On the spectral side of Arthur's trace formula — combinatorial setup

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Abstract

In Arthur's trace formula, a ubiquitous role is played by certain limiting expressions arising from piecewise smooth functions with respect to projections of the Coxeter fan ((G, M)-families). These include terms resulting from intertwining operators on the spectral side and volumes of polytopes on the geometric side. We introduce the combinatorial concept of a compatible family with respect to an arbitrary polyhedral fan and obtain new formulas for the corresponding limiting expressions in this general framework. Our formulas can be regarded as algebraic generalizations of certain volume formulas for convex polytopes. In a companion paper, the results are used to study the spectral side of the trace formula.

Contents

1. Introduction

The volume of a convex polytope P in a real vector space V can be computed in several ways, for example by triangulating the polytope into simplices. On the other hand, a beautiful localization formula due to Brion [Bri97, Prop. 5.3] computes the Fourier-Laplace transform of the characteristic function of P in terms of the piecewise linear support function $H_P(\lambda) = \max_{x \in P} \langle \lambda, x \rangle$ on the dual vector space V^*. As a consequence, the volume

of P can be written as a canonical d-th order derivative $\mathcal{D}_{\Sigma(P)} \exp H_P$ of the function $\exp H_P$ at the origin, where $d = \dim P$. Therefore, every volume formula for polytopes can be interpreted as a derivative formula for the exponentials of certain piecewise linear functions. In this paper we show that two previously known volume identities for polytopes can be generalized to purely algebraic derivative formulas in a noncommutative setup. In this context we are given a finite-dimensional algebra $\mathcal{E}$ over $\mathbb{C}$ and for each vertex of a polytope a d-variable power series with coefficients in $\mathcal{E}$ subject to certain compatibility relations. The crucial qualitative feature of our derivative formulas is that they reduce a d-th order derivative to products of d first-order derivatives in linearly independent directions.

In the scalar case ($\mathcal{E} = \mathbb{C}$) the main examples of compatible families are indeed the exponentials of piecewise linear functions. Interesting examples for the properly noncommutative situation arise in the representation theory of reductive groups over local fields, namely as the normalized intertwining operators associated to representations induced from parabolic subgroups. In the paper [FLM] our results are used to rewrite the spectral side of Arthur's trace formula and to study it from an analytic point of view.

In the case of a simple polytope P (i.e. a polytope for which each vertex belongs to exactly d edges), Brion's formula can be stated as follows:
(1.1)
$$\int_P e^{\langle \lambda, v \rangle} \, dv = \sum_u \mathrm{vol}\{\sum_{i=1}^d \alpha_i(u - u_i) : 0 \le \alpha_i \le 1\} \frac{e^{\langle \lambda, u \rangle}}{\prod_{i=1}^d \langle \lambda, u - u_i \rangle}, \quad \lambda \in V^*,$$

where u ranges over the vertices of P and $u_1, \ldots, u_d$ are the neighboring vertices of u.[1] This expresses an entire function as a sum of meromorphic functions with hyperplane singularities. Following Brion, we can write the right-hand side as $\delta_{\Sigma(P)} \exp H_P$, where $\delta_{\Sigma(P)}$ denotes a certain canonical push-forward map of homogeneous degree $-d$ from piecewise power series on the normal fan $\Sigma(P)$ of P to power series on the vector space V^*. (Since we are dealing with a purely algebraic problem, we are free to work with formal power series rather than analytic functions.) To compute the value at the origin we may use de L'Hôpital's rule to obtain

$$(1.2) \qquad \mathrm{vol}\, P = \sum_u \mathrm{vol}\, \mathrm{conv}\{u, u_1, \ldots, u_d\} \frac{\langle \xi, u \rangle^d}{\prod_{i=1}^d \langle \xi, u - u_i \rangle}$$

for any $\xi \in V^*$ which is not perpendicular to an edge of P. Brion's formula follows from (and is in fact, up to a set of measure zero, equivalent to) the Lawrence-Varchenko decomposition expressing P as an alternating sum of

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[1]While the restriction to simple polytopes allows a more elegant statement, Brion's formula is not restricted to this case.

cones [Var87], [Law91]. Namely for ξ as before the characteristic function χ_P of P can be expressed as

$$(1.3) \qquad \chi_P = \sum_u (-1)^{\#\{i=1,\ldots,d:\langle\xi,u-u_i\rangle<0\}} \chi_{u+\sum_{i=1}^d \mathbb{R}_{\mathrm{sgn}\langle\xi,u-u_i\rangle}(u_i-u)},$$

where $\mathbb{R}_+ = [0,\infty)$ and $\mathbb{R}_- = (-\infty,0)$. The expression (1.1) for $\Re\lambda = \xi$ is obtained by multiplying (1.3) by $e^{\langle\xi,\cdot\rangle}$ (to make the right-hand side rapidly decreasing) and taking the Fourier transform. We note that both Brion's formula and the Lawrence-Varchenko decomposition had been obtained earlier by Arthur in the case where the normal fan of P is a Coxeter fan [Art76, §3] (cf. [Art05, Figure 11.1]). The volumes of such polytopes appear as weight factors in the weighted orbital integrals studied by Arthur. Moreover, piecewise power series and the canonical push-forward map are also ubiquitous in the theory of the trace formula, where they appear under the heading of (G, M)-families ([Art81, §6], [Art05, §17]).

The volume of an arbitrary d-dimensional polytope $P \subset V$ can also be computed in a more geometric way by the inductive formula (cf. [Sch93, Ch. 5])

$$(1.4) \qquad \mathrm{vol}\, P = \frac{1}{d} \sum_F h_F \,\mathrm{vol}\, F,$$

where F ranges over the set of facets (i.e. maximal faces) of P, we represent F as

$$F = \{v \in P : \langle\lambda_F, v\rangle = h_F = \max_{x\in P}\langle\lambda_F, x\rangle\}, \quad \lambda_F \in V^*$$

and use the identification $\lambda_F : \mathcal{V}(P - P)/\mathcal{V}(F - F) \to \mathbb{R}$ (equipped with the standard Lebesgue measure) to normalize the measure on F. Here, $\mathcal{V}(S)$ denotes the linear span of a subset $S \subset V$. This correlates (in the case where the origin lies in the interior of P) to the tessellation of P by the polytopes $\mathrm{conv}(F \cup \{0\})$. Slightly more generally, we have

$$\mathrm{vol}\, P = \frac{1}{d} \sum_F (h_F - \langle\lambda_F, u\rangle) \,\mathrm{vol}\, F$$

for any $u \in V$.

Choosing an auxiliary vector $u(F) \in F$ for *any* face F of P and iterating this procedure we obtain

(1.5)

$$\mathrm{vol}\, P = \frac{1}{d!} \sum_{F_0 \subset \ldots \subset F_d = P} \langle\lambda_1, F_0 - u(F_1)\rangle \langle\lambda_2, F_1 - u(F_2)\rangle \cdots \langle\lambda_d, F_{d-1} - u(F_d)\rangle$$

$$= \sum_{F_0 \subset \ldots \subset F_d = P} \mathrm{vol}\,\mathrm{conv}\{u(F_0), \ldots, u(F_d)\},$$

where the sum is over all ascending chains of faces of P with $\dim F_i = i$ for all i, the maximum of $\lambda_i \in V^*$ on F_i is assumed on F_{i-1}, and the basis of $\mathcal{V}(P - P)^*$ obtained by restricting $\lambda_1, \ldots, \lambda_d$ to $\mathcal{V}(P - P)$ is normalized with

respect to the dual measure. This corresponds to the triangulation of P by the simplices $\operatorname{conv}\{u(F_0), \ldots, u(F_d)\}$. For instance, taking $u(F)$ to be the barycenter of F, (1.5) reflects the barycentric subdivision. Combining (1.2) and (1.5) we obtain a first formula for the d-th order derivative $\delta_{\Sigma(P)} \exp H_P$ evaluated at the origin.

Suppose now that $P_1, \ldots, P_d \subset V$ are d polytopes. Recall that the volume of the polytope $a_1 P_1 + \cdots + a_d P_d$, $a_1, \ldots, a_d \geq 0$, is given by a homogeneous polynomial of degree d in $a_1, \ldots, a_d$. The coefficient of $a_1 \cdots a_d$ is called the *mixed volume* of $P_1, \ldots, P_d$ and is denoted by $\operatorname{mixvol}(P_1, \ldots, P_d)$. In particular, $\operatorname{mixvol}(P, \ldots, P) = d! \operatorname{vol} P$. Fix $\underline{\lambda} = (\lambda_1, \ldots, \lambda_d) \in (V^*)^d$ in general position. To this choice we associate a set $X_{\underline{\lambda}}$ of d-tuples $(e_1, \ldots, e_d)$ where e_i is an edge of P_i. Namely, $X_{\underline{\lambda}}$ is the set of all $(e_1, \ldots, e_d)$ for which there exists $\mu \in V^*$ such that $\langle \mu + \lambda_i, u \rangle = \max_{x \in P_i} \langle \mu + \lambda_i, x \rangle$ for all $u \in e_i$, $i = 1, \ldots, d$. An argument in [MS83] (cf. also [Sch94, Th. 4.1]) based on the Walkup-Wets lifting [WW69] shows that

$$(1.6) \quad \operatorname{mixvol}(P_1, \ldots, P_d) = \sum_{(e_1, \ldots, e_d) \in X_{\underline{\lambda}}} \operatorname{vol}\{\sum_{i=1}^d \alpha_i \vec{e}_i : 0 \leq \alpha_1, \ldots, \alpha_d \leq 1\},$$

where $\vec{e}_i = v_i - u_i$ if e_i connects the vertices u_i and v_i of P_i. Taking $P_1 = \cdots = P_d = P$ we obtain yet another formula for $\operatorname{vol} P$ which has the pleasant feature of expressing it directly in terms of the edge vectors. As it turns out, this formula is closely related to (1.5). We remark that in the case of a zonotope (i.e. a Minkowski sum of line segments) this formula takes a particularly simple form, since then the individual summands in (1.6) do not depend on the choice of $\underline{\lambda}$. This is McMullen's volume formula for zonotopes ([She74]). In the general case the combination of (1.2) and (1.6) provides a second algebraic derivative identity for $\exp H_P$.

In our algebraic generalization of these identities the function $\exp H_P$ is replaced by a general *compatible family* (with respect to $\Sigma(P)$). By this we mean, for an arbitrary finite-dimensional $\mathbb{C}$-algebra $\mathcal{E}$, a collection of $\mathcal{E}$-valued power series $\mathcal{A}_v$ on V^* indexed by the vertices v of P, such that $\mathcal{A}_v(0) = 1_{\mathcal{E}}$ for all v and $\mathcal{A}_{v_2} \mathcal{A}_{v_1}^{-1} \in \mathcal{E}[[v_2 - v_1]]$ for any pair of adjacent vertices v_1 and v_2 (where we view v_1 and v_2 as linear functions on V^*). We consider $\delta_{\Sigma(P)} \mathcal{A}$ and its value at zero $\mathcal{D}_{\Sigma(P)} \mathcal{A}$ and show that the two volume identities generalize to this case. For example, the generalization of the second identity (Theorem 5.1 below) is

$$\mathcal{D}_{\Sigma(P)} \mathcal{A} = \sum_u \operatorname{vol} \operatorname{conv}\{u, u_1, \ldots, u_d\} \frac{D_\xi^d \mathcal{A}_u(0)}{\prod_{i=1}^d \langle \xi, u - u_i \rangle}$$

$$= \frac{1}{d!} \sum_{(e_1, \ldots, e_d) \in X_{\underline{\lambda}}} \operatorname{vol}\{\sum_{i=1}^d \alpha_i \vec{e}_i : 0 \leq \alpha_1, \ldots, \alpha_d \leq 1\} \prod_{i=1}^d \frac{\mathcal{A}_{u_i} \mathcal{A}_{v_i}^{-1} - 1_{\mathcal{E}}}{u_i - v_i}(0).$$

The generalization of the first identity is Corollary 4.3 below. Slightly more general variants are obtained in Sections 6 and 7. Our two derivative identities express $\mathcal{D}_{\Sigma(P)}\mathcal{A}$ in terms of first-order derivatives of the functions $\mathcal{A}_{v_2}\mathcal{A}_{v_1}^{-1}$ for vertices v_1 and v_2 of P. The second identity has the additional feature that it involves only the first-order derivatives of the basic one-variable functions associated to adjacent vertices v_1 and v_2. A very special (but already important) case of this identity has been previously obtained by Arthur in [Art82, Lemma 7.1]. In essence this case is equivalent to McMullen's volume formula for zonotopes specialized to the case of Coxeter zonotopes.

For convenience, instead of working directly with polytopes, we will work with (polyhedral) fans, which provide a slightly more general context. As is well known, in the case of fans defined over the rational numbers the concepts of piecewise polynomial functions, the push-forward operation δ_Σ and Brion's formula have natural interpretations in terms of the equivariant cohomology of toric varieties (cf. [Ful93], [Bri96], [BV97]). However, we do not know if the general concept of a compatible family and the associated derivative formulas have a natural geometric meaning.

To finish this introduction, let us remark that our second derivative identity has been announced in [FLM09] in the case of Coxeter fans, and that the generalization to arbitrary simplicial fans was conjectured there (and formulated in more detail in the note [FL09]).

We now sketch the structure of the main part of this paper. We first review some basic properties of polyhedral fans and set up the notation (§2). Then we recall the notion of piecewise polynomial functions and the push-forward operation (§3). Next, we describe the setup of compatible families and derive the analogue of (1.4) and (1.5) in Section 4. The analogue of (1.6) is given in Section 5. For completeness we extend the results to nonsimplicial fans in Section 6. A slight refinement of the previous results in a relative context (corresponding to the consideration of linear projections of polytopes in the case of volume formulas) is provided in Section 7. Finally, in Section 8 we specialize the previous results to the case of fans defined by hyperplane arrangements (or, dually, zonotopes), which is the case used in [FLM].

