First cohomology groups of Chevalley groups in cross characteristic

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Abstract

Let $G$ be a simple Chevalley group defined over $\mathbb{F}_q$. We show that if $r$ does not divide $q$ and $k$ is an algebraically closed field of characteristic $r$, then very few irreducible $kG$-modules have nonzero $H^1(G, V)$. We also give an explicit upper bound for $\dim H^1(G, V)$ for $V$ an irreducible $kG$-module that does not depend on $q$, but only on the rank of the group. Cline, Parshall and Scott showed that such a bound exists when $r|q$. We obtain extremely strong bounds in the case that a Borel subgroup has no fixed points on $V$.

1. Introduction

Let $G$ be a finite group with $k$ an algebraically closed field of characteristic $r$. Let $V$ be an irreducible $kG$-module that is faithful as a $G$-module (we will abuse notation and say that $V$ is a faithful $kG$-module).

We are interested in how large $H^1(G, V)$ can be. These cohomology groups are critical to understanding primitive permutation groups as well as the module structure of indecomposable modules.

In 1985, the first author [Gur86] made the rather optimistic conjecture:

**Conjecture 1.1.** There exists an absolute constant $C$ such that if $G$ is a finite group and $V$ is an irreducible faithful $kG$-module, then $\dim H^1(G, V) \leq C$.

Indeed, the original conjecture was with $C = 2$. While there were some weak results obtained about $\dim H^1$, very little progress toward the conjecture was made. There have been some examples constructed with $\dim H^1(G, V) = 3$ (in particular, see [Sco03] with examples for arbitrarily large characteristic
and [BW08] for examples in characteristic 2). However, in the last few years, there have been some breakthroughs. In particular, Cline, Parshall and Scott [CPS09, Thms. 7.3, 7.10] proved:

**Theorem 1.2.** Let $G$ be either a simple algebraic group or a finite simple group of Lie type over a field of characteristic $r$. Let $k$ be an algebraically closed field of characteristic $r$. If $V$ is an irreducible $kG$-module, then there is a constant $C$ such that $\dim H^1(G, V) < C$ where $C$ depends only on the rank of $G$.

This result depends on the fact that Lusztig’s conjecture holds in sufficiently large characteristics as well as many other deep results about algebraic groups and the relationship between cohomology of algebraic groups and cohomology of finite groups of Lie type. In particular, our conjecture implies that there are bounds on certain coefficients of the Kazhdan-Lusztig polynomials.

The general problem reduces to the case of simple groups (see [AS85, Th. 3] or [GKKL07, Lemma 5.2]). We then use the classification of finite simple groups. In this paper, we consider the finite simple Chevalley groups. We consider the Chevalley groups to include all the finite simple groups of Lie type including the twisted variations of Steinberg, Suzuki and Ree. See [GLS98, Chap. 2] and [Car89] for the basic facts about Chevalley groups, including the structure of the twisted groups and root subgroups. We consider cross characteristic modules (i.e. modules over fields of characteristic different from the natural characteristic of the Chevalley group). Thus, our results are complementary to those in [CPS09] (and our methods are almost totally disjoint as is to be expected).

So let $k$ be an algebraically closed field of characteristic $r$, and let $G$ be a finite simple Chevalley group over $\mathbb{F}_q$ with $(r, q) = 1$. We prove the same type of result as Theorem 1.2 (but with explicit, albeit almost certainly not best possible, bounds). Indeed, we prove much more — we show that there are very few irreducible $kG$-modules with nonzero $H^1$ and that $\sum \dim H^1(G, V)$ is bounded (in terms of the Weyl group of $G$) — here the sum is over all irreducible $kG$-modules. See [CPS99, Thms. 10.2, 10.5] for a weak version (in particular, there are no explicit bounds) of some of these results for $r$ sufficiently large and $G = \text{GL}_n(q)$ or $\text{SL}_n(q)$ using completely different methods.

Here is a summary of our main results. Let $B$ be a Borel subgroup of $G$, with unipotent radical $Q$, and let $W$ be the Weyl group of $G$. The permutation module $\mathcal{L} := kG^B$ will play a key role in our considerations. We denote by $\text{Irr}_k(G)$ the set of isomorphism classes of irreducible $kG$-modules. Let $e$ denote the twisted rank of $G$. (If $G$ is an untwisted group, this is just the rank of the ambient group. In general, it is the number of orbits of the graph automorphism used in defining the group on the nodes of the Dynkin diagram.
Alternatively, it is the rank of a maximal torus of the Borel subgroup as an abelian group as long as \( q \) is sufficiently large.) If \( V \) is an \( H \)-module, let \( V^H \) denote the fixed point subspace of \( H \) on \( V \); furthermore, if \( H \leq G \), then \( V^G_H \) denotes the corresponding induced module.

**Theorem 1.3.** Let \( V \) be an irreducible \( kG \)-module.

(i) If \( V \) is not a composition factor of \( L \), then \( H^1(G,V) = 0 \).

(ii) If \( V^B = 0 \), then \( \dim H^1(G,V) \leq 1 \) and there are at most four distinct such modules with \( H^1(G,V) \neq 0 \).

(iii) There are at most \( |W| \) isomorphism classes of irreducible \( kG \)-modules \( V \) with \( V^B \neq 0 \), and

\[
\sum_{V \in \text{Irr}_k(G)} \dim V^B \cdot \dim H^1(G,V) \leq |W| + \varepsilon.
\]

(iv) \( \dim H^1(G,V) \leq |W| + \varepsilon \).

Note that the number of irreducible \( kG \)-modules goes to \( \infty \) as \( q \) grows. Thus, for large \( q \), we see that almost every irreducible \( kG \)-module has trivial \( H^1 \).

Combining our theorem with Theorem 1.2 gives:

**Theorem 1.4.** Let \( G \) be a finite simple Chevalley group whose underlying algebraic group has rank \( f \). Let \( V \) be an irreducible \( KG \)-module where \( K \) is an algebraically closed field of arbitrary characteristic. Then there is a function \( C(f) \) such that \( \dim H^1(G,V) \leq C(f) \).

Thus, even if Conjecture 1.1 is false, an obviously very important problem is to study the growth of \( C(f) \).

