Annals of Mathematics **173** (2011), 1841–1882
doi: 10.4007/annals.2011.173.3.12

The local Langlands conjecture for GSp(4)

By Wee Teck Gan and Shuichiro Takeda

Abstract

We prove the local Langlands conjecture for $\mathrm{GSp}_4(F)$ where F is a non-archimedean local field of characteristic zero.

1. Introduction

Let F be a non-archimedean local field of characteristic 0 and residue characteristic p. Let W_F be the Weil group of F and let $WD_F = W_F \times \mathrm{SL}_2(\mathbb{C})$ be the Weil-Deligne group. It was shown by Harris-Taylor [HT01] and Henniart [Hen00] that there is a natural bijection between the set of equivalence classes of irreducible smooth representations of $\mathrm{GL}_n(F)$ and the set of conjugacy classes of L-parameters for GL_n, i.e. admissible homomorphisms

$$\phi : WD_F \longrightarrow \mathrm{GL}_n(\mathbb{C}).$$

This bijection satisfies a number of natural conditions which determine it uniquely.

For a general connected reductive group G over F, which we assume to be split for simplicity, Langlands conjectures that there is a surjective finite-to-one map from the set $\Pi(G)$ of (equivalence classes of) irreducible smooth representations of $G(F)$ to the set $\Phi(G)$ of (equivalence classes of) admissible homomorphisms

$$WD_F \longrightarrow G^\vee,$$

where $G^\vee$ is the Langlands dual group of G and the homomorphisms are taken up to $G^\vee$-conjugacy. This leads to a partition of the set of equivalence classes of irreducible representations of $G(F)$ into a disjoint union of finite subsets, which are the fibers of the map and are called L-packets. Again, this map is supposed to preserve natural invariants which one can attach to both sides. These natural invariants are the γ-factors, L-factors and ε-factors. Unfortunately, on the representation theoretic side, one only has a general theory of these invariants for generic representations of $G(F)$.

The purpose of this paper is to prove the local Langlands conjecture for $G = \mathrm{GSp}_4$.

Main Theorem. There is a *surjective* finite-to-one map

$$L : \Pi(\mathrm{GSp}_4) \longrightarrow \Phi(\mathrm{GSp}_4)$$

with the following properties:

(i) π is a (essentially) discrete series representation of $\mathrm{GSp}_4(F)$ if and only if its L-parameter $\phi_\pi := L(\pi)$ does not factor through any proper Levi subgroup of $\mathrm{GSp}_4(\mathbb{C})$.

(ii) For an L-parameter ϕ, its fiber L_ϕ can be naturally parametrized by the set of irreducible characters of the component group

$$A_\phi = \pi_0(Z(\mathrm{Im}(\phi))/Z_{\mathrm{GSp}_4}).$$

This component group is either trivial or equal to $\mathbb{Z}/2\mathbb{Z}$. When $A_\phi = \mathbb{Z}/2\mathbb{Z}$, exactly one of the two representations in L_ϕ is generic, and it is the one indexed by the trivial character of A_ϕ.

(iii) The similitude character $\mathrm{sim}(\phi_\pi)$ of ϕ_π is equal to the central character ω_π of π (via local class field theory). Here, $\mathrm{sim} : \mathrm{GSp}_4(\mathbb{C}) \longrightarrow \mathbb{C}^\times$ is the similitude character of $\mathrm{GSp}_4(\mathbb{C})$.

(iv) The L-parameter of $\pi \otimes (\chi \circ \lambda)$ is equal to $\phi_\pi \otimes \chi$. Here, $\lambda : \mathrm{GSp}_4(F) \longrightarrow F^\times$ is the similitude character of $\mathrm{GSp}_4(F)$, and we have regarded χ as both a character of $F^\times$ and a character of W_F by local class field theory.

(v) Suppose that π is a generic representation or a nonsupercuspidal representation. Then for any irreducible representation σ of $\mathrm{GL}_r(F)$, with L-parameter ϕ_σ, we have:

$$\begin{cases} \gamma(s, \pi \times \sigma, \psi) = \gamma(s, \phi_\pi \otimes \phi_\sigma, \psi), \\ L(s, \pi \times \sigma) = L(s, \phi_\pi \otimes \phi_\sigma), \\ \varepsilon(s, \pi \times \sigma, \psi) = \varepsilon(s, \phi_\pi \otimes \phi_\sigma, \psi). \end{cases}$$

Here the functions on the right-hand side are the local factors of Artin type associated to the relevant representations of WD_F, whereas those on the left-hand side are the local factors of Shahidi, as defined in Section 4.

(vi) Suppose that π is a nongeneric supercuspidal representation. For any irreducible supercuspidal representation σ of $\mathrm{GL}_r(F)$ with L-parameter ϕ_σ, let $\mu(s, \pi \boxtimes \sigma, \psi)$ denote the Plancherel measure associated to the family of induced representations $I_P(\pi \boxtimes \sigma, s)$ on $\mathrm{GSpin}_{2r+5}(F)$, where we have regarded $\pi \boxtimes \sigma$ as a representation of the Levi subgroup $\mathrm{GSpin}_5(F) \times \mathrm{GL}_r(F) \cong \mathrm{GSp}_4(F) \times \mathrm{GL}_r(F)$. Then $\mu(s, \pi \boxtimes \sigma, \psi)$ is equal to

$$\gamma(s, \phi_\pi^\vee \otimes \phi_\sigma, \psi) \cdot \gamma(-s, \phi_\pi \otimes \phi_\sigma^\vee, \overline{\psi}) \cdot \gamma(2s, \mathrm{Sym}^2 \phi_\sigma \otimes \mathrm{sim}\, \phi_\pi^{-1}, \psi) \cdot \gamma(-2s, \mathrm{Sym}^2 \phi_\sigma^\vee \otimes \mathrm{sim}\, \phi_\pi, \overline{\psi}).$$

(vii) An L-packet L_ϕ contains a generic representation if and only if the adjoint L-factor $L(s, \mathrm{Ad} \circ \phi)$ is holomorphic at $s = 1$. Here, Ad denotes the adjoint representation of $\mathrm{GSp}_4(\mathbb{C})$ on the complex Lie algebra $\mathfrak{sp}_4$. Moreover, L_ϕ contains an essentially tempered generic representation if and only if ϕ is an essentially tempered L-parameter, i.e. $\phi|_{W_F}$ has bounded image in $\mathrm{PGSp}_4(\mathbb{C})$. In this case, the generic representation in the packet is unique and is indexed by the trivial character of A_ϕ.

The map L is uniquely determined by the properties (i), (iii), (v) and (vi), with $r \leq 2$ in (v) and (vi). □

To the best of our knowledge, for nongeneric supercuspidal representations, the theory of local γ-factors, L-factors and ε-factors of pairs has not been fully developed and so at this point, it is not possible for us to say anything regarding these in part (v) of the theorem. However, if one assumes the existence of a theory of γ-factors satisfying the usual properties (such as those listed as the "Ten Commandments" in [LR05]), then we can show that (v) holds for all representations, in which case the map L will be uniquely characterized by (iii) and (v) (with $r \leq 2$ in (v)).

In any case, we substitute the (as yet nonexistent) theory of γ-factors of pairs by the Plancherel measure. The Plancherel measure in (vi) is a coarser invariant than the γ-factor, but has the advantage that it is defined for all representations. For generic representations, the identity in (vi) follows from results of Shahidi [Sha90] and Henniart [Hen10]. Thus, (vi) shows that the Plancherel measure is an invariant of a supercuspidal L-packet. Moreover, it turns out that this coarser invariant is sufficient to distinguish between the nongeneric supercuspidal representations of $\mathrm{GSp}_4(F)$, and this gives the characterization of L by (i), (iii), (v) and (vi).

The proof of the theorem relies on the local Langlands correspondence for GL_2 and GL_4 and a consideration of the following two towers of dual pairs:

$$\begin{array}{ccccc} & & & & \mathrm{GSO}_{3,3} \\ & & & \nearrow & \\ & & \mathrm{GSp}_4 & & \\ & \swarrow & & \searrow & \\ \mathrm{GSO}_{4,0} & & & & \mathrm{GSO}_{2,2}\,. \end{array}$$

Thus, it relies on a study of the local theta correspondence arising between GSp_4 and the orthogonal similitude groups associated to quadratic spaces of rank 4 or 6 with trivial discriminant and the accidental isomorphisms:

$$\begin{cases} \mathrm{GSO}_{2,2} \cong (\mathrm{GL}_2 \times \mathrm{GL}_2)/\{(z, z^{-1}) : z \in \mathrm{GL}_1\}, \\ \mathrm{GSO}_{3,3} \cong (\mathrm{GL}_4 \times \mathrm{GL}_1)/\{(z, z^{-2}) : z \in \mathrm{GL}_1\}, \end{cases}$$

as well as the analogs for the inner forms.

The first attempt to define L-packets for GSp_4 is the paper [Vig86] of Vigneras, who considered the case $p \neq 2$. She defined her L-packets via theta lifts from various forms of GO_4 (including those whose discriminant is not trivial). However, she did not check that her L-packets exhaust $\Pi(\mathrm{GSp}_4)$. In a more recent paper [Rob01], B. Roberts has given a more detailed treatment of Vigneras' construction of local L-packets (including the case $p = 2$).

Our construction of the local L-packets, on the other hand, relies only on the theta lifting from GSp_4 to $\mathrm{GSO}_{2,2}$, $\mathrm{GSO}_{4,0}$ and $\mathrm{GSO}_{3,3}$. It works for any p and also enables us to show that our packets exhaust all representations. We shall also show that, starting from a given L-parameter, the L-packets constructed by Vigneras and Roberts agree with ours. As a consequence, we show that when $p \neq 2$, their construction exhausts all irreducible representations of $\mathrm{GSp}_4(F)$. We also mention that, considering only those representations of $\mathrm{GSp}_4(F)$ whose central character is a square, a section of our map L was constructed by Jiang-Soudry [JS04]. More precisely, for the split group $G = \mathrm{SO}_{2n+1}$, they constructed an injective (but definitely not surjective unless $n = 1$) map from $\Phi(G) \longrightarrow \Pi(G)$ verifying the conditions (i), (iv) and (v) of our main theorem.

Let us mention the various key ingredients used in the proof of the theorem. The first is a paper of Muić-Savin [MS00b] which studies the theta lifting of generic discrete series representations for isometry groups and relates the nonvanishing of these theta lifts to the local L-functions of Shahidi. The second ingredient is a paper of Kudla-Rallis [KR05] which proves the conservation conjecture for the first occurrences of supercuspidal representations of symplectic groups in orthogonal towers. In particular, their results imply that every nongeneric supercuspidal representations of $\mathrm{GSp}_4(F)$ can be obtained as a theta lift from the anisotropic $\mathrm{GSO}_{4,0}$. The third is a recent result of Henniart [Hen10] which shows that the local Langlands correspondence for GL_n respects the twisted exterior square L-factor. Finally, we have a companion paper [GTb] in which we determine completely the three local theta correspondences mentioned above. The detailed study of these theta correspondences is necessary to supplement the results of [KR05] and [MS00b], particularly for the nondiscrete series representations.

The crucial results of [KR05], [MS00b] and [Hen10] are reviewed in Section 5, after we introduce some basic facts about the theta correspondence for similitudes in Sections 2 and 3 and recall Shahidi's construction of certain L-functions in Section 4. Our construction of the L-packets and the proof of exhaustion are given in Section 7, where we also verify the relation of genericity and the adjoint L-factor. The preservation of local factors and Plancherel measures is demonstrated in Sections 8 and 9 respectively, and the characterization of L is given in Section 10. In Section 11, we reconcile our construction with

that given by Vigneras and Roberts. Finally, in Section 12, we give a global application: using the results of this paper, we establish the strong lifting of generic cuspidal representations from GSp_4 to GL_4.