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2. **Polyhedral fans**

In this section we recall the basic definitions and facts concerning poly-
hedral fans which will be needed in the sequel. Let V be a finite-dimensional
real vector space and let V^* be its dual space. For any subset $I \subset V^*$ we write
$\mathcal{V}(I)$ for the linear span of I in V^* and $I^\perp$ for the annihilator of I in V. We
also denote by $\operatorname{relint} I$ the *relative interior* of I, i.e. its interior as a subset of
$\mathcal{V}(I)$. By a *cone* $\mathcal{C}$ in V^* we will always mean a closed convex polyhedral cone,
i.e. the Minkowski sum of a finite number of rays $\mathbb{R}_{\geq 0}\lambda$, $\lambda \in V^*$, or equivalently
the intersection of finitely many half-spaces $\{\lambda \in V^* : \langle \lambda, v \rangle \geq 0\}$, $v \in V$. We
write $\operatorname{codim}\mathcal{C} = \dim \mathcal{C}^\perp$. Note that $\mathcal{C} \cap -\mathcal{C}$ is the maximal subspace of V^*
contained in $\mathcal{C}$. A *face* of $\mathcal{C}$ is a subcone of the form

$$\{\lambda \in \mathcal{C} : \langle \lambda, v \rangle = 0\}$$

for some $v \in V$ such that $\langle \cdot, v \rangle \leq 0$ on $\mathcal{C}$; in particular, $v \in (\mathcal{C} \cap -\mathcal{C})^\perp$.

A (complete) *fan*[2] Σ in V^* is a collection of cones (called the *faces* of Σ)
such that

(1) any face of a cone $\sigma \in \Sigma$ belongs to Σ;
(2) if $\sigma_1, \sigma_2 \in \Sigma$, then $\sigma_1 \cap \sigma_2$ is a face in both σ_1 and σ_2;
(3) $\cup\Sigma = V^*$.

Thus, the relative interiors of the faces of a fan Σ form a partition of V^*.
The minimal face $\mu = \cap\Sigma$ of Σ is a subspace of V^* called the *core* of Σ, and we
have $\mathcal{C} \cap -\mathcal{C} = \mu$ for all $\mathcal{C} \in \Sigma$. The integer $d = \dim \mu^\perp$ is called the dimension
of Σ.

A basic (although not the most general) example of a fan, which provides
most of the intuition, is the *normal fan* $\Sigma(P)$ of a polytope P in V. To define
it, recall that a face F of P is a subset of the form

$$F = \{v \in P : \langle \lambda, v \rangle = \max_{x \in P} \langle \lambda, x \rangle\}$$

for some $\lambda \in V^*$. By definition, the faces of $\Sigma(P)$ are

$$F^\natural := \left\{\lambda \in V^* : \langle \lambda, v \rangle = \max_{x \in P} \langle \lambda, x \rangle \text{ for all } v \in F\right\},$$

where F runs over the faces over P. The map $F \mapsto F^\natural$ is an inclusion reversing
bijection between the sets of faces of P and of $\Sigma(P)$, and we have $\operatorname{codim} F^\natural =$

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[2]We will not consider incomplete fans.

$\dim F$. In particular, the core of $\Sigma(P)$ is $P^\natural$, $P^{\natural\perp}$ is the linear span of $P - P$, and the dimension of $\Sigma(P)$ is the same as the dimension of P.

In general we denote by Σ_i the set of faces of Σ of codimension i. In particular, $\Sigma_d = \{\mu\}$ and we call Σ_0, Σ_1 and Σ_{d-1} the sets of *chambers*, *walls* and *rays* of Σ, respectively.[3] In the case $\Sigma = \Sigma(P)$, the chambers (resp. walls, rays) of Σ correspond to the vertices (resp. edges, facets) of P. For any chamber $\sigma \in \Sigma_0$ we denote by $\mathcal{R}(\sigma)$ the set of rays of Σ contained in σ.

Two chambers are called *adjacent* if they intersect in a wall. Any wall is contained in exactly two chambers (which are adjacent). We will write $\sigma \xrightarrow{\tau} \tilde{\sigma}$ for the ordered pair $(\sigma, \tilde{\sigma})$, if σ and $\tilde{\sigma}$ are adjacent chambers with a common wall $\tau = \sigma \cap \tilde{\sigma}$. We call $\sigma \xrightarrow{\tau} \tilde{\sigma}$ the *directed wall* ω above τ emerging from σ. If $\sigma \xrightarrow{\tau} \tilde{\sigma}$, then

$$(2.1) \qquad \operatorname{relint}\sigma \,\dot\cup\, \operatorname{relint}\tilde{\sigma} \,\dot\cup\, \operatorname{relint}\tau = \operatorname{relint}(\sigma \cup \tilde{\sigma}).$$

Given $\mathcal{C} \in \Sigma$ we say that $v \in V$ is positive with respect to $\mathcal{C}$ if $\langle \lambda, v \rangle > 0$ for any $\lambda \in \operatorname{relint}\mathcal{C}$. Given a directed wall $\omega : \sigma \xrightarrow{\tau} \tilde{\sigma}$, a *directed normal* for ω, or an ω-directed normal, is an element of $\tau^\perp$ which is positive with respect to σ. Such a vector is uniquely determined up to multiplication by a positive scalar.

An element of V^* is called Σ-*regular* (resp. Σ-*strongly regular*) if it lies in the complement of $\cup\Sigma_1$ (resp. $\cup_{\tau \in \Sigma_1} \mathcal{V}(\tau)$). Thus, λ is Σ-regular if and only if it belongs to a unique chamber, or equivalently to the interior of a chamber.

Given a face $\mathcal{C}$ of Σ the cones $\sigma^{\supset \mathcal{C}} := \sigma + \mathcal{V}(\mathcal{C})$, $\mathcal{C} \subset \sigma \in \Sigma$, comprise a fan $\Sigma^{\supset \mathcal{C}}$ in V^* which is called the *restricted fan*. Its core is $\mathcal{V}(\mathcal{C})$. If $\mathcal{C}' \supset \mathcal{C}$, then $(\Sigma^{\supset \mathcal{C}})^{\supset \mathcal{C}'} = \Sigma^{\supset \mathcal{C}'}$. Note that if $\Sigma = \Sigma(P)$ is the normal fan of a polytope $P \subset V$, then the restricted fan $\Sigma^{\supset F^\natural}$ is the normal fan $\Sigma(F)$ of the face F considered as a polytope in V.

An important special case, especially in the theory of piecewise polynomial functions, is the class of simplicial fans. We say that a convex polyhedral cone $\mathcal{C}$ is *simplicial* if there exist $\lambda_1, \ldots, \lambda_k \in V^*$, linearly independent modulo $\mathcal{C} \cap -\mathcal{C}$, such that

$$\mathcal{C} = (\mathcal{C} \cap -\mathcal{C}) + \mathbb{R}_{\geq 0}\lambda_1 + \cdots + \mathbb{R}_{\geq 0}\lambda_k.$$

Equivalently, $\mathcal{C}$ can be written as the intersection of hyperplanes $\{\lambda \in V^* : \langle \lambda, v_i \rangle = 0\}$, $i = 1, \ldots, l$, and half-spaces $\{\lambda \in V^* : \langle \lambda, v_i \rangle \geq 0\}$, $i = l+1, \ldots, m$, for linearly independent vectors $v_1, \ldots, v_m \in V$ and $0 \leq l \leq m$. A face of a simplicial cone is again simplicial. We say that a fan Σ is simplicial if each face of Σ is simplicial; it suffices to check this condition for the chambers of Σ. Equivalently, Σ is simplicial if and only if any $\sigma \in \Sigma_0$ has precisely d directed walls emerging from it. Also, Σ is simplicial if and only if $|\mathcal{R}(\sigma)| = d$ for all

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[3]Note that strictly speaking ρ is a one-dimensional ray only after projection along μ.

$\sigma \in \Sigma_0$. In particular, for a polytope $P \subset V$ the normal fan $\Sigma(P)$ is simplicial if and only if P is simple, i.e. if each vertex of P is contained in exactly $d = \dim P$ edges.

Suppose that Σ is simplicial. Then for any face $\mathcal{C}$ the restricted fan $\Sigma^{\supset \mathcal{C}}$ is also simplicial. Moreover, for any $\sigma \in \Sigma_0$ and $\rho \in \mathcal{R}(\sigma)$ there exists a unique wall $\tau \subset \sigma$ which does not contain ρ. Denote by $\mathrm{pr}_{\sigma;\rho}$ the projection of V^*/μ onto $\mathcal{V}(\rho)/\mu$ along $\mathcal{V}(\tau)/\mu$. Then $\sum_{\rho \in \mathcal{R}(\sigma)} \mathrm{pr}_{\sigma;\rho} = \mathrm{id}$.

3. Piecewise polynomial functions

Next, we recall the definition and the basic properties of the ring of piece-wise polynomial functions with respect to a fan Σ and the canonical push-forward operation δ_Σ of homogeneous degree $-d(\Sigma)$ in the case of simplicial fans following Billera and Brion (cf. [Bil89], [Bri97] for more details). Let $S = \mathrm{Sym}(V)$ be the ring of polynomial functions on V^*. For any $\lambda \in V^*$ we denote by D_λ the differential operator on S defined by λ (the directional derivative along λ). For any face $\mathcal{C} \in \Sigma$ we denote by $\mathfrak{I}_\mathcal{C}$ the ideal of S generated by the subspace $\mathcal{C}^\perp$ of V. In other words, $\mathfrak{I}_\mathcal{C}$ is the ideal of polynomials on V^* vanishing on $\mathcal{C}$.

Definition 3.1. A Σ-*piecewise polynomial function* is a (necessarily continuous) function on V^* whose restriction to any chamber (and hence to any face) of Σ is a polynomial. We denote by $\mathcal{P} = \mathcal{P}_\Sigma$ the graded S-algebra of Σ-piecewise polynomials.

We can view an element of $\mathcal{P}$ as a collection $(X_\sigma)_{\sigma \in \Sigma_0}$ of elements of S (obtained by restricting the function to the chambers), such that $X_{\sigma_1} - X_{\sigma_2} \in \mathfrak{I}_{\sigma_1 \cap \sigma_2}$ for any two chambers σ_1, σ_2. It suffices to check this condition for σ_1, σ_2 adjacent.

Remark 3.2. The algebra $\mathcal{P}$ has especially nice properties in the case where Σ is simplicial. In this case, $\mathcal{P}$ is free over S. Also, $\mathcal{P}$ is generated as an algebra by its degree 1 elements $\mathcal{P}_1$ (the Σ-piecewise linear forms) and it is possible to describe its algebra structure in terms of the combinatorics of Σ. Moreover, $\dim \mathcal{P}_1 = |\Sigma_{d-1}|$ (cf. [Bil89], [Bri97]). We will not use these facts here.

More generally, if M is an S-module, we define $M_\Sigma := M \otimes_S \mathcal{P}_\Sigma$ to be the $\mathcal{P}_\Sigma$-module of Σ-piecewise elements of M. If M is flat over S, then an element of M_Σ can be described as a collection $(m_\sigma)_{\sigma \in \Sigma_0}$ of elements of M such that $m_{\sigma_1} - m_{\sigma_2} \in \mathfrak{I}_{\sigma_1 \cap \sigma_2} M$ for any $\sigma_1, \sigma_2 \in \Sigma_0$.

For any $\mathcal{C} \in \Sigma$ there is a canonical homomorphism

$$\mathcal{P}_\Sigma \to \mathcal{P}_{\Sigma^{\supset \mathcal{C}}}, \quad f \mapsto f^{\supset \mathcal{C}},$$

given by $f^{\supset \mathcal{C}}_{\sigma \supset c} = f_\sigma$ for any $\sigma \in \Sigma_0$, $\sigma \supset \mathcal{C}$. Thus, for any S-module M we get a canonical restriction map

$$M_\Sigma \to M_{\Sigma \supset c}, \quad m \mapsto m^{\supset \mathcal{C}},$$

which is a homomorphism with respect to the pull-back $\mathcal{P}_\Sigma$-structure on $M_{\Sigma \supset c}$.

Let $\Sigma = \Sigma(P)$ be the normal fan of a polytope $P \subset V$. The *support function* H_P on V^* defined by $H_P(\lambda) = \max_{x \in P} \langle \lambda, x \rangle$ is a Σ-piecewise linear form. The function H_P corresponds to the vector $X = (X_\sigma)_{\sigma \in \Sigma_0}$ consisting of the vertices of P indexed by the corresponding chambers of Σ. Conversely, a Σ-piecewise linear form $X = (X_\sigma)_{\sigma \in \Sigma_0}$ defines a polytope with normal fan Σ if and only if $X_\sigma - X_{\tilde\sigma}$ is positive with respect to σ for any directed wall $\omega : \sigma \xrightarrow{\tau} \tilde\sigma$. If we relax the positivity condition to allow the limiting case $X_\sigma = X_{\tilde\sigma}$, we obtain all polytopes P with $H_P \in \mathcal{P}_\Sigma$.

