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Double shuffle relation for associators

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Abstract

It is proved that Drinfel′d's pentagon equation implies the generalized double shuffle relation. As a corollary, an embedding from the Grothendieck-Teichmüller group GRT_1 into Racinet's double shuffle group DMR_0 is obtained, which settles the project of Deligne-Terasoma. It is also proved that the gamma factorization formula follows from the generalized double shuffle relation.

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0. Introduction

This paper shows that Drinfel′d's pentagon equation [Dri90] implies the generalized double shuffle relation. As a corollary, we obtain an embedding from the Grothendieck-Teichmüller (pro-unipotent) group GRT_1 (loc. cit.) to Racinet's double shuffle (pro-unipotent) group DMR_0 ([Rac02]). This realizes the project of Deligne-Terasoma [DT] where a different approach is indicated. Their arguments concern multiplicative convolutions whereas our methods are based on a bar construction calculus. We also prove that the gamma factorization formula follows from the generalized double shuffle relation. It extends the result in [DT], [Iha99] where they show that the GT-relations imply the gamma factorization.

Multiple zeta values $\zeta(k_1,\dots,k_m)$ are the real numbers defined by the following series

$$\zeta(k_1,\dots,k_m) := \sum_{0<n_1<\cdots<n_m} \frac{1}{n_1^{k_1}\cdots n_m^{k_m}}$$

for m, $k_1,\dots,k_m \in \mathbf{N}(=\mathbf{Z}_{>0})$. This converges if and only if $k_m > 1$. They were studied (allegedly) first by Euler [Eul] for $m = 1, 2$. Several types of relations among multiple zeta values have been discussed. In this paper we focus on two types of relations: GT-relations and generalized double shuffle relations. Both of them are described in terms of the Drinfel′d associator [Dri90]

$$\begin{aligned}&\Phi_{\mathrm{KZ}}(X_0,X_1)\\ &\quad = 1 + \sum (-1)^m \zeta(k_1,\dots,k_m) X_0^{k_m-1}X_1\cdots X_0^{k_1-1}X_1 + (\text{regularized terms})\end{aligned}$$

which is a noncommutative formal power series in two variables X_0 and X_1. Its coefficients including regularized terms are explicitly calculated to be linear combinations of multiple zeta values in [Fur03, Prop. 3.2.3] by Le-Murakami's method [LM96]. The Drinfel′d associator is introduced as the connection matrix of the Knizhnik-Zamolodchikov equation in [Dri90].

The GT-*relations* are a kind of geometric relation. They consist of one pentagon equation (1.1) and two hexagon equations (1.2), (1.3) (see below) for group-like (cf. §1) series. An associator means a group-like series φ satisfying (1.1) and for which there exists μ such that the pair (μ,φ) satisfies (1.2) and (1.3). It is shown in [Dri90] that Φ_{KZ} satisfies GT-relations (with $\mu = 2\pi i$) by using symmetry of the KZ-system on configuration spaces. The following is our previous theorem in [Fur10].

Theorem 0.1 ([Fur10, Th. 1]). *For any group-like series φ satisfying* (1.1) *there always exists* (*unique up to signature*) $\mu \in \bar{k}$ *such that* (1.2) *and* (1.3) (*see below*) *hold for* (μ,φ).

In contrast, the *generalized double shuffle relation* is a kind of combinatorial relation. It arises from two ways of expressing multiple zeta values as iterated integrals and as power series. There are several formulations of the relations (see [IKZ06], [Rac02]). In particular, they are formulated as (1.5) (see below) for $\varphi = \Phi_{\mathrm{KZ}}$ in [Rac02].

Theorem 0.2. *Let φ be a noncommutative formal power series in two variables which is group-like. Suppose that φ satisfies Drinfel′d's pentagon equation* (1.1). *Then it also satisfies the generalized double shuffle relation* (1.5) (*see below*).

We note that a similar result is announced by Terasoma in [Ter06]. The essential part of our proof is to use the series shuffle formula (3.1) (see below), a functional relation among complex multiple polylogarithms (2.3) and (2.5). This induces the series shuffle formula (3.2) for the corresponding elements in the bar construction of the moduli space $\mathcal{M}_{0,5}$. We evaluate each term of (3.2) at the product of the last two terms of (1.6) in Section 4 and conclude the series shuffle formula (3.3) for above φ.

The *Grothendieck-Teichmüller group* GRT_1 is a pro-unipotent group introduced by Drinfel′d [Dri90] which is closely related to Grothendieck's philosophy of Teichmüller-Lego in [Gro84]. Its set of k-valued (k: a field with characteristic 0) points is defined to be the set of associators φ with $\mu = 0$. Its multiplication is given by

$$(0.1) \qquad \varphi_1 \circ \varphi_2 := \varphi_1(\varphi_2 X_0 \varphi_2^{-1}, X_1) \cdot \varphi_2 = \varphi_2 \cdot \varphi_1(X_0, \varphi_2^{-1} X_1 \varphi_2).$$

In contrast, the *double shuffle group* DMR_0 is a pro-unipotent group introduced by Racinet [Rac02]. Its set of k-valued points consists of group-like series φ which satisfy (1.5)[1] and $c_{X_0}(\varphi) = c_{X_1}(\varphi) = c_{X_0X_1}(\varphi) = 0$. (For a monic monomial W, $c_W(\varphi)$ is the coefficient of W in φ.) Its multiplication[2] is given by equation (0.1). It is conjectured that both groups are isomorphic to the unipotent part of the motivic Galois group of $\mathbf{Z}$, i.e., the Galois group of unramified mixed Tate motives (as explained in [And04]). The following are direct corollaries of our Theorem 0.2 since equations (1.2) and (1.3) for (μ, φ) imply $c_{X_0X_1}(\varphi) = \frac{\mu^2}{24}$.

Corollary 0.3. *The Grothendieck-Teichmüller group* GRT_1 *is embedded in the double shuffle group* DMR_0 *as pro-algebraic groups.*

Considering their associated Lie algebras, we get an embedding from the Grothendieck-Teichmüller Lie algebra $\mathfrak{grt}_1$ [Dri90] into the double shuffle Lie algebra $\mathfrak{dmr}_0$ [Rac02].

Corollary 0.4. *For $\mu \in k^\times$, the Grothendieck-Teichmüller torsor M_μ is embedded in the double shuffle torsor* DMR_μ *as pro-torsors.*

Here M_μ is the right GRT_1-torsor in [Dri90] whose action is given by equation (0.1) and whose set of k-valued points is defined to be the collection of associators φ with μ, and DMR_μ is the right DMR_0-torsor in [Rac02] whose action is given by equation (0.1) and whose set of k-valued points is defined to be the collection of group-like series φ which satisfy (1.5), $c_{X_0}(\varphi) = c_{X_1}(\varphi) = 0$

[1] For our convenience, we change some signatures in the original definition ([Rac02, Def. 3.2.1]).

[2] Again for our convenience, we change the order of multiplication in [Rac02, (3.1.2.1)].

and $c_{X_0X_1}(\varphi)=\frac{\mu^2}{24}$ respectively. We note that Φ_{KZ} gives an element of $M_\mu(\mathbf{C})$ and $\mathrm{DMR}_\mu(\mathbf{C})$ with $\mu=2\pi i$.

Let $\varphi\in k\langle\langle X_0,X_1\rangle\rangle$ be a noncommutative formal power series in two variables which is group-like with $c_{X_0}(\varphi)=c_{X_1}(\varphi)=0$. It is uniquely expressed as $\varphi=1+\varphi_{X_0}X_0+\varphi_{X_1}X_1$ with $\varphi_{X_0},\ \varphi_{X_1}\in k\langle\langle X_0,X_1\rangle\rangle$. The meta-abelian quotient $B_\varphi(x_0,x_1)$ of φ is defined to be $(1+\varphi_{X_1}X_1)^{\mathrm{ab}}$ where $h\mapsto h^{\mathrm{ab}}$ is the abelianization map $k\langle\langle X_0,X_1\rangle\rangle\to k[[x_0,x_1]]$. The following is our second main theorem.

THEOREM 0.5. *Let φ be a noncommutative formal power series in two variables which is group-like with $c_{X_0}(\varphi)=c_{X_1}(\varphi)=0$. Suppose that it satisfies the generalized double shuffle relation (1.5). Then its meta-abelian quotient is gamma-factorizable; i.e., there exists a unique series $\Gamma_\varphi(s)$ in $1+s^2k[[s]]$ such that*

$$B_\varphi(x_0,x_1)=\frac{\Gamma_\varphi(x_0)\Gamma_\varphi(x_1)}{\Gamma_\varphi(x_0+x_1)}. \tag{0.2}$$

The gamma element Γ_φ gives the correction term φ_{corr} of the series shuffle regularization (1.4) by $\varphi_{\mathrm{corr}}=\Gamma_\varphi(-Y_1)^{-1}$.

This theorem extends the results in [DT], [Iha99] which show that for any group-like series satisfying (1.1), (1.2) and (1.3) its meta-abelian quotient is gamma factorizable. This result might be a step to relate DMR with the set SolKV of solutions of the Kashiwara-Vergne equations which is defined by a certain tangential automorphism condition and a coboundary Jacobian condition [AT], [AET10] since equation (0.2) is regarded as a consequence of the latter condition (cf. [AET10, §2.1]). It is calculated in [Dri90] that $\Gamma_\varphi(s)=\exp\{\sum_{n=2}^{\infty}\frac{\zeta(n)}{n}s^n\}=e^{-\gamma s}\Gamma(1-s)$ (γ: Euler constant, $\Gamma(s)$: the classical gamma function) in particular case of $\varphi=\Phi_{\mathrm{KZ}}$.

Section 1 is a review of the GT-relations and the generalized double shuffle relation. In Section 2 we review the notion of bar constructions which is a main tool of the proof of Theorem 0.2 in Section 3. Auxiliary lemmas which are necessary to the proof are shown in Section 4. Theorem 0.5 is proved in Section 5. Appendix A is a brief review of the essential part of the proof of Racinet's theorem that DMR_0 forms a group.

1. GT-relations and generalized double shuffle relation

Let k be a field with characteristic 0. Let $U\mathfrak{F}_2=k\langle\langle X_0,X_1\rangle\rangle$ be a noncommutative formal power series ring in two variables X_0 and X_1. An element $\varphi=\varphi(X_0,X_1)$ is called *group-like* if it satisfies $\Delta(\varphi)=\varphi\widehat{\otimes}\varphi$ with $\Delta(X_0)=X_0\otimes 1+1\otimes X_0$ and $\Delta(X_1)=X_1\otimes 1+1\otimes X_1$, and its constant term is 1. Here, $\widehat{\otimes}$ means the completed tensor product. For any k-algebra homomorphism $\iota:U\mathfrak{F}_2\to S$, the image $\iota(\varphi)\in S$ is denoted by $\varphi(\iota(X_0),\iota(X_1))$.

