First cohomology groups of Chevalley groups in cross characteristic

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Abstract

Let G be a simple Chevalley group defined over $\mathbb{F}_q$. We show that if r does not divide q and k is an algebraically closed field of characteristic r, then very few irreducible kG-modules have nonzero $H^1(G, V)$. We also give an explicit upper bound for $\dim H^1(G, V)$ for V an irreducible kG-module that does not depend on q, but only on the rank of the group. Cline, Parshall and Scott showed that such a bound exists when $r|q$. We obtain extremely strong bounds in the case that a Borel subgroup has no fixed points on V.

1. Introduction

Let G be a finite group with k an algebraically closed field of characteristic r. Let V be an irreducible kG-module that is faithful as a G-module (we will abuse notation and say that V is a faithful kG-module).

We are interested in how large $H^1(G, V)$ can be. These cohomology groups are critical to understanding primitive permutation groups as well as the module structure of indecomposable modules.

In 1985, the first author [Gur86] made the rather optimistic conjecture:

CONJECTURE 1.1. *There exists an absolute constant C such that if G is a finite group and V is an irreducible faithful kG-module, then* $\dim H^1(G, V) \leq C$.

Indeed, the original conjecture was with $C = 2$. While there were some weak results obtained about $\dim H^1$, very little progress toward the conjecture was made. There have been some examples constructed with $\dim H^1(G, V) = 3$ (in particular, see [Sco03] with examples for arbitrarily large characteristic

and [BW08] for examples in characteristic 2). However, in the last few years, there have been some breakthroughs. In particular, Cline, Parshall and Scott [CPS09, Thms. 7.3, 7.10] proved:

THEOREM 1.2. *Let G be either a simple algebraic group or a finite simple group of Lie type over a field of characteristic r. Let k be an algebraically closed field of characteristic r. If V is an irreducible kG-module, then there is a constant C such that $\dim H^1(G, V) < C$ where C depends only on the rank of G.*

This result depends on the fact that Lusztig's conjecture holds in sufficiently large characteristics as well as many other deep results about algebraic groups and the relationship between cohomology of algebraic groups and cohomology of finite groups of Lie type. In particular, our conjecture implies that there are bounds on certain coefficients of the Kazhdan-Lusztig polynomials.

The general problem reduces to the case of simple groups (see [AS85, Th. 3] or [GKKL07, Lemma 5.2]). We then use the classification of finite simple groups. In this paper, we consider the finite simple Chevalley groups. We consider the Chevalley groups to include all the finite simple groups of Lie type including the twisted variations of Steinberg, Suzuki and Ree. See [GLS98, Chap. 2] and [Car89] for the basic facts about Chevalley groups, including the structure of the twisted groups and root subgroups. We consider cross characteristic modules (i.e. modules over fields of characteristic different from the natural characteristic of the Chevalley group). Thus, our results are complementary to those in [CPS09] (and our methods are almost totally disjoint as is to be expected).

So let k be an algebraically closed field of characteristic r, and let G be a finite simple Chevalley group over $\mathbb{F}_q$ with $(r, q) = 1$. We prove the same type of result as Theorem 1.2 (but with explicit, albeit almost certainly not best possible, bounds). Indeed, we prove much more — we show that there are very few irreducible kG-modules with nonzero H^1 and that $\sum \dim H^1(G, V)$ is bounded (in terms of the Weyl group of G) — here the sum is over all irreducible kG-modules. See [CPS99, Thms. 10.2, 10.5] for a weak version (in particular, there are no explicit bounds) of some of these results for r sufficiently large and $G = \mathrm{GL}_n(q)$ or $\mathrm{SL}_n(q)$ using completely different methods.

Here is a summary of our main results. Let B be a Borel subgroup of G, with unipotent radical Q, and let W be the Weyl group of G. The permutation module $\mathcal{L} := k_B^G$ will play a key role in our considerations. We denote by $\mathrm{Irr}_k(G)$ the set of isomorphism classes of irreducible kG-modules. Let e denote the twisted rank of G. (If G is an untwisted group, this is just the rank of the ambient group. In general, it is the number of orbits of the graph automorphism used in defining the group on the nodes of the Dynkin diagram.

Alternatively, it is the rank of a maximal torus of the Borel subgroup as an abelian group as long as q is sufficiently large.) If V is an H-module, let V^H denote the fixed point subspace of H on V; furthermore, if $H \leq G$, then V_H^G denotes the corresponding induced module.

THEOREM 1.3. *Let V be an irreducible kG-module.*

(i) *If V is not a composition factor of $\mathcal{L}$, then $H^1(G, V) = 0$.*
(ii) *If $V^B = 0$, then $\dim H^1(G, V) \leq 1$ and there are at most four distinct such modules with $H^1(G, V) \neq 0$.*
(iii) *There are at most $|W|$ isomorphism classes of irreducible kG-modules V with $V^B \neq 0$, and*

$$\sum_{V \in \text{ Irr}_k(G)} \dim V^B \cdot \dim H^1(G, V) \leq |W| + e.$$

(iv) $\dim H^1(G, V) \leq |W| + e.$

Note that the number of irreducible kG-modules goes to ∞ as q grows. Thus, for large q, we see that almost every irreducible kG-module has trivial H^1.

Combining our theorem with Theorem 1.2 gives:

THEOREM 1.4. *Let G be a finite simple Chevalley group whose underlying algebraic group has rank f. Let V be an irreducible KG-module where K is an algebraically closed field of arbitrary characteristic. Then there is a function $C(f)$ such that $\dim H^1(G, V) \leq C(f)$.*

Thus, even if Conjecture 1.1 is false, an obviously very important problem is to study the growth of $C(f)$.

There is a version of the previous theorem for any finite group. As Tits has suggested, let us view the alternating group A_n as a Chevalley group of rank $n - 1$ over the field of 1 element. Using Theorem 1.4, the reduction theorems in [AS85, Th. 3] or [GKKL07, Lemma 5.2] and the fact there are only finitely many sporadic simple groups, we immediately obtain:

COROLLARY 1.5. *Let the function $C(f)$ satisfy the conclusion of Theorem 1.4. Then there is an absolute constant C such that, if G is any finite group, K is any algebraically closed field and V is any faithful irreducible KG-module, then one of the following holds:*

(i) $\dim H^1(G, V) < C$;
(ii) *G has a normal subgroup that is a direct product of copies of a simple Chevalley group of rank f and $\dim H^1(G, V) \leq C(f)$.*

These results lead naturally to the question of what one can say about higher cohomology groups. In more recent papers [PS11], [PS], Parshall and

Scott have obtained results about Ext and higher cohomology groups in the context of algebraic and quantum groups (with applications to the finite groups of Lie type). See [GKKL07, §12] for some results and questions for higher cohomology groups for general finite groups. We refer the reader to [AG84], [GH98] for some weaker results about $H^1(G, V)$ that give bounds in terms of $\dim V$. See [KW84] for a nice application of those bounds.