There is a version of the previous theorem for any finite group. As Tits has suggested, let us view the alternating group \( A_n \) as a Chevalley group of rank \( n-1 \) over the field of 1 element. Using Theorem 1.4, the reduction theorems in [AS85, Th. 3] or [GKKL07, Lemma 5.2] and the fact there are only finitely many sporadic simple groups, we immediately obtain:

**Corollary 1.5.** Let the function \( C(f) \) satisfy the conclusion of Theorem 1.4. Then there is an absolute constant \( C \) such that, if \( G \) is any finite group, \( K \) is any algebraically closed field and \( V \) is any faithful irreducible \( KG \)-module, then one of the following holds:

(i) \( \dim H^1(G,V) < C \);

(ii) \( G \) has a normal subgroup that is a direct product of copies of a simple Chevalley group of rank \( f \) and \( \dim H^1(G,V) \leq C(f) \).

These results lead naturally to the question of what one can say about higher cohomology groups. In more recent papers [PS11], [PS], Parshall and
Scott have obtained results about Ext and higher cohomology groups in the context of algebraic and quantum groups (with applications to the finite groups of Lie type). See [GKKL07, §12] for some results and questions for higher cohomology groups for general finite groups. We refer the reader to [AG84], [GH98] for some weaker results about $H^1(G,V)$ that give bounds in terms of $\dim V$. See [KW84] for a nice application of those bounds.

The paper is organized as follows. We first show that if $V$ is an irreducible $kG$-module and $V$ is not a composition factor of $L$, then $H^1(G,V) = 0$. We then obtain bounds for those irreducible $kG$-modules with $V^B \neq 0$ (equivalently those in $\text{soc}(L)$). Next, we analyze $L$ completely for the rank 1 groups. This is used to prove (ii) of Theorem 1.3.

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2. Modules not involved in $L = kG^B$

Let $G$ be a Chevalley group defined over $\mathbb{F}_q$ with Borel subgroup $B$. Let $r$ be a prime not dividing $q$ and let $k$ be an algebraically closed field of characteristic $r$. Let $L := kG^B$.

We first show that most $kG$-modules have trivial $H^1$. Let $L^0$ be the unique submodule of $L$ of codimension 1 (i.e. the “augmentation” submodule).

We begin with the following trivial observation.

**Lemma 2.1.** Let $A := O_r(B)$. For any $kB$-module $V$, the following statements are equivalent:

(i) $V^B \neq 0$;
(ii) $B$ has trivial composition factors on $V$;
(iii) $V^A \neq 0$;
(iv) $(V^*)^B \neq 0$.

**Proof.** Obviously, (i) implies (ii), and (ii) implies (iii) as $A$ acts coprimely on $V$. Next, the $r$-group $B/A$ acts on the $k$-space $V^A$ and so (iii) implies (i). Also, $V^A \neq 0$ if and only if $(V^*)^A \neq 0$ (since $r$ does not divide $|A|$.) Thus, (i)–(iii) hold for $V$ if and only if they hold for $V^*$, whence (iv) is equivalent to (iii). \[\square\]

**Theorem 2.2.** Let $V$ be an irreducible $kG$-module with $V^B = 0$. Then $\dim H^1(G,V)$ is the multiplicity of $V$ in $\text{head}(L^0)$.

By duality, this is also the multiplicity of $V^*$ in $\text{soc}(L/L^G) \cong \text{head}((L^0)^*)$.

Before proving the theorem, we point out a corollary:

**Corollary 2.3.** Let $V$ be an irreducible $kG$-module. If $V$ is not a composition factor of $L$, then $H^1(G,V) = 0$.
Proof. If $V$ is not a composition factor of $\mathcal{L}$, then $V^B = 0$. If $V^B = 0$ and $H^1(G,V) \neq 0$, then the previous result indicates that $V$ is a composition factor of $\mathcal{L}$. \hfill \Box

Proof of Theorem 2.2. Since $V^B = 0$, by Lemma 2.1 we have $V^{O_r(B)} = 0$, whence $H^1(B,V) = 0$ (in the semidirect product $VB$ any complement $B_1$ to $V$ will be the normalizer of $O_r(B_1)$ and by the Schur-Zassenhaus theorem, $O_r(B_1)$ and $O_r(B)$ are conjugate — this also follows by the Hochschild-Serre sequence).

Suppose that $H^1(G,V)$ is $d$-dimensional. By duality, it suffices to show that $V^*$ occurs with multiplicity $d$ in $\text{soc}(\mathcal{L}/\mathcal{L}^G)$.

Since $V^B = (V^*)^B = 0$, by Frobenius reciprocity, $V^*$ does not embed in $\text{soc}(\mathcal{L})$.

Choose a module $Y$ of maximal dimension such that $0 \to k^d \to Y \to V^* \to 0$ is indecomposable. Note that $d = \dim \text{Ext}^1_G(V^*,k) = \dim H^1(G,V)$. Also note that $Y$ is uniquely determined (indeed $Y = D^*$ where $D := \text{Der}(G,V)$ is the space of derivations of $G$ into $V$).

It follows that $\dim \text{Hom}_B(Y,k) = d$ (since $Y \cong V^* \oplus k^d$ as $B$-modules and $\text{Hom}_B(V^*,k) = 0$). So by Frobenius reciprocity, $\text{Hom}_G(Y,\mathcal{L})$ is also $d$-dimensional. If $f \in \text{Hom}_G(Y,\mathcal{L})$ is nonzero, then $f$ cannot vanish on $k^d$ (since otherwise the image of $f$ would be isomorphic to $V^*$ and contained in $\text{soc}(\mathcal{L})$, a contradiction). Thus, $\text{Ker}(f)$ is a hyperplane in $k^d$ and so the image of $f(Y)$ in $\text{soc}(\mathcal{L}/\mathcal{L}^G)$ embeds in $V^*$.

Consider the natural map $\phi : \text{Hom}_G(Y,\mathcal{L}) \to \text{Hom}_G(Y,\mathcal{L}/\mathcal{L}^G)$. If $0 \neq f \in \text{Ker}(\phi)$, then $f(Y) = \mathcal{L}^G$. However, by indecomposability, $k$ is not a homomorphic image of $Y$. Thus, $\phi$ is an injection. Together with the previous paragraph, this implies that $V^*$ has multiplicity at least $d$ in $\text{soc}(\mathcal{L}/\mathcal{L}^G)$. Since $Y$ surjects onto any nonsplit module $N$ with $0 \to k \to N \to V^* \to 0$, we see that $\phi$ is surjective as well. Thus, if $V^*$ occurs with multiplicity $m$ in $\text{soc}(\mathcal{L}/\mathcal{L}^G)$, then $m = \dim \text{Hom}_G(Y,\mathcal{L}) = d$. \hfill \Box

3. Composition factors of $k_B^G$ and modules with $V^B \neq 0$

Keep notation as in the previous section. Let $Q$ be the unipotent radical of $B$, $T$ a maximal torus of $B$ (so $B = QT$), and let $W$ be the Weyl group of $G$ (generically, $W = N_G(T)/T$).

