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On the Castelnuovo-Mumford regularity of rings of polynomial invariants

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Abstract

We show that when a group acts on a polynomial ring over a field the ring of invariants has Castelnuovo-Mumford regularity at most zero. As a consequence, we prove a well-known conjecture that the invariants are always generated in degrees at most $n(|G|-1)$, where $n>1$ is the number of polynomial generators and $|G|>1$ is the order of the group. We also prove some other related conjectures in invariant theory.

The main result of this paper is the following theorem.

Theorem 0.1. *Let k be any field and $S=k[x_1,\dots,x_n]$ a graded polynomial ring. Let the group G act on S by homogeneous linear substitutions. Then the invariant subring S^G has Castelnuovo-Mumford regularity at most zero.*

As a consequence, we can prove a conjecture of Gregor Kemper [19] on the degrees of the generators of S^G.

Corollary 0.2. *The invariant subring S^G is generated in degrees at most $n(|G|-1)$ (provided that $n>1$, $|G|>1$). The relations between the the generators are generated in degrees at most $2n(|G|-1)$.*

That just $|G|$ is a bound on the degrees of the generators when k has characteristic 0 is a result of Noether (for this reason such a bound is called a Noether bound in the literature). This was generalized to the case of coprime characteristic by Fleischmann [13] and Fogarty [14], with a much simplified proof by Benson. However, in general, no bound depending only on $|G|$ is possible, as was shown by Richman [21], [22]. The bound above was shown to hold when S_1 is a trivial source module by Göbel [15]. Some very much weaker bounds have been shown to hold by Derksen and Kemper [8] and by Karagueuzian and the author [18].

This corollary follows from the next corollary, which in turn follows easily from the Main Theorem 0.1 and elementary properties of regularity.

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COROLLARY 0.3. *Let $k[d_1, \ldots, d_n] < S^G$ be a subring of primary invariants. Then*:

(1) *the secondary invariants are bounded in degree by $\sum_i(\deg(d_i) - 1)$,*
(2) *the relations as a ring are generated in degrees at most $2\sum_i(\deg(d_i)-1)$.*

The first part was conjectured by Kemper [19]. It is well known when k has characteristic zero [23] and was proved in the case when the ring of invariants is Gorenstein by Campbell, Geramita, Hughes, Shank and Wehlau [7], then in the Cohen-Macauley case by Broer [5].

The Hilbert series of S^G is the formal power series

$$H(S^G, t) = \sum_i \dim_k(S_i^G)t^i.$$

It is known to be a rational function, by the Hilbert-Serre Theorem. The next result (strictly speaking only a corollary of the proof of Theorem 0.1) was also conjectured by Kemper [19].

COROLLARY 0.4. *The degree of $H(S^G, t)$ as a rational function is at most $-n$.*

This corollary is easily seen to be equivalent to the previous one in the Cohen-Macaulay case. It was proved for a reductive algebraic group over an algebraically closed field of characteristic zero by Knop [20].

There is an excellent survey of results and conjectures on degree bounds by Wehlau [26].

The proof of the Main Theorem 0.1 is quite short, but it depends heavily on the details of the Structure Theorem that we proved with Karagueuzian in [18] and on the relatively projective resolutions of [24]. We also need to develop some of the properties of Castelnuovo-Mumford regularity.

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1. **Castelnuovo-Mumford Regularity**

We will work in categories of $\mathbb{Z}$-graded rings and modules throughout, so $M = \oplus_{i\in\mathbb{Z}}M_i$. We will write $M_{\geq d} = \oplus_{i\geq d}M_i$ and similarly for other inequalities. We also use $M(d)$ to denote a shift down in grading by d, so that $M(d)_i = M_{i+d}$.

Let k be a field, and let R be a finitely generated commutative graded k-algebra in nonnegative degrees with $\dim_k R_0 < \infty$. Let I be a (homogeneous) ideal in R, and let M be an R-module (graded, by assumption). The I-torsion in M is $\Gamma_I(M) = \{m \in M \mid \exists n \in \mathbb{N} \quad I^n m = 0\}$. The local cohomology, $H^i_I(R, M)$, is then defined to be the ith right derived functor of $\Gamma_I(M)$ (in the category of graded R-modules); frequently the ring R is suppressed from the notation and just $H^i_I(M)$ is written. It follows easily from the definitions that $H^i_I(R, M) = H^i_{\sqrt{I}}(R, M)$.

For more information on local cohomology see [6], [4], [11] or [17].

Let $\mathfrak{m} = R_{>0}$ be the ideal of positively graded elements of R; usually we will have $R_0 = k$, so $\mathfrak{m}$ is the unique maximal homogeneous ideal. We will be interested in $H^i_{\mathfrak{m}}(M)$. Let $a_i(R, M)$ denote the maximum degree of a nonzero element of $H^i_{\mathfrak{m}}(R, M)$ (possibly ∞ if unbounded or $-\infty$ if $H^i_{\mathfrak{m}}(R, M) = 0$). The Castelnuovo-Mumford regularity (or just regularity) of M over R is, by definition,

$$\operatorname{reg}(R, M) = \sup_i \{a_i(R, M) + i\}.$$

The number $\operatorname{reg}(R, R)$ is important and we denote it by just $\operatorname{reg}(R)$.

The Independence Theorem for local cohomology ([16, 5.7], [4, 13.1.6]) states that if R' is another ring satisfying the same conditions as R, $I' < R'$ is an ideal and $f\colon R' \to R$ is a ring homomorphism (all graded), then f induces an isomorphism $H^i_{I'}(R', M) \to H^i_{I'R}(R, M)$ (where we regard M as an R'-module via f and $I'R$ is the ideal in R generated by $f(I')$). Let $\mathfrak{m}' = R'_{>0}$; if R is finite over R', then it is easy to see that $\sqrt{\mathfrak{m}'R} = \sqrt{\mathfrak{m}}$. Combining these facts we obtain $H^i_{\mathfrak{m}'}(R', M) \cong H^i_{\mathfrak{m}}(R, M)$, so $\operatorname{reg}(R', M) = \operatorname{reg}(R, M)$.