We conclude this introduction with a number of subsequent developments:

(i) In a sequel [GT10a] to this paper, we deduce the local Langlands correspondence for Sp_4 from the results of this paper;

(ii) In another sequel [GT] to this paper, by the first author and W. Tantono, the local Langlands correspondence is extended to the unique inner form of GSp_4;

(iii) An L-packet is supposed to be "stable" and should satisfy some character identities relative to (twisted) endoscopic transfers. Our method, unfortunately, does not shed any light on these harmonic analytic issues. However, another sequel [CG] to this paper, by P. S. Chan and the first author, establishes these properties of the L-packets constructed here using the Arthur-Selberg trace formula;

(iv) In a recent paper [DR09], Debacker and Reeder have given a construction of L-packets associated to certain tamely ramified L-parameters of an arbitrary reductive group G. The elements in their L-packets are all depth zero supercuspidal representations. One can ask whether their packets agree with ours in the case $G = \mathrm{GSp}_4$. This is shown to be the case in the UCSD Ph.D. thesis of J. Lust;

(v) A theory of L- and ε-factors for $\mathrm{GSp}_4 \times \mathrm{GL}_r$ (for $r = 1$ or 2) which is valid for all representations, including the nongeneric ones, is being developed by N. Townsend in his UCSD Ph.D. thesis.

Acknowledgments. We thank Dipendra Prasad, Brooks Roberts, Gordan Savin and Marie-France Vigneras for their interests, suggestions and encouragements. We also thank Wilhelm Zink for sending us a copy of his letter [Zin] to Vigneras. W. T. Gan's research is partially supported by NSF grant DMS-0500781.

2. **Similitude theta correspondences**

In this section, we shall describe some basic properties of the theta correspondence for similitude groups. The definitive reference for this subject matter is the paper [Ro1] of B. Roberts. However, the results of [Ro1] are not sufficient for our purposes and need to be somewhat extended.

Consider the dual pair $\mathrm{O}(V) \times \mathrm{Sp}(W)$; for simplicity, we assume that $\dim V$ is even. For each nontrivial additive character ψ, let ω_ψ be the Weil representation for $\mathrm{O}(V) \times \mathrm{Sp}(W)$, which can be described as follows. Fix a Witt decomposition $W = X \oplus Y$ and let $P(Y) = \mathrm{GL}(Y) \cdot N(Y)$ be the parabolic

subgroup stabilizing the maximal isotropic subspace Y. Then

$$N(Y) = \{b \in \mathrm{Hom}(X, Y) : b^t = b\},$$

where $b^t \in \mathrm{Hom}(Y^*, X^*) \cong \mathrm{Hom}(X, Y)$. The Weil representation ω_ψ can be realized on the Schwartz space $S(X \otimes V)$ and the action of $P(Y) \times \mathrm{O}(V)$ is given by the usual formulas:

$$\begin{cases} \omega_\psi(h)\phi(x) = \phi(h^{-1}x), & \text{for } h \in \mathrm{O}(V), \\ \omega_\psi(a)\phi(x) = \chi_V(\det_Y(a)) \cdot |\det_Y(a)|^{\frac{1}{2}\dim V} \cdot \phi(a^{-1} \cdot x), & \text{for } a \in \mathrm{GL}(Y), \\ \omega_\psi(b)\phi(x) = \psi(\langle bx, x\rangle) \cdot \phi(x), & \text{for } b \in N(Y), \end{cases}$$

where χ_V is the quadratic character associated to $\mathrm{disc}\, V \in F^\times/F^{\times 2}$ and $\langle -, -\rangle$ is the natural symplectic form on $W \otimes V$. To describe the full action of $\mathrm{Sp}(W)$, one needs to specify the action of a Weyl group element, which acts by a Fourier transform.

If π is an irreducible representation of $\mathrm{O}(V)$ (resp. $\mathrm{Sp}(W)$), then the maximal π-isotypic quotient has the form

$$\pi \boxtimes \Theta_\psi(\pi)$$

for some smooth representation $\Theta_\psi(\pi)$ of $\mathrm{Sp}(W)$ (resp. $\mathrm{O}(V)$). We call $\Theta_\psi(\pi)$ the big theta lift of π. It is known that $\Theta_\psi(\pi)$ is of finite length and hence is admissible. Let $\theta_\psi(\pi)$ be the maximal semisimple quotient of $\Theta_\psi(\pi)$; we call it the small theta lift of π. Then it was a conjecture of Howe that

- $\theta_\psi(\pi)$ is irreducible whenever $\Theta_\psi(\pi)$ is nonzero;
- the map $\pi \mapsto \theta_\psi(\pi)$ is injective on its domain.

This has been proved by Waldspurger when the residual characteristic p of F is not 2 and can be checked in many low-rank cases, regardless of the residual characteristic of F. If the Howe conjecture is true in general, our treatment in the rest of the paper can be somewhat simplified. However, because we would like to include the case $p = 2$ in our discussion, we shall refrain from assuming Howe's conjecture in this paper.

With this in mind, we take note of the following result which was shown by Kudla [Kud86] for any residual characteristic p:

PROPOSITION 2.1. (i) *If π is supercuspidal, $\Theta_\psi(\pi) = \theta_\psi(\pi)$ is irreducible or zero.*

(ii) *If $\theta_\psi(\pi_1) = \theta_\psi(\pi_2) \neq 0$ for two supercuspidal representations π_1 and π_2, then $\pi_1 = \pi_2$.*

One of the main purposes of this section is to extend this result of Kudla to the case of similitude groups.

Let λ_V and λ_W be the similitude factors of $\mathrm{GO}(V)$ and $\mathrm{GSp}(W)$ respectively. We shall consider the group

$$R = \mathrm{GO}(V) \times \mathrm{GSp}(W)^+,$$

where $\mathrm{GSp}(W)^+$ is the subgroup of $\mathrm{GSp}(W)$ consisting of elements g such that $\lambda_W(g)$ is in the image of λ_V. In fact, for the cases of interest in this paper (see the next section), λ_V is surjective, in which case $\mathrm{GSp}(W)^+ = \mathrm{GSp}(W)$.

The group R contains the subgroup

$$R_0 = \{(h, g) \in R : \lambda_V(h) \cdot \lambda_W(g) = 1\}.$$

The Weil representation ω_ψ extends naturally to the group R_0 via

$$\omega_\psi(g, h)\phi = |\lambda_V(h)|^{-\frac{1}{8}\dim V \cdot \dim W} \omega(g_1, 1)(\phi \circ h^{-1}),$$

where

$$g_1 = g \begin{pmatrix} \lambda(g)^{-1} & 0 \\ 0 & 1 \end{pmatrix} \in \mathrm{Sp}(W).$$

Note that this differs from the normalization used in [Rob96]. Observe in particular that the central elements $(t, t^{-1}) \in R_0$ act by the quadratic character $\chi_V(t)^{\frac{\dim W}{2}}$.

Now consider the (compactly) induced representation

$$\Omega = \mathrm{ind}_{R_0}^R \omega_\psi.$$

As a representation of R, Ω depends only on the orbit of ψ under the evident action of $\mathrm{Im}\,\lambda_V \subset F^\times$. For example, if λ_V is surjective, then Ω is independent of ψ. For any irreducible representation π of $\mathrm{GO}(V)$ (resp. $\mathrm{GSp}(W)^+$), the maximal π-isotypic quotient of Ω has the form

$$\pi \otimes \Theta(\pi)$$

where $\Theta(\pi)$ is some smooth representation of $\mathrm{GSp}(W)^+$ (resp. $\mathrm{GO}(V)$). Further, we let $\theta(\pi)$ be the maximal semisimple quotient of $\Theta(\pi)$. Note that though $\Theta(\pi)$ may be reducible, it has a central character $\omega_{\Theta(\pi)}$ given by

$$\omega_{\Theta(\pi)} = \chi_V^{\frac{\dim W}{2}} \cdot \omega_\pi.$$

The extended Howe conjecture for similitudes says that $\theta(\pi)$ is irreducible whenever $\Theta(\pi)$ is nonzero, and the map $\pi \mapsto \theta(\pi)$ is injective on its domain. It was shown by Roberts [Rob96] that this follows from the Howe conjecture for isometry groups, and thus holds if $p \neq 2$. In any case, we have the following lemma which relates the theta correspondence for isometries and similitudes:

LEMMA 2.2. (i) *Suppose that π is an irreducible representation of a similitude group and τ is a constituent of the restriction of π to the isometry group. Then $\theta_\psi(\tau) \neq 0$ implies that $\theta(\pi) \neq 0$.*

(ii) *Suppose that*

$$\mathrm{Hom}_R(\Omega, \pi_1 \boxtimes \pi_2) \neq 0.$$

Suppose further that for each constituent τ_1 *in the restriction of* π_1 *to* $\mathrm{O}(V)$, $\theta_\psi(\tau_1)$ *is irreducible and the map* $\tau_1 \mapsto \theta_\psi(\tau_1)$ *is injective on the set of irreducible constituents of* $\pi_1|_{\mathrm{O}(V)}$. *Then there is a uniquely determined bijection*

$$f : \{\textit{irreducible summands of } \pi_1|_{\mathrm{O}(V)}\} \longrightarrow \{\textit{irreducible summands of } \pi_2|_{\mathrm{Sp}(W)}\}$$

such that for any irreducible summand τ_i *in the restriction of* π_i *to the relevant isometry group*,

$$\tau_2 = f(\tau_1) \Longleftrightarrow \mathrm{Hom}_{\mathrm{O}(V)\times\mathrm{Sp}(W)}(\omega_\psi, \tau_1 \boxtimes \tau_2) \neq 0.$$

One has the analogous statement with the roles of $\mathrm{O}(V)$ *and* $\mathrm{Sp}(W)$ *exchanged.*

(iii) *If* π *is a representation of* $\mathrm{GO}(V)$ (*resp.* $\mathrm{GSp}(W)^+$) *and the restriction of* π *to the relevant isometry group is* $\oplus_i \tau_i$, *then as representations of* $\mathrm{Sp}(W)$ (*resp.* $\mathrm{O}(V)$),

$$\Theta(\pi) \cong \bigoplus_i \Theta_\psi(\tau_i).$$

In particular, $\Theta(\pi)$ *is admissible of finite length. Moreover, if* $\Theta_\psi(\tau_i) = \theta_\psi(\tau_i)$ *for each* i, *then*

$$\Theta(\pi) = \theta(\pi).$$

Proof. (i) Without loss of generality, suppose that π is a representation of $\mathrm{GSp}(W)^+$. As a representation of $\mathrm{GSp}(W)^+$,

$$\Omega = \mathrm{ind}_{\mathrm{Sp}(W)}^{\mathrm{GSp}(W)^+} \omega_\psi.$$

Hence the result follows by Frobenius reciprocity.

(ii) This is [Rob96, Lemma 4.2], taking note of the results of [AP06], where it was shown that restrictions of irreducible representations from similitude groups to isometry groups are multiplicity-free.

