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A remark on the convolution with the box spline

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Abstract

The semi-discrete convolution with the box spline is an important tool in approximation theory. We give a formula for the difference between semi-discrete convolution and convolution with the box spline. This formula involves multiple Bernoulli polynomials.

1. Box splines and semi-discrete convolution

Let V be a n-dimensional real vector space equipped with a lattice Λ. If we choose a basis of the lattice Λ, then we may identify V with $\mathbb{R}^n$ and Λ with $\mathbb{Z}^n$. We choose here the Lebesgue measure dv associated to the lattice Λ.

Let $X = [a_1, a_2, \dots, a_N]$ be a sequence (a multiset) of N nonzero vectors in Λ.

The *zonotope* $Z(X)$ associated with X is the polytope

$$Z(X) := \left\{ \sum_{i=1}^{N} t_i a_i \,;\, t_i \in [0,1] \right\}.$$

In other words, $Z(X)$ is the Minkowski sum of the segments $[0, a_i]$ over all vectors $a_i \in X$.

We denote by $\mathbb{C}[V]$ the space of (complex valued) polynomial functions on V.

Recall that the box spline $B(X)$ is the distribution on V such that, for a test function *test* on V, we have the equality

$$\langle B(X), test\rangle = \int_{t_1=0}^{1} \cdots \int_{t_N=0}^{1} test\left(\sum_{i=1}^{N} t_i a_i\right) dt_1 \cdots dt_N. \tag{1}$$

We will also use the notation $\langle B(X), test\rangle = \int_V B(X)(v)test(v)$.

The distribution $B(X)$ is a probability measure supported on the zonotope $Z(X)$. If X is empty, then $B(X)$ is the δ distribution on V. For the basic properties of the box spline, we refer to [5] (or [6, Chap. 16]) .

If D is any distribution on V, the convolution $B(X) * D$ is well defined and is again a distribution on V. If $D = f(v)dv$ is a smooth density, then

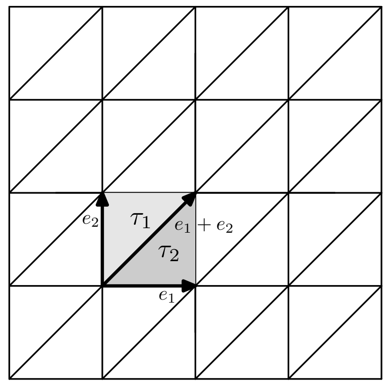

Figure 1. Affine topes for $X = [e_1, e_2, e_1 + e_2]$

$B(X) * D = F(v)dv$ is a smooth density with

$$F(v) = \int_{t_1=0}^{1} \cdots \int_{t_N=0}^{1} f\left(v - \sum_{i=1}^{N} t_i a_i\right) dt_1 \cdots dt_N.$$

If X generates V, the zonotope is a full dimensional polytope, and $B(X)$ is given by integration against a locally L^1-function. Let us describe more precisely where this function is smooth.

We continue to assume that X generates V. A hyperplane of V generated by a subsequence of elements of X is called admissible. An element of V is called (affine) regular if no translate $v+\lambda$ of v by any λ in the lattice Λ lies in an admissible hyperplane. We denote by $V_{\text{reg,aff}}$ the open subset of V consisting of affine regular elements: the set $V_{\text{reg,aff}}$ is the complement of the union of all the translates by Λ of admissible hyperplanes. A connected component τ of the set of regular elements will be called an (affine) tope (see Figure 1).

The choice of the Lebesgue measure dv on V allows us to identify distributions and generalized functions: if F is a generalized function, Fdv is a distribution. If the distribution Fdv is given by $\langle Fdv, test\rangle = \int_V f(v)test(v)dv$, with $f(v)$ locally L^1, we say that F is locally L^1, and we use the same notation for F and the locally L^1 function f.

A generalized function b on V will be called piecewise polynomial (relative to X, Λ) if:

- the function b is locally L^1;
- on each tope τ, there exists a polynomial function $b(\tau)$ on V such that the restriction of b to τ coincides with the restriction of the polynomial $b(\tau)$ to τ.

If F is a piecewise polynomial function, we will say that the distribution Fdv is piecewise polynomial.