The following proposition is crucial for our considerations:

**Proposition 3.1.** For any irreducible $kG$-module $V$, set

$$f_B(V) := \dim(V^B), \quad f_Q(V) := \dim(V^Q),$$

and let $m_\mathcal{L}(V)$ be the multiplicity of $V$ as a composition factor of $\mathcal{L}$. Then the following statements hold:
of X and Frobenius reciprocity, the multiplicity
Thus $\dim(\mathcal{L}) = \dim(\mathcal{L}')$ for any submodule $X$ of $\mathcal{L}$.

(ii) $m_X(V) = |L|$.  

(iii) $f_B(V) = f_B(V')$.  

(iv) $\sum_{V \in \mathrm{Irr}(G)} m_X(V) \cdot f_B(V) \leq \sum_{V \in \mathrm{Irr}(G)} m_X(V) \cdot f_B(V) = |L|$.  

(v) If $|X| = d > 0$, then $m_X(V) \leq |L|/d \leq |L|$.  

(vi) $m_X(k) \leq \sum_{V \in \mathrm{Irr}(G)} \dim(V) \leq |L|$.  

Proof. Note that $G = \bigcup_{w \in W} BwB$. Now $G = QT$, whence $BwB = QTwB = QwB$. Thus, $Q$ and $B$ each have $|W|$ orbits on $G/B$ and so $\dim(\mathcal{L}^Q) = \dim(\mathcal{L}^B) = |L|$. It follows that $X^Q = X^B$ for any submodule $X$ of $\mathcal{L}$.

By Frobenius reciprocity, $\mathrm{Hom}_G(\mathcal{L}, V) \cong \mathrm{Hom}_B(k, V) \cong V^B$, whence (iii) holds. The first inequality of (iv) is obvious. Since $Q$ has order coprime to $\mathrm{char}(k)$, $|W| = \dim(\mathcal{L}^Q) = \sum_{V \in \mathrm{Irr}(G)} m_X(V) \cdot f_B(V)$, yielding (iv). The next two statements are immediate consequences of (iv).

Corollary 3.2. Let $X$ be a submodule of $\mathcal{L}$. Then $\dim H^1(G, X) \leq |W| + e'$, where $e'$ is the r-rank of $B/Q$ and is at most the twisted rank $e$ of $G$.

Proof. Start with the short exact sequence $0 \to X \to \mathcal{L} \to \mathcal{L}/X \to 0$. The long exact sequence in cohomology gives

$0 \to H^0(G, X) \to H^0(G, \mathcal{L}) \to H^0(G, \mathcal{L}/X) \to H^1(G, X) \to H^1(G, \mathcal{L})$.

Note that $H^1(G, \mathcal{L}) \cong H^1(B, k)$ has dimension $e' \leq e$. By the previous result, $H^0(G, \mathcal{L}/X) \leq |W|$. Thus, $\dim H^1(G, X) \leq |W| + e'$.  

Corollary 3.3. $\sum_{V \in \mathrm{Irr}(G)} f_B(V)^* \cdot \dim H^1(G, V) \leq |L| + e$. In particular,

$$\sum_{V \in \mathrm{Irr}(G)} \dim H^1(G, V) \leq |L| + e,$$

and

$$\dim H^1(G, V) \leq \frac{|L| + e}{f_Q(V)}$$

if $V^B \neq 0$.

Proof. Consider $X := \mathrm{soc}(\mathcal{L})$. Then $V \in \mathrm{Irr}(G)$ embeds in $X$ if and only if $(V^*)^B \neq 0$, which is equivalent to $V^B \neq 0$ by Lemma 2.1. By Proposition 3.1 and Frobenius reciprocity, the multiplicity $m_X(V)$ of $V$ as a composition factor of $X$ equals

$$m_X(V) = \dim(V^*)^B = \dim(V^*)^Q = \dim V^Q = f_Q(V) = \dim V^B = f_B(V).$$

Thus $X \cong \oplus_{V \in \mathrm{Irr}(G)} f_B(V)^* \cdot V$. Now apply Corollary 3.2 to $X$.  

\[ \square \]
One can show that \( G \) has exactly \(|W|\) isomorphism classes of irreducible \( kG \)-modules \( V \) with \( V^B \neq 0 \) precisely when \( G \) has rank 1 and \( r \) is coprime to \([G:B]\). In fact, under some mild conditions on \( r \), Geck [Gec06] has shown that there is a canonical injection of the set of isomorphism classes of irreducible \( kG \)-modules with \( V^B \neq 0 \) into \( \text{Irr}(W) \).

4. Rank 1 groups

We are interested in the submodule structure of the module \( \mathcal{L} := kG^B \), where \( G \) is a finite group of Lie type of (twisted) rank 1 and \( B \) a Borel subgroup of \( G \), with unipotent radical \( Q \). In this case, the permutation action of \( G \) on \( \Omega := G/B \) is doubly transitive. One can identify \( \mathcal{L} \) with \( k\Omega \) and consider the \( G \)-submodules

\[
J := \langle \sum_{\omega \in \Omega} \omega \rangle \cong k, \quad J^\perp := \left\{ \sum_{\omega \in \Omega} a_\omega \omega \mid a_\omega \in k, \sum_{\omega \in \Omega} a_\omega = 0 \right\}.
\]

Then the heart of \( \mathcal{L} \) is defined to be \( H := J^\perp/(J \cap J^\perp) \).

First we recall some well-known facts.

**Lemma 4.1.** (i) Assume \( r|[G:B] \), equivalently, \( J \subseteq J^\perp \). Then any nontrivial composition factor \( M \) of \( \mathcal{L} \) has no nonzero \( Q \)-fixed points, and the socle and the head of \( \mathcal{L} \) are simple and trivial.