Suppose that Σ is a fan with core μ. Fix a Haar measure on $\mu^\perp$. We write

$$\beta(v_1 \wedge \cdots \wedge v_d) = \mathrm{vol}\left\{ \sum_{i=1}^{d} \alpha_i v_i : 0 \le \alpha_1, \ldots, \alpha_d \le 1 \right\}, \quad v_1, \ldots, v_d \in \mu^\perp.$$

For our purposes the dual Haar measure on V^*/μ is normalized in such a way that for any basis $v_1, \ldots, v_d$ of $\mu^\perp$ and the corresponding dual basis $\lambda_1, \ldots, \lambda_d$ of V^*/μ we have

$$\beta^*(\lambda_1 \wedge \cdots \wedge \lambda_d)\beta(v_1 \wedge \cdots \wedge v_d) = 1,$$

where

$$\beta^*(\lambda_1 \wedge \cdots \wedge \lambda_d) = \mathrm{vol}\left\{ \sum_{i=1}^{d} \alpha_i \lambda_i : 0 \le \alpha_1, \ldots, \alpha_d \le 1 \right\}.$$

Assume that Σ is simplicial. For $\sigma \in \Sigma_0$ let $\omega_i : \sigma \xrightarrow{\tau_i} \sigma_i$, $i = 1, \ldots, d$, be the directed walls emerging from σ. Set

$$\theta_\sigma = \theta_\sigma^\Sigma = \frac{v_1 \cdots v_d}{\beta(v_1 \wedge \cdots \wedge v_d)} \in S,$$

where v_i is a directed normal of ω_i. As the notation suggests, θ_σ depends only on σ and not on the choice of the v_i's or the order of the ω_i's. (For instance we could take $v_1, \ldots, v_d$ to be the dual basis of λ_ρ, $\rho \in \mathcal{R}(\sigma)$.)

A key elementary property of these polynomials is that for $\sigma \xrightarrow{\tau} \tilde\sigma$ and $0 \ne v \in \tau^\perp$ we have

(3.1) $$\frac{\theta_\sigma}{v} = -\frac{\theta_{\tilde\sigma}}{v} \text{ on } \mathcal{V}(\tau).$$

This property forms the basis for the following definition.

Definition 3.3. For any simplicial fan Σ (and choice of Haar measure on $\mu^\perp$) denote by $\delta = \delta_\Sigma : \mathcal{P}_\Sigma \to S$ the S-linear map, homogeneous of degree

$-d(\Sigma)$, which is defined by

$$X = (X_\sigma)_{\sigma \in \Sigma_0} \mapsto \sum_{\sigma \in \Sigma_0} \frac{X_\sigma}{\theta_\sigma}.$$

We call δ_Σ the *push-forward map* associated to Σ. We also write $\mathcal{D}_\Sigma X = (\delta_\Sigma X)(0)$.

The map δ_Σ is indeed well defined, since by (3.1) the apparent hyperplane singularities cancel in pairs (cf. [Bri97, Th. 2.2]). In particular, $\delta \mathcal{P}_k = 0$ for $k < d$. Extending scalars, we get for any S-module M an S-linear map

$$\delta = \delta_{\Sigma;M} : M_\Sigma \to M.$$

We may use de L'Hôpital's rule to compute

$$\mathcal{D}_\Sigma X = \frac{1}{d!} \sum_{\sigma \in \Sigma_0} \frac{D_\lambda^d X_\sigma(0)}{\theta_\sigma(\lambda)}$$

for any Σ-strongly regular $\lambda \in V^*$. We also remark that

$$(3.2) \qquad \sum_{\sigma \in \Sigma_0} \frac{D_\lambda^k X_\sigma(0)}{\theta_\sigma(\lambda)} = 0$$

for all $k < d$, since by (3.1) the left-hand side is a regular function of λ of degree $k - d$.

4. Compatible families and the first formula

Let $\mathbb{C}[[V]]$ denote the algebra of formal power series in V, i.e. the completion of $S \otimes \mathbb{C}$ at the origin. The constant term homomorphism $\mathbb{C}[[V]] \to \mathbb{C}$ will be denoted by $f \mapsto f(0)$ and for any $\lambda \in V^*$ we denote by $D_\lambda : \mathbb{C}[[V]] \to \mathbb{C}[[V]]$ the (formal) λ-directional derivative. For any subspace $U \subset V$ we will identify $\mathbb{C}[[U]]$ with a subalgebra of $\mathbb{C}[[V]]$. Thus, $\mathbb{C}[[U]] = \{f \in \mathbb{C}[[V]] : D_\lambda f = 0$ for all $\lambda \in U^\perp\}$. Let $\mathcal{E}$ be a finite-dimensional algebra over $\mathbb{C}$. Consider the $\mathbb{C}[[V]]$-algebra $\mathcal{E}[[V]] := \mathbb{C}[[V]] \otimes \mathcal{E}$. As before, the canonical homomorphism $\mathcal{E}[[V]] \to \mathcal{E}$ will be denoted by $f \mapsto f(0)$, and similarly for $D_\lambda : \mathcal{E}[[V]] \to \mathcal{E}[[V]]$, $\lambda \in V^*$. Obviously, in this situation we can consider the module of $\mathcal{E}$-valued Σ-piecewise power series $\mathcal{E}[[V]]_\Sigma$ and the push-forward map $\delta_\Sigma : \mathcal{E}[[V]]_\Sigma \to \mathcal{E}[[V]]$.

We can now define the main object of interest.

Definition 4.1. A Σ-*compatible family* $\mathcal{A} = (\mathcal{A}_\sigma)_{\sigma \in \Sigma_0}$ with values in $\mathcal{E}$ consists of elements $\mathcal{A}_\sigma \in \mathcal{E}[[V]]$, $\sigma \in \Sigma_0$, such that

$$(4.1) \qquad\qquad\qquad \mathcal{A}_\sigma(0) = 1_\mathcal{E} \text{ for all } \sigma \in \Sigma_0,$$

so that in particular, $\mathcal{A}_\sigma$ is invertible in $\mathcal{E}[[V]]$ for all $\sigma \in \Sigma_0$, and

$$(4.2) \qquad \mathcal{A}_{\sigma_1 \to \sigma_2} := \mathcal{A}_{\sigma_1} \mathcal{A}_{\sigma_2}^{-1} \in \mathcal{E}[[(\sigma_1 \cap \sigma_2)^\perp]] \text{ for any } \sigma_1, \sigma_2 \in \Sigma_0.$$

For a directed wall $\omega : \sigma \xrightarrow{\tau} \tilde{\sigma}$ we write $\mathcal{A}_\omega = \mathcal{A}_{\sigma \to \tilde{\sigma}} \in \mathcal{E}[[\tau^\perp]]$.

Usually we suppress $\mathcal{E}$ if it is clear from the context. Note that it suffices to check condition (4.2) for σ_1, σ_2 adjacent.

A compatible family $\mathcal{A}$ is in particular a Σ-piecewise $\mathcal{E}$-valued power series. Indeed, if $\omega : \sigma \xrightarrow{\tau} \tilde{\sigma}$, then $\mathcal{A}_\omega(0) = 1_\mathcal{E}$ and therefore

$$(4.3) \qquad \mathcal{A}_\omega - 1_\mathcal{E} \in \tau^\perp \mathcal{E}[[\tau^\perp]] \subset \tau^\perp \mathcal{E}[[V]].$$

Hence,

$$\mathcal{A}_\sigma - \mathcal{A}_{\tilde{\sigma}} = (\mathcal{A}_\omega - 1_\mathcal{E})\mathcal{A}_{\tilde{\sigma}} \in \tau^\perp \mathcal{E}[[V]].$$

If Σ is simplicial, we can therefore consider $\delta_\Sigma \mathcal{A}$ and the canonical derivative $\mathcal{D}_\Sigma \mathcal{A}$. Since δ_Σ is S-linear, we have $\mathcal{D}_\Sigma \mathcal{A} = \mathcal{D}_\Sigma \mathcal{A}_{\sigma \to \sigma_0}$ for any $\sigma_0 \in \Sigma_0$.

Important examples of scalar-valued compatible families, i.e. with $\mathcal{E} = \mathbb{C}$, are given by $\mathcal{A}_\sigma = \exp X_\sigma$, $\sigma \in \Sigma_0$, for Σ-piecewise linear forms $X \in \mathcal{P}_1 \otimes \mathbb{C}$. In particular, suppose that $\Sigma = \Sigma(P)$ for a polytope $P \subset V$. Then $\mathcal{A}_P = \exp H_P$ is a scalar-valued compatible family with respect to Σ. If P is simple, then by Brion's formula (1.1) the push-forward $\delta_\Sigma \mathcal{A}_P$ is the Taylor series of

$$\lambda \mapsto \int_P e^{\langle \lambda, v \rangle} \, dv,$$

where the measure on P is obtained from the measure on $\mu^\perp$ by translation. In particular, $\mathcal{D}_{\Sigma(P)} \mathcal{A}_P = \operatorname{vol} P$.

We now turn to the task of expressing $\mathcal{D}_\Sigma \mathcal{A}$ in terms of first-order derivatives of the power series $\mathcal{A}_{\sigma_1 \to \sigma_2}$. If $\mathcal{C} \in \Sigma$ and $\lambda \in \mathcal{V}(\mathcal{C})$, then for any $\sigma_0 \in \Sigma_0$ the derivative $[D_\lambda \mathcal{A}_{\sigma \to \sigma_0}](0)$ is independent of the choice of a chamber $\sigma \in \Sigma_0$ containing $\mathcal{C}$. We will write this derivative as $[D_\lambda \mathcal{A}_{\mathcal{C} \to \sigma_0}](0)$.

The following proposition generalizes the inductive formula (1.4) for the computation of volumes of polytopes. Note that if $\mathcal{A}$ is a compatible family with respect to Σ and $\mathcal{C} \in \Sigma$, then $\mathcal{A}^{\supset \mathcal{C}}$ is a compatible family with respect to $\Sigma^{\supset \mathcal{C}}$.

PROPOSITION 4.2. *Suppose that Σ is a simplicial fan with core μ and $\mathcal{A}$ is a compatible family with respect to Σ. For any ray $\rho \in \Sigma_{d-1}$ fix a vector $\lambda_\rho \in \operatorname{relint} \rho$. This choice identifies $\mu^\perp / \rho^\perp$ with $\mathbb{R}$ and determines a Haar measure on $\rho^\perp$ such that the induced quotient measure on $\mu^\perp / \rho^\perp$ is the standard Lebesgue measure on $\mathbb{R}$. Using these measures to normalize $\mathcal{D}_{\Sigma^{\supset \rho}}$, we have*

$$\mathcal{D}_\Sigma \mathcal{A} = \frac{1}{d} \sum_{\rho \in \Sigma_{d-1}} \mathcal{D}_{\Sigma^{\supset \rho}}(\mathcal{A}^{\supset \rho}) \left[D_{\lambda_\rho} \mathcal{A}_{\rho \to \sigma_0} \right](0)$$

for any $\sigma_0 \in \Sigma_0$.

Proof. For a Σ-strongly regular element $\lambda \in V^* / \mu$ we can write

$$\mathcal{D}_\Sigma \mathcal{A} = \frac{1}{d!} \sum_{\sigma \in \Sigma_0} \frac{D_\lambda^d \mathcal{A}_{\sigma \to \sigma_0}}{\theta_\sigma(\lambda)}(0).$$

Note that the numerator is well defined since $\mathcal{A}_{\sigma\to\sigma_0} \in \mathcal{E}[[\mu^\perp]]$. For any ray $\rho \in \Sigma_{d-1}$ fix an auxiliary chamber $\sigma_\rho \in \Sigma_0$ containing ρ. Writing $\lambda = \sum_{\rho\in\mathcal{R}(\sigma)} \mathrm{pr}_{\sigma;\rho}\lambda$ and $\mathcal{A}_{\sigma\to\sigma_0} = \mathcal{A}_{\sigma\to\sigma_\rho}\mathcal{A}_{\sigma_\rho\to\sigma_0}$ we get

$$\frac{1}{d!} \sum_{\sigma\in\Sigma_0} \frac{D_\lambda^{d-1}\sum_{\rho\in\mathcal{R}(\sigma)} D_{\mathrm{pr}_{\sigma;\rho}(\lambda)}(\mathcal{A}_{\sigma\to\sigma_\rho}\mathcal{A}_{\sigma_\rho\to\sigma_0})}{\theta_\sigma(\lambda)}(0).$$

Note that

$$\mathrm{pr}_{\sigma;\rho}(\lambda) = \frac{\theta_\sigma(\lambda)}{\theta_{\sigma\supset\rho}^{\supset\rho}(\lambda)}\lambda_\rho,$$

where we write $\theta_{\sigma\supset\rho}^{\supset\rho} = \theta_{\sigma\supset\rho}^{\Sigma\supset\rho}$. We obtain

$$\frac{1}{d!} \sum_{\sigma\in\Sigma_0} \sum_{\rho\in\mathcal{R}(\sigma)} \frac{D_\lambda^{d-1}D_{\lambda_\rho}(\mathcal{A}_{\sigma\to\sigma_\rho}\mathcal{A}_{\sigma_\rho\to\sigma_0})}{\theta_{\sigma\supset\rho}^{\supset\rho}(\lambda)}(0).$$

Since $\mathcal{A}_{\sigma\to\sigma_\rho} \in \mathcal{E}[[\rho^\perp]]$, this equals

$$\frac{1}{d!} \sum_{\sigma\in\Sigma_0} \sum_{\rho\in\mathcal{R}(\sigma)} \frac{D_\lambda^{d-1}(\mathcal{A}_{\sigma\to\sigma_\rho}D_{\lambda_\rho}\mathcal{A}_{\sigma_\rho\to\sigma_0})}{\theta_{\sigma\supset\rho}^{\supset\rho}(\lambda)}(0),$$

which we can rewrite as the sum over $\rho \in \Sigma_{d-1}$ of

$$\frac{1}{d!} \sum_{\sigma\in\Sigma_0^{\supset\rho}} \frac{D_\lambda^{d-1}(\mathcal{A}_{\sigma\to\sigma_\rho}^{\supset\rho}D_{\lambda_\rho}\mathcal{A}_{\sigma_\rho\to\sigma_0})}{\theta_\sigma^{\supset\rho}(\lambda)}(0).$$

Applying the Leibniz rule we get

$$\frac{1}{d!} \sum_{\rho\in\Sigma_{d-1}} \sum_{i=0}^{d-1} \binom{d-1}{i} \sum_{\sigma\in\Sigma_0^{\supset\rho}} \frac{[D_\lambda^{d-1-i}\mathcal{A}_{\sigma\to\sigma_\rho^{\supset\rho}}^{\supset\rho}](0)}{\theta_\sigma^{\supset\rho}(\lambda)}[D_\lambda^i D_{\lambda_\rho}\mathcal{A}_{\sigma_\rho\to\sigma_0}](0).$$

By (3.2) (applied to $\Sigma^{\supset\rho}$), only $i = 0$ contributes and we obtain

$$\mathcal{D}_\Sigma\mathcal{A} = \frac{1}{d} \sum_{\rho\in\Sigma_{d-1}} [\delta_{\Sigma\supset\rho}\mathcal{A}_{\sigma\to\sigma_\rho^{\supset\rho}}^{\supset\rho}](0)\left[D_{\lambda_\rho}\mathcal{A}_{\sigma_\rho\to\sigma_0}\right](0)$$

$$= \frac{1}{d} \sum_{\rho\in\Sigma_{d-1}} \mathcal{D}_{\Sigma\supset\rho}\mathcal{A}^{\supset\rho}\left[D_{\lambda_\rho}\mathcal{A}_{\rho\to\sigma_0}\right](0)$$

as required. $\qquad\qquad\square$

    A *selector* $\mathfrak{s}$ of a fan Σ is a map $\Sigma \to \Sigma_0$ with $\mathfrak{s}(\mathcal{C}) \supset \mathcal{C}$ for all $\mathcal{C} \in \Sigma$. One way to specify $\mathfrak{s}$ is as follows: fix a Σ-strongly regular $\lambda_0 \in V^*$ and define $\mathfrak{s}_{\lambda_0}$ by the property that $\mathfrak{s}_{\lambda_0}(\mathcal{C})$ is the unique chamber $\sigma \in \Sigma_0$ containing $\mathcal{C}$ such that $\sigma \cap (\lambda_0 + \mathcal{V}(\mathcal{C})) \neq \emptyset$.