The paper is organized as follows. We first show that if V is an irreducible kG-module and V is not a composition factor of $\mathcal{L}$, then $H^1(G, V) = 0$. We then obtain bounds for those irreducible kG-modules with $V^B \neq 0$ (equivalently those in $\mathrm{soc}(\mathcal{L})$). Next, we analyze $\mathcal{L}$ completely for the rank 1 groups. This is used to prove (ii) of Theorem 1.3.

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2. Modules not involved in $\mathcal{L} = k_B^G$

Let G be a Chevalley group defined over $\mathbb{F}_q$ with Borel subgroup B. Let r be a prime not dividing q and let k be an algebraically closed field of characteristic r. Let $\mathcal{L} := k_B^G$.

We first show that most kG-modules have trivial H^1. Let $\mathcal{L}^0$ be the unique submodule of $\mathcal{L}$ of codimension 1 (i.e. the "augmentation" submodule).

We begin with the following trivial observation.

LEMMA 2.1. *Let $A := O_{r'}(B)$. For any kB-module V, the following statements are equivalent*:

  (i) $V^B \neq 0$;
  (ii) B *has trivial composition factors on V*;
  (iii) $V^A \neq 0$;
  (iv) $(V^*)^B \neq 0$.

Proof. Obviously, (i) implies (ii), and (ii) implies (iii) as A acts coprimely on V. Next, the r-group B/A acts on the k-space V^A and so (iii) implies (i). Also, $V^A \neq 0$ if and only if $(V^*)^A \neq 0$ (since r does not divide $|A|$.) Thus, (i)–(iii) hold for V if and only if they hold for V^*, whence (iv) is equivalent to (iii). □

THEOREM 2.2. *Let V be an irreducible kG-module with $V^B = 0$. Then $\dim H^1(G, V)$ is the multiplicity of V in $\mathrm{head}(\mathcal{L}^0)$.*

By duality, this is also the multiplicity of V^* in $\mathrm{soc}(\mathcal{L}/\mathcal{L}^G) \cong \mathrm{head}((\mathcal{L}^0)^*)$. Before proving the theorem, we point out a corollary:

COROLLARY 2.3. *Let V be an irreducible kG-module. If V is not a composition factor of $\mathcal{L}$, then $H^1(G, V) = 0$.*

Proof. If V is not a composition factor of $\mathcal{L}$, then $V^B = 0$. If $V^B = 0$ and $H^1(G, V) \neq 0$, then the previous result indicates that V is a composition factor of $\mathcal{L}$. $\qquad\square$

Proof of Theorem 2.2. Since $V^B = 0$, by Lemma 2.1 we have $V^{O_{r'}(B)} = 0$, whence $H^1(B, V) = 0$ (in the semidirect product VB any complement B_1 to V will be the normalizer of $O_{r'}(B_1)$ and by the Schur-Zassenhaus theorem, $O_{r'}(B_1)$ and $O_{r'}(B)$ are conjugate — this also follows by the Hochschild-Serre sequence).

Suppose that $H^1(G, V)$ is d-dimensional. By duality, it suffices to show that V^* occurs with multiplicity d in $\mathrm{soc}(\mathcal{L}/\mathcal{L}^G)$.

Since $V^B = (V^*)^B = 0$, by Frobenius reciprocity, V^* does not embed in $\mathrm{soc}(\mathcal{L})$.

Choose a module Y of maximal dimension such that $0 \to k^d \to Y \to V^* \to 0$ is indecomposable. Note that $d = \dim \mathrm{Ext}^1_G(V^*, k) = \dim H^1(G, V)$. Also note that Y is uniquely determined (indeed $Y = D^*$ where $D := \mathrm{Der}(G, V)$ is the space of derivations of G into V).

It follows that $\dim \mathrm{Hom}_B(Y, k) = d$ (since $Y \cong V^* \oplus k^d$ as B-modules and $\mathrm{Hom}_B(V^*, k) = 0$). So by Frobenius reciprocity, $\mathrm{Hom}_G(Y, \mathcal{L})$ is also d-dimensional. If $f \in \mathrm{Hom}_G(Y, \mathcal{L})$ is nonzero, then f cannot vanish on k^d (since otherwise the image of f would be isomorphic to V^* and contained in $\mathrm{soc}(\mathcal{L})$, a contradiction). Thus, $\mathrm{Ker}(f)$ is a hyperplane in k^d and so the image of $f(Y)$ in $\mathrm{soc}(\mathcal{L}/\mathcal{L}^G)$ embeds in V^*.

Consider the natural map $\phi : \mathrm{Hom}_G(Y, \mathcal{L}) \to \mathrm{Hom}_G(Y, \mathcal{L}/\mathcal{L}^G)$. If $0 \neq f \in \mathrm{Ker}(\phi)$, then $f(Y) = \mathcal{L}^G$. However, by indecomposability, k is not a homomorphic image of Y. Thus, ϕ is an injection. Together with the previous paragraph, this implies that V^* has multiplicity at least d in $\mathrm{soc}(\mathcal{L}/\mathcal{L}^G)$. Since Y surjects onto any nonsplit module N with $0 \to k \to N \to V^* \to 0$, we see that ϕ is surjective as well. Thus, if V^* occurs with multiplicity m in $\mathrm{soc}(\mathcal{L}/\mathcal{L}^G)$, then $m = \dim \mathrm{Hom}_G(Y, \mathcal{L}) = d$. $\qquad\square$

3. Composition factors of k_B^G and modules with $V^B \neq 0$

Keep notation as in the previous section. Let Q be the unipotent radical of B, T a maximal torus of B (so $B = QT$), and let W be the Weyl group of G (generically, $W = N_G(T)/T$).

The following proposition is crucial for our considerations:

PROPOSITION 3.1. *For any irreducible kG-module V, set*

$$f_B(V) := \dim(V^B), \quad f_Q(V) := \dim(V^Q),$$

and let $m_{\mathcal{L}}(V)$ be the multiplicity of V as a composition factor of $\mathcal{L}$. Then the following statements hold:

(i) $\dim \mathcal{L}^B = \dim \mathcal{L}^Q = |W|$.

(ii) $X^Q = X^B$ for any submodule X of $\mathcal{L}$.

(iii) $m_{\mathcal{L}}(V) \geq f_B(V)$.

(iv) $\sum_{V \in \mathrm{Irr}_k(G)} m_{\mathcal{L}}(V) \cdot f_B(V) \leq \sum_{V \in \mathrm{Irr}_k(G)} m_{\mathcal{L}}(V) \cdot f_Q(V) = |W|$.

(v) If $\dim(V^Q) = d > 0$, then $m_{\mathcal{L}}(V) \leq |W|/d \leq |W|$.