If X generates V, the box spline $B(X)$ is a piecewise polynomial (relative to (X,Λ)) distribution supported on the zonotope $Z(X)$.

Let f be a smooth function on V. Then there are two distributions naturally associated to X,Λ,f:

- the piecewise polynomial distribution $B(X) *_d f$: on a test function *test*,

$$\langle B(X) *_d f, test\rangle = \sum_{\lambda\in\Lambda} f(\lambda)\int_{t_1=0}^{1}\cdots\int_{t_N=0}^{1} test\left(\lambda+\sum_{i=1}^{N} t_i a_i\right)dt_1\cdots dt_N;$$

- the smooth density $B(X) *_c f$: on a test function *test*,

$$\langle B(X) *_c f, test\rangle = \int_V f(v)\int_{t_1=0}^{1}\cdots\int_{t_N=0}^{1} test\left(v+\sum_{i=1}^{N} t_i a_i\right)dt_1\cdots dt_N dv.$$

The notation $*_d$ and $*_c$ means discrete, versus continuous. $B(X) *_d f$ is the convolution of $B(X)$ with the discrete measure $\sum_\lambda f(\lambda)\delta_\lambda$, while $B(X) *_c f$ is the usual convolution of $B(X)$ with the smooth density $f(v)dv$. The subscript $*_c$ is just for emphasis. The operation $*_d$ is denoted $*'$ in [5], [6] and is called semi-discrete convolution.

Our aim is to write an explicit formula for the difference $B(X) *_d f - B(X) *_c f$.

We also associate to $a\in X$ three operators:

- the partial differential operator

$$(\partial_a f)(v) = \frac{d}{d\varepsilon} f(v+\varepsilon a)\Big|_{\varepsilon=0},$$

- the difference operator

$$(\nabla_a f)(v) = f(v) - f(v-a),$$

- the integral operator

$$(I_a f)(v) = \int_0^1 f(v-ta)dt.$$

The operator I_a is the convolution $B([a]) *_c f$ with the box spline associated to the sequence with a single element a.

These three operators respect the space of polynomial functions $\mathbb{C}[V]$ on V. The Taylor series formula implies that, on the space $\mathbb{C}[V]$, the operator I_a is the invertible operator given by

$$I_a = \frac{1-e^{-\partial_a}}{\partial_a} = \sum_{j=0}^{\infty}(-1)^j\frac{1}{(j+1)!}\partial_a^j.$$

In particular, if $f \in \mathbb{C}[V]$ is a polynomial,

$$B(X) *_c f = \left(\left(\prod_{a \in X} \frac{1 - e^{-\partial_a}}{\partial_a} \right) f \right) dv. \tag{2}$$

If I, J are subsequences of X, we define the operators $\partial_I = \prod_{a \in I} \partial_a$ and $\nabla_J = \prod_{b \in J} \nabla_b$. They are defined on distributions.

Recall that $\partial_Y B(X) = \nabla_Y B(X \setminus Y)$ if Y is a subsequence of X. A subsequence Y of X will be called long if the sequence $X \setminus Y$ does not generate the vector space V. A long subsequence Y, minimal along the long subsequences, is also called a cocircuit; then $Y = X \setminus H$ where H is an admissible hyperplane.

In our formula, when f is a polynomial, $B(X) *_d f - B(X) *_c f$ is naturally expressed as a function of the derivatives $\partial_Y f$ with respect to long subsequences Y. More generally, for any smooth function f, products of difference operators ∇_I and differentiation operators ∂_J (with I and J spanning long subsets of X) will appear naturally in the rest $B(X) *_d f - B(X) *_c f$.

2. Piecewise smooth distributions

Our aim is to write an explicit formula for the difference of the two distributions $B(X) *_d f$ and $B(X) *_c f$. As the first one is a piecewise polynomial distribution, the second a smooth density, we will need to introduce an intermediate space of distributions. We will use "piecewise smooth distributions." Let us give a definition.

We continue to assume that X generates V.

Definition 2.1. A generalized function b on V will be called *piecewise smooth* (relative to X, Λ) if:

- the generalized function b is locally L^1;
- on each tope τ, there exists a smooth function $b(\tau)$ on the full space V such that the restriction of b to τ coincides with the restriction of the smooth function $b(\tau)$ to τ.