A *flag* $\mathfrak{f}$ of Σ is a chain $\rho_0 \supset \rho_1 \supset \cdots \supset \rho_d$ with $\rho_i \in \Sigma_i$. In particular, $\rho_d = \mu(\Sigma)$. We denote by $\mathfrak{G} = \mathfrak{G}(\Sigma)$ the set of flags of Σ. For any flag $\mathfrak{f}$ we set

$$\partial_{\mathfrak{f}} \mathcal{A} = \partial_{\mathfrak{f}}^{\mathfrak{s}} \mathcal{A} = \frac{1}{d!} \frac{\left[D_{\lambda_{\rho_0}} \mathcal{A}_{\rho_0 \to \mathfrak{s}(\rho_1)} \right](0) \ldots \left[D_{\lambda_{\rho_{d-1}}} \mathcal{A}_{\rho_{d-1} \to \mathfrak{s}(\rho_d)} \right](0)}{\beta^*(\lambda_{\rho_0} \wedge \cdots \wedge \lambda_{\rho_{d-1}})}$$

with auxiliary vectors $\lambda_{\rho_i} \in \mathrm{relint}\,\rho_i$. Note that since $\mathcal{A}_{\rho_{i-1} \to \mathfrak{s}(\rho_i)} \in \mathcal{E}[[\rho_i^\perp]]$, the directional derivative $[D_{\lambda_{\rho_{i-1}}} \mathcal{A}_{\rho_{i-1} \to \mathfrak{s}(\rho_i)}](0)$ depends only on the image of $\lambda_{\rho_{i-1}}$ in the one-dimensional space $\mathcal{V}(\rho_{i-1})/\mathcal{V}(\rho_i)$. For a different choice of the vectors λ_{ρ_i} these images clearly change by positive factors, and therefore the entire expression $\partial_{\mathfrak{f}}^{\mathfrak{s}} \mathcal{A}$ is independent of the choice of the λ_{ρ_i}'s. However, it depends on the choice of $\mathfrak{s}$.

By applying Proposition 4.2 inductively we infer the following formula for $\mathcal{D}_\Sigma \mathcal{A}$ which generalizes (1.5) (in the case where $u(F)$ is a vertex of F).

COROLLARY 4.3. *For any compatible family $\mathcal{A}$ with respect to a simplicial fan Σ and any choice of a selector $\mathfrak{s} : \Sigma \to \Sigma_0$ we have*

$$\mathcal{D}_\Sigma \mathcal{A} = \sum_{\mathfrak{f} \in \mathfrak{G}(\Sigma)} \partial_{\mathfrak{f}}^{\mathfrak{s}} \mathcal{A}.$$

5. The second formula

Let Σ be a fan. We say that a d-tuple of cones $\underline{\mathcal{C}} = (\mathcal{C}_1, \ldots, \mathcal{C}_d)$ is *transversal* if the sum $\sum_{i=1}^d \mathcal{C}_i^\perp$ is direct. Suppose that $\underline{\tau} = (\tau_1, \ldots, \tau_d) \in \Sigma_1^d$ is a transversal d-tuple of walls of Σ. For each $i = 1, \ldots, d$ choose a directed wall $\omega_i : \sigma_i \xrightarrow{\tau_i} \tilde{\sigma}_i$ and a directed normal v_i for ω_i; by assumption $v_1, \ldots, v_d$ are linearly independent. Let $\mathcal{A}$ be a compatible family with respect to Σ. Using (4.3) we may set

$$\Delta_{\underline{\tau}} \mathcal{A} = \frac{1}{d!} \beta(v_1 \wedge \cdots \wedge v_d) \frac{\mathcal{A}_{\omega_1} - 1_{\mathcal{E}}}{v_1}(0) \cdots \frac{\mathcal{A}_{\omega_d} - 1_{\mathcal{E}}}{v_d}(0) \in \mathcal{E}.$$

Note that $\Delta_{\underline{\tau}} \mathcal{A}$ depends only on $\underline{\tau}$ and not on the choice of the ω_i's or the v_i's. For later use it will be convenient to rewrite this definition as

$$\Delta_{\underline{\tau}} \mathcal{A} = \frac{1}{d!} \frac{\beta(v_1 \wedge \cdots \wedge v_d)}{\langle \lambda_1, v_1 \rangle \ldots \langle \lambda_d, v_d \rangle} D_{\lambda_1} \mathcal{A}_{\omega_1}(0) \cdots D_{\lambda_d} \mathcal{A}_{\omega_d}(0),$$

where $\lambda_i \in V^*$ are arbitrary subject to $\langle \lambda_i, v_i \rangle \neq 0$ for all i.

Given a fan Σ in a vector space U and a linear surjective map $p : U \to U'$, the *quotient fan* in U' is defined as the common refinement of the collection of cones $p(\sigma)$, $\sigma \in \Sigma$ (cf. [KSZ91], [BS94]). In the case where $U = V^*$ and $\Sigma = \Sigma(P)$ for some polytope $P \subset V$, the quotient fan is the normal fan of the *fiber polytope* of P, in the sense of Billera-Sturmfels ([BS92]), with respect to the projection $V \to V/(\mathrm{Ker}\,p)^\perp = (\mathrm{Ker}\,p)^*$ ([KSZ91, Prop. 2.3]).

Consider V^* embedded diagonally in $(V^*)^d$ and let $p : (V^*)^d \to (V^*)^d/V^*$ be the canonical projection. We identify the set Σ^d of d-tuples of faces of

Σ with a fan in $(V^*)^d$ where the face corresponding to $\underline{\mathcal{C}} = (\mathcal{C}_1, \ldots, \mathcal{C}_d)$ is $\pi(\underline{\mathcal{C}}) = \mathcal{C}_1 \times \cdots \times \mathcal{C}_d$. Note that

(5.1)

the faces of a cone $p(\pi(\underline{\mathcal{C}}))$ are of the form $p(\pi(\underline{\mathcal{C}}'))$ for $\mathcal{C}_i' \subset \mathcal{C}_i$, $i = 1, \ldots, d$.

Denote by $\Sigma^\clubsuit$ the quotient fan of Σ^d under p, i.e. the common refinement of the cones $p(\pi(\underline{\mathcal{C}}))$, $\underline{\mathcal{C}} \in \Sigma^d$. It is a fan in the vector space $(V^*)^d/V^*$. An equivalent way to describe this fan is obtained by noting that $\underline{\lambda} = (\lambda_1, \ldots, \lambda_d)$ $(\bmod\ V^*) \in p(\pi(\underline{\mathcal{C}}))$ if and only if $\cap_{i=1}^d (\mathcal{C}_i - \lambda_i) \neq \emptyset$. This means that the partition of $(V^*)^d/V^*$ obtained from $\Sigma^\clubsuit$ is given by the fibers of the map attaching to any $\underline{\lambda} \in (V^*)^d/V^*$ the set $\{\underline{\mathcal{C}} \in \Sigma^d : \cap_{i=1}^d (\mathcal{C}_i - \lambda_i) \neq \emptyset\}$.

We have

$$\operatorname{codim} p(\pi(\underline{\mathcal{C}})) = \sum_{i=1}^d \operatorname{codim} \mathcal{C}_i - \dim \sum_{i=1}^d \mathcal{C}_i^\perp = \sum_{i=1}^d \dim \mathcal{C}_i^\perp - \dim \sum_{i=1}^d \mathcal{C}_i^\perp.$$

Therefore, the chambers of $\Sigma^\clubsuit$ are obtained from the cones $p(\pi(\underline{\mathcal{C}}))$ for transversal $\underline{\mathcal{C}}$ by forming minimal intersections not contained in any linear subspace. Analogously, the walls of $\Sigma^\clubsuit$ are obtained from the d-tuples $\underline{\mathcal{C}} \in \Sigma^d$ such that $\dim \sum_{i=1}^d \mathcal{C}_i^\perp = \sum_{i=1}^d \dim \mathcal{C}_i^\perp - 1$ by forming minimal intersections which span a hyperplane.

For any $\sigma \in \Sigma_0^\clubsuit$ set

$$\mathcal{X}_\sigma = \{\underline{\tau} \in \Sigma_1^d \text{ transversal} : p(\pi(\underline{\tau})) \supset \sigma\}.$$

Equivalently, for any $\underline{\lambda} \in \operatorname{relint} \sigma$ we can write

$$\mathcal{X}_\sigma = \{\underline{\tau} \in \Sigma_1^d \text{ transversal} : \cap_{i=1}^d (\tau_i - \lambda_i) \neq \emptyset\}.$$

Moreover, if $\underline{\tau} \in \mathcal{X}_\sigma$, then $\cap_{i=1}^d (\tau_i - \lambda_i)$ is a translate of μ. Also,

(5.2)

if $\underline{\tau} \in \Sigma_1^d$ is transversal, then p induces an isomorphism of vector spaces

$$\text{between } \mathcal{V}(\pi(\underline{\tau}))/\mu^d \text{ and } (V^*)^d/(V^* + \mu^d).$$

THEOREM 5.1. *Let $\mathcal{A}$ be a compatible family with respect to a simplicial fan Σ. Then for any choice of $\sigma \in \Sigma_0^\clubsuit$ we have*

(5.3)
$$\mathcal{D}_\Sigma \mathcal{A} = \sum_{\underline{\tau} \in \mathcal{X}_\sigma} \Delta_{\underline{\tau}} \mathcal{A}.$$

Remark 5.2. In the case where $\Sigma = \Sigma(P)$ for a polytope $P \subset V$ and $\mathcal{A} = \exp H_P$, (5.3) reduces to the expression (1.6) for the volume of P (for $P_1 = \cdots = P_d = P$).

Denote the right-hand side of (5.3) by $\mathcal{R}_\sigma \mathcal{A}$. We first show the following.

LEMMA 5.3. *$\mathcal{R}_\sigma \mathcal{A}$ is independent of $\sigma \in \Sigma_0^\clubsuit$.*

Proof. Suppose that $\tilde{\sigma} \xrightarrow{\Omega} \hat{\sigma}$ with $\tilde{\sigma}, \hat{\sigma} \in \Sigma_0^{\clubsuit}$ and a wall Ω of $\Sigma^{\clubsuit}$. We will show that $\mathcal{R}_{\tilde{\sigma}}\mathcal{A} = \mathcal{R}_{\hat{\sigma}}\mathcal{A}$. Consider $\mathfrak{C} = \cup_{i=1}^d \mathfrak{C}^i$, where

$$\mathfrak{C}^i = \{\underline{\nu} = (\nu_1, \ldots, \nu_d) \in \Sigma^d : \nu_i \in \Sigma_2, \nu_j \in \Sigma_1 \text{ for all } j \neq i,$$
$$\dim \sum_{j \neq i} \nu_j^\perp = d - 1, \sum_j \nu_j^\perp = \mu^\perp \text{ and } \Omega \subset p(\pi(\underline{\nu}))\}.$$

For any $\underline{\nu} \in \mathfrak{C}$ set

$$\mathfrak{D}(\underline{\nu}) = \{\underline{\tau} \in \Sigma_1^d \text{ transversal} : \tau_j \supset \nu_j \text{ for all } j\}$$

and define

$$\mathfrak{D} = \cup_{\underline{\nu} \in \mathfrak{C}} \mathfrak{D}(\underline{\nu}) = \{\underline{\tau} \in \Sigma_1^d \text{ transversal} : \Omega \text{ lies on a proper face of } p(\pi(\underline{\tau}))\}.$$

Here the second equality follows from (5.1). We first claim that

(5.4) $$\mathcal{X}_{\tilde{\sigma}} \setminus \mathfrak{D} = \mathcal{X}_{\hat{\sigma}} \setminus \mathfrak{D}.$$

(5.5) $$\text{The sets } \mathfrak{D}(\underline{\nu}), \underline{\nu} \in \mathfrak{C}, \text{ are disjoint.}$$

(5.6) $$\mathfrak{D} \text{ is the disjoint union of } \mathcal{X}_{\tilde{\sigma}} \cap \mathfrak{D} \text{ and } \mathcal{X}_{\hat{\sigma}} \cap \mathfrak{D}.$$

Suppose that $\underline{\tau}$ is transversal. To prove (5.4), we will show that $\underline{\tau} \in \mathcal{X}_{\tilde{\sigma}} \cup \mathcal{X}_{\hat{\sigma}} \setminus \mathfrak{D}$ implies $\underline{\tau} \in \mathcal{X}_{\tilde{\sigma}} \cap \mathcal{X}_{\hat{\sigma}}$. Indeed, if $\underline{\tau} \in \mathcal{X}_{\tilde{\sigma}} \cup \mathcal{X}_{\hat{\sigma}}$, then $p(\pi(\underline{\tau})) \supset \Omega$. Moreover, if also $\underline{\tau} \notin \mathfrak{D}$, then $\operatorname{relint} \Omega \subset \operatorname{relint} p(\pi(\underline{\tau}))$. On the other hand, since $\underline{\tau}$ is transversal, $p(\pi(\underline{\tau}))$ is a union of chambers of $\Sigma^{\clubsuit}$. Therefore we have $\tilde{\sigma}, \hat{\sigma} \subset p(\pi(\underline{\tau}))$ as well. Hence, $\underline{\tau} \in \mathcal{X}_{\tilde{\sigma}} \cap \mathcal{X}_{\hat{\sigma}}$. This proves (5.4).