We will use this theory when R' is a Noether normalization of R, that is to say a polynomial subring over which R is finitely generated. These always exist, and their generators are often referred to as a homogeneous system of parameters, or as primary invariants in the case of invariant theory (for more information on this see the references mentioned above or [1]). Restricting to any normalization will yield the same value for the regularity and we will usually write just $\operatorname{reg}(M)$.

Now suppose that $R = k[x_1, \ldots, x_n]$ is a polynomial ring in which the generators have arbitrary positive degree $|x_i| = \deg(x_i)$. We set $\sigma(R) = \sum_{i=1}^n (|x_i| - 1)$ although traditionally one considers the a-invariant, $a(R) = -\sigma(R) - n = -\sum_{i=1}^n |x_i|$.

Let M be an R-module and consider the minimal (graded) projective resolution of M (projective is equivalent to free in this case)

$$\cdots \to P_1 \to P_0 \to M \to 0.$$

Let $\rho_i(R, M)$ be the maximum degree of a nonzero element of $R/\mathfrak{m} \otimes_R P_i$ (possibly ∞ or $-\infty$), which is equal to the maximum degree of a generator

of P_i (Benson [2], [3] uses β_i instead of ρ_i, but this can be confused with the Betti numbers). Define

$$\operatorname{Preg}(R,M) = \sup_i\{\rho_i(R,M) - i\} - \sigma(R).$$

This form of the definition first appeared in a paper of Benson [2]. The usual definition does not contain a σ-term, because all the d_i are supposed to be in degree 1 and so $\sigma(R) = 0$; but the necessity of using this form will become apparent.

We consider Hom-groups between graded modules to be graded modules as well. The homogeneous part in degree i consists of the homomorphisms that increase the grading by i. In this way the Ext groups are also graded modules.

Now define $\varepsilon_i(R,M)$ to be the minimum degree of a nonzero element of $\operatorname{Ext}^i_R(M,R)$ (possibly ∞ or $-\infty$). Define

$$\operatorname{Extreg}(R,M) = \sup_i\{-\varepsilon_i(R,M) - i\} - \sigma(R).$$

LEMMA 1.1. *Assume that M is finitely generated over R (which is still a polynomial ring). Then* $\operatorname{Preg}(R,M) = \operatorname{Extreg}(R,M)$.

Proof. A proof is given in [10, 20.16] (it is assumed there that $\sigma(R) = 0$, but the argument still holds). For the convenience of the reader we sketch the proof.

That $\operatorname{Preg}(R,M) \geq \operatorname{Extreg}(R,M)$ follows easily from the definitions. For the reverse inequality, we will show that $\rho_i(R,M) \leq \operatorname{Extreg}(R,M)+\sigma(R)+i$ by downward induction on i. Since R is a polynomial ring, the minimal resolution has finite length, so the induction certainly starts.

Let $\cdots \to P_r \xrightarrow{d_r} P_{r-1} \to \cdots \to M \to 0$ denote the minimal projective resolution of M. We claim that there is no map $f : P_r \to R(-u)$, for any r or u, such that fd_{r+1} is onto. For then fd_{r+1} would split, and P_{r+1} would contain a summand $R(-u)$ that mapped isomorphically to its image in P_r. We could then factor out the two copies of $R(-u)$ in $P_\bullet$ and obtain a smaller resolution of M, a contradiction.

Suppose that the inequality is established for $i > r$, but that P_r contains a summand $R(-u)$ with $u > \operatorname{Extreg}(R,M) + \sigma(R) + r$. In particular, $u > -\varepsilon_r(R,M)$. Let f be the corresponding projection of P_r onto $R(-u)$ and consider the map $(fd_{r+1})_u : (P_r)_u \to R(-u)_u \cong k$. If it is onto then fd_{r+1} is onto, which is impossible by the discussion above. Thus $(fd_{r+1})_u = 0$ and since, by the induction hypothesis, P_{r+1} contains no summands $R(-v)$ with $v > u$, the map $fd_{r+1} = 0$. Thus f determines an element of $\operatorname{Ext}^r_R(M, R(-u))_0 \cong \operatorname{Ext}^r_R(M,R)_{-u}$.

But this Ext-group is zero by the condition on u, so f factors through d_r, again contradicting the minimality of the resolution. □

Note that we might well have $\rho_i(R,M) \neq \varepsilon_i(R,M)$.

The Local Duality Theorem ([16, 6.3], [6, 3.6.19/3.6.11]) states that for R a polynomial ring in n variables as above and M a finitely generated R-module we have

$$\mathrm{Hom}_k(H^i_{\mathfrak{m}}(M),k) \cong \mathrm{Ext}^{n-i}_R(M, R(a(R))).$$

Recall that k is in degree 0 and that $R(a(R))$ denotes a copy of R that has been shifted down in degree by $a(R)$ or, equivalently, up in degree by $\sigma(R)+n$.

PROPOSITION 1.2. *If R is a polynomial ring over a field and M is a finitely generated R-module, then* $\mathrm{Preg}(R,M) = \mathrm{reg}(R,M)$.

Proof. Combine the Local Duality Theorem with Lemma 1.1. □

This result is well known when $\sigma(R) = 0$ (see e.g. [10, A4.2]). It is stated in this generality in [3, 2.3].

We are really only concerned with Preg in this paper, but we need the connection with local cohomology in order to see that if R is a noetherian ring and R' and R'' are two different Noether normalisations of R, then for any finitely generated R-module M we have $\mathrm{Preg}(R',M) = \mathrm{Preg}(R'',M)$.

For example, if R is a ring of polynomial invariants, then $\mathrm{Preg}(R',R)$ does not depend on the choice of primary invariants R' and it is equal to $\mathrm{reg}(R)$.