(iii) By symmetry, let us suppose that π is a representation of $\mathrm{GSp}(W)^+$. Then we have the following sequence of $\mathrm{O}(V)$-equivariant isomorphisms:

$$\begin{aligned}
\Theta(\pi)^* &\cong \mathrm{Hom}_{\mathrm{GSp}(W)^+}(\Omega, \pi) \\
&\cong \mathrm{Hom}_{\mathrm{Sp}(W)}(\omega_\psi, \pi|_{\mathrm{Sp}(W)}) \quad \text{(by Frobenius reciprocity)} \\
&\cong \bigoplus_i \mathrm{Hom}_{\mathrm{Sp}(W)}(\omega_\psi, \tau_i) \\
&\cong \bigoplus_i \Theta_\psi(\tau_i)^*,
\end{aligned}$$

where $\Theta(\pi)^*$ denotes the full linear dual of $\Theta(\pi)$. Thus, we have an $\mathrm{O}(V)$-equivariant isomorphism of $\mathrm{O}(V)$-smooth vectors

$$\Theta(\pi)^\vee \cong \bigoplus_i \Theta_\psi(\tau_i)^\vee.$$

Note that since $\Theta(\pi)$ has a central character, the subspace of $\mathrm{GO}(V)$-smooth vectors in $\Theta(\pi)^*$ is the same as its subspace of $\mathrm{O}(V)$-smooth vectors. In other words, the contragredient of $\Theta(\pi)$ as a representation of $\mathrm{GO}(V)$ is the same as its contragredient as a representation of $\mathrm{O}(V)$. Using the fact that the $\Theta_\psi(\tau_i)$'s are admissible of finite length, the above identity implies that $\Theta(\pi)$ is admissible of finite length. The desired result then follows by taking contragredient. Moreover, if $\Theta_\psi(\tau_i)$ is semisimple for each i, then it is clear from the above that $\Theta(\pi)$ is semisimple as well. □

PROPOSITION 2.3. *Suppose that π is a supercuspidal representation of* $\mathrm{GO}(V)$ (*resp.* $\mathrm{GSp}(W)^+$). *Then we have*:

(i) $\Theta(\pi)$ *is either zero or is an irreducible representation of* $\mathrm{GSp}(W)^+$ (*resp.* $\mathrm{GO}(V)$).

(ii) *If π' is another supercuspidal representation such that $\Theta(\pi') = \Theta(\pi) \neq 0$, then $\pi' = \pi$.*

Proof. (i) Without loss of generality, suppose that π is a supercuspidal representation of $\mathrm{GO}(V)$ and $\Theta(\pi)$ is nonzero. By Lemma 2.2(iii), if $\pi|_{\mathrm{O}(V)} = \bigoplus_i \tau_i$, we have:

$$\Theta(\pi) = \theta(\pi) = \bigoplus_i \theta_\psi(\tau_i).$$

By Lemma 2.2(ii) and Proposition 2.1, we see that any irreducible constituent Π of $\theta(\pi)$ satisfies:

$$\Pi|_{\mathrm{Sp}(W)} = \bigoplus_i \theta_\psi(\tau_i).$$

Thus we see that $\Theta(\pi)$ is irreducible. This proves (i).

(ii) Proposition 2.1(ii) implies that if $\Theta(\pi') = \Theta(\pi) \neq 0$, then $\pi'|_{\mathrm{O}(V)} \cong \pi|_{\mathrm{O}(V)}$. Since π and π' must have the same central character, we see that $\pi' = \pi \otimes (\chi \circ \lambda_V)$ for some quadratic character χ. Moreover, it is easy to see that

$$\Theta(\pi \otimes (\chi \circ \lambda_V)) = \Theta(\pi) \otimes (\chi \circ \lambda_W).$$

Hence we would be done if we can show that for any quadratic character χ,

$$\pi \otimes \chi = \pi \Longleftrightarrow \Theta(\pi) \otimes \chi = \Theta(\pi).$$

Of course, the implication ($\Longrightarrow$) is clear from the above. To show the converse, let us set

$$I(\pi) = \{\text{quadratic characters } \chi\colon\ \pi \otimes \chi = \pi\},$$

and let $I(\Theta(\pi))$ be the analogous group of quadratic characters. As we noted above,

$$I(\pi) \subset I(\Theta(\pi)),$$

and we need to show the reverse inclusion. Now the size of the group $I(\pi)$ is equal to the number of irreducible constituents in $\pi|_{\mathrm{O}(V)}$. By Lemma 2.2(ii),

however, the number of irreducible constituents in $\pi|_{\mathrm{O}(V)}$ and $\Theta(\pi)|_{\mathrm{Sp}(W)}$ are equal. Hence

$$I(\pi) = I(\Theta(\pi)),$$

as desired. □

3. **Theta correspondences for** GSp_4

In this section, we specialize to the cases of interest in this paper. Let D be a (possibly split) quaternion algebra over F and let $\mathbb{N}_D$ be its reduced norm. Then $(D, \mathbb{N}_D)$ is a rank 4 quadratic space. We have an isomorphism

$$\mathrm{GSO}(D, \mathbb{N}_D) \cong (D^\times \times D^\times)/\{(z, z^{-1}) : z \in \mathrm{GL}_1\}$$

via the action of the latter on D given by

$$(\alpha, \beta) \mapsto \alpha x \overline{\beta}.$$

Moreover, an element of $\mathrm{GO}(D, \mathbb{N}_D)$ of determinant -1 is given by the conjugation action $c : x \mapsto \overline{x}$ on D. An irreducible representation of $\mathrm{GSO}(D)$ is thus of the form $\tau_1 \boxtimes \tau_2$ where τ_i are representations of $D^\times$ with equal central character. Moreover, the action of c on representations of $\mathrm{GSO}(D)$ is given by $\tau_1 \boxtimes \tau_2 \mapsto \tau_2 \boxtimes \tau_1$. One of the dual pairs which will be of interest to us is $\mathrm{GSp}_4 \times \mathrm{GO}(D)$. In the companion paper [GTb], we determine the associated theta correspondence completely. Some of the results are summarized in Theorem 5.6 below.

Now consider the rank 6 quadratic space:

$$(V_D, q_D) = (D, \mathbb{N}_D) \oplus \mathbb{H}$$

where $\mathbb{H}$ is the hyperbolic plane. Then one has an isomorphism

$$\mathrm{GSO}(V_D) \cong (\mathrm{GL}_2(D) \times \mathrm{GL}_1)/\{(z \cdot \mathrm{Id},\ z^{-2}) : z \in \mathrm{GL}_1\}.$$

To see this, note that the quadratic space V_D can also be described as the space of 2×2-Hermitian matrices with entries in D, so that a typical element has the form

$$(a, d; x) = \begin{pmatrix} a & x \\ \overline{x} & d \end{pmatrix}, \qquad a, d \in F \text{ and } x \in D,$$

equipped with the quadratic form $-\det(a, d; x) = -ad + \mathbb{N}_D(x)$. The action of $\mathrm{GL}_2(D) \times \mathrm{GL}_1$ on this space is given by

$$(g, z)(X) = z \cdot g \cdot X \cdot \overline{g}^t.$$

Observe that an irreducible representation of $\mathrm{GSO}(V_D)$ is of the form $\pi \boxtimes \mu$ where π is a representation of $\mathrm{GL}_2(D)$ and μ is a square root of the central character of π. The similitude factor of $\mathrm{GSO}(V_D)$ is given by $\lambda_D(g, z) =$

$N(g) \cdot z^2$, where N is the reduced norm on the central simple algebra $\mathrm{M}_2(D)$. Thus,

$$\mathrm{SO}(V_D) = \{(g, z) \in \mathrm{GSO}(V_D) : N(g) \cdot z^2 = 1\}.$$

We can now consider the theta correspondence in this case. Since we only need to consider V_D when D is split, we shall suppress D from the notations. Thus we specialize the results of the previous section to the case when $\dim W = 4$ and V is the split quadratic space of dimension 6, so that λ_V is surjective and the induced Weil representation Ω is a representation of $R = \mathrm{GSp}(W) \times \mathrm{GO}(V)$. In fact, we shall only consider the theta correspondence for $\mathrm{GSp}(W) \times \mathrm{GSO}(V)$. There is no significant loss in restricting to $\mathrm{GSO}(V)$ because of the following lemma:

LEMMA 3.1. *Let π (resp. τ) be an irreducible representation of* $\mathrm{GSp}(W)$ (*resp.* $\mathrm{GO}(V)$) *and suppose that*

$$\mathrm{Hom}_{\mathrm{GSp}(W)\times\mathrm{GO}(V)}(\Omega, \pi \otimes \tau) \neq 0.$$

Then the restriction of τ to $\mathrm{GSO}(V)$ *is irreducible. If $\nu_0 = \lambda_V^{-3} \cdot \det$ is the unique nontrivial quadratic character of* $\mathrm{GO}(V)/\mathrm{GSO}(V)$, *then $\tau \otimes \nu_0$ does not participate in the theta correspondence with* $\mathrm{GSp}(W)$.

Proof. The analogous result for isometry groups is a well-known result of Rallis [Ral84, App.] (see also [Pra93, §5, p. 282]). The lemma follows easily from this and we omit the details. □

We now collect some results concerning the theta correspondence for $\mathrm{GSp}(W) \times \mathrm{GSO}(V)$. Firstly, we have:

THEOREM 3.2. *Let π be an irreducible representation of* $\mathrm{GSp}(W)$. *Then $\theta(\pi)$ is irreducible or zero as a representation of* $\mathrm{GSO}(V)$. *Moreover, if $\theta(\pi) = \theta(\pi') \neq 0$, then $\pi = \pi'$.*

Proof. For supercuspidal representations, the result follows by Proposition 2.3 and the previous lemma. For nonsupercuspidal representations, the result follows by the explicit determination of the theta correspondence for $\mathrm{GSp}(W) \times \mathrm{GSO}(V)$ given in the companion paper [GTb, Th. 8.3]. □

Suppose now that U (resp. U_0) is the unipotent radical of a Borel subgroup of $\mathrm{GSp}(W)$ (resp. $\mathrm{GSO}(V)$) and χ (resp. χ_0) is a generic character of U (resp. U_0). One may compute the twisted Jacquet module Ω_{U_0,χ_0}. The following lemma (see [GT10b, Prop. 7.4] and [MS00b, Prop. 4.1]) describes the result:

LEMMA 3.3. *As a representation of* $\mathrm{GSp}(W)$,

$$\Omega_{U_0,\chi_0} = \mathrm{ind}_U^{\mathrm{GSp}(W)} \chi.$$

In particular, if π is an irreducible generic representation of $\mathrm{GSp}(W)$, *then* $\Theta(\pi)$ *is nonzero.*

Finally we describe the functoriality of the above theta correspondence for unramified representations. The L-group of $\mathrm{GSp}(W)$ is $\mathrm{GSp}_4(\mathbb{C})$ and so an unramified representation of $\mathrm{GSp}(W)$ corresponds to a semisimple class in $\mathrm{GSp}_4(\mathbb{C})$. On the other hand, the L-group of $\mathrm{GSO}(V)$ is the subgroup of $\mathrm{GL}_4(\mathbb{C}) \times \mathrm{GL}_1(\mathbb{C})$ given by

$$^L\mathrm{GSO}(V) = \{(g, z) \in \mathrm{GL}_4(\mathbb{C}) \times \mathrm{GL}_1(\mathbb{C}) : \det(g) = z^2\},$$

which is isomorphic to the group $\mathrm{GSpin}_6(\mathbb{C})$. There is a natural map

$$\iota : {}^L\mathrm{GSp}_4 \longrightarrow {}^L\mathrm{GSO}(V)$$

given by

$$g \mapsto (g, \mathrm{sim}(g))$$

where $\mathrm{sim} : \mathrm{GSp}_4(\mathbb{C}) \to \mathbb{C}^\times$ is the similitude factor. The following is shown in the companion paper [GTb, Cor. 12.3]:

PROPOSITION 3.4. *Let $\pi = \pi(s)$ be an unramified representation of* $\mathrm{GSp}(W)$ *corresponding to the semisimple class $s \in \mathrm{GSp}_4(\mathbb{C})$. Then $\theta(\pi(s))$ is the unramified representation of* $\mathrm{GSO}(V)$ *corresponding to the semisimple class $\iota(s) \in \mathrm{GL}_4(\mathbb{C}) \times \mathrm{GL}_1(\mathbb{C})$.*

4. **On certain L-functions**

In this section, we introduce certain L-functions, ε-factors and γ-factors which we will need. These local factors were defined by Shahidi [Sha90]. To specify them more precisely, we need to consider certain representations of the relevant L-groups.

