1}$ effectively by the addition of walls $\mathfrak{p}$ with base a joint $\mathfrak{j}_\mathfrak{p}$ of $\mathscr{S}_{k-1}$, the other walls having $c_\mathfrak{p} = 0$ and thus being irrelevant. We now discuss the contributions of such $\mathfrak{p}$ to the computation of $\theta^k_{\mathfrak{D}_\mathfrak{j},\mathrm{id}_{\sigma_\mathfrak{j}}}$. There are the following possibilities for the relative position of $\mathfrak{j}$ inside $\mathfrak{p}$:

(1) $\mathfrak{j} \subseteq \operatorname{Base}(\mathfrak{p}) = \mathfrak{j}_\mathfrak{p}$.
(2) $\mathfrak{j} \subseteq \operatorname{Top}(\mathfrak{p})$.
(3) $\mathfrak{j} \not\subseteq \partial\mathfrak{p}$.
(4) $\mathfrak{j} \subseteq \operatorname{Sides}(\mathfrak{p})$.

In (1), $\mathfrak{j}_\mathfrak{p}$ is a joint of $\mathscr{S}_{k-1}$ and $\mathfrak{p}$ arose from an outgoing ray produced in Proposition 3.9. These precisely lead to the desired form (3.4) of $\theta^k_{\mathfrak{D}_\mathfrak{j},\mathrm{id}_{\sigma_\mathfrak{j}}}$.

In Case (2), $\mathrm{ord}_\mathfrak{j}(m_\mathfrak{p})>k$ by Lemma 3.11. These walls do not make any contribution in order k at $\mathfrak{j}$.

Case (3) can only happen in the codimension-zero case, that is, if $\sigma_\mathfrak{j}\in\mathscr{P}_{\max}$. Then $\mathrm{ord}_\mathfrak{j}(m)=\mathrm{ord}_{\sigma_\mathfrak{j}}(m)=k$ and hence $\exp\big(-\log(1+cz^m)\partial_n\big)$ commutes with any log automorphism of $R^k_{\mathrm{id}_{\sigma_\mathfrak{j}},\sigma_\mathfrak{j}}$; see (2.10). Since the automorphism associated to $\mathfrak{p}$ occurs twice with opposite signs in $\theta^k_{\mathfrak{D}_\mathfrak{j},\mathrm{id}_{\sigma_\mathfrak{j}}}$, it makes no contribution.

In (4) with $\sigma_\mathfrak{j}$ maximal, the automorphism associated to $\mathfrak{p}$ lies in $O^k\big((\Lambda_\mathfrak{j}\setminus\{0\})\otimes\Lambda_\mathfrak{j}^\perp\big)$. Hence this wall preserves (3.4). Finally, in (4) with $\mathrm{codim}\,\sigma_\mathfrak{j}>0$, Proposition 3.10(2) shows that the insertion of $\mathfrak{p}$ preserves the form of (3.4). $\square$

3.4. *Interstices and consistency in codimension* 0. The remaining terms in (3.7) all involve exponents tangent to joints or slabs. The topology at intersections of joints now imply strong compatibility conditions that are the subject of this subsection. Among other things, these restrictions already imply consistency at codimension-zero joints.

Definition 3.13. An *interstice* of a structure $\mathscr{S}$ is an $(n-3)$-cell $\mathfrak{d}\in\mathscr{P}_\mathscr{S}$ with $\mathfrak{d}\not\subseteq\partial B$.

Analogous to the situation for a joint, for a vertex $v\in\sigma_\mathfrak{d}$ we obtain a normal space $\mathcal{Q}^v_{\mathfrak{d},\mathbb{R}}\simeq\mathbb{R}^3$, defined as $\Lambda_{v,\mathbb{R}}/\Lambda_{\mathfrak{d},\mathbb{R}}$. Again we write $\overline{\overline{m}}\in\mathcal{Q}^v_{\mathfrak{d},\mathbb{R}}$ for the image of an exponent on any $\sigma\in\mathscr{P}_{\max}$ containing $\mathfrak{d}$, and $\overline{\overline{\tau}}\subseteq\mathcal{Q}^v_{\mathfrak{d},\mathbb{R}}$ for the image of the tangent wedge along $\mathfrak{d}$ of a cell $\tau\subseteq B$, $\tau\supseteq\mathfrak{d}$.

To study the topology of the situation along $\mathfrak{d}$ we look at the associated 2-sphere

$$S_\mathfrak{d}:=\big(\mathcal{Q}^v_{\mathfrak{d},\mathbb{R}}\setminus\{0\}\big)/\mathbb{R}_{>0},$$

which we orient arbitrarily. This 2-sphere comes with the following cell-decomposition. For the joints $\mathfrak{j}_1,\ldots,\mathfrak{j}_s$ containing $\mathfrak{d}$ we have 0-cells $\overline{\overline{\mathfrak{j}_i}}/\sim$, the 1-cells are given by $\overline{\overline{\mathfrak{v}}}/\sim$ for $\mathfrak{v}\in\mathscr{P}^{[n-1]}_\mathscr{S}$, $\mathfrak{v}\supseteq\mathfrak{d}$, and the 2-cells are $\overline{\overline{\mathfrak{u}}}/\sim$ for $\mathfrak{u}\in\mathrm{Chambers}(\mathscr{S})$, $\mathfrak{u}\supseteq\mathfrak{d}$. Let $\Sigma_\mathfrak{d}$ be the dual cell complex, with 0, 1 and 2-cells $\widehat{\mathfrak{u}}$, $\widehat{\mathfrak{v}}$ and $\widehat{\mathfrak{j}}$ defined by chambers, elements of $\mathscr{P}^{[n-1]}_\mathscr{S}$ and joints containing $\mathfrak{d}$, respectively. Note that this is a subcomplex of the cell-complex Σ studied in Step 3 of the proof of Lemma 2.30 for the case $g=\mathrm{id}_{\sigma_\mathfrak{d}}$. It is then clear that for each edge path β in $\Sigma_\mathfrak{d}$ from $\widehat{\mathfrak{u}}$ to $\widehat{\mathfrak{u}}'$, we obtain a log isomorphism

$$\theta_\beta: P_{\sigma_\mathfrak{d},\sigma_\mathfrak{u}}\longrightarrow\big(R^k_{\mathrm{id}_{\sigma_\mathfrak{d}},\sigma_{\mathfrak{u}'}}\big)^\times$$

from $R^k_{\mathrm{id}_{\sigma_\mathfrak{d}},\sigma_\mathfrak{u}}$ to $R^k_{\mathrm{id}_{\sigma_\mathfrak{d}},\sigma_{\mathfrak{u}'}}$. The underlying monoid homomorphism is obtained by parallel transport through v.

Next choose a base vertex $\widehat{\mathfrak{u}_0} \in \Sigma_{\mathfrak{d}}$ and let γ_i be a closed loop covering the edges of $\widehat{\mathfrak{j}_i}$ in counterclockwise direction. Because the 1-skeleton $\Sigma_{\mathfrak{d}}^1$ of $\Sigma_{\mathfrak{d}}$ has the homotopy type of S^2 minus s points, there exist paths β_i on $\Sigma_{\mathfrak{d}}$, with β_i connecting $\widehat{\mathfrak{u}_0}$ with the base point of γ_i, such that $\beta_i\gamma_i\beta_i^{-1}$ is a standard generating set of $\pi_1(\Sigma_{\mathfrak{d}}^1, \widehat{\mathfrak{u}_0})$:

$$\beta_1\gamma_1\beta_1^{-1}\beta_2\gamma_2\beta_2^{-1}\dots\beta_s\gamma_s\beta_s^{-1} = 1. \tag{3.8}$$

Note that such β_i exist regardless of the given order $\mathfrak{j}_1, \dots, \mathfrak{j}_s$ of the joints. For the corresponding sequence of log isomorphisms, we conclude that

$$\theta_{\beta_s}^{-1} \circ \theta_{\gamma_s} \circ \theta_{\beta_s} \circ \dots \circ \theta_{\beta_1}^{-1} \circ \theta_{\gamma_1} \circ \theta_{\beta_1} = 1. \tag{3.9}$$

Note that we may impose additional conditions on the choices of β_i, γ_i as long as (3.8) holds.

For $\mathscr{S} = \mathscr{S}_k^{\mathrm{I}}$ constructed in Step I, (3.9) implies the following result for interstices $\mathfrak{d}$ with $\operatorname{codim}\sigma_{\mathfrak{d}} = 0$.

PROPOSITION 3.14. *Assume that* $\mathfrak{d}$ *is an interstice of* $\mathscr{S}_k^{\mathrm{I}}$ *with* $\operatorname{codim}\sigma_{\mathfrak{d}} = 0$ *and write, using* (3.7),

$$\theta_{\gamma_i} = \exp\Big(\sum_m a_{m,i} z^m \partial_{n_i(m)}\Big)$$

as log automorphism of $R^k_{\mathrm{id}_{\sigma_{\mathfrak{d}}},\sigma_{\mathfrak{d}}}$, *where the sum runs over those exponents* m *on* $\sigma_{\mathfrak{d}}$ *with* $\overline{m} \in \Lambda_{\mathfrak{j}_i}$, $\operatorname{ord}_{\sigma_{\mathfrak{d}}}(m) = k$, *and* $a_{m,i} \in \Bbbk$, $n_i(m) \in \Lambda_{\mathfrak{j}_i}^{\perp}$. *Then for any* $m \in P_{\sigma_{\mathfrak{d}},\sigma_{\mathfrak{d}}}$ *with* $\operatorname{ord}_{\sigma_{\mathfrak{d}}}(m) = k$, *in* $\Lambda^*_{\sigma_{\mathfrak{d}}}$ *it holds that*

$$\sum_i a_{m,i} n_i(m) = 0. \tag{3.10}$$

Proof. Since θ_{γ_i} only involves monomials of order k, it commutes with θ_{β_i} and any θ_{γ_j}. Hence (3.9) shows that

$$1 = \theta_{\gamma_s} \circ \dots \circ \theta_{\gamma_1} = \exp\Big(\sum_{m,i} a_{m,i} z^m \partial_{n_i(m)}\Big),$$

which readily implies the result. □

To deduce analogous restrictions from higher codimension interstices we need to understand how θ_{γ_i} transforms by commutation with log isomorphisms changing chambers $\mathfrak{u} \supseteq \mathfrak{d}$.

LEMMA 3.15. *Let* θ_γ *be the log isomorphism from* $R^k_{\mathrm{id}_{\sigma_{\mathfrak{d}}},\sigma_{\mathfrak{u}'}}$ *to* $R^k_{\mathrm{id}_{\sigma_{\mathfrak{d}}},\sigma_{\mathfrak{u}}}$ *associated to an edge path* γ *in* $\Sigma_{\mathfrak{d}}$ *connecting* $\hat{\mathfrak{u}}'$ *with* $\hat{\mathfrak{u}}$. *Then for any* $m \in P_{\sigma_{\mathfrak{d}},\sigma_{\mathfrak{u}'}}$ *with* $z^m \in J_{l-1} := I_0^{l-1} \cdot I_{k-1} + I_k$, $n \in \Lambda^*_\sigma$ *and* $a \in \Bbbk(\Lambda_{\sigma_{\mathfrak{d}}}) \cap R^k_{\mathrm{id}_{\sigma_{\mathfrak{d}}},\sigma_{\mathfrak{u}'}}$, *there exist* $b_i \in \Bbbk(\Lambda_{\sigma_{\mathfrak{d}}}) \cap R^k_{\mathrm{id}_{\sigma_{\mathfrak{d}}},\sigma_{\mathfrak{u}}}$ *and* $n_i \in \Lambda^*_{\sigma_{\mathfrak{u}}}$ *such that*

$$\theta_\gamma \circ \exp(a z^m \partial_n) \circ \theta_\gamma^{-1} = \exp\Big(\sum_i b_i z^m \partial_{n_i}\Big) = \prod_i \exp\big(b_i z^m \partial_{n_i}\big) \mod J_l$$

as log automorphisms of $R^k_{\mathrm{id}_{\sigma_{\mathfrak{d}}},\sigma_{\mathfrak{u}'}}$*. Here we identify* $\Lambda_{\sigma_{\mathfrak{u}}}$ *and* $\Lambda_{\sigma_{\mathfrak{u}'}}$ *by parallel transport through some vertex* $v\in\sigma_{\mathfrak{u}}\cap\sigma_{\mathfrak{u}'}$*.*

Proof. By induction on the number of edges passed by γ it suffices to consider the case that $\mathfrak{u}$ and $\mathfrak{u}'$ are adjacent chambers and θ_γ is the associated basic gluing morphism. Thus, up to choosing an isomorphism $R^k_{\mathrm{id}_{\sigma_{\mathfrak{d}}},\sigma_{\mathfrak{u}}}\to R^k_{\mathrm{id}_{\sigma_{\mathfrak{d}}},\sigma_{\mathfrak{u}'}}$ by parallel transport through a point in $\mathrm{Int}(\mathfrak{d})\setminus\Delta$, we are in the situation of Lemma 2.15. Since $I_0\cdot J_{l-1}\subseteq J_l$ this shows first that we can ignore all expressions in θ_γ involving monomials $z^{m'}$ with $\mathrm{ord}_{\sigma_{\mathfrak{d}}}(m')>0$. Thus we may assume θ_γ to be of the form $m'\mapsto f^{\langle\overline{m'},n_0\rangle}$ with $f\in\Bbbk(\Lambda_{\sigma_{\mathfrak{d}}})\cap(R^k_{\mathrm{id}_{\sigma_{\mathfrak{d}}},\sigma_{\mathfrak{u}}})^\times$, and then Lemma 2.15 gives the claimed result. □

Remark 3.16. The reason for introducing $J_{l-1}=I_0^{l-1}\cdot I_{k-1}+I_k$ here is that the order function from Definition 2.3 is not in general additive. For example, for a vertex v in a one-dimensonal B with adjacent maximal cells σ_1,σ_2 and $\varphi_v|_{\sigma_1}=0$, $\varphi_v|_{\sigma_2}$ having slope 1, we have $k[P_{\omega,\sigma}]\simeq\Bbbk[\mathbb{N}^2]$, $t=z^{(1,1)}$ and $\mathrm{ord}_v\big((1,0)\big)=1$, $\mathrm{ord}_v\big((0,1)\big)=1$, but also $\mathrm{ord}_v\big((1,1)\big)=1$. Similar ideals as J_l will occur repeatedly in the following.

Note that one exception where $\mathrm{ord}_\tau(m+m')=\mathrm{ord}_\tau(m)+\mathrm{ord}_\tau(m')$ is when $m\in\Lambda_\tau$, because then $\mathrm{ord}_\sigma(m)=\mathrm{ord}_\tau(m)$ for any $\sigma\in\mathscr{P}_{\max}$ containing τ.

We can now deduce an analogue of Proposition 3.14 for interstices of higher codimension.

PROPOSITION 3.17. *Assume that* $\mathfrak{d}$ *is an interstice of* $\mathscr{S}_k^{\mathrm{I}}$ *with* $\operatorname{codim}\sigma_{\mathfrak{d}}\ge 1$ *and let* $\sigma\in\mathscr{P}_{\max}$, $\sigma\supseteq\mathfrak{d}$*. For any joint* $\mathfrak{j}\subseteq\sigma$ *of* $\mathscr{S}_k^I$ *containing* $\mathfrak{d}$*, take the base chamber of the loop* $\gamma_{\mathfrak{j}}$ *around* $\mathfrak{j}$ *to be contained in* σ *and oriented according to the chosen orientation of* $\Sigma_{\mathfrak{d}}$*. Write*

$$\theta_{\gamma_{\mathfrak{j}}}=\exp\Big(\sum_{\{(m,\nu)\,|\,m\in A,\,\mathrm{ord}_{\sigma_{\mathfrak{d}}}(m)=k\}}a_{\mathfrak{j},m,\nu}z^m\partial_{n_{\mathfrak{j},m,\nu}}\Big)$$

as log automorphism of $R^k_{\mathrm{id}_{\sigma_{\mathfrak{d}}},\sigma}$*, where* $a_{\mathfrak{j},m,\nu}\in\Bbbk(\Lambda_{\sigma_{\mathfrak{d}}})\cap R^k_{\mathrm{id}_{\sigma_{\mathfrak{d}}},\sigma}$*,* $n_{\mathfrak{j},m,\nu}\in\Lambda_{\mathfrak{d}}^\perp$ *and* A *is a set of representatives of* $P_{\sigma_{\mathfrak{d}},\sigma}/\Lambda_{\sigma_{\mathfrak{d}}}$*.*

(1) *If* $\sigma_{\mathfrak{j}}=\sigma$ *and* $-\overline{m}\in\mathrm{Int}\,K_{\sigma_{\mathfrak{d}}}\sigma$*, then* $\sum_\nu a_{\mathfrak{j},m,\nu}\partial_{n_{\mathfrak{j},m,\nu}}=0$*.*

(2) *Let* $\rho\in\mathscr{P}^{[n-1]}$*,* $\mathfrak{d}\subseteq\rho\subseteq\sigma$*.*
   - (a) *If* $\operatorname{codim}\sigma_{\mathfrak{d}}\ge 2$ *and* $-\overline{m}\in\mathrm{Int}\,K_{\sigma_{\mathfrak{d}}}\rho$*, then* $\sum_{\nu,\{\mathfrak{j}\supseteq\mathfrak{d}\,|\,\sigma_{\mathfrak{j}}=\rho\}}a_{\mathfrak{j},m,\nu}\partial_{n_{\mathfrak{j},m,\nu}}=0$*.*
   - (b) *If* $\operatorname{codim}\sigma_{\mathfrak{d}}=1$*, assume in addition that* $\theta_{\gamma_{\mathfrak{j}}}=1$ *for any joint* $\mathfrak{j}\supseteq\mathfrak{d}$ *with* $\operatorname{codim}\sigma_{\mathfrak{j}}=0$*. Then for any* $m\in A$*,* $\sum_{\nu,\{\mathfrak{j}\supseteq\mathfrak{d}\,|\,\sigma_{\mathfrak{j}}=\rho\}}a_{\mathfrak{j},m,\nu}\partial_{n_{\mathfrak{j},m,\nu}}=0$*.*

(3) *Assume in addition that* $\theta_{\gamma_{\mathfrak{j}}}=1$ *for any joint* $\mathfrak{j}\supseteq\mathfrak{d}$ *with* $\operatorname{codim}\sigma_{\mathfrak{j}}\le 1$*.*

(a) *If* $\operatorname{codim}\sigma_{\mathfrak{d}} = 2$ *and* $\mathfrak{j}, \mathfrak{j}'$ *are the unique joints in* $\sigma_{\mathfrak{d}}$ *containing* $\mathfrak{d}$, *then* $\theta_{\gamma_{\mathfrak{j}'}} = \theta_{\gamma_{\mathfrak{j}}}^{-1}$.

(b) *If* $\operatorname{codim}\sigma_{\mathfrak{d}} = 3$ *and* $\mathfrak{j} \supseteq \mathfrak{d}$ *is a joint with* $\operatorname{codim}\sigma_{\mathfrak{j}} = 2$, *then* $\theta_{\gamma_{\mathfrak{j}}} \in O^k(\Lambda_{\mathfrak{d}} \otimes \Lambda_{\mathfrak{d}}^{\perp})$.

Proof. We proceed inductively, proving the statement for any $\sigma \in \mathscr{P}_{\max}$, $\sigma \supseteq \mathfrak{d}$, and those exponents m with $z^m \in J_l \setminus J_{l-1}$, $J_l = I_0^l \cdot I_{k-1} + I_k$. For $l = 0$ there is nothing to prove.

The key ingredient is (3.9) with a particular choice of γ_i, β_i. The additional requirement is that for any $\tau \in \mathscr{P}$ containing $\mathfrak{d}$, the loops around joints $\mathfrak{j}$ with $\sigma_{\mathfrak{j}} = \tau$ are numbered consecutively $\gamma_i, \gamma_{i+1}, \ldots, \gamma_{i+r}$, and are based on chambers contained in the same maximal cell $\sigma(\tau) \supseteq \tau$; furthermore,

$$\beta_i\gamma_i\beta_i^{-1} \ldots \beta_{i+r}\gamma_{i+r}\beta_{i+r}^{-1}$$

shall be freely homotopic to an edge path γ_τ passing along the boundary of $\bigcup_{j=i}^{i+r} \hat{\mathfrak{j}}_j$ once. There are two exceptional cases. First, if $\sigma_{\mathfrak{d}} = \tau = \rho \in \mathscr{P}^{[n-1]}$, then $\bigcup_{j=i}^{i+r} \hat{\mathfrak{j}}_j$ is an annulus; in this case we want a homotopy to an edge path first following one boundary component, then an edge $\hat{\mathfrak{v}}$ to the other boundary component, then following the other boundary component, and finally back along $\hat{\mathfrak{v}}$. The other exceptional case occurs for $\sigma_{\mathfrak{d}} = \tau \in \mathscr{P}^{[n-2]}$, where τ contains exactly two joints $\mathfrak{j}, \mathfrak{j}'$, as in (3)(a). We then take γ_τ to consist of the composition of two loops with some common base point, denoted $\hat{\mathfrak{u}}_\tau$, and which go around $\hat{\mathfrak{j}}$ and $\hat{\mathfrak{j}}'$, respectively. In any case, following γ_τ defines a log automorphism $\theta_\tau := \theta_{\gamma_\tau}$ of $R^k_{\mathrm{id}_{\sigma_{\mathfrak{d}}},\sigma(\tau)}$.

We note at this point that with this selection of paths, parts (2)(b) and (3)(a) of this proposition follow immediately from (3.9), observing that each $\theta_{\gamma_{\mathfrak{j}}}$ commutes with automorphisms attached to any wall containing $\mathfrak{d}$.

Continuing with the other cases, note that for any τ, the $\theta^k_{\gamma_{\mathfrak{j}}}$ with $\sigma_{\mathfrak{j}} = \tau$ commute mutually and with any automorphism associated to a wall $\mathfrak{p} \subseteq \sigma(\tau)$. This shows that

$$\theta_\tau = \prod_{\{\mathfrak{j}\,|\,\sigma_{\mathfrak{j}}=\tau\}} \theta_{\gamma_{\mathfrak{j}}} = \exp\Big(\sum_{\left\{(m,\nu)\,\middle|\, \substack{m\in A,\, \operatorname{ord}_{\sigma_{\mathfrak{d}}}(m)=k \\ \operatorname{ord}_\tau(m)=k}\right\}} a_{\tau,m,\nu} z^m \partial_{n_{\tau,m,\nu}}\Big) \tag{3.11}$$

as log automorphism of $R^k_{\mathrm{id}_{\sigma_{\mathfrak{d}}},\sigma(\tau)}$, with $a_{\tau,m,\nu} \in \Bbbk(\Lambda_{\sigma_{\mathfrak{d}}}) \cap R^k_{\mathrm{id}_{\sigma_{\mathfrak{d}}},\sigma(\tau)}$ and $n_{\tau,m,\nu} \in \Lambda_{\mathfrak{d}}^{\perp}$. By (3.7) and Proposition 2.6, the sum runs only over those m with $-\overline{m} \in K_{\sigma_{\mathfrak{d}}}\tau$. Note also that since the elements of A are not congruent modulo $\Lambda_{\sigma_{\mathfrak{d}}}$, $\sum_\nu a_{\tau,m,\nu} n_{\tau,m,\nu}$ is uniquely determined.

In the proof special care has to be taken for codimension-one joints $\mathfrak{j}$, because these potentially involve monomials z^m with the only restriction $-\overline{m} \in K_{\sigma_{\mathfrak{d}}}\rho$, $\rho = \sigma_{\mathfrak{j}}$. If $\operatorname{codim}\sigma_{\mathfrak{d}} = 3$, these may interact with terms arising from codimension-two joints $\mathfrak{j}' \subseteq \partial\rho$ in a way spoiling the induction process. We deal with this problem as follows. If $\operatorname{codim}\sigma_{\mathfrak{d}} = 3$, there are exactly two cells $\tau_1 \neq \tau_2$ of $\mathscr{P}$ of codimension two with $\mathfrak{d} \subseteq \tau_\mu \subseteq \rho$. Let $\mathfrak{v}_\mu \in \mathscr{P}^{[n-1]}_{\mathscr{S}_k^{\mathrm{I}}}$ be the unique cell (support of a slab) with $\mathfrak{d} \subseteq \mathfrak{v}_\mu \subseteq \rho$, and $\dim \tau_\mu \cap \mathfrak{v}_\mu = n-2$, $\mu = 1, 2$. Note that (3.6) from Step I implies $\mathfrak{v}_1 \neq \mathfrak{v}_2$. Thus $\hat{\mathfrak{v}}_\mu$ separates the 2-cell $\hat{\mathfrak{j}}_\mu \in \Sigma_{\mathfrak{d}}$, $\mathfrak{j}_\mu$ the unique joint with $\mathfrak{d} \subseteq \mathfrak{j}_\mu \subseteq \tau_\mu$, from another 2-cell $\hat{\mathfrak{j}}$ with $\mathfrak{j}$ a joint with $\sigma_{\mathfrak{j}} = \rho$. Now change $\theta(\mathfrak{v}_\mu)$, the log isomorphism associated to following the edge $\hat{\mathfrak{v}}_\mu$ in the same direction as $\gamma_{\mathfrak{j}_\mu}$, by composition with

$$\theta_\mu := \exp\Big(\sum_{\{(m,\nu)\,|\, m\in A,\, -\overline{m}\in \operatorname{Int}(K_{\sigma_{\mathfrak{d}}}\tau_\mu)\}} a_{\rho,m,\nu} z^m \partial_{n_{\rho,m,\nu}}\Big).$$

This has the effect of composing $\theta_{\tau_\mu} = \theta_{\mathfrak{j}_\mu}$ with θ_μ and θ_ρ with θ_μ^{-1}. Thus this change cancels all terms $a_{\rho,m,\nu}\partial_{n_{\rho,m,\nu}}$ on the right-hand side of (3.11) whenever $-\overline{m} \in (\partial K_{\sigma_{\mathfrak{d}}}\rho) \setminus \Lambda_{\sigma_{\mathfrak{d}}}$. With this reinterpreted $\theta(\mathfrak{v}_\mu)$, formula (3.9) still holds, and the conclusions of the proposition remain unchanged. We henceforth assume these terms do not arise in (3.11) for any $\tau \in \mathscr{P}^{[n-1]}$ in the first place.

After having established this property, if $\operatorname{codim}\sigma_{\mathfrak{d}} = 3$, we add to the induction hypothesis the following analogue of (1) and (2)(a) for codimension-two joints:

(4) If $\tau \in \mathscr{P}^{[n-2]}$, $\mathfrak{d} \subseteq \tau \subseteq \sigma$ and $-\overline{m} \in \operatorname{Int} K_{\sigma_{\mathfrak{d}}}\tau$, then $\sum_\nu a_{\mathfrak{j},m,\nu}\partial_{n_{\mathfrak{j},m,\nu}} = 0$ for the unique joint $\mathfrak{j} \supseteq \mathfrak{d}$ with $\sigma_{\mathfrak{j}} = \tau$.

Now for any $\tau \supseteq \mathfrak{d}$, if $\mathfrak{j}_i, \ldots, \mathfrak{j}_{i+r}$ are the joints with $\sigma_{\mathfrak{j}} = \tau$ then by the above choice of $\gamma_i, \ldots, \gamma_{i+r}$ there exists an edge path β_τ from $\hat{\mathfrak{u}}_0$ to the base point $\hat{\mathfrak{u}}_\tau$ of γ_τ such that

$$\beta_\tau\gamma_\tau\beta_\tau^{-1} = \beta_i\gamma_i\beta_i^{-1}\ldots\beta_{i+r}\gamma_{i+r}\beta_{i+r}^{-1}.$$

Therefore,

$$\theta_{\beta_\tau}^{-1}\circ\theta_\tau\circ\theta_{\beta_\tau} = \theta_{\beta_{i+r}}^{-1}\circ\theta_{\gamma_{i+r}}\circ\theta_{\beta_{i+r}}\circ\cdots\circ\theta_{\beta_i}^{-1}\circ\theta_{\gamma_i}\circ\theta_{\beta_i}. \tag{3.12}$$

In particular, we can now rewrite (3.9) in the form

$$\theta_{\beta'_{s'}}^{-1}\circ\theta_{s'}\circ\theta_{\beta'_{s'}}\circ\cdots\circ\theta_{\beta'_1}^{-1}\circ\theta_1\circ\theta_{\beta'_1} = 1, \tag{3.13}$$

where for any i we have $\theta_i = \theta_\tau$, $\beta'_i = \beta_\tau$ for some $\tau = \tau(i)$, and each $\tau \supseteq \mathfrak{d}$ occurs exactly once.

We now do the inductive step from $l-1$ to l. The plan is to deduce (1), (2)(a) and (4) for m with $z^m \in J_l \setminus J_{l-1}$ by looking at (3.13) modulo J_{l+1}. For $\tau \supsetneq \sigma_{\mathfrak{d}}$, consistency of $\mathscr{S}_k^{\mathrm{I}}$ to order $k-1$ and the present induction hypothesis show that

$$\theta_\tau = \exp\Big(\sum_{\{(m,\nu)\,|\,m\in A\}} a_{\tau,m,\nu} z^m \partial_{n_{\tau,m,\nu}}\Big) \circ \exp\Big(\sum_{\nu'} a_{\tau,\nu'} t^k \partial_{n_{\tau,\nu'}}\Big) \mod J_{l+1} \tag{3.14}$$

with $z^m \in J_l \setminus J_{l-1}$, $-\overline{m} \in \operatorname{Int} K_{\sigma_{\mathfrak{d}}}\tau$ and $a_{\tau,\nu'}, a_{\tau,m,\nu} \in \Bbbk(\Lambda_{\sigma_{\mathfrak{d}}}) \cap R^k_{\mathrm{id}_{\sigma_{\mathfrak{d}}},\sigma(\tau)}$. Note that if $\tau \in \mathscr{P}^{[n-2]}$ and $\operatorname{codim} \sigma_{\mathfrak{d}} = 3$, then (3.14) follows from (4). In (3.13), θ_τ occurs conjugated by the log isomorphism associated to the edge path β_τ. Now since $I_0 \cdot J_l \subseteq J_{l+1}$, the conjugation by a log automorphism associated to crossing a wall does not have any effect modulo J_{l+1} and can thus be ignored in the following. For the conjugation by a log automorphism associated to crossing a slab, Lemma 3.15 shows that likewise

$$\theta_{\beta_\tau}^{-1} \circ \theta_\tau \circ \theta_{\beta_\tau} = \exp\Big(\sum_{m,\nu} a'_{\tau,m,\nu} z^m \partial_{n'_{\tau,m,\nu}}\Big) \circ \exp\Big(\sum_{\nu'} a'_{\tau,\nu'} t^k \partial_{n'_{\tau,\nu'}}\Big) \mod J_{l+1}$$

with $z^m \in J_l \setminus J_{l-1}$, $-\overline{m} \in \operatorname{Int} K_{\sigma_{\mathfrak{d}}}\tau$ and $a'_{\tau,\nu'}, a'_{\tau,m,\nu} \in \Bbbk(\Lambda_{\sigma_{\mathfrak{d}}}) \cap R^k_{\mathrm{id}_{\sigma_{\mathfrak{d}}},\sigma(\tau)}$. On the other hand, if $\tau = \sigma_{\mathfrak{d}}$, we readily obtain a similar expansion without the first factor, that is, with all $a'_{\tau,m,\nu} = 0$. Thus in any case, for any τ, m, ν with $z^m \in J_l \setminus J_{l-1}$, the expression $a'_{\tau,m,\nu} z^m \partial_{n'_{\tau,m,\nu}}$ cannot cancel with any term from $\theta_{\tau'}$ for any $\tau' \neq \tau$. In view of (3.13) we therefore conclude that

$$\theta_{\beta_\tau}^{-1} \circ \theta_\tau \circ \theta_{\beta_\tau} = \exp\Big(\sum_{\nu'} a'_{\tau,\nu'} t^k \partial_{n'_{\tau,\nu'}}\Big) \mod J_{l+1}$$

for any τ, with $a'_{\tau,\nu'} \in \Bbbk(\Lambda_{\sigma_{\mathfrak{d}}}) \cap R^k_{\mathrm{id}_{\sigma_{\mathfrak{d}}},\sigma(\tau)}$. Hence also $\theta_\tau = \exp(\sum_{\nu'} a_{\tau,\nu'} t^k \partial_{n_{\tau,\nu'}})$, which in turn gives that modulo J_{l+1}, $\sum_{m,\nu} a_{\tau,m,\nu} z^m \partial_{n_{\tau,m,\nu}} = 0$, where the sum is over those m such that $-\overline{m} \in \operatorname{Int} K_{\sigma_{\mathfrak{d}}}\tau$. Expanding out the definition of θ_τ now proves the claimed formulae in (1), (2)(a) and (4) for m with $z^m \in J_l \setminus J_{l-1}$. This shows (1) and (2)(a) in the statement of the proposition. (3)(b) follows easily from the fact that under the additional hypothesis we only have $a_{\tau,m,\nu} \neq 0$ if $\tau \in \mathscr{P}^{[n-2]}$. □

We are now in position to prove consistency to order k along codimension-zero joints.