In this definition, given a tope τ, the function b restricted to τ (as well as all its derivatives) extends continuously to the closure of τ. However, these extensions do not always coincide on intersections of the closures of topes.

If b is piecewise smooth, we then say that the distribution $B := b(v)dv$ (given by integration against the locally L^1 function b) is piecewise smooth.

It is clear that if we multiply a piecewise polynomial distribution B by a smooth function, we obtain a piecewise smooth distribution. Note that the space of piecewise smooth distributions is stable by the operators ∇_a and by convolution with box splines $B(Y)$ (Y any subsequence of X). However, it is not stable under operators ∂_a. For example, $\partial_X B(X) = \nabla_X B(\emptyset)$ is a linear combination of δ distributions.

3. Multiple Bernoulli periodic polynomials

Let U be the dual vector space to V and $\Gamma \subset U$ be the dual lattice to Λ. If Y is a subsequence of X, we define

$$U_{\mathrm{reg}}(Y) = \{u \in U\,;\, \langle a, u\rangle \neq 0, \text{ for all } a \in Y\}$$

and

$$\Gamma_{\mathrm{reg}}(Y) = \Gamma \cap U_{\mathrm{reg}}(Y).$$

Consider the periodic function on V given by the (oscillatory) sum

$$W(X)(v) = \sum_{\gamma \in \Gamma_{\mathrm{reg}}(X)} \frac{e^{2i\pi\langle v,\gamma\rangle}}{\prod_{a\in X} 2i\pi\langle a,\gamma\rangle}. \tag{3}$$

This is well defined as a generalized function on V. In the sense of generalized functions, we have

$$\partial_X W(X)(v) = \sum_{\gamma \in \Gamma_{\mathrm{reg}}(X)} e^{2i\pi\langle \gamma, v\rangle}. \tag{4}$$

We will use this equation to construct "primitives" of parts of the Poisson formula.

We will call the series $W(X)$ a multiple Bernoulli series. Multiple Bernoulli series have been extensively studied by A. Szenes [7]. They are natural generalizations of Bernoulli series: for $\Lambda = \mathbb{Z}\omega$ and $X_k := [\omega, \omega, \ldots, \omega]$, where ω is repeated k times with $k > 0$, the series

$$W(X_k)(t\omega) = \sum_{n \neq 0} \frac{e^{2i\pi n t}}{(2i\pi n)^k}$$

is equal to $-\frac{1}{k!}B(k, t-[t])$, where $B(k,t)$ denotes the k^{th} Bernoulli polynomial in variable t. In particular, for $k=1$, we have $W(X_1)(t\omega) = \frac{1}{2} - t + [t]$ (see Figure 2).

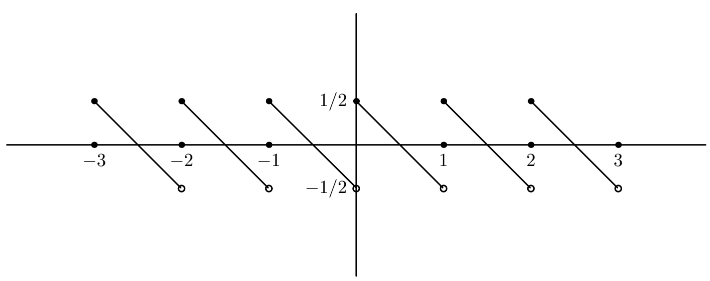

Figure 2. Graph of $W(X_1)(t\omega) = \frac{1}{2} - t + [t]$

We recall the following proposition [7] (see also [2], [1]).

PROPOSITION 3.1. *If X generates V, the generalized function $W(X)$ is piecewise polynomial* (*relative to* (X, Λ)).

Thus we will also call $W(X)$ a multiple periodic Bernoulli polynomial.

The above proposition is proved by reduction to the one variable case. Indeed, the function $\frac{1}{\prod_{a\in X}\langle a,z\rangle}$ can be decomposed in a sum of functions $\frac{1}{\prod_{i=1}^{n}\langle a_{j_i},z\rangle^{n_i}}$ with respect to a basis a_{j_i} of V extracted from X. This reduces the computation to the one-dimensional case. A. Szenes [7] gave an efficient multidimensional explicit residue formula to compute $W(X)$.