To prove (5.5), assume on the contrary that $\underline{\tau} \in \mathfrak{D}(\underline{\nu}^{(1)}) \cap \mathfrak{D}(\underline{\nu}^{(2)})$ with distinct $\underline{\nu}^{(1)}, \underline{\nu}^{(2)} \in \mathfrak{C}$. Then $\Omega \subset p(\pi(\underline{\nu}^{(1)})) \cap p(\pi(\underline{\nu}^{(2)}))$ and since $\pi(\underline{\nu}^{(1)}), \pi(\underline{\nu}^{(2)}) \subset \pi(\underline{\tau})$ we get from (5.2) that $\Omega \subset p(\pi(\underline{\tilde{\nu}}))$ for $\tilde{\nu}_j = \nu_j^{(1)} \cap \nu_j^{(2)}$. However, $\operatorname{codim} p(\pi(\underline{\tilde{\nu}})) \geq 2$, which is a contradiction.

Finally, suppose that $\underline{\tau} \in \mathfrak{D}$. Then (2.1) implies that

$$p(\pi(\underline{\tau})) \cap \operatorname{relint}(\tilde{\sigma} \cup \hat{\sigma}) \neq \emptyset.$$

But $\operatorname{relint}(\tilde{\sigma} \cup \hat{\sigma})$ is open in $(V^*)^d/V^*$, and therefore also $\operatorname{relint} p(\pi(\underline{\tau})) \cap \operatorname{relint}(\tilde{\sigma} \cup \hat{\sigma}) \neq \emptyset$. However, $\operatorname{relint} p(\pi(\underline{\tau})) \cap \Omega = \emptyset$ because Ω is assumed to lie on a proper face of $p(\pi(\underline{\tau}))$. Hence, $\operatorname{relint} p(\pi(\underline{\tau}))$ intersects $\operatorname{relint} \tilde{\sigma}$ or $\operatorname{relint} \hat{\sigma}$. It follows that $p(\pi(\underline{\tau}))$, being a union of chambers of $\Sigma^{\clubsuit}$, necessarily contains $\tilde{\sigma}$ or $\hat{\sigma}$, or equivalently that $\underline{\tau} \in \mathcal{X}_{\tilde{\sigma}} \cup \mathcal{X}_{\hat{\sigma}}$. On the other hand, $p(\pi(\underline{\tau}))$ cannot contain both $\tilde{\sigma}$ and $\hat{\sigma}$, since otherwise $\operatorname{relint} p(\pi(\underline{\tau})) \supset \operatorname{relint}(\tilde{\sigma} \cup \hat{\sigma}) \supset \operatorname{relint} \Omega$. Hence, $\mathcal{X}_{\tilde{\sigma}} \cap \mathcal{X}_{\hat{\sigma}} \cap \mathfrak{D} = \emptyset$ and we obtain (5.6).

In order to prove the lemma it remains to show that

(5.7) $$\sum_{\underline{\tau} \in \mathcal{X}_{\tilde{\sigma}} \cap \mathfrak{D}(\underline{\nu})} \Delta_{\underline{\tau}}\mathcal{A} = \sum_{\underline{\tau} \in \mathcal{X}_{\hat{\sigma}} \cap \mathfrak{D}(\underline{\nu})} \Delta_{\underline{\tau}}\mathcal{A}$$

for all $\underline{\nu} \in \mathfrak{C}$. For this fix $1 \leq i \leq d$ and $\underline{\nu} \in \mathfrak{C}^i$ and consider $\tilde{\underline{\lambda}} \in \operatorname{relint} \tilde{\sigma}$ and $\hat{\underline{\lambda}} \in \operatorname{relint} \hat{\sigma}$. The image of the intersection $\tilde{I} = \cap_{j \neq i}(\nu_j - \tilde{\lambda}_j)$ in V^*/μ is

either a ray or an interval. Also, $\tilde{I} + \tilde{\lambda}_i$ does not intersect any face of Σ of codimension ≥ 2 since $\tilde{\underline{\lambda}}$ is $\Sigma^{\clubsuit}$-regular. Choose a direction $\delta \in \cap_{j \neq i} \mathcal{V}(\nu_j) \backslash \mu$ for $\tilde{I}$ and let $\tilde{\sigma}_1, \ldots, \tilde{\sigma}_k$ be the chambers of Σ intersecting $\tilde{I} + \tilde{\lambda}_i$, enumerated in the order of intersection with respect to δ. Thus, $\tilde{\sigma}_n$ and $\tilde{\sigma}_{n+1}$ are adjacent for $n = 1, \ldots, k - 1$, the walls $\tilde{\tau}$ intersecting $\tilde{I} + \tilde{\lambda}_i$ are precisely the intersections $\tilde{\sigma}_{n+1} \cap \tilde{\sigma}_n$, and $\langle \delta, w_n \rangle > 0$ for a directed normal w_n for $\tilde{\sigma}_{n+1} \to \tilde{\sigma}_n$.

Note that $\tilde{I} + \tilde{\lambda}_i$ intersects a chamber containing ν_i. In fact, $\mathcal{C}(\nu_i) := \cup_{\sigma \in \Sigma_0 : \sigma \supset \nu_i} \sigma$ is a convex cone in V^* with $\operatorname{relint} \nu_i \subset \operatorname{relint} \mathcal{C}(\nu_i)$. Since $\Omega \subset p(\pi(\underline{\nu}))$, we have

$$\operatorname{relint} \Omega \subset p(\operatorname{relint} \nu_1 \times \cdots \times \operatorname{relint} \nu_d)$$
$$\subset p(\operatorname{relint} \nu_1 \times \cdots \times \operatorname{relint} \mathcal{C}(\nu_i) \times \cdots \times \operatorname{relint} \nu_d)$$
$$\subset \operatorname{relint} p(\nu_1 \times \cdots \times \mathcal{C}(\nu_i) \times \cdots \times \nu_d).$$

Since $p(\nu_1 \times \cdots \times \mathcal{C}(\nu_i) \times \cdots \times \nu_d)$ is a union of chambers of $\Sigma^{\clubsuit}$, it therefore has to contain $\tilde{\sigma}$, which means that $\tilde{\sigma}$ is contained in $p(\nu_1 \times \cdots \times \sigma \times \cdots \times \nu_d)$ for some chamber $\sigma \supset \nu_i$. This is in turn equivalent to $(\tilde{I} + \tilde{\lambda}_i) \cap \sigma \neq \emptyset$.

If now $\tilde{n}_1$ (resp. $\tilde{n}_2$) is the first (resp. last) index n with $\sigma_n \supset \nu_i$, then

$$\mathcal{X}_{\tilde{\sigma}} \cap \mathfrak{D}(\underline{\nu}) = \{(\nu_1, \ldots, \nu_{i-1}, \tilde{\sigma}_{n+1} \cap \tilde{\sigma}_n, \nu_{i+1}, \ldots, \nu_d) \text{ transversal} : \tilde{n}_1 \leq n < \tilde{n}_2\}.$$

For $j \neq i$ let ω_j be a directed wall with underlying wall ν_j, v_j a directed normal for ω_j and $\xi_j \in V^*$ with $\langle \xi_j, v_j \rangle = 1$. Then there exists a unique positive multiple ξ of δ with

$$|\langle \xi, v \rangle| = \beta(v_1 \wedge \cdots \wedge v_{i-1} \wedge v \wedge v_{i+1} \wedge \cdots \wedge v_d), \quad v \in V.$$

We therefore have

$$\Delta_{\underline{\tau}} \mathcal{A} = \frac{1}{d!} D_{\xi_1} \mathcal{A}_{\omega_1}(0)$$
$$\cdots D_{\xi_{i-1}} \mathcal{A}_{\omega_{i-1}}(0) D_\xi \mathcal{A}_{\tilde{\sigma}_{n+1} \to \tilde{\sigma}_n}(0) D_{\xi_{i+1}} \mathcal{A}_{\omega_{i+1}}(0) \cdots D_{\xi_d} \mathcal{A}_{\omega_d}(0)$$

for $\underline{\tau} = (\nu_1, \ldots, \nu_{i-1}, \tilde{\sigma}_{n+1} \cap \tilde{\sigma}_n, \nu_{i+1}, \ldots, \nu_d)$ transversal, $\tilde{n}_1 \leq n < \tilde{n}_2$. By the Leibniz rule we can sum this to

$$\sum_{\underline{\tau} \in \mathcal{X}_{\tilde{\sigma}} \cap \mathfrak{D}(\underline{\nu})} \Delta_{\underline{\tau}} \mathcal{A} = \frac{1}{d!} D_{\xi_1} \mathcal{A}_{\omega_1}(0)$$
$$\cdots D_{\xi_{i-1}} \mathcal{A}_{\omega_{i-1}}(0) D_\xi \mathcal{A}_{\tilde{\sigma}_{\tilde{n}_2} \to \tilde{\sigma}_{\tilde{n}_1}}(0) D_{\xi_{i+1}} \mathcal{A}_{\omega_{i+1}}(0) \cdots D_{\xi_d} \mathcal{A}_{\omega_d}(0).$$

We define $\hat{I}$, $\hat{\sigma}_1, \ldots, \hat{\sigma}_l$ and $\hat{n}_1$, $\hat{n}_2$ analogously with respect to $\hat{\sigma}$ and $\hat{\underline{\lambda}}$ and obtain similarly

$$\sum_{\underline{\tau} \in \mathcal{X}_{\hat{\sigma}} \cap \mathfrak{D}(\underline{\nu})} \Delta_{\underline{\tau}} \mathcal{A} = \frac{1}{d!} D_{\xi_1} \mathcal{A}_{\omega_1}(0)$$
$$\cdots D_{\xi_{i-1}} \mathcal{A}_{\omega_{i-1}}(0) D_\xi \mathcal{A}_{\hat{\sigma}_{\hat{n}_2} \to \hat{\sigma}_{\hat{n}_1}}(0) D_{\xi_{i+1}} \mathcal{A}_{\omega_{i+1}}(0) \cdots D_{\xi_d} \mathcal{A}_{\omega_d}(0).$$

Consider the chambers of Σ containing ν_i, which correspond to the chambers of the two-dimensional restricted fan $\Sigma^{\supset \nu_i}$. Among them are the chambers $\tilde{\sigma}_n$, $\tilde{n}_1 \leq n \leq \tilde{n}_2$, and $\hat{\sigma}_n$, $\hat{n}_1 \leq n \leq \hat{n}_2$, intersecting $\tilde{I} + \tilde{\lambda}_i$ and $\hat{I} + \hat{\lambda}_i$, respectively. Note that the images of $\tilde{I} + \tilde{\lambda}_i$ and $\hat{I} + \hat{\lambda}_i$ in the plane $V^*/\mathcal{V}(\nu_i)$ are two parallel segments lying on different sides of the line $\cap_{j \neq i} \mathcal{V}(\nu_j)$ (mod $\mathcal{V}(\nu_i)$). By (5.6), any wall $\tau_i \supset \nu_i$ for which $(\nu_1, \ldots, \nu_{i-1}, \tau_i, \nu_{i+1}, \ldots, \nu_d)$ is transversal intersects either $\tilde{I} + \tilde{\lambda}_i$ or $\hat{I} + \hat{\lambda}_i$. It follows that every chamber of Σ containing ν_i intersects $\tilde{I} + \tilde{\lambda}_i$ or $\hat{I} + \hat{\lambda}_i$, and that moreover $\tilde{\sigma}_{\tilde{n}_1}$ and $\hat{\sigma}_{\hat{n}_1}$ are either equal or adjacent along a wall τ such that $\tau^\perp \subset \sum_{j \neq i} \nu_j^\perp$, and similarly for $\tilde{\sigma}_{\tilde{n}_2}$ and $\hat{\sigma}_{\hat{n}_2}$. We infer that $D_\xi \mathcal{A}_{\hat{\sigma}_{\hat{n}_1} \to \hat{\sigma}_{\tilde{n}_1}}(0) = D_\xi \mathcal{A}_{\tilde{\sigma}_{\tilde{n}_2} \to \hat{\sigma}_{\hat{n}_2}}(0) = 0$ and hence $D_\xi \mathcal{A}_{\tilde{\sigma}_{\tilde{n}_2} \to \tilde{\sigma}_{\tilde{n}_1}}(0) = D_\xi \mathcal{A}_{\hat{\sigma}_{\hat{n}_2} \to \hat{\sigma}_{\hat{n}_1}}(0)$, which implies (5.7). The lemma follows. $\square$

Proof of Theorem 5.1. We use induction on d, the case $d = 1$ being trivial. Assume that (5.3) is true in dimension $d - 1$. By Lemma 5.3 it is enough to prove (5.3) in dimension d for a particular choice of σ, or equivalently for a particular choice of $\Sigma^\clubsuit$-regular $\underline{\lambda} \in (V^*)^d/V^*$. Fix $\lambda_1, \ldots, \lambda_{d-1}$ in general position and analyze the limiting case where λ_d is sufficiently regular (i.e. far away from all walls) in a fixed chamber $\sigma_0 \in \Sigma_0$. Let σ be the chamber of $\Sigma^\clubsuit$ containing $\underline{\lambda}$. The sum defining $\mathcal{R}_{\underline{\lambda}} = \mathcal{R}_\sigma$ breaks up as follows. For any $\rho \in \Sigma_{d-1} \cup \{\mu\}$ let

$$Z_\rho = \left\{ (\tau_1, \ldots, \tau_{d-1}) \in \Sigma_1^{d-1} : \dim \sum_{i=1}^{d-1} \tau_i^\perp = d - 1, \ \bigcap_{i=1}^{d-1} \tau_i = \rho \right\}$$

and set

$$Y_\rho = \sum_{(\tau_1, \ldots, \tau_{d-1}) \in Z_\rho} X_{(\tau_1, \ldots, \tau_{d-1})}$$

with

(5.8) $$X_{(\tau_1, \ldots, \tau_{d-1})} = \sum_{\tau_d \in \Sigma_1 : \cap_{i=1}^d (\tau_i - \lambda_i) \neq \emptyset} \Delta_{(\tau_1, \ldots, \tau_d)} \mathcal{A}.$$

Then we can write

$$\mathcal{R}_{\underline{\lambda}} \mathcal{A} = Y_\mu + \sum_{\rho \in \Sigma_{d-1}} Y_\rho.$$

First note that $Y_\mu = 0$. Indeed, for any $(\tau_1, \ldots, \tau_{d-1}) \in Z_\mu$ the set $I := \cap_{i=1}^{d-1} (\tau_i - \lambda_i)$ is compact modulo μ. Hence, by our condition on λ_d the translate $I + \lambda_d$ is contained in the interior of σ_0, and therefore does not intersect any wall. So, in this case the sum in (5.8) is empty.