Proposition 3.18. *$\mathscr{S}_k^{\mathrm{I}}$ is consistent to order k at any joint $\mathfrak{j}$ with* $\operatorname{codim} \sigma_{\mathfrak{j}} = 0$ *provided $\sigma_{\mathfrak{j}}$ is bounded.*

Proof. Let $\sigma \in \mathscr{P}_{\max}$. For a joint $\mathfrak{j} \subseteq \sigma$ intersecting $\operatorname{Int}(\sigma)$, a loop around $\mathfrak{j}$ defines a log automorphism

$$\theta_{\mathfrak{j}}^k = \exp\Big(\sum_i c_i z^{m_i} \partial_{n_i}\Big)$$

of $R^k_{\mathrm{id}_\sigma,\sigma}$ with $\mathrm{ord}_\sigma(m_i) = k$ and $\overline{m_i} \in \Lambda_{\mathfrak{j}}$ for all i. This depends only on the sense of orientation of the loop, since changing the base chamber leads to conjugation by automorphisms associated to walls, and these only involve monomials of higher order.

Now fix an exponent m on $\sigma := \sigma_{\mathfrak{j}}$. We have to show that $c^m_{\mathfrak{j}} := \sum_{m_i=m} c_i n_i \in \Lambda^*_\sigma \otimes \Bbbk$ vanishes. This is clear if $\overline{m} = 0$ because $\theta^k_{\mathfrak{j}} \in H^{I_k}_{\mathfrak{j}}$. Otherwise $L_x = (x + \mathbb{R}\overline{m}) \cap \sigma$ for $x \in \mathrm{Int}\,\mathfrak{j}$ is a line segment, varying in an $(n-3)$-dimensional family with parameter x. Since the boundaries of interstices in σ have dimension $n-4$, we may choose $x \in \mathfrak{j}$ in such a way that the intersection of L_x with any $\mathfrak{v} \in \mathscr{P}^{[n-3]}_{\mathscr{S}^{\mathrm{I}}_k}$ lies in $\mathrm{Int}\,\mathfrak{v}$. Then there exist real numbers $0 < \lambda_1 < \lambda_2 < \cdots < \lambda_s < \infty$ and joints $\mathfrak{j}_1 = \mathfrak{j}, \mathfrak{j}_2, \ldots, \mathfrak{j}_s$ with $x + \lambda\overline{m} \in \mathrm{Int}\,\mathfrak{j}_i$ for $\lambda \in (\lambda_{i-1}, \lambda_i)$, and s and λ_s maximal with this property. By the choice of x we see that $x + \lambda_i\overline{m} \in \mathfrak{j}_i \cap \mathfrak{j}_{i+1}$, for $1 \le i < s$, must be an interior point of an interstice $\mathfrak{d}_i$ intersecting $\mathrm{Int}\,\sigma$, and $\overline{m} \notin \Lambda_{\mathfrak{d}_i}$. In this situation only the two joints $\mathfrak{j}_i$, $\mathfrak{j}_{i+1}$, containing $\overline{m}$ in their tangent spaces contribute to the sum in (3.10). Thus Proposition 3.14 implies $c^m_{\mathfrak{j}_i} = c^m_{\mathfrak{j}_{i+1}}$, provided we orient the loops around $\mathfrak{j}_i$ and $\mathfrak{j}_{i+1}$ in the same way.

Inductively, we thus see that $c^m_{\mathfrak{j}} = c^m_{\mathfrak{j}_s}$. Now the maximality of λ_s implies that $x + \lambda_s\overline{m} \in \partial\mathfrak{j}_s$ is an interior point of some $\mathfrak{v} \in \mathscr{P}^{[n-3]}_{\mathscr{S}^{\mathrm{I}}_k}$. If $\mathfrak{v} \subseteq \partial B$, noting in any event that $-\overline{m} \in K_{\sigma_\mathfrak{v}}\sigma$, $\overline{m}$ cannot be in the support of the fan $\Sigma_{\sigma_\mathfrak{v}}$, which is convex. So $m \notin P_{\sigma_\mathfrak{v},\sigma}$; a contradiction. Thus $\mathfrak{v} = \mathfrak{d}$ is an interstice, and $\overline{m} \notin \Lambda_\mathfrak{d}$. If $\mathfrak{d} \not\subseteq \partial\sigma$, we can run Proposition 3.14 again to conclude that $c^m_{\mathfrak{j}} = c^m_{\mathfrak{j}_s} = 0$. If $\mathfrak{d}_s \subseteq \partial\sigma$, $\mathfrak{d}_s \not\subseteq \partial B$, Proposition 3.17(1) applies since $-\overline{m} \in \mathrm{Int}_{\sigma_{\mathfrak{d}_s}}\sigma$. Hence $c^m_{\mathfrak{j}} = c^m_{\mathfrak{j}_s} = 0$ also in this case. $\square$

3.5. *Step* II: *Homological modification of slabs.* The next step of the algorithm achieves consistency for codimension-one joints. At a single joint this can be done by modifying the functions associated to the two slabs containing this joint. There is then a problem of whether or not this can be done consistently, as changes to slabs dictated by one joint may conflict with changes to slabs dictated by another joint. Furthermore, monomials in the functions associated to slabs do not propagate in the same way monomials associated to walls do, because of (2.11). Thus it is impossible to emulate the arguments used for codimension-zero joints, and instead, we need to use homological arguments which will fix the corrections to all slabs in a given codimension-one $\rho \in \mathscr{P}$ simultaneously.

Throughout the following discussion we therefore fix $\rho \in \mathscr{P}^{[n-1]}$ and a reference cell $\sigma \in \mathscr{P}_{\max}$, $\rho \subseteq \sigma$. The discussion in this subsection will only apply to the case that ρ is bounded. To fix signs, orient σ and each joint $\mathfrak{j} \subseteq \rho$ arbitrarily. This distinguishes a sense of orientation of each loop around any joint in ρ that we tacitly assume in the following. Let $\check{d}_\rho \in \Lambda^\perp_\rho \simeq \mathbb{Z}$

be the generator that is positive on σ. If $\mathfrak{j}$ is a joint with $\operatorname{Int}\mathfrak{j} \subseteq \rho$, then by (3.7) the log automorphism $\theta_\mathfrak{j}^k$ of $R^k_{\mathrm{id}_\rho,\sigma}$ associated to a loop around $\mathfrak{j}$ lies in $O^k\big(\Lambda_\mathfrak{j}\otimes\Lambda_\mathfrak{j}^\perp + (\Lambda_\rho/f_\rho)\otimes\Lambda_\rho^\perp\big)$. Thus for any vertex $v\in\sigma_\mathfrak{j}=\rho$, we may write

$$\theta_\mathfrak{j}^k = \exp\Big(\textstyle\sum_i c_i z^{m_i}\partial_{n_i} + \dfrac{\sum_i d_i z^{m_i'}}{f_{\rho,v}}\partial_{\check{d}_\rho}\Big)$$

with $m_i\in\Lambda_\mathfrak{j}$, $m_i'\in\Lambda_\rho$ and $\operatorname{ord}_\rho(m_i)=\operatorname{ord}_\rho(m_i')=k$. The expression $\sum_i d_i z^{m_i'}$ is unique up to adding multiples of $f_{\rho,v}$. Note that in view of (3.3) a different choice v' of vertex leads to the expression $\sum_i d_i z^{m_i'+m^\rho_{v'v}}$.

Thus $\sum_i d_i z^{m_i'}$ defines a well-defined element $d_\mathfrak{j}=(d_{\mathfrak{j},v})_v$ in the $\Bbbk$-vector space W_ρ that is defined as follows. For a vertex $v\in\rho$ let

$$W_{\rho,v} := \widetilde{W}_{\rho,v}/\big(\widetilde{W}_{\rho,v}\cdot f_{\rho,v}\big)$$

with

$$\widetilde{W}_{\rho,v} := \Big\{\textstyle\sum_i a_i z^{m_i}\in\Bbbk[P_{\rho,\sigma}]\,\Big|\,\overline{m_i}\in\Lambda_\rho,\ \operatorname{ord}_\rho(m_i)=k\Big\}.$$

Note that $\widetilde{W}_{\rho,v}\cdot f_{\rho,v}\subseteq\widetilde{W}_{\rho,v}$ because $f_{\rho,v}$ involves only exponents m with $\operatorname{ord}_\rho(m)=0$ and $\overline{m}\in\Lambda_\rho$. For another vertex $v'\in\rho$ we have the isomorphism

$$W_{\rho,v}\longrightarrow W_{\rho,v'},\quad h\longmapsto h\cdot z^{m^\rho_{v'v}}.$$

Then W_ρ is defined as the set of tuples $(h_v)_{v\in\rho}$ with $h_v\in W_{\rho,v}$ and

$$h_{v'} = z^{m^\rho_{v'v}}\cdot h_v.$$

The plan is now to achieve $d_\mathfrak{j}=0$ by changing the slabs contained in ρ. Fix a vertex $v\in\rho$ and let $\mathfrak{B}\subseteq P_{\rho,\sigma}$ be a set of exponents such that $\big(z^{m^\rho_{v'v}+m}\big)_{v'}$, $m\in\mathfrak{B}$, forms a basis of W_ρ. For $m\in\mathfrak{B}$ and $x\in\rho\setminus\Delta$, write $m[x]$ for the parallel transport of $m+m^\rho_{v'v}$ to x inside $\rho\setminus\Delta$, where $v'=v[x]\in\rho$. Thus we can write

$$d_{\mathfrak{j},v'} = \sum_{m\in\mathfrak{B}} d_\mathfrak{j}^m z^{m[v']} \tag{3.15}$$

for some $d_\mathfrak{j}^m\in\Bbbk$. We now follow a procedure to adjust the functions $f_{\mathfrak{b},x}$, for $\mathfrak{b}\subseteq\rho$ a slab, by multiples of $z^{m[x]}$ for a single $m\in\mathfrak{B}$. Write $\mathscr{P}_\rho$ for the polyhedral decomposition of ρ given by $\mathscr{P}_{\mathscr{S}_k^\mathrm{I}}$. For the function $f_{\mathfrak{b},x}$ of a slab $\mathfrak{b}\subseteq\rho$ to receive a correction by a multiple of $z^{m[x]}$, we require that

(1) $m[x]\in P_x$,
(2) for any joint $\mathfrak{j}\subseteq\mathfrak{b}$ with $x\in\mathfrak{j}$ and any $\sigma'\in\mathscr{P}_{\max}$ with $\sigma'\supseteq\mathfrak{j}$ it holds that $\operatorname{ord}_{\sigma'}(m[x])\geq k$.

The second condition ensures that such a correction does not influence lower order computations. Thus we consider only the polyhedral complex $\mathscr{P}_m \subseteq \mathscr{P}_\rho$ defined as the complement of the open star of

$$\Omega := \left\{x \in \partial\rho \,\middle|\, m[x] \notin P_x\right\} \cup \textstyle\bigcup_{\mathfrak{j}} \operatorname{Int}\mathfrak{j} \subseteq \partial\rho, \tag{3.16}$$

where the union runs over all joints $\mathfrak{j} \subseteq \partial\rho$ such that $\operatorname{ord}_{\sigma'}(m[x]) < k$ for some $\sigma' \in \mathscr{P}_{\max}$ containing $\mathfrak{j}$ and $x \in \mathfrak{j} \setminus \Delta$. Said differently, $\mathscr{P}_m$ consists of all cells of $\mathscr{P}_\rho$ not intersecting Ω.

On the other hand, a codimension-two joint $\mathfrak{j} \subseteq \partial\rho$ does not impose any conditions on changing a slab function $f_{\mathfrak{b},x}$, $\mathfrak{b} \supseteq \mathfrak{j}$, by $cz^{m[x]}$ if (i) $\operatorname{ord}_{\mathfrak{j}}(m[x]) > k$, or (ii) $\overline{m[x]} \in \Lambda_{\mathfrak{j}}$, $x \in (\operatorname{Int}\mathfrak{j}) \setminus \Delta$. In fact, Proposition 3.10(1) shows that such changes keep the form (3.7) for $\theta^k_{\mathfrak{D}_\mathfrak{j}}$. We therefore work relative to the subcomplex $\widetilde{\mathscr{A}_m} \subseteq \mathscr{P}_\rho$ consisting of faces of $(n-2)$-cells $\mathfrak{j} \subseteq \partial\rho$ of $\mathscr{P}_\rho$ obeying (i) or (ii). ($\widetilde{\mathscr{A}_m}$ may include $(n-2)$-cells $\mathfrak{j} \subseteq \partial B$, which are not joints, but these do not impose conditions anyway.) Note that by Proposition 2.6, an $(n-2)$-cell $\mathfrak{j} \subseteq \partial\rho$ of $\mathscr{P}_\rho$ is contained in $\widetilde{A}_m$ if and only if $\overline{m}$ is contained in the half-plane tangent wedge to $\mathfrak{j}$ in ρ. In particular, the underlying topological space $\widetilde{A}_m \subseteq \partial\rho$ of $\widetilde{\mathscr{A}_m}$ is a union of facets of ρ, and an alternative description of $\widetilde{A}_m$ is

$$\widetilde{A}_m = \operatorname{cl}\left\{x \in \partial\rho \setminus \Delta \,\middle|\, \overline{m[x]} \in K_x\rho\right\}. \tag{3.17}$$

Finally denote

$$\mathscr{A}_m := \widetilde{\mathscr{A}_m} \cap \mathscr{P}_m.$$

Note that $\widetilde{A}_m \cap \Omega$ is contained in the relative boundary of $\widetilde{A}_m$, and hence $\mathscr{A}_m$ is also obtained by removing the open star of a subset of $\partial\widetilde{\mathscr{A}_m}$. In fact, if $\mathfrak{v} \in \mathscr{P}_\rho$ is contained in the relative interior of $\widetilde{A}_m$, then $\overline{m[x]} \in K_x\rho$ for all $x \in \mathfrak{v} \setminus \Delta$. This implies $\operatorname{ord}_{\sigma'}(m[x]) \geq \operatorname{ord}_\rho(m[x]) = k$ for all $x \in \mathfrak{v} \setminus \Delta$ and $\sigma' \in \mathscr{P}_{\max}$ containing x, and hence $\mathfrak{v} \in \mathscr{P}_m$.

Our interest in $(\mathscr{P}_m, \mathscr{A}_m)$ comes from the following result.

Lemma 3.19. *The cellular $(n-2)$-chain $(d^m_\mathfrak{j})_{\mathfrak{j} \in \mathscr{P}_m^{[n-2]}}$ with $d^m_\mathfrak{j} = 0$ for $\mathfrak{j} \subseteq \partial\rho$ and as in* (3.15) *otherwise, is a relative cycle for $(\mathscr{P}_m, \mathscr{A}_m)$.*

Proof. Orient each interstice $\mathfrak{d} \subseteq \rho$ arbitrarily. Then for any $\mathfrak{d} \subseteq \mathfrak{j} \subseteq \rho$, the comparison of the chosen orientation of $\mathfrak{d}$ with the one induced from $\mathfrak{j}$ defines a sign $\operatorname{sgn}(\mathfrak{d}, \mathfrak{j}) \in \{\pm 1\}$ such that the coefficient of $\mathfrak{d}$ in the boundary of a cellular $(n-2)$-chain $(c_\mathfrak{j})_\mathfrak{j}$ is

$$\sum_{\mathfrak{j} \supseteq \mathfrak{d}} \operatorname{sgn}(\mathfrak{d}, \mathfrak{j}) c_\mathfrak{j}.$$

Now let $\mathfrak{v} \in \mathscr{P}_m^{[n-3]}$. Then either $\mathfrak{v} \subseteq \partial B$ or $\mathfrak{v} = \mathfrak{d}$ is an interstice. In the first case, if $\mathfrak{v} \notin \mathscr{A}_m$, then by (3.17), $\overline{m[x]} \notin K_x\rho$ for x in a neighbourhood of $\mathfrak{d}$ in $\partial\rho$. In particular, $-\overline{m[x]} \in \operatorname{Int} K_{\sigma_\mathfrak{v}}\rho$, but also $\overline{m[x]}$ maps to $|\Sigma_{\sigma_\mathfrak{v}}|$ since

$m[x] \in P_x$. This contradicts convexity of $|\Sigma_{\sigma_{\mathfrak{v}}}|$. Thus $\mathfrak{v} \in \mathscr{A}_m$ and there is nothing to check.

In the case of an interstice with $\mathfrak{d} \subseteq \partial\rho$ and $\mathfrak{d} \notin \mathscr{A}_m$, we have $-\overline{m[x]} \in \operatorname{Int} K_{\sigma_{\mathfrak{d}}}\rho$ as before. Proposition 3.17(2a) now shows $\sum_{\mathfrak{j}\supseteq\mathfrak{d},\, \mathfrak{j}\notin\mathscr{A}_m} \operatorname{sgn}(\mathfrak{d},\mathfrak{j}) d^m_{\mathfrak{j}} = 0$. The sign arises from the difference in orientation conventions for loops around joints.

If $\mathfrak{d} \not\subseteq \partial\rho$, Proposition 3.17(2b) implies $\sum_{\mathfrak{j}\supseteq\mathfrak{d}} \operatorname{sgn}(\mathfrak{d},\mathfrak{j}) d^m_{\mathfrak{j}} = 0$, again observing the different orientation conventions. □

We now prove two lemmas concerning the topology of this situation.

Lemma 3.20. *The pair $(\mathscr{P}_m, \mathscr{A}_m)$ is a deformation retract of $(\mathscr{P}_\rho, \widetilde{\mathscr{A}_m})$.*

Proof. We want to retract the cells in $\mathscr{P}_\rho \setminus \mathscr{P}_m$ successively, in a way compatible with $\widetilde{\mathscr{A}_m}$. For this we use the following elementary result. If $\Xi \subseteq \mathbb{R}^k$ is a bounded convex polytope and $x \in \partial\Xi$, then the projection from a point $x' \in \mathbb{R}^k \setminus \Xi$ sufficiently close to x and with $x - x' \in K_x\Xi$ defines a deformation retraction of Ξ onto the union of facets of Ξ not containing x. Explicitly, for $y \in \Xi$, define

$$\alpha(y) = \max\{\alpha \in \mathbb{R}_{\geq 0} \,|\, y + \alpha \cdot (y - x') \in \Xi\}.$$

Then

$$[0,1] \times \Xi \longrightarrow \Xi, \quad (\lambda, y) \longmapsto y + \lambda\alpha(y)(y - x')$$

is the desired deformation retraction.

We apply this result first to successively retract $(\mathscr{P}_\rho, \widetilde{\mathscr{A}_m})$ to $(\mathscr{P}_m \cup \widetilde{\mathscr{A}_m}, \widetilde{\mathscr{A}_m})$. Let $\mathscr{P}' \subseteq \mathscr{P}_\rho$ be a subcomplex obtained inductively. Let $\mathscr{P}'_\partial \subseteq \mathscr{P}'$ consist of subcells of cells $\mathfrak{w} \in \mathscr{P}' \setminus (\mathscr{P}_m \cup \widetilde{\mathscr{A}_m})$ with the property that there is a unique $\mathfrak{v} \in \mathscr{P}' \setminus (\mathscr{P}_m \cup \widetilde{\mathscr{A}_m})$ with $\mathfrak{w} \subsetneq \mathfrak{v}$. This $\mathscr{P}'_\partial$ is the subset of cells that can be taken as center for the next retraction. We will assume inductively that $\mathscr{P}'_\partial \neq \emptyset$ as long as $\mathscr{P}' \neq \mathscr{P}_m \cup \widetilde{\mathscr{A}_m}$. To see this is true initially, note first that if $(\mathscr{P}_m \cup \widetilde{\mathscr{A}_m}) \cap \partial\mathscr{P}_\rho = \partial\mathscr{P}_\rho$, then $\mathscr{P}_m = \mathscr{P}_\rho$ and $\widetilde{\mathscr{A}_m} = \partial\mathscr{P}_\rho$ anyway and there is nothing to do. Otherwise, there is a slab $\mathfrak{b} \in \mathscr{P}_\rho \setminus (\mathscr{P}_m \cup \widetilde{\mathscr{A}_m})$ with $\dim \mathfrak{b} \cap \partial\rho = n - 1$, and then $\mathfrak{b} \cap \partial\rho \in \mathscr{P}'_\partial$.

Given $\mathscr{P}'_\partial \neq \emptyset$, choose a point x in the interior of a maximal cell $\mathfrak{w} \in \mathscr{P}'_\partial$, contained properly in a unique cell $\mathfrak{v} \in \mathscr{P}' \setminus (\mathscr{P}_m \cup \widetilde{\mathscr{A}_m})$. Now apply the above deformation retraction of $\mathfrak{v}$ using the chosen x. Since x is disjoint from any cell of $\mathscr{P}_m \cup \widetilde{\mathscr{A}_m}$, this deformation retraction is trivial on this subcomplex of $\mathscr{P}'$. We now note that after making this retraction, we continue to have $\mathscr{P}'_\partial \neq \emptyset$. In fact, if $\tilde{\mathfrak{w}} \in \mathscr{P}' \setminus (\mathscr{P}_m \cup \widetilde{\mathscr{A}_m})$, then $\mathfrak{w} := \tilde{\mathfrak{w}} \cap \Omega$ is a nonempty cell of $\mathscr{P}_\rho$. Moreover, by the inductive construction, the link of $\mathfrak{w}$ in $\mathscr{P}'$ is a retraction of the link of $\mathfrak{w}$ in $\mathscr{P}_\rho$. From this one can see that the link of $\mathfrak{w}$ contains a cell in $\mathscr{P}'_\partial$, and hence $\mathscr{P}'_\partial$ continues to be nonempty. The process stops when $\mathscr{P}' = \mathscr{P}_m \cup \widetilde{\mathscr{A}_m}$. An analogous argument then retracts $(\widetilde{\mathscr{A}_m}, \mathscr{A}_m)$ onto $(\mathscr{A}_m, \mathscr{A}_m)$, and hence $(\mathscr{P}_m \cup \widetilde{\mathscr{A}_m}, \widetilde{\mathscr{A}_m})$ onto $(\mathscr{P}_m, \mathscr{A}_m)$. $\square$

LEMMA 3.21. $H_{n-2}(\mathscr{P}_m, \mathscr{A}_m) = 0$.

Proof. By Lemma 3.20 this follows once we prove $H_{n-2}(\mathscr{P}_\rho, \widetilde{\mathscr{A}_m}) = 0$. Let $\Delta(\rho) \subseteq \Lambda_{\rho,\mathbb{R}}$ be the convex hull of $\{m^\rho_{vv'} \,|\, v' \in \rho \text{ vertex}\}$ and $\psi_{\check\rho}$ be the corresponding PL-function on the normal fan $\check\Sigma_\rho$ of ρ. Recall from (3.17) that $\widetilde{A}_m$ is the union of facets $\tau \subseteq \rho$ with $\overline{m[x]} \in K_x\rho$ for some $x \in \operatorname{Int}\tau$. If $n \in \Lambda^*_\rho$ is the inward normal vector to τ generating the ray in $\check\Sigma_\rho$ dual to τ, this is equivalent to

$$0 \le \langle \overline{m[x]}, n\rangle = \langle \overline{m}, n\rangle + \langle m^\rho_{v[x]v}, n\rangle = \langle \overline{m}, n\rangle + \psi_{\check\rho}(n).$$

Thus $\widetilde{A}_m \subseteq \partial\rho$ is dual to the subset of rays of $\Sigma_{\check\rho}$ on which the convex function $\psi_{\check\rho} + \overline{m}$ is nonnegative. Thus if $\overline{m} \in \Delta(\rho)$, we obtain $\widetilde{A}_m = \partial\rho$, and otherwise ρ deformation retracts to $\widetilde{A}_m$. In any case it follows that $H_{n-2}(\mathscr{P}_\rho, \widetilde{\mathscr{A}_m}) = H_{n-2}(\rho, \widetilde{A}_m) = 0$. $\square$

II.1. *First homological modification of slabs.* By Lemma 3.21 we can find an $(n-1)$-chain $(b^m_{\mathfrak{b}})_{\mathfrak{b}\in\mathscr{P}_m^{[n-1]}}$ whose boundary is $(d^m_{\mathfrak{j}})_{\mathfrak{j}}$ modulo chains in $\mathscr{A}_m$. Then for any slab $\mathfrak{b} \subseteq \rho$, subtract the term $D(s_e,\rho,v)s_e(b^m_{\mathfrak{b}} z^{m[x]})$ from $f_{\mathfrak{b},x}$, where $v = v[x]$ and $e : v \to \rho$. By construction of $\mathscr{P}_m$ we have $m[x] \in P_x$ for all $x \in \mathfrak{b} \setminus \Delta$ and the change of vertex formula (2.11) continues to hold, so this makes sense. Proposition 3.10(1) shows that doing so removes the terms $d^m_{\mathfrak{j}} z^{m[v]}/f_{\rho,v}$ from $\theta^k_{\mathfrak{j}}$, whenever $\mathfrak{j} \in \mathscr{P}_m$, $\mathfrak{j} \not\subseteq \partial\rho$. Furthermore, if $m[v]$ does appear in $\theta^k_{\mathfrak{j}}$ for some joint $\mathfrak{j} \subseteq \rho$, $\mathfrak{j} \not\subseteq \partial\rho$, then $\mathfrak{j} \in \mathscr{P}_m$, so this process removes the term involving $m[v]$ from $\theta^k_{\mathfrak{j}}$ whenever such a term appears.

Repeat this for all exponents $m \in \mathfrak{B} = \mathfrak{B}(\rho)$ and for all $\rho \in \mathscr{P}^{[n-1]}$.

For $\mathfrak{j} \subseteq \rho$ with $\operatorname{codim}\sigma_{\mathfrak{j}} = 1$, we can now write

$$\theta^k_{\mathfrak{j}} = \exp\left(\sum c_i z^{m_i}\partial_{n_i}\right)$$

with $m_i \in \Lambda_\rho$, $n_i \in \Lambda^\perp_{\mathfrak{j}}$. Next we would like to achieve $\theta^k_{\mathfrak{j}} \in O^k(\Lambda_{\mathfrak{j}} \otimes \Lambda^\perp_{\mathfrak{j}})$. This is possible by a further, straightforward modification of slabs.

II.2. *Further subdivision of slabs to achieve $\theta^k_{\mathfrak{j}} \in O^k(\Lambda_{\mathfrak{j}} \otimes \Lambda^\perp_{\mathfrak{j}})$.* For every $\mathfrak{j}$ with $\operatorname{codim}\sigma_{\mathfrak{j}} = 1$ and every m_i appearing in $\theta^k_{\mathfrak{j}}$ with $\overline{m_i} \notin \Lambda_{\mathfrak{j}}$, we note that n_i must be proportional to $\check d_\rho$, and thus we can assume after changing c_i that $n_i = \check d_\rho$. Viewing $\mathfrak{j} \subseteq \rho$ as a subset of $\Lambda_{\rho,\mathbb{R}}$, define

$$\mathfrak{b}(m_i) := (\mathfrak{j} - \mathbb{R}_{\ge 0}\overline{m_i}) \cap \rho \subseteq \rho.$$

We then modify slabs contained in ρ by adding $\pm c_i s_e(m_i) z^{m_i} f_e$ to $f_{\mathfrak{b},y}$ for $y \in \mathfrak{b}(m_i) \cap \mathfrak{b} \setminus \Delta$, $\mathfrak{b} \subseteq \rho$ a slab and $e : v[y] \to \sigma_{\mathfrak{j}} = \rho$. This of course might mean subdividing the slabs further. By doing so, it follows again from Proposition 3.10(1) that, with proper choice of sign, the term $c_i z^{m_i} \partial_{n_i}$ disappears from $\theta_{\mathfrak{j}}^k$. Note that this process might introduce new joints $\mathfrak{j}'$ contained in the sides of $\mathfrak{b}(m_i)$. But for such $\mathfrak{j}'$, $\overline{m_i'} \in \Lambda_{\mathfrak{j}'}$, and so by a simple calculation analogous to Proposition 3.10(2), $\theta_{\mathfrak{j}'}^k$ satisfies

$$\theta_{\mathfrak{j}'}^k = \exp\left(\sum c_i z^{m_i} \partial_{n_i}\right)$$

with $\overline{m_i} \in \Lambda_{\mathfrak{j}'}$. In fact, after carrying this out for every joint $\mathfrak{j}$ with $\operatorname{codim} \sigma_{\mathfrak{j}} = 1$, we see that this now holds for all joints with $\operatorname{codim} \sigma_{\mathfrak{j}} = 1$. We write $\mathscr{S}_k^{\mathrm{II,pre}}$ for the structure thus obtained.

The arguments of Proposition 3.18 for codimension-zero joints now imply that the remaining terms of $\theta_{\mathfrak{j}}^k$ are undirectional. Recall that a log automorphism lies in $O^k(0 \otimes \Lambda_{\mathfrak{j}}^{\perp})$ if it is of the form $\exp(\sum_n a_n \partial_n)$ with $a_n \in \Bbbk[t]$ and $n \in \Lambda_{\mathfrak{j}}^{\perp}$.

Proposition 3.22. *For $\mathfrak{j} \in \operatorname{Joints}(\mathscr{S}_k^{\mathrm{II}})$ with $\sigma_{\mathfrak{j}} = \rho \in \mathscr{P}^{[n-1]}$, it holds that*

$$\theta_{\mathfrak{j}}^k \in O^k(0 \otimes \Lambda_\rho^{\perp}).$$

Proof. The construction of $\mathscr{S}_k^{\mathrm{II,pre}}$ was designed to achieve $\theta_{\mathfrak{j}}^k \in O^k(\Lambda_{\mathfrak{j}} \otimes \Lambda_{\mathfrak{j}}^{\perp})$ for codimension-one joints. It then follows exactly as in Proposition 3.18 that there are no contributions of exponents m with $\overline{m} \neq 0$. In this argument Proposition 3.17(2) replaces both Proposition 3.14 and Proposition 3.17(1). Thus $\theta_{\mathfrak{j}}^k \in O^k(0 \otimes \Lambda_{\mathfrak{j}}^{\perp})$.

To see furthermore that even $\theta_{\mathfrak{j}}^k \in O^k(0 \otimes \Lambda_\rho^{\perp})$, we have to show that $\theta_{\mathfrak{j}}^k(m) = 1$ for all $m \in \Lambda_\rho$. This follows easily with the notion of tlog that comes out naturally of our discussion of the higher order normalization procedure in Step III. We therefore postpone the rest of the proof to Section 3.6; see the discussion after Lemma 3.29. □

To remove the remaining undirectional terms we now run a homological argument again. For $\rho \in \mathscr{P}^{[n-1]}$ and a joint $\mathfrak{j} \in \mathscr{S}_k^{\mathrm{II,pre}}$ with $\sigma_{\mathfrak{j}} = \rho$, Proposition 3.22 shows we can write uniquely

$$\theta_{\mathfrak{j}}^k = \exp\left(c_{\mathfrak{j}} t^k \partial_{\check{d}_\rho}\right)$$

for some $c_{\mathfrak{j}} \in \Bbbk$. Let $\mathfrak{d}$ be an interstice of $\mathscr{S}_k^{\mathrm{II,pre}}$ with $\sigma_{\mathfrak{d}} = \rho$ and $\mathfrak{j}_1, \ldots, \mathfrak{j}_r \subseteq \rho$ be the codimension-one joints containing $\mathfrak{d}$. Then the $\theta_{\mathfrak{j}_i}^k$ commute and (3.9), now interpreted for $\mathscr{S}_k^{\mathrm{II,pre}}$, implies

$$1 = \theta^k_{\mathfrak{j}_r} \circ \cdots \circ \theta^k_{\mathfrak{j}_1} = \exp\Big(\sum_{i=1}^r \mathrm{sgn}(\mathfrak{d}, \mathfrak{j}_i) c_{\mathfrak{j}_i} t^k \partial_{\check{d}_\rho}\Big).$$

Here we use the signs $\mathrm{sgn}(\mathfrak{d}, \mathfrak{j}_i)$ which were introduced above. This shows that $\sum_{i=1}^r \mathrm{sgn}(\mathfrak{d}, \mathfrak{j}_i) c_{\mathfrak{j}_i} = 0$, and hence $(c_\mathfrak{j})_{\mathfrak{j}\in\mathscr{P}_\rho^{n-2}}$ can be viewed as a relative cellular $(n-2)$-cycle for $(\mathscr{P}_\rho, \partial\mathscr{P}_\rho)$. Again we simply set $c_\mathfrak{j} = 0$ whenever $\mathfrak{j} \subseteq \partial\mathscr{P}_\rho$.