Recall that we have an inclusion of L-groups:

$$\iota : {}^L\mathrm{GSp}_4 \longrightarrow {}^L\mathrm{GSO}(V) \cong \mathrm{GSpin}_6(\mathbb{C}) \subset \mathrm{GL}_4(\mathbb{C}) \times \mathrm{GL}_1(\mathbb{C}).$$

The projection of $\mathrm{GL}_4(\mathbb{C}) \times \mathrm{GL}_1(\mathbb{C})$ onto the first factor thus defines a natural 4-dimensional representation of ${}^L\mathrm{GSO}(V)$ (one of the half-spin representations of $\mathrm{GSpin}_6(\mathbb{C})$) whose restriction to ${}^L\mathrm{GSp}_4 \cong \mathrm{GSp}_4(\mathbb{C})$ is the natural 4-dimensional representation of $\mathrm{GSp}_4(\mathbb{C})$. Following a terminology common in the literature, we call this representation the Spin representation of ${}^L\mathrm{GSp}_4$ and ${}^L\mathrm{GSO}(V)$.

Now, corresponding to the inclusion $\mathrm{SO}(V) \hookrightarrow \mathrm{GSO}(V)$, one has a map of L-groups

$$\mathrm{std} : {}^L\mathrm{GSO}(V) \longrightarrow {}^L\mathrm{SO}(V) = \mathrm{SO}_6(\mathbb{C}).$$

Indeed, one has the map

$$\mathrm{GL}_4(\mathbb{C}) \times \mathrm{GL}_1(\mathbb{C}) \longrightarrow \mathrm{GSO}_6(\mathbb{C})$$

given by

$$(g, z) \mapsto z^{-1} \cdot \wedge^2 g,$$

and the map std is simply the restriction of this map to the subgroup ${}^L\mathrm{GSO}(V) \subset \mathrm{GL}_4(\mathbb{C}) \times \mathrm{GL}_1(\mathbb{C})$. Similarly, corresponding to the inclusion $\mathrm{Sp}(W) \hookrightarrow \mathrm{GSp}(W)$, we have a map of L-groups

$$\mathrm{std} : \mathrm{GSp}_4(\mathbb{C}) \longrightarrow \mathrm{SO}_5(\mathbb{C}) \cong \mathrm{PGSp}_4(\mathbb{C})$$

and a commutative diagram

$$\begin{array}{ccccc} \mathrm{GSp}_4(\mathbb{C}) & \xrightarrow{\iota} & {}^L\mathrm{GSO}(V) & \longrightarrow & \mathrm{GL}_4(\mathbb{C}) \times \mathrm{GL}_1(\mathbb{C}) \\ \downarrow{\scriptstyle \mathrm{std}} & & \downarrow{\scriptstyle \mathrm{std}} & & \downarrow{\scriptstyle \mathrm{std}} \\ \mathrm{SO}_5(\mathbb{C}) & \xrightarrow{\iota_0} & \mathrm{SO}_6(\mathbb{C}) & \longrightarrow & \mathrm{GSO}_6(\mathbb{C}) \end{array} .$$

We regard std as a 5-dimensional (resp. 6-dimensional) representation of ${}^L\mathrm{GSp}_4$ (resp. ${}^L\mathrm{GSO}(V)$) and call it the standard representation.

Thus, for a representation π of $\mathrm{GSp}_4(F)$, one expects to be able to define a standard degree 5 L-function $L(s, \pi, \mathrm{std})$ and a degree 4 Spin L-function $L(s, \pi, \mathrm{spin})$. More generally, for representations π of $\mathrm{GSp}_4(F)$ and σ of $\mathrm{GL}_r(F)$, one expects to have the L-functions

$$L(s, \pi \times \sigma, \mathrm{std} \boxtimes \mathrm{std}) \quad \text{and} \quad L(s, \pi \times \sigma, \mathrm{spin} \boxtimes \mathrm{std}),$$

which are associated to the representations std$\boxtimes$std and spin$\boxtimes$std respectively.

Similarly, given a representation Σ of $\mathrm{GSO}(V)$, one expects to have the degree 6 L-function $L(s, \Sigma, \mathrm{std})$ and the degree 4 L-function $L(s, \Sigma, \mathrm{spin})$. More generally, for representations Σ of $\mathrm{GSO}(V)$ and σ of $\mathrm{GL}_r(F)$, one expects to have the L-functions

$$L(s, \Sigma \times \sigma, \mathrm{std} \boxtimes \mathrm{std}) \quad \text{and} \quad L(s, \Sigma \times \sigma, \mathrm{spin} \boxtimes \mathrm{std}).$$

Moreover, if we regard Σ as a representation $\Pi \boxtimes \mu$ of $\mathrm{GL}_4(F) \times \mathrm{GL}_1(F)$, then $L(s, \Sigma \times \sigma, \mathrm{spin} \boxtimes \mathrm{std})$ should be nothing but the Rankin-Selberg L-function $L(s, \Pi \times \sigma)$ of the representation $\Pi \boxtimes \sigma$ of $\mathrm{GL}_4(F) \times \mathrm{GL}_r(F)$. Further, one expects that

$$L(s, \Sigma, \mathrm{std}) = L\Big(s, \Pi, \bigwedge^2 \otimes \mu^{-1}\Big),$$

where the L-function on the right-hand side is the twisted exterior square L-function.

In the important paper [Sha90], the above L-functions and their associated ε-factors were defined by Shahidi when the representation $\pi \boxtimes \sigma$ or Σ is generic. More precisely, suppose that

- $M \subset G$ is the Levi subgroup of a maximal parabolic subgroup $P = M \cdot N$;
- τ is an irreducible generic representation of $M(F)$;

Table 1. On certain L-functions

	L-function	M	G	r_i	τ
a	$L(s, \pi \times \sigma, \text{spin} \boxtimes \text{std})$	$\text{GSpin}_5 \times \text{GL}_r$	GSpin_{2r+5}	$r_1 = \text{spin}^\vee \boxtimes \text{std}$	$\pi^\vee \boxtimes \sigma$
b	$L(s, \sigma, \text{Sym}^2 \otimes \omega_\pi)$			$r_2 = \text{sim}^{-1} \otimes (\text{Sym}^2 \text{std})$	
c	$L(s, \pi \times \sigma, \text{std} \boxtimes \text{std})$	$\text{GSp}_4 \times \text{GL}_r$	GSp_{2r+4}	$r_1 = \text{std}^\vee \boxtimes \text{std}$	$\pi^\vee \boxtimes \sigma$
d	$L(s, \Sigma, \text{std})$	$\text{GSO}_6 \times \text{GL}_1$	GSO_8	$r_1 = \text{std}^\vee \boxtimes \text{std}$	$\Sigma^\vee \boxtimes \mathbf{1}$
e	$L(s, \Pi \times \sigma)$	$\text{GL}_4 \times \text{GL}_r$	GL_{r+4}	$r_1 = \text{std} \boxtimes \text{std}^\vee$	$\Pi \boxtimes \sigma^\vee$
f	$L(s, \Pi, \bigwedge^2 \otimes \mu^{-1})$	$\text{GL}_4 \times \text{GL}_1$	GSpin_8	$r_2 = (\bigwedge^2 \text{std}) \boxtimes \text{std}^\vee$	$\Pi \boxtimes \mu$

- the adjoint action of the dual group $M^\vee$ on $\mathfrak{n}^\vee = Lie(N^\vee)$ decomposes as $r_1 \oplus r_2 \oplus ... \oplus r_k$, where each r_i is a maximal isotypic component for the action of the central torus in $M^\vee$.

Then Shahidi defined the local factors $\gamma(s, \tau, r_i, \psi)$, $L(s, \tau, r_i)$, $\varepsilon(s, \tau, r_i, \psi)$ which satisfy

$$\gamma(s, \tau, r_i, \psi) = \varepsilon(s, \tau, r_i, \psi) \cdot \frac{L(1-s, \tau^\vee, r_i)}{L(s, \tau, r_i)}.$$

In Table 1, we list the various L-functions that we will use and the data (M, G, τ) which are used in their definition via the Shahidi machinery.

Though Shahidi defined the L-functions in Table 1 only for generic representations, the definition can be extended to nongeneric nonsupercuspidal representations of the simple factors of the groups M which occur in the table. This uses the Langlands classification, which says that every irreducible admissible representation can be expressed as a quotient of a standard module, i.e. one induced from a nonnegative twist of a discrete series representation of a Levi subgroup. For the groups M occurring in Table 1, their simple factors have proper Levi subgroups which are essentially products of GL_k's, so that a discrete series representation of such a proper Levi subgroup is generic. Moreover, the restriction of each representation r_i to a proper Levi subgroup of M decomposes into the sum of irreducible constituents, all of which appears in the setup of Shahidi's theory. Thus, one may extend the definition of the local factors to all nongeneric nonsupercuspidal representations of each simple factor of M by multiplicativity (with respect to the standard module under consideration). Thus, the local factors given in Table 1 are defined except when the representation π of $\text{GSp}_4(F) \cong \text{GSpin}_5(F)$ is nongeneric supercuspidal.

The L-function $L(s, \pi \times \sigma, \text{spin} \boxtimes \text{std})$ in (a) of Table 1 is the main one which intervenes in our main theorem. Hence, we shall simplify notations by

writing it as $L(s, \pi \times \sigma)$, suppressing the mention of spin $\boxtimes$ std. The same comment applies to the ε- and γ-factors.

Finally, we note the following two lemmas:

LEMMA 4.1. *Let Σ be an irreducible generic representation of $\mathrm{GSO}(V)$ which we may identify with a representation $\Pi \boxtimes \mu$ of $\mathrm{GL}_4(F) \times \mathrm{GL}_1(F)$ via the isomorphism*

$$\mathrm{GSO}(V) \cong \mathrm{GL}_4(F) \times \mathrm{GL}_1(F)/\{(t, t^{-2}) : t \in F^\times\}.$$

Then we have:

$$L(s, \Sigma, \mathrm{std}) = L\Big(s, \Pi, \bigwedge^2 \otimes \mu^{-1}\Big),$$

where the L-function on the left-hand side (resp. right-hand side) is that in (d) (resp. (f)) of the above table. Moreover, one has the analogous identity for the ε- and γ-factors.

LEMMA 4.2. *Suppose that Σ is an irreducible generic representation of a similitude group $\mathrm{GSp}(W)$ or $\mathrm{GSO}(V)$ and Σ_0 is an irreducible constituent of the restriction of Σ to the isometry group $\mathrm{Sp}(W)$ or $\mathrm{SO}(V)$. Then one has*:

$$L(s, \Sigma, \mathrm{std}) = L(s, \Sigma_0, \mathrm{std}) \quad \textit{and} \quad \varepsilon(s, \Sigma, \mathrm{std}, \psi) = \varepsilon(s, \Sigma_0, \mathrm{std}, \psi).$$

Both of these lemmas follow from the characterization of Shahidi's local factors given in [Sha90, Th. 3.5].

5. **The results of Kudla-Rallis, Muić-Savin and Henniart**

In this section, we review some crucial general results of Kudla-Rallis [KR05], Muić-Savin [MS00b] and Henniart [Hen10] before specializing them to the cases of interest in this paper.