(vi) $m_{\mathcal{L}}(k) \leq \sum_{V \in \mathrm{Irr}_k(G),\ V^B \neq 0} m_{\mathcal{L}}(V) \leq |W|$.

Proof. Note that $G = \cup_{w \in W} BwB$. Now $B = QT$, whence $BwB = QTwB = QwB$. Thus, Q and B each have $|W|$ orbits on G/B and so $\dim(\mathcal{L}^Q) = \dim(\mathcal{L}^B) = |W|$. It follows that $X^Q = X^B$ for any submodule X of $\mathcal{L}$.

By Frobenius reciprocity, $\mathrm{Hom}_G(\mathcal{L}, V) \cong \mathrm{Hom}_B(k, V) \cong V^B$, whence (iii) holds. The first inequality of (iv) is obvious. Since Q has order coprime to $\mathrm{char}(k)$, $|W| = \dim(\mathcal{L}^Q) = \sum_{V \in \mathrm{Irr}_k(G)} m_{\mathcal{L}}(V) \cdot f_Q(V)$, yielding (iv). The next two statements are immediate consequences of (iv). $\qquad\square$

COROLLARY 3.2. *Let X be a submodule of $\mathcal{L}$. Then $\dim H^1(G, X) \leq |W| + e'$, where e' is the r-rank of B/Q and is at most the twisted rank e of G.*

Proof. Start with the short exact sequence $0 \to X \to \mathcal{L} \to \mathcal{L}/X \to 0$. The long exact sequence in cohomology gives

$$0 \to H^0(G, X) \to H^0(G, \mathcal{L}) \to H^0(G, \mathcal{L}/X) \to H^1(G, X) \to H^1(G, \mathcal{L}).$$

Note that $H^1(G, \mathcal{L}) \cong H^1(B, k)$ has dimension $e' \leq e$. By the previous result, $H^0(G, \mathcal{L}/X) \leq |W|$. Thus, $\dim H^1(G, X) \leq |W| + e'$. $\qquad\square$

COROLLARY 3.3. $\sum_{V \in \mathrm{Irr}_k(G)} f_B(V) \cdot \dim H^1(G, V) \leq |W| + e$. *In particular,*

$$\sum_{V \in \mathrm{Irr}_k(G),\ V^B \neq 0} \dim H^1(G, V) \leq |W| + e,$$

and

$$\dim H^1(G, V) \leq \frac{|W| + e}{f_Q(V)}$$

if $V^B \neq 0$.

Proof. Consider $X := \mathrm{soc}(\mathcal{L})$. Then $V \in \mathrm{Irr}_k(G)$ embeds in X if and only if $(V^*)^B \neq 0$, which is equivalent to $V^B \neq 0$ by Lemma 2.1. By Proposition 3.1 and Frobenius reciprocity, the multiplicity $m_X(V)$ of V as a composition factor of X equals

$$m_X(V) = \dim(V^*)^B = \dim(V^*)^Q = \dim V^Q = f_Q(V) = \dim V^B = f_B(V).$$

Thus $X \cong \oplus_{V \in \mathrm{Irr}_k(G)} f_B(V) \cdot V$. Now apply Corollary 3.2 to X. $\qquad\square$

One can show that G has exactly $|W|$ isomorphism classes of irreducible kG-modules V with $V^B \neq 0$ precisely when G has rank 1 and r is coprime to $[G : B]$. In fact, under some mild conditions on r, Geck [Gec06] has shown that there is a canonical injection of the set of isomorphism classes of irreducible kG-modules with $V^B \neq 0$ into $\mathrm{Irr}(W)$.

4. Rank 1 groups

We are interested in the submodule structure of the module $\mathcal{L} := k_B^G$, where G is a finite group of Lie type of (twisted) rank 1 and B a Borel subgroup of G, with unipotent radical Q. In this case, the permutation action of G on $\Omega := G/B$ is doubly transitive. One can identify $\mathcal{L}$ with $k\Omega$ and consider the G-submodules

$$J := \langle \sum_{\omega \in \Omega} \omega \rangle \cong k, \ J^\perp := \left\{ \sum_{\omega \in \Omega} a_\omega \omega \mid a_\omega \in k, \sum_{\omega \in \Omega} a_\omega = 0 \right\}.$$

Then the *heart* of $\mathcal{L}$ is defined to be $\mathcal{H} := J^\perp/(J \cap J^\perp)$.

First we recall some well-known facts.

LEMMA 4.1. (i) *Assume $r|[G : B]$, equivalently, $J \subseteq J^\perp$. Then any nontrivial composition factor M of $\mathcal{L}$ has no nonzero Q-fixed points, and the socle and the head of $\mathcal{L}$ are simple and trivial.*

(ii) *Assume the heart $\mathcal{H}$ of $\mathcal{L}$ is simple. Then the only submodules of $\mathcal{L}$ are 0, J, $J^\perp$, and $\mathcal{L}$. Furthermore, one of the following holds*:

  (a) $(r, [G : B]) = 1$ *and* $\mathcal{L} \cong J \oplus \mathcal{H}$;
  (b) $r|[G : B]$ *and $\mathcal{L}$ is uniserial with composition factors k, $\mathcal{H}$, k (in that order).*

(iii) *Assume the heart $\mathcal{H}$ of $\mathcal{L}$ is not simple. Then $r = 2$ (and q is odd) if $G = \mathrm{PSL}_2(q)$, $r|(q^2 + 1)$ if $G = {}^2B_2(q)$, $r|(q^3 + 1)$ if $G = {}^2G_2(q)$, and $r|(q + 1)$ if $G = \mathrm{SU}_3(q)$.*

Proof. (i) Note that the permutation module $\mathbb{C}\Omega$ affords the character $1_G + \mathrm{St}$, where St is the Steinberg character of G, and furthermore, $[\mathrm{St}_Q, 1_Q]_Q = 1$. Since Q acts coprimely on $\mathcal{L}$, we get $\dim \mathcal{L}^Q = 2$. Also, $\mathcal{L}$ has the filtration $0 \subset J \subset J^\perp \subset \mathcal{L}$, and $J \cong \mathcal{L}/J^\perp \cong k$, whence $\mathcal{H}^Q = 0$. Now, if M is a nontrivial composition factor of $\mathcal{L}$, then we may assume that M is a subquotient of $\mathcal{H}$, and so $M^Q = 0$.

Next, since $\mathcal{L}$ is self-dual and $\mathrm{soc}(\mathcal{L}) \supseteq J$, it suffices to prove that any simple submodule M of $\mathcal{L}$ must equal J. Assume the contrary: $M \neq J$. Now, $0 \neq \mathrm{Hom}_G(M, \mathcal{L}) \cong \mathrm{Hom}_B(M, k)$ implies that $0 \neq \mathrm{Hom}_Q(M, k)$, and so $M^Q \neq 0$. By the above, $M \cong k$, $\mathrm{soc}(\mathcal{L}) \supseteq k \oplus k$, and so

$$2 \leq \dim \mathrm{Hom}_G(k, \mathcal{L}) = \dim \mathrm{Hom}_B(k, k) = 1,$$

a contradiction.