II.3. *Second homological modification of slabs.* Since $H_{n-2}(\mathscr{P}_\rho, \partial\mathscr{P}_\rho) = 0$ there exists an $(n-1)$-chain $(b_\mathfrak{b})_{\mathfrak{b}\in\mathscr{P}_\rho^{n-1}}$ with boundary $(c_\mathfrak{j})_\mathfrak{j}$. Then for a slab $\mathfrak{b} \subseteq \rho$ and $x \in \mathfrak{b}\setminus\Delta$, add $b_\mathfrak{b} t^k f_e$ to $f_{\mathfrak{b},x}$ for $v := v[x]$ and $e : v[x] \to \rho$. Proposition 3.10(1) now shows that after these changes, $\theta^k_\mathfrak{j} = 1$ holds for any codimension one-joint $\mathfrak{j} \subseteq \rho$. Repeat this process for all $\rho \in \mathscr{P}^{[n-1]}$. The structure thus obtained is denoted as $\mathscr{S}_k^{\mathrm{II}}$.

We have now arrived at a structure $\mathscr{S}_k^{\mathrm{II}}$ that is consistent up to codimension one. Moreover, for codimension-two joints essentially the same arguments together with local rigidity (Definition 1.26) give further restrictions.

PROPOSITION 3.23. *Let* $\mathfrak{j} \in \mathrm{Joints}(\mathscr{S}_k^{\mathrm{II}})$.

(1) *If* $\mathrm{codim}\,\sigma_\mathfrak{j} \le 1$, *then* $\theta^k_\mathfrak{j} = 1$.
(2) *If* $\mathrm{codim}\,\sigma_\mathfrak{j} = 2$, *let* $\tau = \sigma_\mathfrak{j}$ *and* $v \in \tau$ *a vertex. Then we can write*

$$\theta^k_\mathfrak{j} = \exp\Big(ct^k\partial_n + \sum_{\substack{\rho\supseteq\mathfrak{j}\\ v'\in\tau}} c_{\rho,v'} t^k \frac{z^{m^\rho_{vv'}}}{f_{\rho,v}} \partial_{\check{d}_\rho}\Big)$$

with $c, c_{\rho,v'} \in \Bbbk$ *and* $n \in \Lambda_\tau^\perp$.

Proof. (1) As we have not changed anything at joints $\mathfrak{j}$ of $\mathscr{S}_k^{\mathrm{I}}$ with $\mathrm{codim}\,\sigma_\mathfrak{j} = 0$ this case follows from Proposition 3.18, while the constructions in Step II were designed to achieve $\theta^k_\mathfrak{j} = 1$ if $\mathrm{codim}\,\sigma_\mathfrak{j} = 1$.

(2) By the definition of the polyhedral complex $\mathscr{P}_m \subseteq \mathscr{P}_\rho$ and Proposition 3.10(1), the changes from $\mathscr{S}_k^{\mathrm{I}}$ to $\mathscr{S}_k^{\mathrm{II}}$ do not affect the form (3.7) of $\theta^k_\mathfrak{j}$ at codimension-two joints.

Now note that the joints $\mathfrak{j} \subseteq \tau$ are the maximal cells of the polyhedral decomposition of τ given by $\mathscr{P}_{\mathscr{S}_k^{\mathrm{II}}}$. Thus if $\mathfrak{d}$ is an interstice with $\sigma_\mathfrak{d} = \tau$ there are precisely two codimension-two joints $\mathfrak{j}, \mathfrak{j}' \subseteq \tau$ containing $\mathfrak{d}$. In this case Proposition 3.17(3)(a) shows that $\theta^k_\mathfrak{j} \circ \big(\theta^k_{\mathfrak{j}'}\big)^{-1} = 1$, assuming the normal spaces $\mathcal{Q}^v_{\mathfrak{j},\mathbb{R}} = \mathcal{Q}^v_{\tau,\mathbb{R}} = \mathcal{Q}^v_{\mathfrak{j}',\mathbb{R}}$ are oriented in the same way. Thus all $\theta^k_\mathfrak{j}$ with $\mathfrak{j} \subseteq \tau$ agree. Thus there exist $c_j, d_{\rho,j} \in \Bbbk$ and $m_j, m'_{\rho,j}$ with $\overline{m_j}, \overline{m'_{\rho,j}} \in \Lambda_\tau$,

$\mathrm{ord}_\tau(m_j) = \mathrm{ord}_\tau(m'_{\rho,j}) = k$, $n_j \in \Lambda_\tau^\perp$, such that for any $\mathfrak{j} \subseteq \tau$,

$$\theta_{\mathfrak{j}}^k = \exp\Big(\sum_j c_j z^{m_j}\partial_{n_j} + \sum_{j,\,\rho\supseteq\tau} d_{\rho,j}\frac{z^{m'_{\rho,j}}}{f_{\rho,v}}\partial_{\check{d}_\rho}\Big).$$

Now by looking at an interstice $\mathfrak{d} \subseteq \partial\tau$ containing v, Proposition 3.17(3)(b) implies that we may assume all $\overline{m_j}$ and $\overline{m'_{\rho,j}}$ to be contained in the half-plane tangent wedge $K_{\mathfrak{d}}\tau$. Indeed, otherwise their contribution vanishes by the proposition. Taking a different vertex v' transforms $z^{m'_{\rho,j}}/f_{\rho,v}$ into $z^{m'_{\rho,j}-m^\rho_{vv'}}/f_{\rho,v'}$. Thus we may take $c_j = 0$ for all j, and $d_{\rho,j} = 0$ unless $\overline{m'_{\rho,j}} - m^\rho_{vv'} \in K_{v'}\tau$ for every vertex $v' \in \tau$. As in the proof of Lemma 3.21 one sees that, in the latter case, $\overline{m'_{\rho,j}}$ must be contained in the convex hull of $\{m^\rho_{vv'} \,|\, v' \in \tau \text{ vertex}\}$. This is a face of $\Delta(\rho)$, and by Definition 1.26(i) any integral point of this face is a vertex. Hence $\overline{m'_{\rho,j}} = m^\rho_{vv'}$ for some $v' \in \tau$. Because $\mathrm{ord}_\tau(m'_{\rho,j}) = k$, it then follows that

$$z^{m'_{\rho,j}} = t^k \cdot z^{m^\rho_{vv'}}.$$

This proves the claimed formula for $\theta^k_{\mathfrak{j}}$. □

3.6. *Step* III: *Normalization.* For a joint $\mathfrak{j}$ of $\mathscr{S}_k^{\mathrm{II}}$ the remaining terms in $\theta^k_{\mathfrak{j}}$ do not propagate — either they are undirectional (z^m with $\overline{m} = 0$), or they are of the form $z^m/f_{\rho,v}$ and $-\overline{m}$ points into the tangent wedge of ρ at v for any choice of vertex $v \in \rho$. Step III removes these terms by a normalization procedure.

The normalization condition asks that there be no pure t-terms in the logarithm of the functions $s_e^{-1}(f_{\mathfrak{b},x})$ that occur in changing chambers separated by a slab $\mathfrak{b}$, up to order k. Because this expression may contain exponents m with $\mathrm{ord}_\rho(m) = 0$, we need to take appropriate completions of our rings $R^k_{g,\sigma}$ to make sense of the logarithm.

Again, in this section, we assume all cells of B are bounded.

Construction 3.24. Let $(g : \omega \to \tau) \in \mathrm{Hom}(\mathscr{P})$, $\sigma \in \mathscr{P}_{\max}$, a reference cell containing τ and $v \in \tau$ a vertex. Because $K_v\tau$ is a strictly convex cone, the subset

$$E = \Big\{m \in P_{\omega,\sigma} \,\Big|\, \overline{m} \in K_v\tau \setminus \{0\}\Big\}$$

of $P_{\omega,\sigma}$ is additively closed and

$$\bigcap_{\nu\geq 0} \nu E = \emptyset.$$

We can thus define a Hausdorff topology on $\Bbbk[P_{\omega,\sigma}]$ by taking

$$U_\nu := \Big\{ \textstyle\sum_{m\in\nu E} a_m z^m \in \Bbbk[P_{\omega,\sigma}]\Big\}, \qquad \nu \geq 1$$

as fundamental neighbourhood system of 0. Note that if $\Lambda_\omega \cap (K_v\tau \setminus \{0\}) \neq \emptyset$, then E generates the unit ideal, and hence this is not the I-adic topology for any ideal I. Denote the completion of this topological ring by $\Bbbk[\![P_{\omega,\sigma}]\!]_v$.

Similarly, one defines a topology on $R^k_{g,\sigma}$, with associated completion ${}^v\widehat{R}^k_{g,\sigma}$. Because the localizing functions $f_{\rho,v}$ are invertible in $\Bbbk[\![P_{\omega,\sigma}]\!]_v/\hat{I}^{>k}_{g,\sigma}$, for any $\rho \supseteq \tau$, we have

$${}^v\widehat{R}^k_{g,\sigma} = \Bbbk[\![P_{\omega,\sigma}]\!]_v/\hat{I}^{>k}_{g,\sigma},$$

where $\hat{I}^{>k}_{g,\sigma} \subseteq \Bbbk[\![P_{\omega,\sigma}]\!]_v$ is the ideal generated by $I^{>k}_{g,\sigma} \subseteq \Bbbk[P_{\omega,\sigma}]$.

Definition 3.25. Let $f = \sum_{m\in P_{\omega,\sigma}} a_m z^m \in \Bbbk[\![P_{\omega,\sigma}]\!]_v/\hat{I}^{>k}_{g,\sigma}$.

(1) The *t-content* of f is defined by

$$\operatorname{cont}_t f := \sum_{m\in P_{\omega,\sigma}:\overline{m}=0} a_m z^m = \sum_{m\in P_{\omega,\sigma}:\overline{m}=0} a_m t^{\operatorname{ord}_\sigma(m)} \in \Bbbk[\![t]\!]/(t^{k+1}).$$

(2) We say that f *fulfills the cone condition* if $a_m \neq 0$, $\operatorname{ord}_\omega(m) = 0$, implies $\overline{m} \in K_v\tau$.

(3) If $a_0 \neq 0$ and f fulfills the cone condition, we define

$$\operatorname{tlog} f := \operatorname{cont}_t\left(-\sum_{i=1}^\infty \frac{1}{i}\Big(1-\frac{f}{a_0}\Big)^i\right) \in \Bbbk[\![t]\!]/(t^{k+1}).$$

If $f \in \Bbbk[P_{\omega,\sigma}]$, we indicate which completion to work in by writing $\operatorname{tlog}_v f$.

Remarks 3.26. The sum in the definition of $\operatorname{tlog} f$ is the power series of $\log f$ without the constant term. Thus since cont_t is additive, the usual power series identity implies that

$$\operatorname{tlog}\big(f_1\cdot f_2\big) = \operatorname{tlog} f_1 + \operatorname{tlog} f_2 \mod t^{k+1}$$

for $f_i \in \Bbbk[\![P_{\omega,\sigma}]\!]_v/\hat{I}^{>k}_{g,\sigma}$ with nonvanishing constant terms and fulfilling the cone condition.

We are now in position to formulate the normalization condition.

Definition 3.27. A slab $\mathfrak{b}$ is called *normalized to order k* if for any $x \in \mathfrak{b}\setminus\Delta$ and $v' \in \rho_{\mathfrak{b}}$ a vertex, it holds that

$$\operatorname{tlog}_{v'}\Big(z^{m^{\rho_{\mathfrak{b}}}_{v'v[x]}} s_e^{-1}(f_{\mathfrak{b},x})\Big) \in (t^{k+1}),$$

where $e: v[x] \to \rho_{\mathfrak{b}}$, and we consider $z^{m^{\rho_{\mathfrak{b}}}_{v'v[x]}} s_e^{-1}(f_{\mathfrak{b},x})$ as an element of $\Bbbk[P_{\rho_{\mathfrak{b}},\sigma}]$. A structure is *normalized to order k* if each of its slabs is normalized to order k.

The point of normalization is the following.

PROPOSITION 3.28. *Assume that $\mathfrak{j}$ is a joint of a structure $\mathscr{S}$ such that each slab $\mathfrak{b} \supseteq \mathfrak{j}$ is normalized to order k, and let $\theta^k_{\mathfrak{j}}$ be the log automorphism* *associated to a loop around* $\mathfrak{j}$ *based on* $\sigma \in \mathscr{P}_{\max}$. *Then for any* $m \in P_{\sigma_{\mathfrak{j}},\sigma}$ *and* $v \in \sigma_{\mathfrak{j}}$, *it holds that*

$$\mathrm{tlog}_v\left(\theta_{\mathfrak{j}}^k(m)\right) = 0 \mod t^{k+1}.$$

Proof. By parallel transport through v view the basic gluing morphisms associated to slabs and walls containing $\mathfrak{j}$ as log automorphisms of $R^k_{\mathrm{id}_{\sigma_{\mathfrak{j}}},\sigma}$. These all have the form

$$m \longmapsto f^{\langle \overline{m}, n\rangle}$$

for some $n \in \Lambda_{\mathfrak{j}}^{\perp} \setminus \{0\}$ and $f \in \Bbbk[P_{\sigma_{\mathfrak{j}},\sigma}]$ with $\partial_n f = 0$. Moreover, $\mathrm{tlog}_v f = 0$ for automorphisms associated to walls in any case and for slabs by the normalization condition. Note that the factors $D(s_e,\rho,v)^{-1} \in \Bbbk \setminus \{0\}$ occurring for slabs have no influence on $\mathrm{tlog}_v f$. Hence taking into account the composition formula for log morphisms (2.4) and Remark 3.26, the result follows readily from Lemma 3.29 below by induction on the number of such automorphisms. □

LEMMA 3.29. *Let* θ *be a log automorphism of* $R^k_{g,\sigma}$, $g : \omega \to \tau$, *of the form*

$$m \longmapsto f^{\langle \overline{m}, n\rangle}$$

with $n \in \Lambda^*_\sigma \setminus 0$ *and* $f \in \Bbbk[P_{\omega,\sigma}]$, $\partial_n f = 0$. *Assume that* $v \in \tau$ *is a vertex such that* $a \in R^k_{g,\sigma}$ *and* f, *viewed as elements of* $\Bbbk[\![P_{\omega,\sigma}]\!]_v$, *have nonvanishing constant terms and fulfill the cone condition. Then*

$$\mathrm{tlog}_v\left(\overline{\theta}(a)\right) = \mathrm{tlog}_v\, a \in \Bbbk[t]/(t^{k+1}).$$

Proof. Observe that

$$\overline{\theta}(z^m) = f^{\langle \overline{m}, n\rangle} \cdot z^m$$

has vanishing t-content unless $-m$ occurs as an exponent in $f^{\langle \overline{m}, n\rangle}$. But then $\partial_n f = 0$ implies $\langle \overline{m}, n\rangle = 0$, and hence $\overline{\theta}(z^m) = z^m$. This shows that for any $b \in \Bbbk[\![P_{\omega,\sigma}]\!]_v$,

$$\mathrm{cont}_t(b^i) = \mathrm{cont}_t\left(\overline{\theta}(b^i)\right) = \mathrm{cont}_t\left(\overline{\theta}(b)^i\right),$$

and hence, if b fulfills the cone condition and has vanishing constant term,

$$\mathrm{cont}_t\left(\sum\nolimits_{i\ge 1}\frac{1}{i}b^i\right) = \mathrm{cont}_t\left(\sum\nolimits_{i\ge 1}\frac{1}{i}\overline{\theta}(b)^i\right).$$

The statement follows from this by setting $b = 1 - (a/a_0)$ for $a_0 \in \Bbbk$ the constant term of a. □

With the notion of tlog at hand it is now easy to complete the proof of Proposition 3.22 left unfinished in Step II.

Proof of Proposition 3.22 — *finish.* We have seen that $\theta_{\mathfrak{j}}^k \in O^k(0 \otimes \Lambda_{\mathfrak{j}}^{\perp})$; that is,

$$\theta_{\mathfrak{j}}^k = \exp\Big(\sum_i c_i t^k \partial_{n_i}\Big)$$

with $c_i \in \Bbbk$ and $n_i \in \Lambda_{\mathfrak{j}}^{\perp}$. If $m \in P_{\sigma_{\mathfrak{j}},\sigma}$ for some $\sigma \in \mathscr{P}_{\max}$ containing $\mathfrak{j}$ and $v \in \sigma_{\mathfrak{j}}$ is a vertex, then

$$\operatorname{tlog}_v\left(\theta_{\mathfrak{j}}^k(m)\right) = \left\langle \overline{m}, \textstyle\sum_i c_i n_i \right\rangle t^k \qquad \operatorname{mod} t^{k+1}.$$

Thus $\theta_{\mathfrak{j}}^k \in O^k(0 \otimes \Lambda_\rho^\perp)$ if this expression vanishes for all m with $\overline{m} \in \Lambda_\rho$.

Now $\theta_{\mathfrak{j}}^k$ is the composition of log isomorphisms associated to walls and to two slabs contained in ρ. Arguing as in Proposition 3.28 we see that the former do not make any contribution to $\operatorname{tlog}_v\left(\theta_{\mathfrak{j}}^k(m)\right)$ for any m, while the log isomorphisms associated to the two slabs are trivial on those $m \in P_{\sigma_{\mathfrak{j}},\sigma}$ with $\overline{m} \in \Lambda_\rho$. Thus indeed $\operatorname{tlog}_v\left(\theta_{\mathfrak{j}}^k(m)\right) = 0$ for m with $\overline{m} \in \Lambda_\rho$. □

We will now impose the additional inductive assumption that $\mathscr{S}_{k-1}$ is normalized. This is an empty statement for $k = 0$. For the inductive step from $k-1$ to k note that since no terms of order $k-1$ have been added to slabs to obtain $\mathscr{S}_k^{\mathrm{II}}$, all slabs in this latter structure are also normalized to order $k-1$. It is then easy to normalize $\mathscr{S}_k^{\mathrm{II}}$ to order k:

III. *Normalization of slabs.* For any slab $\mathfrak{b} \in \mathscr{S}_k^{\mathrm{II}}$ and $x \in \mathfrak{b} \setminus \Delta$, the inductive assumption shows for any vertex $v' \in \rho_{\mathfrak{b}}$

$$\operatorname{tlog}_{v'}\left(z^{m^{\rho_{\mathfrak{b}}}_{v'v}} s_e^{-1}(f_{\mathfrak{b},x})\right) = c_{v'} t^k \qquad \operatorname{mod} t^{k+1}$$

for some $c_{v'} \in \Bbbk$. Here $v = v[x] \in \rho_{\mathfrak{b}}$ and $e: v \to \rho_{\mathfrak{b}}$. Fix a set of vertices $\mathscr{V}_{\mathfrak{b}}$ of $\rho_{\mathfrak{b}}$ such that

$$\mathscr{V}_{\mathfrak{b}} \longrightarrow \left\{ m^{\rho_{\mathfrak{b}}}_{v'v} \,\middle|\, v' \in \rho_{\mathfrak{b}} \text{ vertex} \right\}, \quad v' \longmapsto m^{\rho_{\mathfrak{b}}}_{v'v}$$

is a bijection. Now replace $f_{\mathfrak{b},x}$ by

$$f_{\mathfrak{b},x} - \sum_{v' \in \mathscr{V}_{\mathfrak{b}}} c_{v'} t^k s_e\left(z^{-m^{\rho_{\mathfrak{b}}}_{v'v}}\right).$$

Noting that $f_{\mathfrak{b},x}$ already contains the monomial $z^{-m^{\rho_{\mathfrak{b}}}_{v'v}} = z^{m^{\rho_{\mathfrak{b}}}_{vv'}}$ for each $v' \in \rho_{\mathfrak{b}}$, as follows from Equations (1.10) and (1.13), we must have $-m^{\rho_{\mathfrak{b}}}_{v'v} \in P_x$. Thus the new collection $\{f_{\mathfrak{b},x} \,|\, x \in \mathfrak{b} \setminus \Delta\}$ satisfies the definition of a slab. Note that $c_{v'}$ for $v' \in \rho_{\mathfrak{b}}$ depends only $m^{\rho_{\mathfrak{b}}}_{v'v}$. This shows that the new $f_{\mathfrak{b},x}$ is independent of the particular choice of representative vertices $\mathscr{V}_{\mathfrak{b}}$. By construction the slab $\mathfrak{b}$ is now normalized to order k.

After modifying each slab in $\mathscr{S}_k^{\mathrm{II}}$ in this way, we obtain $\mathscr{S}_k^{\mathrm{III}}$.

We can now complete the proof of Theorem 3.1.

Proposition 3.30. *The structure $\mathscr{S}_k^{\mathrm{III}}$ is consistent to order k.*

Proof. Since the normalization procedure does not change walls, Proposition 3.23(1) still shows consistency to order k for codimension-zero joints.

If $\operatorname{codim}\sigma_{\mathfrak{j}} = 2$, local rigidity (Definition 1.26) provides a partition of the set of codimension-one cells $\rho \supseteq \mathfrak{j}$ with $Z_\rho \cap X_{\sigma_{\mathfrak{j}}} \neq \emptyset$ into subsets of cardinalities 2 and 3. Let ρ_i, $i = 1, \ldots, s$, be a choice of one representative for each such subset. Let $f_i \in \Bbbk[P_{\sigma_{\mathfrak{j}},\sigma}]$ be the sum of the terms $a_m z^m$ of $f_{\rho_i,v}$ with $\operatorname{ord}_{\mathfrak{j}}(m) = 0$. Note that this is independent of the choice of representative by Definition 1.26(ii) and the normalization condition (1.13). By local rigidity the Newton polytope of f_i is

$$\Xi_i := \operatorname{conv}\left\{m^{\rho_i}_{vv'} \,|\, v' \in \sigma_{\mathfrak{j}}\right\} \subseteq \Lambda_{\sigma_{\mathfrak{j}},\mathbb{R}},$$

and any integral point of this polytope is a vertex. Now Proposition 3.23(2), which by Proposition 3.10(1) continues to hold after normalization, gives

$$\begin{aligned}\theta^k_{\mathfrak{j}} &= \exp\Big(ct^k\partial_n + \textstyle\sum_{i,v'\in\sigma_{\mathfrak{j}}} c_{i,v'}t^k \dfrac{z^{m^{\rho_i}_{vv'}}}{f_i}\partial_{n_{i,v'}}\Big)\\ &= \exp\Big(t^k \dfrac{\big(\prod_j f_j\big)\partial_{cn} + \sum_i \big(\prod_{j\neq i} f_j\big)\sum_{m\in\Xi_i} z^m \partial_{n_i(m)}}{\prod_j f_j}\Big),\end{aligned}$$

where $n_i(m) = \sum_{\{v'\in\sigma_{\mathfrak{j}}\,|\, m^{\rho_i}_{vv'}=m\}} c_{i,v'} n_{i,v'} \in \Bbbk\otimes\Lambda^*_\sigma$. Writing $f_i = \sum_{m\in\Xi_i} d_{i,m}z^m$, $d_{i,m} \in \Bbbk\setminus\{0\}$, this leads to

$$\theta^k_{\mathfrak{j}} = \exp\Big(\frac{t^k}{\prod_i f_i}\sum_{m\in\Delta(\sigma_{\mathfrak{j}})} z^m\partial_{n(m)}\Big)$$

with $\Delta(\sigma_{\mathfrak{j}}) = \sum_i \Xi_i$ and

$$n(m) = \sum_{\sum_j m_j = m,\, m_j\in\Xi_j} d_{1,m_1}\cdot\ldots\cdot d_{s,m_s}\Big(cn + \sum_i \frac{n_i(m_i)}{d_{i,m_i}}\Big). \tag{3.18}$$

Now Proposition 3.28 applies and we obtain

$$0 = \operatorname{tlog}_v\left(\theta^k_{\mathfrak{j}}(m')\right) = t^k\langle m', n(0)\rangle.$$

Thus $n(0) = 0$ by the normalization condition. Expanding at a different vertex $v' \in \sigma_{\mathfrak{j}}$ changes f_i to $z^{m^{\rho_i}_{v'v}} f_i$ by (3.3), and hence $n(0)$ becomes $n(m_{v'})$ with $m_{v'} = \sum_i m^{\rho_i}_{vv'}$. Because the normal fan of $\sigma_{\mathfrak{j}}$ is a refinement of the normal fan of $\Delta(\sigma_{\mathfrak{j}})$, the $m_{v'}$ run over all vertices of $\Delta(\sigma_{\mathfrak{j}})$. Now any vertex m of $\Delta(\sigma_{\mathfrak{j}})$ may uniquely be written $m = \sum_i m_i$ with $m_i \in \Xi_i$. Therefore the sum in (3.18) has only one term. This shows that for any vertex $m = \sum_i m_i \in \Delta(\sigma_{\mathfrak{j}})$,

$$cn + \sum_i \frac{n_i(m_i)}{d_{i,m_i}} = 0.$$

Now apply Definition 1.26(iii) to the tuple of functions associating $\frac{cn}{s} + \frac{n_i(m)}{d_{i,m}}$ to a vertex $m \in \Xi_i$. It then follows that there are $n_i \in \Lambda_{\sigma_{\mathfrak{j}}}^{\perp} \otimes \Bbbk$ with

$$n_i = cn + \frac{n_i(m)}{d_{i,m}} \quad \text{for all } m \in \Xi_i,$$

and $\sum_i n_i = 0$. Thus $n(m) = 0$ for all $m \in \Delta(\sigma_{\mathfrak{j}}) \cap \Lambda_{\sigma_{\mathfrak{j}}}$. Hence $\theta_{\mathfrak{j}}^k = 1$, as desired.

If $\operatorname{codim} \sigma_{\mathfrak{j}} = 1$, we have to show that the normalization procedure does not spoil the consistency from Proposition 3.23(1). Indeed, Proposition 3.10(1) shows that we can write

$$\theta_{\mathfrak{j}}^k = \exp\Big(\sum_{v' \in \rho} c_{v'} t^k \frac{z^{m^{\rho}_{vv'}}}{f_{\rho,v}} \partial_{\check{d}_\rho} \Big).$$

Now as in the codimension-two case,

$$0 = \operatorname{tlog}_{v'}\big(\theta_{\mathfrak{j}}^k(m)\big) = t^k \langle m, c_{v'} \check{d}_\rho \rangle,$$

and hence $c_{v'} = 0$ for all v'. □

4. **Higher codimension scattering diagrams**

This section fills in the remaining parts of the proof of Proposition 3.9. Nothing here requires boundedness of cells in $\mathscr{P}$. We first establish a stronger uniqueness theorem and then show existence for the two cases $\operatorname{codim} \sigma_{\mathfrak{j}} = 1, 2$ separately. Throughout this section we fix the following notation. Let $\mathfrak{j} \subseteq B$ be an $(n-2)$-dimensional polyhedral subset contained in a reference cell $\sigma \in \mathscr{P}_{\max}$ and let $g : \omega \to \sigma_{\mathfrak{j}}$ with $\mathfrak{j} \cap \operatorname{Int} \omega \neq \emptyset$. Furthermore, choose $x \in (\mathfrak{j} \cap \operatorname{Int} \omega) \setminus \Delta$ and let $v = v[x] \in \sigma_{\mathfrak{j}}$ be a vertex in the same connected component of $\sigma_{\mathfrak{j}} \setminus \Delta$ as x. We work in various rings $R^I_{g,\sigma}$ for $\sigma \in \mathscr{P}_{\max}$ containing $\mathfrak{j}$, for ideals I with radical $I_0 = I^{>0}_{g,\sigma}$. We keep the standard notation $I_l = I^{>l}_{g,\sigma}$ from before. Different choices of σ are identified by parallel transport through v without further notice. We also fix an orientation of $\mathcal{Q}^v_{\mathfrak{j},\mathbb{R}} = \Lambda_{v,\mathbb{R}}/\Lambda_{\mathfrak{j},\mathbb{R}}$ and write $\mathcal{Q} := \mathcal{Q}^v_{\mathfrak{j},\mathbb{R}}$ for brevity. Recall that a *cut* $\mathfrak{c} \subseteq \mathcal{Q}$ is a one-dimensional cone contained in $\overline{\overline{\rho}} \subseteq \mathcal{Q}$ for some $\rho \in \mathscr{P}^{[n-1]}$, $\rho \supseteq \mathfrak{j}$. Moreover, as before, $m \mapsto \overline{\overline{m}}$ denotes the quotient maps $P_{\omega,\sigma} \to \mathcal{Q}$ and $P_x \to \mathcal{Q}$.

4.1. *Uniqueness.*

PROPOSITION 4.1. *Let* $\mathfrak{j} \in \operatorname{Joints}(\mathscr{S}_{k-1})$ *and* $J, J' \subseteq \Bbbk[P_{\mathrm{id}_{\sigma_{\mathfrak{j}}},\sigma}]$ *ideals with* $I_k \subseteq J \subseteq J'$ *and* $I_0 \cdot J' \subseteq J$, *where* $\sigma \in \mathscr{P}_{\max}$, $\sigma \supseteq \mathfrak{j}$. *Let* $\mathfrak{D}, \mathfrak{D}'$, *be scattering* *diagrams for* $\mathfrak{j}$ *such that* $\theta^k_{\mathfrak{D},\mathrm{id}_{\sigma_\mathfrak{j}}} = \theta^k_{\mathfrak{D}',\mathrm{id}_{\sigma_\mathfrak{j}}} = 1 \mod J'$, *and modulo* J,

$$\theta^k_{\mathfrak{D},\mathrm{id}_{\sigma_\mathfrak{j}}}, \theta^k_{\mathfrak{D}',\mathrm{id}_{\sigma_\mathfrak{j}}} \in \begin{cases} O^k\big((\Lambda_\mathfrak{j}\setminus\{0\})\otimes\Lambda_\mathfrak{j}^\perp\big), & \operatorname{codim}\sigma_\mathfrak{j}=0,\\ O^k\Big(\Lambda_\mathfrak{j}\otimes\Lambda_\mathfrak{j}^\perp+\dfrac{\Lambda_\rho}{f_\rho}\otimes\Lambda_\rho^\perp\Big), & \operatorname{codim}\sigma_\mathfrak{j}=1\quad(\rho=\sigma_\mathfrak{j}),\\ O^k\Big(\Lambda_\mathfrak{j}\otimes\Lambda_\mathfrak{j}^\perp+\sum_{\rho\supseteq\mathfrak{j}}\dfrac{\Lambda_\mathfrak{j}}{f_\rho}\otimes\Lambda_\rho^\perp\Big), & \operatorname{codim}\sigma_\mathfrak{j}=2;\end{cases}$$

that is, as log automorphisms of $R^k_{\mathrm{id}_{\sigma_\mathfrak{j}},\sigma}/J$. *Assume that* $\mathfrak{D},\mathfrak{D}'$ *only differ by outgoing rays* $(\mathfrak{r}, m_\mathfrak{r}, c_\mathfrak{r})$ *not contained in any cut, with* $z^{m_\mathfrak{r}}\in J'$ *and, if* $\operatorname{codim}\sigma_\mathfrak{j}=2$, *by changing the functions* $f_{\mathfrak{c},x}$ *by multiples of some* $z^m\in J'$ *with* $-\overline{\overline{m}}\in\mathfrak{c}\setminus\{0\}$. *Then* $\mathfrak{D}$ *and* $\mathfrak{D}'$ *are equivalent modulo* J.

Moreover, if $\operatorname{codim}\sigma_\mathfrak{j}=1$, *then the same conclusion holds if we also allow adding outgoing rays contained in* $\overline{\overline{\rho}}$, *provided* $\theta^k_{\mathfrak{D},\mathrm{id}_{\sigma_\mathfrak{j}}} = \theta^k_{\mathfrak{D}',\mathrm{id}_{\sigma_\mathfrak{j}}} = 1 \mod J$.