(ii) Assume the heart \( H \) of \( \mathcal{L} \) is simple. Then the only submodules of \( \mathcal{L} \) are 0, \( J \), \( J^\perp \), and \( \mathcal{L} \). Furthermore, one of the following holds:

(a) \( (r,[G:B]) = 1 \) and \( \mathcal{L} \cong J \oplus H \);

(b) \( r|[G:B] \) and \( \mathcal{L} \) is uniserial with composition factors \( k \), \( H \), \( k \) (in that order).

(iii) Assume the heart \( H \) of \( \mathcal{L} \) is not simple. Then \( r = 2 \) (and \( q \) is odd) if \( G = \text{PSL}_2(q) \), \( r|(q^2+1) \) if \( G = 2B_2(q) \), \( r|(q^3+1) \) if \( G = 2G_2(q) \), and \( r|(q+1) \) if \( G = \text{SU}_3(q) \).

**Proof.** (i) Note that the permutation module \( \mathbb{C}\Omega \) affords the character \( 1_G + \text{St} \), where \( \text{St} \) is the Steinberg character of \( G \), and furthermore, \([\text{St}_Q,1_Q]_Q = 1 \). Since \( Q \) acts coprimely on \( \mathcal{L} \), we get \( \dim \mathcal{L}^Q = 2 \). Also, \( \mathcal{L} \) has the filtration \( 0 \subset J \subset J^\perp \subset \mathcal{L} \), and \( J \cong \mathcal{L}/J^\perp \cong k \), whence \( H^Q = 0 \). Now, if \( M \) is a nontrivial composition factor of \( \mathcal{L} \), then we may assume that \( M \) is a subquotient of \( H \), and so \( M^Q = 0 \).

Next, since \( \mathcal{L} \) is self-dual and \( \text{soc}(\mathcal{L}) \supseteq J \), it suffices to prove that any simple submodule \( M \) of \( \mathcal{L} \) must equal \( J \). Assume the contrary: \( M \neq J \). Now, \( 0 \neq \text{Hom}_G(M,\mathcal{L}) \cong \text{Hom}_B(M,k) \) implies that \( 0 \neq \text{Hom}_Q(M,k) \), and so \( M^Q \neq 0 \). By the above, \( M \cong k \), \( \text{soc}(\mathcal{L}) \supseteq k \oplus k \), and so

\[
2 \leq \dim \text{Hom}_G(k,\mathcal{L}) = \dim \text{Hom}_B(k,k) = 1,
\]
a contradiction.
(ii) In the case of (a), \( J \cap J^\perp = 0 \) and \( \mathcal{H} \cong J^\perp \cong \mathcal{L}/J \), whence \( \mathcal{L} \cong J \oplus \mathcal{H} \). Next, (b) follows from (i).

(iii) is recorded in [Mor80, Table 1]. \( \square \)

**Lemma 4.2.** Assume \( r \) divides \( |G| \) but not \( |B| \), and the Sylow \( r \)-subgroups of \( G \) are cyclic. Then \( \mathcal{L} \) is uniserial, with simple, trivial, socle and head. Furthermore, \( M^\mathcal{Q} = 0 \) for each nontrivial composition factor \( M \) of the \( G \)-module \( \mathcal{L} \).

**Proof.** The assumptions imply that \( \mathcal{L} \) is projective. By Lemma 4.1(i), \( \text{soc}(\mathcal{L}) \cong \text{head}(\mathcal{L}) \cong k \), so \( \mathcal{L} \) is a projective cover of the trivial \( kG \)-module \( k \).

Next, let \( D \) be a defect group of the principal block \( B_0 \) of \( G \), \( D_1 \) the unique subgroup of order \( r \) of \( D \), \( N := N_G(D_1) \), and \( b \) the unique \( kN \)-block with \( b^G = B_0 \). Then the Green correspondence \( (G, B_0) \rightarrow (N, b) \) sends the trivial \( kG \)-module \( k \) to the trivial \( kN \)-module; in particular, the Green correspondent of \( k \) has composition length 1. Hence [Pea75, theorem, pp. 234–236] implies that the projective covers of \( k \) are uniserial. \( \square \)

### 4.1. \( \text{PSL}_2(q) \).

In this case, Lemmas 4.1 and 4.2 imply that \( \mathcal{L} \) has the following structure, where \( V, V_1, V_2 \) are certain simple \( kG \)-modules:

- \( \mathcal{L} = k \oplus V \) if \( (r, q + 1) = 1 \);
- \( \mathcal{L} \) is uniserial of length 3 with composition factors \( k, V, k \), if \( G = \text{PSL}_2(q) \) and \( 2 < r | (q + 1) \), or if \( G = \text{PGL}_2(q) \) and \( r | (q + 1) \);
- \( \mathcal{L} \) has socle series \( V_1 \oplus V_2 \) if \( G = \text{PSL}_2(q) \) and \( r = 2 | (q + 1) \).

### 4.2. \( \text{PSL}_2(q) \), \( q = 2^{2a+1} \geq 8 \).

Here, \( \mathcal{L} \) has the following structure where \( V \) and \( U \cong U^* \) are simple, and \( \dim U = (q - 1)\sqrt{q/2} \):

- \( \mathcal{L} = k \oplus V \) if \( (r, q^2 + 1) = 1 \);
- \( \mathcal{L} \) is uniserial of length 3 with composition factors \( k, V, k \), if \( r | (q - \sqrt{2q} + 1) \);
- \( \mathcal{L} \) is uniserial of length 5 with composition factors \( k, U, V, U^*, k \) (in this order), if \( r | (q + \sqrt{2q} + 1) \).

Indeed, by Lemma 4.2, \( \mathcal{L} \) is uniserial if \( r | (q^2 + 1) \). Furthermore, the Brauer tree for the principal block of \( G \), and the possible structure of \( \mathcal{L}/k \) as a uniserial reduction modulo \( r \) of the Steinberg character \( St \), are described in [His93, Anhang D.1].