(ii) In the case of (a), $J \cap J^\perp = 0$ and $\mathcal{H} \cong J^\perp \cong \mathcal{L}/J$, whence $\mathcal{L} \cong J \oplus \mathcal{H}$. Next, (b) follows from (i).

(iii) is recorded in [Mor80, Table 1]. $\square$

LEMMA 4.2. *Assume r divides $|G|$ but not $|B|$, and the Sylow r-subgroups of G are cyclic. Then $\mathcal{L}$ is uniserial, with simple, trivial, socle and head. Furthermore, $M^Q = 0$ for each nontrivial composition factor M of the G-module $\mathcal{L}$.*

Proof. The assumptions imply that $\mathcal{L}$ is projective. By Lemma 4.1(i), $\mathrm{soc}(\mathcal{L}) \cong \mathrm{head}(\mathcal{L}) \cong k$, so $\mathcal{L}$ is a projective cover of the trivial kG-module k. Next, let D be a defect group of the principal block B_0 of G, D_1 the unique subgroup of order r of D, $N := N_G(D_1)$, and b the unique kN-block with $b^G = B_0$. Then the Green correspondence $(G, B_0) \to (N, b)$ sends the trivial kG-module k to the trivial kN-module; in particular, the Green correspondent of k has composition length 1. Hence [Pea75, theorem, pp. 234–236] implies that the projective covers of k are uniserial. $\square$

4.1. $\mathrm{PSL}_2(q)$. In this case, Lemmas 4.1 and 4.2 imply that $\mathcal{L}$ has the following structure, where V, V_1, V_2 are certain simple kG-modules:

- $\mathcal{L} = k \oplus V$ if $(r, q + 1) = 1$;
- $\mathcal{L}$ is uniserial of length 3 with composition factors k, V, k, if $G = \mathrm{PSL}_2(q)$ and $2 < r|(q+1)$, or if $G = \mathrm{PGL}_2(q)$ and $r|(q+1)$;
- $\mathcal{L}$ has socle series $\begin{matrix} k \\ V_1 \oplus V_2 \\ k \end{matrix}$ if $G = \mathrm{PSL}_2(q)$ and $r = 2|(q+1)$.

4.2. $^2B_2(q)$, $q = 2^{2a+1} \geq 8$. Here, $\mathcal{L}$ has the following structure where V and $U \not\cong U^*$ are simple, and $\dim U = (q-1)\sqrt{q/2}$:

- $\mathcal{L} = k \oplus V$ if $(r, q^2 + 1) = 1$;
- $\mathcal{L}$ is uniserial of length 3 with composition factors k, V, k, if $r|(q-\sqrt{2q}+1)$;
- $\mathcal{L}$ is uniserial of length 5 with composition factors k, U, V, U^*, k (in this order), if $r|(q+\sqrt{2q}+1)$.

Indeed, by Lemma 4.2, $\mathcal{L}$ is uniserial if $r|(q^2+1)$. Furthermore, the Brauer tree for the principal block of G, and the possible structure of $\mathcal{L}/k$ as a uniserial reduction modulo r of the Steinberg character St, are described in [His93, Anhang D.1].

4.3. $^2G_2(q)$, $q = 3^{2a+1} \geq 27$. In this case, $\mathcal{L}$ has the following structure, where V, $U \not\cong U^*$, $M \not\cong M^*$, T, are simple kG-modules:

- $\mathcal{L} = k \oplus V$ if $(r, q^3 + 1) = 1$;
- $\mathcal{L}$ is uniserial of length 3 with composition factors k, V, k, if $r|(q-\sqrt{3q}+1)$;
- $\mathcal{L}$ is uniserial of length 5 with composition factors k, U, V, U^*, k (in this order, and $\dim U = (q-1)(q+\sqrt{3q}+1)\sqrt{q/12}$), if $2 < r|(q+1)$;

- $\mathcal{L}$ is uniserial of length 7 with composition factors k, M, U, V, U^*, M^*, k (in this order, and $\dim U = (q-1)(q-\sqrt{3q}+1)\sqrt{q/12}$, $\dim M = (q^2-1)\sqrt{q/3}$), if $r|(q+\sqrt{3q}+1)$.

Indeed, by Lemma 4.2, $\mathcal{L}$ is uniserial if $2 < r|(q^3+1)$. Again, in this case we use the structure of the Brauer tree for the principal block of G as described in [His93, Anhang D.2]:

$$k$$
$$T$$
- $\mathcal{L}$ has socle series $\quad U \oplus V \oplus U^* \quad$ if $r = 2$, by [LM80, Prop. 3.8].
$$T$$
$$k$$

4.4. $\mathrm{PSU}_3(q)$. In this case, $\mathcal{L}$ has the following structure, where V and U are simple, and $\dim U = q(q-1)$:

- $\mathcal{L} = k \oplus V$ if $(r, q^3+1) = 1$;
- $\mathcal{L}$ is uniserial of length 3 with composition factors k, V, k, if r divides q^3+1 but not $q+1$. These first two cases follow from Lemmas 4.1 and 4.2.
- Assume $G = \mathrm{PGU}_3(q)$, and either $2 < r|(q+1)$, or $r = 2$ and $q \equiv 3\,(\mathrm{mod}\,4)$. Then $\mathcal{L}$ is uniserial of length 5 with composition factors k, U, M, U, k (in this order). Here, M is irreducible over $\mathrm{PGU}_3(q)$, and a sum of 1 or 3 irreducible $\mathrm{PSU}_3(q)$-modules, according to $r \neq 3$ or $r = 3$. This case is analyzed in [His04, Th. 4.1].
$$k$$
- $\mathcal{L}$ has socle series $U \oplus V$ if $r = 2$ and $q \equiv 1\,(\mathrm{mod}\,4)$, by [Erd79, Lemma (4.2)].
$$k$$

Now we are ready to prove the main result of this section:

THEOREM 4.3. *Let G be one of the following rank 1 groups: $\mathrm{PSL}_2(q)$, $\mathrm{PGL}_2(q)$, $\mathrm{PSU}_3(q)$, $\mathrm{PGU}_3(q)$, $^2B_2(q)$ with $q \geq 8$, or $^2G_2(q)$ with $q \geq 27$. Then $\dim H^1(G, V) \leq 1$ for any irreducible kG-module V. Moreover, $H^1(G, V) \neq 0$ for at most two irreducible kG-modules.*

Proof. 1) First we claim that $H^1(G, k) = 0$ unless
(a) $r = 2|(q+1)$ and $G = \mathrm{PGL}_2(q)$, or
(b) $r = 3$ and $G = \mathrm{PGU}_3(2)$,
in which cases $H^1(G, k) \cong k$. Indeed, $H^1(G, k) \cong \mathrm{Ext}^1_G(k, k)$. Let U be any 2-dimensional kG-module with k as the unique composition factor (with multiplicity 2). The action of G on U induces a group homomorphism $f \in \mathrm{Hom}(G, (k, +)) \cong \mathrm{Hom}(G/G', (k, +))$. The latter hom-space is 0 unless (a) or (b) occurs. In the case of (a), $r = 2 = |G/G'|$. In the case of (b), $r = 3 = |G/G'|$. (Note that G/G' is a 2-group if $G = \mathrm{PSL}_2(2)$ or $\mathrm{PSU}_3(2)$, but here $r \neq 2$.) In either case, the hom-space is 1-dimensional, and the claim follows.