Let $\mathfrak{a}_4$ be the completion (with respect to the natural grading) of the pure braid Lie algebra with 4-strings, i.e., the Lie algebra over k with generators t_{ij} $(1 \leqslant i,j \leqslant 4)$ and defining relations $t_{ii}=0$, $t_{ij}=t_{ji}$, $[t_{ij}, t_{ik}+t_{jk}]=0$ (i,j,k: all distinct) and $[t_{ij},t_{kl}]=0$ (i,j,k,l: all distinct). Let $\varphi=\varphi(X_0,X_1)$ be a group-like element of $U\mathfrak{F}_2$ and $\mu \in k$. The GT-*relations* for (μ,φ) consist of one pentagon equation

(1.1) $$\varphi(t_{12},t_{23}+t_{24})\varphi(t_{13}+t_{23},t_{34})=\varphi(t_{23},t_{34})\varphi(t_{12}+t_{13},t_{24}+t_{34})\varphi(t_{12},t_{23})$$

and two hexagon equations

(1.2)
$$\exp\left\{\frac{\mu(t_{13}+t_{23})}{2}\right\}=\varphi(t_{13},t_{12})\exp\left\{\frac{\mu t_{13}}{2}\right\}\varphi(t_{13},t_{23})^{-1}\exp\left\{\frac{\mu t_{23}}{2}\right\}\varphi(t_{12},t_{23}),$$

(1.3)
$$\exp\left\{\frac{\mu(t_{12}+t_{13})}{2}\right\}=\varphi(t_{23},t_{13})^{-1}\exp\left\{\frac{\mu t_{13}}{2}\right\}\varphi(t_{12},t_{13})\exp\left\{\frac{\mu t_{12}}{2}\right\}\varphi(t_{12},t_{23})^{-1}.$$

Let $\pi_Y : k\langle\langle X_0,X_1\rangle\rangle \to k\langle\langle Y_1,Y_2,\dots\rangle\rangle$ be the k-linear map between non-commutative formal power series rings that sends all the words ending in X_0 to zero and the word $X_0^{n_m-1}X_1\cdots X_0^{n_1-1}X_1$ $(n_1,\dots,n_m \in \mathbf{N})$ to $(-1)^m Y_{n_m}\cdots Y_{n_1}$. Define the coproduct Δ_* on $k\langle\langle Y_1,Y_2,\dots\rangle\rangle$ by $\Delta_* Y_n=\sum_{i=0}^{n} Y_i\otimes Y_{n-i}$ with $Y_0:=1$. For $\varphi=\sum_{W:\text{word}} c_W(\varphi)W \in k\langle\langle X_0,X_1\rangle\rangle$, define the series shuffle regularization $\varphi_*=\varphi_{\text{corr}}\cdot\pi_Y(\varphi)$ with the correction term

(1.4) $$\varphi_{\text{corr}}=\exp\left(\sum_{n=1}^{\infty}\frac{(-1)^n}{n}c_{X_0^{n-1}X_1}(\varphi)Y_1^n\right).$$

For a group-like series $\varphi \in U\mathfrak{F}_2$ the *generalised double shuffle relation* means the equality

(1.5) $$\Delta_*(\varphi_*)=\varphi_*\widehat{\otimes}\varphi_*.$$

Let $\mathfrak{P}_5$ stand for the completion (with respect to the natural grading) of the pure sphere braid Lie algebra with 5 strings; the Lie algebra over k generated by X_{ij} $(1 \leqslant i,j \leqslant 5)$ with relations $X_{ii}=0$, $X_{ij}=X_{ji}$, $\sum_{j=1}^{5} X_{ij}=0$ $(1 \leqslant i,j \leqslant 5)$ and $[X_{ij},X_{kl}]=0$ if $\{i,j\}\cap\{k,l\}=\emptyset$. Denote its universal enveloping algebra by $U\mathfrak{P}_5$. (Note: $X_{45}=X_{12}+X_{13}+X_{23}$, $X_{51}=X_{23}+X_{24}+X_{34}$.) There is a surjection $\tau:\mathfrak{a}_4\to\mathfrak{P}_5$ sending t_{ij} to X_{ij} $(1 \leqslant i,j \leqslant 4)$. Its kernel is the center of $\mathfrak{a}_4$ generated by $\sum_{1\leqslant i,j\leqslant 4} t_{ij}$. By [Fur10, Lemma 5], Theorem 0.2 is reduced to the following.

THEOREM 1.1. *Let φ be a group-like element of $U\mathfrak{F}_2$ with $c_{X_0}(\varphi)=c_{X_1}(\varphi)=0$. Suppose that φ satisfies the* 5*-cycle relation in $U\mathfrak{P}_5$*

(1.6) $$\varphi(X_{34},X_{45})\varphi(X_{51},X_{12})\varphi(X_{23},X_{34})\varphi(X_{45},X_{51})\varphi(X_{12},X_{23})=1.$$

Then it also satisfies the generalized double shuffle relation, i.e. $\Delta_*(\varphi_*) = \varphi_* \widehat{\otimes} \varphi_*$.

2. Bar constructions

In this section we review the notion of bar construction and multiple polylogarithm functions which are essential to prove our main theorem.

Let $\mathcal{M}_{0,4}$ be the moduli space

$$\{(x_1, \dots, x_4) \in (\mathbf{P}^1_k)^4 | x_i \neq x_j (i \neq j)\}/\mathrm{PGL}_2(k)$$

of 4 different points in $\mathbf{P}^1$. It is identified with $\{z \in \mathbf{P}^1 | z \neq 0, 1, \infty\}$ by sending $[(0, z, 1, \infty)]$ to z. Let $\mathcal{M}_{0,5}$ be the moduli space

$$\{(x_1, \dots, x_5) \in (\mathbf{P}^1_k)^5 | x_i \neq x_j (i \neq j)\}/\mathrm{PGL}_2(k)$$

of 5 different points in $\mathbf{P}^1$. It is identified with

$$\{(x, y) \in \mathbf{G}_m^2 | x \neq 1, \ \ y \neq 1, \ \ xy \neq 1\}$$

by sending $[(0, xy, y, 1, \infty)]$ to (x, y).

For $\mathcal{M} = \mathcal{M}_{0,4}/k$ or $\mathcal{M}_{0,5}/k$, we consider Brown's variant $V(\mathcal{M})$ [Bro09] of Chen's reduced bar construction [Che77]. This is a graded Hopf algebra $V(\mathcal{M}) = \oplus_{m=0}^{\infty} V_m$ $(\subset TV_1 = \oplus_{m=0}^{\infty} V_1^{\otimes m})$ over k. Here $V_0 = k$, $V_1 = H^1_{DR}(\mathcal{M})$, and V_m is the totality of linear combinations (finite sums) $\sum_{I=(i_m,\dots,i_1)} c_I [\omega_{i_m} | \cdots | \omega_{i_1}] \in V_1^{\otimes m}$ $(c_I \in k,\ \omega_{i_j} \in V_1,\ [\omega_{i_m} | \cdots | \omega_{i_1}] := \omega_{i_m} \otimes \cdots \otimes \omega_{i_1})$ satisfying the integrability condition

$$\sum_{I=(i_m,\dots,i_1)} c_I [\omega_{i_m} | \omega_{i_{m-1}} | \cdots | \omega_{i_{j+1}} \wedge \omega_{i_j} | \cdots | \omega_{i_1}] = 0 \tag{2.1}$$

in $V_1^{\otimes m-j-1} \otimes H^2_{DR}(\mathcal{M}) \otimes V_1^{\otimes j-1}$ for all j $(1 \leqslant j < m)$.

For $\mathcal{M} = \mathcal{M}_{0,4}$, $V(\mathcal{M}_{0,4})$ is generated by $\omega_0 = d\log(z)$ and $\omega_1 = d\log(z-1)$ with the relation $\omega_0 \wedge \omega_1 = 0$. We identify $V(\mathcal{M}_{0,4})$ with the graded k-linear dual of $U\mathfrak{F}_2$ by $\mathrm{Exp}\Omega_4 = \mathrm{Exp}\Omega_4(x; X_0, X_1) := \sum X_{i_m} \cdots X_{i_1} \otimes [\omega_{i_m} | \cdots | \omega_{i_1}] \in U\mathfrak{F}_2 \widehat{\otimes} V(\mathcal{M}_{0,4})$. Here the sum is taken over $m \geqslant 0$ and $i_1, \dots, i_m \in \{0, 1\}$. It is easy to see that the identification is compatible with Hopf algebra structures. We note that the product $l_1 \cdot l_2 \in V(\mathcal{M}_{0,4})$ for $l_1,\ l_2 \in V(\mathcal{M}_{0,4})$ is given by $l_1 \cdot l_2(f) := \sum_i l_1(f_1^{(i)}) l_2(f_2^{(i)})$ for $f \in U\mathfrak{F}_2$ with $\Delta(f) = \sum_i f_1^{(i)} \otimes f_2^{(i)}$. Occasionally we regard $V(\mathcal{M}_{0,4})$ as the regular function ring of $F_2(k) = \{g \in U\mathfrak{F}_2 | g :$ group-like$\} = \{g \in U\mathfrak{F}_2 | g(0) = 1, \Delta(g) = g \otimes g\}$.

For $\mathcal{M} = \mathcal{M}_{0,5}$, $V(\mathcal{M}_{0,5})$ is generated by $\omega_{k,l} = d\log(x_k - x_l)$ $(1 \leqslant k, l \leqslant 5)$ with relations $\omega_{ii} = 0$, $\omega_{ij} = \omega_{ji}$, $\sum_{j=1}^5 \omega_{ij} = 0$ $(1 \leqslant i, j \leqslant 5)$ and $\omega_{ij} \wedge \omega_{kl} = 0$ if $\{i, j\} \cap \{k, l\} = \emptyset$. By definition, $V(\mathcal{M}_{0,5})$ is naturally identified with the graded k-linear dual of $U\mathfrak{P}_5$. The identification is induced from

$$\mathrm{Exp}\Omega_5 := \sum X_{J_m} \cdots X_{J_1} \otimes [\omega_{J_m} | \cdots | \omega_{J_1}] \in U\mathfrak{P}_5 \widehat{\otimes} TV_1,$$

where the sum is taken over $m \geqslant 0$ and $J_1, \dots, J_m \in \{(k, l) | 1 \leqslant k < l \leqslant 5\}$.

Remark 2.1. Here every monomial in $U\mathfrak{P}_5$ appears in the left-hand tensor factor of $\mathrm{Exp}\Omega_5$. But when these monomials are gathered in terms of a linear basis of $U\mathfrak{P}_5$, the right-hand tensor factors automatically gather into linear combinations of a basis of $V(\mathcal{M}_{0,5})$. Hence $\mathrm{Exp}\Omega_5$ lies on $U\mathfrak{P}_5\widehat{\otimes}V(\mathcal{M}_{0,5})$.