Example 3.2. Let $V = \mathbb{R}e_1 \oplus \mathbb{R}e_2$ with lattice $\Lambda = \mathbb{Z}e_1 \oplus \mathbb{Z}e_2$. Let $X = [e_1, e_2, e_1 + e_2]$. We write $v \in V$ as $v = v_1e_1 + v_2e_2$.

We compute the generalized function

$$W(v_1, v_2) \quad = \sum_{n_1\neq 0, n_2\neq 0, n_1+n_2\neq 0} \frac{e^{2i\pi(n_1v_1+n_2v_2)}}{(2i\pi n_1)(2i\pi n_2)(2i\pi(n_1+n_2))}.$$

Then W is a locally L^1-function on V, periodic with respect to $\mathbb{Z}e_1 + \mathbb{Z}e_2$. To describe this, it is sufficient to write the formulae of $W(v_1, v_2)$ for $0 < v_1 < 1$ and $0 < v_2 < 1$, which we compute (for example, using the relation $\frac{1}{n_1n_2(n_1+n_2)} = \frac{1}{n_1(n_1+n_2)^2} + \frac{1}{n_2(n_1+n_2)^2}$) as

$$W(v_1, v_2) = \begin{cases} -\frac{1}{6}(1 + v_1 - 2v_2)(v_1 - 1 + v_2)(2v_1 - v_2), & v_1 < v_2, \\ -\frac{1}{6}(v_1 - 2v_2)(v_1 - 1 + v_2)(2v_1 - 1 - v_2), & v_1 > v_2. \end{cases}$$

Thus we see that W is a piecewise polynomial function.

Remark 3.3. If X does not generate V, $W(X)$ is not locally L^1: Take $X = \emptyset$; then, by the Poisson formula, $W(\emptyset)$ is the delta distribution of the lattice Λ.

Definition 3.4. A subspace $\mathbf{s}$ of V generated by a subsequence of elements of X is called X-admissible. We denote by $\mathcal{R}$ the set of X-admissible subspaces of V. We denote by $\mathcal{R}'$ the set of proper X-admissible subspaces.

The spaces $\mathbf{s} = V$ and $\mathbf{s} = \{0\}$ are among the admissible subspaces of V. The set $\mathcal{R}'$ consists of all admissible subspaces of V, except $\mathbf{s} = V$.

Let $\mathbf{s}$ be an admissible subspace of V. Let us consider the list $X \setminus \mathbf{s}$, where we have removed from the list X all elements belonging to $\mathbf{s}$. The projection of the list $X \setminus \mathbf{s}$ on $V/\mathbf{s}$ will be denoted by $X/\mathbf{s}$. The image, in $V/\mathbf{s}$, of the lattice Λ is again a lattice. If X generates V, $X/\mathbf{s}$ generates $V/\mathbf{s}$. Using the projection $V \to V/\mathbf{s}$, we identify the piecewise polynomial function $W(X/\mathbf{s})$ on $V/\mathbf{s}$ to a piecewise polynomial function on V constant along the affine spaces $v + \mathbf{s}$.

Define $U_{\mathrm{reg}}(X/\mathbf{s}) = U_{\mathrm{reg}}(X \setminus \mathbf{s}) \cap \mathbf{s}^{\perp}$. Thus $\Gamma_{\mathrm{reg}}(X/\mathbf{s}) := \Gamma \cap U_{\mathrm{reg}}(X/\mathbf{s})$ is the set of elements $\gamma \in \Gamma$ such that:

$$\langle \gamma, s\rangle = 0 \ \text{ for all } s \in \mathbf{s}; \quad \langle \gamma, a\rangle \neq 0 \ \text{ for all } a \in X \setminus \mathbf{s}.$$

Identifying the dual space to $V/\mathbf{s}$ to the space $\mathbf{s}^\perp$, we see that the function $W(X/\mathbf{s})$ is the function on V given by the series (convergent in the sense of generalized functions)

$$W(X/\mathbf{s})(v) := \sum_{\gamma\in\Gamma_{\mathrm{reg}}(X/\mathbf{s})} \frac{e^{2i\pi\langle v,\gamma\rangle}}{\prod_{a\in X\setminus\mathbf{s}} 2i\pi\langle a,\gamma\rangle}.$$

This function is periodic with respect to the lattice Λ, piecewise polynomial on V (relative to X,Λ) and constant along $v+\mathbf{s}$.