### 4.3. \( \text{PSL}_2(q) \), \( q = 3^{2a+1} \geq 27 \).

In this case, \( \mathcal{L} \) has the following structure, where \( V, U \not\cong U^*, M \not\cong M^*, T \), are simple \( kG \)-modules:

- \( \mathcal{L} = k \oplus V \) if \( (r, q^3 + 1) = 1 \);
- \( \mathcal{L} \) is uniserial of length 3 with composition factors \( k, V, k \), if \( r | (q - \sqrt{3q} + 1) \);
- \( \mathcal{L} \) is uniserial of length 5 with composition factors \( k, U, V, U^*, k \) (in this order, and \( \dim U = (q - 1)(q + \sqrt{3q} + 1)\sqrt{q/12} \), if \( 2 < r | (q + 1) \));
• \( \mathcal{L} \) is uniserial of length 7 with composition factors \( k, M, U, V, U^*, M^* \), \( k \) (in this order, and \( \dim U = (q - 1)(q - \sqrt{3q} + 1)\sqrt{q/12}, \dim M = (q^2 - 1)\sqrt{q/3} \), if \( r|(q + \sqrt{3q} + 1) \).

Indeed, by Lemma 4.2, \( \mathcal{L} \) is uniserial if \( 2 < r |(q^3 + 1) \). Again, in this case we use the structure of the Brauer tree for the principal block of \( G \) as described in [His93, Anhang D.2]:

\[
\begin{array}{c}
k \\
T \\
U \\
V \\
M^* \\
M \\
k \\
\end{array}
\]

• \( \mathcal{L} \) has socle series \( U \oplus V \oplus U^* \) if \( r = 2 \), by [LM80, Prop. 3.8].

4.4. PSU\(_3\)(q). In this case, \( \mathcal{L} \) has the following structure, where \( V \) and \( U \) are simple, and \( \dim U = q(q - 1) \):

• \( \mathcal{L} = k \oplus V \) if \( r, q^3 + 1 = 1 \);
• \( \mathcal{L} \) is uniserial of length 3 with composition factors \( k, V, k \), if \( r \) divides \( q^3 + 1 \) but not \( q + 1 \). These first two cases follow from Lemmas 4.1 and 4.2.
• Assume \( G = \text{PGU}_3(q) \), and either \( 2 < r |(q + 1) \), or \( r = 2 \) and \( q \equiv 3(\text{mod } 4) \). Then \( \mathcal{L} \) is uniserial of length 5 with composition factors \( k, U, M, U, k \) (in this order). Here, \( M \) is irreducible over \( \text{PGU}_3(q) \), and a sum of 1 or 3 irreducible \( \text{PSU}_3(q) \)-modules, according to \( r \neq 3 \) or \( r = 3 \). This case is analyzed in [His04, Th. 4.1].

• \( \mathcal{L} \) has socle series \( U \oplus V \) if \( r = 2 \) and \( q \equiv 1 (\text{mod } 4) \), by [Erd79, Lemma (4.2)].

Now we are ready to prove the main result of this section:

**Theorem 4.3.** Let \( G \) be one of the following rank 1 groups: \( \text{PSL}_2(q), \text{PGL}_2(q), \text{PSU}_3(q), \text{PGU}_3(q), 2B_2(q) \) with \( q \geq 8 \), or \( 2G_2(q) \) with \( q \geq 27 \). Then \( \dim H^1(G, V) \leq 1 \) for any irreducible \( kG \)-module \( V \). Moreover, \( H^1(G, V) \neq 0 \) for at most two irreducible \( kG \)-modules.

**Proof.** 1) First we claim that \( H^1(G, k) = 0 \) unless

(a) \( r = 2 |(q + 1) \) and \( G = \text{PGL}_2(q) \), or

(b) \( r = 3 \) and \( G = \text{PGU}_3(2) \),

in which cases \( H^1(G, k) \cong k \). Indeed, \( H^1(G, k) \cong \text{Ext}_G^1(k, k) \). Let \( U \) be any 2-dimensional \( kG \)-module with \( k \) as the unique composition factor (with multiplicity 2). The action of \( G \) on \( U \) induces a group homomorphism \( f \in \text{Hom}(G, (k, +)) \cong \text{Hom}(G/G', (k, +)) \). The latter hom-space is 0 unless (a) or (b) occurs. In the case of (a), \( r = 2 = |G/G'| \). In the case of (b), \( r = 3 = |G/G'| \). (Note that \( G/G' \) is a 2-group if \( G = \text{PSL}_2(2) \) or \( \text{PSU}_3(2) \), but here \( r \neq 2 \).) In either case, the hom-space is 1-dimensional, and the claim follows.
2) Next we assume that $H^1(G,V) \neq 0$ for some nontrivial $V \in \text{Irr}_k(G)$. By Corollary 2.3, $V$ is a composition factor of $\mathcal{L} = kG_B$.

Consider the case $r \mid [G : B]$. By Lemma 4.1(i), $V^B = 0$; hence $d := \dim H^1(G,V)$ is just the multiplicity of $V$ in $\text{head}(\mathcal{L}^0)$ by Theorem 2.2. The structure of $\text{head}(\mathcal{L}^0)$ has already been described above. It follows that, if $G = \text{PGL}_2(q)$ or $\text{PGU}_3(2)$ then $V$ is unique (up to isomorphism) and $d = 1$. In all other cases, $d = 1$ and there are at most two isomorphism classes for such $V$.

Finally, we consider the case $(r, [G : B]) = 1$. By Lemma 4.1, $\mathcal{L} \cong k \oplus \mathcal{H}$; hence we may assume that $V \cong \mathcal{H}$ and $H^1(G,V)$ embeds in $H^1(G,\mathcal{L})$. But $H^1(G,\mathcal{L}) \cong H^1(B,k)$ has dimension $e' \leq e = 1$, where $e'$ is the $r$-rank of $B$.

It follows that $H^1(G,V)$ is 1-dimensional, and $V$ is unique in this case. □

Note that if $G = 2B_2(q)$, $r \mid (q + \sqrt{q} + 1)$ or $2G_2(q)$, $2 < r \mid (q + 1)(q + \sqrt{3q} + 1)$, then there is an irreducible $kG$-module $V$ with $H^1(G,V) \neq 0$ but $H^1(G,V^*) = 0$.

Also note that if $G = \text{PSL}_2(q)$ with $r = 2|(q + 1)$ or $\text{PSU}_3(q)$ with $r = 2$, $q \equiv 1(\text{mod } 4)$, then there are two irreducible modules $V$ with $V^B = 0$ and $H^1(G,V) \neq 0$. In fact, more generally if $G = \text{Sp}_{2n}(q)$ with $q$ odd and $r = 2$, then each of the two irreducible Weil modules of dimension $(q^n - 1)/2$ has 1-dimensional $H^1$.