Especially the identification between degree 1 terms is given by

$$\Omega_5 = \sum_{1\leqslant k<l\leqslant 5} X_{kl} d\log(x_k - x_l) \in \mathfrak{P}_5 \otimes H^1_{DR}(\mathcal{M}_{0,5}).$$

In terms of the coordinate (x, y),

$$\begin{aligned}\Omega_5 &= X_{12} d\log(xy) + X_{13} d\log y + X_{23} d\log y(1-x)\\ &\quad + X_{24} d\log(1-xy) + X_{34} d\log(1-y)\\ &= X_{12} d\log x + X_{23} d\log(1-x) + (X_{12}+X_{13}+X_{23}) d\log y\\ &\quad\ + X_{34} d\log(1-y) + X_{24} d\log(1-xy)\\ &= X_{12}\frac{dx}{x} + X_{23}\frac{dx}{x-1} + X_{45}\frac{dy}{y} + X_{34}\frac{dy}{y-1} + X_{24}\frac{ydx+xdy}{xy-1}.\end{aligned}$$

It is easy to see that the identification is compatible with Hopf algebra structures. We note again that the product $l_1 \cdot l_2 \in V(\mathcal{M}_{0,5})$ for l_1, $l_2 \in V(\mathcal{M}_{0,5})$ is given by $l_1 \cdot l_2(f) := \sum_i l_1(f_1^{(i)}) l_2(f_2^{(i)})$ for $f \in U\mathfrak{P}_5$ with $\Delta(f) = \sum_i f_1^{(i)} \otimes f_2^{(i)}$ (Δ: the coproduct of $U\mathfrak{P}_5$). Occasionally we also regard $V(\mathcal{M}_{0,5})$ as the regular function ring of $P_5(k) = \{g \in U\mathfrak{P}_5 | g : \text{group-like}\}$.

For the moment, assume that k is a subfield of $\mathbf{C}$. We have an embedding (called a realization in [Bro09, §§1.2 and 3.6]) $\rho : V(\mathcal{M}) \hookrightarrow I_o(\mathcal{M})$ as an algebra over k which sends $\sum_{I=(i_m,\dots,i_1)} c_I[\omega_{i_m}|\cdots|\omega_{i_1}]$ ($c_I \in k$) to $\sum_I c_I \mathrm{It}\int_o \omega_{i_m} \circ \cdots \circ \omega_{i_1}$. Here $\sum_I c_I \mathrm{It}\int_o \omega_{i_m} \circ \cdots \circ \omega_{i_1}$ means the iterated integral defined by

$$(2.2)\qquad \sum_I c_I \int_{0<t_1<\cdots<t_{m-1}<t_m<1} \omega_{i_m}(\gamma(t_m)) \cdot \omega_{i_{m-1}}(\gamma(t_{m-1})) \cdots \omega_{i_1}(\gamma(t_1))$$

for all analytic paths $\gamma : (0,1) \to \mathcal{M}(\mathbf{C})$ starting from the tangential basepoint o (defined by $\frac{d}{dz}$ for $\mathcal{M} = \mathcal{M}_{0,4}$ and defined by $\frac{d}{dx}$ and $\frac{d}{dy}$ for $\mathcal{M} = \mathcal{M}_{0,5}$) at the origin in $\mathcal{M}$ (for its treatment, see also [Del89, §15]) and $I_o(\mathcal{M})$[3] denotes the $\mathcal{O}^{\mathrm{an}}_{\mathcal{M}}$-module generated by all such homotopy invariant iterated integrals with $m \geqslant 1$ and holomorphic 1-forms $\omega_{i_1}, \dots, \omega_{i_m} \in \Omega^1(\mathcal{M})$.

For $\mathbf{a} = (a_1, \dots, a_k) \in \mathbf{Z}^k_{>0}$, its weight and its depth are defined to be $wt(\mathbf{a}) = a_1 + \cdots + a_k$ and $dp(\mathbf{a}) = k$ respectively. Put $z \in \mathbf{C}$ with $|z| < 1$. Consider the following complex function which is called the *one variable multiple polylogarithm*:

$$(2.3)\qquad Li_{\mathbf{a}}(z) := \sum_{0<m_1<\cdots<m_k} \frac{z^{m_k}}{m_1^{a_1}\cdots m_k^{a_k}}.$$

[3] In [Bro09] it is denoted by $L_o(\mathcal{M})$.

It satisfies the following differential equation:

$$\frac{d}{dz}Li_{\mathbf{a}}(z)=\begin{cases}\frac{1}{z}Li_{(a_1,\dots,a_{k-1},a_k-1)}(z) & \text{if } a_k\neq 1,\\ \frac{1}{1-z}Li_{(a_1,\dots,a_{k-1})}(z) & \text{if } a_k=1, k\neq 1,\\ \frac{1}{1-z} & \text{if } a_k=1, k=1.\end{cases}$$

It gives an iterated integral starting from o, which lies on $I_o(\mathcal{M}_{0,4})$. Actually it corresponds to an element of $V(\mathcal{M}_{0,4})$ denoted by $l_{\mathbf{a}}$. It is expressed as

$$(2.4)\qquad l_{\mathbf{a}}=(-1)^k[\underbrace{\omega_0|\cdots|\omega_0}_{a_k-1}|\omega_1|\underbrace{\omega_0|\cdots|\omega_0}_{a_{k-1}-1}|\omega_1|\omega_0|\cdots\cdots|\omega_1|\underbrace{\omega_0|\cdots|\omega_0}_{a_1-1}|\omega_1]$$

and is calculated by $l_{\mathbf{a}}(\varphi)=(-1)^k c_{X_0^{a_k-1}X_1X_0^{a_{k-1}-1}X_1\cdots X_0^{a_1-1}X_1}(\varphi)$ for a series $\varphi=\sum_{W:\text{word}}c_W(\varphi)W$.

For $\mathbf{a}=(a_1,\dots,a_k)\in\mathbf{Z}^k_{>0}$, $\mathbf{b}=(b_1,\dots,b_l)\in\mathbf{Z}^l_{>0}$ and $x,y\in\mathbf{C}$ with $|x|<1$ and $|y|<1$, consider the following complex function which is called the *two variables multiple polylogarithm*:

$$(2.5)\qquad Li_{\mathbf{a},\mathbf{b}}(x,y):=\sum_{\substack{0<m_1<\cdots<m_k\\<n_1<\cdots<n_l}}\frac{x^{m_k}\qquad y^{n_l}}{m_1^{a_1}\cdots m_k^{a_k}n_1^{b_1}\cdots n_l^{b_l}}.$$

It satisfies the following differential equations in [BF06, §5]:

(2.6)

$$\frac{d}{dx}Li_{\mathbf{a},\mathbf{b}}(x,y)$$
$$=\begin{cases}\frac{1}{x}Li_{(a_1,\dots,a_{k-1},a_k-1),\mathbf{b}}(x,y) & \text{if } a_k\neq 1,\\ \frac{1}{1-x}Li_{(a_1,\dots,a_{k-1}),\mathbf{b}}(x,y)-\left(\frac{1}{x}+\frac{1}{1-x}\right)Li_{(a_1,\dots,a_{k-1},b_1),(b_2,\dots,b_l)}(x,y) & \\ & \text{if } a_k=1,k\neq 1,l\neq 1,\\ \frac{1}{1-x}Li_{\mathbf{b}}(y)-\left(\frac{1}{x}+\frac{1}{1-x}\right)Li_{(b_1),(b_2,\dots,b_l)}(x,y) & \text{if } a_k=1,k=1,l\neq 1,\\ \frac{1}{1-x}Li_{(a_1,\dots,a_{k-1}),\mathbf{b}}(x,y)-\left(\frac{1}{x}+\frac{1}{1-x}\right)Li_{(a_1,\dots,a_{k-1},b_1)}(xy) & \\ & \text{if } a_k=1,k\neq 1,l=1,\\ \frac{1}{1-x}Li_{\mathbf{b}}(y)-\left(\frac{1}{x}+\frac{1}{1-x}\right)Li_{\mathbf{b}}(xy) & \text{if } a_k=1,k=1,l=1,\end{cases}$$

$$\frac{d}{dy}Li_{\mathbf{a},\mathbf{b}}(x,y)=\begin{cases}\frac{1}{y}Li_{\mathbf{a},(b_1,\dots,b_{l-1},b_l-1)}(x,y) & \text{if } b_l\neq 1,\\ \frac{1}{1-y}Li_{\mathbf{a},(b_1,\dots,b_{l-1})}(x,y) & \text{if } b_l=1,l\neq 1,\\ \frac{1}{1-y}Li_{\mathbf{a}}(xy) & \text{if } b_l=1,l=1.\end{cases}$$

By analytic continuation, the functions $Li_{\mathbf{a},\mathbf{b}}(x,y)$, $Li_{\mathbf{a},\mathbf{b}}(y,x)$, $Li_{\mathbf{a}}(x)$, $Li_{\mathbf{a}}(y)$ and $Li_{\mathbf{a}}(xy)$ give iterated integrals starting from o, which lie on $I_o(\mathcal{M}_{0,5})$. They correspond to elements of $V(\mathcal{M}_{0,5})$ by the map ρ denoted by $l^{x,y}_{\mathbf{a},\mathbf{b}}$, $l^{y,x}_{\mathbf{a},\mathbf{b}}$, $l^x_{\mathbf{a}}$, $l^y_{\mathbf{a}}$ and $l^{xy}_{\mathbf{a}}$ respectively. Note that they are expressed as

$$(2.7)\qquad \sum_{I=(i_m,\dots,i_1)}c_I[\omega_{i_m}|\cdots|\omega_{i_1}]$$

for some $m \in \mathbf{N}$ with $c_I \in \mathbf{Q}$ and $\omega_{i_j} \in \{\frac{dx}{x}, \frac{dx}{1-x}, \frac{dy}{y}, \frac{dy}{1-y}, \frac{xdy+ydx}{1-xy}\}$. It is easy to see that they indeed lie on $V(\mathcal{M}_{0,5})$; i.e., they satisfy the integrability condition (2.1). It is proved by induction on weights: suppose that $l^{x,y}_{\mathbf{a},\mathbf{b}}$ is expressed as above. Then by our induction assumption, (2.1) holds for $1 \leqslant j \leqslant m-2$. By the integrability of the complex analytic function $Li_{\mathbf{a},\mathbf{b}}(x,y)$, we have $d \circ dLi_{\mathbf{a},\mathbf{b}}(x,y) = 0$. This implies (2.1) for $j = m-1$. The same arguments also work for $l^{y,x}_{\mathbf{a},\mathbf{b}}$, $l^x_{\mathbf{a}}$, $l^y_{\mathbf{a}}$ and $l^{xy}_{\mathbf{a}}$.