Proof. We have to investigate how $\theta^k_{\mathfrak{D},\mathrm{id}_{\sigma_\mathfrak{j}}}$ changes when $\mathfrak{D}$ is modified. We first derive formulae for the effect of adding a single ray $(\mathfrak{r}, m_\mathfrak{r}, c_\mathfrak{r})$ or of adding $cz^{m_\mathfrak{c}}$ to $f_{\mathfrak{c},x}$ for some cut $\mathfrak{c}$, where $z^{m_\mathfrak{r}}, z^{m_\mathfrak{c}}\in J'\setminus J$. In the case of adding a ray, since $I_0\cdot J'\subseteq J$, the associated log automorphism

$$\theta_\mathfrak{r}=\exp\big(-\log(1+c_\mathfrak{r}z^{m_\mathfrak{r}})\partial_{n_\mathfrak{r}}\big): m'\mapsto(1+c_\mathfrak{r}z^{m_\mathfrak{r}})^{-\langle\overline{m'},n_\mathfrak{r}\rangle}$$

commutes with any other log automorphism associated to a ray of $\mathfrak{D}$, modulo J. For the commutation with the log automorphism $\theta_\mathfrak{c}$ associated to a cut $\mathfrak{c}$, Lemma 2.15 shows that

$$\theta_\mathfrak{c}\circ\theta_\mathfrak{r}\circ\theta_\mathfrak{c}^{-1}=\exp\big(-c_\mathfrak{r}z^{m_\mathfrak{r}}(f_{\mathfrak{c},x}^{-\langle\overline{m_\mathfrak{r}},n_\mathfrak{c}\rangle}\partial_{n_\mathfrak{r}}+f_{\mathfrak{c},x}^{-\langle\overline{m_\mathfrak{r}},n_\mathfrak{c}\rangle-1}(\partial_{n_\mathfrak{r}}f_{\mathfrak{c},x})\partial_{n_\mathfrak{c}})\big).$$

If $\operatorname{codim}\sigma_\mathfrak{j}=2$, then any monomial z^m in $\partial_{n_\mathfrak{r}}f_{\mathfrak{c},x}$ fulfills $\operatorname{ord}_\mathfrak{j}(m)>0$, and hence we may write

$$\theta_\mathfrak{c}\circ\theta_\mathfrak{r}\circ\theta_\mathfrak{c}^{-1}=\exp\big(a_\mathfrak{r}z^{m_\mathfrak{r}}\partial_{n_\mathfrak{r}}\big)$$

for some $a_\mathfrak{r}\in\Bbbk(\Lambda_\mathfrak{j})\cap(R^k_{\mathrm{id}_{\sigma_\mathfrak{j}},\sigma'})^\times$, $\sigma'\in\mathscr{P}_{\max}$, the relevant reference cell. Note that $a_\mathfrak{r}$ depends, up to a constant factor, only on $\overline{\overline{m_\mathfrak{r}}}$. If $\operatorname{codim}\sigma_\mathfrak{j}=1$, this needs not be true because $\partial_{n_\mathfrak{r}}f_{\mathfrak{c},x}$ may contain monomials z^m with $\operatorname{ord}_\rho(m)=0$. This term, however, becomes irrelevant after restriction to

$$P_\rho:=\Big\{m\in P_{\mathrm{id}_{\sigma_\mathfrak{j}},\sigma'}\,\Big|\,\overline{m}\in\Lambda_\rho\Big\}$$

because it occurs in combination with $\partial_{\check{d}_\rho}$. We can thus nevertheless write

$$\theta_\mathfrak{c}\circ\theta_\mathfrak{r}\circ\theta_\mathfrak{c}^{-1}|_{P_\rho}=\exp\big(a_\mathfrak{r}z^{m_\mathfrak{r}}\partial_{n_\mathfrak{r}}\big)|_{P_\rho}$$

with $a_\mathfrak{r}\in\Bbbk(\Lambda_{\sigma_\mathfrak{j}})\cap(R^k_{\mathrm{id}_{\sigma_\mathfrak{j}},\sigma'})^\times$.

As for the change of $f_{\mathfrak{c},x}$ for $\operatorname{codim}\sigma_{\mathfrak{j}} = 2$ by adding $c_{\mathfrak{c}} z^{m_{\mathfrak{c}}} \in J'$ to $f_{\mathfrak{c},x}$, we note that modulo J this is equivalent to composing the log isomorphism $\theta_{\mathfrak{c}}$ by $\exp\big(-(c_{\mathfrak{c}} z^{m_{\mathfrak{c}}}/f_{\mathfrak{c},x})\partial_{n_{\mathfrak{c}}}\big)$. Then analogous arguments show

$$\theta_{\mathfrak{c}'} \circ \exp\big(-(c_{\mathfrak{c}} z^{m_{\mathfrak{c}}}/f_{\mathfrak{c},x})\partial_{n_{\mathfrak{c}}}\big) \circ \theta_{\mathfrak{c}'}^{-1} = \exp\big(a_{\mathfrak{c}} z^{m_{\mathfrak{c}}}\partial_{n_{\mathfrak{c}}}\big)$$

for some $a_{\mathfrak{c}} \in \Bbbk(\Lambda_{\mathfrak{j}}) \cap (R^k_{\mathrm{id}_{\sigma_{\mathfrak{j}}},\sigma'})^\times$ depending only on $\overline{\overline{m_{\mathfrak{c}}}}$ and $c_{\mathfrak{c}}$.

Now assume without loss of generality that $\mathfrak{D}'$ is obtained from $\mathfrak{D}$ by adding rays $(\mathfrak{r}_i, m_{\mathfrak{r}_i}, c_{\mathfrak{r}_i})$ and addition of $c_j z^{m_j}$ to $f_{\mathfrak{c}(j),x}$ with $c_{\mathfrak{r}_i} z^{m_{\mathfrak{r}_i}}$, $c_j z^{m_j} \in J'$, $-\overline{\overline{m_j}} \in \mathfrak{c}(j) \setminus \{0\}$ and all $m_{\mathfrak{r}_i}, m_{\mathfrak{c}_j}$ pairwise distinct. Then if $\operatorname{codim}\sigma_{\mathfrak{j}} \neq 1$, the above computations show that

$$\theta^k_{\mathfrak{D}',\mathrm{id}_{\sigma_{\mathfrak{j}}}} = \theta^k_{\mathfrak{D},\mathrm{id}_{\sigma_{\mathfrak{j}}}} \circ \exp\Big(\sum\nolimits_i a_{\mathfrak{r}_i} z^{m_{\mathfrak{r}_i}}\partial_{n_{\mathfrak{r}_i}} + \sum\nolimits_j a_{\mathfrak{c}_j} z^{m_j}\partial_{n_{\mathfrak{c}(j)}}\Big) \mod J.$$

Under the hypotheses on $\theta_{\mathfrak{D},\mathrm{id}_{\sigma_{\mathfrak{j}}}}, \theta_{\mathfrak{D}',\mathrm{id}_{\sigma_{\mathfrak{j}}}}$ this is only possible if both sums are empty. Indeed, as the $a_{\mathfrak{r}_i}$'s and $a_{\mathfrak{c}_j}$'s are determined, up to constant factors, by the $\overline{\overline{m_{\mathfrak{r}_i}}}$'s and $\overline{\overline{m_{\mathfrak{c}_j}}}$'s, there is no way nonzero terms in these sums can cancel.

If $\operatorname{codim}\sigma_{\mathfrak{j}} = 1$, the line $\overline{\overline{\rho}}$ separates $\mathcal{Q}$ into two half-planes. By symmetry it suffices to show that $\mathfrak{D}'$ differs from $\mathfrak{D}$ at most by adding rays in the half-plane not containing $\overline{\overline{\sigma}}$. Letting $\mathfrak{r}_1, \ldots, \mathfrak{r}_s$ be the rays with $\mathfrak{r}_i \subseteq \overline{\overline{\sigma}}$ we obtain

$$\theta^k_{\mathfrak{D}',\mathrm{id}_{\sigma_{\mathfrak{j}}}}|_{P_\rho} = \theta^k_{\mathfrak{D},\mathrm{id}_{\sigma_{\mathfrak{j}}}}|_{P_\rho} \circ \exp\Big(-\sum\nolimits_{i\le s} c_{\mathfrak{r}_i} z^{m_{\mathfrak{r}_i}}\partial_{n_{\mathfrak{r}_i}} + \sum\nolimits_{i>s} a_{\mathfrak{r}_i} z^{m_{\mathfrak{r}_i}}\partial_{n_{\mathfrak{r}_i}}\Big)\Big|_{P_\rho} \mod J.$$

Now for any monomial z^m occurring in the second sum, $\overline{\overline{m}}$ is contained in the interior of the half-plane not containing $\overline{\overline{\sigma}}$. So these cannot cancel with any term from the first sum. As before we can thus conclude that the first sum must be empty. This finishes the proof of the first paragraph of the proposition.

To prove the second paragraph we just need to add that if $\sigma_{\mathfrak{j}} = \rho \in \mathscr{P}^{[n-1]}$, then the log automorphism $\theta_{\mathfrak{r}}$ for an outgoing ray $\mathfrak{r}$ with $\mathfrak{r} \subseteq \overline{\overline{\rho}}$ commutes with $\theta_{\mathfrak{c}}$ for the two cuts $\mathfrak{c}$ present in this case. One then sees easily that adding such rays destroys the condition $\theta^k_{\mathfrak{D},\mathrm{id}_{\sigma_{\mathfrak{j}}}} = 1$ unless the change leaves the equivalence class of $\mathfrak{D}$ unchanged. □

4.2. *Infinitesimal scattering diagrams.* One basic idea in the existence proofs of the next two paragraphs is to "perturb" a scattering diagram in order to simplify the type of scatterings to be considered. For the case of codimension two we also need to consider more general log automorphisms and more general functions asssociated to cuts than before. This leads to a decoration of the elements of the deformed scattering diagram by group elements. We obtain the following notion of "infinitesimal scattering diagram."

Definition 4.2. A *squiggly ray* or *s-ray* $\mathfrak{l}$ in $\mathcal{Q}$ is the image of a C^∞-embedding $i : [0, +\infty) \to \mathcal{Q}$, such that for $t \gg 0$, $i(t) = (t - t_0)\overline{\overline{m}} + i(t_0)$ for some $m \in \Lambda_\sigma \setminus \Lambda_{\mathfrak{j}}$. We call $i(0)$ its *endpoint*, and denote by $\mathfrak{r}(\mathfrak{l}) := \mathbb{R}_{\ge 0} \cdot \overline{\overline{m}}$

the associated *asymptotic half-line*. A *segment* $\mathfrak{l}$ in $\mathcal{Q}$ is the image of a C^∞-embedding $i:[0,1]\to\mathcal{Q}$ with distinguished *initial* and *final endpoint* $i(0)$ and $i(1)$, respectively. An *orientation* of an s-ray or segment is an orientation of its tangent bundle.

An *infinitesimal scattering diagram* for a group G of log automorphisms of $R^I_{g,\sigma}$ is a collection $\mathfrak{D}=\{(\mathfrak{l},\theta_\mathfrak{l}),f_\mathfrak{c}\}$, where (1) $\mathfrak{l}\subseteq\mathcal{Q}$ is an s-ray or segment, either oriented or unoriented; (2) $\theta_\mathfrak{l}$ is an element of G of the form $\exp\big(\sum_i c_{\mathfrak{l},i}z^{m_{\mathfrak{l},i}}\partial_{n_{\mathfrak{l},i}}\big)$; (3) for each cut $\mathfrak{c}\subseteq\mathcal{Q}$ and any $p\in\mathfrak{c}\setminus\{0\}$ not contained in any $\mathfrak{l}$ with $\dim\mathfrak{l}\cap\mathfrak{c}=0$, we have a polynomial $f_{\mathfrak{c},p}=\sum_{m\in P_x,\,\overline{m}\in\Lambda_{\rho_\mathfrak{c}}}c_m z^m\in\Bbbk[P_x]$ defining an invertible element of $R^I_{g,\sigma}$.

We have the following additional conditions imposed on this data:

(i) If $\mathfrak{l}$ is an oriented s-ray or segment, then $-\mathbb{R}_{\ge0}\cdot\overline{\overline{m_{\mathfrak{l},i}}}\subseteq\mathcal{Q}$ is independent of i and, in the case of an s-ray, is parallel to $\mathfrak{l}$ outside a compact subset, extending in the direction of the orientation of $\mathfrak{l}$.

(ii) If $\mathfrak{l}$ is an unoriented s-ray or segment, then for any i, $\overline{\overline{m_{\mathfrak{l},i}}}=0$ and, in the case of an s-ray, $n_{\mathfrak{l},i}\in\Lambda_\mathfrak{j}^\perp\cap\mathfrak{r}(\mathfrak{l})^\perp$.

(iii) If $G\subseteq\widetilde{H}^I_\mathfrak{j}$, then in addition we will assume that for any s-ray or segment $(\mathfrak{l},\theta_\mathfrak{l})\in\mathfrak{D}$, $\theta_\mathfrak{l}=\exp\big(-\log(1+cz^{m_\mathfrak{l}})\partial_{n_\mathfrak{l}}\big)$ for some $m_\mathfrak{l}\in P_x$, $n_\mathfrak{l}\in m_\mathfrak{l}^\perp$, $c\in\Bbbk$.

(iv) For s-rays or segments $(\mathfrak{l},\theta_\mathfrak{l})$, $(\mathfrak{l}',\theta_{\mathfrak{l}'})\in\mathfrak{D}$, either $\mathfrak{l}\cap\mathfrak{l}'$ is a finite set of points, or $\theta_\mathfrak{l}\circ\theta_{\mathfrak{l}'}=\theta_{\mathfrak{l}'}\circ\theta_\mathfrak{l}$ and $\mathfrak{l}\subseteq\mathfrak{l}'$ or $\mathfrak{l}'\subseteq\mathfrak{l}$.

(v) If $p,p'\in\mathfrak{c}\setminus\{0\}$ lie in the same connected component of $\mathfrak{c}\setminus\bigcup_{\dim\mathfrak{l}\cap\mathfrak{c}=0}\mathfrak{l}$, then $f_{\mathfrak{c},p}=f_{\mathfrak{c},p'}$.

An oriented s-ray is called *incoming* if it is oriented towards its endpoint; otherwise it is called *outgoing*. An oriented segment is *outgoing* from its initial endpoint and *incoming* into its final endpoint.

Remark 4.3. Recall from Section 3.2 that, for any rational half-line $\mathfrak{l}\subseteq\mathcal{Q}$, the chosen orientation of $\mathcal{Q}$ determines uniquely a primitive *normal vector* $n_\mathfrak{l}\in\Lambda^*_\sigma$. A polynomial $f_{\mathfrak{c},p}$ of an infinitesimal scattering diagram thus defines unambigously the log isomorphism

$$\theta_{\mathfrak{c},p}:m\longmapsto f_{\mathfrak{c},p}^{-\langle\overline{m},n_\mathfrak{c}\rangle}$$

from R^I_{g,σ_-} to R^I_{g,σ_+}, if $\sigma_\pm$ are the maximal cells with $\overline{\overline{\sigma}}_+\cap\overline{\overline{\sigma}}_-=\mathfrak{c}$, ordered appropriately. Note that this differs slightly from the convention for ordinary scattering diagrams since here we have already taken into account the effect of the open gluing data (s_e). Conversely, this log isomorphism determines $f_{\mathfrak{c},p}$ uniquely modulo I. For uniformity of notation we will thus describe both s-rays and the polynomials $f_{\mathfrak{c},p}$ for cuts by pairs $(\mathfrak{z},\theta_\mathfrak{z})$ consisting of a locally closed submanifold ($=\mathfrak{l}$ or a connected component of $\mathfrak{c}\setminus\bigcup_{\dim\mathfrak{l}\cap\mathfrak{c}=0}\mathfrak{l}$) of $\mathcal{Q}$ and a log isomorphism between rings $R^I_{g,\sigma}$. We then call $(\mathfrak{z},\theta_\mathfrak{z})$ *foundational* if it comes from a cut and *nonfoundational* otherwise.

We will also sometimes confuse an element of a scattering diagram $(\mathfrak{z},\theta_\mathfrak{z})$ with its *support*, $\mathfrak{z}$.

An element $\mathfrak{z}\in\mathfrak{D}$ comes with an orientation of the normal bundle. For s-rays or cuts it is defined by the normal vector $n_\mathfrak{z}$ (for s-rays, of the asymptotic half-line). For a segment $\mathfrak{z}=\operatorname{im}\big(i:[0,1]\to\mathcal{Q}\big)$, take the orientation in such a way that if $b\in\mathfrak{z}$ and $\xi\in T_{\mathcal{Q},b}$ maps to a positive normal vector, then $i_*\partial_t,\xi$ forms an oriented basis of $T_{\mathcal{Q},b}$.

Construction 4.4. Given a smooth immersion $\gamma:[0,1]\to\mathcal{Q}$ which intersects elements of an infinitesimal scattering diagram $\mathfrak{D}$ for a group G transversally, with endpoints disjoint from any element of $\mathfrak{D}$, and which does not pass through any point of

$$\operatorname{Sing}(\mathfrak{D}):=\bigcup_{\mathfrak{z}\in\mathfrak{D}}\partial\mathfrak{z}\cup\bigcup_{\substack{\mathfrak{z}_1,\mathfrak{z}_2\in\mathfrak{D}\\ \dim\mathfrak{z}_1\cap\mathfrak{z}_2=0}}\mathfrak{z}_1\cap\mathfrak{z}_2,$$

we now define $\theta_{\gamma,\mathfrak{D}}\in G$, the γ-ordered product of those $\theta_\mathfrak{z}$ with $\mathfrak{z}$ crossed by γ. Explicitly, we can find numbers

$$0<t_1\le t_2\le\cdots\le t_s<1$$

and elements $\mathfrak{z}_i\in\mathfrak{D}$ such that $\gamma(t_i)\in\mathfrak{z}_i$, and $\mathfrak{z}_i\neq\mathfrak{z}_j$ if $t_i=t_j$, $i\neq j$, with s taken to be as large as possible. Then we set

$$\theta_{\gamma,\mathfrak{D}}=\theta_{\mathfrak{z}_s}^{\varepsilon_s}\circ\cdots\circ\theta_{\mathfrak{z}_1}^{\varepsilon_1}$$

with the sign $\varepsilon_i=\pm1$ positive if and only if $\gamma_*\partial_t|_{t_i}$ maps to a positive normal vector along $\mathfrak{z}$.

Note that if $t_i=t_{i+1}$, then $\dim\mathfrak{z}_i\cap\mathfrak{z}_{i+1}=1$; hence $\theta_{\mathfrak{z}_i}$ and $\theta_{\mathfrak{z}_{i+1}}$ commute according to (iv) in Definition 4.2. Thus the γ-ordered product is well defined.

Construction 4.5. An infinitesimal scattering diagram $\mathfrak{D}$ for $H^I_\mathfrak{j}$ has an associated *asymptotic scattering diagram* $\mathfrak{D}_{\mathrm{as}}$ constructed as follows. Take $\omega\in\mathscr{P}$ in Definition 3.3(1) as fixed throughout this section. For each s-ray $(\mathfrak{l},\theta_\mathfrak{l})$, according to Definition 4.2(iii), we can write uniquely

$$\theta_\mathfrak{l}=\exp\big(-\log(1+c_\mathfrak{l}z^{m_\mathfrak{l}})\partial_{n_\mathfrak{l}}\big).$$

Define $\mathfrak{D}_{\mathrm{as}}$ as the collection of rays $(\mathfrak{r}(\mathfrak{l}),m_\mathfrak{l},s^{-1}_{\omega\to\sigma_\mathfrak{l}}(m_\mathfrak{l})\cdot c_\mathfrak{l})$, where $\mathfrak{l}\in\mathfrak{D}$ is an s-ray and $\sigma_\mathfrak{l}\in\mathscr{P}_{\max}$ is such that $\mathfrak{r}(\mathfrak{l})\subseteq\overline{\overline{\sigma_\mathfrak{l}}}$, together with the functions

$$f_{\mathfrak{c},y}:=D(s_{e_y},\rho_\mathfrak{c},v[y])\cdot s_{e_y}\Big(z^{m^{\rho_\mathfrak{c}}_{v[y]v}}f_{\mathfrak{c},p}\Big)$$

for cuts $\mathfrak{c}$ and $y\in(\mathfrak{j}\cap\operatorname{Int}\omega)\setminus\Delta$, $e_y:v[y]\to\omega$. Here p is any point in the unbounded connected component of $\mathfrak{c}\setminus\operatorname{Sing}(\mathfrak{D})$, and we use parallel transport through a maximal cell σ containing $\rho_{\mathfrak{c}}$ to interpret $f_{\mathfrak{c},p} \in \Bbbk[P_x]$ as an element of $\Bbbk[P_y]$.

Remark 4.6. If $\mathfrak{D} = \{(\mathfrak{r}, m_{\mathfrak{r}}, c_{\mathfrak{r}}), f_{\mathfrak{c},x}\}$ is an ordinary scattering diagram, we obtain an infinitesimal scattering diagram for $H_{\mathfrak{j}}^I$ for any ideal I, with one straight s-ray $\mathfrak{l}$ with endpoint the origin for each ray $\mathfrak{r}$ and $f_{\mathfrak{c},p} := D(s_e, \rho_{\mathfrak{c}}, v)^{-1} \cdot s_e^{-1}(f_{\mathfrak{c},x})$, $e : v \to \omega$ for all $p \in \mathfrak{c} \setminus \{0\}$. The associated asymptotic scattering diagram is equivalent to $\mathfrak{D}$ modulo I.

The point of the definition of infinitesimal scattering diagrams is of course that if $\theta_{\gamma,\mathfrak{D}} \in H$ for all small loops γ around points of $\mathrm{Sing}(\mathfrak{D})$, where $H \subseteq G$ is a normal subgroup, say of log automorphisms of $R_{g,\sigma}^k$, then also $\theta_{\mathfrak{D}_{\mathrm{as}},g}^k \in H$.

4.3. *Existence in codimension one.* This subsection is devoted to the proof of the existence statement in Proposition 3.9 in the case $\operatorname{codim} \sigma_{\mathfrak{j}} = 1$; that is, $\rho := \sigma_{\mathfrak{j}} \in \mathscr{P}^{[n-1]}$. In this case $\mathcal{Q}$ has two cuts that we denote by $\mathfrak{c}_+, \mathfrak{c}_-$, and we take $\check{d}_\rho = n_{\mathfrak{c}_+}$. Denote by K_+, K_- the two connected components of $\mathcal{Q} \setminus \overline{\rho}$ such that $\check{d}_\rho$ is positive on K_+. We are given a scattering diagram $\mathfrak{D}' = \{\mathfrak{r}, f_{\mathfrak{c},y}\}$ for $\mathfrak{j}$ with $\theta_{\mathfrak{D}',g}^{k-1} = 1$.

Interpreting $\mathfrak{D}'$ as an infinitesimal scattering diagram for $H_{\mathfrak{j}}^k$ according to Remark 4.6, let $\mathfrak{D}_{\mathrm{in}}$ be the set of all elements of $\mathfrak{D}'$ which are not outgoing s-rays. We will construct an infinitesimal scattering diagram $\mathfrak{D}$ for $H_{\mathfrak{j}}^k$ containing $\mathfrak{D}_{\mathrm{in}}$ such that $\mathfrak{D}_{\mathrm{as}}$ is obtained from $\mathfrak{D}_{\mathrm{in}}$ by adding only outgoing s-rays. This $\mathfrak{D}$ will be such that $\theta_{\gamma,\mathfrak{D}} = \theta_{\mathfrak{D}_{\mathrm{as}},g}^k$ for γ a large counterclockwise loop around the origin, and it satisfies the conditions of the Proposition, except that *a priori* it is not constructed to contain $\mathfrak{D}'$. But since in particular $\theta_{\gamma,\mathfrak{D}} = 1 \mod I_{k-1}$, it then follows by the uniqueness result of Proposition 4.1, used inductively, that in fact $\mathfrak{D}'$ and $\mathfrak{D}_{\mathrm{as}}$ are equivalent to order $k-1$. Thus $\mathfrak{D}_{\mathrm{as}}$ will be the desired extension of $\mathfrak{D}'$.

Write $\mathfrak{D}_{\mathrm{in}} = \mathfrak{D}_{\mathrm{no}} \cup \mathfrak{D}_{\mathrm{in},+} \cup \mathfrak{D}_{\mathrm{in},-}$, where $\mathfrak{D}_{\mathrm{no}}$ consists of the two foundational elements and the nonoriented s-rays of $\mathfrak{D}'$, and $\mathfrak{D}_{\mathrm{in},\pm}$ consists of those incoming s-rays contained in $\mathrm{cl}(K_\pm)$. As a starting point for the construction of $\mathfrak{D}$, deform $\mathfrak{D}_{\mathrm{in}}$ to an infinitesimal scattering diagram $\mathfrak{D}_0$ with $(\mathfrak{D}_0)_{\mathrm{as}} = \mathfrak{D}_{\mathrm{in}}$, by moving the s-rays in $\mathfrak{D}_{\mathrm{in},\pm}$ so that their endpoints are points $p_\pm \in \mathrm{Int}(\mathfrak{c}_\pm)$, as illustrated in Figure 4.1.

Set

$$J_l := I_0^{l+1} + I_k,$$

and let s be the smallest integer such that $J_s = I_k$. We then construct infinitesimal scattering diagrams $\mathfrak{D}_1, \dots, \mathfrak{D}_s$ for $H_{\mathfrak{j}}^k$, always with the same functions associated to cuts as $\mathfrak{D}'$; that is,

$$f_{\mathfrak{c}_\pm} = f_{\mathfrak{c}_\pm,p} := D(s_e, \rho_{\mathfrak{c}_\pm}, v)^{-1} \cdot s_e^{-1}(f_{\mathfrak{c}_\pm,x}),$$

$e : v \to \omega$, for any $p \in \mathfrak{c} \setminus \mathrm{Sing}(\mathfrak{D}_l)$, enjoying the following properties:

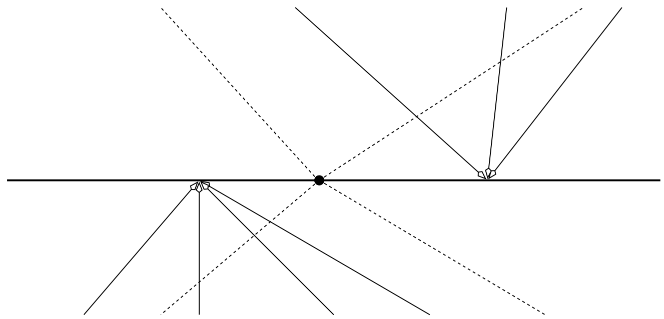


Figure 4.1. The deformation $\mathfrak{D}_0$ of $\mathfrak{D}_{\rm in}$.

(1) $\theta_{\delta,\mathfrak{D}_l} = 1 \mod J_l$ for each loop δ around a singular point of $\mathfrak{D}_l$ *except* the origin, and $\theta_{\delta_0,\mathfrak{D}_l} = 1 \mod J_l + I_{k-1}$ for δ_0 a small loop around the origin.
(2) Each $\mathfrak{z} \in \mathfrak{D}_{l+1} \setminus \mathfrak{D}_l$ is either oriented and fulfills $\mathrm{Int}(\mathfrak{z}) \cap \mathrm{Sing}(\mathfrak{D}_l) = \emptyset$ and $\mathrm{Int}(\mathfrak{z}) \cap \overline{\overline{\rho}} = \emptyset$, or has support equal to $\mathfrak{c}_-$ or $\mathfrak{c}_+$.
(3) If $\mathfrak{z} \in \mathfrak{D}_{l+1} \setminus \mathfrak{D}_l$ is a segment, then the final endpoint of $\mathfrak{z}$ is in $\overline{\overline{\rho}}$ and $\overline{\overline{m_{\mathfrak{z}}}}$ is in the same half-plane $K_\pm$ as $\mathrm{Int}(\mathfrak{z})$.
(4) If $\mathfrak{z} \in \mathfrak{D}_{\rm no} \setminus \{\mathfrak{c}_+, \mathfrak{c}_-\}$ and $p \in \mathrm{Int}(\mathfrak{z})$ is a point of $\mathrm{Sing}(\mathfrak{D}_l)$, then there is exactly one oriented segment or s-ray $\mathfrak{z}' \in \mathfrak{D}_l$ with $p \in \mathrm{Int}(\mathfrak{z}')$. Furthermore $\mathfrak{z}'$ intersects $\mathfrak{z}$ transversally at p and there is an open neighbourhood U of p such that, if U_+ and U_- are the connected components of $U \setminus \mathfrak{z}$ such that $U_+ \cap \mathfrak{z}'$ is oriented away from p and $U_- \cap \mathfrak{z}'$ is oriented towards p, then for any other $\mathfrak{z}'' \in \mathfrak{D}_l$ containing p, p is an endpoint of $\mathfrak{z}''$, and $\mathrm{Int}(\mathfrak{z}'') \cap U \subseteq U_+$.
(5) If $p \in \overline{\overline{\rho}} \cap \mathrm{Sing}(\mathfrak{D}_l) \setminus \{0\}$, then for any nonfoundational $\mathfrak{z} \in \mathfrak{D}_l$ with endpoint p, $\langle \overline{m}_{\mathfrak{z}}, \check{d}_\rho \rangle < 0$ or $\langle \overline{m}_{\mathfrak{z}}, \check{d}_\rho \rangle > 0$, independently of $\mathfrak{z}$.

Note here, in general, that the elements of $\mathfrak{D}_l$ are not straight! We need this so that we do not get "nongeneral" collision points. Figure 4.2 demonstrates allowable behaviour at nonoriented s-rays and along $\overline{\overline{\rho}}$, illustrating (4) and (5).

Note that $\mathfrak{D}_0$ satisfies properties (1)–(5). We shall now construct $\mathfrak{D}_{l+1}$ from $\mathfrak{D}_l$ with these properties in several steps, adding new s-rays and segments for each singular point of $\mathfrak{D}_l$.

Construction 4.7. *Step* 1. If $p \in \mathrm{Sing}(\mathfrak{D}_l) \setminus \overline{\overline{\rho}}$, there are two cases. If p is not contained in a nonoriented s-ray, we follow essentially the same process as in the proof of the codimension-zero case. If δ is a small counterclockwise loop around p, then since all $\mathfrak{z} \in \mathfrak{D}_l$ containing p have $\theta_{\mathfrak{z}} \in {}^{\perp}H^k_{\mathfrak{j}}$, we can write

$$\theta_{\delta,\mathfrak{D}_l} = \exp\Big(\sum c_i z^{m_i} \partial_{n_i}\Big) \mod J_{l+1}$$

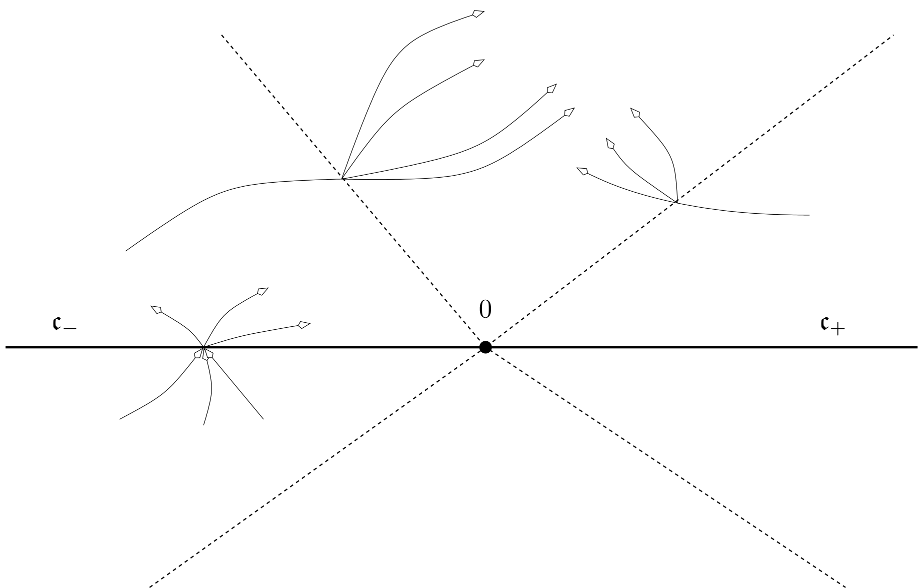


Figure 4.2. Behaviour of $\mathfrak{D}_l$ at nonoriented s-rays.

with $z^{m_i} \in J_l \setminus J_{l+1}$, $\overline{\overline{m_i}} \neq 0$ and $\langle \overline{m}_i, n_i \rangle = 0$. We can then take

$$\mathfrak{D}(p) = \Big\{ \big(\mathfrak{z}_i, \exp\big(-\log(1 + c_i z^{m_i}) \partial_{n_i} \big)\big) \Big\},$$

where

(1) If $-\overline{\overline{m_i}}$ is in the closure of the same connected component $K_\pm$ of $\mathcal{Q} \setminus \overline{\overline{\rho}}$ as p, then we take $\mathfrak{z}_i$ to be an outgoing s-ray (not necessarily straight!) disjoint from $\overline{\overline{\rho}}$, with endpoint p.
(2) If $-\overline{\overline{m_i}}$ is in the other connected component, then we take $\mathfrak{z}_i$ to be a segment with initial endpoint p and final endpoint on $\overline{\overline{\rho}}$, but not a singular point of $\mathfrak{D}_l$.

In either event, we choose $\mathfrak{z}_i$ so that it satisfies the relevant constraints (2)–(5) listed above. It now holds that $\theta_{\delta, \mathfrak{D}_l \cup \mathfrak{D}(p)} = 1 \mod J_{l+1}$, just as in the proof of the codimension-zero case of the Proposition.