2) Next we assume that $H^1(G, V) \neq 0$ for some nontrivial $V \in \mathrm{Irr}_k(G)$. By Corollary 2.3, V is a composition factor of $\mathcal{L} = k_B^G$.

Consider the case $r|[G : B]$. By Lemma 4.1(i), $V^B = 0$; hence $d := \dim H^1(G, V)$ is just the multiplicity of V in $\mathrm{head}(\mathcal{L}^0)$ by Theorem 2.2. The structure of $\mathrm{head}(\mathcal{L}^0)$ has already been described above. It follows that, if $G = \mathrm{PGL}_2(q)$ or $\mathrm{PGU}_3(2)$ then V is unique (up to isomorphism) and $d = 1$. In all other cases, $d = 1$ and there are at most two isomorphism classes for such V.

Finally, we consider the case $(r, [G : B]) = 1$. By Lemma 4.1, $\mathcal{L} \cong k \oplus \mathcal{H}$; hence we may assume that $V \cong \mathcal{H}$ and $H^1(G, V)$ embeds in $H^1(G, \mathcal{L})$. But $H^1(G, \mathcal{L}) \cong H^1(B, k)$ has dimension $e' \leq e = 1$, where e' is the r-rank of B. It follows that $H^1(G, V)$ is 1-dimensional, and V is unique in this case. $\square$

Note that if $G = {}^2B_2(q)$, $r|(q + \sqrt{2q} + 1)$ or ${}^2G_2(q)$, $2 < r|(q + 1)(q + \sqrt{3q} + 1)$, then there is an irreducible kG-module V with $H^1(G, V) \neq 0$ but $H^1(G, V^*) = 0$.

Also note that if $G = \mathrm{PSL}_2(q)$ with $r = 2|(q + 1)$ or $\mathrm{PSU}_3(q)$ with $r = 2$, $q \equiv 1 \pmod 4$, then there are two irreducible modules V with $V^B = 0$ and $H^1(G, V) \neq 0$. In fact, more generally if $G = \mathrm{Sp}_{2n}(q)$ with q odd and $r = 2$, then each of the two irreducible Weil modules of dimension $(q^n - 1)/2$ has 1-dimensional H^1.

5. Quasi-equivalence and duality

Let k be an algebraically closed field of characteristic $r > 0$ and G a group.

We say that two G-modules X and Y are *quasi-equivalent* if Y can be obtained from X by a twist by an automorphism of G. If X is a kG-module, we denote X^σ as the twist of X by σ (in particular if the character of X is χ, then the character of X^σ is $\chi \circ \sigma$). Clearly, modules that are quasi-equivalent have the same cohomology.

The next result will be very useful.

LEMMA 5.1. *Let G be a normal subgroup of a finite group H such that every r'-element of G is conjugate to its inverse in H. If V is any irreducible kG-module, then V and V^* are quasi-equivalent.*

Proof. Note the hypothesis implies that any H-invariant semisimple kG-module is self-dual. Consider $M := V_G^H$. Then M is a semisimple kG-module, where the simple summands are precisely all the twists of V by elements in H/G. By our observation, M is a self-dual kG-module, and so both V and V^* are kG-summands of M. Hence the result follows. $\square$

We have seen examples with $H^1(G, V) \neq 0$ and $H^1(G, V^*) = 0$. Note that M_{11} provides another example.

If G is a Chevalley group (or an algebraic group) in the same characteristic as k, the dual of any irreducible module is quasi-equivalent. If G is finite, this can be seen via the previous result because every semisimple element of G is inverted by an automorphism of G. In the case G is an algebraic group, there is always an automorphism τ of G which acts as inversion on a maximal torus $\mathcal{T}$. Thus, $V^* \cong V^\tau$ and so we see directly that V^* and V are quasi-equivalent.

If G is any finite classical group of simply connected type (no matter what is the characteristic of k), then any irreducible kG-module is quasi-equivalent to its dual. For it is again well known that every element of G is inverted by an automorphism of G. For instance, every element of $\mathrm{SL}_n(q)$ is conjugate to its inverse via an element of $\mathrm{GL}_n(q).\langle \tau \rangle$ where τ is the transpose-inverse map.

We record the result we need in the next section.

COROLLARY 5.2. *Let $G = \mathrm{SL}_n(q)$. Every irreducible kG-module is quasi-equivalent to its dual.*

6. Modules with $V^B = 0$

In this section we assume that G has twisted rank $e > 1$. Fix a Borel subgroup B. We keep notation as above.

Let V be an irreducible kG-module with $V^B = 0$. We will show that any such irreducible has at most 1-dimensional H^1, and moreover there are at most four such modules (the example $G = \mathrm{Sp}_4(q)$ with q odd and $r = 2$ show that three is possible). In fact, in many cases we will see that there is at most one such module.

A very easy case is the following:

THEOREM 6.1. *Let V be a kG-module with $V^B = 0$. Let P_i, $1 \leq i \leq n$, denote the minimal parabolic subgroups of G containing B. If r does not divide $[P_i : B]$ for any i, then $H^1(G, V) = 0$.*

Proof. Since $V^B = 0$, it follows by Lemma 2.1 that $V^{O_{r'}(B)} = 0$, whence $H^1(B, V) = 0$. Since r does not divide $[P_i : B]$ for any i, the restriction map $H^1(P_i, V) \to H^1(B, V)$ is injective, whence $H^1(P_i, V) = 0$ for all i. Since $G = \langle P_1, \ldots, P_n \rangle$, the result follows by [AG72] (or see [GK90]). $\square$

We now give some examples of modules with $H^1(G, V) \neq 0$ and $V^B = 0$. This construction will be used in the proof of Theorem 6.6.