Examples 2.2. Put $\alpha_0 = \frac{dx}{x}$, $\alpha_1 = \frac{dx}{1-x}$, $\beta_0 = \frac{dy}{y}$, $\beta_1 = \frac{dy}{1-y}$ and $\gamma = \frac{xdy+ydx}{1-xy}$. The function $Li_{2,1}(x,y)$ corresponds to $l^{x,y}_{2,1} = [\beta_1|\alpha_0|\gamma] + [\beta_1|\beta_0|\gamma] + [\alpha_0|\beta_1|\gamma] + [\alpha_0|\alpha_1|\beta_1] - [\alpha_0|\alpha_0|\gamma] - [\alpha_0|\alpha_1|\gamma]$. By a direct computation it can be checked that $l^{x,y}_{2,1}$ lies on $V(\mathcal{M}_{0,5})$. The analytic continuation $\rho(l^{x,y}_{2,1})$ of $Li_{2,1}(x,y)$ is calculated by (2.2). In particular, when we take any path $\gamma(t)$ starting from o to (x,y) in the open unit disk of $\mathcal{M}_{0,5}(\mathbf{C})$ we get the expression (2.5) of $Li_{2,1}(x,y)$ for $|x|, |y| < 1$.

3. Proof of Theorem 0.2

Suppose that φ is an element as in Theorem 1.1. Recall that multiple polylogarithms satisfy the analytic identity, the series shuffle formula in $I_o(\mathcal{M}_{0,5})$

$$Li_{\mathbf{a}}(x) \cdot Li_{\mathbf{b}}(y) = \sum_{\sigma \in Sh^{\leqslant}(k,l)} Li_{\sigma(\mathbf{a},\mathbf{b})}(\sigma(x,y)). \tag{3.1}$$

Here $Sh^{\leqslant}(k,l) := \cup_{N=1}^{\infty}\{\sigma : \{1,\dots,k+l\} \to \{1,\dots,N\} | \sigma \text{ is onto}, \sigma(1) < \cdots < \sigma(k), \sigma(k+1) < \cdots < \sigma(k+l)\}$, $\sigma(\mathbf{a},\mathbf{b}) := ((c_1,\dots,c_j),(c_{j+1},\dots,c_N))$ with $\{j,N\} = \{\sigma(k),\sigma(k+l)\}$,

$$c_i = \begin{cases} a_s + b_{t-k} & \text{if } \sigma^{-1}(i) = \{s,t\} \text{ with } s < t, \\ a_s & \text{if } \sigma^{-1}(i) = \{s\} \quad \text{with } s \leqslant k, \\ b_{s-k} & \text{if } \sigma^{-1}(i) = \{s\} \quad \text{with } s > k, \end{cases}$$

and

$$\sigma(x,y) = \begin{cases} xy & \text{if } \sigma^{-1}(N) = k, k+l, \\ (x,y) & \text{if } \sigma^{-1}(N) = k+l, \\ (y,x) & \text{if } \sigma^{-1}(N) = k. \end{cases}$$

Since ρ is an embedding of algebras, the above analytic identity immediately implies the algebraic identity, the series shuffle formula in $V(\mathcal{M}_{0,5})$

$$l^x_{\mathbf{a}} \cdot l^y_{\mathbf{b}} = \sum_{\sigma \in Sh^{\leqslant}(k,l)} l^{\sigma(x,y)}_{\sigma(\mathbf{a},\mathbf{b})}. \tag{3.2}$$

Examples 3.1. Under the notation in Example 2.2, $l^x_2 = [\alpha_0|\alpha_1]$, $l^y_1 = [\beta_1]$, $l^{xy}_3 = [\alpha_0+\beta_0|\alpha_0+\beta_0|\gamma]$ and $l^{y,x}_{1,2} = [\beta_1|\alpha_0|\alpha_1] - [\beta_0|\alpha_0|\gamma] - [\beta_0|\beta_0|\gamma] - [\beta_1|\alpha_0|\gamma] - [\beta_1|\beta_0|\gamma] + [\alpha_0|\beta_1|\alpha_1] - [\alpha_0|\beta_0|\gamma] - [\alpha_0|\beta_1|\gamma] + [\alpha_0|\alpha_1|\gamma]$. A direct calculation shows the validity of equation (3.2) in the case of $\mathbf{a}=(2)$ and $\mathbf{b}=(1)$. (For $l_{2,1}^{x,y}$, see Example 2.2.)

Evaluation of equation (3.2) at the group-like element $\varphi_{451}\varphi_{123}$[4] gives the series shuffle formula

$$l_{\mathbf{a}}(\varphi)\cdot l_{\mathbf{b}}(\varphi)=\sum_{\sigma\in Sh^{\leqslant}(k,l)} l_{\sigma(\mathbf{a},\mathbf{b})}(\varphi) \tag{3.3}$$

for admissible[5] indices $\mathbf{a}$ and $\mathbf{b}$ because of Lemmas 4.1 and 4.2 below.

Define the integral regularized value $l^I_{\mathbf{a}}(\varphi)$ and series regularized value $l^S_{\mathbf{a}}(\varphi)$ in $k[T]$ (T: a parameter which stands for $\log x$) for all indices $\mathbf{a}$ as follows: $l^I_{\mathbf{a}}(\varphi)$ is defined by $l^I_{\mathbf{a}}(\varphi)=l_{\mathbf{a}}(e^{TX_1}\varphi)$. (Equivalently, $l^I_{\mathbf{a}}(\varphi)$ for any index $\mathbf{a}$ is uniquely defined in such a way that the iterated integral shuffle formulae (cf.[BF06]§7) remain valid for all indices $\mathbf{a}$ with $l^I_1(\varphi):=-T$ and $l^I_{\mathbf{a}}(\varphi):=l_{\mathbf{a}}(\varphi)$ for all admissible indices $\mathbf{a}$.) Similarly, put $l^S_1(\varphi):=-T$ and put $l^S_{\mathbf{a}}(\varphi):=l_{\mathbf{a}}(\varphi)$ for all admissible indices $\mathbf{a}$. Then by the results in [Hof97], $l^S_{\mathbf{a}}(\varphi)$ for a nonadmissible index $\mathbf{a}$ is also uniquely defined in such a way that the series shuffle formulae (3.3) remain valid for $l^S_{\mathbf{a}}(\varphi)$ with all indices $\mathbf{a}$.

Let $\mathbb{L}$ be the k-linear map from $k[T]$ to itself defined via the generating function:

$$\begin{aligned}\mathbb{L}(\exp Tu)&=\sum_{n=0}^{\infty}\mathbb{L}(T^n)\frac{u^n}{n!}\\&=\exp\Big\{-\sum_{n=1}^{\infty}l^I_n(\varphi)\frac{u^n}{n}\Big\}\left(=\exp\Big\{Tu-\sum_{n=1}^{\infty}l_n(\varphi)\frac{u^n}{n}\Big\}\right).\end{aligned}\tag{3.4}$$

PROPOSITION 3.2. *Let φ be an element as in Theorem 1.1. Then the regularization relation holds, i.e. $l^S_{\mathbf{a}}(\varphi)=\mathbb{L}\big(l^I_{\mathbf{a}}(\varphi)\big)$ for all indices $\mathbf{a}$.*

Proof. We may assume that $\mathbf{a}$ is nonadmissible because the proposition is trivial if $\mathbf{a}$ is admissible. When $\mathbf{a}$ is of the form $(1,1,\dots,1)$, the proof is given as follows, using the same argument as in [Gon02, Lemma 7.9]: By the series shuffle formulae,

$$\sum_{k=0}^{m}(-1)^k l^S_{k+1}(\varphi)\cdot l^S_{\underbrace{1,1,\dots,1}_{m-k}}(\varphi)=(m+1)l^S_{\underbrace{1,1,\dots,1}_{m+1}}(\varphi)$$

for $m\geqslant 0$. Here we put $l^S_{\emptyset}(\varphi)=1$. This means

$$\sum_{k,l\geqslant 0}(-1)^k l^S_{k+1}(\varphi)\cdot l^S_{\underbrace{1,1,\dots,1}_{l}}(\varphi)u^{k+l}=\sum_{m\geqslant 0}(m+1)l^S_{\underbrace{1,1,\dots,1}_{m+1}}(\varphi)u^m.$$

[4] For simplicity, we mean φ_{ijk} for $\varphi(X_{ij},X_{jk})\in U\mathfrak{P}_5$.

[5] An index $\mathbf{a}=(a_1,\dots,a_k)$ is called *admissible* if $a_k>1$.

Put $f(u)=\sum_{n\geqslant 0} l^S_{\underbrace{1,1,\dots,1}_{n}}(\varphi)u^n$. Then the above equality can be read as

$$\sum_{k\geqslant 0}(-1)^k l^S_{k+1}(\varphi)u^k=\frac{d}{du}\log f(u).$$

Integrating and adjusting constant terms gives

$$\sum_{n\geqslant 0} l^S_{\underbrace{1,1,\dots,1}_{n}}(\varphi)u^n=\exp\left\{-\sum_{n\geqslant 1}(-1)^n l^S_n(\varphi)\frac{u^n}{n}\right\}=\exp\left\{-\sum_{n\geqslant 1}(-1)^n l^I_n(\varphi)\frac{u^n}{n}\right\}$$

because $l^S_n(\varphi)=l^I_n(\varphi)=l_n(\varphi)$ for $n>1$ and $l^S_1(\varphi)=l^I_1(\varphi)=-T$. Since $l^I_{\mathbf{a}}(\varphi)=\frac{(-T)^m}{m!}$ for $\mathbf{a}=(\underbrace{1,1,\dots,1}_{m})$, we get $l^S_{\mathbf{a}}(\varphi)=\mathbb{L}\big(l^I_{\mathbf{a}}(\varphi)\big)$.