Remark. If R is a polynomial ring, then $\mathrm{reg}(R) = \mathrm{Preg}(R;R) = -\sigma(R)$, so the regularity of a ring of invariants can certainly be negative. However, in characteristic zero the ring of invariants has regularity zero if and only if the representation in degree one has trivial determinant (see [23, 3.9]).

This is in contrast to the case of the cohomology of a finite group, where Benson shows in [3] that $\mathrm{reg}(H^*(G,\mathbb{F}_p)) \geq 0$ and conjectures that equality holds. This has now been proved [25].

2. **Generators and relations**

Given a finitely generated graded k-algebra S in nonnegative degrees and an integer N, let $\tau^k_N S$ be the k-algebra determined by the generators and relations of S that occur in degrees at most N. We will normally write just $\tau_N S$. There is a canonical map $\tau_N S \to S$, which is an isomorphism in degrees up to and including N.

For a more abstract setting, consider the functor $S \mapsto S/S_{>N}$ on graded k-algebras in nonnegative degrees; τ_N is its left adjoint.

It is not hard to see that if ℓ is an extension field of k then $\tau^\ell_N(\ell \otimes_k S) \cong \ell \otimes_k \tau^k_N S$.

PROPOSITION 2.1. *Let $R = k[d_1, \dots, d_m]$, and suppose that there is a map $f\colon R \to S$ such that S is finitely generated over R (e.g. if R is a Noether normalisation of S). Then*:

(1) *if $N \geq \max\{\operatorname{reg}(S) + \sigma(R),\ \deg(d_i)\}$, then $\tau_N S \to S$ is a surjection*;
(2) *if $N \geq \max\{2(\operatorname{reg}(S)+\sigma(R)), \operatorname{reg}(S)+\sigma(R)+1, \deg(d_i)\}$, then $\tau_N S \to S$ is an isomorphism*;
(3) *if $N \geq \max\{\operatorname{reg}(S) + \sigma(R) + 1,\ \deg(d_i)\}$ and if $\tau_N S$, considered an R-module, is generated in degrees at most N, then $\tau_N S \to S$ is an isomorphism.*

Proof. Let $\cdots \to P_1 \to P_0 \to S \to 0$ be the minimal projective resolution of S as an R-module.

It is clear from the definitions that S, considered as an R-module, is generated in degrees at most $\rho_0(R, S)$ and is presented in degrees at most $\max\{\rho_0(R, S),\ \rho_1(R, S)\}$.

If $N \geq \max\{\deg(d_i)\}$, then f can be lifted uniquely to $\tau_N S$, making $\tau_N S$ into a finitely generated R-module.

Let $\{v_i\}$ be a set of homogeneous generators of S as an R-module with minimum degrees. These have degrees not exceeding $\rho_0(R, M)$, which is bounded by $\operatorname{reg}(S) + \sigma(R)$ according to the definition of Preg. This proves part (1).

The R-module relations between the v_i are generated in degrees at most $\operatorname{reg}(S) + \sigma(R) + 1$. The only information still needed in order to determine the structure of S as a ring is an expression for each of the products $v_j v_k$ as an R-linear combination of the v_i. Such a formula will lie in degree at most $2(\operatorname{reg}(S) + \sigma(R))$. This proves part (2).

For part (3), consider the following commutative diagram of R-modules with exact rows, where the vertical arrows can be filled in since the P_i are projective:

$$\begin{array}{ccccccccc}
 & \longrightarrow & P_1 & \longrightarrow & P_0 & \longrightarrow & S & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & K & \longrightarrow & \tau_N S & \longrightarrow & S & \longrightarrow & 0.
\end{array}$$

We know that $K_{\leq N} = 0$ and that P_1 is generated in degrees at most $\rho_1(R, S) \leq N$. Thus the composite map $P_1 \to \tau_N S$ is zero. It follows that the bottom row is split as a sequence of R-modules. Since $\tau_N S$ is generated as an R-module in degrees at most N, by hypothesis, so is K, which implies that $K = 0$. □

Remark. There is a similar result when both S and $\tau_N S$ are taken to be graded commutative rings (although R remains strictly commutative).

Proofs of 0.2 *and* 0.3. Corollary 0.3 now follows directly from the Main Theorem 0.1 and Proposition 2.1. It is easy to check that the maximum in the

formulas of Proposition 2.1 is achieved by the first term, except in the trivial case when all the d_i have degree 1.

The remark about τ_N commuting with field extensions shows that, in order to prove Corollary 0.2, we may extend the field. But then a result of Dade in [23] shows that for some finite field extension we can find a set of primary invariants of degree at most the order of the group, so from Corollary 0.3 we obtain the bound $\max\{n(|G|-1), |G|\}$ for the degrees of the generators. But the $|G|$ term is only larger than the other in the trivial cases that are excluded in the statement of the corollary. Similarly, the bound obtained on the degrees of the relations is always $2n(|G|-1)$, except in the case when $G = 1$ and there are no relations anyway.

Remark. In the case when G is a p-group, one of the primary invariants can be taken to be in degree 1 and we obtain the bound $(n-1)(|G|-1)$.

Remark. Benson [2, §10] defines $\tau_N S$ and proves a version of Proposition 2.1 in the case when S is the cohomology of a group.

3. **Relatively projective resolutions**

Let M be an RG-module for some finite group G. A relatively projective resolution $P_\bullet$ of M relative to kG is a complex of RG-modules

$$\cdots \to P_2 \to P_1 \to P_0 \to M \to 0$$

that is split exact over kG and in which each P_i is a sum of terms $R \otimes_k V(-d)$, where $V(-d)$ is a kG-module considered to be in degree d. For brevity we will call this just an RG/kG-resolution. Here R acts on the first term of $R \otimes_k V(-d)$ and G acts on the second. We could also write $R \otimes_k V(-d)$ as $RG \otimes_{kG} V(-d)$, with RG acting on the left in the usual way.