REMARK 6.2. Let St denote the Steinberg module for $\mathbb{Q}G$ in characteristic 0. Note that St is a free rank 1 module for Q and so V^Q is 1-dimensional. It is well known that $\dim V^B = 1$ as well. Thus, St embeds in $\mathbb{Q}_B^G$ and we view it as such. Let M be the standard permutation module for 1_B^G over $\mathbb{Z}$. Then $L := \mathrm{St} \cap M$ is a $\mathbb{Z}G$-lattice in St with L a pure submodule of M.

Now reducing modulo r, we have a submodule X of $\mathcal{L}$ with X the reduction of L. Clearly, $\dim X^Q = 1$ and so X has exactly one composition factor Y

with $Y^Q \neq 0$. Since $\mathrm{soc}(X) \leq \mathrm{soc}(\mathcal{L})$, it follows by Frobenius reciprocity that $Y = \mathrm{soc}(X)$. If P is a parabolic subgroup containing B with radical R, then Y^R is a reduction of the Steinberg module for P/R. So by induction and inspection of the rank one cases, it follows that $Y \cong k$ if and only if r divides $[P : B]$ for every nontrivial parabolic subgroup P (and so it is enough to check the minimal ones). See [His90, Th. A]. In particular, $\mathrm{Ext}^1_G(Y, Z) \neq 0$ for each composition factor Z in $\mathrm{soc}(X/Y)$. So if $Y = k$ (i.e. r divides $[P_i : B]$ for each minimal parabolic P_i), we see that $H^1(G, Z^*) \neq 0$ and $Z^B = (Z^*)^B = 0$.

Next, we have:

LEMMA 6.3. *Let P be a parabolic subgroup properly containing B with radical R. Then $\mathcal{L}^R \cong \oplus_{w \in W_0} k^P_{P \cap B^w R} \cong \oplus_{w \in W_0} k^P_B$ as kP-modules where W_0 is a set of double coset representatives for $P \backslash G / B$.*

Proof. Let Δ be a set of positive roots corresponding to the root system which determines B. Recall that Q is the product of various root subgroups U_β, β a positive root (see [Car89, 13.6] or [GLS98, 2.3]).

We view W_0 as a subset of W. By Mackey decomposition, $\mathcal{L} \cong \oplus_{w \in W_0} k^P_{P \cap B^w}$ as kP-modules. Now take R-fixed points. Observe that since $R \leq P$, $P \cap B^w R = (P \cap B^w)R$. Note that the space of R-fixed points of $k^P_{P \cap B^w}$ is isomorphic to $k^P_{P \cap B^w R}$. Let H be the standard Levi subgroup of P; so H is generated by a maximal torus of B and all the root subgroups U_β where the roots β are linear combinations of a subset of the simple roots. Note for each positive root subgroup $U_\beta < G$ and for each $w \in W$, we have that precisely one of U_β or $U_{-\beta}$ is contained in $B^w \cap H$. Thus, $P \cap B^w R = B_1 R$ where B_1 is a Borel subgroup of H and so $B_1 R$ is a Borel subgroup contained in P. This completes the proof. $\square$

THEOREM 6.4. *Let P be a minimal parabolic subgroup containing B with radical R. Assume that r divides $[P : B]$. Let X be a submodule of $\mathcal{L}$ containing $\mathcal{L}^G$ such that $X^Q = \mathcal{L}^G$ and $\dim X^R > 1$. Then the following statements hold:*

(i) *Either $\mathrm{soc}(X^R/X^G)$ is an irreducible kP-module, or $r = 2$ and $[P, P]/R \cong \mathrm{SL}_2(q)$ with q odd or $\mathrm{SU}_3(q)$ with $q \equiv 1 \,(\mathrm{mod}\ 4)$. In the latter two cases $\mathrm{soc}(X^R/X^G)$ is either irreducible or a direct sum of two nonisomorphic irreducible kP-modules.*

(ii) *If $X/\mathcal{L}^G$ is irreducible, then $\mathrm{Ext}^1_G(X, k) = 0$ and $\dim \mathrm{Ext}^1_G(X/k, k) = 1$.*

Proof. It is clear that $Y := X^R$ is a kP-submodule of $M := \mathcal{L}^R$, and, by Lemma 6.3, M is a direct sum of copies of k^P_B as kP-modules. Since X^B is 1-dimensional, the same is true for X^P and so Y^P is 1-dimensional.

We claim that Y embeds in k^P_B. Let π be a projection of Y into one of the direct factors of M with $\pi(Y) \neq 0$. Now $Z := \mathrm{Ker}(\pi)$ is a submodule

of Y with $Z^Q = 0$. In particular, $\mathrm{soc}(Z)$ contains no trivial kP-composition factors. Since $\mathrm{soc}(M)$ is a direct sum of trivial composition factors, it follows that $\mathrm{soc}(Z) = 0$, whence $Z = 0$ and the claim follows.

Now (i) follows by our results on the rank 1 groups. Set $U := X/k$. If U is semisimple, then by (i), U is multiplicity free (since $\mathrm{soc}(U^R)$ is multiplicity free as a kP-module). Thus, by Theorem 2.2 if V is any simple summand of U, $\dim H^1(G, V) = \dim \mathrm{Ext}^1_G(V^*, k) = 1$, whence the second part of (ii) holds.

Now assume that U is simple. Let X^* denote the dual of X. So it has socle U^* and is indecomposable of length 2. Then we have $0 \to U^* \to X^* \to k \to 0$, giving rise to:

$$0 \to k \to H^1(G, U^*) \to H^1(G, X^*) \to H^1(G, k) = 0.$$

As we have already noted, $H^1(G, U^*)$ is 1-dimensional, whence (ii) follows. $\square$

COROLLARY 6.5. *Let V be an irreducible kG-module with $V^B = 0$ and $H^1(G, V) \neq 0$.*

(i) $\dim H^1(G, V) = 1$.
(ii) *If $r \neq 2$, then V is defined over $\mathbb{F}_r$.*
(iii) *If $r \neq 2$ and $H^1(G, V^*) \neq 0$, then $V \cong V^*$.*

Proof. Let X be a submodule of $\mathcal{L}$ such that $X^Q = \mathcal{L}^G$. Assume first that $Z := X/X^G$ is simple.

We claim that $\dim X^R > 1$ for R the radical of some minimal parabolic P containing B with r dividing $[P : B]$. If not, then $\mathrm{Ext}^1_P(X/k, k) = 0$ for each minimal parabolic subgroup P and so $\mathrm{Ext}^1_G(X/k, k) = 0$ by [AG72], a contradiction.

Now apply the previous result to see that $\dim \mathrm{Ext}^1_G(Z, k) = 1$.

Now take X a submodule of $\mathcal{L}$ with $X^Q = \mathcal{L}^G$ and $Z := X/X^G$ semisimple and as large as possible. Suppose that R is the radical of a minimal parabolic subgroup P. Note that $\mathrm{soc}(X^R/X^G)$ is a simple kP-module if $r \neq 2$ or at worst a direct sum of 2 distinct irreducibles. Thus, for $r \neq 2$, there is at most one summand of Z with $Z^R \neq 0$ (and at most two if $r = 2$).