5. Quasi-equivalence and duality

Let $k$ be an algebraically closed field of characteristic $r > 0$ and $G$ a group.

We say that two $G$-modules $X$ and $Y$ are quasi-equivalent if $Y$ can be obtained from $X$ by a twist by an automorphism of $G$. If $X$ is a $kG$-module, we denote $X^\sigma$ as the twist of $X$ by $\sigma$ (in particular if the character of $X$ is $\chi$, then the character of $X^\sigma$ is $\chi \circ \sigma$). Clearly, modules that are quasi-equivalent have the same cohomology.

The next result will be very useful.

**Lemma 5.1.** Let $G$ be a normal subgroup of a finite group $H$ such that every $r'$-element of $G$ is conjugate to its inverse in $H$. If $V$ is any irreducible $kG$-module, then $V$ and $V^*$ are quasi-equivalent.

**Proof.** Note the hypothesis implies that any $H$-invariant semisimple $kG$-module is self-dual. Consider $M := V^H_G$. Then $M$ is a semisimple $kG$-module, where the simple summands are precisely all the twists of $V$ by elements in $H/G$. By our observation, $M$ is a self-dual $kG$-module, and so both $V$ and $V^*$ are $kG$-summands of $M$. Hence the result follows. □

We have seen examples with $H^1(G,V) \neq 0$ and $H^1(G,V^*) = 0$. Note that $M_{11}$ provides another example.
If $G$ is a Chevalley group (or an algebraic group) in the same characteristic as $k$, the dual of any irreducible module is quasi-equivalent. If $G$ is finite, this can be seen via the previous result because every semisimple element of $G$ is inverted by an automorphism of $G$. In the case $G$ is an algebraic group, there is always an automorphism $\tau$ of $G$ which acts as inversion on a maximal torus $T$. Thus, $V^* \cong V^\tau$ and so we see directly that $V^*$ and $V$ are quasi-equivalent.

If $G$ is any finite classical group of simply connected type (no matter what is the characteristic of $k$), then any irreducible $kG$-module is quasi-equivalent to its dual. For it is again well known that every element of $G$ is inverted by an automorphism of $G$. For instance, every element of $\text{SL}_n(q)$ is conjugate to its inverse via an element of $\text{GL}_n(q).\langle \tau \rangle$ where $\tau$ is the transpose-inverse map.

We record the result we need in the next section.

**Corollary 5.2.** Let $G = \text{SL}_n(q)$. Every irreducible $kG$-module is quasi-equivalent to its dual.

### 6. Modules with $V^B = 0$

In this section we assume that $G$ has twisted rank $e > 1$. Fix a Borel subgroup $B$. We keep notation as above.

Let $V$ be an irreducible $kG$-module with $V^B = 0$. We will show that any such irreducible has at most 1-dimensional $H^1$, and moreover there are at most four such modules (the example $G = \text{Sp}_4(q)$ with $q$ odd and $r = 2$ show that three is possible). In fact, in many cases we will see that there is at most one such module.

A very easy case is the following:

**Theorem 6.1.** Let $V$ be a $kG$-module with $V^B = 0$. Let $P_i$, $1 \leq i \leq n$, denote the minimal parabolic subgroups of $G$ containing $B$. If $r$ does not divide $[P_i : B]$ for any $i$, then $H^1(G, V) = 0$.

**Proof.** Since $V^B = 0$, it follows by Lemma 2.1 that $V^{O_r(B)} = 0$, whence $H^1(B, V) = 0$. Since $r$ does not divide $[P_i : B]$ for any $i$, the restriction map $H^1(P_i, V) \to H^1(B, V)$ is injective, whence $H^1(P_i, V) = 0$ for all $i$. Since $G = \langle P_1, \ldots, P_n \rangle$, the result follows by [AG72] (or see [GK90]).

We now give some examples of modules with $H^1(G, V) \neq 0$ and $V^B = 0$. This construction will be used in the proof of Theorem 6.6.

**Remark 6.2.** Let $\text{St}$ denote the Steinberg module for $QG$ in characteristic 0. Note that $\text{St}$ is a free rank 1 module for $Q$ and so $V^Q$ is 1-dimensional. It is well known that $\dim V^B = 1$ as well. Thus, $\text{St}$ embeds in $Q^G_B$ and we view it as such. Let $M$ be the standard permutation module for $1^Q_B$ over $\mathbb{Z}$. Then $L := \text{St} \cap M$ is a $\mathbb{Z}G$-lattice in $\text{St}$ with $L$ a pure submodule of $M$.

Now reducing modulo $r$, we have a submodule $X$ of $L$ with $X$ the reduction of $L$. Clearly, $\dim X^Q = 1$ and so $X$ has exactly one composition factor $Y$
with $Y^Q \neq 0$. Since $\soc(X) \leq \soc(L)$, it follows by Frobenius reciprocity that $Y = \soc(X)$. If $P$ is a parabolic subgroup containing $B$ with radical $R$, then $Y^R$ is a reduction of the Steinberg module for $P/R$. So by induction and inspection of the rank one cases, it follows that $Y \cong k$ if and only if $r \mid [P:B]$ for every nontrivial parabolic subgroup $P$ (and so it is enough to check the minimal ones). See [His90, Th. A]. In particular, $\Ext^1_G(Y,Z) \neq 0$ for each composition factor $Z$ in $\soc(X/Y)$. So if $Y = k$ (i.e. $r \mid [P_i:B]$ for each minimal parabolic $P_i$), we see that $H^1(G,Z^*) \neq 0$ and $Z^B = (Z^*)^B = 0$.

Next, we have:

**Lemma 6.3.** Let $P$ be a parabolic subgroup properly containing $B$ with radical $R$. Then $\mathcal{L}^R \cong \bigoplus_{w \in W_0} k_{P \cap B^w}^P$ as $kP$-modules where $W_0$ is a set of double coset representatives for $P \backslash G/B$.

**Proof.** Let $\Delta$ be a set of positive roots corresponding to the root system which determines $B$. Recall that $Q$ is the product of various root subgroups $U_\beta$, $\beta$ a positive root (see [Car89, 13.6] or [GLS98, 2.3]).