Let W_n be the $2n$-dimensional symplectic vector space with associated symplectic group $\mathrm{Sp}(W_n)$ and consider the two towers of orthogonal groups attached to the quadratic spaces with trivial discriminant. More precisely, let

$$V_m = \mathbb{H}^m \quad \text{and} \quad V_m^\# = D \oplus \mathbb{H}^{m-2}$$

and denote the orthogonal groups by $\mathrm{O}(V_m)$ and $\mathrm{O}(V_m^\#)$ respectively. For an irreducible representation π of $\mathrm{Sp}(W_n)$, one may consider the theta lifts $\theta_m(\pi)$ and $\theta_m^\#(\pi)$ to $\mathrm{O}(V_m)$ and $\mathrm{O}(V_m^\#)$ respectively (with respect to a fixed nontrivial additive character ψ). Set

$$\begin{cases} m(\pi) = \inf\{m : \theta_m(\pi) \neq 0\}, \\ m^\#(\pi) = \inf\{m : \theta_m^\#(\pi) \neq 0\}. \end{cases}$$

Then Kudla and Rallis [KR05, Ths. 3.8 and 3.9] showed:

THEOREM 5.1. (i) *For any irreducible representation* π *of* $\mathrm{Sp}(W_n)$,

$$m(\pi) + m^{\#}(\pi) \geq 2n + 2.$$

(ii) *If* π *is a supercuspidal representation of* $\mathrm{Sp}(W_n)$, *then*

$$m(\pi) + m^{\#}(\pi) = 2n + 2.$$

If we specialize this result to the case $\dim W_n = 4$ and take into account the results of the companion paper [GTb], we obtain:

THEOREM 5.2. *Let* π *be an irreducible representation of* $\mathrm{GSp}_4(F)$. *Then one has the following two mutually exclusive possibilities*:

(A) π *participates in the theta correspondence with* $\mathrm{GSO}(D) = \mathrm{GSO}_{4,0}(F)$, *where* D *is nonsplit*;

(B) π *participates in the theta correspondence with* $\mathrm{GSO}(V) = \mathrm{GSO}_{3,3}(F)$.

Another way of describing this result is that one of the following two possibilities holds:

(I) π *participates in the theta correspondence with either* $\mathrm{GSO}(D)$ *or* $\mathrm{GSO}(V_2) = \mathrm{GSO}_{2,2}(F)$ (*but necessarily not both*);

(II) π *does not participate in the theta correspondence with* $\mathrm{GSO}(D)$ *or* $\mathrm{GSO}(V_2)$, *in which case it must participate in the theta correspondence with* $\mathrm{GSO}(V)$.

Proof. Theorem 5.1(i) implies that any representation π participates in the theta correspondence with at most one of $\mathrm{GSO}(D)$ or $\mathrm{GSO}(V)$. Hence it remains to show that any π does participate in the theta correspondence with $\mathrm{GSO}(D)$ or $\mathrm{GSO}(V)$. If π is supercuspidal, this is an immediate consequence of Theorem 5.1(ii). For generic representations, it follows by Lemma 3.3 that π has nonzero theta lift to $\mathrm{GSO}(V)$; in particular, this implies the theorem for essentially discrete series representations which are not supercuspidal, since these are generic. For the remaining nongeneric representations, the result follows by an explicit determination of theta correspondences for GSp_4 given in the companion paper [GTb], especially [GTb, Ths. 8.1 and 8.3]. □

Now we come to the results of Muić-Savin [MS00b]. In the setting above, they considered a discrete series representation of $\mathrm{Sp}(W_n)$ which is generic with respect to a character χ and determine the value of $m(\pi)$. Similarly, starting with a discrete series representation τ of $\mathrm{SO}(V_m)$, one may define $n(\tau)$ analogously. Here is the result of Muić-Savin:

THEOREM 5.3. (i) *Suppose that* π *is a discrete series representation of* $\mathrm{Sp}(W_n)$ *which is generic with respect to a character* χ.

(a) *If the standard L-factor* $L(s,\pi,\mathrm{std})$ *has a pole at* $s=0$, *then* $m(\pi)=n$ *and all the irreducible constituents (as* $\mathrm{SO}(V_{m(\pi)})$*-modules) of* $\theta_{m(\pi)}(\pi)$ *are discrete series representations which are generic with respect to* χ.

(b) *If the standard L-factor* $L(s,\pi,\mathrm{std})$ *does not have a pole at* $s=0$, *then* $m(\pi)=n+1$ *and* $\theta_{m(\pi)}(\pi)$ *has a unique* χ*-generic constituent. Moreover, this* χ*-generic constituent is a discrete series representation.*

(ii) *Suppose that* τ *is a discrete series representation of* $\mathrm{SO}(V_m)$ *which is generic with respect to a character* χ.

(a) *If the standard L-factor* $L(s,\tau,\mathrm{std})$ *has a pole at* $s=0$, *then* $n(\tau)=m-1$ *and all the irreducible constituents of* $\theta_{n(\tau)}(\tau)$ *are discrete series representations which are generic with respect to* χ.

(b) *If the standard L-factor* $L(s,\tau,\mathrm{std})$ *does not have a pole at* $s=0$, *then* $n(\tau)=m$ *and* $\theta_{n(\tau)}(\tau)$ *has a unique* χ*-generic* $\mathrm{SO}(V_{m(\pi)})$*-constituent. Moreover, this* χ*-generic constituent is a discrete series representation.*

COROLLARY 5.4. *Let* $\Sigma=\Pi\boxtimes\mu$ *be a (necessarily generic) discrete series representation of* $\mathrm{GSO}(V)$ *and suppose that the twisted exterior square L-function* $L(s,\Pi,\bigwedge^2\otimes\mu^{-1})$ *has a pole at* $s=0$. *Then the theta lift of* $\Sigma=\Pi\boxtimes\mu$ *to* $\mathrm{GSp}_4(F)$ *is a nonzero generic discrete series representation.*

Proof. Let Σ_0 be an irreducible constituent of the restriction of $\Sigma=\Pi\boxtimes\mu$ to $\mathrm{SO}(V)$, so that Σ_0 is χ-generic with respect to some χ. By Lemmas 4.1 and 4.2, the standard L-function of Σ_0 is

$$L(s,\Sigma_0,\mathrm{std})=L\Big(s,\Pi,\bigwedge^2\otimes\mu^{-1}\Big)$$

and thus has a pole at $s=0$. By Theorem 5.3(ii)(a), the theta lift $\theta(\Sigma)$ of Σ to $\mathrm{GSp}_4(F)$ is nonzero and all its constituents are χ-generic discrete series representations. This shows that $\theta(\Pi\boxtimes\mu)$ is a nonzero generic discrete series representation, which is irreducible by Theorem 3.2. □

Finally, in our application of Corollary 5.4 later on, we need the following crucial result of Henniart [Hen10]:

THEOREM 5.5. *The local Langlands correspondence for* GL_n *respects the twisted exterior square L-function. In other words, if* Π *is an irreducible representation of* $\mathrm{GL}_n(F)$ *with L-parameter* ϕ_Π *and* μ *is a* 1*-dimensional character of* $F^\times$, *then*

$$L\Big(s,\Pi,\bigwedge{}^2\otimes\mu^{-1}\Big)=L\Big(s,\bigwedge{}^2\phi_\Pi\otimes\mu^{-1}\Big),$$

where the L-function on the left-hand side is that of Shahidi.

The dichotomy given in Theorem 5.2 is qualitative in nature, but can be made more concrete by the explicit determination of the three theta correspondences in question. This detailed study of theta correspondences is given in the companion paper [GTb]. We conclude this section by assembling the above results together with those of [GTb].

THEOREM 5.6. (i) *The set of irreducible representations of* $\mathrm{GSp}_4(F)$ *which are of Type* (A) *is precisely*

$$\Pi(\mathrm{GSp}_4)_{ng}^{\mathrm{temp}} := \{\textit{nongeneric essentially tempered representations of } \mathrm{GSp}_4(F)\}.$$

More precisely, under the theta correspondence for $\mathrm{GSO}(D) \times \mathrm{GSp}_4$, *the map*

$$\tau_1^D \boxtimes \tau_2^D \mapsto \theta(\tau_1^D \boxtimes \tau_2^D)$$

defines a bijection

$$\Pi(\mathrm{GSO}(D)) \textit{ modulo action of } \mathrm{GO}(D) \longleftrightarrow \Pi(\mathrm{GSp}_4)_{ng}^{\mathrm{temp}}.$$

Moreover, the image of the subset of $\tau_1^D \boxtimes \tau_2^D$*'s, with* $\tau_1^D \neq \tau_2^D$, *is precisely the subset of nongeneric supercuspidal representations of* $\mathrm{GSp}_4(F)$. *The other representations in the image are the nondiscrete series representations in* [GTb, Table 1, NDS(c)].

(ii) *The theta correspondence for* $\mathrm{GSO}(V_2) \times \mathrm{GSp}_4$ *defines an injection*

$$\Pi(\mathrm{GSO}(V_2)) \textit{ modulo action of } \mathrm{GO}(V_2) \longrightarrow \Pi(\mathrm{GSp}_4).$$

The image is disjoint from $\Pi(\mathrm{GSp}_4)_{ng}^{\mathrm{temp}}$ *and consists of*:

(a) *the generic discrete series representations* (*including the supercuspidal ones*) *such that* $L(s, \pi, \mathrm{std})$ *has a pole at* $s = 0$.

(b) *the nondiscrete series representations in* [GTb, Table 1, NDS(b, d, e)].

Moreover, the images of the representations $\tau_1 \boxtimes \tau_2$*'s, with* $\tau_1 \neq \tau_2$ *discrete series representations of* $\mathrm{GL}_2(F)$, *are precisely the representations in* (a).

(iii) *The theta correspondence for* $\mathrm{GSp}_4 \times \mathrm{GSO}(V)$ *defines an injection*

$$\Pi(\mathrm{GSp}_4) \smallsetminus \Pi(\mathrm{GSp}_4)_{ng}^{\mathrm{temp}} \longrightarrow \Pi(\mathrm{GSO}(V)) \subset \Pi(\mathrm{GL}_4) \times \Pi(\mathrm{GL}_1).$$

Moreover, the representations of $\mathrm{GSp}_4(F)$ *which are of Type* (*II*), *i.e. those not accounted for by* (i) *and* (ii) *above, are*

(a) *the generic discrete series representations* π *whose standard factor* $L(s, \pi, \mathrm{std})$ *is holomorphic at* $s = 0$. *The images of these representations under the above map are precisely the discrete series representations* $\Pi \boxtimes \mu$ *of* $\mathrm{GL}_4(F) \times \mathrm{GL}_1(F)$ *such that* $L(s, \bigwedge^2 \phi_\Pi \otimes \mu^{-1})$ *has a pole at* $s = 0$.

(b) *the nondiscrete series representations in* [GTb, Table 1, NDS(a)]. *The images of these under the above map consists of nondiscrete series representations* $\Pi \boxtimes \mu$ *such that*

$$\phi_\Pi = \rho \oplus \rho \cdot \chi \quad \textit{and} \quad \mu = \det \rho \cdot \chi,$$

for an irreducible two dimensional ρ *and a character* $\chi \neq 1$.

(iv) *If a representation* π *of* $\mathrm{GSp}_4(F)$ *with central character* μ *participates in the theta correspondence with* $\mathrm{GSO}(V_2)$, *so that*

$$\pi = \theta(\tau_1 \boxtimes \tau_2) = \theta(\tau_2 \boxtimes \tau_1),$$

then π *has a nonzero theta lift to* $\mathrm{GSO}(V)$. *If* $\Pi \boxtimes \mu$ *is the small theta lift of* π *to* $\mathrm{GSO}(V)$, *with* Π *a representation of* $\mathrm{GL}_4(F)$, *then*

$$\phi_\Pi = \phi_{\tau_1} \oplus \phi_{\tau_2} \quad \textit{and} \quad \mu = \det \phi_{\tau_1} = \det \phi_{\tau_2}.$$

The proof of Theorem 5.6 is given in [GTb], where more complete and explicit information can be found.