If N is another RG-module, then we obtain R-modules $\operatorname{Ext}^i_{RG/kG}(M, N)$ by applying $\operatorname{Hom}_{RG}(-, N)$ to the resolution of M and taking homology.

For more information on general relative homological algebra see [9], [12], [27]. We will closely follow our treatment in [24]. In particular, the next result is taken from [24, 4.2].

THEOREM 3.1. *If M is a finitely generated RG-module, then M has a unique minimal RG/kG-resolution $P_\bullet$. For any indecomposable kG-module V and any integer d, the number of summands of the form $R \otimes_k V(-d)$ in any given term P_i is finite and is zero if V is not a summand of M as a kG-module. If R is polynomial, then this resolution has finite length.*

The question of minimality and uniqueness is not explicitly addressed in [24], but the construction used there proceeds by changing the problem to one about projective resolutions in another category, in which modules bounded

below in degree have projective covers, so the existence of a unique minimal projective resolution is guaranteed. For a theoretical framework in the context of relative homological algebra see [12, Chap. 8].

From now on, assume that R is polynomial and that M is finitely generated over R. For any indecomposable kG-module V, define $\rho_i(R, G; M, V)$ to be the largest d for which $R \otimes_k V(-d)$ appears in the P_i term of the minimal RG/kG-resolution of M. Define

$$\rho_i(R, G; M) = \sup_V \{\rho_i(R, G; M, V)\},$$
$$\operatorname{Preg}(R, G; M) = \sup_i \{\rho_i(R, G; M) - i\} - \sigma(R).$$

It is clear that if H is a subgroup of G, then an RG/kG-resolution of M restricts to an RH/kH-resolution of $M\downarrow_H^G$. This fact together with the next lemma is key to our strategy.

LEMMA 3.2. *If $\cdots \to P_1 \to P_0 \to M \to 0$ is an RG/kG-resolution of M, then $\cdots \to P_1^G \to P_0^G \to M^G \to 0$ is a projective resolution of the invariants M^G over R. As a consequence,*

$$\rho_i(R, M^G) \le \rho_i(R, G; M),$$
$$\operatorname{Preg}(R, M^G) \le \operatorname{Preg}(R, G; M).$$

Proof. Since the resolution is split over kG, taking fixed points preserves exactness. Each P_i is a sum of terms of the form $R \otimes_k V(-d)$; thus P_i^G is a sum of terms of the form $R \otimes_k V^G(-d)$, so is free over R. □

The next lemma is a sort of generalization of Lemma 1.1, but notice that the bound is horizontal instead of diagonal.

LEMMA 3.3. *For any integer N, the following are equivalent*:

(1) $\rho_i(R, G; M) \le N$ *for all* i;
(2) $\operatorname{Ext}^i_{RG/kG}(M, R \otimes_k V(-d))_0 = 0$ *for all kG-modules V, all i and all $d > N$*;
(3) $\operatorname{Ext}^i_{RG/kG}(M, R \otimes_k V(-d))_0 = 0$ *for all indecomposable kG modules V that occur as a summand of M, all i and all $d > N$*;
(4) *for each indecomposable kG-module V, either $\rho_i(R, G; M, V) \le N$ for all i or $\operatorname{Ext}^i_{RG/kG}(M, R \otimes_k V(-d))_0 = 0$ for all i and all $d > N$.*

Proof. This is essentially what is proved in [24, §5], where it is shown that condition (3′) there is equivalent to the other conditions, although the given proof does not explicitly keep track of N, so we do so here.

Condition (4) is implied by each of the other conditions and, using the definition of $\operatorname{Ext}^*_{RG/kG}$, we see that condition (1) implies all the others, so we concentrate on (4)⇒(1). We assume (4) and prove (1) by downward induction

on i. Since the minimal projective resolution is of finite length, the induction starts.

We suppose that $\rho_i(R,G;M) \leq N$ for all $i > r$. Consider an indecomposable kG-module V; if $\rho_r(R,G;M,V) \leq N$, then there is nothing to prove, so we assume that there is a summand $R \otimes_k V(-u)$ of P_r for some $u > N$. Let f be a projection of P_r onto $R \otimes_k V(-u)$. Since P_{r+1} is generated as an R-module in degrees at most N, the map fd_{r+1} is zero. Thus f determines an element of $\mathrm{Ext}^r_{RG/kG}(M, R \otimes_k V(-u))_0$.

But this Ext-group is zero, by hypothesis, so f factors through d_r. It follows that $R \otimes_k V(-u) \xrightarrow{d_r} d_r(R \otimes_k V(-u))$ is a summand of the minimal resolution as a complex of RG-modules, a contradiction. We must have $\rho_r(R,G;M,V) \leq N$, as required. □

LEMMA 3.4. *If $H < G$ and W is a kH-module, then*

$$\mathrm{Ext}^i_{RG/kG}(M, R \otimes_k (W(-d)\uparrow^G_H)) \cong \mathrm{Ext}^i_{RH/kH}(M\downarrow^G_H, R \otimes_k W(-d)).$$

As usual, $\uparrow$ denotes induction and $\downarrow$ denotes restriction.