When $\mathbf{a}$ is of the form $(\mathbf{a}',\underbrace{1,1,\dots,1}_{l})$ with $\mathbf{a}'$ admissible, the proof is given by the following induction on l. By (3.2),

$$l^x_{\mathbf{a}'}(e^{TX_{51}}\varphi_{451}\varphi_{123})\cdot l^y_{\underbrace{1,1,\dots,1}_{l}}(e^{TX_{51}}\varphi_{451}\varphi_{123})$$
$$=\sum_{\sigma\in Sh^{\leqslant}(k,l)} l^{\sigma(x,y)}_{\sigma(\mathbf{a}',\underbrace{(1,1,\dots,1)}_{l})}(e^{TX_{51}}\varphi_{451}\varphi_{123})$$

with $k=dp(\mathbf{a}')$. By Lemmas 4.3 and 4.4,

$$l_{\mathbf{a}'}(\varphi)\cdot l^I_{\underbrace{1,1,\dots,1}_{l}}(\varphi)=\sum_{\sigma\in Sh^{\leqslant}(k,l)} l^I_{\sigma(\mathbf{a}',\underbrace{(1,1,\dots,1)}_{l})}(\varphi).$$

Then by our induction assumption, taking the image by the map $\mathbb{L}$ gives

$$l_{\mathbf{a}'}(\varphi)\cdot l^S_{\underbrace{1,1,\dots,1}_{l}}(\varphi)=\mathbb{L}\big(l^I_{\mathbf{a}',\underbrace{1,1,\dots,1}_{l}}(\varphi)\big)+\sum_{\sigma\neq id\in Sh^{\leqslant}(k,l)} l^S_{\sigma(\mathbf{a}',\underbrace{(1,1,\dots,1)}_{l})}(\varphi).$$

Since $l^S_{\mathbf{a}'}(\varphi)$ and $l^S_{1,\dots,1}(\varphi)$ satisfy the series shuffle formula, $\mathbb{L}\big(l^I_{\mathbf{a}}(\varphi)\big)$ must be equal to $l^S_{\mathbf{a}}(\varphi)$. □

Embed $k\langle\langle Y_1,Y_2,\dots\rangle\rangle$ into $k\langle\langle X_0,X_1\rangle\rangle$ by sending Y_m to $-X_0^{m-1}X_1$. Then by the above proposition,

$$\begin{aligned}
l^S_{\mathbf{a}}(\varphi)&=\mathbb{L}(l^I_{\mathbf{a}}(\varphi))=\mathbb{L}(l_{\mathbf{a}}(e^{TX_1}\varphi))=l_{\mathbf{a}}\left(\mathbb{L}(e^{TX_1}\pi_Y(\varphi))\right)\\
&=l_{\mathbf{a}}(\exp\left\{-\sum_{n=1}^{\infty} l^I_n(\varphi)\frac{X_1^n}{n}\right\}\cdot\pi_Y(\varphi))\\
&=l_{\mathbf{a}}(\exp\left\{-TY_1+\sum_{n=1}^{\infty}\frac{(-1)^n}{n}c_{X_0^{n-1}X_1}(\varphi)Y_1^n\right\}\cdot\pi_Y(\varphi))=l_{\mathbf{a}}(e^{-TY_1}\varphi_*)
\end{aligned}$$

for all $\mathbf{a}$ because $l_1(\varphi)=0$. As for the third equality we use $(\mathbb{L}\otimes_k \mathrm{id})\circ(\mathrm{id}\otimes_k l_{\mathbf{a}}) = (\mathrm{id}\otimes_k l_{\mathbf{a}})\circ(\mathbb{L}\otimes_k \mathrm{id})$ on $k[T]\otimes_k k\langle\langle X_0,X_1\rangle\rangle$. All $l^S_{\mathbf{a}}(\varphi)$'s satisfy the series shuffle formulae (3.3), so the $l_{\mathbf{a}}(e^{-TY_1}\varphi_*)$'s do also. By putting $T=0$, we get that $l_{\mathbf{a}}(\varphi_*)$'s also satisfy the series shuffle formulae for all $\mathbf{a}$. Therefore $\Delta_*(\varphi_*)=\varphi_*\widehat{\otimes}\varphi_*$. This completes the proof of Theorem 1.1, which implies Theorem 0.2. □

4. Auxiliary lemmas

We prove all lemmas which are required to prove Theorem 0.2 in the previous section.

Lemma 4.1. *Let φ be a group-like element in $k\langle\langle X_0,X_1\rangle\rangle$ with $c_{X_0}(\varphi)=c_{X_1}(\varphi)=0$. Then $l^x_{\mathbf{a}}(\varphi_{451}\varphi_{123})=l_{\mathbf{a}}(\varphi)$, $l^y_{\mathbf{a}}(\varphi_{451}\varphi_{123})=l_{\mathbf{a}}(\varphi)$, $l^{xy}_{\mathbf{a}}(\varphi_{451}\varphi_{123})=l_{\mathbf{a}}(\varphi)$ and $l^{x,y}_{\mathbf{a},\mathbf{b}}(\varphi_{451}\varphi_{123})=l_{\mathbf{ab}}(\varphi)$ for any indices $\mathbf{a}$ and $\mathbf{b}$.*

Proof. Consider the map $\mathcal{M}_{0,5}\to\mathcal{M}_{0,4}$ sending

$$[(x_1,\dots,x_5)]\mapsto[(x_1,x_2,x_3,x_5)].$$

This induces the projection $p_4: U\mathfrak{P}_5\twoheadrightarrow U\mathfrak{F}_2$ sending $X_{12}\mapsto X_0$, $X_{23}\mapsto X_1$ and $X_{i4}\mapsto 0$ $(i\in\mathbf{Z}/5)$. Express $l_{\mathbf{a}}$ as (2.4). Since $(p_4\otimes\mathrm{id})(\mathrm{Exp}\Omega_5)=\mathrm{Exp}\Omega_4(x)\in U\mathfrak{F}_2\widehat{\otimes}V(\mathcal{M}_{0,5})$, it induces the map $p_4^*: V(\mathcal{M}_{0,4})\to V(\mathcal{M}_{0,5})$ which gives $p_4^*([\frac{dz}{z}])=[\frac{dx}{x}]$ and $p_4^*([\frac{dz}{1-z}])=[\frac{dx}{1-x}]$. Hence $p_4^*(l_{\mathbf{a}})=l^x_{\mathbf{a}}$. Then

$$l^x_{\mathbf{a}}(\varphi_{451}\varphi_{123})=l_{\mathbf{a}}(p_4(\varphi_{451}\varphi_{123}))=l_{\mathbf{a}}(\varphi)$$

because $p_4(\varphi_{451})=0$ by our assumption $c_{X_1}(\varphi)=0$.

Next consider the map $\mathcal{M}_{0,5}\to\mathcal{M}_{0,4}$ sending

$$[(x_1,\dots,x_5)]\mapsto[(x_1,x_3,x_4,x_5)].$$

This induces the projection $p_2: U\mathfrak{P}_5\twoheadrightarrow U\mathfrak{F}_2$ sending $X_{45}\mapsto X_0$, $X_{51}\mapsto X_1$ and $X_{i2}\mapsto 0$ $(i\in\mathbf{Z}/5)$. Since $(p_2\otimes\mathrm{id})(\mathrm{Exp}\Omega_5)=\mathrm{Exp}\Omega_4(y)\in U\mathfrak{F}_2\widehat{\otimes}V(\mathcal{M}_{0,5})$, it induces the map $p_2^*: V(\mathcal{M}_{0,4})\to V(\mathcal{M}_{0,5})$ which gives $p_2^*([\frac{dz}{z}])=[\frac{dy}{y}]$ and $p_2^*([\frac{dz}{1-z}])=[\frac{dy}{1-y}]$. Hence $p_2^*(l_{\mathbf{a}})=l^y_{\mathbf{a}}$. Then

$$l^y_{\mathbf{a}}(\varphi_{451}\varphi_{123})=l_{\mathbf{a}}(p_2(\varphi_{451}\varphi_{123}))=l_{\mathbf{a}}(\varphi)$$

because $p_2(\varphi_{123})=0$.

Similarly consider the map $\mathcal{M}_{0,5}\to\mathcal{M}_{0,4}$ sending

$$[(x_1,\dots,x_5)]\mapsto[(x_1,x_2,x_4,x_5)].$$

This induces the projection $p_3: U\mathfrak{P}_5\twoheadrightarrow U\mathfrak{F}_2$ sending $X_{12}\mapsto X_0$, $X_{24}\mapsto X_1$ and $X_{i3}\mapsto 0$ $(i\in\mathbf{Z}/5)$. Since $(p_3\otimes\mathrm{id})(\mathrm{Exp}\Omega_5)=\mathrm{Exp}\Omega_4(xy)\in U\mathfrak{F}_2\widehat{\otimes}V(\mathcal{M}_{0,5})$, it induces the map $p_3^*: V(\mathcal{M}_{0,4})\to V(\mathcal{M}_{0,5})$ which gives $p_3^*([\frac{dz}{z}])=[\frac{dx}{x}+\frac{dy}{y}]$ and $p_3^*([\frac{dz}{1-z}])=[\frac{xdy+ydx}{1-xy}]$. Hence $p_3^*(l_{\mathbf{a}})=l^{xy}_{\mathbf{a}}$. Then

$$l^{xy}_{\mathbf{a}}(\varphi_{451}\varphi_{123})=l_{\mathbf{a}}(p_3(\varphi_{451}\varphi_{123}))=l_{\mathbf{a}}(\varphi)$$

because $p_3(\varphi_{123})=0$ by our assumption $c_{X_0}(\varphi)=0$.

Consider the embedding of Hopf algebra $i_{123} : U\mathfrak{F}_2 \hookrightarrow U\mathfrak{P}_5$ sending $X_0 \mapsto X_{12}$ and $X_1 \mapsto X_{23}$. (Geometrically it is explained by the residue map in [BF06] along the divisor $\{y = 0\}$.) Since $(i_{123} \otimes \mathrm{id})(\mathrm{Exp}\Omega_4) = \mathrm{Exp}\Omega_4(z; X_{12}, X_{23}) \in U\mathfrak{P}_5 \widehat{\otimes} V(\mathcal{M}_{0,4})$, it induces the map $i^*_{123} : V(\mathcal{M}_{0,5}) \to V(\mathcal{M}_{0,4})$ which gives $i^*_{123}([\frac{dy}{y}]) = i^*_{123}([\frac{dy}{1-y}]) = i^*_{123}([\frac{xdy+ydx}{1-xy}]) = 0$. Express $l^{x,y}_{\mathbf{a},\mathbf{b}}$ and $l^{xy}_{\mathbf{a}}$ as (2.7). In the expression each term contains at least one $\frac{dy}{y}$, $\frac{dy}{1-y}$ or $\frac{xdy+ydx}{1-xy}$. Therefore we have $i^*_{123}(l^{x,y}_{\mathbf{a},\mathbf{b}}) = 0$ and $i^*_{123}(l^{xy}_{\mathbf{a}}) = 0$. Thus

$$l^{x,y}_{\mathbf{a},\mathbf{b}}(\varphi_{123}) = 0 \text{ and } l^{xy}_{\mathbf{a}}(\varphi_{123}) = 0.$$