We view $W_0$ as a subset of $W$. By Mackey decomposition, $\mathcal{L} \cong \bigoplus_{w \in W_0} k_{P \cap B^w}^P$ as $kP$-modules. Now take $R$-fixed points. Observe that since $R \leq P$, $P \cap B^w R = (P \cap B^w) R$. Note that the space of $R$-fixed points of $k_{P \cap B^w}^P$ is isomorphic to $k_{P \cap B^w}^P$. Let $H$ be the standard Levi subgroup of $P$; so $H$ is generated by a maximal torus of $B$ and all the root subgroups $U_\beta$ where the roots $\beta$ are linear combinations of a subset of the simple roots. Note for each positive root subgroup $U_\beta < G$ and for each $w \in W$, we have that precisely one of $U_\beta$ or $U_{-\beta}$ is contained in $B^w \cap H$. Thus, $P \cap B^w R = B_1 R$ where $B_1$ is a Borel subgroup of $H$ and so $B_1 R$ is a Borel subgroup contained in $P$. This completes the proof.

**Theorem 6.4.** Let $P$ be a minimal parabolic subgroup containing $B$ with radical $R$. Assume that $r \mid [P:B]$. Let $X$ be a submodule of $\mathcal{L}$ containing $\mathcal{L}^G$ such that $X^Q = \mathcal{L}^G$ and $\dim X^R > 1$. Then the following statements hold:

(i) Either $\soc(X^R/X^G)$ is an irreducible $kP$-module, or $r = 2$ and $[P,P]/R \cong \mathrm{SL}_2(q)$ with $q$ odd or $\mathrm{SU}_3(q)$ with $q \equiv 1 \pmod 4$. In the latter two cases $\soc(X^R/X^G)$ is either irreducible or a direct sum of two nonisomorphic irreducible $kP$-modules.

(ii) If $X/\mathcal{L}^G$ is irreducible, then $\Ext^1_G(X,k) = 0$ and $\dim \Ext^1_G(X,k) = 1$.

**Proof.** It is clear that $Y := X^R$ is a $kP$-submodule of $M := \mathcal{L}^R$, and, by Lemma 6.3, $M$ is a direct sum of copies of $k_B^P$ as $kP$-modules. Since $X^B$ is 1-dimensional, the same is true for $X^P$ and so $Y^P$ is 1-dimensional.

We claim that $Y$ embeds in $k_B^P$. Let $\pi$ be a projection of $Y$ into one of the direct factors of $M$ with $\pi(Y) \neq 0$. Now $Z := \Ker(\pi)$ is a submodule
of $Y$ with $Z^Q = 0$. In particular, $\text{soc}(Z)$ contains no trivial $kP$-composition factors. Since $\text{soc}(M)$ is a direct sum of trivial composition factors, it follows that $\text{soc}(Z) = 0$, whence $Z = 0$ and the claim follows.

Now (i) follows by our results on the rank 1 groups. Set $U := X/k$. If $U$ is semisimple, then by (i), $U$ is multiplicity free (since $\text{soc}(U_R)$ is multiplicity free as a $kP$-module). Thus, by Theorem 2.2 if $V$ is any simple summand of $U$, $\dim H^1(G, V) = \dim \text{Ext}_G^1(V^*, k) = 1$, whence the second part of (ii) holds.

Now assume that $U$ is simple. Let $X^*$ denote the dual of $X$. So it has socle $U^*$ and is indecomposable of length 2. Then we have $0 \to U^* \to X^* \to k \to 0$, giving rise to:

$$0 \to k \to H^1(G, U^*) \to H^1(G, X^*) \to H^1(G, k) = 0.$$

As we have already noted, $H^1(G, U^*)$ is 1-dimensional, whence (ii) follows. □

**Corollary 6.5.** Let $V$ be an irreducible $kG$-module with $V^B = 0$ and $H^1(G, V) \neq 0$.

(i) $\dim H^1(G, V) = 1$.

(ii) If $r \neq 2$, then $V$ is defined over $\mathbb{F}_r$.

(iii) If $r \neq 2$ and $H^1(G, V^*) \neq 0$, then $V \cong V^*$.

**Proof.** Let $X$ be a submodule of $L$ such that $X^Q = L^G$. Assume first that $Z := X/X^G$ is simple.

We claim that $\dim X_R > 1$ for $R$ the radical of some minimal parabolic $P$ containing $B$ with $r$ dividing $[P : B]$. If not, then $\text{Ext}_P^1(X/k, k) = 0$ for each minimal parabolic subgroup $P$ and so $\text{Ext}_G^1(X/k, k) = 0$ by [AG72], a contradiction.

Now apply the previous result to see that $\dim \text{Ext}_G^1(Z, k) = 1$.

Now take $X$ a submodule of $L$ with $X^Q = L^G$ and $Z := X/X^G$ semisimple and as large as possible. Suppose that $R$ is the radical of a minimal parabolic subgroup $P$. Note that $\text{soc}(X_R/X^G)$ is a simple $kP$-module if $r \neq 2$ or at worst a direct sum of 2 distinct irreducibles. Thus, for $r \neq 2$, there is at most one summand of $Z$ with $Z_R \neq 0$ (and at most two if $r = 2$).

The previous result also implies that $Z^R$ is multiplicity free as a $kP$-module. This implies that $\dim \text{Ext}_G^1(U, k) = 1$ for any simple summand $U$ of $Z$. So (i) follows by Theorem 2.2.

Now assume that $r \neq 2$. If $\sigma$ is an element of the Galois group of $\mathbb{F}_r$, then $H^1(G, V^\sigma) \neq 0$, and $V^\sigma$ is also a summand of $Z$. Moreover, $\dim V^R = \dim (V^*)^R$. As we have seen above, there is at most one simple summand of $Z$ having nontrivial $R$-fixed points. Thus, $V \cong V^\sigma$. Thus, $V$ is defined over $\mathbb{F}_r$.

A similar argument gives the last result. □

We want to show that there are very few irreducible modules with $V^B = 0 \neq H^1(G, V)$. The proof of the previous result easily gives that there are most
e (for \(r \neq 2\)) or 2e such modules. We will show that there are at most four such modules (one can check that for \(r = 2\), there are three such modules for \(\text{Sp}_4(q)\) with \(q\) odd).

We first classify such modules for \(G = \text{SL}_n(q)\). By Theorem 6.1, we may assume that \(r\) divides \(q + 1\).