If p is contained in a nonoriented s-ray $(\mathfrak{z}, \theta_\mathfrak{z})$, we apply the same process, but now have to argue that $\theta_{\delta, \mathfrak{D}_l} \in {}^\perp H^k_\mathfrak{j}$ rather than just $H^k_\mathfrak{j}$. In fact, it follows from constraint (4) that there is exactly one other s-ray or segment $\mathfrak{z}' \in \mathfrak{D}_l$ with $p \in \mathrm{Int}(\mathfrak{z}')$, and any other $\mathfrak{z}'' \in \mathfrak{D}_l$ containing p has endpoint p and is initially contained on the same side of $\mathfrak{z}$ as the outgoing part of $\mathfrak{z}'$. By abuse of notation, denote by $\theta_\mathfrak{z}$ the composition of all log automorphisms associated to nonoriented s-rays with support $\mathfrak{z}$. It then suffices to check $\theta_\mathfrak{z} \circ \theta_{\mathfrak{z}'}^{\pm 1} \circ \theta_\mathfrak{z}^{-1} \in {}^\perp H^k_\mathfrak{j}$. By Definition 4.2(ii) and (iii), we can write

$$\theta_\mathfrak{z} = \exp\Big(\sum c_j z^{m_j} \partial_{n_\mathfrak{z}} \Big)$$

with $\overline{\overline{m_j}} = 0$, and $\theta_{\mathfrak{z}'} = \exp\big(-\log(1+cz^{m_{\mathfrak{z}'}})\partial_{n_{\mathfrak{z}'}}\big) = \exp\big(\sum_j c'_j z^{jm_{\mathfrak{z}'}}\partial_{n_{\mathfrak{z}'}}\big)$. It then follows from Lemma 2.15 with $h = \exp(-\sum c_j z^{m_j})$ and $n_0 = n_{\mathfrak{z}}$ that

$$\theta_{\mathfrak{z}} \circ \theta_{\mathfrak{z}'}^{\pm 1} \circ \theta_{\mathfrak{z}}^{-1} = \exp\Big(\pm \sum\nolimits_j \mathrm{Ad}_{\theta_{\mathfrak{z}}}(c'_j z^{jm_{\mathfrak{z}'}}\partial_{n_{\mathfrak{z}'}})\Big)$$

does not contain any monomials m' with $\overline{\overline{m'}} \in \Lambda_{\mathrm{j}}$.

We can thus obtain $\mathfrak{D}(p) = \big\{(\mathfrak{z}_i, \exp\big(-\log(1+c_i z^{m_i})\partial_{n_i})\big)\big\}$ as before, with the further property that each $\mathfrak{z}_i$ lies on the same side of $\mathfrak{z}$ as all the other outgoing s-rays or segments with endpoint p, giving the inductive requirement (4).

Now set

$$\mathfrak{D}_l^{(1)} = \mathfrak{D}_l \cup \bigcup_{p\in \mathrm{Sing}(\mathfrak{D}_l)\setminus\overline{\overline{\rho}}} \mathfrak{D}(p).$$

We can make all the choices of s-rays and segments so that $\mathfrak{D}_l^{(1)}$ satisfies the constraints (2)–(5).

Step 2. If $p \in \overline{\overline{\rho}} \cap \mathrm{Sing}(\mathfrak{D}_l^{(1)}) \setminus \{0\}$, we can construct $\mathfrak{D}(p)$ consisting of outgoing s-rays with endpoint p such that $\theta_{\gamma,\mathfrak{D}_l^{(1)}\cup\mathfrak{D}(p)} = 1 \mod J_{l+1}$. Indeed, by constraint (5), all incoming s-rays or line segments at p not contained in $\overline{\overline{\rho}}$ are contained in, without loss of generality, K_+, and outgoing s-rays are contained in K_-. We claim that $\theta_{\delta,\mathfrak{D}_l} \in H^k_{\mathrm{j},K_-\cup\{0\}}$, the group defined in Definition 3.6. In fact, it follows again from Lemma 2.15 that any monomial z^m occurring in $\theta_{\delta,\mathfrak{D}_l}$ obeys $-\overline{\overline{m}} \in K_-$. To check the claim it remains to verify that the commutation of an element $\exp\big(cz^m\partial_n\big)$ of H^k_{j} with the log automorphism associated to crossing $\mathfrak{c}_-$ preserves Ω_{std}; see Remark 2.16(3). If one writes $\Omega_{\mathrm{std}} = \alpha \cdot \bigwedge_i \mathrm{dlog}(m_i)$ with $\alpha \in \Bbbk$ and $m_i \in \Lambda_{\mathrm{j}}$ for $i \geq 3$, and $\langle m_1, \check{d}_\rho\rangle = \langle m_2, n\rangle = 0$, this follows by a straightforward computation from Lemma 2.15.

Thus we can now follow the same procedure as in Step 1, defining $\mathfrak{D}(p)$ to consist of a finite number of outgoing s-rays with endpoint p and with interior contained in K_-. (No segments are necessary since these s-rays need never cross $\overline{\overline{\rho}}$.) This can be done so that $\theta_{\gamma,\mathfrak{D}_l\cup\mathfrak{D}(p)} = 1 \mod J_{l+1}$. Doing this for each such p, we set

$$\mathfrak{D}_l^{(2)} = \mathfrak{D}_l^{(1)} \cup \bigcup_{p\in\mathrm{Sing}(\mathfrak{D}_l^{(1)})\cap\overline{\overline{\rho}},\, p\neq 0} \mathfrak{D}(p).$$

Again, we can make these choices so that $\mathfrak{D}_l^{(2)}$ satisfies the constraints (2)–(5). One then sees easily that

$$\theta_{\delta,\mathfrak{D}_l^{(2)}} = 1 \mod J_{l+1}$$

for δ a loop around any point of $\mathrm{Sing}(\mathfrak{D}_l^{(2)}) \setminus \{0\}$, as follows from the fact that $\ker\big(H_{\mathrm{j}}^{J_{l+1}} \to H_{\mathrm{j}}^{J_l}\big)$ is contained in the center of $H_{\mathrm{j}}^{J_{l+1}}$.

Step 3. It remains to address the situation at the origin. Without loss of generality, we will take δ_0 to have a base point immediately to one side of $\mathfrak{c}_+$

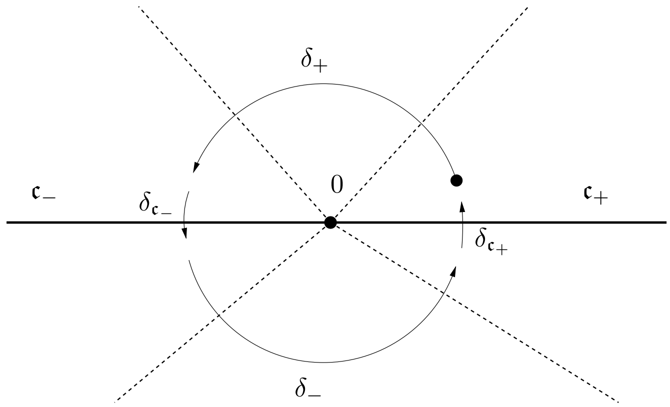


Figure 4.3. Decomposition of δ_0.

and split δ_0 into four semi-circular arcs in such a way that $\delta_{\mathfrak{c}_-}$ and $\delta_{\mathfrak{c}_+}$ only cross s-rays with support $\mathfrak{c}_+$ or $\mathfrak{c}_-$; see Figure 4.3. Then

$$\theta_{\delta_0,\mathfrak{D}_l^{(2)}} = \theta_{\mathfrak{c}_+} \circ \theta_- \circ \theta_{\mathfrak{c}_-} \circ \theta_+,$$

where we have written $\theta_{\mathfrak{c}_\pm} := \theta_{\delta_{\mathfrak{c}_\pm},\mathfrak{D}_l^{(2)}}$, $\theta_\pm := \theta_{\delta_\pm,\mathfrak{D}_l^{(2)}}$. Using (2.5) we calculate

$$\theta_{\delta_0,\mathfrak{D}_l^{(2)}}(m) = \theta_{\mathfrak{c}_+}(m) \cdot \overline{\theta}_{\mathfrak{c}_+}(\theta_-(m)) \cdot \overline{\theta}_{\mathfrak{c}_+}(\overline{\theta}_-(\theta_{\mathfrak{c}_-}(m))) \cdot \overline{\theta}_{\mathfrak{c}_+}(\overline{\theta}_-(\overline{\theta}_{\mathfrak{c}_-}(\theta_+(m)))).$$

Now $\theta_\pm$ only involves monomials $z^{m'}$ with $\overline{\overline{m'}} = 0$, and these are left invariant by $\overline{\theta}_{\mathfrak{c}_\pm}$. In addition, $\theta_{\mathfrak{c}_\pm}$ take the form

$$m \mapsto (f_\pm)^{-\langle \overline{m}, \check{d}_\rho\rangle}.$$

(Note that we may not have $f_\pm = f_{\mathfrak{c}_\pm}$ since it is possible that some outgoing rays with support $\mathfrak{c}_\pm$ may have been added in this Step for a smaller l.) This allows us to simplify to get

$$\begin{aligned} \theta_{\delta_0,\mathfrak{D}_l^{(2)}}(m) &= (f_+)^{-\langle \overline{m},\check{d}_\rho\rangle} \cdot \theta_-(m) \cdot \overline{\theta}_-(f_-)^{\langle \overline{m},\check{d}_\rho\rangle} \cdot \overline{\theta}_-(\theta_+(m)) \\ &= (\theta_- \circ \theta_+)(m) \cdot \left(\frac{\overline{\theta}_-(f_-)}{f_+}\right)^{\langle \overline{m},\check{d}_\rho\rangle}. \end{aligned} \tag{4.1}$$

Assuming, say, that $\langle \overline{m}, \check{d}_\rho\rangle = 1$, we can write this as

$$\theta_{\delta_0,\mathfrak{D}_l^{(2)}}(m) = \frac{f_\rho + \cdots}{f_\rho + \cdots}, \tag{4.2}$$

where $\cdots$ denotes expressions only involving monomials $z^{m'} \in I_0$ with $\langle \overline{m'}, \check{d}_\rho\rangle = 0$. Indeed, $f_\pm = f_\rho \mod I_0$ and $\theta_- \circ \theta_+ = 1 \mod I_0$. Since by induction $\theta_{\delta_0,\mathfrak{D}_l^{(2)}} = 1 \mod J_l + I_{k-1}$, the numerator and denominator must agree up to terms in $J_l + I_{k-1}$. Thus modulo $J_{l+1} + I_{k-1}$, $\theta_{\delta_0,\mathfrak{D}_l^{(2)}}(m) = 1 + \sum(a_j z^{m'_j}/f_\rho)$ with $z^{m'_j} \in J_l$, $\overline{m'_j} \in \Lambda_\rho$. On the other hand, if $\langle \overline{m}, \check{d}_\rho\rangle = 0$, then $\theta_{\delta_0,\mathfrak{D}_l^{(2)}}(m) =$

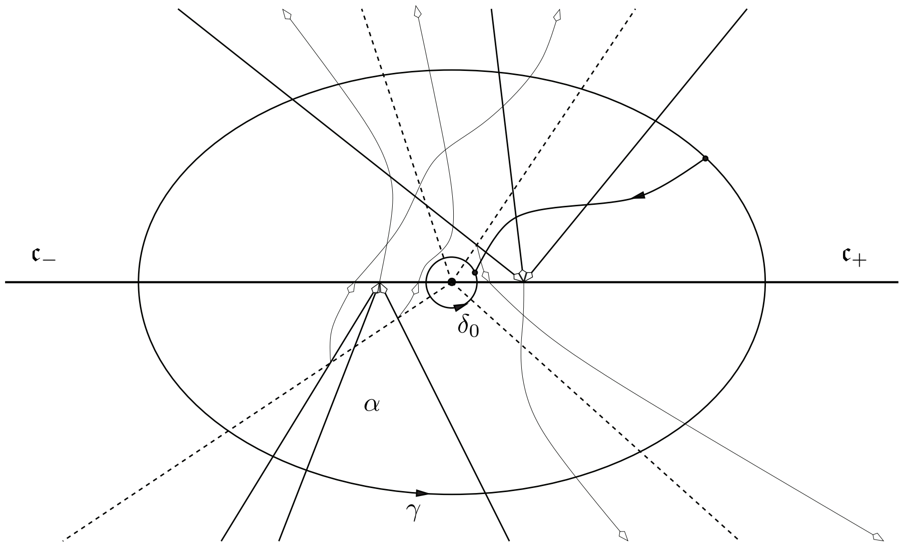


Figure 4.4. Illustration of α,γ and δ_0.

$(\theta_- \circ \theta_+)(m)$, and we can write $\theta_{\delta_0,\mathfrak{D}_l^{(2)}}(m) = 1 + \sum b_j z^{m_j}$ with $z^{m_j} \in I_0$ and $\overline{\overline{m_j}} = 0$. Thus

$$\theta_{\delta_0,\mathfrak{D}_l^{(2)}} = \exp\Big(\sum c_j z^{m_j}\partial_{n_j} + \sum d_j \frac{z^{m'_j}}{f_\rho}\partial_{\check{d}_\rho}\Big) \mod J_{l+1} + I_{k-1} \tag{4.3}$$

for some coefficients $c_j, d_j \in \Bbbk$ and exponents m_j, m'_j with $\overline{m_j} \in \Lambda_{\mathrm{j}}$, $\overline{m'_j} \in \Lambda_\rho$ (possibly with a different set of m_j's and m'_j's) and $z^{m_j}, z^{m'_j} \in J_l$.

Now taking γ to be a large loop around the origin, enclosing all singular points and segments of $\mathfrak{D}_l^{(2)}$ and with base point in the same connected component of $\mathcal{Q} \setminus \overline{\overline{\rho}}$ as δ_0, let α be a path disjoint from $\overline{\overline{\rho}}$ joining the base point of γ to the base point of δ_0, so that $\alpha\delta_0\alpha^{-1}$ is homotopic to γ in $\mathcal{Q} \setminus \{0\}$. Then modulo J_{l+1},

$$\theta_{\gamma,\mathfrak{D}_l^{(2)}} = \theta^{-1}_{\alpha,\mathfrak{D}_l^{(2)}} \circ \theta_{\delta_0,\mathfrak{D}_l^{(2)}} \circ \theta_{\alpha,\mathfrak{D}_l^{(2)}}.$$

Since $\theta_{\delta_0,\mathfrak{D}_l^{(2)}}$ only involves monomials z^m in $J_l + I_{k-1}$, one sees that modulo $J_{l+1} + I_{k-1}$, $\theta_{\alpha,\mathfrak{D}_l^{(2)}}$ commutes with $\theta_{\delta_0,\mathfrak{D}_l^{(2)}}$. Thus if we let $\mathfrak{D}'_l = (\mathfrak{D}_l^{(2)})_{\mathrm{as}}$, then

$$\theta^k_{\mathfrak{D}'_l,g} = \theta_{\gamma,\mathfrak{D}_l^{(2)}} = \theta_{\delta_0,\mathfrak{D}_l^{(2)}} \mod J_{l+1} + I_{k-1}.$$

This shows that $\theta^k_{\mathfrak{D}'_l,g}$ equals the right-hand side of (4.3) modulo $J_{l+1} + I_{k-1}$. Note that $\mathfrak{D}'_l$ has the same incoming and nondirectional rays as $\mathfrak{D}'$ and the same functions $f_\mathfrak{c}$. Since $\theta^k_{\mathfrak{D}',g} = 1 \mod I_{k-1}$ and $\theta^k_{\mathfrak{D}'_l,g} = 1 \mod J_l + I_{k-1}$, and furthermore $\mathfrak{D}'$ and $\mathfrak{D}'_l$ agree on cuts, incoming rays and nonoriented rays, it follows from the last sentence of Proposition 4.1 applied inductively that $\mathfrak{D}'$ and $\mathfrak{D}'_l$ are equivalent modulo $J_l + I_{k-1}$. To compare $\mathfrak{D}'$ and $\mathfrak{D}'_l$ modulo $J_{l+1} + I_{k-1}$, let $\mathfrak{D}_l''$ be obtained from $\mathfrak{D}_l'$ by removing all outgoing rays $\mathfrak{r}$ contained in $\overline{\overline{\rho}}$ with $z^{m_\mathfrak{r}} \in J_l$. Then $\theta_{\mathfrak{D}_l'',g} = 1 \mod J_l + I_{k-1}$ and $\mathfrak{D}'$, $\mathfrak{D}_l''$, satisfy the hypothesis of Proposition 4.1 modulo $J_{l+1} + I_{k-1}$. Thus $\mathfrak{D}_l''$ and $\mathfrak{D}'$ are equivalent modulo $J_{l+1} + I_{k-1}$. Thus by adding a number of outgoing rays contained in $\overline{\overline{\rho}}$ to $\mathfrak{D}_l'$, observing Proposition 3.10(1) once more, we can insure that $\theta^k_{\mathfrak{D}_l',g} = 1 \mod J_{l+1} + I_{k-1}$. Adding these same s-rays to $\mathfrak{D}_l^{(2)}$ to obtain $\mathfrak{D}_{l+1}$ completes the third step of the construction, so that $\theta_{\delta_0,\mathfrak{D}_{l+1}} = 1 \mod J_{l+1} + I_{k-1}$. Note that this step does not destroy the result of Step 2.

We have now obtained $\mathfrak{D}_1, \ldots, \mathfrak{D}_s$, and from $\mathfrak{D}_s$ we take $\overline{\mathfrak{D}} = (\mathfrak{D}_s)_{\mathrm{as}}$, which we modify by throwing out all outgoing rays in $\overline{\mathfrak{D}}$ contained in $\overline{\overline{\rho}}$ to get a scattering diagram $\mathfrak{D}$. Now compare $\mathfrak{D}'$, $\overline{\mathfrak{D}}$ and $\mathfrak{D}$. All three scattering diagrams share the same incoming and undirectional rays by construction, and the functions $f_{\mathfrak{c}_\pm}$ are the same. In addition, neither $\mathfrak{D}'$ nor $\mathfrak{D}$ have outgoing rays contained in $\overline{\overline{\rho}}$. We will use uniqueness (Proposition 4.1) to argue inductively that $\mathfrak{D}'$, $\overline{\mathfrak{D}}$ and $\mathfrak{D}$ are equivalent modulo $J_l + I_{k-1}$ for every l, hence modulo I_{k-1}. This is trivially true for $l = 0$. Assume that it is true for a given l. Now by assumption on $\mathfrak{D}'$ and by construction of $\overline{\mathfrak{D}}$

$$\theta^k_{\mathfrak{D}',g} = \theta^k_{\overline{\mathfrak{D}},g} = 1 \mod J_{l+1} + I_{k-1}.$$

Thus by Proposition 4.1, $\mathfrak{D}'$ and $\overline{\mathfrak{D}}$ are equivalent modulo $J_{l+1} + I_{k-1}$. On the other hand, inductively $\overline{\mathfrak{D}}$ and $\mathfrak{D}$ are equivalent modulo $J_l + I_{k-1}$, so up to equivalence, these two scattering diagrams only disagree by outgoing rays in $\overline{\overline{\rho}}$ with attached monomial in $J_l + I_{k-1}$. It then follows that modulo $J_{l+1} + I_{k-1}$, $\theta^k_{\mathfrak{D},g}$ takes the form given in the codimension-one case of Proposition 4.1, and hence by the uniqueness of the first paragraph of that proposition, $\mathfrak{D}'$ and $\mathfrak{D}$ are equivalent modulo $J_{l+1} + I_{k-1}$.

Now since $\theta_{\delta_0,\mathfrak{D}_s} = 1 \mod I_{k-1}$, (4.3) shows that

$$\theta_{\delta_0,\mathfrak{D}_s} \in O^k\Big(\Lambda_\mathfrak{j} \otimes \Lambda_\mathfrak{j}^\perp + \frac{\Lambda_\rho}{f_\rho} \otimes \Lambda_\rho^\perp\Big).$$

It then follows as before that $\theta^k_{\mathfrak{D},g}$ has the same form (but is not equal to $\theta_{\delta_0,\mathfrak{D}_s}$ as we threw out s-rays contained in $\overline{\overline{\rho}}$). This finishes the proof of the existence of scattering diagrams in codimension one. $\square$

4.4. *Existence in codimension two.*

4.4.1. *The denominator problem.* We now want to prove the existence part of Proposition 3.9 for the case that $\tau := \sigma_\mathfrak{j}$ is of codimension two. Unlike the cases of lower codimension this requires the use of our hypothesis of local rigidity, specifically, (ii) in Definition 1.26. We continue to use the notation set up at the beginning of this section. In addition, we write $P = P_x$, which we also identify with $P_{\omega,\sigma}$ via parallel transport for any $\sigma \in \mathscr{P}_{\max}$ containing τ.

Using this convention we drop the reference cell σ in the notation for the rings, so R^k_g means $R^k_{g,\sigma}$ for any appropriate σ.

Before we embark on the proof, let us explain the basic difficulty, why we call this the denominator problem, and why we need some hypotheses.

The naive approach to proving this result is to simply emulate the argument of Lemma 3.7, proceeding inductively and at each stage calculating $\theta^k_{\mathfrak{D},g}$. When doing so, because of the automorphisms associated to slabs, we obtain terms of the form $\left(\prod_\mu f^{a_\mu}_{\mathfrak{c}_\mu,x}\right) z^m\partial_n$ for some $a_\mu\in\mathbb{Z}$; in particular, the $f_{\mathfrak{c}_\mu}$'s can appear as denominators in this expression. We can attempt to get rid of these terms by adding outgoing rays with support $-\mathbb{R}_{\geq 0}\overline{m}$ or modifying cuts so as to produce automorphisms of the form $\left(\prod_\mu f^{a'_\mu}_{\mathfrak{c}_\mu,x}\right) z^m\partial_n$. Here a'_μ and a_μ may differ because this automorphism might need to be commuted past some cuts before it can be cancelled with the original troubling term. However, we are not allowed to have terms with denominators appearing in the automorphisms in $\mathfrak{D}$. If $-\mathbb{R}_{\geq 0}\overline{m}$ does not coincide with a cut, we need to have $a'_\mu\geq 0$ for all μ, while if $-\mathbb{R}_{\geq 0}\overline{m}$ does coincide with a cut $\mathfrak{c}_\mu$, then we need $a'_\mu\geq -1$ and $a'_{\mu'}\geq 0$ for all $\mu'\neq\mu$. This means we need to carefully control the powers a'_μ which appear. This is what we call the *denominator problem*. If one attempts this direct approach, then in sufficiently complex examples, the absence of denominators seems like a miracle. This direct approach can be carried out using a computer algebra package, and when item (ii) of the definition of local rigidity is satisfied, this naive algorithm works. Unfortunately, we have been unable to prove this directly. Instead, we have to resort to a more indirect solution to this problem. However, the fact that Proposition 3.9 holds, as proved in this section, implies this naive algorithm does work.

It does not always work if item (ii) of local rigidity fails.

Example 4.8. Take $\dim B=3$, $\tau\in\mathscr{P}^{[1]}$, $g=\mathrm{id}_\tau$, and write $\mathcal{Q}=\mathbb{Z}^2$ with cuts $\mathfrak{c}_i\subseteq\mathcal{Q}_\mathbb{R}$ generated by $(-1,0)$, $(0,-1)$ and $(1,2)$. Suppose the polarization φ is given by $\varphi(-1,0)=\varphi(0,-1)=0$ and $\varphi(1,2)=2$. Writing $\Lambda_x=\mathcal{Q}\oplus\mathbb{Z}$ and $\mathcal{A}\!f\!f(\check B,\mathbb{Z})_x=\mathcal{Q}\oplus\mathbb{Z}\oplus\mathbb{Z}$ we obtain

$$P=P_x=\{(a_1,a_2,a_3,a_4)\in\mathbb{Z}^4\,|\,\varphi(a_1,a_2)\leq a_4\}.$$

We then consider the (infinitesimal) scattering diagram $\mathfrak{D}'$ without any rays, thus given by the functions $f_{\mathfrak{c}_\mu}\in\Bbbk[P_x]$, which we take as follows:

$$\begin{aligned} f_{\mathfrak{c}_1}&=1+z^{(0,0,1,0)},\\ f_{\mathfrak{c}_2}&=(1+z^{(0,0,1,0)})^2+z^{(0,-1,0,0)},\\ f_{\mathfrak{c}_3}&=1+z^{(0,0,1,0)}. \end{aligned}$$

Noting that $\mathrm{ord}_\tau(0,-1,0,0)=1$, we see that $z^{(0,-1,0,0)}\in I_0$ and $\theta^0_{\mathfrak{D}',g}=1$. On the other hand, an elementary calculation shows that the unique choice of lifting $\mathfrak{D}$ of $\mathfrak{D}'$ so that $\theta^1_{\mathfrak{D},g} = 1$, is obtained by adding the outgoing s-ray $(\mathbb{R}_{\geq 0} \cdot (0,1), \theta)$ with

$$\theta(m) = \Big(1 + \frac{z^{(0,-1,0,0)}}{1 + z^{(0,0,1,0)}}\Big)^{\langle \overline{m}, (1,0,0) \rangle}.$$

Of course, this is not permitted, as we cannot allow denominators in our automorphisms attached to s-rays.

Note that the restriction of $f_{\mathfrak{c}_2}$ to X_τ is (necessarily) not reduced, and hence this example does not fulfill Definition 1.26(ii).

Of course, the foundational elements in this example are completely determined by the initial data defining the log smooth structure on X, and the point is that there may be local obstructions to smoothability. In this particular example, one can check that there is no suitable smoothing. Thus additional hypotheses are required. The key point of Definition 1.26(ii) is that if this hypothesis holds, there exists a sufficiently rigid local model for a smoothing, which we now describe.

4.4.2. *Model deformations.* Let $\rho_1, \ldots, \rho_r = \rho_0$ be a cyclic numbering of the codimension-one cells containing τ, inducing a counterclockwise ordering of the corresponding cuts $\mathfrak{c}_\mu = \overline{\overline{\rho_\mu}}$ in $\mathcal{Q}$. Let $\check{d}_{\rho_\mu} = n_{\mathfrak{c}_\mu}$ be as defined in Remark 4.3. Recall from Definition 1.26 that equality of the subschemes $Z_{\rho_\mu} \cap X_\tau$ distinguishes subsets of $\{\rho_\mu\}$ of cardinalities 2 and 3. Moreover, for each such subset we have a convex PL-function φ on Σ_τ with Newton polygon $\check{\Xi}$ an integral line segment or triangle with edges of integral length one; see Remark 1.27. We number these functions arbitrarily $\varphi_1, \ldots, \varphi_s$, and write $i(\rho)$ for the index given by $\rho \supseteq \tau$, provided $Z_\rho \cap X_\tau \neq \emptyset$. We call such ρ *singular*, while if $Z_\rho \cap X_\tau = \emptyset$, we put $i(\rho) = 0$ and say ρ is *nonsingular*. Furthermore, let φ_0 be the pull-back to Λ_x of a convex PL-function on Σ_τ defining the polarization. Similarly, we view φ_i as PL-functions on Λ_x via composition with $\Lambda_x \to \mathcal{Q} = |\Sigma_\tau|$.

For $\sigma \in \mathscr{P}_{\max}$ containing τ, denote by $n_{i,\sigma} \in \Lambda_x^*$, $1 \leq i \leq s$, the linear function defining φ_i on $K_\tau\sigma$. We also need the partial linear extension of φ_i along $\rho \in \mathscr{P}^{[n-1]}$, $\rho \supseteq \tau$, defined as follows:

$$\varphi_{i,\rho}(\overline{m}) := \max\Big\{\langle \overline{m}, n_{i,\sigma}\rangle \,\Big|\, \sigma \in \mathscr{P}_{\max},\, \sigma \supseteq \rho\Big\}.$$

Let $e_1, \ldots, e_s$ be the standard generating set for $\mathbb{Z}^s$. Now define $\widetilde{P} \subseteq \mathit{Aff}(\check{B}, \mathbb{Z})_x \oplus \mathbb{Z}^s$ to be the monoid

$$\widetilde{P} = \Big\{m + \sum\nolimits_{i=1}^s a_i e_i \,\Big|\, m \in P,\, a_i \geq \varphi_i(\overline{m}),\, i = 1, \ldots, s\Big\}.$$

Then $k[\widetilde{P}]$ is a $\Bbbk[t]$-algebra by setting $t = z^{\mathbb{1}}$ for $\mathbb{1} \in P$ the distinguished element as before.

To define the ideal of our local model in $\Bbbk[\widetilde{P}]$ we furthermore use the following notion of t-divisibility of a monomial $z^m \in \Bbbk[P]$ along the τ-stratum.

For $m \in P$, define

$$\operatorname{ht}(m) = \min\left\{ \operatorname{ord}_\sigma(m) \,\middle|\, \sigma \in \mathscr{P}_{\max},\, \sigma \supseteq \tau \right\}.$$

Note that $\operatorname{ht}(m)$ is the integral height of m above the graph of φ_0 for an identification $\mathcal{A}\!f\!f(\check{B},\mathbb{Z})_x = \Lambda_x \oplus \mathbb{Z}$. Note also that by Proposition 2.6, $\operatorname{ht}(m) = \operatorname{ord}_\sigma(m)$ if and only if $\overline{\overline{m}} \in \overline{\overline{\sigma}}$.

For the moment, we will assume that we are given functions $\tilde{f}_i \in \Bbbk[\widetilde{P}]$ of the form

$$\tilde{f}_i = \sum_m c_{i,m} z^{m+\sum_j \varphi_j(\overline{m})e_j} \tag{4.4}$$

such that if $\operatorname{ht}(m) = 0$ and $\overline{m} \notin K_\tau\rho$ for some ρ with $i(\rho) = i$, then $c_{i,m} = 0$, and

$$f_\rho = \sum_{\overline{m} \in K_\tau\rho,\, \operatorname{ht}(m)=0} c_{i,m} z^m.$$

In particular, for any i we require the functions f_ρ with $i(\rho) = i$ to have a common extension to $V(\omega)$. This is not always possible. Our construction of $\tilde{f}_i$ in Section 4.4.6 indeed requires the hypothesis of Definition 1.26(ii). For example, it is impossible to achieve this in Example 4.8.

Now write $t_i = z^{e_i} \in \Bbbk[\widetilde{P}]$, $1 \le i \le s$, and consider the ideal

$$\tilde{J}^{>k} = \left(t_1 - \tilde{f}_1, \dots, t_s - \tilde{f}_s\right) + \left(z^{m+\sum_i a_i e_i} \in \Bbbk[\widetilde{P}] \,\middle|\, \operatorname{ht}(m) > k\right)$$

in $\Bbbk[\widetilde{P}]$. For later use it will be convenient to formally define $t_0 := 1$.

We will also use the following related ideals, with $A = \sigma, \rho, \tau$ and $l \ge 0$:

$$\begin{aligned} \tilde{J}^{>k}_l &= \tilde{J}^{>k} + \left(z^{m+\sum_i a_i e_i} \in \Bbbk[\widetilde{P}] \,\middle|\, z^m \in I_0^{l+1} + I_k\right) \subseteq \Bbbk[\widetilde{P}], \\ \tilde{J}^{>k}_A &= \tilde{J}^{>k} + \left(z^{m+\sum_i a_i e_i} \in \Bbbk[\widetilde{P}] \,\middle|\, \operatorname{ord}_A(m) > k\right) \subseteq \Bbbk[\widetilde{P}]. \end{aligned} \tag{4.5}$$

The ideal $\tilde{J}^{>k}$ defines the local model, while $\tilde{J}^{>k}_l$, $\tilde{J}^{>k}_A$ provide the reduction modulo $I_0^{l+1} + I_k$ and the various primary components, respectively. To study the situation along a codimension-one cell $\rho \supseteq \tau$ it is natural to forget the nonstandard behaviour on all other codimension-one cells and work in $\Bbbk[\widetilde{P}_\rho]$ with

$$\widetilde{P}_\rho = \left\{ m + \sum a_i e_i \,\middle|\, m \in P,\, a_i \ge \varphi_{i,\rho}(\overline{m}),\, i = 1, \dots, s \right\}.$$

The analogue of $\tilde{J}^{>k}$ in $\Bbbk[\widetilde{P}_\rho]$ is

$${}_\rho\tilde{J}^{>k} = \left(t_1 - \tilde{f}_1, \dots, t_s - \tilde{f}_s\right) + \left(z^{m+\sum_i a_i e_i} \in \Bbbk[\widetilde{P}_\rho] \,\middle|\, \operatorname{ht}_\rho(m) > k\right),$$

where

$$\operatorname{ht}_\rho(m) = \min\left\{ \operatorname{ord}_\sigma(m) \,\middle|\, \sigma \in \mathscr{P}_{\max},\, \sigma \supseteq \rho \right\}.$$

The formulae for the analogues ${}_\rho\tilde{J}^{>k}_l$, ${}_\rho\tilde{J}^{>k}_A$ of $\tilde{J}^{>k}_l$, $\tilde{J}^{>k}_A$ are defined as in (4.5) with ${}_\rho\tilde{J}^{>k}$ replacing $\tilde{J}^{>k}$.

Remark 4.9. The basic idea of what we are going to do is that

$$\operatorname{Spec} \mathbb{k}[\widetilde{P}]/\tilde{J}^{>k} \longrightarrow \operatorname{Spec} \mathbb{k}[t]/(t^{k+1})$$

gives a good local model for the k-th order deformation of $V(\tau)$. Clearly, the reduction of the central fibre is canonically isomorphic to the closure of $V(\tau)$ in $V(\omega) = \operatorname{Spec} \mathbb{k}[P]/(t)$, and the central fibre is reduced on $V(\tau)$. To describe the map at the generic point of a codimension-one stratum $V_{\tau\to\rho} \subseteq V(\tau)$, find m, m', in a localization of P along the face corresponding to ρ such that $\operatorname{ht}_\rho(m) = \operatorname{ht}_\rho(m') = 0$ and $\overline{m} = -\overline{m'}$ is a generator of Λ_x/Λ_ρ. Then $x = z^m$, $y = z^{m'}$, fulfill the relation $xy = t^e$, while their canonical lifts $\tilde{x} = z^{m+\sum_i \varphi_{i,\rho}(\overline{m})e_i}$, $\tilde{y} = z^{m'+\sum_i \varphi_{i,\rho}(\overline{m'})e_i}$, fulfill

$$\tilde{x}\tilde{y} = z^{e_{i(\rho)}} \cdot t^e = t_{i(\rho)} \cdot t^e = \tilde{f}_i \cdot t^e \mod \tilde{J}^{>k}.$$

This is just a slight perturbation of the standard local model as derived in the proof of Lemma 2.34. As a result, we can essentially describe this local model by gluings of the standard thickenings via certain automorphisms, which can be written down explicitly. Some effort is then required to massage these automorphisms into a standard form.