Next consider the embedding of Hopf algebra $i_{451} : U\mathfrak{F}_2 \hookrightarrow U\mathfrak{P}_5$ sending $X_0 \mapsto X_{45}$ and $X_1 \mapsto X_{51} = X_{23} + X_{24} + X_{34}$ (geometrically caused by the divisor $\{x = 1\}$). Since $(i_{451} \otimes \mathrm{id})(\mathrm{Exp}\Omega_4) = \mathrm{Exp}\Omega_4(z; X_{45}, X_{23} + X_{24} + X_{34}) \in U\mathfrak{P}_5 \widehat{\otimes} V(\mathcal{M}_{0,4})$, it induces the map $i^*_{451} : V(\mathcal{M}_{0,5}) \to V(\mathcal{M}_{0,4})$ which gives $i^*_{451}([\frac{dx}{x}]) = 0$, $i^*_{451}([\frac{dx}{1-x}]) = [\frac{dz}{1-z}]$, $i^*_{451}([\frac{dy}{y}]) = [\frac{dz}{z}]$, $i^*_{451}([\frac{dy}{1-y}]) = [\frac{dz}{1-z}]$ and $i^*_{451}([\frac{xdy+ydx}{1-xy}]) = [\frac{dz}{1-z}]$. By induction on weight, $i^*_{451}(l^{x,y}_{\mathbf{a},\mathbf{b}}) = l_{\mathbf{ab}}$ and $i^*_{451}(l^{xy}_{\mathbf{a}}) = l_{\mathbf{a}}$ can be deduced from the differential equations (2.6): For instance if $a_k = 1$, $k \neq 1$, $b_l \neq 1$ and $l \neq 1$, then by (2.6)

$$\begin{aligned} i^*_{451}(l^{x,y}_{\mathbf{a},\mathbf{b}}) = &\left[\frac{dz}{1-z}\middle| i^*_{451}(l^{x,y}_{(a_1,\dots,a_{k-1}),\mathbf{b}})\right] \\ &- \left[\frac{dz}{1-z}\middle| i^*_{451}(l^{x,y}_{(a_1,\dots,a_{k-1},b_1),(b_2,\dots,b_l)})\right] + \left[\frac{dz}{z}\middle| i^*_{451}(l^{x,y}_{\mathbf{a},(b_1,\dots,b_l-1)})\right]. \end{aligned}$$

However, by our induction assumption it is equal to $[\frac{dz}{z}|l_{\mathbf{a},(b_1,\dots,b_l-1)}] = l_{\mathbf{ab}}$. Thus

$$l^{x,y}_{\mathbf{a},\mathbf{b}}(\varphi_{451}) = l_{\mathbf{ab}}(\varphi).$$

Let δ be the coproduct of $V(\mathcal{M}_{0,5})$. Express $\delta(l^{x,y}_{\mathbf{a},\mathbf{b}}) = \sum_i l'_i \otimes l''_i$ with $l'_i \in V_{m'_i}$ and $l''_i \in V_{m''_i}$ for some m'_i and m''_i. If $m''_i \neq 0$, $l''_i(\varphi_{123}) = 0$ because l''_i is a combination of elements of the form $l^{x,y}_{\mathbf{c},\mathbf{d}}$ and $l^{xy}_{\mathbf{e}}$ for some indices $\mathbf{c}$, $\mathbf{d}$ and $\mathbf{e}$. Since $\delta(l^{x,y}_{\mathbf{a},\mathbf{b}})(1 \otimes \varphi_{451}\varphi_{123}) = \delta(l^{x,y}_{\mathbf{a},\mathbf{b}})(\varphi_{451} \otimes \varphi_{123})$, $l^{x,y}_{\mathbf{a},\mathbf{b}}(\varphi_{451}\varphi_{123}) = \sum_i l'_i(\varphi_{451}) \otimes l''_i(\varphi_{123}) = l^{x,y}_{\mathbf{a},\mathbf{b}}(\varphi_{451}) = l_{\mathbf{ab}}(\varphi)$. □

LEMMA 4.2. *Let φ be an element as in Theorem 1.1. Suppose that* $\mathbf{a}$ *and* $\mathbf{b}$ *are admissible. Then $l^{y,x}_{\mathbf{a},\mathbf{b}}(\varphi_{451}\varphi_{123}) = l_{\mathbf{ab}}(\varphi)$.*

Proof. Because (1.6) implies $\varphi(X_0, X_1)\varphi(X_1, X_0) = 1$ (take a projection $U\mathfrak{P}_5 \twoheadrightarrow U\mathfrak{F}_2$), we have $\varphi_{451}\varphi_{123} = \varphi_{432}\varphi_{215}\varphi_{543}$. Put $\delta(l^{y,x}_{\mathbf{a},\mathbf{b}}) = \sum_i l'_i \otimes l''_i$. By the same arguments as in the last paragraph of the proof of Lemma 4.1, $l^{y,x}_{\mathbf{c},\mathbf{d}}(\varphi_{543}) = 0$ and $l^{xy}_{\mathbf{e}}(\varphi_{543}) = 0$ for any indices $\mathbf{c}$, $\mathbf{d}$ and $\mathbf{e}$. So we have

$$l^{y,x}_{\mathbf{a},\mathbf{b}}(\varphi_{451}\varphi_{123}) = l^{y,x}_{\mathbf{a},\mathbf{b}}(\varphi_{432}\varphi_{215}\varphi_{543}) = l^{y,x}_{\mathbf{a},\mathbf{b}}(\varphi_{432}\varphi_{215}) = \sum_i l'_i(\varphi_{432}) \otimes l''_i(\varphi_{215}).$$

Consider the embedding of Hopf algebra $i_{432}: U\mathfrak{F}_2 \hookrightarrow U\mathfrak{P}_5$ sending $X_0 \mapsto X_{43}$ and $X_1 \mapsto X_{32}$ (geometrically caused by the exceptional divisor obtained by blowing up at $(x,y)=(1,1)$). Since $(i_{432}\otimes\mathrm{id})(\mathrm{Exp}\Omega_4) = \mathrm{Exp}\Omega_4(z; X_{34}, X_{23}) \in U\mathfrak{P}_5\widehat{\otimes}V(\mathcal{M}_{0,4})$, it induces the morphism $i_{432}^*: V(\mathcal{M}_{0,5}) \to V(\mathcal{M}_{0,4})$ which gives $i_{432}^*([\frac{dx}{x}])=0$, $i_{432}^*([\frac{dx}{x-1}])=[\frac{dz}{z-1}]$, $i_{432}^*([\frac{dy}{y}])=0$, $i_{432}^*([\frac{dy}{y-1}])=[\frac{dz}{z}]$ and $i_{432}^*([\frac{xdy+ydx}{1-xy}])=0$. In each term of the expression

$$l_{\mathbf{a},\mathbf{b}}^{y,x} = \sum_{I=(i_m,\dots,i_1)} c_I[\omega_{i_m}|\cdots|\omega_{i_1}],$$

the first component ω_{i_m} is always $\frac{dx}{x}$ or $\frac{dy}{y}$ because $\mathbf{a}$ and $\mathbf{b}$ are admissible. So $i_{432}^*(l_i')=0$ unless $m_i'=0$. Therefore $\sum_i l_i'(\varphi_{432})\otimes l_i''(\varphi_{215}) = l_{\mathbf{a},\mathbf{b}}^{y,x}(\varphi_{215})$. By the same arguments as in Lemma 4.1, $i_{215}^*(l_{\mathbf{a},\mathbf{b}}^{y,x}) = l_{\mathbf{ab}}$. Thus

$$l_{\mathbf{a},\mathbf{b}}^{y,x}(\varphi_{215}) = l_{\mathbf{ab}}(\varphi). \qquad \square$$

Lemma 4.3. *Let φ be a group-like element in $k\langle\langle X_0,X_1\rangle\rangle$ with $c_{X_0}(\varphi) = c_{X_1}(\varphi)=0$. Then $l_{\mathbf{a}}^{x}(e^{TX_{51}}\varphi_{451}\varphi_{123}) = l_{\mathbf{a}}^{I}(\varphi)$, $l_{\mathbf{a}}^{y}(e^{TX_{51}}\varphi_{451}\varphi_{123}) = l_{\mathbf{a}}^{I}(\varphi)$, $l_{\mathbf{a}}^{xy}(e^{TX_{51}}\varphi_{451}\varphi_{123}) = l_{\mathbf{a}}^{I}(\varphi)$ and $l_{\mathbf{a},\mathbf{b}}^{x,y}(e^{TX_{51}}\varphi_{451}\varphi_{123}) = l_{\mathbf{ab}}^{I}(\varphi)$ for any index $\mathbf{a}$ and $\mathbf{b}$.*

Proof. By the arguments in Lemma 4.1,

$$\begin{aligned} l_{\mathbf{a}}^{x}(e^{TX_{51}}\varphi_{451}\varphi_{123}) &= l_{\mathbf{a}}(p_4(e^{TX_{51}}\varphi_{451}\varphi_{123})) = l_{\mathbf{a}}(e^{TX_1}\varphi) = l_{\mathbf{a}}^{I}(\varphi),\\ l_{\mathbf{a}}^{y}(e^{TX_{51}}\varphi_{451}\varphi_{123}) &= l_{\mathbf{a}}(p_2(e^{TX_{51}}\varphi_{451}\varphi_{123})) = l_{\mathbf{a}}(e^{TX_1}\varphi) = l_{\mathbf{a}}^{I}(\varphi),\\ l_{\mathbf{a}}^{xy}(e^{TX_{51}}\varphi_{451}\varphi_{123}) &= l_{\mathbf{a}}(p_3(e^{TX_{51}}\varphi_{451}\varphi_{123})) = l_{\mathbf{a}}(e^{TX_1}\varphi) = l_{\mathbf{a}}^{I}(\varphi) \end{aligned}$$

and

$$l_{\mathbf{a},\mathbf{b}}^{x,y}(e^{TX_{51}}\varphi_{451}\varphi_{123}) = l_{\mathbf{a},\mathbf{b}}^{x,y}(e^{TX_{51}}\varphi_{451}) = l_{\mathbf{ab}}(e^{TX_1}\varphi) = l_{\mathbf{ab}}^{I}(\varphi). \qquad \square$$

Lemma 4.4. *Let φ be an element as in Theorem 1.1. Suppose that $\mathbf{b}$ is admissible. Then $l_{\mathbf{a},\mathbf{b}}^{y,x}(e^{TX_{51}}\varphi_{451}\varphi_{123}) = l_{\mathbf{ab}}(\varphi)$.*