If $\mathbf{s}=V$, the function $W(X/\mathbf{s})$ is identically equal to 1, while if $\mathbf{s}=\{0\}$, we obtain again our series $W(X)$.

4. **A formula**

Let us now state our formula. We assume, as before, that X generates V.

For each $\mathbf{s}\in\mathcal{R}$, we consider all possible decompositions of the list $X\setminus\mathbf{s}$ in disjoint lists $I\sqcup J$. If f is a smooth function, the function

$$F(v) = W(X/\mathbf{s})(v)(\partial_I\nabla_J f)(v)$$

is a piecewise smooth function on V. If Y is a subsequence of X, the convolution $B(Y)*Fdv$ is well defined and the result is a piecewise smooth distribution on V that we denote by $B(Y)*_c(W(X/\mathbf{s})\partial_I\nabla_J f)$.

THEOREM 4.1. *Let f be a smooth function on V. We have*

$$B(X)*_d f - B(X)*_c f = \sum_{\mathbf{s}\in\mathcal{R}'}\sum_{I\subset X\setminus\mathbf{s}} (-1)^{|I|} B((X\cap\mathbf{s})\sqcup I)*_c(W(X/\mathbf{s})\partial_I\nabla_J f).$$

In this formula J is the complement of the sequence I in $X\setminus\mathbf{s}$. This equality holds in the space of piecewise (*relative to* (X,Λ)) *smooth distributions on V, relative to* (X,Λ).

Remark 4.2. If f is a polynomial, the term $B(X)*_c f$ is a polynomial density and all terms of the difference formula are locally polynomial distributions on V.

Before proceeding, let us comment on the proof. As in [3] (see also [5]), we use the Poisson formula to compute $B(X)*_d f$. Then we group the terms in the dual lattice Γ in strata according to the hyperplane arrangement $\cup_{a\in X}\{a=0\}$. We then use the Bernoulli series as primitives of the corresponding sums. This way, we introduce the needed derivatives of the function f.

Proof. Let $\mathcal{R}$ be the set of admissible subspaces of V. We have the disjoint decomposition

$$U = \bigsqcup_{\mathbf{s}\in\mathcal{R}} U_{\mathrm{reg}}(X/\mathbf{s}). \qquad (5)$$

Let *test* be a test function on V. We compute

$$S := \int_V (B(X) *_d f)(v) test(v) = \sum_{\lambda\in\Lambda} f(\lambda) \int_V B(X)(v) test(\lambda+v).$$

We apply the Poisson formula to the compactly supported smooth function

$$q(w) = f(w) \int_V B(X)(v) test(w+v)$$

as our sum S is equal to $\sum_{\lambda\in\Lambda} q(\lambda)$. We obtain

$$S = \sum_{\gamma\in\Gamma} \int_V e^{2i\pi\langle w,\gamma\rangle} q(w) dw.$$

The lattice Γ is a disjoint union of the sets $\Gamma_{\mathrm{reg}}(X/\mathbf{s}) = \Gamma \cap U_{\mathrm{reg}}(X/\mathbf{s})$ associated to the admissible subspaces $\mathbf{s}$. Note that the set associated to $\mathbf{s} = V$ is $\{\gamma = 0\}$. The term in S corresponding to $\gamma = 0$ is $\int_V q(w)dw$; that is $\langle B(X) *_c f, test\rangle$.

As in the generalized function sense

$$\sum_{\gamma\in\Gamma_{\mathrm{reg}}(X/\mathbf{s})} e^{2i\pi\langle w,\gamma\rangle} = \partial_{X\setminus\mathbf{s}} W(X/\mathbf{s})(w),$$

we obtain

$$S = \sum_{\mathbf{s}\in\mathcal{R}} \int_V W(X/\mathbf{s})(w)(-1)^{|X\setminus\mathbf{s}|}\partial_{X\setminus\mathbf{s}} q(w) dw.$$

The function $q(w)$ is a product of the two smooth functions

$$f(w) \quad \text{and} \quad \int_V B(X)(v) test(w+v).$$

By the Leibniz rule,

$$S := \sum_{\mathbf{s}} (-1)^{|X\setminus\mathbf{s}|} \sum_{I\sqcup J = X\setminus\mathbf{s}} \int_V\int_V W(X/\mathbf{s})(w)\partial_I f(w) B(X)(v)\partial_J test(w+v) dw.$$