The previous result also implies that Z^R is multiplicity free as a kP-module. This implies that $\dim \mathrm{Ext}^1_G(U, k) = 1$ for any simple summand U of Z. So (i) follows by Theorem 2.2.

Now assume that $r \neq 2$. If σ is an element of the Galois group of $\overline{\mathbb{F}}_r$, then $H^1(G, V^\sigma) \neq 0$, and V^σ is a also a summand of Z. Moreover, $\dim V^R = \dim(V^*)^R$. As we have seen above, there is at most one simple summand of Z having nontrivial R-fixed points. Thus, $V \cong V^\sigma$. Thus, V is defined over $\mathbb{F}_r$.

A similar argument gives the last result. $\square$

We want to show that there are very few irreducible modules with $V^B = 0 \neq H^1(G, V)$. The proof of the previous result easily gives that there are most

e (for $r \neq 2$) or $2e$ such modules. We will show that there are at most four such modules (one can check that for $r = 2$, there are three such modules for $\mathrm{Sp}_4(q)$ with q odd).

We first classify such modules for $G = \mathrm{SL}_n(q)$. By Theorem 6.1, we may assume that r divides $q + 1$.

THEOREM 6.6. *Let $G = \mathrm{SL}_n(q)$ with $n > 2$ and $r|(q + 1)$. There is a unique irreducible kG-module V with $V^B = 0$ and $H^1(G, V) \neq 0$, and this V can be found as in Remark 6.2.*

Proof. 1) Let W be any irreducible kG-module with $W^B = 0$, and so $(W^*)^B = 0$ by Lemma 2.1. By Theorem 2.2, $H^1(G, W^*) \neq 0$ precisely when W is a submodule of $\mathrm{soc}(\mathcal{L}/\mathcal{L}^G)$. By Corollary 5.2, W^* is quasi-equivalent to W; hence $H^1(G, W^*) \neq 0$ if and only if $H^1(G, W) \neq 0$. It follows that $H^1(G, W) \neq 0$ exactly when W is a submodule of $\mathrm{soc}(\mathcal{L}/\mathcal{L}^G)$. Observe that in this case, W^τ is also a submodule of $\mathrm{soc}(\mathcal{L}/\mathcal{L}^G)$, where $\tau = \tau^{-1}$ is the transpose-inverse map. (Indeed, $\dim H^1(G, W^\tau) = \dim H^1(G, W)$. Next, if T is an irreducible kH-module lying above W for $H := \mathrm{GL}_n(q)$, then it is well known that $T^\tau \cong T^*$, and $T^B = 0$ if and only if $W^B = 0$ as $H = N_H(B)G$. Hence we also have $(W^\tau)^B = 0$.)

2) We have seen in Remark 6.2 that such a module V exists. Let $\bar{X}$ be the sum of all simple summands W of $\mathrm{soc}(\mathcal{L}/\mathcal{L}^G)$ with $W^B = 0$, and let X be the complete inverse image of $\bar{X}$ in $\mathcal{L}$; in particular, $X^G = \mathcal{L}^G \cong k$. By the observations in 1), it suffices to show that X/X^G is simple. We will proceed by induction on n. When $n = 2$, the claim follows by Section 4.1 unless $r = 2$ and q is odd, but even in that case, the result holds for $\mathrm{PGL}_2(q)$.

For the induction step, let $B \leq P$ be a maximal end node parabolic subgroup of G with unipotent radical R. Let P_1 be the other end node parabolic subgroup containing B and set R_1 to be its unipotent radical. Note that $R_1 = R^{\tau g}$ for some $g \in G$.

Set $Y = X^R$. By Lemma 6.3, Y embeds in a direct sum of copies of k_B^P. Since $Y^B = X^G \cong k$, it follows that Y embeds in k_B^P. By induction on n (notice that when $n = 3$, $P/R \cong \mathrm{GL}_2(q)$), the socle of Y/k as a kP-module is simple. It follows that there is at most one summand U of X/X^G with $U^R \neq 0$. Since R is unipotent, for such a U we also have $(U^*)^R \neq 0$. As in 1), if T is an irreducible kH-module lying above U, then $T^\tau \cong T^*$, $(T^\tau)^R \neq 0$, and so $(U^\tau)^R \neq 0$ (since $H = N_H(R)G$). According to 1), U^τ is a summand of X/X^G, so the uniqueness of U implies that $U \cong U^\tau$. Now assume that W is a summand of X/X^G with $W^{R_1} \neq 0$. Then

$$\dim(W^\tau)^R = \dim(W^{\tau^{-1}})^R = \dim W^{R^\tau} = \dim W^{R_1},$$

and so $(W^\tau)^R \neq 0$. By 1), W^τ is a summand of X/X^G, so the uniqueness of U again implies that $W^\tau \cong U \cong U^\tau$, whence $W \cong U$. It follows that

$W^R \cong U^R \neq 0$ and so $W = U$. Consequently, U is also the unique summand of X/X^G with $U^{R_1} \neq 0$.

Now, if $Y \neq U$ is any summand of X/X^G, then $Y^R = Y^{R_1} = 0$, whence $H^1(P, Y^*) = H^1(P_1, Y^*) = H^0(B, Y^*) = 0$. Thus, by [AG72], $H^1(G, Y^*) = 0$, contradicting Theorem 2.2. Hence U exists and $X/X^G = U$ is simple, as required. $\square$

COROLLARY 6.7. *There are at most four irreducible kG-modules V with $V^B = 0$ and $H^1(G, V) \neq 0$.*

Proof. The proof is similar to the previous case. Let X be any submodule of $\mathcal{L}$ containing $\mathcal{L}^G = X^G$ such that $Y := X/X^G$ is semisimple and every composition factor V of X/X^G satisfies $V^B = 0$. Choose two parabolic subgroups P_1 and P_2 such that $G = \langle P_1, P_2 \rangle$. Let R_i denote the unipotent radical of P_i. Let m_i be the number of summands V of Y such that R_i has fixed points on V. It suffices to show that $m_1 + m_2 \leq 4$. If P_i is a minimal parabolic, then we have seen that $m_i \leq 2$ (and indeed $m_i \leq 1$ if $r \neq 2$). This gives the result if G has rank 2. So we assume that G has rank at least 3.

If possible choose P_1 with Levi subgroup of type SL, and so by the previous result $m_1 \leq 1$. If we take P_1 to be maximal, then we can choose P_2 to be minimal. We have seen that $m_2 \leq 2$. Thus, there are at most 3 summands.