Remark 4.10. We will use repeatedly the following observation. *Any monomial in* $\mathbb{k}[\widetilde{P}]/\tilde{J}^{>k}$ *is equal to an expression only involving monomials of the form* $z^{m+\sum_i \varphi_i(\overline{m})e_i}$. Indeed, we can show this by downward induction on $\operatorname{ht}(m)$, starting with those monomials with $\operatorname{ht} = k+1$, in which case such a monomial is already in $\tilde{J}^{>k}$. Now suppose the result is true for all monomials with $\operatorname{ht} > k'$. Then for a monomial $z^{m+\sum_i a_i e_i}$ with $\operatorname{ht}(m) = k'$, in $\mathbb{k}[\widetilde{P}]$ we can write

$$z^{m+\sum_i a_i e_i} = z^{m+\sum_i \varphi_i(\overline{m})e_i} \prod_{i=1}^{s} t_i^{a_i - \varphi_i(\overline{m})}. \tag{4.6}$$

We can then substitute $t_i = \tilde{f}_i$ for each i. Now, by design, $\tilde{f}_i$ only contains terms of the desired form $z^{m'+\sum_j \varphi_j(\overline{m'})e_j}$; however, in making the substitution, cross terms will arise which are not of this form. These cross terms in $\prod_{i=1}^s \tilde{f}_i^{a_i-\varphi_i(\overline{m})}$ are of the form

$$\prod_j z^{m_j+\sum_i \varphi_i(\overline{m_j})e_i} = z^{\sum_j m_j + \sum_{i,j} \varphi_i(\overline{m_j})e_i}.$$

By convexity of φ_i, $\sum_j \varphi_i(\overline{m_j}) \geq \varphi_i(\sum_j \overline{m_j})$, and if we have inequality, then there is no $\sigma \in \mathscr{P}_{\max}$ with $\overline{m_j} \in K_\tau\sigma$ for all j; strict convexity of φ_0 on the fan Σ_τ then implies $\sum_j \varphi_0(\overline{m_j}) > \varphi_0(\sum_j \overline{m_j})$. Thus any term arising in the expansion of (4.6) of the form $z^{m'+\sum_i a'_i e_i}$ with $a'_i > \varphi_i(\overline{m})$ for some $1 \leq i \leq s$ must have $\operatorname{ht}(m') > k'$. By the induction hypothesis, these terms can be written in the desired form. We call this process *reduction* and say a monomial in $\Bbbk[\widetilde{P}]/\tilde{J}^{>k}$ is in *reduced form* if it is of the form $z^{m+\sum_{i=1}^{s}\varphi_i(\overline{m})e_i}$.

The same argument works in the ring $\Bbbk[\widetilde{P}_\rho]/_\rho\tilde{J}^{>k}$ if φ_i is replaced by $\varphi_{i,\rho}$, for $0\le i\le s$. There is one slight difference here. The terms $z^{m'+\sum_i a_ie_i}$ appearing in $\tilde{f}_j$ may not satisfy $a_i=\varphi_{i,\rho}(\overline{m'})$, but if they do not, then we also have $\mathrm{ht}_\rho(m')>0$, allowing the induction process. Again, we call a monomial in this ring in *reduced form* if it is of the form $z^{m+\sum\varphi_{i,\rho}(\overline{m})e_i}$.

4.4.3. *Comparison with the standard model.* We now want to decompose $\operatorname{Spec}\Bbbk[\widetilde{P}]/\tilde{J}^{>k}$ into standard pieces. To make the comparison with the standard piece $\operatorname{Spec}R^k_{\omega\to\sigma}$ for some $\sigma\in\mathscr{P}_{\max}$ containing τ, we need to localize at the product of

$$f_{i,\sigma}:=\sum_{\{m\,|\,\mathrm{ht}(m)=0,\,\overline{m}\in K_\omega\sigma\}}c_{i,m}z^m\in\Bbbk[P].$$

Let $\sigma_1,\ldots,\sigma_r$ be the maximal cells containing τ, labelled modulo r in such a way that $\rho_\mu=\sigma_\mu\cap\sigma_{\mu+1}$. Since we often need to consider neighouring cells we write $\mu^-:=\mu-1$, $\mu^+:=\mu$ for $1\le\mu\le r$, interpreted modulo r.

Proposition 4.11. *For* $1\le\mu\le r$,

$$z^m\longmapsto z^{m+\sum_i\langle\overline{m},n_{i,\sigma_\mu}\rangle e_i}\tag{4.7}$$

induces ring isomorphisms

$$\beta^\pm_\mu:(R^k_{\omega\to\sigma_\mu})_{\prod_i f_{i,\sigma_\mu}}\longrightarrow\left(\Bbbk[\widetilde{P}_{\rho_{\mu^\pm}}]/_{\rho_{\mu^\pm}}\tilde{J}^{>k}_{\sigma_\mu}\right)_{\prod_i t_i},$$
$$\kappa_\mu:\left(R^k_{\omega\to\tau}\right)_{\prod_i f_{i,\sigma_\mu}}\longrightarrow\left(\Bbbk[\widetilde{P}]/\tilde{J}^{>k}_\tau\right)_{\prod_i t_i}\simeq\left(\Bbbk[\widetilde{P}_{\rho_{\mu^\pm}}]/_{\rho_{\mu^\pm}}\tilde{J}^{>k}_\tau\right)_{\prod_i t_i}.$$

Proof. Let us first consider the case of β^+_μ. First we note that

$$\beta^+_\mu(f_{i,\sigma_\mu})=\sum_{\{m\,|\,\mathrm{ht}(m)=0,\,\overline{m}\in K_\omega\sigma_\mu\}}c_{i,m}z^{m+\sum_j\langle\overline{m},n_{j,\sigma_\mu}\rangle e_j}=\tilde{f}_i\mod{}_{\rho_\mu}\tilde{J}^{>0}_{\sigma_\mu}.$$

Because ${}_{\rho_\mu}\tilde{J}^{>0}_{\sigma_\mu}$ is nilpotent in $\Bbbk[\widetilde{P}_{\rho_\mu}]/_{\rho_\mu}\tilde{J}^{>k}_{\sigma_\mu}$, we see that if we can invert $t_i=\tilde{f}_i$, we can invert $\beta^+_\mu(f_{i,\sigma_\mu})$, and vice versa. Thus β^+_μ is defined. Now set

$$\widetilde{R}^k_\mu:=\left(\Bbbk[P]\otimes_\Bbbk\Bbbk[t_1^{\pm1},\ldots,t_s^{\pm1}]\right)\Big/\left(z^m\otimes1\,\Big|\,m\in P,\,\mathrm{ord}_{\sigma_\mu}(m)>k\right).$$

The formula for β^+_μ induces an obvious identification of $\widetilde{R}^k_\mu$ with

$$\left(\Bbbk[\widetilde{P}_{\rho_\mu}]/\langle z^{m+\sum_j a_je_j}\mid\mathrm{ord}_{\sigma_\mu}(m)>k\rangle\right)_{\prod t_i}.$$

This identification yields an isomorphism

$$\widetilde{R}^k_\mu/(t_1-\tilde{f}_1,\ldots,t_s-\tilde{f}_s)\simeq\left(\Bbbk[\widetilde{P}_{\rho_\mu}]/_{\rho_\mu}\tilde{J}^{>k}_{\sigma_\mu}\right)_{\prod t_i}.$$

Now using the same reduction process as in Remark 4.10, $\tilde{f}_i \in \widetilde{R}^k_\mu$ is equivalent modulo $(t_1 - \tilde{f}_1, \ldots, t_s - \tilde{f}_s)$ to a $\tilde{f}'_i$ only containing monomials of the form $z^m \otimes 1$. In particular, in $\widetilde{R}^k_\mu$, we can write

$$t_i - \tilde{f}'_i = \sum g_{ij}(t_j - \tilde{f}_j),$$

and we can assume the image of g_{ij} in $\widetilde{R}^0_\mu$ is δ_{ij}. Thus the matrix (g_{ij}) is invertible in $\widetilde{R}^k_\mu$, and the ideals $(t_1 - \tilde{f}_1, \ldots, t_s - \tilde{f}_s)$ and $(t_1 - \tilde{f}'_1, \ldots, t_s - \tilde{f}'_s)$ coincide. Thus

$$\begin{aligned}(R^k_{\omega\to\sigma_\mu})_{\prod f_{i,\sigma_\mu}} &\simeq \left(\Bbbk[P]/I^{>k}_{\omega\to\sigma_\mu}\right)_{\prod \tilde{f}'_i}\\ &\simeq \widetilde{R}^k_\mu/(t_1 - \tilde{f}'_1, \ldots, t_s - \tilde{f}'_s) \;\simeq\; \left(\Bbbk[\widetilde{P}_{\rho_\mu}]/_{\rho_\mu}\widetilde{J}^{>k}_{\sigma_\mu}\right)_{\prod t_i},\end{aligned}$$

the first isomorphism since the localizing elements only differ by nilpotent monomials. Furthermore, this isomorphism is induced by β^+_μ, giving the result.

The proofs for β^-_μ and κ_μ run identically. For the target of κ_μ note that the inclusion $\Bbbk[\widetilde{P}] \to \Bbbk[\widetilde{P}_{\rho_{\mu\pm}}]$ is an isomorphism after localizing at $\prod t_i$. $\square$

For the following proposition recall that we defined $t_{i(\rho)} = t_0 = 1$ whenever ρ is nonsingular.

Proposition 4.12. *For each $1 \le \mu \le r$ define the log automorphism*

$$\theta_\mu : P \longrightarrow (R^k_g)^\times, \quad \theta_\mu(m) = \kappa^{-1}_{\mu+1}(t_{i(\rho_\mu)})^{-\langle \overline{m}, \check{d}_{\rho_\mu}\rangle}.$$

Then $\theta_r \circ \cdots \circ \theta_1 = 1$.

Proof. First note that from the definition of the convex PL-functions φ_j and their defining linear functions $n_{j,\sigma}$,

$$t_{i(\rho_\mu)}^{-\langle \overline{m}, \check{d}_{\rho_\mu}\rangle} = \prod_{j=1}^s t_j^{\langle \overline{m}, n_{j,\sigma_\mu} - n_{j,\sigma_{\mu+1}}\rangle}.$$

Thus

$$\begin{aligned}\kappa_{\mu+1}(\overline{\theta}_\mu(z^m)) &= \kappa_{\mu+1}(\theta_\mu(m))\kappa_{\mu+1}(z^m)\\ &= \left(\prod_{i=1}^s t_i^{\langle \overline{m}, n_{i,\sigma_\mu} - n_{i,\sigma_{\mu+1}}\rangle}\right) z^{m + \sum_i \langle \overline{m}, n_{i,\sigma_{\mu+1}}\rangle e_i}\\ &= z^{m + \sum \langle \overline{m}, n_{i,\sigma_\mu}\rangle e_i} \;=\; \kappa_\mu(z^m),\end{aligned}$$

and hence $\overline{\theta}_\mu = \kappa^{-1}_{\mu+1} \circ \kappa_\mu$.

From this one easily sees that

$$\begin{aligned}\theta_r \circ \cdots \circ \theta_1(m) &= \prod_{\mu=1}^r \left(\kappa_1^{-1} \circ \kappa_{\mu+1} \circ \theta_\mu\right)(m) = \kappa_1^{-1}\Big(\prod_{\mu=1}^r t_{i(\rho_\mu)}^{-\langle \overline{m}, \check{d}_{\rho_\mu}\rangle}\Big)\\ &= \kappa_1^{-1}\Big(\prod_{i=1}^s \prod_{\{\rho \,|\, i(\rho) = i\}} t_i^{-\langle \overline{m}, \check{d}_\rho\rangle}\Big) = 1,\end{aligned}$$

the last equality by Definition 1.26(ii) and Remark 1.27. $\square$

4.4.4. *The factorization lemma.* The problem now is that when ρ_μ is singular, θ_μ is not a very well-behaved log automorphism. Our next task is to factor it into manageable log automorphisms. The first step achieves a product decomposition of $\tilde{f}_{i(\rho_\mu)}$ of the form $t_i(1+t_i^{-1}\widetilde{G}^+_\mu)(1+t_i^{-1}\widetilde{G}^-_\mu)$, with $\widetilde{G}^\pm_\mu$ gathering all monomials propagating into $\sigma_{\mu^\pm+1}$ and $i=i(\rho_\mu)$. There are two problems with this. First, this is generally only possible modulo monomials of higher order or propagating along ρ away from τ. And second, $\widetilde{G}^\pm_\mu$ will only become divisible by t_i after pulling back by $\beta^\pm_\mu$. From this factorization Proposition 4.15 constructs the desired factorization of θ_μ.

Recall that $c_{i,m}$ denotes a coefficient of $\tilde{f}_i$ (4.4).

LEMMA 4.13. *Let $\rho=\rho_\mu$ be singular and $i=i(\rho_\mu)$.*

(1) *There exist $\widetilde{F}_\mu, \widetilde{G}^\pm_\mu \in \Bbbk[\widetilde{P}_\rho]/{}_\rho\tilde{J}^{>k}$ with the following properties:*

(i) *One has*
$$\widetilde{F}_\mu = \sum_{m\in P,\,\overline{m}\in\Lambda_\rho} d_{\mu,m} z^{m+\sum_j \varphi_{j,\rho}(\overline{m})e_j}$$
with $d_{\mu,m}=c_{i,m}$ if $m\in K_\omega\rho$ and $\operatorname{ord}_\rho(m)=0$.

(ii) *$\widetilde{G}^\pm_\mu$ is a linear combination of monomials of the form $z^{m+\sum_j \varphi_{j,\rho}(\overline{m})e_j}$ with $\pm\langle\overline{m},\check{d}_\rho\rangle<0$.*

(iii) *$\widetilde{G}^-_\mu\widetilde{G}^+_\mu$ is divisible by t_i in $\Bbbk[\widetilde{P}_\rho]/{}_\rho\tilde{J}^{>k}$, and in this ring,*

$$t_i + \widetilde{G}^-_\mu + \widetilde{G}^+_\mu + t_i^{-1}\widetilde{G}^-_\mu\widetilde{G}^+_\mu - \widetilde{F}_\mu = 0. \tag{4.8}$$

(2) *Suppose that for some i', $\tilde{f}_{i'}$ is replaced by $\tilde{f}_{i'}+cz^{m+\sum_j\varphi_j(\overline{m})e_j}$, where $z^m\in I_0^{l+1}+I_k$.*

(i) *If $i'\neq i$, then $\widetilde{F}_\mu$ and $\widetilde{G}^\pm_\mu$ are unchanged modulo ${}_\rho\tilde{J}^{>k}_{l+1}$.*

(ii) *If $i'=i$ and $\overline{\overline{m}}\in\overline{\overline{\rho}}$, then $\widetilde{F}_\mu$ is replaced by $\widetilde{F}_\mu+cz^{m+\sum_j\varphi_{j,\rho}(\overline{m})e_j}$ modulo ${}_\rho\tilde{J}^{>k}_{l+1}$, while $\widetilde{G}^\pm_\mu$ are unchanged modulo ${}_\rho\tilde{J}^{>k}_{l+1}$.*

(iii) *If we are not in case* (i) *or* (ii), *then modulo ${}_\rho\tilde{J}^{>k}_{l+1}$, $\widetilde{F}_\mu$ and $\widetilde{G}^\pm_\mu$ are modified by expresssions of the form $az^{m+\sum_j\varphi_{j,\rho}(\overline{m})e_j}$, where $a\in\Bbbk[\widetilde{P}_\rho]\setminus{}_\rho\tilde{J}^{>k}_0$.*

Proof. We first note that if $\widetilde{G}^\pm_\mu$ satisfy condition (ii) in (1), then $\widetilde{G}^-_\mu\widetilde{G}^+_\mu$ consists entirely of cross-terms of the form
$$z^{m+\sum_j\varphi_{j,\rho}(\overline{m})e_j}\cdot z^{m'+\sum_j\varphi_{j,\rho}(\overline{m'})e_j}$$
with $\langle\overline{m},\check{d}_\rho\rangle>0$, $\langle\overline{m'},\check{d}_\rho\rangle<0$. Then $\varphi_{i,\rho}(\overline{m})+\varphi_{i,\rho}(\overline{m'})>\varphi_{i,\rho}(\overline{m}+\overline{m'})$, so $\widetilde{G}^-_\mu\widetilde{G}^+_\mu$ is divisible by t_i.

We will construct $\widetilde{G}^\pm_\mu$ and $\widetilde{F}_\mu$ by induction on k. Begin initially with $\widetilde{F}_\mu=\widetilde{G}^\pm_\mu=0$ for $k=-1$.

Now assume $\widetilde{G}^{\pm}_{\mu}$ and $\widetilde{F}_{\mu}$ have been constructed so that (4.8) holds in $\Bbbk[\widetilde{P}_\rho]/{}_\rho\tilde{J}^{>k-1}$. Now as in Remark 4.10, the left-hand side of (4.8) can be rewritten, modulo ${}_\rho\tilde{J}^{>k}$, entirely in terms of monomials of the form $cz^{m+\sum_j \varphi_{j,\rho}(m)e_j}$, necessarily with $\mathrm{ht}_\rho(m) = k$, by the induction hypothesis. For each such term, we have three cases.

(1) If $\langle \overline{m}, \check{d}_\rho\rangle > 0$, we subtract this term from $\widetilde{G}^-_\mu$.
(2) If $\langle \overline{m}, \check{d}_\rho\rangle < 0$, we subtract this term from $\widetilde{G}^+_\mu$.
(3) If $\langle \overline{m}, \check{d}_\rho\rangle = 0$, we add this term to $\widetilde{F}_\mu$.

After making these changes to $\widetilde{G}^{\pm}_{\mu}$ and $\widetilde{F}_{\mu}$, it is then clear that (4.8) holds modulo ${}_\rho\tilde{J}^{>k}$. Proceeding inductively, we construct $\widetilde{F}_{\mu}$ and $\widetilde{G}^{\pm}_{\mu}$, such that conditions (ii) and (iii) hold.

Note that when this procedure is carried out for $k = 0$, for the left-hand side of (4.8) we get just t_i, which is equivalent to $\tilde{f}_i$. The only terms of the form $z^{m+\sum_j a_j e_j}$ appearing in $\tilde{f}_i$ with $\langle \overline{m}, \check{d}_\rho\rangle = 0$ that are not in ${}_\rho\tilde{J}^{>0}$ must have $\mathrm{ord}_\rho(m) = 0$; hence $\overline{m} \in K_\omega\rho$ and $a_j = \varphi_j(\overline{m}) = \varphi_{j,\rho}(\overline{m})$. This makes it clear that condition (i) holds.

For (2), we just need to look at the terms in (4.8) and investigate what effect the reduction process of Remark 4.10 has on these. Terms in $\widetilde{G}^{\pm}_{\mu}$ and $\widetilde{F}_{\mu}$ are already in reduced form by conditions (i) and (ii) in (1). On the other hand, any cross-term $z^{m'+\sum_j a_j e_j}$ in $t_i^{-1}\widetilde{G}^-_\mu \widetilde{G}^+_\mu$ is necessarily in ${}_\rho\tilde{J}^{>0}_\tau$, so the effect of the change to $\tilde{f}_{i'}$ to the reduction process only affects this term by something in ${}_\rho\tilde{J}^{>0}_\tau \cdot {}_\rho\tilde{J}^{>k}_l \subseteq {}_\rho\tilde{J}^{>k}_{l+1}$. Finally, reducing $t_i = \tilde{f}_i$, we see that if $i \neq i'$, any term appearing in $\tilde{f}_i$ which is not in ${}_\rho\tilde{J}^{>0}_\tau$ is already reduced, so similarly the change to this term from the change in $\tilde{f}_{i'}$ is in ${}_\rho\tilde{J}^{>k}_{l+1}$. This gives Case (i).

Now if $i = i'$, then $t_i = \tilde{f}_i$ acquires an additional term $cz^{m+\sum_j \varphi_j(\overline{m})e_j}$. If $\overline{\overline{m}} \in \overline{\overline{\rho}}$, then this is already in reduced form as $\varphi_j(\overline{m}) = \varphi_{j,\rho}(\overline{m})$, giving Case (ii). Otherwise, the reduction process will replace this term modulo ${}_\rho\tilde{J}^{>k}_{l+1}$ with an expression

$$az^{m+\sum_j \varphi_{j,\rho}(\overline{m})e_j}$$

with $a \notin {}_\rho\tilde{J}^{>k}_0$, giving Case (iii). □

The following lemma is the key point for showing that no denominators will occur in our factorization.

LEMMA 4.14. *For $1 \leq \mu \leq r$ and $\rho = \rho_\mu$ singular there are chains of surjections and inclusions*

$$\begin{array}{ccccc} \Bbbk[\widetilde{P}_\rho]/{}_\rho\tilde{J}^{>k} & \twoheadrightarrow & \Bbbk[\widetilde{P}_\rho]/{}_\rho\tilde{J}^{>k}_{\sigma_\mu} & \overset{(\beta^+_\mu)^{-1}}{\hookrightarrow} & R^k_{\omega\to\sigma_\mu}, \\ \Bbbk[\widetilde{P}_\rho]/{}_\rho\tilde{J}^{>k} & \twoheadrightarrow & \Bbbk[\widetilde{P}_\rho]/{}_\rho\tilde{J}^{>k}_{\sigma_{\mu+1}} & \overset{(\beta^-_{\mu+1})^{-1}}{\hookrightarrow} & R^k_{\omega\to\sigma_{\mu+1}}, \end{array}$$

where in each instance the first map is the obvious one. Furthermore, there exist elements $G_\mu^- \in R^k_{\omega\to\sigma_\mu}$, $G_\mu^+ \in R^k_{\omega\to\sigma_{\mu+1}}$ *such that*

$$\beta_\mu^+(G_\mu^-) = t_i \cdot \widetilde{G}_\mu^-,$$
$$\beta_{\mu+1}^-(G_\mu^+) = t_i \cdot \widetilde{G}_\mu^+,$$

where $i = i(\rho_\mu)$. *In addition, the image of* $G_\mu^\pm$ *in* R^k_g *is in* I_0.

Proof. We give the proof for G_μ^- and β_μ^+, the other case being completely analogous.

The first point is to show that $(\beta_\mu^+)^{-1}$ takes $\Bbbk[\widetilde{P}_\rho]/_\rho\tilde{J}^{>k}_{\sigma_\mu}$ into $R^k_{\omega\to\sigma_\mu}$ rather than, *a priori*, into $(R^k_{\omega\to\sigma_\mu})_{\prod_i f_{i,\sigma_\mu}}$. To see this, let $\widetilde{P}_{\sigma_\mu}$ be defined by

$$\widetilde{P}_{\sigma_\mu} = \Big\{ m + \textstyle\sum_i a_i e_i \,\Big|\, m \in P_{\mathrm{id}_{\sigma_\mu}},\, a_i \geq \langle \overline{m}, n_{i,\sigma_\mu}\rangle,\, i = 1,\ldots,s \Big\}.$$

This can be viewed as the localization of $\widetilde{P}$ along the face corresponding to σ_μ. Beware that this notation is not in strict analogy with $\widetilde{P}_\rho$ because now we localize at elements of P. If we denote

$${}_{\sigma_\mu}\tilde{J}^{>k} := \big(t_1 - \tilde{f}_1, \ldots, t_s - \tilde{f}_s\big) + \Big(z^{m+\sum_i a_i e_i} \in \Bbbk[\widetilde{P}_{\sigma_\mu}] \,\Big|\, \mathrm{ord}_{\sigma_\mu}(m) > k\Big),$$

then $z^m \mapsto z^{m+\sum_j \langle \overline{m}, n_{j,\sigma_\mu}\rangle e_j}$ also defines a map

$$\alpha'_\mu : R^k_{\mathrm{id}_{\sigma_\mu}} \longrightarrow \Bbbk[\widetilde{P}_{\sigma_\mu}]/_{\sigma_\mu}\tilde{J}^{>k}.$$

This is an isomorphism. Indeed, $\Bbbk[\widetilde{P}_{\sigma_\mu}] \simeq \Bbbk[P_{\mathrm{id}_{\sigma_\mu}}] \otimes_\Bbbk \Bbbk[t_1,\ldots,t_s]$, and as in Remark 4.10, modulo ${}_{\sigma_\mu}\tilde{J}^{>k}$, every t_i is equivalent to an element of $\Bbbk[P_{\mathrm{id}_{\sigma_\mu}}]$ under this isomorphism. Thus

$$\Bbbk[\widetilde{P}_{\sigma_\mu}]/_{\sigma_\mu}\tilde{J}^{>k} \simeq \Bbbk[P_{\mathrm{id}_{\sigma_\mu}}]/(t^{k+1}) = R^k_{\mathrm{id}_{\sigma_\mu}}.$$

Now $R^k_{\mathrm{id}_{\sigma_\mu}}$ is the localization of $R^k_{\omega\to\sigma_\mu}$ at any element z^m with $\mathrm{ord}_{\sigma_\mu}(m) = 0$ and $\overline{m} \in \mathrm{Int}\, K_\omega\sigma_\mu$. Thus $U := \mathrm{Spec}\, R^k_{\mathrm{id}_{\sigma_\mu}}$ and $V := \mathrm{Spec}(R^k_{\omega\to\sigma_\mu})_{\prod f_{i,\sigma_\mu}}$ are both open subsets of $X = \mathrm{Spec}\, R^k_{\omega\to\sigma_\mu}$, and $X \setminus (U \cup V)$ is a closed subset of X of codimension at least two and not contained in a toric stratum. Hence the restriction map $\Gamma(X, \mathcal{O}_X) \to \Gamma(U \cup V, \mathcal{O}_X)$ is an isomorphism by Lemma 2.35.

Similarly, $\widetilde{U} = \mathrm{Spec}\,\Bbbk[\widetilde{P}_{\sigma_\mu}]/_{\sigma_\mu}\tilde{J}^{>k}$ and $\widetilde{V} = \mathrm{Spec}\,\big(\Bbbk[\widetilde{P}_\rho]/_\rho\tilde{J}^{>k}_{\sigma_\mu}\big)_{\prod t_i}$ are open subschemes of $\widetilde{X} = \mathrm{Spec}\,\Bbbk[\widetilde{P}_\rho]/_\rho\tilde{J}^{>k}_{\sigma_\mu}$. The maps β_μ^+ and α'_μ induce isomorphisms $\widetilde{V} \to V$ and $\tilde{U} \to U$ respectively, which from their explicit form are clearly compatible on overlaps, defining an isomorphism $\alpha : \widetilde{U} \cup \widetilde{V} \to U \cup V$. Thus any $\xi \in \Bbbk[\widetilde{P}_\rho]/_\rho\tilde{J}^{>k}_{\sigma_\mu}$ defines a regular function on $\widetilde{U} \cup \widetilde{V}$ by restriction from $\tilde{X}$, and $(\alpha^{-1})^*(\xi) \in \Gamma(U \cup V, \mathcal{O}_X) = \Gamma(X, \mathcal{O}_X)$. As $(\alpha^{-1})^*$ coincides with $(\beta_\mu^+)^{-1}$ for functions on $\widetilde{V}$, this shows that $(\beta_\mu^+)^{-1}$ maps $\Bbbk[\widetilde{P}_\rho]/_\rho\tilde{J}^{>k}_{\sigma_\mu}$ into $R^k_{\omega\to\sigma_\mu}$.

Now (a) obviously $(\beta_\mu^+)^{-1}(t_i^{-1}\widetilde{G}_\mu^-) \in (R^k_{\omega\to\sigma_\mu})_{\prod f_{i,\sigma_\mu}}$; (b) any monomial in $\widetilde{G}_\mu^-$ is of the form $z^{m+\sum_j \varphi_{j,\rho}(\overline{m})e_j}$ with $\langle \overline{m}, \check{d}_\rho\rangle > 0$, so $\varphi_{j,\rho}(\overline{m}) = \langle \overline{m}, n_{j,\sigma_\mu}\rangle$ for $j \neq i$, but $\varphi_{i,\rho}(\overline{m}) > \langle \overline{m}, n_{i,\sigma_\mu}\rangle$. This shows that

$$t_i^{-1} z^{m+\sum_j \varphi_{j,\rho}(\overline{m})e_j} = z^{m-e_i+\sum_j \varphi_{j,\rho}(\overline{m})e_j} \in \Bbbk[\widetilde{P}_{\sigma_\mu}],$$

so $(\beta_\mu^+)^{-1}\big(t_i^{-1}\widetilde{G}_\mu^-\big) \in R^k_{\mathrm{id}_{\sigma_\mu}}$.

Thus from (a) and (b), we see that

$$G_\mu^- := (\beta_\mu^+)^{-1}\big(t_i^{-1}\widetilde{G}_\mu^-\big) \in R^k_{\omega\to\sigma_\mu}$$

as desired. For the statement that $G_\mu^- \in I_0$, we note that if $z^m \notin I_0$, then $\overline{m} \in \Lambda_\tau$, but such monomials do not occur in G_μ^- by the properties of $\widetilde{G}_\mu^-$. $\square$

Proposition 4.15. *For singular $\rho = \rho_\mu$, define log automorphisms*

$$\theta_\mu^\pm : P \longrightarrow (R_g^k)^\times,\ \theta_\mu^\pm(m) = (1+G_\mu^\pm)^{\pm\langle \overline{m}, \check{d}_\rho\rangle}$$
$$\theta'_\mu : P \longrightarrow (R_g^k)^\times,\ \theta'_\mu(m) = F_\mu^{-\langle \overline{m}, \check{d}_\rho\rangle},$$

where

$$F_\mu = \kappa_\mu^{-1}(\widetilde{F}_\mu) = \kappa_{\mu+1}^{-1}(\widetilde{F}_\mu).$$

Then $\theta_\mu^\pm \in G_{\mathrm{j}}^k$ and

$$\theta_\mu^+ \circ \theta'_\mu \circ (\theta_\mu^-)^{-1} = \theta_\mu. \tag{4.9}$$

Proof. The fact that $\theta_\mu^\pm \in G_{\mathrm{j}}^k$ follows from the fact that $G_\mu^\pm$ are in I_0 and are induced by elements of $\Bbbk[P]$, by Lemma 4.14. Similarly, by condition (i) of Lemma 4.13(1), $F_\mu \in (R_g^k)^\times$.