We first integrate in v and use the equation satisfied by the box spline

$$\langle B(X), \partial_b h\rangle = -\langle B(X\setminus\{b\}), \nabla_{-b} h\rangle. \tag{6}$$

Thus we obtain

$$S = \sum_{\mathbf{s}} \sum_{I\sqcup J = X\setminus\mathbf{s}} (-1)^{|I|} \int_V\int_V W(X/\mathbf{s})(w)\partial_I f(w) B(X\setminus J)(v)(\nabla_{-J} test)(w+v) dw.$$

Let us integrate in w. The integral being invariant by ∇_b, we have

$$\int_V (\nabla_b f_1)(w) f_2(w) dw = \int_V f_1(w)(\nabla_{-b} f_2)(w) dw.$$

As $b \in J$ is in Λ and $W(X/\mathbf{s})(w)$ is periodic,

$$S = \sum_{\mathbf{s}\in\mathcal{R}} \sum_{I\sqcup J = X\setminus\mathbf{s}} (-1)^{|I|} \int_V\int_V B(X\setminus J)(v) W(X/\mathbf{s})(w)\nabla_J\partial_I f(w) test(w+v) dw.$$

Writing $\mathcal{R} = \{V\} \sqcup \mathcal{R}'$, we obtain the formula of the theorem. □

On the space of polynomials, one has

$$\nabla_J \partial_I f = \left(\prod_{b \in J} \frac{1 - e^{-\partial_b}}{\partial_b} \right) \partial_{X \setminus \mathbf{s}} f$$

if $I \sqcup J = X \setminus \mathbf{s}$.

Recall that the space $D(X)$ of Dahmen-Micchelli polynomials is the space of polynomials on V such that $\partial_Y f = 0$ for all long subsequences Y. In particular, if $\mathbf{s}$ is a proper subspace, the sequence $X \setminus \mathbf{s}$ is a long subsequence. So if I and J are such that $I \sqcup J = X \setminus \mathbf{s}$ and $f \in D(X)$, then $\nabla_J \partial_I f = 0$.

As a corollary of our formula, if $p \in D(X)$, we see that $B(X) *_d p = B(X) *_c p$. Let us state more precisely this result of Dahmen-Micchelli [4] (see also [6, Chap. 17]).

Corollary 4.3. *If* $p \in D(X)$, *then*

$$P(v) := B(X) *_d p = \sum_\lambda p(\lambda) B(X)(v - \lambda)$$

is a polynomial function on V, *equal to* $(\prod_{a \in X} \frac{1-e^{-\partial_a}}{\partial_a}) p = B(X) *_c p$.

In this formula, we have identified $B(X)$, $B(X) *_d p$, and $B(X) *_c p$ to piecewise polynomial functions.

5. Vertices of the arrangement and semi-discrete convolutions

We now give a twisted version of Theorem 4.1, where we twist f by an exponential function $e^{2i\pi \langle G, v \rangle}$.

The set of characters on Λ is the torus $T := U/\Gamma$. If $g \in T$, we denote by g^λ the corresponding character on Λ. More precisely, if g has representative $G \in U$, then by definition $g^\lambda = e^{2i\pi \langle G, \lambda \rangle}$. Define

$$X(g) := \{a \in X; g^a = 1\}.$$

If $g \in T = U/\Gamma$ has representative $G \in U$, we denote by $g + \Gamma$ the set $G + \Gamma$.

For $a \in X$, introduce the operator

$$(\nabla(a, g) f)(v) = f(v) - g^{-a} f(v - a).$$

If Y is a subsequence of X, define

$$\nabla_Y^g = \prod_{a \in Y} \nabla(a, g).$$

We introduce a subset $\mathcal{R}(g)$ of admissible subspaces, depending on g.

Definition 5.1. The admissible space $\mathbf{s}$ is in $\mathcal{R}(g)$ if the space $(g + \Gamma) \cap \mathbf{s}^\perp$ is nonempty

Note that if G is not in Γ, then V is not in the set $\mathcal{R}(g)$.