Fix $\rho \in \Sigma_{d-1}$ and $(\tau_1, \ldots, \tau_{d-1}) \in Z_\rho$. Also fix linearly independent directed normals $v_1, \ldots, v_{d-1}$ for $\tau_1, \ldots, \tau_{d-1}$ and let I be as before. If I is nonempty, then it is a translate of ρ. Hence the chambers intersecting $I + \lambda_d$ form a sequence $\sigma_0, \ldots, \sigma_k$ starting at σ_0, where σ_{i-1} and σ_i are adjacent, a

directed normal w_i for $\sigma_i \to \sigma_{i-1}$ is positive with respect to ρ, and $\sigma_k \supset \rho$. Therefore, the sum in (5.8) is over the walls $\sigma_i \cap \sigma_{i-1}$, $i = 1, \dots, k$. Fix a vector $\xi \in \rho$, which is unique modulo μ, such that

$$|\langle \xi, v \rangle| = \beta(v_1 \wedge \cdots \wedge v_{d-1} \wedge v), \quad v \in V.$$

Then we can write

$$\beta(v_1 \wedge \cdots \wedge v_{d-1} \wedge w_i) \frac{\mathcal{A}_{\sigma_i \to \sigma_{i-1}} - 1_\mathcal{E}}{w_i}(0) = D_\xi \mathcal{A}_{\sigma_i \to \sigma_{i-1}}(0).$$

Therefore

$$X_{(\tau_1, \dots, \tau_{d-1})} = \frac{1}{d!} \frac{\mathcal{A}_{\omega_1} - 1_\mathcal{E}}{v_1}(0) \cdots \frac{\mathcal{A}_{\omega_{d-1}} - 1_\mathcal{E}}{v_{d-1}}(0) \sum_{i=1}^{k} D_\xi \mathcal{A}_{\sigma_i \to \sigma_{i-1}}(0).$$

Note that

$$\sum_{i=1}^{k} D_\xi \mathcal{A}_{\sigma_i \to \sigma_{i-1}}(0) = D_\xi \mathcal{A}_{\sigma_k \to \sigma_0}(0) = D_\xi \mathcal{A}_{\rho \to \sigma_0}(0).$$

We also remark that for $(\tau_1, \dots, \tau_{d-1}) \in Z_\rho$ the condition $\cap_{i=1}^{d-1}(\tau_i - \lambda_i) \neq \emptyset$ is equivalent to $\cap_{i=1}^{d-1}(\tau_i^{\supset \rho} - \lambda_i) \neq \emptyset$. Therefore,

$$Y_\rho = \frac{1}{d!} \sum_{\substack{\tau_1, \dots, \tau_{d-1} \in \Sigma_1^{\supset \rho}: \\ \cap_{i=1}^{d-1}(\tau_i - \lambda_i) \neq \emptyset}} \beta^{\supset \rho}(v_1 \wedge \cdots \wedge v_{d-1}) \frac{\mathcal{A}_{\omega_1}^{\supset \rho} - 1_\mathcal{E}}{v_1}(0)$$

$$\cdots \frac{\mathcal{A}_{\omega_{d-1}}^{\supset \rho} - 1_\mathcal{E}}{v_{d-1}}(0) \left[D_{\lambda_\rho} \mathcal{A}_{\rho \to \sigma_0} \right](0),$$

where $\lambda_\rho \in \operatorname{relint} \rho$ is arbitrary and $\beta^{\supset \rho}$ is determined by
(5.9)
$$\beta(w_1 \wedge \cdots \wedge w_d) = \beta^{\supset \rho}(w_1 \wedge \cdots \wedge w_{d-1}) |\langle \lambda_\rho, w_d \rangle|, \quad w_1, \dots, w_{d-1} \in \rho^\perp, w_d \in V.$$

We can now apply the induction hypothesis and Proposition 4.2 to complete the proof. $\qquad\square$

6. Nonsimplicial fans

We now generalize the results of the previous sections to arbitrary (not necessarily simplicial) fans by considering simplicial refinements. We say that a fan $\widetilde{\Sigma}$ is a *refinement* of a fan Σ if they have the same core and every chamber of $\widetilde{\Sigma}$ is contained in a (necessarily unique) chamber of Σ. In other words, every chamber of Σ is a union of chambers of $\widetilde{\Sigma}$. In this case we write $\widetilde{\Sigma} \prec \Sigma$ and observe that every face of Σ_i is the union of the faces in $\widetilde{\Sigma}_i$ contained in it. It is well known that any fan admits a simplicial refinement. One way to see this is by barycentric subdivision of the faces, starting with the chambers and proceeding in order of increasing codimension (cf. [Ewa96, §III.2], for example).

If $\widetilde{\Sigma} \prec \Sigma$, then any Σ-piecewise polynomial is a $\widetilde{\Sigma}$-piecewise polynomial and we therefore obtain a canonical injection $\kappa_{\Sigma;\widetilde{\Sigma}} : \mathcal{P}_\Sigma \to \mathcal{P}_{\widetilde{\Sigma}}$. Thus, for any $f \in \mathcal{P}_\Sigma$ we have $(\kappa_{\Sigma;\widetilde{\Sigma}}f)_{\sigma'} = f_\sigma$ whenever $\sigma' \subset \sigma$. If M is any S-module, then $\kappa_{\Sigma;\widetilde{\Sigma};M} : M_\Sigma \to M_{\widetilde{\Sigma}}$ is defined by extension of scalars. Likewise, it is clear that if $\mathcal{A}$ is a compatible family with respect to Σ, then $\kappa_{\Sigma;\widetilde{\Sigma};\mathcal{E}[[V]]}(\mathcal{A})$ is a compatible family with respect to $\widetilde{\Sigma}$.

By [Bri97, §2.3] we have the basic compatibility relation

(6.1) $$\delta_\Sigma = \delta_{\widetilde{\Sigma}} \circ \kappa_{\Sigma;\widetilde{\Sigma}}$$

if both Σ and $\widetilde{\Sigma}$ are simplicial. Therefore, for any fan Σ we can now define

$$\delta_\Sigma := \delta_{\widetilde{\Sigma}} \circ \kappa_{\Sigma;\widetilde{\Sigma}},$$

where $\widetilde{\Sigma}$ is an arbitrary simplicial refinement of Σ and (6.1) continues to hold without any restriction on Σ and $\widetilde{\Sigma}$. Similarly, we define $\delta_{M;\Sigma} : M_\Sigma \to M$ for any fan Σ and S-module M. We remark that with this definition Brion's formula generalizes to arbitrary polytopes P: the push-forward $\delta_{\Sigma(P)} \exp H_P$ is the Taylor series of $\lambda \mapsto \int_P e^{\langle \lambda, v \rangle} \, dv$ for any polytope P ([Bri97, Prop. 5.3]).

Suppose that $\widetilde{\Sigma} \prec \Sigma$. For any face $\widetilde{\mathcal{C}}$ of $\widetilde{\Sigma}$ let $\iota(\widetilde{\mathcal{C}})$ be the smallest face of Σ containing $\widetilde{\mathcal{C}}$. Clearly $\operatorname{codim} \iota(\widetilde{\mathcal{C}}) \leq \operatorname{codim} \widetilde{\mathcal{C}}$ for all $\widetilde{\mathcal{C}} \in \widetilde{\Sigma}$ and ι is monotonic with respect to inclusion. Let $\widetilde{\Sigma}^\flat = \{\widetilde{\mathcal{C}} \in \widetilde{\Sigma} : \operatorname{codim} \widetilde{\mathcal{C}} = \operatorname{codim} \iota(\widetilde{\mathcal{C}})\}$ and let $\mathfrak{G}^\flat(\widetilde{\Sigma})$ be the set of flags $\tilde{\mathfrak{f}} : \tilde{\rho}_0 \supset \cdots \supset \tilde{\rho}_d = \mu$ such that $\tilde{\rho}_i \in \widetilde{\Sigma}_i^\flat$ for all i. Given a selector $\mathfrak{s} : \Sigma \to \Sigma_0$ we can choose a selector $\tilde{\mathfrak{s}} : \widetilde{\Sigma} \to \widetilde{\Sigma}_0$ such that $\widetilde{\mathcal{C}} \subset \tilde{\mathfrak{s}}(\widetilde{\mathcal{C}}) \subset \mathfrak{s}(\iota(\widetilde{\mathcal{C}}))$ for any face $\widetilde{\mathcal{C}}$ of $\widetilde{\Sigma}$. In this case we write $\tilde{\mathfrak{s}} \prec \mathfrak{s}$.

LEMMA 6.1. *Suppose that* $\widetilde{\Sigma} \prec \Sigma$ *and* $\tilde{\mathfrak{s}} \prec \mathfrak{s}$. *Then*

(1) $\widetilde{\Sigma}^\flat \supset \widetilde{\Sigma}_0$ *and* $\mu \in \widetilde{\Sigma}^\flat$.
(2) *Suppose that* $\widetilde{\mathcal{C}}_1 \in \widetilde{\Sigma}_i^\flat$ *and let* $\mathcal{C}_1 = \iota(\widetilde{\mathcal{C}}_1)$. *Then for any* $\mathcal{C}_2 \in \Sigma_{i-1}$ *containing* $\mathcal{C}_1$ *there exists a unique* $\widetilde{\mathcal{C}}_2 \in \widetilde{\Sigma}_{i-1}^\flat$ *containing* $\widetilde{\mathcal{C}}_1$ *such that* $\iota(\widetilde{\mathcal{C}}_2) = \mathcal{C}_2$.
(3) *The map*

$$\tilde{\mathfrak{f}} : \tilde{\rho}_0 \supset \cdots \supset \tilde{\rho}_d \mapsto \iota(\tilde{\mathfrak{f}}) : \iota(\tilde{\rho}_0) \supset \cdots \supset \iota(\tilde{\rho}_d)$$

defines a bijection between $\mathfrak{G}^\flat(\widetilde{\Sigma})$ *and* $\mathfrak{G}(\Sigma)$.
(4) *If* $\tilde{\mathfrak{f}} \notin \mathfrak{G}^\flat(\widetilde{\Sigma})$, *then there exists* $1 \leq i \leq d$ *such that* $\iota(\tilde{\rho}_i) = \iota(\tilde{\rho}_{i-1})$ *and hence* $\tilde{\mathfrak{s}}(\tilde{\rho}_i), \tilde{\mathfrak{s}}(\tilde{\rho}_{i-1}) \subset \mathfrak{s}(\iota(\tilde{\rho}_i))$.
(5) *If* $\mathcal{A}$ *is a compatible family with respect to* Σ, *then*

$$\partial_{\tilde{\mathfrak{f}}}^{\tilde{\mathfrak{s}}}(\kappa_{\Sigma;\widetilde{\Sigma}}\mathcal{A}) = \begin{cases} \partial_{\iota(\tilde{\mathfrak{f}})}^{\mathfrak{s}}\mathcal{A} & \text{if } \tilde{\mathfrak{f}} \in \mathfrak{G}^\flat(\widetilde{\Sigma}), \\ 0 & \text{otherwise.} \end{cases}$$

Proof. The first part is clear. By considering $\widetilde{\Sigma}^{\supset \widetilde{\mathcal{C}}_1} \prec \Sigma^{\supset \mathcal{C}_1}$, the second part reduces to the case where $\mathcal{C}_1 = \widetilde{\mathcal{C}}_1$ is the core. In this case every ray of Σ is also a ray of $\widetilde{\Sigma}$.

To prove the third part we construct the inverse map as follows. Given $\mathfrak{f} : \rho_0 \supset \cdots \supset \rho_d \in \mathfrak{G}(\Sigma)$ we define $\tilde{\rho}_i$ inductively by setting $\tilde{\rho}_d = \mu$ and letting $\tilde{\rho}_{i-1}$ be the unique face in $\widetilde{\Sigma}_{i-1}$ containing $\tilde{\rho}_i$ such that $\iota(\tilde{\rho}_{i-1}) = \rho_{i-1}$.