We are going to verify (4.9) in the form

$$\theta_\mu^+ \circ \theta'_\mu = \theta_\mu \circ \theta_\mu^-. \tag{4.10}$$

The proof relies on (4.8) and the fact that $\overline{\theta_\mu}$ transforms $\kappa_{\mu+1}$ into κ_μ:

(4.11)
$$\begin{aligned}\kappa_{\mu+1}\big(\overline{\theta_\mu}(z^m)\big) &= \kappa_{\mu+1}\Big(\kappa_{\mu+1}^{-1}\big(t_i^{-\langle \overline{m}, \check{d}_\rho\rangle}\big)z^m\Big) \;=\; t_i^{-\langle \overline{m}, \check{d}_\rho\rangle} \cdot z^{m+\sum_j \langle \overline{m}, n_{j,\sigma_{\mu+1}}\rangle} \\ &= z^{m+\sum_j \langle \overline{m}, n_{j,\sigma_\mu}\rangle} \;=\; \kappa_\mu(z^m).\end{aligned}$$

Here $i = i(\rho_\mu)$. Now if $\overline{m} \in \Lambda_\rho$, then $\big(\theta_\mu^+ \circ \theta'_\mu\big)(m) = 1 = \big(\theta_\mu \circ \theta_\mu^-\big)(m)$. It thus suffices to evaluate both sides of (4.10) at one $m \in P$ with $\langle \overline{m}, \check{d}_\rho\rangle = 1$. Note that by (4.8),

$$\widetilde{F}_\mu = t_i(1+t_i^{-1}\widetilde{G}_\mu^+)(1+t_i^{-1}\widetilde{G}_\mu^-) \tag{4.12}$$

holds in $\big(\Bbbk[\widetilde{P}_\rho]/_\rho\tilde{J}^{>k}\big)_{\prod_i t_i}$. Moreover, since F_μ only involves monomials $z^{m'}$ with $\langle\overline{m'},\check{d}_\rho\rangle=0$, multiplication with F_μ^{-1} commutes with $\overline{\theta_\mu^+}$. Thus

$$\overline{\theta_\mu^+}\big(\theta'_\mu(m)\big)=F_\mu^{-1}.$$

Using the composition formula for log automorphisms (2.4) we now compute

$$\begin{aligned}\kappa_{\mu+1}\Big(\big(\theta_\mu^+\circ\theta'_\mu\big)(m)\Big)&=\kappa_{\mu+1}\Big(\overline{\theta_\mu^+}\big(\theta'_\mu(m)\big)\cdot\theta_\mu^+(m)\Big)\;=\;\kappa_{\mu+1}\Big(F_\mu^{-1}\cdot(1+G_\mu^+)\Big)\\&=\widetilde{F}_\mu^{-1}\cdot\Big(1+t_i^{-1}\widetilde{G}_\mu^+\Big)\;\overset{(4.12)}{=}\;\Big(1+t_i^{-1}\widetilde{G}_\mu^-\Big)^{-1}\cdot t_i^{-1}\\&=\kappa_\mu\Big(1+G_\mu^-\Big)^{-1}\cdot t_i^{-1}\\&\overset{(4.11)}{=}\kappa_{\mu+1}\Big(\overline{\theta_\mu}\big(\theta_\mu^-(m)\big)\Big)\cdot\kappa_{\mu+1}\big(\theta_\mu(m)\big)\\&=\kappa_{\mu+1}\Big(\big(\theta_\mu\circ\theta_\mu^-\big)(m)\Big).\end{aligned}$$

By Proposition 4.11, this implies (4.10) after localization at $\prod_i f_{i,\sigma_{\mu+1}}$, which is enough because $\operatorname{Spec} R_g^k$ has no embedded components. □

Letting $\theta_\mu^\pm=\theta'_\mu=1$ if ρ is nonsingular, Proposition 4.12 and Proposition 4.15 now show

$$\theta_r^+\circ\theta'_r\circ(\theta_r^-)^{-1}\circ\cdots\circ\theta_2^+\circ\theta'_2\circ(\theta_2^-)^{-1}\circ\theta_1^+\circ\theta'_1\circ(\theta_1^-)^{-1}=1. \tag{4.13}$$

4.4.5. *Construction of the scattering diagram.* Recall from the hypothesis of Proposition 3.9 that we are given a scattering diagram $\mathfrak{D}'$ for $\mathfrak{j}$ with $\theta^{k-1}_{\mathfrak{D}',g}=1$. Following Remark 4.6 view $\mathfrak{D}'$ as an infinitesimal scattering diagram for $G_{\mathfrak{j}}^k$ and write $\mathfrak{D}'_{\mathrm{in}}$ and $\mathfrak{D}'_{\mathrm{no}}$ for the sets of incoming and unoriented rays of $\mathfrak{D}'$, respectively. In this step we will explain how to produce a scattering diagram $\mathfrak{D}$ with certain properties given $\mathfrak{D}'_{\mathrm{in}}$ and $\mathfrak{D}'_{\mathrm{no}}$ and an additional choice of elements $\tilde{f}_1,\ldots,\tilde{f}_s\in\Bbbk[\widetilde{P}]$ as was considered above. This choice of functions will determine the functions associated to singular ρ containing τ. To determine the functions for nonsingular ρ let

$$\mathrm{NS}(\tau)=\{\mu\,|\,\rho_\mu\text{ is nonsingular}\}.$$

Now we assume we are given a collection of functions $\{F_\mu\in\Bbbk[P]\,|\,\mu\in\mathrm{NS}(\tau)\}$ with $F_\mu\in 1+I_0$. In the final step below we will then show how to choose $\tilde{f}_i$ and $\{F_\mu\,|\,\mu\in\mathrm{NS}(\tau)\}$ in such a way that $\mathfrak{D}$ is equivalent to $\mathfrak{D}'$ to order $k-1$. In doing this it turns out that we need more flexibility for terms cz^m with $\overline{\overline{m}}=0$. Hence we also add, as auxiliary input, polynomials $h_1,\ldots,h_r\in\Bbbk[P]$ with all occurring exponents m fulfilling $\operatorname{ord}_\tau(m)>0$ and $\overline{\overline{m}}=0$. These should be thought of as potential perturbations of $f_{\mathfrak{c}_\mu}$ by undirectional terms. As compatibility condition for the h_μ, we require

$$\prod_{\{\rho_\mu\,|\,i(\rho_\mu)=i\}}(1+h_\mu)\otimes\check{d}_{\rho_\mu}=1 \tag{4.14}$$

in $(R_g^k)^\times\otimes\Lambda_x^*$ for $i=1,\ldots,s$.

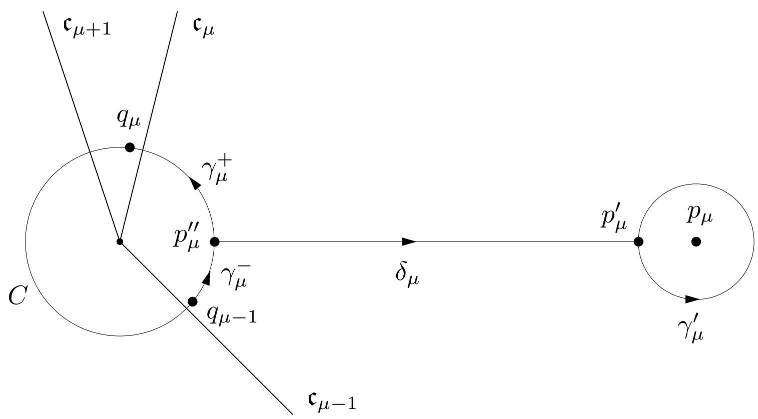


Figure 4.5. Choice of paths. Only a part of the scattering diagram is shown.

Let $\mathfrak{D}_f$ be the infinitesimal scattering diagram without s-rays or segments, and with

$$f_{\mathfrak{c}_\mu,p} = F_\mu$$

for any $p \in \mathfrak{c}_\mu \setminus \{0\}$. Here if $\mu \in \mathrm{NS}(\tau)$, then F_μ is the chosen element of $\Bbbk[P]$, and if $\mu \notin \mathrm{NS}(\tau)$, then F_μ is as constructed in Proposition 4.15 from the data $\tilde{f}_1, \ldots, \tilde{f}_s$.

For $\mu = 1, \ldots, r$ pick a point $p_\mu \in \mathrm{Int}\,\overline{\overline{\sigma_\mu}}$. As illustrated in Figure 4.5 let p'_μ be a point very close to p_μ on the line joining the origin and p_μ, and let p''_μ be a point on this same line, but very close to the origin. We can assume $p''_1, \ldots, p''_n$ are on a small circle C centered at the origin. Let $q_{\mu-1} \in C \cap \mathrm{Int}\,\overline{\overline{\sigma_\mu}}$ and close to $\mathfrak{c}_{\mu-1}$, so that there are no s-rays of $\mathfrak{D}'_{\mathrm{no}}$ intersecting C between $C \cap \mathfrak{c}_{\mu-1}$ and $q_{\mu-1}$. Let γ_μ^-, γ_μ^+ be the arcs of C running from $q_{\mu-1}$ to p''_μ and from p''_μ to q_μ, respectively. Let δ_μ be a path connecting p''_μ to p'_μ along the line joining them, and let γ'_μ be a small loop around p_μ based at p'_μ, oriented in the same direction as γ, which we take here to be a big loop around the origin. The point of these choices is that γ is freely homotopic inside $\mathcal{Q} \setminus \{0, p_1, \ldots, p_r\}$ to

$$\tilde{\gamma} := \prod_{\mu=1}^{r} \gamma_\mu^- \delta_\mu \gamma'_\mu \delta_\mu^{-1} \gamma_\mu^+. \tag{4.15}$$

Furthermore, let $q \in \mathcal{Q} \setminus \left(\{p_1, \ldots, p_r\} \cup \bigcup_\mu \mathfrak{c}_\mu\right)$ be a point not on any of the chosen paths and encircled by γ but not by $\tilde{\gamma}$.

For each $\mathfrak{z} \in \mathfrak{D}'_{\mathrm{in}}$, choose a point $q_\mathfrak{z} \in \mathrm{Int}(\mathfrak{c}_\mu)$ for μ such that $\mathfrak{z} \subseteq \overline{\overline{\sigma_\mu}}$. Taking all $q_\mathfrak{z}$'s distinct and outside C, we set

$$\mathfrak{D}_{\mathrm{in}} := \{(\mathfrak{z} + q_\mathfrak{z}, \theta_\mathfrak{z}) \,|\, (\mathfrak{z}, \theta_\mathfrak{z}) \in \mathfrak{D}'_{\mathrm{in}}\}.$$

This has the effect of translating each incoming s-ray so its endpoint lies on a cut, but not the origin. Choose the $q_{\mathfrak{z}}$'s so no element of $\mathfrak{D}_{\mathrm{in}}$ passes through q or one of the p_μ's. Also write $\mathfrak{D}_{\mathrm{no}} := \mathfrak{D}'_{\mathrm{no}}$.

Set

$$J_l := I_0^{l+1} + I_k,$$

so for sufficiently large l, $J_l = I_k$. We now construct inductively infinitesimal scattering diagrams $\mathfrak{D}_0 \subseteq \mathfrak{D}_1 \subseteq \cdots$ for G_{j}^k with the following properties:

(1) Modulo J_l,

$$\theta_{\gamma'_\mu,\mathfrak{D}_l} = \theta_{\delta_\mu,\mathfrak{D}_l} \circ \theta^{-1}_{\gamma_\mu^+,\mathfrak{D}_{\mathrm{no}}} \circ (\theta_\mu^{\mathrm{ns}})^{-1} \circ (\theta_\mu^-)^{-1} \circ \theta^+_{\mu-1} \circ \theta^{-1}_{\gamma_\mu^-,\mathfrak{D}_{\mathrm{no}}} \circ \theta^{-1}_{\delta_\mu,\mathfrak{D}_l},$$

where $\theta_\mu^\pm$ are the log automorphisms of R_g^k constructed in Proposition 4.15 (the identity if $\mu \in \mathrm{NS}(\tau)$), and

$$\theta_\mu^{\mathrm{ns}} = \begin{cases} m \mapsto F_\mu^{-\langle \overline{m}, \check{d}_{\rho_\mu}\rangle} & \mu \in \mathrm{NS}(\tau), \\ 1 & \mu \notin \mathrm{NS}(\tau). \end{cases}$$

(2) For γ' any loop around a singular point of $\mathfrak{D}_l$ *other* than the origin or any p_μ, $\theta_{\gamma',\mathfrak{D}_l} = 1 \mod J_l$.

(3) No nonfoundational element of $\mathfrak{D}_l \setminus \mathfrak{D}_{\mathrm{no}}$ intersects C or its interior.

(4) For each $\mathfrak{z} \in \mathfrak{D}_{l+1} \setminus \mathfrak{D}_l$, $\theta_{\mathfrak{z}}$ is congruent to $1 \mod J_l$.

(5) If $p \in \mathrm{Sing}(\mathfrak{D}_l) \cap (\mathfrak{c}_\mu \setminus \{0\})$ for some μ, then either (a) there is an undirectional s-ray $\mathfrak{z}$ with $p \in \mathrm{Int}(\mathfrak{z})$ and $\mathfrak{z}$ is the only nonfoundational element containing p, or (b) there is exactly one incoming segment or s-ray with endpoint p, and all other nonfoundational elements $\mathfrak{z} \in \mathfrak{D}$ containing p are oriented s-rays or segments with endpoint p which lie on the other side of $\mathfrak{c}_\mu$, with $\overline{\overline{m_{\mathfrak{z}}}}$ in the same connected component of $\mathcal{Q} \setminus \mathbb{R}\mathfrak{c}_\mu$ as $\mathrm{Int}\,\mathfrak{z}$.

(6) Given $\mathfrak{z} \in \mathfrak{D}_{\mathrm{in}}$, all s-rays in $\mathfrak{D}_l \setminus \mathfrak{D}_{\mathrm{in}}$ asymptotically parallel to $\mathfrak{z}$ are encountered by γ (the large loop around the origin) before encountering $\mathfrak{z}$.

(7) The only elements $\mathfrak{z}$ of $\mathfrak{D}_l$ containing q are unoriented s-rays $\mathfrak{l}_\mu$, $\mu = 1, \ldots, r$, with endpoint q, $\mathfrak{r}(\mathfrak{l}_\mu) = \mathfrak{c}_\mu$ and $\theta_{\mathfrak{l}_\mu} = \exp\big(-\log(1+h_\mu)\partial_{\check{d}_{\rho_\mu}}\big)$.

Note that (4.14) together with (7) implies (2) for small loops around q.

Construction 4.16. To start the inductive construction of $\mathfrak{D}_l$ consider $\mathfrak{D}_f \cup \mathfrak{D}_{\mathrm{no}} \cup \mathfrak{D}_{\mathrm{in}}$. This infinitesimal scattering diagram fulfills (1)–(6) for $l = 0$. To achieve (7) insert unoriented s-rays with endpoint q as demanded, observing (3), and crossing any cut at most once. Note that we cannot in general avoid crossing cuts, so we need to adjust the functions $f_{\mathfrak{c}_\mu,p}$ to achieve (2) for small loops around such intersection points. The adjustments are made inductively along each cut $\mathfrak{c}_\mu$, treating the points in $\mathfrak{c}_\mu \cap \bigcup_{\mu'} \mathfrak{l}_{\mu'}$ in the order encountered along $\mathfrak{c}_\mu$ starting from 0.

At such a point $p \in \mathfrak{c}_\mu \cap \mathfrak{l}_{\mu'}$, denote by $\mathfrak{c}_\pm$ the connected components of $\mathfrak{c}_\mu \setminus \bigcup_{\mu''} \mathfrak{l}_{\mu''}$ adjacent to p such that $\mathfrak{c}_+$ is contained in the unbounded part of

$\mathfrak{c}_\mu \setminus \{p\}$, and let $f_{\mathfrak{c}_\pm}$ and $\theta_{\mathfrak{c}_\pm}$ be the associated polynomials and log automorphisms, respectively. Then for a small counterclockwise loop γ' around p with appropriate base point, we find

$$\theta_{\gamma',\mathfrak{D}_l} = \theta_{\mathfrak{c}_+} \circ \theta_{\mathfrak{l}_{\mu'}}^{\mp 1} \circ \theta_{\mathfrak{c}_-}^{-1} \circ \theta_{\mathfrak{l}_{\mu'}}^{\pm 1}.$$

The signs for $\theta_{\mathfrak{l}_{\mu'}}$ are determined by the orientation of the normal bundle of $\mathfrak{l}_{\mu'}$. Then the same arguments as in Step 3 of the codimension-one case (§ 4.3) give

$$\theta_{\gamma',\mathfrak{D}_l}(m) = \left(\frac{(\overline{\theta_{\mathfrak{l}_{\mu'}}})^{\mp 1}(f_{\mathfrak{c}_-})}{f_{\mathfrak{c}_+}} \right)^{\langle \overline{m}, \check{d}_{\rho_\mu} \rangle}; \tag{4.16}$$

see (4.1) with $\theta_\pm = \theta_{\mathfrak{l}_{\mu'}}^{\pm 1}$. Now replace all polynomials $f_{\mathfrak{c}_\mu,p'}$ for p' in the unbounded part of $\mathfrak{c}_\mu \setminus \{p\}$ by $(\overline{\theta_{l_{\mu'}}})^{\mp 1}(f_{\mathfrak{c}_-})$. Continuing in this fashion along $\mathfrak{c}_\mu$ and for all μ defines $\mathfrak{D}_0$. Note that $\mathfrak{D}_0$ fulfills (2) for loops around $\mathfrak{c}_\mu \cap \mathfrak{l}_{\mu'}$ for any l.

We now construct $\mathfrak{D}_{l+1}$ from $\mathfrak{D}_l$ by adding new s-rays and segments at the singular points of $\mathfrak{D}_l$ and the p_μ.

Step 1. Let p either be a singular point of $\mathfrak{D}_l$ with $p \in \operatorname{Int} \overline{\overline{\sigma_\mu}}$ or $p = p_\mu$ for some μ. Let v^-, v^+ be primitive generators of $\mathfrak{c}_{\mu-1}$ and $\mathfrak{c}_\mu$, respectively. Let γ_p be a small loop around p, oriented in the same direction as γ. Write

$$\theta_p := \begin{cases} \theta_{\delta_\mu,\mathfrak{D}_l} \circ \theta^{-1}_{\gamma_\mu^+,\mathfrak{D}_{\mathrm{no}}} \circ (\theta_\mu^{\mathrm{ns}})^{-1} \circ (\theta_\mu^-)^{-1} \circ \theta_{\mu-1}^+ \circ \theta^{-1}_{\gamma_\mu^-,\mathfrak{D}_{\mathrm{no}}} \circ \theta^{-1}_{\delta_\mu,\mathfrak{D}_l} & p = p_\mu, \\ 1 & p \neq p_\mu. \end{cases}$$

By the inductive assumption (1),

$$\theta_p \circ \theta^{-1}_{\gamma_p,\mathfrak{D}_l} = 1 \mod J_l,$$

so we can write

$$\theta_p \circ \theta^{-1}_{\gamma_p,\mathfrak{D}_l} = \exp\left(\sum c_i z^{m_i} \partial_{n_i}\right) \mod J_{l+1}$$

with $\sum c_i z^{m_i} \partial_{n_i} \in \ker\left(\mathfrak{g}_{\mathfrak{j}}^{J_{l+1}} \to \mathfrak{g}_{\mathfrak{j}}^{J_l}\right)$. Let S be the set of indices $S = \{i \mid \overline{\overline{m_i}} \neq 0\}$. For each $i \in S$, take $\mathfrak{z}_i$ to be a suitably chosen outgoing s-ray with endpoint p, asymptotically parallel to $-\mathbb{R}_{\geq 0}\overline{\overline{m_i}}$. It will need to be chosen so that $\mathfrak{z} \cap \operatorname{Sing}(\mathfrak{D}_l) = \{p\}$, and so that it passes through the maximal cones $\overline{\overline{\sigma}}_\mu$ of Σ_τ in the same order $p - \mathbb{R}_{\geq 0}\overline{\overline{m_i}}$ passes through these cones. In addition, for $i \notin S$, we can write

$$c_i z^{m_i} \partial_{n_i} = c_i^- z^{m_i} \partial_{\check{v}^-} + c_i^+ z^{m_i} \partial_{\check{v}^+},$$

where $\check{v}^\pm \in \Lambda_{\mathfrak{j}}^\perp \otimes \mathbb{Q}$ are the dual basis to $v^\pm$. Define *unoriented* s-rays $\mathfrak{z}^\pm \subseteq \operatorname{Int} \overline{\overline{\sigma_\mu}}$ with endpoint p, asymptotically parallel to $\mathfrak{c}_{\mu-1}$ and $\mathfrak{c}_\mu$, respectively, and with

$$\theta_{\mathfrak{z}^\pm} := \exp\left(\sum_{i \notin S} c_i^\pm z^{m_i} \partial_{\check{v}^\pm}\right).$$

Setting

$$\mathfrak{D}(p) = \Big\{\big(\mathfrak{z}_i, \exp(c_i z^{m_i}\partial_{n_i})\big)\,\Big|\, i \in S\Big\} \cup \Big\{\big(\mathfrak{z}^{\pm}, \theta_{\mathfrak{z}^{\pm}}\big)\Big\},$$

we then have $\theta_{\gamma_p, \mathfrak{D}_l \cup \mathfrak{D}(p)} = \theta_p \mod J_{l+1}$, as desired.

However, we cannot just add $\mathfrak{D}(p)$ to $\mathfrak{D}_l$, because while the s-rays of $\mathfrak{D}(p)$, being the identity modulo J_l, commute modulo J_{l+1} with all nonfoundational elements of $\mathfrak{D}_l$, they do not commute with foundational elements.

To rectify this, we modify $\mathfrak{D}(p)$ as follows. Replace an s-ray $(\mathfrak{z}, \theta_{\mathfrak{z}}) \in \mathfrak{D}(p)$ with $\mathfrak{z}_1, \ldots, \mathfrak{z}_b$, where $\mathfrak{z}_1, \ldots, \mathfrak{z}_b$ are the closures of the connected components of $\mathfrak{z} \setminus \bigcup_\mu \mathfrak{c}_\mu$. Of course there are a finite number of such components, and if they are ordered so that $p \in \mathfrak{z}_1$ and $\mathfrak{z}_i \cap \mathfrak{z}_{i+1} \neq \emptyset$, then $\mathfrak{z}_i$ is a segment for $i < b$ while $\mathfrak{z}_b$ is an s-ray. Note that if $\mathfrak{z}$ was unoriented, then by the construction of $\mathfrak{D}(p)$, $\mathfrak{z}$ is in fact contained in the interior of $\overline{\overline{\sigma_\mu}}$ anyway, and no modification of $\mathfrak{z}$ is necessary. Otherwise, $\mathfrak{z}$ is an outgoing s-ray, and we define $\theta_{\mathfrak{z}_i}$ inductively as follows, starting with $\theta_{\mathfrak{z}_1} = \theta_{\mathfrak{z}}$. Let μ' be chosen so that $\mathfrak{z}_i \cap \mathfrak{z}_{i+1} \in \mathfrak{c}_{\mu'}$. If $\theta_{\mathfrak{z}_i} = \exp\big(\sum_{m,n} c_m z^m \partial_n\big)$, then by our choice of $\mathfrak{z}$, $-\overline{\overline{m}}$ is contained in the connected component of $\mathcal{Q} \setminus \mathbb{R}\mathfrak{c}_{\mu'}$ containing $\mathfrak{z}_{i+1}$. Then we take

$$\theta_{\mathfrak{z}_{i+1}} = \exp\Big(\sum_{m,n} c_m z^m f_{\rho_{\mu'}}^{|\langle \overline{m}, \check{d}_{\rho_{\mu'}}\rangle|}\partial_n\Big).$$

Note that $f_{\mathfrak{c}_{\mu'}, p'} = f_{\rho_{\mu'}} \mod I_0$ for any $p' \in \mathfrak{c}_{\mu'} \setminus \mathrm{Sing}(\mathfrak{D}_l)$ and $\partial_n f_{\rho_{\mu'}} = 0 \mod I_0$ for $n \in \Lambda_{\mathrm{j}}^\perp$, so that by Lemma 2.15, for $z^m \in J_l$,

$$\mathrm{Ad}_{\theta^{-1}_{\mathfrak{c}_{\mu'}}}(z^m \partial_n) = z^m f_{\rho_{\mu'}}^{\langle \overline{m}, \check{d}_{\rho_{\mu'}}\rangle}\partial_n$$

for $\langle \overline{m}, \check{d}_{\rho_{\mu'}}\rangle > 0$ and

$$\mathrm{Ad}_{\theta_{\mathfrak{c}_{\mu'}}}(z^m \partial_n) = z^m f_{\rho_{\mu'}}^{-\langle \overline{m}, \check{d}_{\rho_{\mu'}}\rangle}\partial_n$$

for $\langle \overline{m}, \check{d}_{\rho_{\mu'}}\rangle < 0$. From this we conclude that $\theta_{\gamma', \{\mathfrak{c}_{\mu'}, \mathfrak{z}_i, \mathfrak{z}_{i+1}\}} = 1 \mod J_{l+1}$, where γ' is a loop around the point $\mathfrak{z} \cap \mathfrak{c}_{\mu'}$. Applying this procedure to each s-ray in $\mathfrak{D}(p)$, we get a modified $\mathfrak{D}(p)$, consisting of a collection of segments and s-rays.

Step 2. Assume that $p \in \mathrm{Sing}(\mathfrak{D}_l) \cap (\mathfrak{c}_\mu \setminus \{0\})$. Since singular points on cuts contained in an unoriented s-ray or segment lie on some $\mathfrak{l}_\mu$, which we have already discussed for all l, it suffices to consider the case that p is contained in an *oriented* s-ray or segment. We then follow a similar but simpler procedure than in Step 1. By condition (5)(b), there is precisely one incoming segment or s-ray $\mathfrak{z}$ with endpoint p. Then by Lemma 2.15, $\theta_{\gamma_p, \{\mathfrak{z}, f_{\mathfrak{c}_\mu}\}} \in G^{J_{l+1}}_{\mathrm{j}, K \cup \{0\}}$ (Definition 3.6), where K is the connected component of $\mathcal{Q} \setminus \mathbb{R}\mathfrak{c}_\mu$ disjoint from $\mathfrak{z}$. Thus, using the same technique as that of the proof of Lemma 3.7 and the previous case, one can construct a collection of outgoing s-rays $\mathfrak{D}(p)$ with endpoint p and interiors contained in K such that $\theta_{\gamma_p,\mathfrak{D}_l\cup\mathfrak{D}(p)} = 1 \mod J_{l+1}$. We then modify $\mathfrak{D}(p)$ by subdividing the s-rays as we did above.

We now see if we take

$$\mathfrak{D}_{l+1} = \mathfrak{D}_l \cup \bigcup_p \mathfrak{D}(p),$$

then $\mathfrak{D}_{l+1}$ satisfies the inductive properties (1), (2) and (4). For (1) note that because all monomials occuring in $\mathfrak{D}_{\rm no}$ and $\theta_\mu^\pm$ are in I_0, we may replace $\theta_{\delta_\mu,\mathfrak{D}_{l+1}}$ by $\theta_{\delta_\mu,\mathfrak{D}_l}$ in the required equation for $\theta_{\gamma'_\mu,\mathfrak{D}_{l+1}}$. Thus (1) follows from the definition of θ_{p_μ} in Step 1 above. With a further bit of care in making the choices of s-rays above sufficiently general, all other conditions can be satisfied also. This completes the construction of the $\mathfrak{D}_l$'s.

LEMMA 4.17. *For any l, $\theta_{\gamma,\mathfrak{D}_l} = 1 \mod J_l$.*

Proof. By (3) the only elements of $\mathfrak{D}_l$ that $\gamma_\mu^\pm$ crosses are foundational or in $\mathfrak{D}_{\rm no}$. Hence

$$\theta_{\gamma_\mu^-,\mathfrak{D}_l} = \theta_{\gamma_\mu^-,\mathfrak{D}_{\rm no}}, \qquad \theta_{\gamma_\mu^+,\mathfrak{D}_l} = \begin{cases} \theta'_\mu \circ \theta_{\gamma_\mu^+,\mathfrak{D}_{\rm no}} & \mu \notin \mathrm{NS}(\tau), \\ \theta_\mu^{\rm ns} \circ \theta_{\gamma_\mu^+,\mathfrak{D}_{\rm no}} & \mu \in \mathrm{NS}(\tau), \end{cases}$$

where $\theta'_\mu = \theta_{\mathfrak{c}_\mu,p}$ is the log automorphism from Proposition 4.15 and p is the intersection point of γ_μ^+ with $\mathfrak{c}_\mu$. In view of property (1) of $\mathfrak{D}_l$, the definition of $\tilde\gamma$ (4.15) and (4.13), this shows that

$$\begin{aligned}
\theta_{\tilde\gamma,\mathfrak{D}_l} &= \prod_{\mu=r}^{1} \theta_{\gamma_\mu^+,\mathfrak{D}_l} \circ \theta^{-1}_{\delta_\mu,\mathfrak{D}_l} \circ \theta_{\gamma'_\mu,\mathfrak{D}_l} \circ \theta_{\delta_\mu,\mathfrak{D}_l} \circ \theta_{\gamma_\mu^-,\mathfrak{D}_l} \\
&= \prod_{\mu=r}^{1} \theta_{\gamma_\mu^+,\mathfrak{D}_l} \circ \theta^{-1}_{\gamma_\mu^+,\mathfrak{D}_{\rm no}} \circ (\theta_\mu^{\rm ns})^{-1} \circ \left(\theta_\mu^-\right)^{-1} \circ \theta_{\mu-1}^+ \circ \theta^{-1}_{\gamma_\mu^-,\mathfrak{D}_{\rm no}} \circ \theta_{\gamma_\mu^-,\mathfrak{D}_l} \\
&= \prod_{\mu=r}^{1} \theta'_\mu \circ \left(\theta_\mu^-\right)^{-1} \circ \theta_{\mu-1}^+ \;=\; 1.
\end{aligned}$$

By property (2), we conclude that

$$\theta_{\gamma,\mathfrak{D}_l} = \theta_{\tilde\gamma,\mathfrak{D}_l} = 1 \mod J_l$$

because γ, being a big loop around the origin, is freely homotopic to $\tilde\gamma$ in $\mathcal{Q} \setminus \{0, p_1, \ldots, p_r\}$. □

LEMMA 4.18. *For a rational half-line $\mathfrak{r} \subseteq \mathcal{Q}$, let $\theta_\mathfrak{r}$ be the contribution to $\theta_{\gamma,\mathfrak{D}_l}$ from outgoing s-rays asymptotically parallel to $\mathfrak{r}$. Then $\theta_\mathfrak{r} \in \widetilde{H}_{\rm j}^{J_l}$, and if $\mathfrak{r}$ is not a cut, then $\theta_\mathfrak{r} \in {}^\perp H_{\rm j}^{J_l}$.*

Proof. Note that automorphisms attached to nonoriented or incoming s-rays are in $H_{\mathrm{j}}^{J_l}$ in any event, and hence preserve Ω_{std}. Automorphisms attached to foundational elements preserve Ω_{std} as follows from their explicit form. Together with Proposition 4.17 this shows

$$\Omega_{\mathrm{std}} = \theta_{\gamma,\mathfrak{D}_l}(\Omega_{\mathrm{std}}) = (\theta_{\mathfrak{r}_t} \circ \cdots \circ \theta_{\mathfrak{r}_1})(\Omega_{\mathrm{std}}) \mod J_l, \tag{4.17}$$

where $\{\mathfrak{r}_1, \ldots, \mathfrak{r}_t\}$ is the finite set of asymptotic directions of outgoing rays in $\mathfrak{D}_l$ in the order encountered by γ. Assume we have shown inductively that each $\theta_{\mathfrak{r}_i}$ preserves Ω_{std} modulo J_l, the base case $l = 0$ being trivial. Then modulo J_{l+1}, we can factor $\theta_{\mathfrak{r}_i} = \theta_{i,1} \circ \theta_{i,2}$, where $\theta_{i,1} \in \widetilde{H}_{\mathrm{j}}^{J_{l+1}}$, and $\theta_{i,2} = \exp\big(\sum_m z^m \partial_{n_m}\big)$ with $n_m \in \Lambda_{\mathrm{j}}^{\perp} \otimes \Bbbk$, $z^m \in J_l$ and $-\overline{\overline{m}} \in \mathfrak{r}_i$. Then by Remark 2.16(3),

$$\theta_{\mathfrak{r}_i}(\Omega_{\mathrm{std}}) = \theta_{i,2}(\Omega_{\mathrm{std}}) = \Big(1 + \sum\nolimits_m \langle \overline{m}, n_m\rangle z^m\Big)\Omega_{\mathrm{std}} \mod J_{l+1}.$$

However, monomials $z^{m'}$, $z^{m''}$, with $-\mathbb{R}_{\geq 0}\overline{\overline{m'}} \neq -\mathbb{R}_{\geq 0}\overline{\overline{m''}}$ can never cancel, so in order for (4.17) to hold modulo J_{l+1}, we in fact must have $\langle \overline{m}, n_m\rangle = 0$ for each m. Thus $\theta_{\mathfrak{r}_\mu} \in \widetilde{H}_{\mathrm{j}}^{J_{l+1}}$. Furthermore, if $\mathfrak{r}_\mu$ is not a cut, then $\theta_{\mathfrak{r}_\mu} \in {}^{\perp}H_{\mathrm{j}}^{J_{l+1}}$ since only outgoing s-rays contribute to $\theta_{\mathfrak{r}_\mu}$, and these only involve monomials z^m with $\overline{\overline{m}} \neq 0$. □

Now take l sufficiently large so that $J_l = I_k$. Similar to Construction 4.5 we now construct an asymptotic scattering diagram $\mathfrak{D}$ by following a procedure for each rational half-line $\mathfrak{r} \subseteq \mathcal{Q}$.

First suppose $\mathfrak{r}$ is not a cut. Consider the contribution $\theta_{\mathfrak{r}}$ to $\theta_{\gamma,\mathfrak{D}_l}$ from all s-rays asymptotically parallel to $\mathfrak{r}$. By property (6) of $\mathfrak{D}_l$ we can write

$$\theta_{\mathfrak{r}} = \theta_{\mathrm{in},\mathfrak{r}} \circ \theta_{\mathrm{no},\mathfrak{r}} \circ \theta'_{\mathfrak{r}},$$

where $\theta_{\mathrm{in},\mathfrak{r}}, \theta_{\mathrm{no},\mathfrak{r}}$ and $\theta'_{\mathfrak{r}}$ are the contributions from elements of $\mathfrak{D}_{\mathrm{in}}$ and from nonoriented and outgoing s-rays, respectively. Note that by the explicit commutator formula (2.10), automorphisms attached to nonoriented s-rays asymptotically parallel to $\mathfrak{r}$ commute with automorphisms attached to oriented s-rays asymptotically parallel to $\mathfrak{r}$. Now by Lemma 4.18, $\theta'_{\mathfrak{r}} \in {}^{\perp}H^k_{\mathrm{j},\mathfrak{r}}$, so by Lemma 3.7 we can write $\theta'_{\mathfrak{r}} = \prod_{\mathfrak{r}' \in \mathfrak{D}_{\mathfrak{r}}} \theta_{\mathfrak{r}'}$ for $\mathfrak{D}_{\mathfrak{r}}$ a set of rays with support $\mathfrak{r}$ (in particular $\theta_{\mathfrak{r}'} \in {}^{\perp}H^k_{\mathrm{j},\mathfrak{r}}$).