Proof. We have

$$\begin{aligned} l_{\mathbf{a},\mathbf{b}}^{y,x}(e^{TX_{51}}\varphi_{451}\varphi_{123}) &= l_{\mathbf{a},\mathbf{b}}^{y,x}(e^{TX_{51}}\varphi_{432}\varphi_{215}\varphi_{543})\\ &= l_{\mathbf{a},\mathbf{b}}^{y,x}(\varphi_{432}e^{TX_{51}}\varphi_{215}) = \sum_i l_i'(\varphi_{432})\otimes l_i''(e^{TX_{51}}\varphi_{215}) \end{aligned}$$

because $e^{TX_{51}}\varphi_{432} = \varphi_{432}e^{TX_{51}}$. Since $\mathbf{b}$ is admissible, $i_{432}^*(l_i')$ is of the form $\alpha[\frac{dz}{z}|\cdots|\frac{dz}{z}]$ with $\alpha\in k$. By our assumption, $c_{X_0\cdots X_0}(\varphi)=0$. So $l_i'(i_{432}(\varphi)) = l_i'(\varphi_{432}) = 0$ unless $m_i'=0$. Thus $\sum_i l_i'(\varphi_{432})\otimes l_i''(e^{TX_{51}}\varphi_{215}) = l_{\mathbf{a},\mathbf{b}}^{y,x}(e^{TX_{51}}\varphi_{215}) = l_{\mathbf{ab}}(e^{TX_1}\varphi)$. Since $\mathbf{b}$ is admissible, $l_{\mathbf{ab}}(e^{TX_1}\varphi) = l_{\mathbf{ab}}(\varphi)$. $\square$

5. Proof of Theorem 0.5

Put $F_2'(k) = \{\varphi \in U\mathfrak{F}_2 | \varphi : \text{group-like}, c_{X_0}(\varphi) = c_{X_1}(\varphi) = 0\}$. It forms a group with respect to (0.1) and contains $\mathrm{DMR}_0(k)$ as a subgroup. Consider the map $m : (F_2'(k), \circ) \to k[[x_0, x_1]]^\times$ sending φ to its meta-abelian quotient $B_\varphi(x_0, x_1)$. By a direct calculation we see that it is a group homomorphism, i.e. $m(\varphi_1 \circ \varphi_2) = m(\varphi_1) \cdot m(\varphi_2)$. Put $B(k) = \left\{ b \in k[[x_0, x_1]]^\times \middle| b(x_0, x_1) = \frac{c(x_0)c(x_1)}{c(x_0+x_1)} \text{ for } c(s) \in 1 + s^2 k[[s]] \right\}$. It is a subgroup of $k[[x_0, x_1]]^\times$. The first statement of Theorem 0.5 claims $m(\mathrm{DMR}_\mu(k)) \subset B(k)$ for all $\mu \in k$.

Proposition 5.1. $m(\mathrm{DMR}_0(k)) \subset B(k)$.

Proof. Let M be the Lie algebra homomorphism, associated with $m|_{\mathrm{DMR}_0}$, from the Lie algebra $\mathfrak{dmr}_0$ (see Appendix A) of DMR_0 to the trivial Lie algebra $k[[x_0, x_1]]$. In order to prove our proposition it is enough to show $M(\mathfrak{dmr}_0) \subset \mathfrak{B}$, where $\mathfrak{B}$ is the Lie subalgebra

$$\left\{ \beta \in k[[x_0, x_1]] \middle| \beta(x_0, x_1) = \gamma(x_0) + \gamma(x_1) - \gamma(x_0 + x_1) \text{ for } \gamma(s) \in s^2 k[[s]] \right\}$$

with trivial Lie structure. □

Lemma 5.2. *For* $\psi \in \mathfrak{dmr}_0$ *with* $\psi = \psi_{X_0} X_0 + \psi_{X_1} X_1$, $M(\psi) = (\psi_{X_1} X_1)^{ab}$.

Proof. The exponential map $\mathrm{Exp} : \mathfrak{dmr}_0 \to \mathrm{DMR}_0$ is given by the formula $\psi \mapsto \sum_{i=0}^{\infty} \frac{1}{i!}(\mu_\psi + d_\psi)^i(1) = 1 + \psi + \frac{1}{2}(\psi^2 + d_\psi(\psi))^2 + \frac{1}{6}(\psi^3 + 2\psi d_\psi(\psi) + d_\psi(\psi)\psi + d_\psi^2(\psi)) + \cdots$ (μ_ψ: the left multiplication by ψ and d_ψ: see our appendix) in [DG05, Remark 5.15]. By a direct calculation, it can be checked that $m \circ \mathrm{Exp}(\psi) = \exp(\psi_{X_1} X_1)^{\mathrm{ab}}$, which implies our lemma. □

By the above lemma, the proof of $M(\mathfrak{dmr}_0) \subset \mathfrak{B}$ is reduced to the following:

Lemma 5.3. *For* $\psi \in \mathfrak{dmr}_0$,

$$\sum_{\substack{wt(\mathbf{a})=w \\ dp(\mathbf{a})=m}} l_{\mathbf{a}}(\psi) = \begin{cases} (-1)^{m-1} \binom{w}{m} \frac{l_w(\psi)}{w} & \text{for } m < w, \\ 0 & \text{for } m = w. \end{cases}$$

Proof. By (A.1) we have $\sum_{\sigma \in Sh^{\leqslant}(k,l)} l_{\sigma(\mathbf{a},\mathbf{b})}(\psi_*) = 0$ with $dp(\mathbf{a}) = k$ and $dp(\mathbf{b}) = l$. Summing up all pairs $(\mathbf{a}, \mathbf{b})$ with $wt(\mathbf{a}) = k$, $dp(\mathbf{a}) = 1$ and $wt(\mathbf{a}) + wt(\mathbf{b}) = w$, we get

$$\sum_{wt(\mathbf{a})=w, dp(\mathbf{a})=k+1} (k+1) l_{\mathbf{a}}(\psi_*) + \sum_{wt(\mathbf{a})=w, dp(\mathbf{a})=k} (w-k) l_{\mathbf{a}}(\psi_*) = 0.$$

Then the lemma follows by induction because $l_{\mathbf{a}}(\psi_*) = l_{\mathbf{a}}(\psi)$ for $dp(\mathbf{a}) \neq wt(\mathbf{a})$. □

The above lemma implies Proposition 5.1. □

To complete the proof of the first statement of Theorem 0.5, we may assume that $\mu = 1$ and $k = \mathbf{Q}$. Let φ be any element of $\mathrm{DMR}_1(\mathbf{Q})$. Put $\varphi_{\mathrm{KZ}} = \Phi_{\mathrm{KZ}}(\frac{X_0}{2\pi i}, \frac{X_1}{2\pi i})$. It gives an element of $\mathrm{DMR}_1(\mathbf{C})$. Because DMR_1 is a right DMR_0-torsor, there exists uniquely $\varphi' \in \mathrm{DMR}_0(\mathbf{C})$ such that $\varphi = \varphi_{\mathrm{KZ}} \circ \varphi'$. In [Dri90] it is shown that $m(\varphi_{\mathrm{KZ}}) \in B(\mathbf{C})$. By Proposition 5.1, $m(\varphi') \in B(\mathbf{C})$. Thus $m(\varphi) = m(\varphi_{\mathrm{KZ}})m(\varphi')$ must lie on B. The proof of the second statement is easy. Express $\log \Gamma_\varphi(s) = \sum_{n=2}^\infty d_n(\varphi)s^n$. Then by (0.2), $d_n(\varphi) = \frac{-1}{n} c_{X_0^{n-1}X_1}(\varphi)$. This completes the proof of Theorem 0.5. □

Appendix A. Review of the proof of Racinet's theorem

In [Rac02, Th. I], Racinet shows a highly nontrivial result that DMR_0 is closed under the multiplication (0.1). However his proof looks too complicated. The aim of this appendix is to review the essential part ([Rac02, Prop. 4.A.i]) of his proof clearly in the case of $\Gamma = \{1\}$, in order to help the readers to catch his arguments.

In [Rac02, 3.3.1], $\mathfrak{dmr}_0$ is introduced to be the set of formal Lie series $\psi \in \mathfrak{F}_2 = \{\psi \in U\mathfrak{F}_2 | \Delta(\psi) = 1 \otimes \psi + \psi \otimes 1\}$ satisfying $c_{X_0}(\psi) = c_{X_1}(\psi) = 0$ and

$$\Delta_*(\psi_*) = 1 \otimes \psi_* + \psi_* \otimes 1 \tag{A.1}$$

with $\psi_* = \psi_{\mathrm{corr}} + \pi_Y(\psi)$ and $\psi_{\mathrm{corr}} = \sum_{n=1}^\infty \frac{(-1)^n}{n} c_{X_0^{n-1}X_1}(\psi) Y_1^n$. It is the tangent vector space at the origin of DMR_0.

Theorem A.1 ([Rac02, Prop. 4.A.i]). *The set $\mathfrak{dmr}_0$ has a structure of Lie algebra with the Lie bracket[6] given by*

$$\{\psi_1, \psi_2\} = d_{\psi_2}(\psi_1) - d_{\psi_1}(\psi_2) - [\psi_1, \psi_2],$$

where d_ψ ($\psi \in U\mathfrak{F}_2$) is the derivation of $U\mathfrak{F}_2$ given by $d_\psi(X_0) = 0$ and $d_\psi(X_1) = [X_1, \psi]$.

Proof. Put $U\mathfrak{F}_Y = k\langle\langle Y_1, Y_2, \dots \rangle\rangle$. It is the universal enveloping algebra of the Lie algebra $\mathfrak{F}_Y = \{\psi_* \in U\mathfrak{F}_Y | \psi_* \text{ satisfies (A.1)}\}$ with the coproduct Δ_*. By the algebraic map sending Y_m to $-X_0^{m-1}X_1$ $(m = 1, 2, \dots)$, we frequently regard $U\mathfrak{F}_Y$ as a k-linear (or algebraic) subspace of $U\mathfrak{F}_2$. Define S_X to be the antipode, the *anti*-automorphism of $U\mathfrak{F}_2$ such that $S_X(X_0) = -X_0$ and $S_X(X_1) = -X_1$. Define R_Y to be the *anti*-automorphism of $U\mathfrak{F}_Y$ such that $R_Y(Y_n) = Y_n$ $(n \geqslant 1)$. It is easily proved that $S_X(x_1 w) = (-1)^{p+1} R_Y(x_1 w)$ for $w \in U\mathfrak{F}_Y$ with degree p and that $S_X(f) = -f$ (resp. $R_Y(f) = -f$) for $f \in \mathfrak{F}_2$ (resp. $f \in \mathfrak{F}_Y$).

[6] For our convenience, we change the order of bracket in [Rac02, (3.1.10.2)].

For a homogeneous element of $U\mathfrak{F}_2$ with degree p, we denote $\{f_i\}_{i=0}^p$ to be elements of $U\mathfrak{F}_Y$ characterized by $f=\sum_{i=0}^p f_iX_0^i$ and define $f_{i,j}=f_iY_j+Y_j\bar{f_i}\in U\mathfrak{F}_2$ $(0\leqslant i\leqslant p,\ 0\leqslant j)$ with $\bar{f_i}=(-1)^pR_Y(f_i)$.