6. **Langlands parameters**

In this section, we record some facts about the Langlands parameters for GSp_4. Given such a parameter

$$\phi : WD_F \longrightarrow \mathrm{GSp}_4(\mathbb{C}),$$

we may consider its composition with the similitude factor sim to obtain a 1-dimensional character $\mathrm{sim}(\phi)$ of WD_F. We call $\mathrm{sim}(\phi)$ the similitude character of ϕ.

Now consider the composite of ϕ with the inclusion

$$\iota : \mathrm{GSp}_4(\mathbb{C}) \hookrightarrow \mathrm{GL}_4(\mathbb{C}) \times \mathrm{GL}_1(\mathbb{C})$$

to obtain an L-parameter for $\mathrm{GL}_4 \times \mathrm{GL}_1$. We note the following crucial lemma (cf. also [V, §7]), which is the analog of Theorem 5.6(iii) for L-parameters.

LEMMA 6.1. *The map* $\phi \mapsto \iota \circ \phi$ *gives an injection*

$$\Phi(\mathrm{GSp}_4) \hookrightarrow \Phi(\mathrm{GL}_4) \times \Phi(\mathrm{GL}_1).$$

The discrete series L-parameters $\phi \times \mu$ *of* $\mathrm{GL}_4 \times \mathrm{GL}_1$ *which are in the image of the map are precisely those such that* $L(s, \bigwedge^2 \phi \otimes \mu^{-1})$ *has a pole at* $s = 0$.

Proof. We shall prove the lemma in the context of GSp_{2n}. For any $n \geq 1$, we shall show that the natural map

$$\Phi(\mathrm{GSp}_{2n}) \longrightarrow \Phi(\mathrm{GL}_{2n}) \times \Phi(\mathrm{GL}_1)$$

is injective. This injectivity is equivalent to the following assertion: if $\phi : WD_F \longrightarrow \mathrm{GL}(V)$ is an L-parameter, with V a $2n$-dimensional vector space

over $\mathbb{C}$, and B and B' are two nondegenerate symplectic forms on V which are preserved by ϕ up to a similitude character μ, then B and B' are conjugate by an element of $\mathrm{GL}(V)$ which centralizes ϕ.

We now prove the above statement. Since (ϕ, V) is semisimple, we may write

$$V = \bigoplus_i M_i \otimes V_i$$

where V_i is irreducible and M_i is its multiplicity space. Since $V^\vee \otimes \mu \cong V$, we see that, for each i, either $V_i^\vee \otimes \mu \cong V_i$, or else $V_i^\vee \otimes \mu = V_j$ for some $j \neq i$. Corresponding to these two possibilities, we may decompose V as:

$$V = \left(\bigoplus_i M_i \otimes V_i \right) \oplus \left(\bigoplus_j M_j \otimes (W_j \oplus W_j^\vee \cdot \mu) \right).$$

Since the nondegenerate forms B and B' remain nondegenerate on each summand above, we are reduced to showing the statement on each summand.

First examine $M_i \otimes V_i$. Since V_i is irreducible, it follows by Schur's lemma that there is a unique (up to scaling) bilinear form on V_i which is WD_F-equivariant with similitude character μ. Any such nonzero form is necessarily nondegenerate and has a sign ε_i. Thus, giving a nondegenerate (WD_F, μ)-equivariant symplectic form on $M_i \otimes V_i$ is equivalent to giving a nondegenerate bilinear form on M_i of sign $-\varepsilon_i$. But any two such forms on M_i are conjugate under $\mathrm{GL}(M_i)$, which commutes with the action of WD_F. This proves the statement for the summand $M_i \otimes V_i$.

Now examine the summand $W = M_j \otimes (W_j \oplus W_j^\vee \cdot \mu)$. In this case, the subspaces $M_j \otimes W_j$ and $M_j \otimes W_j^\vee \cdot \mu$ are necessarily totally isotropic (with respect to any (WD_F, μ)-equivariant symplectic form). Moreover, there is a unique (WD_F, μ)-equivariant pairing on $W_j \times W_j^\vee \cdot \mu$, up to scaling. Hence, giving a nondegenerate (WD_F, μ)-equivariant symplectic form on W is equivalent to giving a nondegenerate bilinear form on M_j. But any two such are conjugate under the natural action of $\mathrm{GL}(M_j) \times \mathrm{GL}(M_j)$, which commutes with the action of WD_F. This proves the statement for the summand W.

Finally, suppose that $\phi \times \mu \in \Phi(\mathrm{GL}_{2n}) \times \Phi(\mathrm{GL}_1)$ is such that ϕ is irreducible as a $2n$-dimensional representation. Then it is not difficult to see that $L(s, \bigwedge^2 \phi \otimes \mu^{-1})$ has a pole at $s = 0$ if and only if $(\bigwedge^2 \phi) \otimes \mu^{-1}$ contains the trivial representation as a summand. In other words, the action of WD_F via ϕ preserves a nonzero symplectic form up to scaling by the character μ. This symplectic form is necessarily nondegenerate, so that ϕ factors through $\mathrm{GSp}_{2n}(\mathbb{C})$ after conjugation, for otherwise its kernel would be a nontrivial submodule. $\square$

The theory of endoscopy shows that GSp_4 has a unique endoscopic group which is isomorphic to $\mathrm{GSO}_{2,2}$. The dual group of $\mathrm{GSO}_{2,2}$ is

$$\mathrm{GSpin}_4(\mathbb{C}) \cong (\mathrm{GL}_2(\mathbb{C}) \times \mathrm{GL}_2(\mathbb{C}))^0 = \{(g_1, g_2) : \det g_1 = \det g_2\},$$

so that there is a distinguished conjugacy class of embeddings of dual groups

$$(\mathrm{GL}_2(\mathbb{C}) \times \mathrm{GL}_2(\mathbb{C}))^0 \hookrightarrow \mathrm{GSp}_4(\mathbb{C}).$$

This gives rise to a natural map

$$\Phi(\mathrm{GSO}_{2,2}) \longrightarrow \Phi(\mathrm{GSp}_4).$$

We say that an L-parameter $\phi \in \Phi(\mathrm{GSp}_4)$ is **endoscopic** if it is in the image of this map. More concretely, ϕ is endoscopic if

$$\phi = \phi_1 \oplus \phi_2 \quad \text{with } \dim \phi_i = 2 \text{ and } \mathrm{sim}\, \phi = \det \phi_1 = \det \phi_2.$$

Note that the ϕ_i's are not necessarily distinct and may be reducible. Observe further that the outer automorphism group $\mathrm{Out}(\mathrm{SO}_4) \cong \mathbb{Z}/2\mathbb{Z}$ of SO_4 acts on $\Phi(\mathrm{GSO}_{2,2})$ via $(\phi_1, \phi_2) \mapsto (\phi_2, \phi_1)$. It is clear that the natural map above descends to give

$$\Phi(\mathrm{GSO}_{2,2}) \text{ modulo action of } \mathrm{Out}(\mathrm{SO}_4) \longrightarrow \Phi(\mathrm{GSp}_4).$$

The following lemma is the analog of Theorem 5.6(i) and (ii) for L-parameters.

LEMMA 6.2. (i) *The map*

$$\Phi(\mathrm{GSO}_{2,2}) \textit{ modulo action of } \mathrm{Out}(\mathrm{SO}_4) \longrightarrow \Phi(\mathrm{GSp}_4)$$

is injective.

(ii) *If ϕ is a discrete series parameter of GSp_4, then ϕ is either endoscopic or is irreducible as a 4-dimensional representation. If ϕ is endoscopic, then $\phi = \phi_1 \oplus \phi_2$ with $\phi_1 \ncong \phi_2$ irreducible, and the component group A_ϕ is $\mathbb{Z}/2\mathbb{Z}$. Otherwise, A_ϕ is trivial.*

(iii) *If ϕ is a nondiscrete series parameter of GSp_4, then ϕ is either endoscopic or $\phi = \rho \oplus \rho \cdot \chi$ with $\mathrm{sim}\, \phi = \det \rho \cdot \chi$ and $\chi \neq 1$. The component group A_ϕ is trivial unless $\phi = \phi_1 \oplus \phi_2$ is endoscopic with $\phi_1 \cong \phi_2$ irreducible, in which case $A_\phi = \mathbb{Z}/2\mathbb{Z}$.*

In particular, under the injection of Lemma 6.1, *the nonendoscopic L-parameters consists precisely of those pairs $(\phi, \mu) \in \Phi(\mathrm{GL}_4) \times \Phi(\mathrm{GL}_1)$ which arise as (ϕ_Π, μ) in Theorem* 5.6(iii)(a) *and* (b).

Proof. (i) By Lemma 6.1, it suffices to show that the natural map

$$\Phi(\mathrm{GSO}_{2,2}) \text{ modulo action of } \mathrm{Out}(\mathrm{SO}_4) \longrightarrow \Phi(\mathrm{GL}_4) \times \Phi(\mathrm{GL}_1)$$

is injective. This is a simple exercise which we leave to the reader.

(ii) If ϕ is irreducible, then the centralizer in $\mathrm{GSp}_4(\mathbb{C})$ of the image of ϕ is $Z_{\mathrm{GSp}_4}(\mathrm{Im}\,\phi) = Z_{\mathrm{GSp}_4}$, so that A_ϕ is trivial. If ϕ is reducible, then the hypothesis that ϕ does not factor through any proper parabolic subgroup implies that ϕ does not stabilize any nonzero isotropic subspaces. Thus ϕ cannot contain any 1-dimensional summand and any 2-dimensional summand must be a nondegenerate symplectic subspace. This shows that $\phi = \phi_1 \oplus \phi_2$ with ϕ_i irreducible 2-dimensional and $\mathrm{sim}\,\phi = \det \phi_i$. Moreover, $\phi_1 \ncong \phi_2$, for otherwise ϕ would stabilize a 2-dimensional isotropic subspace. Thus, ϕ has the form given in (i). Moreover,

$$Z_{\mathrm{GSp}_4}(\mathrm{Im}(\phi)) \cong \{(a,b) \in \mathbb{C}^\times \times \mathbb{C}^\times : a^2 = b^2\} \subset (\mathrm{GL}_2(\mathbb{C}) \times \mathrm{GL}_2(\mathbb{C}))^0$$

with Z_{GSp_4} sitting diagonally as a subgroup. Thus, $A_\phi \cong \mathbb{Z}/2\mathbb{Z}$,

(iii) If ϕ stabilizes an isotropic line L which affords the character χ, then ϕ must stabilize another line L' which has nonzero pairing with L and affords the character $\chi^{-1} \cdot \mathrm{sim}\,\phi$. The subspace spanned by L and L' supports a 2-dimensional submodule ϕ_1 of ϕ with $\mathrm{sim}\,\phi = \det \phi_1$. It follows that ϕ is endoscopic. On the other hand, if ϕ stabilizes an isotropic plane, but not a line, then

$$\phi = \rho \oplus \rho^\vee \cdot \mathrm{sim}\,\phi = \rho \oplus \rho \cdot \chi$$

for an irreducible 2-dimensional ρ and with $\mathrm{sim}\,\phi = \chi \cdot \det \rho$. If $\chi = 1$, then $\phi = 2 \cdot \rho$ is endoscopic. If $\chi \neq 1$, then ϕ is nonendoscopic of the type given in (ii). We leave the determination of A_ϕ in the various cases to the reader. □

Remarks. In a letter [Zin] to Vigneras (dated November 25, 1984 and mentioned at the end of the introduction of [Vig86]), W. Zink gave an argument that there do not exist primitive symplectic representations of W_F. However, this is not the case, as one can construct such L-parameters when the residue characteristic p of F is 2; see [GT10a, Prop. 5.3].

7. Construction of L-packets and exhaustion

In this section, we shall construct the map L, show that it is surjective and verify (i), (ii), (iii), (iv) and (vii) of the Main Theorem. Since we are working with reductive but nonsemisimple groups G, let us mention that for the rest of the paper, by a discrete series (resp. tempered) representation of $G(F)$, we mean a representation which is equal to a unitary discrete series (resp. tempered) representation after twisting by a 1-dimensional character.