Proof. This is an easy adaptation of the usual Eckmann-Shapiro lemma. □

LEMMA 3.5. *If $R' < R$ and both are polynomial rings, R is finitely generated over R' and V is an indecomposable kG-module, then*

$$\rho_i(R',G;M,V) - \sigma(R') \leq \rho_i(R,G;M,V) - \sigma(R).$$

As a consequence,

$$\mathrm{Preg}(R',G;M) \leq \mathrm{Preg}(R,G;M).$$

Proof. Take a minimal RG/kG-resolution of M and restrict it to R'. By the basic theory of Cohen-Macauley rings (see e.g. [1] or [6]), R is free of finite rank over R' with a basis of homogeneous elements $\{z_j\}$ bounded in degree by $\sigma(R') - \sigma(R)$. For each summand $R \otimes_k V(-d)$ in P_i in the original resolution, we now have $\oplus_j R'z_j \otimes_k V(-d)$ and the result follows. □

LEMMA 3.6. *If $R' < R$, M is a finitely generated $R'G$-module and V is an indecomposable kG-module, then*

$$\rho_i(R,G;R \otimes_{R'} M,V) \leq \rho_i(R',G;M,V).$$

Proof. Tensoring an $R'G/kG$-resolution of M with R yields an RG/kG-resolution of $R \otimes_{R'} M$. □

LEMMA 3.7. *If $H < G$, then $\rho_i(R,H;M\downarrow^G_H) \leq \rho_i(R,G;M)$ and so*

$$\mathrm{Preg}(R,H;M\downarrow^G_H) \leq \mathrm{Preg}(R,G;M).$$

Proof. Take an RG/kG-resolution of M and restrict it to H. □

LEMMA 3.8. *If P is a Sylow p-subgroup of G (where $p = \operatorname{char} k$), then $\rho_i(R, P; M{\downarrow}_P^G) = \rho_i(R, G; M)$ and so $\operatorname{Preg}(R, P; M{\downarrow}_P^G) = \operatorname{Preg}(R, G; M)$.*

Proof. The inequality $\leq$ follows from 3.7.

For $\geq$, note that M is a summand of $M{\downarrow}_P{\uparrow}^G$as an RG-module; thus $\rho_i(R, G; M{\downarrow}_P{\uparrow}^G) \geq \rho_i(R, G; M)$. But an RP/kP-resolution of $M{\downarrow}_P$ induces to an RG/kG-resolution of $M{\downarrow}_P{\uparrow}^G$ and so

$$\rho_i(R, P; M{\downarrow}_P) \geq \rho_i(R, G; M{\downarrow}_P{\uparrow}^G). \qquad \square$$

LEMMA 3.9. *Let $0 \to A \to B \to C \to 0$ be a short exact sequence of RG-modules that is split over kG and let V be an indecomposable kG-module. Then*

$$\rho_i(R, G; B, V) \leq \max\{\rho_i(R, G; A, V), \rho_i(R, G; C, V)\}.$$

As a consequence,

$$\operatorname{Preg}(R, G; B) \leq \max\{\operatorname{Preg}(R, G; A), \operatorname{Preg}(R, G; C)\}.$$

Proof. A relatively projective resolution for B can be constructed from ones for A and C (see [12, 8.2.1]) just as in the Horseshoe Lemma for ordinary projective resolutions (cf. [27, 2.2.8]). □

LEMMA 3.10. *If M is a finitely generated RG-module and d is an integer such that $M_{>d} = 0$, then $\operatorname{Preg}(R, G; M) \leq d$.*

Proof. Filter M by its submodules $M_{\geq r}$; since M is finitely generated and $M_{>d} = 0$, this is a finite filtration and the composition factors are just the homogeneous pieces M_r. Clearly M is the sum of its composition factors over kG. By repeated use of 3.9, we see that $\operatorname{Preg}(R, G; M) \leq \max_r\{\operatorname{Preg}(R, G; M_r)\}$; hence it will be sufficient to show that $\operatorname{Preg}(R, G; M_r) \leq r$.

Let $R = k[d_1, \ldots, d_n]$, where $|d_1| \geq |d_2| \geq \cdots \geq |d_n| \geq 1$. Since the d_i annihilate M_r, we can resolve M_r by tensoring it with the Koszul resolution on the d_i; this is an RG/kG-resolution.

It is now easy to calculate that $\rho_i(R, G; M_r) \leq r + \sum_{j=1}^{i} |d_j|$. But $\sum_{j=1}^{i} |d_j| \leq \sum_{j=1}^{i} |d_j| + \sum_{j=i+1}^{n} (|d_j| - 1) = \sigma(R) + i$. Hence $\rho_i(R, G; M_r) - i - \sigma(R) \leq r$ and so $\operatorname{Preg}(R, G; M_r) \leq r$. □

4. **The Structure Theorem**

Here we summarize the material that we will require from our paper with Karagueuzian [18].

From now until Section 6, k will always be a finite field. Let $S = k[x_1, \ldots, x_n]$ be a polynomial ring with all the generators in degree 1. For

any subset $I \subseteq \{1, 2, \ldots, n-1\}$, let U_I denote the group of upper-triangular matrices over k with 1's on the diagonal and nonzero off-diagonal entries only in rows corresponding to the elements of I. The group U_I acts on S in the natural way, i.e. so that S_1 is the canonical module. The invariants form a polynomial ring generated by the orbit powers of the x_i; we denote the latter by $d_i(I)$ (here our notation differs slightly from that of [18]).

The Main Theorem of [18] is as follows.

THEOREM 4.1. *As a graded kU_I-module,*

$$S \cong \bigoplus_{J \subseteq I} k[d_i(I); i \notin J] \otimes_k \bar{X}_J(I),$$

where $\bar{X}_J(I)$ is a finite-dimensional graded kU_I-submodule of S and the map from right to left is induced by multiplication in S.

We also have some information about the modules $\bar{X}_J(I)$. Let us write $k[d(I)]$ for $k[d_i(I); i = 1, \ldots, n]$ and $\sigma(I)$ for $\sigma(k[d(I)])$.

PROPOSITION 4.2. (1) *$\bar{X}_J(I)$ is induced from U_J.*
(2) *$\bar{X}_\emptyset(I)$ is homogeneous of degree $\sigma(I)$.*
(3) *$\bar{X}_I(I)$ lies in degrees at most $\sigma(I) - \sum_{i \in I} |d_i(I)|$.*

Proof. We refer by numbers in parentheses to statements in [18].

Part (1) is by construction (10.1(2)).