If $\mathfrak{r} = \mathfrak{c}_\mu$ for some μ, we are not allowed to add outgoing rays with support $\mathfrak{r}$ and rather need to modify $f_{\mathfrak{c}_\mu}$. In this case, the contribution to $\theta_{\gamma,\mathfrak{D}_l}$ from s-rays asymptotically parallel to $\mathfrak{r}$ takes the form $\theta_1 \circ \theta_{\mathfrak{c}_\mu} \circ \theta_2$, where $\theta_1, \theta_2 \in G_{\mathrm{j}}^{k+1}$ and $\theta_1 \circ \theta_2 \in \tilde{H}^k_{\mathrm{j},\mathfrak{c}_\mu}$, by Lemma 4.18. Recall that $\theta_{\mathfrak{c}_\mu,p}$ is given by $m \mapsto f_{\mathfrak{c}_\mu,p}^{-\langle \overline{m}, \check{d}_{\rho_\mu}\rangle}$ for $p \in \mathfrak{c}_\mu$ far away from the origin. Note in addition that for $m \in P$, $\theta_1(m)$ and $\theta_2(m)$ can be written as a sum of terms $cz^{m'}$ with $-\overline{\overline{m'}} \in \mathfrak{c}_\mu$. It then follows that if $\theta'_{\mathfrak{c}_\mu}$ is defined by $m \mapsto \overline{\theta_1}(f_{\mathfrak{c}_\mu,p})^{-\langle \overline{m}, \check{d}_{\rho_\mu}\rangle}$, then

$$\begin{aligned}\theta_1 \circ \theta_{\mathfrak{c}_\mu}(m) &= \overline{\theta_1}(f_{\mathfrak{c}_\mu,p}^{-\langle \overline{m}, \check{d}_{\rho_\mu}\rangle}) \cdot \theta_1(m)\\ &= \theta'_{\mathfrak{c}_\mu}(m) \cdot \overline{\theta'_{\mathfrak{c}_\mu}}\big(\theta_1(m)\big) \;=\; \big(\theta'_{\mathfrak{c}_\mu} \circ \theta_1\big)(m).\end{aligned}$$

Thus $\theta_1 \circ \theta_{\mathfrak{c}_\mu} \circ \theta_2 = \theta'_{\mathfrak{c}_\mu} \circ \theta_1 \circ \theta_2$, and since $\theta_1 \circ \theta_2 \in \widetilde{H}^k_{\mathfrak{j},\mathfrak{c}_\mu}$, we can in fact write $\theta_1 \circ \theta_2$ as $m \mapsto g_\mu^{-\langle \overline{m}, \check{d}_{\rho_\mu}\rangle}$ for some g_μ only involving monomials z^m with $\langle \overline{m}, \check{d}_{\rho_\mu}\rangle = 0$. Thus the log automorphism $m \mapsto \big(\overline{\theta_1}(f_{\mathfrak{c}_\mu,p})\cdot g_\mu\big)^{-\langle \overline{m}, \check{d}_{\rho_\mu}\rangle}$ coincides with $\theta_1 \circ \theta_{\mathfrak{c}_\mu} \circ \theta_2$.

Now set

$$\mathfrak{D} := \mathfrak{D}'_{\text{in}} \cup \mathfrak{D}'_{\text{no}} \cup \bigcup\nolimits_{\mathfrak{r}} \mathfrak{D}_{\mathfrak{r}} \cup \big\{\overline{\theta_1}(f_{\mathfrak{c}_\mu,p}) \cdot g_\mu\big\},$$

where the union is over all rational half-lines $\mathfrak{r} \subseteq \mathcal{Q}$ that are not cuts. By construction, $\theta^k_{\mathfrak{D}} = \theta^k_{\mathfrak{D}_l} = 1$.

4.4.6. *Construction of* $\tilde{f}_i$, h_μ *and* $\{F_\mu \,|\, \mu \in \mathrm{NS}(\tau)\}$. The diagram $\mathfrak{D}$ constructed in 4.4.5 depends on the choices of $\tilde{f}_1, \ldots, \tilde{f}_s$, $h_1, \ldots, h_r$ and $\{F_\mu \,|\, \mu \in \mathrm{NS}(\tau)\}$, and needs not be an extension of $\mathfrak{D}'$. We now explain how to make these choices so that it is. We will construct a sequence of choices for the $\tilde{f}_i$'s, F_μ's and h_μ's, $(\tilde{f}^l_i)$, (F^l_μ), (h^l_μ), $l = 0, 1, \ldots$, each yielding via Construction 4.16 a scattering diagram $\mathfrak{D}(l) = \big\{(\mathfrak{r}, m_{\mathfrak{r}}, c_{\mathfrak{r}}), f^l_{\mathfrak{c}_\mu}\big\}$ with the properties

1. $\mathfrak{D}(l)$ is equivalent to $\mathfrak{D}'$ modulo $J_l + I_{k-1}$;
2. for each μ, any monomial z^m appearing in $f^l_{\mathfrak{c}_\mu} - f_{\mathfrak{c}_\mu} \mod J_l$ fulfills $-\overline{\overline{m}} \in \mathfrak{c}_\mu$.

To begin, take $h^0_\mu = 0$, $F^0_\mu = 1$ for $\mu \in \mathrm{NS}(\tau)$, and $\tilde{f}^0_i$ to be given by the condition that $c_{i,m} = 0$ unless $\mathrm{ht}(m) = 0$ and $\overline{\overline{m}} \in \mathfrak{c}_\mu$ for some μ with $i = i(\rho_\mu)$. Note that $\tilde{f}^0_i$ is uniquely determined by the f_ρ's. Furthermore, modulo $J_0 = I_0$, $\mathfrak{D}(0)$ is equivalent to the scattering diagram with functions f_ρ and no rays. Indeed, all nonfoundational elements in $\mathfrak{D}(0)$ are irrelevant modulo I_0, and the statement for the foundational elements follows from Lemma 4.13(1)(i) and $F_\mu = \kappa_\mu^{-1}(\widetilde{F}_\mu)$ for $\mu \notin \mathrm{NS}(\tau)$.

Now suppose we have constructed $\tilde{f}^l_i$, F^l_μ and h^l_μ with the desired properties. For each μ, let $g^l_{\rho_\mu} \in \Bbbk[P]$ be the sum of terms in $f^l_{\mathfrak{c}_\mu} - f_{\mathfrak{c}_\mu}$ of the form $cz^m \in J_l \setminus J_{l+1}$ with $\overline{\overline{m}} \in \mathfrak{c}_\mu \setminus \{0\}$. These are the terms we need to remove from $f_{\mathfrak{c}_\mu}$ to obtain (2). Denote by $\tilde{g}^l_\rho \in \Bbbk[\widetilde{P}]$ what we get by replacing each term cz^m in g^l_ρ by $cz^{m+\sum_j \varphi_j(\overline{m})e_j}$. We then take

$$\begin{aligned}\tilde{f}^{l+1}_i &= \tilde{f}^l_i - \sum_{\{\rho \,|\, i(\rho)=i\}} \tilde{g}^l_\rho,\\ F^{l+1}_\mu &= F^l_\mu - g^l_{\rho_\mu}, \quad \mu \in \mathrm{NS}(\tau).\end{aligned}$$

At this point, we do not change the h_μ's. We now look carefully at how the new scattering diagram $\mathfrak{D}(l+1)$ differs from $\mathfrak{D}(l)$ modulo J_{l+1}.

To do so, we first use Lemma 4.13(2) to see how $\widetilde{F}_\mu$ and $\widetilde{G}_\mu^\pm$ change for $\mu \notin \mathrm{NS}(\tau)$. Modulo ${}_{\rho_\mu}\tilde{J}_{l+1}^{>k}$, $\widetilde{F}_\mu$ is replaced by

$$\widetilde{F}_\mu - \tilde{g}_{\rho_\mu}^l + \sum_{\{m \,|\, -\overline{\overline{m}} \in \mathfrak{c}_\mu \setminus \{0\}\}} a_m z^{m+\sum_j \varphi_{j,\rho_\mu}(m) e_l},$$

where $a_m \in \Bbbk[\widetilde{P}_{\rho_\mu}] \setminus {}_{\rho_\mu}\tilde{J}_0^{>k}$. As $F_\mu = \kappa_\mu^{-1}(\widetilde{F}_\mu)$, this has the effect, modulo J_{l+1}, of replacing F_μ with

$$F_\mu - g_{\rho_\mu}^l + \text{terms of the form } cz^m \text{ with } -\overline{\overline{m}} \in \mathfrak{c}_\mu \setminus \{0\}.$$

Similarly, $\widetilde{G}_\mu^\pm$ are only changed by terms in ${}_{\rho_\mu}\tilde{J}_l^{>k} \setminus {}_{\rho_\mu}\tilde{J}_{l+1}^{>k}$, so $G_\mu^\pm$ are only changed by terms in $J_l \setminus J_{l+1}$. Tracing through Construction 4.16 one sees that this has the effect of producing $\mathfrak{D}(l+1)$ with the property that modulo J_{l+1},

$$f_{\mathfrak{c}_\mu}^{l+1} - f_{\mathfrak{c}_\mu} = \text{sum of terms of the form } cz^m \text{ with } -\overline{\overline{m}} \in \mathfrak{c}_\mu.$$

The same holds for $\mu \in \mathrm{NS}(\tau)$ directly from the construction of F_μ^{l+1}. This yields the desired condition (2). Unfortunately, we cannot use the uniqueness statement Proposition 4.1 yet to deduce (1) for $l+1$ because $f_{\mathfrak{c}_\mu}^{l+1} - f_{\mathfrak{c}_\mu}$ may contain monomials z^m with $\overline{m} \in \Lambda_{\mathrm{j}}$. Thus further modification is necessary.

Modify $\mathfrak{D}(l+1)$ to get an auxiliary scattering diagram $\widehat{\mathfrak{D}}(l+1)$ as follows. Let $\hat{f}_{\mathfrak{c}_\mu}^{l+1}$ be obtained from $f_{\mathfrak{c}_\mu}^{l+1}$ by subtracting those terms cz^m in $f_{\mathfrak{c}_\mu}^{l+1} - f_{\mathfrak{c}_\mu}$ with $\overline{m} \in \Lambda_{\mathrm{j}}$. Note that by induction hypothesis (1) for $\mathfrak{D}(l)$, $\hat{f}_{\mathfrak{c}_\mu}^{l+1} - f_{\mathfrak{c}_\mu} \in J_l + I_{k-1}$. We then replace each foundational element in $\mathfrak{D}(l+1)$ by replacing $f_{\mathfrak{c}_\mu}^{l+1}$ with $\hat{f}_{\mathfrak{c}_\mu}^{l+1}$ to get $\widehat{\mathfrak{D}}(l+1)$. Since by construction and induction hypothesis (1), $\mathfrak{D}(l), \mathfrak{D}(l+1)$ and $\hat{\mathfrak{D}}(l+1)$ are all equivalent to $\mathfrak{D}'$ modulo $J_l + I_{k-1}$, we have

$$\theta_{\hat{\mathfrak{D}}(l+1)}^{k-1} = \theta_{\mathfrak{D}(l)}^{k-1} = \theta_{\mathfrak{D}'}^{k-1} = 1 \mod J_l + I_{k-1}.$$

Moreover, by construction $\hat{f}_{\mathfrak{c}_\mu}^{l+1} - f_{\mathfrak{c}_\mu}$ consists only of terms $z^m \in J_l \setminus J_{l+1}$ with $-\overline{\overline{m}} \in \mathfrak{c}_\mu \setminus \{0\}$. On the other hand, using Proposition 3.10(1) to compare $\theta_{\hat{\mathfrak{D}}(l+1)}^{k-1}$ and $\theta_{\mathfrak{D}(l+1)}^{k-1} = 1$, we see that

$$\theta_{\hat{\mathfrak{D}}(l+1)}^{k-1} = \exp\Big(-\sum_\mu \big[(\hat{f}_{\mathfrak{c}_\mu}^{l+1} - f_{\mathfrak{c}_\mu}^{l+1})/f_{\rho_\mu}\big]\partial_{\check{d}_{\rho_\mu}}\Big) \mod J_{l+1} + I_{k-1}.$$

Hence Proposition 4.1 tells us that $\hat{\mathfrak{D}}(l+1)$ must be equivalent to $\mathfrak{D}'$ modulo $J_{l+1} + I_{k-1}$. Thus, in particular,

$$\theta_{\hat{\mathfrak{D}}(l+1)}^{k-1} = 1 \mod J_{l+1} + I_{k-1}.$$

By the normalization condition, if $i(\rho) = i(\rho')$, or equivalently $Z_\rho \cap X_\tau = Z_{\rho'} \cap X_\tau$, then $f_\rho = f_{\rho'} \mod I_0$. Thus again modulo $J_{l+1} + I_{k-1}$,

$$1 = \theta^{k-1}_{\hat{\mathfrak{D}}(l+1)} = \exp\Bigg(- \sum_{\mu \in \mathrm{NS}(\tau)} \big(\hat{f}^{l+1}_{\mathfrak{c}_\mu} - f^{l+1}_{\mathfrak{c}_\mu}\big)\partial_{\check{d}_{\rho_\mu}} - \sum_{i=1}^{s} f_i^{-1}\Bigg(\sum_{\{\mu \,|\, i(\rho_\mu)=i\}} \big(\hat{f}^{l+1}_{\mathfrak{c}_\mu} - f^{l+1}_{\mathfrak{c}_\mu}\big)\partial_{\check{d}_{\rho_\mu}}\Bigg)\Bigg).$$

Here $f_i := f_\rho$ for some ρ with $i(\rho) = i$. Now since $f_1, \ldots, f_s$ are relatively prime modulo I_0 by Definition 1.26(ii), this is only possible if f_i divides $\sum_{\{\mu|i(\rho_\mu)=i\}} \big(\hat{f}^{l+1}_{\mathfrak{c}_\mu} - f^{l+1}_{\mathfrak{c}_\mu}\big)\partial_{\check{d}_{\rho_\mu}}$ modulo $J_{l+1} + I_{k-1}$ for $i = 1, \ldots, s$. We now use changes of h^l_μ to turn to zero each term for $\mu \in \mathrm{NS}(\tau)$ and each sum over μ in the second summation. If $\mu \in \mathrm{NS}(\tau)$, take

$$h^{l+1}_\mu = h^l_\mu + (\hat{f}^{l+1}_{\mathfrak{c}_\mu} - f^{l+1}_{\mathfrak{c}_\mu}).$$

If $\mu \notin \mathrm{NS}(\tau)$, the polygon $\check{\Xi}_i$ belonging to μ according to Remark 1.27 is either a line segment or a triangle. Let μ_ν, $\nu = 1, 2$ or $\nu = 1, 2, 3$ be the corresponding indices, that is, with $i(\rho_{\mu_\nu}) = i$. In the case of a line segment we have $\check{d}_{\rho_{\mu_1}} = -\check{d}_{\rho_{\mu_2}}$, while in the case of a triangle the $\check{d}_{\rho_{\mu_\nu}}$ generate $\Lambda^\perp_\tau$ as $\mathbb{Q}$-vector space. In any case we can write, modulo $J_{l+1} + I_{k-1}$,

$$\sum_\nu \big(\hat{f}^{l+1}_{\mathfrak{c}_{\mu_\nu}} - f^{l+1}_{\mathfrak{c}_{\mu_\nu}}\big)\partial_{\check{d}_{\rho_{\mu_\nu}}} = f_i \sum_\nu a_{i,\nu}\partial_{\check{d}_{\rho_{\mu_\nu}}} \tag{4.18}$$

for some $a_{i,\nu} \in \Bbbk[P]$ containing only monomials z^m with $\overline{m} \in \Lambda_{\mathfrak{j}}$. Now take

$$h^{l+1}_{\mu_\nu} = h^l_{\mu_\nu} + a_{i,\nu}.$$

We can now run Construction 4.16 again, with the same $\tilde{f}^{l+1}_i$, F^{l+1}_μ, as previously, but now with the newly defined h^{l+1}_μ rather than h^l_μ. Since we only changed h^l_μ by terms cz^m in J_l and with $\overline{m} \in \Lambda_{\mathfrak{j}}$, the infinitesimal scattering diagram $\mathfrak{D}_{l+1}$ remains unchanged modulo J_l, while modulo J_{l+1} it differs only on the automorphisms associated to the undirectional s-rays emanating from q as given by the modification of h^l_μ. In fact, the definition of the $f_{\mathfrak{c}_\mu,p}$ from (4.16) remains unchanged because $\overline{\theta_{\mathfrak{l}_{\mu'}}}$ acts trivially on z^m if $\overline{m} \in \Lambda_{\mathfrak{j}}$, while if $\overline{m} \notin \Lambda_{\mathfrak{j}}$, then $\overline{\theta_{\mathfrak{l}_{\mu'}}}(z^m)$ differs from z^m by terms in J_{l+1}.

The effect to the scattering diagram $\mathfrak{D}(l+1)$ is that modulo $J_{l+1} + I_{k-1}$, for $\mu \in \mathrm{NS}(\tau)$, $f^{l+1}_{\mathfrak{c}_\mu}$ gets replaced by $\hat{f}^{l+1}_{\mathfrak{c}_\mu}$. For $\mu_\nu \notin \mathrm{NS}(\tau)$, we add $f_i a_{i,\nu}$ to $f^{l+1}_{\mathfrak{c}_{\mu_\nu}}$. Hence by the definition of $a_{i,\nu}$ in (4.18) we now obtain

$$\sum_{\{\mu \,|\, i(\rho_\mu)=i\}} \big(\hat{f}^{l+1}_{\mathfrak{c}_\mu} - f^{l+1}_{\mathfrak{c}_\mu}\big)\partial_{\check{d}_{\rho_\mu}} = 0 \mod J_{l+1} + I_{k-1}$$

for any i. From the condition that each $\check{\Xi}_i$ is a line segment or triangle with every edge of unit affine length, this can only hold if $\hat{f}^{l+1}_{\mathfrak{c}_\mu} - f^{l+1}_{\mathfrak{c}_\mu}$ is independent modulo $J_{l+1} + I_{k-1}$ of the choice of μ with $i(\rho_\mu) = i$.

We are now in position to modify $\tilde{f}^{l+1}_i$ a second time by, for each term cz^m in $\hat{f}^{l+1}_{\mathfrak{c}_\mu} - f^{l+1}_{\mathfrak{c}_\mu}$ for any μ with $i(\rho_\mu) = i$, adding

$$cz^{m+\sum_j \varphi_j(m)e_j}$$

to $\tilde{f}^{l+1}_i$. By applying Lemma 4.13(2) and the argument already made, $\mathfrak{D}(l+1)$ is modified so that now $f^{l+1}_{\mathfrak{c}_\mu} - f_{\mathfrak{c}_\mu}$ contains no terms in $J_l \setminus J_{l+1}$ of the form z^m with $-\overline{\overline{m}} \notin \mathfrak{c}_\mu \setminus \{0\}$. Thus by the same uniqueness arguments, $\mathfrak{D}(l+1)$ coincides with $\mathfrak{D}'$ modulo $J_{l+1} + I_{k-1}$. This completes the inductive construction of $\tilde{f}^{l+1}_i$ and h^l_μ.

Now take l sufficiently large so that $J_l = I_k$. Then $\theta^k_{\mathfrak{D}(l)} = 1 \mod I_k$, and $\mathfrak{D}(l)$ is equivalent to $\mathfrak{D}'$ modulo I_k. The diagram $\mathfrak{D}(l)$ is almost what we want; the functions $f_{\mathfrak{c}_\mu}$ however may still contain terms of the form $cz^m \in I_{k-1} \setminus I_k$ with $\overline{\overline{m}} \in \Lambda_{\mathfrak{j}}$, which we do not wish to allow in Proposition 3.9. We simply discard these terms to get $\mathfrak{D}$; by Proposition 3.10(1), $\theta^k_{\mathfrak{D}}$ takes the desired form. $\square$

5. **Concluding remarks**

We will end with a number of short remarks and observations about our construction.

Remark 5.1. The first point to emphasize is the importance of the normalization procedure carried out in Step III of the algorithm. Observe that given $\mathscr{S}_{k-1}$ consistent to order $k-1$, we actually can produce many liftings to obtain a structure $\mathscr{S}_k$ consistent to order k. We can do so by modifying the structure $\mathscr{S}_k$ produced in our algorithm as follows. Change all the slabs in a given $\rho \in \mathscr{P}^{[n-1]}$ in the same way. For each vertex v' of ρ choose $c_{v'} \in \Bbbk$, and add to $f_{\mathfrak{b},x}$ for $x \in \mathfrak{b} \setminus \Delta$ the expression

$$D(s_e, \rho, v[x]) \cdot s_e\left(\sum_{v'} c_{v'} t^k z^{m^\rho_{v[x]v'}}\right)$$

with $e : v[x] \to \rho$. Doing so does not destroy consistency for codimension-one joints. If such modifications are made for each $\rho \in \mathscr{P}^{[n-1]}$, then consistency at codimension-two joints is a cocycle condition which, with proper choices of coefficients $c_{v'}$, can be satisfied. In the case when B is simple, it turns out that this gives all "well-behaved" logarithmic k-th order liftings of the $(k-1)$-st order logarithmic deformation of X specified by $\mathscr{S}_{k-1}$. In fact, one can develop a form of logarithmic deformation theory for log Calabi-Yau spaces, which is done in [GS10], which explains what "well-behaved" means. In this context,

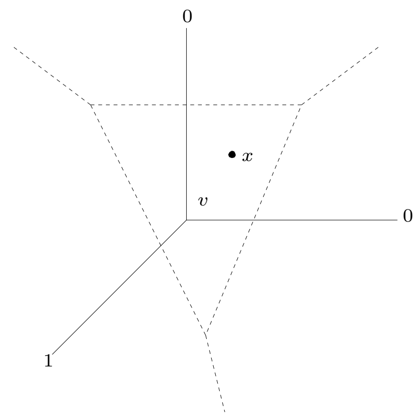


Figure 5.1.

the set of these well-behaved k-th order liftings modulo a suitable equivalence relation is in fact a vector space of the expected dimension, defined as H^1 of a sheaf of logarithmic derivations of X. What might seem surprising at first about our construction is that we construct a canonical choice of lifting; normally one expects the set of liftings to be a torsor over this H^1, without a canonical choice of origin.

Of course, in mirror symmetry, there is a natural set of coordinates on the moduli space of Calabi-Yau varieties near a large complex structure limit point, namely canonical coordinates. The expectation is that our canonical choice of lifting makes t into a canonical coordinate. We do not wish to make this statement precise here, but just illustrate in a simple example why this might arise. Consider a local three-dimensional example. Suppose we have a vertex $v \in \mathscr{P}$ contained in a two-dimensional monodromy invariant affine subspace, a plane, as depicted in Figure 5.1. The figure only shows a neighbourhood of v and only those cells of $\mathscr{P}$ contained in the plane. The dotted lines represent the discriminant locus, and the numbers indicate a choice of representative for φ near v; the value given is that on a primitive vector in the direction of the labelled cell. Assuming all monodromy vectors $m^{\rho}_{v'v}$ appearing in this example are primitive and the gluing data is trivial, the slab function at the point x, as depicted, takes the form

$$f_{\mathfrak{b},x} = 1 + z^{(1,0,0,0)} + z^{(0,1,0,0)} + z^{(-1,-1,0,1)} + \sum_{k\geq 1} a_k t^k,$$

where $t = z^{(0,0,0,1)}$. The normalization condition dictates the values of the coefficients a_k, which are easily seen to give the sum

$$-2t + 5t^2 - 32t^3 + 286t^4 - 3038t^5 + \cdots .$$

This can be compared with the mirror of the anti-canonical bundle of $\mathbb{P}^2$, as described, e.g., in [GZ02, §4.2]; the extra power series in t means t is a canonical coordinate for this family.

The next observation is that our construction is integral in a precise sense. This may be related to some of the observed arithmetic properties of mirror phenomena.

THEOREM 5.2. *Let $A \subseteq \Bbbk$ be a subring. Given a proper, locally rigid, positive, toric log* CY*-pair given by open gluing data $s = (s_e)_e$ with s_e taking values in $A^\times$ for all e, then the structures $\mathscr{S}_k$ produced by our algorithm are also defined over A; i.e. for each slab $\mathfrak{b} \in \mathscr{S}_k$ and wall $\mathfrak{p} \in \mathscr{S}_k$,*

$$f_{\mathfrak{b},x} \in A[P_x],$$
$$c_{\mathfrak{p}} \in A.$$

Proof. We will give a sketch of the argument. One needs to check that at each step of the algorithm, all coefficients are in A. In Step I, we need to know that in Proposition 3.9, if $\mathfrak{D}'$ is defined over A, so is $\mathfrak{D}$. To check this, we need to check it is never necessary to divide by an element of A. It is easy to see that this is the case if $\operatorname{codim}\sigma_{\mathfrak{j}} = 0$ directly from the proof of that case. Indeed, in the exponentials which appear in the proof, only a first order expansion is necessary as terms of the form $z^{m_i} \in I_{k-1} \setminus I_k$ have square zero modulo I_k. Thus no denominators appear in the expansion of the expressions used in the proof. Note that here it is important that the log automorphisms associated to walls take the form $\exp(-\log(1+cz^m)\partial_n)$ rather than $\exp(-cz^m\partial_n)$, to guarantee that no denominators appear in the automorphisms attached to walls.

When $\operatorname{codim}\sigma_{\mathfrak{j}} = 1$, a similar analysis of the argument in Section 4.3 shows the same integrality. However, this is not true of the argument given for $\operatorname{codim}\sigma_{\mathfrak{j}} = 2$, but it is faster in both cases to argue directly. Once one knows that Proposition 3.9 is true, one knows that the naive algorithm which works for codimension-zero joints also works for the other types of joints, as described at the beginning of Section 4.4.1. We omit the details.

Step II presents no additional problems; the relevant relative homology groups are zero whether the coefficient ring is $\Bbbk$ or A, and thus we only need to modify slabs by adding terms with coefficients in A. Finally, integrality in Step III requires understanding the normalization condition better. Consider the following situation. Suppose that we have $\rho \in \mathscr{P}^{[n-1]}$, $v \in \rho \subseteq \sigma \in \mathscr{P}_{\max}$, so we have the set

$$E = \{m \in P_{\rho,\sigma} | \overline{m} \in K_v\rho \setminus \{0\}\}.$$

Suppose $f = f_0 + g \in A[P_{\rho,\sigma}]$ such that f_0 consists of all terms with zero ord_ρ appearing in f, and g contains only exponents m with $\overline{m} \in \Lambda_\rho$ and with $\operatorname{ord}_\rho m > 0$. Furthermore, assume f_0 has constant term $a_0 \in A^\times$, all other exponents appearing in f_0 are contained in E, and $\mathrm{tlog}_v(f) \in (t^k)$. Then we need to show that $\mathrm{tlog}_v(f) = at^k \mod t^{k+1}$ for $a \in A$. To do this, we can first of all replace f by f/a_0 (recalling that $a_0 \in A^\times$) without changing $\mathrm{tlog}_v f$, so we can assume the constant term is 1. We will show how to expand f in an infinite product expansion

$$f = \prod_{m \in P_{\rho,\sigma}} \left(1 + a_m z^m\right)$$

with $a_m \in A$ and which converges in the completed ring ${}^v\widehat{R}^k_{\mathrm{id}_\rho,\sigma}$. A sufficient condition to guarantee this convergence is that for any ν, there are only a finite number of m with $\overline{m} \notin \nu E$ with $a_m \neq 0$. Then for any given ν, all but a finite number of the cross-terms in the expansion of the product are in νE.

We will construct the infinite product expansion in two steps. First, write $f = f_0(1 + g/f_0)$. This factorization can be performed in ${}^v\widehat{R}^k_{\mathrm{id}_\rho,\sigma}$, since f_0 is invertible in this ring. We then express both f_0 and $1 + g/f_0$ as infinite products of the desired form.

To express f_0 as an infinite product, we proceed inductively, for each $\nu \geq 1$, writing f_0 as a product $\prod_m (1 + a_m z^m)$ with $m \notin \nu E$, up to terms in νE. For $\nu = 1$, the product is taken to be empty. For $\nu > 1$, suppose that the product $\prod_m (1 + a_m z^m)$ agrees with f_0 up to terms in $(\nu - 1)E$. Then $f_0 - \prod_m (1 + a_m z^m)$ contains only a finite number of terms $\sum b_{m'} z^{m'}$ with $m \in (\nu - 1)E \setminus \nu E$. We then can replace $\prod (1 + a_m z^m)$ with

$$\prod \left(1 + a_m z^m\right) \prod \left(1 + b_{m'} z^{m'}\right)$$

to obtain a product which works for ν.

For $1 + g/f_0$, we proceed similarly, but now not all the exponents occuring are in νE for some ν. This time we proceed order-by-order. Suppose we have writtten $1 + g/f_0 = \prod_m (1 + a_m z^m)$, an infinite product defined in the ring ${}^v\widehat{R}^l_{\mathrm{id}_\rho,\sigma}$, for some $0 \leq l < k$, and we wish to extend the infinite product to work in the ring ${}^v\widehat{R}^{l+1}_{\mathrm{id}_\rho,\sigma}$. It is not difficult to see that after expanding $1 + g/f_0$ in any of these rings, writing $f_0 = 1 + \text{terms in } E$, that for any given ν, $1 + g/f_0$ contains only a finite number of terms not in νE. The same is true of $\prod_m (1 + a_m z^m)$. Thus the same is true of

$$1 + g/f_0 - \prod_m (1 + a_m z^m) = \sum_{m'} b_{m'} z^{m'}$$

in ${}^v\widehat{R}^{l+1}_{\mathrm{id}_\rho,\sigma}$, with $\mathrm{ord}_\rho m' = l + 1$ for each m'. Then replace $\prod (1 + a_m z^m)$ with

$$\prod \left(1 + a_m z^m\right) \prod \left(1 + b_{m'} z^{m'}\right).$$

All new cross-terms now have ord_ρ larger than $l + 1$. Proceeding for l up to k, we obtain the full expansion.

We now observe that if $f = \prod_m (1 + a_m z^m)$ as above, then $\mathrm{tlog}_v f \in \Bbbk[t]/(t^{k+1})$ coincides with

$$\sum_m \mathrm{tlog}_v(1 + a_m z^m) = \sum_{\{m \,|\, \overline{m}=0\}} \mathrm{tlog}_v(1 + a_m z^m).$$

Of course, this is true if this is a finite product; for the case of an infinite product, we observe that for any given f and k, there is some $\nu \geq 1$ such that if f and f' agree up to terms in νE, then $\mathrm{tlog}_v f = \mathrm{tlog}_v f'$. Hence the infinite product can be truncated to a finite product without changing the value of tlog_v.

Now note that in this construction, since we only took products and subtracted, all of the a_m's are in the ring A, given that all the coefficients of f were. Furthermore, by assumption, $\mathrm{tlog}_v f \in (t^k)$. Since a term of the form $1 + a_l t^l$ appears at most once in the infinite product expansion of f, the only way this can happen is if the factors of the form $1+a_l t^l$ appearing in the product expansion have $a_l = 0$ for $l < k$. Thus $\mathrm{tlog}_v f = \mathrm{tlog}_v(1 + at^k) = at^k \mod t^{k+1}$ for some $a \in A$, as desired. This shows integrality in Step III of the algorithm. □

Remark 5.3. Finally, we would like to comment on the dependence on the choice of discriminant locus Δ in our construction. In fact, our construction is independent of this choice. The easiest way to see this is to run our algorithm on $X_0 \times \mathbb{P}^1$, with intersection complex $B \times [0,1]$ with the product affine structure. We take the discriminant locus in $B \times [0,1]$ to be an isotopy between two choices of discriminant locus $\Delta_0 \subseteq B \times \{0\}$ and $\Delta_1 \subseteq B \times \{1\}$, chosen so Δ contains no rational points. We then run our algorithm for $B \times [0,1]$, and it is not difficult to see that the structures $\mathscr{S}_k$ on $B \times [0,1]$ restrict to the structures on $B \times \{0\}$ and $B \times \{1\}$. In particular, the structures given by the two choices of discriminant locus in fact lead to the same formal toric degeneration of CY-pairs.

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University of California, San Diego, La Jolla, CA
E-mail: mgross@math.ucsd.edu

Albert-Ludwigs-Universität Freiburg, Freiburg, Germany
Current address: Universität Hamburg, Hamburg, Germany
E-mail: bernd.siebert@math.uni-hamburg.de