Remark 5.2. If $\mathbf{s} \in \mathcal{R}(g)$, then all elements of $X \cap \mathbf{s}$ are in $X(g)$. Thus $\mathcal{R}(g)$ is contained in the set of admissible spaces for $X(g)$. However the converse does not hold: take $V = \mathbb{R}\omega$, $X = [2\omega]$, $\Lambda = \mathbb{Z}\omega$, and $G = \frac{1}{2}\omega^*$. Then $X(g) = X$, so that V is an admissible subspace for $X(g)$. However, V is not in $\mathcal{R}(g)$.

If $\mathbf{s} \in \mathcal{R}(g)$, take $g_{\mathbf{s}} \in (g+\Gamma) \cap \mathbf{s}^\perp$. Then $(g+\Gamma) \cap \mathbf{s}^\perp$ is the translate by $g_{\mathbf{s}}$ of the lattice $\Gamma \cap \mathbf{s}^\perp$.

Define

$$\Gamma_{\mathrm{reg}}(X/\mathbf{s}, g) = (g+\Gamma) \cap U_{\mathrm{reg}}(X/\mathbf{s})^\perp.$$

Thus $\Gamma_{\mathrm{reg}}(X/\mathbf{s}, g)$ consists of elements $\xi \in g+\Gamma$ such that

$$\langle \xi, s\rangle = 0 \quad \text{for all } s \in \mathbf{s}; \qquad \langle \xi, a\rangle \neq 0 \quad \text{for all } a \in X \setminus \mathbf{s}.$$

The following series

$$W(X/\mathbf{s}, g)(v) = \sum_{\xi \in \Gamma_{\mathrm{reg}}(X/\mathbf{s},g)} \frac{e^{2i\pi\langle v,\xi\rangle}}{\prod_{a\in X} 2i\pi\langle a, \xi\rangle} \tag{7}$$

is well defined as a generalized function on V.

The function $W(X/\mathbf{s}, g)(v)$ is not periodic with respect to Λ. We have instead the covariance formula

$$W(X/\mathbf{s}, g)(v-\lambda) = g^{-\lambda} W(X/\mathbf{s}, g)(v). \tag{8}$$

In the sense of generalized functions, we have

$$\partial_{X\setminus\mathbf{s}} W(X/\mathbf{s}, g)(v) = \sum_{\xi \in \Gamma_{\mathrm{reg}}(X/\mathbf{s},g)} e^{2i\pi\langle \xi, v\rangle}. \tag{9}$$

We recall the following proposition [7] (see also [2], [1]).

Proposition 5.3. *The generalized function $W(X/\mathbf{s}, g)$ is a piecewise polynomial* (*relative to* (X, Λ)) *function on V.*

This is proved similarly by reduction to one variable.

Example 5.4. Let $V = \mathbb{R}\omega$, $\Lambda = \mathbb{Z}\omega$, and $X_k := [\omega, \omega, \ldots, \omega]$, where ω is repeated k times with $k > 0$. Then $\Gamma = \mathbb{Z}\omega^*$, and if z is not an integer, we have

$$W(X_k, z\omega^*)(t\omega) = \sum_{n\in\mathbb{Z}} \frac{e^{2i\pi(n+z)t}}{(2i\pi(n+z))^k}.$$

We have, for example (see [2]),

$$W(X_1, z\omega^*)(t\omega) = \frac{e^{2i\pi[t]z}}{1-e^{-2i\pi z}},$$

$$W(X_2, z\omega^*)(t\omega) = e^{2i\pi[t]z}\left(\frac{t-[t]}{1-e^{-2i\pi z}} - \frac{1}{(1-e^{-2i\pi z})(1-e^{2i\pi z})}\right).$$

Here $[t]$ is the integral part of t. This function $[t]$ is a constant on each interval $]\ell, \ell+1[$, and $W(X_k, z\omega^*)$ is locally a polynomial function of t.

THEOREM 5.5. *Let* $G \in U$, *and* g *its image in* U/Γ. *Let* $f(v) = e^{2i\pi\langle v,G\rangle}h(v)$, *where* h *is a smooth function. Then*

$$B(X) *_d f = \sum_{\mathbf{s}\in\mathcal{R}(g)} \sum_{I\subset X\setminus\mathbf{s}} (-1)^{|I|} B((X\cap\mathbf{s})\sqcup I) *_c (W(X/\mathbf{s}, g)\partial_I \nabla_J^g h).$$

In this formula, J *is the complement of* I *in* $X \setminus \mathbf{s}$.