For part (2), notice that $\bar{X}_\emptyset(I)^{U_I}$ is 1-dimensional in a degree that is denoted by $\deg_I(\vec{p})$ (10.1(3) and 5.14). The fact that $\deg_I(\vec{p}) = \sigma(I)$ can be verified by direct calculation from the definitions or, more conceptually, by observing that $\deg \bar{X}_\emptyset(I) = \deg(G(I, \emptyset))$ by construction (10.1(2)), where $G(I, \emptyset)$ is a polynomial that clearly has degree $\sigma(I)$ from its definition (9.1).

Part (3) follows from part (2) and (5.22). □

We also record one other fact.

PROPOSITION 4.3. *The ring S contains a $k[d_i(I); i \in I]U_I$-submodule $T(I)$ such that*

$$S \cong k[d_i(I); i \notin I] \otimes_k T(I) \cong k[d(I)] \otimes_{k[d_i(I); i \in I]} T(I)$$

as $k[d(I)]U_I$-modules.

This is (6.4) in [18]; although there the map is only stated to be a $k[d_i(I); i \in I]U_I$-module isomorphism, it is clearly a $k[d(I)]U_I$-isomorphism, by construction.

5. **Proof of the Main Theorem**

What we would like to do is to construct an explicit $k[d(I)]U_I/kU_I$-resolution of S. There is an obvious candidate for a description of what the modules in this resolution ought to be. For each $X_J(I) = k[d_i(I); i \notin J] \otimes_k \bar{X}_J(I)$ in the Structure Theorem, there should be a contribution that looks like $X_J(I)$ tensored with the Koszul complex on the $d_i(I), i \in J$. If we knew that there existed a filtration of S by $k[d(I)]U_I$-modules with the $X_J(I)$ as composition factors, then this could be verified. However, the existence of such a filtration is not clear.

We will content ourselves with proving some of the bounds that would be implied by the existence of such a resolution.

Our key result is the following proposition. Notice that the bound is not the diagonal one that might be expected.

PROPOSITION 5.1. *In the context of the Structure Theorem,*

$$\rho_i(k[d(I)], U_I; S) \leq \sigma(I)$$

for all $i \geq 0$. *As a consequence,*

$$\mathrm{Preg}(k[d(I)], U_I; S) \leq 0.$$

Proof. Use induction on $|I|$; the case $I = \emptyset$ is clear since $k[d(\emptyset)] = S$ and $\sigma(\emptyset) = 0$.

Let V be an indecomposable kU_I-module; by Lemma 3.3, it is sufficient to verify condition 3.3(4) with $N = \sigma(I)$. First consider the case when V is projective relative to some proper subgroup U_J, $J \subsetneqq I$; say V is a summand of $W\uparrow_{U_J}^{U_I}$ for some indecomposable kU_J-module W. Then, by 3.4,

$$\mathrm{Ext}^i_{k[d(I)]U_I/kU_I}(S, k[d(I)] \otimes_k V)$$

is a summand of

$$\mathrm{Ext}^i_{k[d(I)]U_J/kU_J}(S, k[d(I)] \otimes_k W).$$

By the induction hypothesis, $\rho_i(k[d(J)], U_J; S, W) \leq \sigma(J)$, so, by 3.5, $\rho_i(k[d(I)], U_J; S, W) \leq \sigma(I)$.

This implies that the first Ext group above vanishes in degrees less than $-\sigma(I)$ for all i, so the same is true for the second one. Since

$$\mathrm{Ext}^i_{k[d(I)]U_I/kU_I}(S, k[d(I)] \otimes_k V(-d))_0 \cong \mathrm{Ext}^i_{k[d(I)]U_I/kU_I}(S, k[d(I)] \otimes_k V)_{-d},$$

this Ext-group vanishes for $d > \sigma(I)$, as required.

In the case that V is not projective relative to any proper subgroup U_J, we use 4.3 to write

$$S \cong k[d(I)] \otimes_{k[d(I)']} T(I),$$

where $k[d(I)'] = k[d_i(I); i \in I]$. Thus

$$\rho_i(k[d(I)], U_I; S, V) \leq \rho_i(k[d(I)'], U_I; T(I), V),$$

by 3.6. We need to show that the right-hand side is bounded by $\sigma(I)$.

Since V is not projective relative to any proper subgroup U_J, 4.2(1) shows that the only $\bar{X}_J(I)$ in which it can appear is $\bar{X}_I(I)$. As a consequence, V does not appear in $T(I)_{>b}$, where b is the maximum degree of an element of $\bar{X}_I(I)$ (finite, by 4.2(3)).

From the short exact sequence, split over kU_I,

$$0 \to T(I)_{>b} \to T(I) \to T(I)/T(I)_{>b} \to 0,$$

it follows, by 3.9, that

$$\rho_i(k[d(I)'], U_I; T(I), V) \leq \rho_i(k[d(I)'], U_I; T(I)/T(I)_{>b}, V).$$

But, since $T(I)/T(I)_{>b}$ is bounded in degree by b, we know, from 3.10, that

$$\mathrm{Preg}(k[d(I)'], U_I; T(I)/T(I)_{>b}) \leq b.$$

Thus we have

$$\rho_i(k[d(I)'], U_I; T(I)/T(I)_{>b}, V) - i \leq b + \sum_{i \in I}(|d_i(I)| - 1).$$

But, by 4.2(3), $b \leq \sigma(I) - \sum_{i \in I} |d_i(I)|$. Thus $\rho_i(k[d(I)'], U_I; T(I), V) \leq \sigma(I)$, as required. □

We can now finish the proof of the Main Theorem 0.1 in the case of a finite field.

Since the field k is finite, the largest possible group that can act on S is $G\ell_n = G\ell_n(k)$, which is finite. Its Sylow p-subgroup is $U_n = U_{\{1,2,\ldots,n-1\}}$, in the notation of Section 4. Let $k[c]$ denote the ring of Dickson invariants in S for the action of $G\ell_n$, and let $k[d]$ be the invariants for U_n.