**Theorem 6.6.** Let \(G = \text{SL}_n(q)\) with \(n > 2\) and \(r | (q + 1)\). There is a unique irreducible \(kG\)-module \(V\) with \(V^B = 0\) and \(H^1(G, V) \neq 0\), and this \(V\) can be found as in Remark 6.2.

**Proof.** 1) Let \(W\) be any irreducible \(kG\)-module with \(W^B = 0\), and so \((W^*)^B = 0\) by Lemma 2.1. By Theorem 2.2, \(H^1(G, W^*) \neq 0\) precisely when \(W\) is a submodule of \(\text{soc}(L/L^G)\). By Corollary 5.2, \(W^*\) is quasi-equivalent to \(W\); hence \(H^1(G, W^*) \neq 0\) if and only if \(H^1(G, W) \neq 0\). It follows that \(H^1(G, W) \neq 0\) exactly when \(W\) is a submodule of \(\text{soc}(L/L^G)\). Observe that in this case, \(W^\tau\) is also a submodule of \(\text{soc}(L/L^G)\), where \(\tau = \tau^{-1}\) is the transpose-inverse map. (Indeed, \(\dim H^1(G, W^\tau) = \dim H^1(G, W)\). Next, if \(T\) is an irreducible \(kH\)-module lying above \(W\) for \(H := \text{GL}_n(q)\), then it is well known that \(T^\tau \cong T^*\), and \(T^B = 0\) if and only if \(W^B = 0\) as \(H = N_H(B)G\). Hence we also have \((W^\tau)^B = 0\).

2) We have seen in Remark 6.2 that such a module \(V\) exists. Let \(\bar{X}\) be the sum of all simple summands \(X\) of \(\text{soc}(L/L^G)\) with \(W^B = 0\), and let \(X\) be the complete inverse image of \(\bar{X}\) in \(L\); in particular, \(X^G = L^G \cong k\). By the observations in 1), it suffices to show that \(X/X^G\) is simple. We will proceed by induction on \(n\). When \(n = 2\), the claim follows by Section 4.1 unless \(r = 2\) and \(q\) is odd, but even in that case, the result holds for \(\text{PGL}_2(q)\).

For the induction step, let \(B \leq P\) be a maximal end node parabolic subgroup of \(G\) with unipotent radical \(R\). Let \(P_1\) be the other end node parabolic subgroup containing \(B\) and set \(R_1\) to be its unipotent radical. Note that \(R_1 = R^{r^g}\) for some \(g \in G\).

Set \(Y = X^R\). By Lemma 6.3, \(Y\) embeds in a direct sum of copies of \(k_B^P\). Since \(Y^B = X^G \cong k\), it follows that \(Y\) embeds in \(k_B^P\). By induction on \(n\) (notice that when \(n = 3\), \(P/R \cong \text{GL}_2(q)\)), the socle of \(Y/k\) as a \(kP\)-module is simple. It follows that there is at most one summand \(U\) of \(X/X^G\) with \(U^R \neq 0\). Since \(R\) is unipotent, for such a \(U\) we also have \((U^*)^R \neq 0\). As in 1), if \(T\) is an irreducible \(kH\)-module lying above \(U\), then \(T^\tau \cong T^*\), \((T^*)^R \neq 0\), and so \((U^*)^R \neq 0\) (since \(H = N_H(R)G\)). According to 1), \(U^\tau\) is a summand of \(X/X^G\), so the uniqueness of \(U\) implies that \(U \cong U^\tau\). Now assume that \(W\) is a summand of \(X/X^G\) with \(WR_1 \neq 0\). Then

\[\dim(W^\tau)^R = \dim(W^\tau^{-1})^R = \dim W^R = \dim W^{R_1},\]

and so \((W^\tau)^R \neq 0\). By 1), \(W^\tau\) is a summand of \(X/X^G\), so the uniqueness of \(U\) again implies that \(W^\tau \cong U \cong U^\tau\), whence \(W \cong U\). It follows that
$W^R \cong U^R \neq 0$ and so $W = U$. Consequently, $U$ is also the unique summand of $X/X^G$ with $U^R \neq 0$.

Now, if $Y \neq U$ is any summand of $X/X^G$, then $Y^R = Y^{R_1} = 0$, whence $H^1(P,Y^*) = H^1(P_1,Y^*) = H^0(B,Y^*) = 0$. Thus, by [AG72], $H^1(G,Y^*) = 0$, contradicting Theorem 2.2. Hence $U$ exists and $X/X^G = U$ is simple, as required.

Corollary 6.7. There are at most four irreducible $kG$-modules $V$ with $V^B = 0$ and $H^1(G,V) \neq 0$.

Proof. The proof is similar to the previous case. Let $X$ be any submodule of $L$ containing $L^G = X^G$ such that $Y := X/X^G$ is semisimple and every composition factor $V$ of $X/X^G$ satisfies $V^B = 0$. Choose two parabolic subgroups $P_1$ and $P_2$ such that $G = \langle P_1, P_2 \rangle$. Let $R_i$ denote the unipotent radical of $P_i$. Let $m_i$ be the number of summands $V$ of $Y$ such that $R_i$ has fixed points on $V$. It suffices to show that $m_1 + m_2 \leq 4$. If $P_i$ is a minimal parabolic, then we have seen that $m_i \leq 2$ (and indeed $m_i \leq 1$ if $r \neq 2$). This gives the result if $G$ has rank 2. So we assume that $G$ has rank at least 3.

If possible choose $P_1$ with Levi subgroup of type $SL$, and so by the previous result $m_1 \leq 1$. If we take $P_1$ to be maximal, then we can choose $P_2$ to be minimal. We have seen that $m_2 \leq 2$. Thus, there are at most 3 summands.

This can be done in all cases except $G = F_4(q)$ or $^2E_6(q)$. If $G = F_4$, we take $P_1$ to have a Levi of type $SL_3(q)$ for $i = 1, 2$. If $G = ^2E_6(q)$, we take $P_1$ to have a Levi of type $SU_6(q)$ and $P_2$ to be a minimal parabolic. Modifying the argument above for $SU$, we see that $m_1 \leq 2$, whence the result.

In many cases, the argument above can be used to give better bounds on the number of such modules. For example, it is not hard to see that if $G$ is untwisted of types $D$ or $E$, then there is a unique such module (when $r$ divides $q + 1$).

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First cohomology groups of Chevalley groups


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