The fourth part is evident. The last part follows from parts (3) and (4). $\square$

We can now generalize our previous results.

PROPOSITION 6.2. *Proposition* 4.2, *Corollary* 4.3 *and Theorem* 5.1 *hold for compatible families with respect to arbitrary fans.*

Proof. By passing to a simplicial refinement $\widetilde{\Sigma} \prec \Sigma$ and considering $\kappa_{\Sigma;\widetilde{\Sigma}}\mathcal{A}$ we infer that Corollary 4.3 holds without restriction on Σ. From this we deduce that the same is true for Proposition 4.2. Then the proof of Theorem 5.1 shows that it too holds for any Σ.

Alternatively, one easily sees that the right-hand side of (5.3) is compatible with refinement. Indeed, if $\tilde{\mathcal{A}} = \kappa_{\Sigma;\widetilde{\Sigma}}\mathcal{A}$, then $(\tilde{\tau}_1, \ldots, \tilde{\tau}_d) \in \mathcal{X}_{\underline{\lambda}}^{\widetilde{\Sigma}}$ contributes only if $\tilde{\tau}_i \in \widetilde{\Sigma}^{\flat}$ for all i. (Otherwise the two chambers of $\widetilde{\Sigma}$ which are adjacent along $\tilde{\tau}_i$ are both contained in the same chamber of Σ.) On the other hand, for any $(\tau_1, \ldots, \tau_d) \in \mathcal{X}_{\underline{\lambda}}^{\Sigma}$ there exists a unique $(\tilde{\tau}_1, \ldots, \tilde{\tau}_d) \in \mathcal{X}_{\underline{\lambda}}^{\widetilde{\Sigma}}$ with $\tilde{\tau}_i \subset \tau_i$ for all i. $\square$

7. Generalization to the relative case

In this section we provide a relative setup for Corollary 4.3 and Theorem 5.1. Instead of the push-forward of a compatible family, we consider more generally the push-forward of its restriction to a subspace of V^*.

Let U be an arbitrary subspace of V and $^\sharp : V \to V^\sharp := V/U$ be the canonical projection. We extend $^\sharp$ to homomorphisms $^\sharp : S = \mathrm{Sym}(V) \to S^\sharp := \mathrm{Sym}(V/U)$ and $^\sharp : \mathbb{C}[[V]] \to \mathbb{C}[[V^\sharp]]$. Thus $^\sharp$ corresponds to the restriction of linear forms (respectively, polynomial functions or power series) from V^* to $U^\perp \simeq (V^\sharp)^*$. If $V' \subset V$ is a subspace, we denote its image under $^\sharp$ by $V'^\sharp$. The image of $\mathrm{Sym}(V')$ (resp., $\mathbb{C}[[V']]$) under $^\sharp$ is $\mathrm{Sym}(V'^\sharp)$ (resp., $\mathbb{C}[[V'^\sharp]]$).

The following is standard.

LEMMA 7.1. *Suppose that Σ is a fan in V^*. Then*

(1) *The set of cones $\Sigma^\sharp = \Sigma^\sharp(U) := \{\mathcal{C}^\sharp := \mathcal{C} \cap U^\perp : \mathcal{C} \in \Sigma\}$ forms a fan in the subspace $U^\perp$ of V^*. We call it the* induced fan.
(2) *We have $\mu(\Sigma^\sharp) = \mu(\Sigma)^\sharp$ and $\mu(\Sigma^\sharp)^\perp = (\mu(\Sigma)^\perp)^\sharp$ inside $V^\sharp = V/U$.*

(3) *The map $\mathcal{C} \mapsto \mathcal{C}^\sharp$ is a bijection between*

$$\mathcal{S} := \{\mathcal{C} \in \Sigma : \mathcal{C} \cap U^\perp \supsetneq \mathcal{C}' \cap U^\perp \text{ for all } \mathcal{C} \supsetneq \mathcal{C}' \in \Sigma\}$$

and $\Sigma^\sharp$. The inverse map is given by $\widetilde{\mathcal{C}} \mapsto \cap_{\mathcal{C} \in \Sigma : \mathcal{C} \supset \widetilde{\mathcal{C}}} \mathcal{C}$ for $\widetilde{\mathcal{C}} \in \Sigma^\sharp$.

(4) *For any $\mathcal{C} \in \mathcal{S}$ we have $(\Sigma^{\supset \mathcal{C}})^\sharp = (\Sigma^\sharp)^{\supset \mathcal{C}^\sharp}$.*

(5) *The image of Σ_0 under $\sharp$ contains $\Sigma_0^\sharp$ (the chambers of $\Sigma^\sharp$).*

For instance, if $\Sigma = \Sigma(P)$ for a polytope $P \subset V$, then $\Sigma^\sharp = \Sigma(P^\sharp)$ where $P^\sharp$ is the image of P under $\sharp$ and for any face F of P, the face $F^{\sharp\sharp}$ of $\Sigma(P^\sharp)$ corresponds to the minimal face of $P^\sharp$ containing the image $F^\sharp$ of F under $\sharp$. In particular, $\mathcal{S}$ corresponds to the set of preimages of faces of $P^\sharp$.

We have a canonical homomorphism

$$f \mapsto f^\sharp := f|_{U^\perp}, \quad \mathcal{P}_\Sigma \to \mathcal{P}_{\Sigma^\sharp}.$$

Thus, for any $\sigma \in \Sigma_0$ such that $\sigma^\sharp \in \Sigma_0^\sharp$ we have $f_{\sigma^\sharp}^\sharp = (f_\sigma)^\sharp$. Given an $S^\sharp$-module $M^\sharp$ and a homomorphism of S-modules $M \to M^\sharp$ we get a canonical map $M_\Sigma \to M_{\Sigma^\sharp}^\sharp$ which we also denote by $m \mapsto m^\sharp$. In particular, applying this to the restriction map $\mathcal{E}[[V]] \to \mathcal{E}[[V^\sharp]]$ we obtain from any Σ-compatible family $\mathcal{A}$ a $\Sigma^\sharp$-compatible family $\mathcal{A}^\sharp$. Clearly, for any $\mathcal{C} \in \mathcal{S}$ we have $(\mathcal{A}^\sharp)^{\supset \mathcal{C}^\sharp} = (\mathcal{A}^{\supset \mathcal{C}})^\sharp$.

Set

$$m = \dim \mu(\Sigma^\sharp)^\perp = \dim U^\perp - \dim \mu(\Sigma) \cap U^\perp.$$

Fix a Haar measure on $U^\perp/\mu^\sharp$ and define $\beta^\sharp : \bigwedge^m(\mu^{\perp\sharp}) \to \mathbb{R}_{\geq 0}$ and $\beta^{\sharp*} : \bigwedge^m(U^\perp/\mu^\sharp) \to \mathbb{R}_{\geq 0}$ as in §3. Let $\underline{\tau} = (\tau_1, \ldots, \tau_m) \in \Sigma_1^m$. For each $i = 1, \ldots, m$ choose a directed wall $\omega_i : \sigma_i \xrightarrow{\tau_i} \tilde{\sigma}_i$ and a directed normal v_i for ω_i. We say that $\underline{\tau}$ is *relatively transversal* if $(\sum_{i=1}^m \tau_i^\perp)^\sharp = (\mu^\perp)^\sharp$, i.e. if $v_1, \ldots, v_m$ are linearly independent modulo U. Let $\mathcal{A}$ be a compatible family with respect to Σ. Then $\frac{\mathcal{A}_{\omega_i} - 1_\mathcal{E}}{v_i} \in \mathcal{E}[[\tau_i^\perp]]$ for all i and we set

$$\Delta_{\underline{\tau}}^\sharp \mathcal{A} = \frac{1}{m!} \beta^\sharp(v_1^\sharp \wedge \cdots \wedge v_m^\sharp) \frac{\mathcal{A}_{\omega_1} - 1_\mathcal{E}}{v_1}(0) \cdots \frac{\mathcal{A}_{\omega_m} - 1_\mathcal{E}}{v_m}(0) \in \mathcal{E}.$$

Note that $\Delta_{\underline{\tau}}^\sharp \mathcal{A}$ depends only on $\underline{\tau}$ and not on the choice of the ω_i's or the v_i's.

Consider $U^\perp$ embedded diagonally in $(V^*)^m$. Let $p^\sharp : (V^*)^m \to (V^*)^m/U^\perp$ be the canonical projection. We identify the set Σ^m of m-tuples of faces of Σ with a fan in $(V^*)^m$ where the face corresponding to $\underline{\mathcal{C}} = (\mathcal{C}_1, \ldots, \mathcal{C}_m)$ is $\pi(\underline{\mathcal{C}}) = \mathcal{C}_1 \times \cdots \times \mathcal{C}_m$. Denote by $\Sigma^\spadesuit$ the quotient fan in $(V^*)^m/U^\perp$. Note that $\underline{\lambda} = (\lambda_1, \ldots, \lambda_m) \pmod{V^*} \in p^\sharp(\pi(\underline{\mathcal{C}}))$ if and only if $\cap_{i=1}^m(\mathcal{C}_i - \lambda_i) \cap U^\perp \neq \emptyset$. The partition of $(V^*)^m/U^\perp$ obtained from $\Sigma^\spadesuit$ is given by the fibers of the map sending $\underline{\lambda} \in (V^*)^m/U^\perp$ to the set $\{\underline{\mathcal{C}} \in \Sigma^m : \cap_{i=1}^m(\mathcal{C}_i - \lambda_i) \cap U^\perp \neq \emptyset\}$. For $\sigma \in \Sigma_0^\spadesuit$ set

$$\mathcal{X}_\sigma^\sharp = \{\underline{\tau} \in \Sigma_1^m \text{ relatively transversal} : p^\sharp(\pi(\underline{\tau})) \supset \sigma\}.$$

Equivalently, if $\underline{\lambda} \in \operatorname{relint} \sigma$, then

$$\mathcal{X}_\sigma^\sharp = \{\underline{\tau} \in \Sigma_1^m \text{ relatively transversal} : \cap_{i=1}^m (\mathcal{C}_i - \lambda_i) \cap U^\perp \neq \emptyset\}.$$

Note that if $\underline{\tau} \in \mathcal{X}_\sigma^\sharp$, then $\cap_{i=1}^m (\tau_i - \lambda_i) \cap U^\perp$ is a translate of $\mu^\sharp$.

THEOREM 7.2. *Let $\mathcal{A}$ be a compatible family with respect to a fan Σ and $U \subset V$ a subspace. Then for any $\sigma \in \Sigma_0^{\spadesuit}$ we have*

$$(7.1) \qquad\qquad \mathcal{D}_{\Sigma^\sharp} \mathcal{A}^\sharp = \sum_{\underline{\tau} \in \mathcal{X}_\sigma^\sharp} \Delta_{\underline{\tau}}^\sharp \mathcal{A}.$$

Proof. The proof is very similar to the proof of Theorem 5.1. Denote the right-hand side of (7.1) by $\mathcal{R}_\sigma \mathcal{A}$. We first observe that the analogue of Lemma 5.3 applies, namely that $\mathcal{R}_\sigma \mathcal{A}$ is independent of $\sigma \in \Sigma_0^{\spadesuit}$. Since the proof of this fact is exactly the same as before, we omit it.

We can now prove the theorem by induction on m, starting with the trivial case $m = 0$. Assume that (7.1) is true in dimension $m-1$. It is enough to prove (7.1) in dimension m for a particular choice of $\sigma \in \Sigma_0^{\spadesuit}$. Let σ be the unique chamber of $\Sigma^{\spadesuit}$ containing $\underline{\lambda} \in (V^*)^m/U^\perp$, where $\lambda_1, \ldots, \lambda_{m-1} \in V^*$ are fixed in general position and λ_m is sufficiently regular (i.e. far away from all walls) in a fixed chamber σ_0 of Σ. Given $\tau_1, \ldots, \tau_{m-1} \in \Sigma_1$ such that directed normals $v_1, \ldots, v_{m-1}$ are linearly independent modulo U and $I = \cap_{i=1}^{m-1}(\tau_i - \lambda_i) \cap U^\perp \neq \emptyset$, we have to sum over all $\tau_m \in \Sigma_1$ which intersect $I + \lambda_m$. Only $\tau_1, \ldots, \tau_{m-1}$ such that $\cap_{i=1}^{m-1} \tau_i^\sharp \in \Sigma_{m-1}^\sharp$ contribute. Otherwise, $\cap_{i=1}^{m-1} \tau_i^\sharp = \mu^\sharp$ and therefore I is compact modulo $\mu^\sharp$. By our condition on λ_m, $I + \lambda_m$ is contained in the interior of σ_0, and hence does not intersect any wall.