For our given group $G < G\ell_n$ we may compute $\mathrm{reg}(S^G)$ by treating S^G as a $k[c]$-module:

$$\begin{aligned}
\mathrm{reg}(S^G) &= \mathrm{Preg}(k[c], S^G) && \text{by 1.2} \\
&\leq \mathrm{Preg}(k[c], G; S) && \text{by 3.2} \\
&\leq \mathrm{Preg}(k[c], G\ell_n; S) && \text{by 3.7} \\
&= \mathrm{Preg}(k[c], U_n; S) && \text{by 3.8} \\
&\leq \mathrm{Preg}(k[d], U_n; S) && \text{by 3.5} \\
&\leq 0 && \text{by 5.1.}
\end{aligned}$$

This completes the proof.

The last result that remains to be proved is Corollary 0.4. Let $\cdots \to P_1 \to P_0 \to S^G \to 0$ be the minimal $k[c]$-resolution of S^G. Then

$$H(S^G, t) = \sum_i (-1)^i H(P_i, t).$$

Now

$$H(P_i, t) = \frac{f_i(t)}{\prod_j (1 - t^{|c_j|})},$$

where $f_i(t)$ is a polynomial in which the coefficient of t^u is the multiplicity of $k[c](-u)$ in P_i. Thus the degree of $f_i(t)$ is equal to $\rho_i(k[c], S^G)$.

Because

$$H(S^G, t) = \frac{\sum_i (-1)^i f_i(t)}{\prod_j (1 - t^{|c_j|})},$$

we see that it suffices to show that $\deg(f_i) \leq \sigma(k[c])$. But we have just seen that $\deg(f_i) = \rho_i(k[c], S^G)$, and $\rho_i(k[c], S^G) \leq \rho_i(k[c], G; S)$, by 3.2. The proof concludes with the next lemma.

LEMMA 5.2. $\rho_i(k[c], G; S) \leq \sigma(k[c])$.

Proof.

$$\begin{aligned}
\rho_i(k[c], G; S) &\leq \rho_i(k[c], G\ell_n; S) && \text{by 3.7} \\
&= \rho_i(k[c], U_n; S) && \text{by 3.8} \\
&\leq \rho_i(k[d], U_n; S) - \sigma(k[d]) + \sigma(k[c]) && \text{by 3.5} \\
&\leq \sigma(k[c]) && \text{by 5.1.}
\end{aligned}$$

□

6. **Infinite fields**

We now explain how the case of Theorem 0.1 for finite fields implies the result for all fields. This argument was shown to us by Burt Totaro, and we are grateful to him for permission to include it here.

Of course, if the representation of G on S_1 can be written in the algebraic closure of the prime field, then it can be written in a finite field, so our results for finite fields still hold. It is not so clear what might happen if the field contains transcendental elements.

We have a finite group G that acts on $S_K = K[x_1, \ldots, x_n]$ for some infinite field K. The representation of G on $(S_K)_1$, the part in degree 1, can be written over a finitely generated subring A of K (the $\mathbb{Z}$-subalgebra generated by the coefficients of the matrices with respect to some basis); hence the same is true in all degrees and G acts on $S_A = A[x_1, \ldots, x_n]$.

Let $S_A^{(G)}$ denote $\prod_{g \in G} S_A$, and let $\Delta_A^{(G)} : S_A \to S_A^{(G)}$ be the map that is multiplication by $g - 1$ on the g-coordinate. Then $S_A^G = \ker(\Delta_A^{(G)})$ and we

have an exact sequence of S_A^G-modules

$$0 \to S_A^G \to S_A \overset{\Delta_A^{(G)}}{\to} S_A^{(G)} \to C \to 0.$$

We know, by the Hilbert-Noether Theorem (see e.g. [1, 1.3.1], [8, 3.0.6]), that S_A^G is noetherian and that S_A is finite over it. By Grothendieck's Generic Freeness (or Flatness) Lemma (see e.g. [10, 14.4]), there is a nonzero element $f \in A$ such that $C \otimes_A A[f^{-1}]$ is free as an $A[f^{-1}]$-module. Let M be a maximal ideal of $A[f^{-1}]$ with residue field k, and let B denote the localization of $A[f^{-1}]$ at M.

LEMMA 6.1. *As a B-module, S_B^G is free and $S_B^G \otimes_B k \cong S_k^G$.*

Proof. As an A-module, B is flat, so if we apply $- \otimes_A B$ to the exact sequence above, it remains exact and becomes

$$0 \to S_A^G \otimes_A B \to S_B \overset{\Delta_B^{(G)}}{\to} S_B^{(G)} \to C \otimes_A B \to 0.$$

All the terms but the first are certainly free over B, so the sequence splits over B and the first term must also be free over B. This first term is $\ker(\Delta_B^{(G)})$; thus it is isomorphic to S_B^G. If we now apply $- \otimes_B k$, we obtain

$$0 \to S_B^G \otimes_B k \to S_k \overset{\Delta_k^{(G)}}{\to} S_k^{(G)} \to C \otimes_A B \otimes_B k \to 0.$$

But $\ker(\Delta_k^{(G)}) = S_k^G$, by the discussion above. □

By the Hilbert-Noether Theorem again, S_B^G is finitely generated by homogeneous elements as a B-algebra, and thus it is finite over some polynomial ring $B[d_1, \ldots, d_m]$. Consequently, S_k^G is finite over $k[d_1, \ldots, d_m]$.

The next lemma is standard.

LEMMA 6.2. *Any projective resolution $P_\bullet$ of S_k^G over $k[d_1, \ldots, d_m]$ can be lifted to a projective resolution $Q_\bullet$ of S_B^G over $B[d_1, \ldots, d_m]$ such that $P_\bullet \cong Q_\bullet \otimes_B k$.*

Proof. We lift the resolution step by step, starting at the 0-term. Clearly P_0 can be lifted to a projective module Q_0, and the map $P_0 \to S_k^G$ can be lifted to a map $Q_0 \to S_B^G$, by 6.1. Furthermore, this map is surjective, by Nakayama's Lemma in each degree; hence it is split over B, since S_B^G is free over B, by 6.1 again.