Define the k-linear endomorphism s_f $(f\in U\mathfrak{F}_2)$ of $U\mathfrak{F}_2$ by $s_f(v)=fv+d_f(v)$. It induces the k-linear endomorphism s_f^Y of $U\mathfrak{F}_Y$ such that $\pi_Y\circ s_f=s_f^Y\circ\pi_Y$. Define the k-linear endomorphism D_f^Y of $U\mathfrak{F}_Y$ by $D_f^Y(w)=s_f^Y(w)-w\pi_Y(f)$. Direct computations show that D_f^Y forms a derivation of the noncommutative algebra $U\mathfrak{F}_Y$, and actually it is the restriction into $U\mathfrak{F}_Y$ of the derivation D_f of $U\mathfrak{F}_2$ defined by $X_0\mapsto[f,X_0]$ and $X_1\mapsto 0$.

LEMMA A.2. *Let f be a homogeneous element of $U\mathfrak{F}_2$ with degree $p(\geqslant 1)$ such that $S_X(f)=-f$. Then for $n\geqslant 1$,*

$$D_f^Y(Y_n)=X_0^{n-1}fX_1-fX_0^{n-1}X_1=\sum_{i=0}^p f_{i,i+n}.$$

Proof. The first equality is easy. Denote $f_i=\sum_{j=0}^{p-1-i}X_0^jX_1f_i^j$ $(0\leqslant i\leqslant p-1)$. By $S_X(f)=-f$, we have $X_1f_i^j=(-1)^{p+1}R_Y(X_1f_j^i)$ $(0\leqslant i\leqslant p-1,\ 0\leqslant j\leqslant p-1-i)$ and $\{1+(-1)^p\}f_p=0$. Hence $\bar{f_i}=-\sum_{j=0}^{p-1-i}f_j^iX_0^jX_1$ $(0\leqslant i\leqslant p-1)$ and $\bar{f_p}=(-1)^pf_p$. Then direct calculation using them yields the second equality. ☐

Put ∂_0 as the derivation of $U\mathfrak{F}_2$ defined by $\partial_0(X_0)=1$ and $\partial_0(X_1)=0$. Define the map $\sec:U\mathfrak{F}_Y\to U\mathfrak{F}_2$ by $\sec(w)=\sum_{i\geqslant 0}\frac{(-1)^i}{i!}\partial_0^i(w)X_0^i$, which actually maps to $\ker\partial_0\subset U\mathfrak{F}_2$. It is easy to see that the map $\sec:U\mathfrak{F}_Y\to\ker\partial_0$ is the inverse of $\pi_Y|_{\ker\partial_0}$.

LEMMA A.3. *Let g be a homogeneous element of $U\mathfrak{F}_2$ with degree $p(\geqslant 1)$. Assume $g\in\mathfrak{F}_Y$. Set $f=\sec g$. Then for $n\geqslant 1$*

$$\Delta_*\circ D_f^Y(Y_n)-(\mathrm{id}\otimes D_f^Y+D_f^Y\otimes\mathrm{id})\Delta_*(Y_n)=\sum_{k=0}^p\sum_{i=k}^p\{f_{i,i-k}\otimes Y_{n+k}+Y_{n+k}\otimes f_{i,i-k}\}.$$

Proof. It is well known that if for $i\geqslant 1$ we define U_i to be the weight i part of $\log(1+Y_1+Y_2+\cdots)$ (so $U_1=Y_1$, $U_2=Y_2-\frac{Y_1^2}{2}$, etc), then $g\in U\mathfrak{F}_Y$ lies on $\mathfrak{F}_Y$ if and only if g lies on the free Lie algebra generated by $U_1,U_2,\dots$. By direct calculation, $\partial_0(U_n)=(n-1)U_{n-1}$ $(n\geqslant 1)$. So by induction, we get $\partial_0(w)\in\mathfrak{F}_Y$ if $w\in\mathfrak{F}_Y$. Because $f_i=\frac{(-1)^i}{i!}\partial_0^i(g)$ and $\bar{f_i}=(-1)^pR_Y(f_i)$ $(0\leqslant i\leqslant p)$, both f_i and $\bar{f_i}$ satisfy (A.1) (belong to $\mathfrak{F}_Y$). Then it follows that (the first term)$=\sum_{i=0}^p\Delta_*(f_iY_{i+n}+Y_{i+n}\bar{f_i})=\sum_{i=0}^p\sum_{k=0}^{n+i}(f_{i,i+n-k}\otimes Y_k+Y_k\otimes f_{i,i+n-k})$ and (the second term)$=\sum_{k=0}^{n-1}\sum_{i=0}^p(f_{i,i+n-k}\otimes Y_k+Y_k\otimes f_{i,i+n-k})$. Therefore (l.h.s)$=\sum_{i=0}^p\sum_{k=0}^i(f_{i,i-k}\otimes Y_{n+k}+Y_{n+k}\otimes f_{i,i-k})=\sum_{k=0}^p\sum_{i=k}^p(f_{i,i-k}\otimes Y_{n+k}+Y_{n+k}\otimes f_{i,i-k})$. ☐

LEMMA A.4. *Let g be a homogeneous element of $U\mathfrak{F}_2$ with degree $p(\geqslant 1)$. Set $f = \sec g$. Assume $g \in \mathfrak{F}_Y$ and $S_X(f) = -f$. Then*

$$\sum_{i=k}^{p} f_{i,i-k} = \begin{cases} (-1)^{p-k-1}\binom{p-1}{k}\{1+(-1)^p\}c_{X_0^{p-1}X_1}(g)Y_{p-k} & (0 \leqslant k \leqslant p-1), \\ 0 & (k=p). \end{cases}$$

Proof. Let ∂_{U_i} $(i \geqslant 1)$ be the derivation defined by 1 on U_i and 0 on U_j $(j \neq i)$. It is not difficult to show that $\partial_{U_i}(w) = c_{Y_i}(w)$ for $w \in \mathfrak{F}_Y$ (cf. [Rac02, Prop. 2.3.8]). By $g \in \mathfrak{F}_Y$, we have $f_i, \bar{f}_i \in \mathfrak{F}_Y$, whence direct calculation yields for $k \geqslant 0$

$$\partial_{U_{k+1}}\left(\sum_{i=0}^{p} f_{i,i+1}\right) = \sum_{i=0}^{p} c_{Y_{k+1}}(f_i + \bar{f}_i)Y_{i+1} + \sum_{i=k}^{p} f_{i,i-k}.$$

By $\deg f_i = p - i$, $\bar{f}_i = (-1)^p R_Y(f_i)$ and $f_i = \frac{(-1)^i}{i!}\partial_0^i(g)$,

$$\begin{aligned}\sum_{i=0}^{p} c_{Y_{k+1}}(f_i+\bar{f}_i)Y_{i+1} &= -\{1+(-1)^p\}c_{X_0^kX_1}(f_{p-k-1})Y_{p-k} \\ &= \{1+(-1)^p\}\frac{(-1)^{p-k}}{(p-k-1)!}\frac{(p-1)!}{k!}c_{X_0^{p-1}X_1}(f)Y_{p-k}\end{aligned}$$

for $0 \leqslant k \leqslant p-1$. By the definitions above it is immediate that $D_f(Y_1) = 0$. So by Lemma A.2, $\sum_{i=0}^{p} f_{i,i+1} = 0$. It implies the desired equality for $0 \leqslant k \leqslant p-1$. The case for $k = p$ is immediate. □

PROPOSITION A.5 ([Rac02, Prop. 4.3.1]). *For $\psi \in \mathfrak{dmr}_0$, $s^Y_{\sec\psi_*}$ forms a coderivation of $U\mathfrak{F}_Y$ with respect to Δ_*.*

Proof. Because $\mathfrak{F}_2^{\geqslant 1} \subset \ker\partial_0$ and $\sec(Y_1^n) = Y_1^n$ $(n \geqslant 1)$, we have

$$\sec\psi_* = \psi + \psi_{\mathrm{corr}} \tag{A.2}$$

for $\psi \in \mathfrak{dmr}_0$. Since ψ is an element of $\mathfrak{F}_2$, $S_X(\psi) = -\psi$. Because ψ lies in $\mathfrak{dmr}_0$, it is known that $c_{X_0^{n-1}X_1}(\psi) = 0$ for even n (cf. [Rac02, Prop. 3.3.3]). So $S_X(\psi_{\mathrm{corr}}) = -\psi_{\mathrm{corr}}$. Therefore, $S_X(\sec\psi_*) = -\sec\psi_*$ by (A.2) and we can apply Lemmas A.3 and A.4 with $g = \psi_*$ and $f = \sec\psi_*$ to obtain the identity $\Delta_* \circ D^Y_{\sec\psi_*}(Y_n) = (\mathrm{id}\otimes D^Y_{\sec\psi_*} + D^Y_{\sec\psi_*}\otimes\mathrm{id})\Delta_*(Y_n)$ for all $n \geqslant 1$. This implies that the derivation $D^Y_{\sec\psi_*}$ forms a coderivation. Since $\psi_* = \pi_Y \circ \sec\psi_*$ lies on $\mathfrak{F}_Y$, the right multiplication by ψ_* forms a coderivation. Therefore $s^Y_{\sec\psi_*}$ must form a coderivation. □

Recall that $s^Y_\psi(1) = \pi_Y(\psi)$ and $[s_\psi, s_{X_1^n}] = 0$ for any ψ and $n > 0$ and $s_{\{\psi_1,\psi_2\}} = [s_{\psi_2}, s_{\psi_1}]$ for any ψ_1 and ψ_2. Assume that ψ_1 and ψ_2 be elements of $\mathfrak{dmr}_0$. Then $s_{\{\psi_1,\psi_2\}} = [s_{\psi_2}, s_{\psi_1}] = [s_{\sec\psi_{2*}}, s_{\sec\psi_{1*}}]$ by (A.2). Therefore $\pi_Y(\{\psi_1,\psi_2\}) = s^Y_{\{\psi_1,\psi_2\}}(1) = [s^Y_{\sec\psi_{2*}}, s^Y_{\sec\psi_{1*}}](1)$. By Proposition A.5, $s^Y_{\sec\psi_{i*}}$ $(i = 1,2)$ forms a coderivation. So $[s^Y_{\sec\psi_{2*}}, s^Y_{\sec\psi_{1*}}]$ forms a coderivation. Since

$\pi_Y(\{\psi_1, \psi_2\})$ is the image of 1 by the coderivation, it must be Lie-like with respect to Δ_*. That means (A.1) holds for $\psi = \{\psi_1, \psi_2\}$ because $\psi_{\text{corr}} = 0$. It completes the proof of Theorem A.1. □

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