This can be done in all cases except $G = F_4(q)$ or $^2E_6(q)$. If $G = F_4$, we take P_i to have a Levi of type $\mathrm{SL}_3(q)$ for $i = 1, 2$. If $G = {}^2E_6(q)$, we take P_1 to have a Levi of type $\mathrm{SU}_6(q)$ and P_2 to be a minimal parabolic. Modifying the argument above for SU, we see that $m_1 \leq 2$, whence the result. $\square$

In many cases, the argument above can be used to give better bounds on the number of such modules. For example, it is not hard to see that if G is untwisted of types D or E, then there is a unique such module (when r divides $q + 1$).

References

[AG72]   J. L. ALPERIN and D. GORENSTEIN, A vanishing theorem for cohomology, *Proc. Amer. Math. Soc.* **32** (1972), 87–88. MR 0291293. Zbl 0249.18026. doi: 10.2307/2038312.

[AG84]   M. ASCHBACHER and R. GURALNICK, Some applications of the first cohomology group, *J. Algebra* **90** (1984), 446–460. MR 0760022. Zbl 0554.20017. doi: 10.1016/0021-8693(84)90183-2.

[AS85]   M. ASCHBACHER and L. SCOTT, Maximal subgroups of finite groups, *J. Algebra* **92** (1985), 44–80. MR 0772471. Zbl 0549.20011. doi: 10.1016/0021-8693(85)90145-0.

[BW08]   J. N. BRAY and R. A. WILSON, Examples of 3-dimensional 1-cohomology for absolutely irreducible modules of finite simple groups, *J. Group Theory*

11 (2008), 669–673. MR 2446148. Zbl 1160.20045. doi: 10.1515/JGT. 2008.042.

[Car89] R. W. CARTER, *Simple Groups of Lie Type*, *Wiley Classics Library*, John Wiley & Sons, New York, 1989, reprint of the 1972 original. MR 1013112. Zbl 0723.20006.

[CPS99] E. T. CLINE, B. J. PARSHALL, and L. L. SCOTT, Generic and q-rational representation theory, *Publ. Res. Inst. Math. Sci.* **35** (1999), 31–90. MR 1681236. Zbl 0998.20017. doi: 10.2977/prims/1195144189.

[CPS09] _____, Reduced standard modules and cohomology, *Trans. Amer. Math. Soc.* **361** (2009), 5223–5261. MR 2515810. Zbl 1183.20051. doi: 10.1090/ S0002-9947-09-04633-9.

[Erd79] K. ERDMANN, On 2-blocks with semidihedral defect groups, *Trans. Amer. Math. Soc.* **256** (1979), 267–287. MR 0546918. Zbl 0388.20007. doi: 10. 2307/1998111.

[Gec06] M. GECK, Modular principal series representations, *Int. Math. Res. Not.* **2006** (2006), Art. ID 41957, 20. MR 2219230. Zbl 1118.20017. doi: 10. 1155/IMRN/2006/41957.

[GLS98] D. GORENSTEIN, R. LYONS, and R. SOLOMON, *The Classification of the Finite Simple groups. Number 3. Part I. Chapter A. Almost Simple K-Groups*, *Math. Surveys Monogr.* **40**, Amer. Math. Soc., Providence, RI, 1998. MR 1490581. Zbl 0890.20012.

[Gur86] R. M. GURALNICK, The dimension of the first cohomology group, in *Representation Theory*, II (Ottawa, Ont., 1984), *Lecture Notes in Math.* **1178**, Springer-Verlag, New York, 1986, pp. 94–97. MR 0842479. Zbl 0625. 20039. doi: 10.1007/BFb0075290.

[GH98] R. M. GURALNICK and C. HOFFMAN, The first cohomology group and generation of simple groups, in *Groups and Geometries* (Siena, 1996), *Trends Math.*, Birkhäuser, Basel, 1998, pp. 81–89. MR 1644977. Zbl 0894. 20039.

[GKKL07] R. M. GURALNICK, W. M. KANTOR, M. KASSABOV, and A. LUBOTZKY, Presentations of finite simple groups: profinite and cohomological approaches, *Groups Geom. Dyn.* **1** (2007), 469–523. MR 2357481. Zbl 1135. 20024. doi: 10.4171/GGD/22.

[GK90] R. M. GURALNICK and W. KIMMERLE, On the cohomology of alternating and symmetric groups and decomposition of relation modules, *J. Pure Appl. Algebra* **69** (1990), 135–140. MR 1086556. Zbl 0756.20008. doi: 10.1016/0022-4049(90)90038-J.

[His90] G. HISS, The number of trivial composition factors of the Steinberg module, *Arch. Math.* (*Basel*) **54** (1990), 247–251. MR 1037613. Zbl 0669. 20036. doi: 10.1007/BF01188519.

[His93] _____, *Zerlegungszahlen endlicher Gruppen vom Lie-Typ in nicht-definierender Characteristik*, *Habilitationsschrift, Lehrstuhl D für Mathematik*, RWTH Aachen, Germany, 1993.

[His04]   G. HISS, Hermitian function fields, classical unitals, and representations of 3-dimensional unitary groups, *Indag. Math.* **15** (2004), 223–243. MR 2071863. Zbl 1139.05308. doi: 10.1016/S0019-3577(04)90016-X.

[KW84]   W. KIMMERLE and J. WILLIAMS, On minimal relation modules and 1-cohomology of finite groups, *Arch. Math. (Basel)* **42** (1984), 214–223. MR 0751498. Zbl 0553.20003. doi: 10.1007/BF01191178.

[LM80]   P. LANDROCK and G. O. MICHLER, Principal 2-blocks of the simple groups of Ree type, *Trans. Amer. Math. Soc.* **260** (1980), 83–111. MR 0570780. Zbl 0438.20007. doi: 10.2307/1999877.

[Mor80]   B. MORTIMER, The modular permutation representations of the known doubly transitive groups, *Proc. London Math. Soc.* **41** (1980), 1–20. MR 0579714. Zbl 0393.20002. doi: 10.1112/plms/s3-41.1.1.

[PS11]   B. J. PARSHALL and L. L. SCOTT, Bounding Ext for modules for algebraic groups, finite groups and quantum groups, *Adv. Math.* **226** (2011), 2065–2088. MR 2739773. Zbl 05848696. doi: 10.1016/j.aim.2010.09.021.

[PS]   _____, Cohomological growth rates and Kazhdan-Lusztig polynomials, *Israel J. Math.*, to appear.

[Pea75]   R. M. PEACOCK, Blocks with a cyclic defect group, *J. Algebra* **34** (1975), 232–259. MR 0427443. Zbl 0303.20008. doi: 10.1016/0021-8693(75)90181-7.

[Sco03]   L. L. SCOTT, Some new examples in 1-cohomology, *J. Algebra* **260** (2003), 416–425, special issue celebrating the 80th birthday of Robert Steinberg. MR 1976701. Zbl 1019.20020. doi: 10.1016/S0021-8693(02)00667-1.

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