Let Z_0 denote the kernel of the lifted map. It is free of finite rank over B and, because of the splitting, $Z_0 \otimes_B k \cong \ker(P_0 \to S_k^G)$. We can now repeat the procedure at the 1-term and continue. □

The field k is finite, since any field that is finitely generated as a ring is finite (cf. [10, 4.19]). Thus our previous results give us bounds on the degrees

of the generators of the terms of the minimal projective resolution $P_\bullet$ of S_k^G over $k[d_1, \ldots, d_m]$. These bounds are inherited by $Q_\bullet$, and even by $Q_\bullet \otimes_B K$, which is a projective resolution of S_K^G over $K[d_1, \ldots, d_m]$.

It follows that the results 0.1, 0.2, 0.3, 0.4 are all valid for arbitrary fields. The same will be true for 7.2 and 8.2.

Remark. Experts will recognize that the argument can be summarized by saying that regularity is upper semicontinuous on flat families.

7. **Horizontal bounds**

The proofs yield more precise information than can be stated in terms of regularity, although this is not useful for bounding the degrees of the generators of the invariants, which is why we only mention it here.

Instead of using the usual diagonal bound in the definition of regularity we can use a horizontal one: we set

$$\begin{aligned}\operatorname{hreg}(R, M) &= \sup_i\{a_i(R, M)\},\\ \operatorname{Phreg}(R, M) &= \sup_i\{\rho_i(R, M)\} - \sum_i |d_i|,\\ \operatorname{Exthreg}(R, M) &= \sup_i\{\varepsilon_i(R, M)\} - \sum_i |d_i|.\end{aligned}$$

It is still true that these numbers coincide. The same proof still works, the key point being that the proof of Lemma 1.1 is still valid; in fact, what we need is Lemma 3.3 in the case of the trivial group (see too [2, 5.7(i)]). Also hreg is still clearly invariant under change of ring.

It follows from the definitions that $\operatorname{hreg}(R, M) \leq \operatorname{reg}(R, M)$.

Similarly, we can define

$$\operatorname{Phreg}(R, G; M) = \sup_i\{\rho_i(R, G; M)\} - \sum_i |d_i|.$$

Lemma 5.2 now becomes:

PROPOSITION 7.1. *In the context of the Structure Theorem*,

$$\operatorname{Phreg}(k[c], G; S) \leq -n.$$

From which we deduce, as before:

THEOREM 7.2. *We have*

$$\operatorname{hreg}(S^G) \leq -n.$$

Our bound on ρ_i is thus improved by i. The statement of Proposition 2.1 now has both the $\operatorname{reg}(S) + \sigma(R)$ and the $\operatorname{reg}(S) + \sigma(R) + 1$ terms replaced by $\operatorname{hreg}(S) + \sum |d_i|$.

Remark. All that we need for a version of Lemma 1.1 to hold is that the bound on ρ_{i+1} should not exceed the desired bound on ρ_i by more than 1 or, at any rate, not by more than $\min\{i > 0 \mid R_i \neq 0\}$. This allows the definition of many different well-behaved regularities between reg and hreg; cf. [2, §5].

8. **Polynomial tensor exterior algebras**

One sometimes encounters invariants of algebras of the form $k[V]\otimes_k\Lambda(V^*)$, where V is a kG-module for some group G; V^* is its contragredient (the dual module considered as a left kG-module); $k[V]$ is the symmetric algebra on V^*, but graded so that the elements of V^* are in degree 2, and $\Lambda(V^*)$ is the exterior algebra on V^*, graded with V^* in degree 1.

More generally, let S be our usual polynomial ring with an action of G; for any positive integer r, let $S^{\langle r\rangle}$ denote the dilated ring with $S^{\langle r\rangle}_{ri} = S_i$ (0 in degrees not divisible by r). If S is a module over $k[d_1, \dots, d_m]$, then $S^{\langle r\rangle}$ is a module over $k[d_1^{\langle r\rangle}, \dots, d_m^{\langle r\rangle}]$, where $|d_i^{\langle r\rangle}| = r|d_i|$.

Let X be a finite-dimensional graded kG-module; we will write $\operatorname{reg}(X)$ for the top nonzero degree (this is consistent with the definition of $\operatorname{reg}(k, X)$, and we could just as well use $\operatorname{hreg}(X)$).

Proposition 8.1. *In the context of the Structure Theorem,*

$$\operatorname{reg}((S^{\langle r\rangle}\otimes X)^G) \leq \operatorname{reg}(X) - (r-1)n,$$
$$\operatorname{hreg}((S^{\langle r\rangle}\otimes X)^G) \leq \operatorname{hreg}(X) - rn.$$

Proof. We work over a finite field k and consider S as a module over the ring of Dickson invariants $k[c]$. It is easily verified that $\sigma(k[c^{\langle r\rangle}]) = r\sigma(k[c]) + (r-1)n$.

From Lemma 5.2, we know that $\rho_i(k[c], G; S) \leq \sigma(k[c])$. By dilating the minimal relatively projective resolution of S we find that $\rho_i(k[c^{\langle r\rangle}], G; S^{\langle r\rangle}) \leq r\sigma(k[c])$; by tensoring this dilated resolution with X we see that

$$\rho_i(k[c^{\langle r\rangle}], G; S^{\langle r\rangle}\otimes X) \leq r\sigma(k[c]) + \operatorname{reg}(X).$$

The proposition now follows in the usual way from the definition of regularity. □

Corollary 8.2. *For a polynomial tensor exterior algebra of the type discussed above,*

$$\operatorname{reg}((k[V]\otimes\Lambda(V^*))^G) \leq 0,$$
$$\operatorname{hreg}((k[V]\otimes\Lambda(V^*))^G) \leq -\dim V.$$

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