Definition of the map L. According to Theorem 5.2, the irreducible representations π of $\mathrm{GSp}_4(F)$ fall into two disjoint families of Type (I) or (II).

Type (I). If π is of Type (I), then there is an irreducible representation $\tau_1^D \boxtimes \tau_2^D$ of $\mathrm{GSO}(D)$ (where D is possibly split) such that

$$\pi = \theta(\tau_1^D \boxtimes \tau_2^D) = \theta(\tau_2^D \boxtimes \tau_1^D).$$

By the Jacquet-Langlands correspondence and the local Langlands correspondence for GL_2, each τ_i^D gives rise to an irreducible 2-dimensional representation ϕ_i of WD_F such that $\det\phi_1 = \det\phi_2$. We define $L(\pi)$ to be the parameter

$$\phi = \phi_1 \oplus \phi_2 : WD_F \longrightarrow (\mathrm{GL}_2(\mathbb{C}) \times \mathrm{GL}_2(\mathbb{C}))^0 \subset \mathrm{GSp}_4(\mathbb{C}).$$

By Theorem 5.6(i), (ii) and Lemma 6.2, we see that $L(\pi)$ is a discrete series parameter if and only if π is a discrete series representation.

Type (II). If π is of type (II), then the theta lift of π to $\mathrm{GSO}(V)$ is nonzero. Regarding $\mathrm{GSO}(V)$ as a quotient of $\mathrm{GL}_4(F) \times \mathrm{GL}_1(F)$, we may write

$$\theta(\pi) = \Pi \boxtimes \mu.$$

Note that the central character of π is necessarily equal to μ. Then we set

$$L(\pi) = \phi_\Pi \times \mu : WD_F \longrightarrow \mathrm{GL}_4(\mathbb{C}) \times \mathrm{GL}_1(\mathbb{C}),$$

where ϕ_Π is the Langlands parameter of Π. We need to show that $L(\pi)$ factors through the inclusion $\iota : \mathrm{GSp}_4(\mathbb{C}) \hookrightarrow \mathrm{GL}_4(\mathbb{C}) \times \mathrm{GL}_1(\mathbb{C})$.

If π is a discrete series representation as in Theorem 5.6(iii)(a), then Π is a discrete series representation of $\mathrm{GL}_4(F)$ such that $L(s, \wedge^2\phi_\Pi \otimes \mu^{-1})$ has a pole at $s = 0$. By Lemma 6.1, we conclude that $\phi_\Pi \times \mu$ factors through $\mathrm{GSp}_4(\mathbb{C})$ and is a discrete series parameter. On the other hand, if π is a nondiscrete series representation as in Theorem 5.6(iii)(b), then we have

$$\phi_\Pi = \phi \oplus \phi \cdot \chi \quad \text{and} \quad \mu = \det\phi \cdot \chi, \quad \text{with } \chi \neq 1.$$

One may consider the nondegenerate symplectic form which is totally isotropic on ϕ and $\phi \cdot \chi$ and such that the pairing between ϕ and $\phi \cdot \chi$ is given by the natural map

$$\phi \otimes \phi \cdot \chi \longrightarrow \bigwedge^2 \phi \cdot \chi = \mu.$$

It is clear that this last map is WD_F-equivariant, so that $L(\pi)$ is a nondiscrete series L-parameter of GSp_4. Thus, we see again that $L(\pi)$ is a discrete series parameter if and only if π is a discrete series representation.

Using the fact that the theta correspondence preserves central characters and the basic properties of the local Langlands correspondence for GL_n, one easily checks that the central character of π is equal to the similitude character of $L(\pi)$, and for any character χ, one has

$$L(\pi \otimes \chi) = L(\pi) \otimes \chi.$$

We have thus completed the definition of the map L and verified properties (i), (iii) and (iv) of the Main Theorem.

Observe that, in Theorem 5.2, there is another partition of the set of irreducible representations of $\mathrm{GSp}_4(F)$ into two sets , namely those of Type (A) or (B). One could have defined the map L using this partition as follows.

Type (A). If π is of Type (A), then $\pi = \theta(\tau_1^D \boxtimes \tau_2^D)$ where now D is the quaternion division algebra. One then defines $L(\pi)$ as in the Type (I) case above.

Type (B). If π is of Type (B), then the theta lift of π to $\mathrm{GSO}(V)$ is nonzero and has the form $\Pi \boxtimes \mu$. One then defines $L(\pi)$ following the Type (II) case above.

The only potential difference in these two definitions is for those representations π which have nonzero theta lifts to $\mathrm{GSO}(V_2) = \mathrm{GSO}_{2,2}(F)$. However, it follows from Theorem 5.6(iv) that the two definitions are in fact the same.

Surjectivity and fibers. For a given L-parameter

$$\phi : WD_F \longrightarrow \mathrm{GSp}_4(\mathbb{C}),$$

with $\mathrm{sim}(\phi) = \mu$, we must now determine the fiber of the map L over ϕ, and in particular show that it is nonempty. From the construction of L, observe that the parameter $L(\pi)$ is endoscopic if and only if π is of Type I. Thus, we see that $\#L_\phi \leq 2$ if ϕ is endoscopic, and $\#L_\phi \leq 1$ otherwise. We want to show that $\#L_\phi = \#A_\phi$, and we consider the endoscopic and nonendoscopic cases separately.

Endoscopic case: If $\phi = \phi_1 \oplus \phi_2$ is endoscopic, then ϕ gives rise to an L-parameter of $\mathrm{GSO}(V_2)$ and thus determines a representation $\tau_1 \boxtimes \tau_2$ of $\mathrm{GSO}(V_2)$, where τ_i has L-parameter ϕ_i. If the ϕ_i's are irreducible, then ϕ is also an L-parameter of $\mathrm{GSO}(D)$ and thus determines a representation $\tau_1^D \boxtimes \tau_2^D$ of $\mathrm{GSO}(D)$, with τ_i^D the Jacquet-Langlands lift of τ_i. By Theorem 5.6(i) and (ii), both $\tau_1 \boxtimes \tau_2$ and $\tau_1^D \boxtimes \tau_2^D$ have nonzero theta lifts to GSp_4, and it follows from the construction of L that

$$L_\phi = \{\theta(\tau_1 \boxtimes \tau_2), \theta(\tau_1^D \boxtimes \tau_2^D)\},$$

where the latter representation is regarded as zero if one of the ϕ_i's is reducible. In view of Lemma 6.2, we see that $\#L_\phi = \#A_\phi$. Moreover, when $\#L_\phi = 2$, we have $A_\phi = \mathbb{Z}/2\mathbb{Z}$, which has two irreducible characters $\mathbf{1}$ and $\mathbf{sign}$. In that case, we set

$$\pi_{\mathbf{1}} = \theta(\tau_1 \boxtimes \tau_2) \quad \text{and} \quad \pi_{\mathbf{sign}} = \theta(\tau_1^D \boxtimes \tau_2^D).$$

Note that the representation $\pi_{\mathbf{1}}$ is generic, whereas $\pi_{\mathbf{sign}}$ is nongeneric.

Nonendoscopic case: If ϕ is nonendoscopic with similitude character μ, then ϕ is as described in Lemma 6.2 or Theorem 5.6(iii). In this case, Theorem 5.6(iii) implies immediately that L_ϕ is nonempty, so that $\#L_\phi = 1$. Indeed,

if Π is the representation of $\mathrm{GL}_4(F)$ with L-parameter ϕ, then the representation $\Pi\boxtimes\mu$ of $\mathrm{GSO}(V)$ has nonzero theta lift to GSp_4 by Theorem 5.6(iii), so that

$$L_\phi = \{\theta(\Pi \boxtimes \mu)\}.$$

Thus, for each L-parameter ϕ, we have determined the fiber L_ϕ, which consists of one or two representations parametrized by the irreducible characters of A_ϕ. Indeed, in the companion paper [GTb, Prop. 13.1], we explicitly write down the L-parameters of all nonsupercuspidal representations. We should mention that for a nondiscrete series parameter ϕ, a construction of the L-packet has been carried out by B. Roberts and R. Schmidt in their monograph [RS07] using the results of Sally-Tadic [ST93]. The reader can refer to the table in [RS07, §A.5] for the explicit description. Using [GTb, Prop. 13.1], it is easy to check that our definition of L_ϕ agrees with theirs.

We have thus verified property (ii) in the Main Theorem.

Genericity. Let us conclude this section by verifying property (vii) of the Main Theorem, which relates generic representations and the adjoint L-factor. For those L-packets which contain a nonsupercuspidal representation, this is a straightforward calculation using the companion paper [GTb] and the structure of principal series representations of $\mathrm{GSp}_4(F)$ [ST93]. The verification in this case has been carried out in a recent paper of Asgari-Schmidt [AS08]. However, we give an independent proof in [GTb, Prop. 13.2], as our verification is much more concise than that of [AS08].

Suppose on the other hand that L_ϕ contains only supercuspidal representations. From our construction of the discrete series L-packets, it is clear that such an L-packet always contains a generic supercuspidal representation, since it contains an element which is the theta lift of a (necessarily generic) supercuspidal representation of $\mathrm{GSO}_{2,2}(F)$ or $\mathrm{GSO}(V) = \mathrm{GSO}_{3,3}(F)$. Thus, we need to show that $L(s, \mathrm{Ad}\circ\phi)$ is holomorphic at $s = 1$. But we know that ϕ is either an irreducible 4-dimensional representation of W_F, or else $\phi = \phi_1 \oplus \phi_2$ where $\phi_1 \neq \phi_2$ are irreducible 2-dimensional representations of W_F with $\det\phi_1 = \det\phi_2$. Thus, up to a twist by an unramified character, ϕ is of Galois type and has bounded image in $\mathrm{GL}_4(\mathbb{C})$. Thus, $\mathrm{Ad}\circ\phi$ is a representation of W_F with bounded image. It follows that $L(s, \mathrm{Ad}\circ\phi)$ is holomorphic in $\mathrm{Re}(s) > 0$.

8. **Preservation of local factors**

In this section, we shall check that the local Langlands correspondence we defined in the previous section respects the L-, γ- and ε-factors of pairs for generic representations and nongeneric nonsupercuspidal representations. This will prove property (v) of the Main Theorem. More precisely, we have:

THEOREM 8.1. *Let π be an irreducible generic or nonsupercuspidal representation of $\mathrm{GSp}_4(F)$ and σ an irreducible representation of $\mathrm{GL}_r(F)$. Suppose that ϕ_π and ϕ_σ are the L-parameters of π and σ respectively. Then*

$$\begin{cases} L(s, \pi \times \sigma) = L(s, \phi_\pi \otimes \phi_\sigma), \\ \varepsilon(s, \pi \times \sigma, \psi) = \varepsilon(s, \phi_\pi \otimes \phi_\sigma, \psi), \\ \gamma(s, \pi \times \sigma, \psi) = \gamma(s, \phi_\pi \otimes \phi_\sigma, \psi). \end{cases}$$

Proof. The proof is entirely similar to that of [MS00b, Prop. 5.4]. By using the multiplicativity of γ-factors and the explicit determination of the local theta correspondences given in the companion paper [GTb], we are reduced by a standard argument to the case when π and σ are generic supercuspidal representations.