Remark 5.6. If $G \in \Gamma$, then $B(X) *_d f = B(X) *_d h$, and the formula of the theorem above coincides with the formula of Theorem 4.1 for h: The set $\mathcal{R}(g)$ coincides with the set $\mathcal{R}$, and the term corresponding to V in the formula of Theorem 5.5 is $B(X) *_c h$.

Proof. We proceed in the same way as in the proof of Theorem 4.1. Let *test* be a test function on V. We compute $S := \int_V (B(X) *_d f)(v) test(v)$ by the Poisson formula. If

$$q(w) = h(w) \int_V B(X)(v) test(w+v),$$

we obtain

$$S = \sum_{\gamma\in\Gamma} \int_V e^{2i\pi\langle w,\gamma\rangle} e^{2i\pi\langle w,G\rangle} q(w) dw.$$

Thus

$$S = \sum_{\xi\in(g+\Gamma)} \int_V e^{2i\pi\langle w,\xi\rangle} q(w) dw.$$

The set $g+\Gamma$ is a disjoint union over $\mathbf{s} \in \mathcal{R}(g)$ of the sets $\Gamma_{\mathrm{reg}}(X/\mathbf{s}, g) = (g+\Gamma) \cap U_{\mathrm{reg}}(X/\mathbf{s})$, so that

$$S = \sum_{\mathbf{s}\in\mathcal{R}(g)} \int_V W(X/\mathbf{s}, g)(w)(-1)^{|X\setminus\mathbf{s}|} \partial_{X\setminus\mathbf{s}} q(w) dw.$$

Then, using the Leibniz rule for ∂_a and equation (6) for the box spline, we obtain that S is equal to

$$\sum_{\mathbf{s}\in\mathcal{R}(g)} \sum_{I\sqcup J=X\setminus\mathbf{s}} (-1)^{|I|} \int_V\int_V W(X\setminus\mathbf{s}, g)(w) \partial_I f(w) B(X\setminus J)(v)(\nabla_{-J} test)(w+v) dw.$$

Using the covariance formula (8) for $W(X\setminus\mathbf{s}, g)$, we see that

$$\int_V W(X\setminus\mathbf{s}, g)(w) f_1(w)(\nabla_{-b} f_2)(w) dw = \int_V W(X\setminus\mathbf{s}, g)(w)(\nabla(b,g) f_1)(w) f_2(w) dw$$

and we obtain the formula of the theorem. □

Let us point out a corollary of this formula.

Definition 5.7. We say that a point $g \in U/\Gamma$ is a *toric vertex* of the arrangement X if $X(g)$ generates V. We denote by $\mathcal{V}(X)$ the *set of toric vertices* of the arrangement X.

If g is a vertex, there is a basis σ of V extracted from X such that $g^a = 1$, for all $a \in \sigma$. We thus see that the set $\mathcal{V}(X)$ is finite. If X is unimodular, then $\mathcal{V}(X)$ is reduced to $g = 0$.

Corollary 5.8 (Dahmen-Micchelli). *Let $g \in \mathcal{V}(X)$ be a toric vertex of the arrangement X, and let $p \in D(X(g))$ be a polynomial in the Dahmen-Micchelli space for $X(g)$. Assume that $g \neq 0$. Let $f(\lambda) = g^\lambda p(\lambda)$. Then $B(X) *_d f = 0$.*

Proof. We apply the formula of Theorem 5.5 with $h = p$. As $g \neq 0$, all terms $\mathbf{s} \in \mathcal{R}(g)$ are proper subspaces of V. Let us show that all the terms in our formula are 0. Indeed let $I \sqcup J = X \setminus \mathbf{s}$. Let $I' = I \cap X(g)$ and $J' = J \cap X(g)$. Then $I' \sqcup J' = X(g) \setminus \mathbf{s}$ is a long subset of $X(g)$. As $\nabla^g_{I'} = \nabla_{I'}$, we see that $\partial_{I'} \nabla^p_{J'}$ is already equal to 0. $\square$

A simple proof of this corollary is also given in ([6, Th. 17.15]).

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