We can therefore suppose that $\cap_{i=1}^{m-1} \tau_i^\sharp = \rho^\sharp \in \Sigma_{m-1}^\sharp$ for $\rho \in \mathcal{S}$. If I is nonempty, it is a translate of $\rho^\sharp$ and hence the chambers intersecting $I + \lambda_m$ form a sequence $\sigma_0, \ldots, \sigma_k$ starting at σ_0 where σ_{i-1} and σ_i are adjacent, a directed normal for $\sigma_i \to \sigma_{i-1}$ is positive with respect to $\rho^\sharp$, and $\sigma_k \supset \rho^\sharp$. Since $\rho \in \mathcal{S}$, we get $\sigma_k \supset \rho$. Therefore the τ_m's which contribute are the precisely the walls $\sigma_i \cap \sigma_{i-1}$, $i = 1, \ldots, k$, and their total contribution is

$$\frac{1}{m!} \frac{\mathcal{A}_{\omega_1} - 1_{\mathcal{E}}}{v_1}(0) \cdots \frac{\mathcal{A}_{\omega_{m-1}} - 1_{\mathcal{E}}}{v_{m-1}}(0) \sum_{i=1}^k D_\xi \mathcal{A}_{\sigma_i \to \sigma_{i-1}}(0),$$

where $\xi \in \rho^\sharp$ is given by $|\langle \xi, v \rangle| = \beta^\sharp(v_1^\sharp \wedge \cdots \wedge v_{m-1}^\sharp \wedge v^\sharp)$, $v \in V$. Note that

$$\sum_{i=1}^k D_\xi \mathcal{A}_{\sigma_i \to \sigma_{i-1}}(0) = D_\xi \mathcal{A}_{\sigma_k \to \sigma_0}(0) = D_\xi \mathcal{A}_{\rho \to \sigma_0}(0)$$

since $\xi \in \rho^\sharp \subset \rho$. We also remark that given $\tau_1, \ldots, \tau_{m-1}$ with $\cap_{i=1}^{m-1} \tau_i^\sharp = \rho^\sharp$ we have $\cap_{i=1}^{m-1} \tau_i \supset \rho$. Therefore the condition $\cap_{i=1}^{m-1}(\tau_i - \lambda_i) \cap U^\perp \neq \emptyset$ is equivalent

to $\cap_{i=1}^{m-1}(\tau_i^{\supset\rho} - \lambda_i) \cap U^\perp \neq \emptyset$. All in all, $\mathcal{R}_\lambda \mathcal{A} = \mathcal{R}_\sigma \mathcal{A}$ is equal to the sum over $\rho^\sharp \in \Sigma_{m-1}^\sharp$ of

$$\frac{1}{m!} \sum_{\substack{\tau_1,\dots,\tau_{m-1}\in\Sigma_1^{\supset\rho}: \\ \cap_{i=1}^{m-1}(\tau_i-\lambda_i)\cap U^\perp\neq\emptyset}} (\beta^\sharp)^{\supset\rho^\sharp}(v_1^\sharp \wedge \cdots \wedge v_{m-1}^\sharp)\frac{\mathcal{A}_{\omega_1}^{\supset\rho} - 1_\mathcal{E}}{v_1}(0)$$

$$\cdots \frac{\mathcal{A}_{\omega_{m-1}}^{\supset\rho} - 1_\mathcal{E}}{v_{m-1}}(0)D_{\lambda_{\rho^\sharp}}\mathcal{A}_{\rho\to\sigma_0}(0),$$

where $\lambda_{\rho^\sharp} \in \operatorname{relint}\rho^\sharp$ is arbitrary and $(\beta^\sharp)^{\supset\rho^\sharp}$ is obtained from $\lambda_{\rho^\sharp}$ as in (5.9) above. On the other hand, take $\sigma_0 \in \Sigma_0$ such that $\sigma_0^\sharp \in \Sigma_0^\sharp$. Then Proposition 4.2 applied to $\mathcal{A}^\sharp$ yields

$$(7.2) \quad \mathcal{D}_{\Sigma^\sharp}\mathcal{A}^\sharp = \frac{1}{m} \sum_{\rho^\sharp\in\Sigma_{m-1}^\sharp} \mathcal{D}_{(\Sigma^\sharp)^{\supset\rho^\sharp}}\left((\mathcal{A}^\sharp)^{\supset\rho^\sharp}\right)\left[D_{\lambda_{\rho^\sharp}}\mathcal{A}_{\rho^\sharp\to\sigma_0^\sharp}^\sharp\right](0)$$

$$= \frac{1}{m} \sum_{\rho^\sharp\in\Sigma_{m-1}^\sharp} \mathcal{D}_{(\Sigma^{\supset\rho})^\sharp}(\mathcal{A}^{\supset\rho})^\sharp\left[D_{\lambda_{\rho^\sharp}}\mathcal{A}_{\rho\to\sigma_0}\right](0).$$

Comparing with the above we deduce the induction step. $\qquad\square$

Remark 7.3. The proof also shows that the relation (7.2) holds for any $\sigma_0 \in \Sigma_0$ (regardless of whether or not $\sigma_0^\sharp \in \Sigma_0^\sharp$).

Note that if $\mathcal{C}^\sharp \in \Sigma^\sharp$ and $\lambda \in \mathcal{V}(\mathcal{C}^\sharp)$, then for any $\sigma_0 \in \Sigma_0$, $[D_\lambda\mathcal{A}_{\sigma\to\sigma_0}](0)$ does not depend on $\sigma \in \Sigma_0$ with $\sigma \supset \mathcal{C}^\sharp$. We denote this derivative by $[D_\lambda\mathcal{A}_{\mathcal{C}^\sharp\to\sigma_0}](0)$.

Fix a selector $\mathfrak{s}^\sharp : \Sigma^\sharp \to \Sigma_0$ such that $\mathfrak{s}^\sharp(\mathcal{C}^\sharp) \supset \mathcal{C}^\sharp$ for all $\mathcal{C}^\sharp \in \Sigma^\sharp$. For any flag $\mathfrak{f}^\sharp : \rho_0^\sharp \supset \cdots \supset \rho_m^\sharp$ of $\Sigma^\sharp$ define

$$\partial_{\mathfrak{f}^\sharp}\mathcal{A} = \partial_{\mathfrak{f}^\sharp}^{\mathfrak{s}^\sharp}\mathcal{A} = \frac{1}{m!}\frac{\left[D_{\lambda_{\rho_0^\sharp}}\mathcal{A}_{\rho_0^\sharp\to\mathfrak{s}^\sharp(\rho_1^\sharp)}\right](0)\dots\left[D_{\lambda_{\rho_{m-1}^\sharp}}\mathcal{A}_{\rho_{m-1}^\sharp\to\mathfrak{s}^\sharp(\rho_m^\sharp)}\right](0)}{\beta^{\sharp*}(\lambda_{\rho_0^\sharp}\wedge\cdots\wedge\lambda_{\rho_{m-1}^\sharp})}$$

with auxiliary vectors $\lambda_{\rho_i^\sharp} \in \operatorname{relint}\rho_i^\sharp$. Note that since $\mathcal{A}_{\rho_{i-1}^\sharp\to\mathfrak{s}^\sharp(\rho_i^\sharp)} \in \mathcal{E}[[(\rho_i^\sharp)^\perp]]$ for all i, $\partial_{\mathfrak{f}^\sharp}^{\mathfrak{s}^\sharp}\mathcal{A}$ is independent of the choice of the $\lambda_{\rho_i^\sharp}$'s. However, it depends on the choice of $\mathfrak{s}^\sharp$.

Using (7.2) and the remark above we obtain by induction on m the following.

THEOREM 7.4. *For any Σ-compatible family $\mathcal{A}$, any subspace $U \subset V$ and any choice of selector $\mathfrak{s}^\sharp : \Sigma^\sharp \to \Sigma_0$ as above we have*

$$\mathcal{D}_{\Sigma^\sharp}\mathcal{A}^\sharp = \sum_{\mathfrak{f}^\sharp\in\mathfrak{G}(\Sigma^\sharp)} \partial_{\mathfrak{f}^\sharp}^{\mathfrak{s}^\sharp}\mathcal{A}.$$

8. **Hyperplane arrangements**

An important class of fans is obtained by *hyperplane arrangements.* We will now specialize Theorems 5.1 and 7.2 to this situation. Suppose that $\mathfrak{H}$ is a finite set of hyperplanes in V^*. Each hyperplane $H \in \mathfrak{H}$ defines two closed half-spaces $H_\pm$ in V^* with $H_+ \cap H_- = H$. For any $f \in \prod_{H \in \mathfrak{H}} \{H_+, H_-, H\}$ we can form the intersection $\cap_{H \in \mathfrak{H}} f(H)$, which is a cone in V^*. These cones (for all possible choices of f) form a fan $\Sigma = \Sigma(\mathfrak{H})$ in V^* with $\mu(\Sigma) = \cap \mathfrak{H}$. The chambers of Σ are the closures of the connected components of $V^* \setminus \cup \mathfrak{H}$. For any $H \in \mathfrak{H}$ we have $H = \cup \{\tau \in \Sigma_1 : \tau \subset H\}$. We remark that $\Sigma(\mathfrak{H})$ is the normal fan of any zonotope which is the Minkowski sum of arbitrary intervals in $H^\perp$ for each $H \in \mathfrak{H}$ (cf. [Zie95, Ch. 7]).

In the case $\Sigma = \Sigma(\mathfrak{H})$ we may rewrite Theorem 5.1 (or more precisely, its generalization provided by Proposition 6.2) as follows. Denote by $\mathfrak{B} = \mathfrak{B}(\mathfrak{H})$ the set of d-tuples $\underline{H} = (H_1, \dots, H_d)$ of hyperplanes in $\mathfrak{H}$ such that $\cap_{i=1}^d H_i = \mu$. For any $\underline{H} \in \mathfrak{B}$ let

$$\Xi(\underline{H}) = \{\underline{\tau} \in \Sigma_1^d : \tau_i \subset H_i, \, i = 1, \dots, d\}.$$

Then we have

$$(V^*)^d / V^* = p(H_1 \times \cdots \times H_d) = \bigcup_{\underline{\tau} \in \Xi(\underline{H})} p(\underline{\tau}).$$

Thus for any $\underline{\lambda} \in (V^*)^d / V^*$ and $\underline{H} \in \mathfrak{B}$ there exists $\underline{\tau} \in \Xi(\underline{H})$ such that $\underline{\lambda} \in p(\underline{\tau})$. Moreover, if $\underline{\lambda}$ is $\Sigma^\clubsuit$-regular, then this d-tuple $\underline{\tau} \in \Xi(\underline{H})$ is uniquely determined for all $\underline{H} \in \mathfrak{B}$, and we may denote it by $\underline{\tau}_{\underline{\lambda}}(\underline{H})$. We get

THEOREM 8.1. *Let $\mathcal{A}$ be a compatible family with respect to $\Sigma(\mathfrak{H})$. Then for any choice of $\Sigma^\clubsuit$-regular $\underline{\lambda}$ we have*

$$\mathcal{D}_{\Sigma(\mathfrak{H})} \mathcal{A} = \sum_{\underline{H} \in \mathfrak{B}(\mathfrak{H})} \Delta_{\underline{\tau}_{\underline{\lambda}}(\underline{H})} \mathcal{A}.$$

Suppose that $U = \sum_{H \in S} H^\perp$ for some subset $S \subset \mathfrak{H}$ and let $m = d - \dim U$. Let $^\sharp : V \to V/U$ be the canonical projection as in the previous section. Let $\mathfrak{H}^\sharp$ be the hyperplane arrangement $\{H^\sharp = H \cap U^\perp : H \not\supset U^\perp\}$ in $U^\perp$. Then $\Sigma^\sharp = \Sigma(\mathfrak{H}^\sharp)$. Let

$$\mathfrak{B}^\sharp = \{\underline{H} = (H_1, \dots, H_m) : H_i \in \mathfrak{H}, H_i \not\supset U^\perp, (H_1^\sharp, \dots, H_m^\sharp) \in \mathfrak{B}(\mathfrak{H}^\sharp)\}$$

$$= \{\underline{H} = (H_1, \dots, H_m) : H_i \in \mathfrak{H}, H_i \not\supset U^\perp, \bigcap_{i=1}^m H_i \cap U^\perp = \mu\}$$

and for any $\underline{H} \in \mathfrak{B}^\sharp$ let

$$\Xi(\underline{H}) = \{\underline{\tau} \in \Sigma_1^m : \tau_i \subset H_i, \, i = 1, \dots, m\}.$$

As before, for any $\underline{\lambda} \in (V^*)^m/U^\perp$ and $\underline{H} \in \mathfrak{B}^\sharp$ there exists $\underline{\tau} \in \Xi(\underline{H})$ such that $\underline{\lambda} \in p^\sharp(\underline{\tau})$. Moreover, if $\underline{\lambda}$ is $\Sigma^\spadesuit$-regular, this m-tuple $\underline{\tau} \in \Xi(\underline{H})$ is unique and we denote it by $\underline{\tau}_{\underline{\lambda}}^\sharp(\underline{H})$. Theorem 7.2 becomes

THEOREM 8.2. *For any $\Sigma(\mathfrak{H})$-compatible family $\mathcal{A}$, any subspace of V of the form $U = \sum_{H \in S} H^\perp$ for some $S \subset \mathfrak{H}$ and any $\Sigma^\spadesuit$-regular $\underline{\lambda}$ we have*

$$(8.1) \qquad \mathcal{D}_{\Sigma^\sharp(\mathfrak{H})} \mathcal{A}^\sharp = \sum_{\underline{H} \in \mathfrak{B}^\sharp(\mathfrak{H})} \Delta_{\underline{\tau}_{\underline{\lambda}}^\sharp(\underline{H})}^\sharp \mathcal{A}.$$

Remark 8.3. The condition of $\Sigma^\spadesuit$-regularity can be explicitly described as follows. By a *minimal dependency* in $\mathfrak{H}$ modulo U we mean a linear relation of the form

$$v_1 + \cdots + v_k \in U, \quad k \geq 1,$$

where $v_1, \ldots, v_k \in \cup_{H \in \mathfrak{H}} H^\perp$ and any proper subsequence of $(v_1, \ldots, v_k)$ is linearly independent modulo U. Clearly, up to scaling there are only finitely many minimal dependencies. (In the language of matroids, the minimal dependencies correspond to the circuits in the matroid associated with $\mathfrak{H}^\sharp$, where the latter is regarded as a multiset.) Then $\underline{\lambda} \in (V^*)^m/U^\perp$ is $\Sigma^\spadesuit$-regular if and only if for any minimal dependency $v := v_1 + \cdots + v_k \in U$ we have

(1) $\sum_{i=1}^k \langle \lambda_{\pi(i)}, v_i \rangle \neq 0$ for any nonconstant function $\pi : \{1, \ldots, k\} \to \{1, \ldots, m\}$, and,

(2) if $v \neq 0$, then $\langle \lambda_j, v \rangle \neq 0$ for $j = 1, \ldots, m$.

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