We first consider the case when the theta lift of π to $\mathrm{GSO}_{2,2}(F)$ is zero, so that its theta lift to $\mathrm{GSO}(V)$ is nonzero supercuspidal. By [MS00b, Lemma 5.2], one can find a totally imaginary number field $\mathbb{F}$ such that $\mathbb{F}_{v_0} = F$ for some place v_0. We may consider the split group GSp_4 over $\mathbb{F}$. By [Sha90, Prop. 5.1], one can find a globally generic cuspidal representation Π of $\mathrm{GSp}_4(\mathbb{A}_\mathbb{F})$ such that $\Pi_{v_0} \cong \pi$ and for all other finite places $v \neq v_0$, Π_v is unramified. By [GRS97], the global theta lift $\theta(\Pi)$ of Π to $\mathrm{GSO}(V)$ is nonzero and cuspidal (since its local component Π_{v_0} does not participate in the local theta correspondence with $\mathrm{GSO}_{2,2}(F)$). Similarly, let Σ be a cuspidal representation of $\mathrm{GL}_r(\mathbb{A}_\mathbb{F})$ such that $\Sigma_{v_0} = \sigma$ and Σ_v is unramified for all finite places $v \neq v_0$.

By the functoriality of the theta correspondence for unramified representations at finite places (see Proposition 3.4), we see that for all finite $v \neq v_0$,

$$\begin{cases} L(s, \Pi_v \times \Sigma_v) = L(s, \theta(\Pi_v) \times \Sigma_v) = L(s, \phi_{\Pi_v} \otimes \phi_{\Sigma_v}), \\ \varepsilon(s, \Pi_v \times \Sigma_v, \psi_v) = \varepsilon(s, \theta(\Pi_v) \times \Sigma_v) = \varepsilon(s, \phi_{\Pi_v} \otimes \phi_{\Sigma_v}, \psi_v), \\ \gamma(s, \Pi_v \times \Sigma_v, \psi_v) = \gamma(s, \theta(\Pi_v) \times \Sigma_v) = \gamma(s, \phi_{\Pi_v} \otimes \phi_{\Sigma_v}, \psi_v). \end{cases}$$

Here, if we regard $\theta(\Pi_v)$ as a representation $\Pi'_v \boxtimes \mu_v$ of $\mathrm{GL}_4(\mathbb{F}_v) \times \mathrm{GL}_1(\mathbb{F}_v)$, then the L-factor $L(s, \theta(\Pi_v) \times \Sigma_v)$ is simply the Rankin-Selberg L-factor $L(s, \Pi'_v \times \Sigma_v)$ (cf. (e) of Table 1). Moreover, the theta correspondences over $\mathbb{C}$ have been completely determined by Adams-Barbasch [AB95]. Though they work with isometry groups, there is no subtlety in passing from isometry groups to similitude groups over $\mathbb{C}$. Thus, from their results, one sees that the theta correspondence from GSp_4 to $\mathrm{GSO}(V)$ is functorial with respect to the inclusion

$$\iota : \mathrm{GSp}_4(\mathbb{C}) \longrightarrow \mathrm{GL}_4(\mathbb{C}) \times \mathrm{GL}_1(\mathbb{C}).$$

Thus the above identities of local factors also hold over the archimedean places of $\mathbb{F}$.

Now by [Sha90, Th. 3.5 (3.14)], we see that for some finite set S of places including all the archimedean ones, we have

$$\begin{cases} L^S(s,\Pi\times\Sigma) = (\prod_{v\in S}\gamma(s,\Pi_v\times\Sigma_v,\psi_v))\cdot L^S(1-s,\Pi^\vee\times\Sigma^\vee), \\ L^S(s,\Theta(\Pi)\times\Sigma) = (\prod_{v\in S}\gamma(s,\Theta(\Pi_v)\times\Sigma_v,\psi_v))\cdot L^S(1-s,\Theta(\Pi^\vee)\times\Sigma^\vee). \end{cases}$$

Taking everything into account, one deduces that at $v = v_0$

$$\gamma(s,\Pi_{v_0}\times\Sigma_{v_0},\psi_{v_0}) = \gamma(s,\Theta(\Pi_{v_0})\times\Sigma_{v_0},\psi_{v_0}).$$

From the definition of the parameter ϕ_π and the fact that the local Langlands correspondence for GL_4 respects local factors of pairs, we conclude that

$$\gamma(s,\pi\otimes\sigma,\psi) = \gamma(s,\phi_\pi\otimes\phi_\sigma,\psi).$$

Then as in [MS00b, Cor. 5.1], since one knows that L-functions of discrete series representations are holomorphic in $\mathrm{Re}(s) > 0$, we deduce the desired identity of L-factors from that of the γ-factors. From this, the identity of ε-factors also follows. This proves the theorem for those supercuspidal π which do not lift to $\mathrm{GSO}_{2,2}(F)$.

Consider now the case when the theta lift of π to $\mathrm{GSO}_{2,2}(F)$ is nonzero. Thus, there is a supercuspidal representation $\tau_1\boxtimes\tau_2$ of $\mathrm{GSO}_{2,2}(F)$ such that $\theta(\tau_1\boxtimes\tau_2) = \pi$. Now we can globalize $\tau_1\boxtimes\tau_2$ as above and repeat the same argument to get

$$\gamma(s,\pi\times\sigma,\psi) = \gamma(s,(\tau_1\boxtimes\tau_2)\times\sigma,\psi) = \gamma(s,\phi_\pi\otimes\phi_\sigma,\psi).$$

From this, the desired identities of L-factors and ε-factors follow. The theorem is proven. □

PROPOSITION 8.2. *Suppose that one has a theory of γ-factors for all irreducible representations of $\mathrm{GSp}_4(F)\times\mathrm{GL}_r(F)$ satisfying the expected properties, for example those listed in* [LR05, Th. 4]. *Then the conclusion of Theorem* 8.1 *holds for all irreducible representations.*

Proof. We shall only give a sketch proof. As before, one is reduced to the case when π is a nongeneric supercuspidal representation and σ is a supercuspidal representation of GL_r. In this case, $\pi = \theta(\tau_1^D\boxtimes\tau_2^D)$ for a representation $\tau_1^D\boxtimes\tau_2^D$ of $\mathrm{GSO}(D)$ (where D is the quaternion division algebra over F).

Now as in the proof of Theorem 8.1, let $\mathbb{F}$ be a totally imaginary number field with $\mathbb{F}_{v_0} = F$. Let v_1 be another finite place of $\mathbb{F}$ and let $\mathbb{D}$ be the quaternion algebra over $\mathbb{F}$ ramified at precisely v_0 and v_1. Pick an irreducible representation τ of $\mathbb{D}^\times(\mathbb{F}_{v_1})$ of dimension > 1, so that $\tau\boxtimes\tau$ is a representation of $\mathrm{GSO}(\mathbb{D})(\mathbb{F}_{v_1})$. One can then find a cuspidal representation $\Pi^{\mathbb{D}}$ of $\mathrm{GSO}(\mathbb{D})(\mathbb{A}_{\mathbb{F}})$ whose local components at v_0 and v_1 are $\tau_1^D\boxtimes\tau_2^D$ and $\tau\boxtimes\tau$ respectively. Moreover, by the Jacquet-Langlands correspondence, one knows that $\mathrm{JL}(\Pi^{\mathbb{D}})$

is cuspidal and hence the local components of $\Pi^{\mathbb{D}}$ are generic at all places outside of $\{v_0, v_1\}$.

One can show that the global theta lift $\Theta(\Pi^{\mathbb{D}})$ of $\Pi^{\mathbb{D}}$ to $\mathrm{GSp}_4(\mathbb{A}_{\mathbb{F}})$ is nonzero, irreducible and cuspidal (cf. [GTa]). Moreover, the local component of $\Theta(\Pi^{\mathbb{D}})$ is π at v_0, nonsupercuspidal at v_1 and generic at all other places. Thus, one knows the desired equalities of local factors at every place except v_0, and the same argument as in the proof of Theorem 8.1 gives the desired result at v_0. □

One can also consider the standard L-function of $\mathrm{GSp}_4 \times \mathrm{GL}_r$. By the same argument as above, one has the following theorem; we omit the details.

THEOREM 8.3. *Let π be a generic representation or a nonsupercuspidal representation of $\mathrm{GSp}_4(F)$ and σ a representation of $\mathrm{GL}_r(F)$. Suppose that ϕ_π and ϕ_σ are the L-parameters of π and σ respectively. Then*

$$\begin{cases} L(s, \pi \times \sigma, \mathrm{std} \boxtimes \mathrm{std}) = L(s, \mathrm{std}\, \phi_\pi \otimes \phi_\sigma), \\ \varepsilon(s, \pi \times \sigma, \mathrm{std} \boxtimes \mathrm{std}, \psi) = \varepsilon(s, \mathrm{std}\, \phi_\pi \otimes \phi_\sigma, \psi), \\ \gamma(s, \pi \times \sigma, \mathrm{std} \boxtimes \mathrm{std}, \psi) = \gamma(s, \mathrm{std}\, \phi_\pi \otimes \phi_\sigma, \psi). \end{cases}$$

If one has a theory of these standard γ-factors for all irreducible representations of $\mathrm{GSp}_4(F) \times \mathrm{GL}_4(F)$, then the above identities hold for all irreducible representations.

9. **Conservation of Plancherel measure**

In this section, we prove (vi) of the Main Theorem, which expresses the Plancherel measure $\mu(s, \pi \times \sigma, \psi)$ in terms of the product of various gamma factors of the corresponding L-parameters. Let us briefly recall the definition of the relevant Plancherel measure.

Let π be an irreducible representation of $\mathrm{GSp}_4(F)$ and σ a representation of $\mathrm{GL}_r(F)$, so that $\pi \boxtimes \sigma$ is a representation of

$$M_r(F) := \mathrm{GSp}_4(F) \times \mathrm{GL}_r(F) \cong \mathrm{GSpin}_5(F) \times \mathrm{GL}_r(F).$$

Now M_r is the Levi factor of a maximal parabolic subgroup $P_r = M_r \cdot N_r$ of $G_r = \mathrm{GSpin}_{2r+5}$. One can thus form the generalized principal series representation

$$I_{P_r}(s, \pi \boxtimes \sigma) = \mathrm{Ind}_{P_r}^{G_r} \delta_{P_r}^{1/2} \cdot \pi \boxtimes \sigma |\det|^s,$$

where det is the determinant character of $\mathrm{GL}_r(F)$. If $\bar{P}_r = M_r \cdot \bar{N}_r$ is the opposite parabolic, then we similarly have the induced representation $I_{\bar{P}_r}(s, \pi \boxtimes \sigma)$. The additive character ψ determines a Haar measure on N_r, which induces a dual measure on $\bar{N}_r$. Then there is a standard intertwining operator

$$A_\psi(s, \pi \boxtimes \sigma, N_r, \bar{N}_r) : I_{P_r}(s, \pi \boxtimes \sigma) \longrightarrow I_{\bar{P}_r}(s, \pi \boxtimes \sigma).$$

Then the composite $A_\psi(s, \pi \boxtimes \sigma, \bar{N}_r, N_r) \circ A_\psi(s, \pi \boxtimes \sigma, N_r, \bar{N}_r)$ is a scalar operator on $I_{P_r}(s, \pi \boxtimes \sigma)$ and the Plancherel measure is the scalar-valued meromorphic function defined by

$$\mu(s, \pi \boxtimes \sigma, \psi)^{-1} = A_\psi(s, \pi \boxtimes \sigma, \bar{N}_r, N_r) \circ A_\psi(s, \pi \boxtimes \sigma, N_r, \bar{N}_r).$$

When π and σ are both supercuspidal, the analytic properties of $\mu(s, \pi \times \sigma, \psi)$ determine the points of reducibility of the principal series $I_{P_r}(s, \pi \boxtimes \sigma)$. More precisely, we have [Sil79, §§5.3–5.4]:

PROPOSITION 9.1. *Suppose that $\pi \boxtimes \sigma$ is a unitary supercuspidal representation of $\mathrm{GSp}_4(F) \times \mathrm{GL}_r(F)$.*

(i) *On the imaginary axis $i\mathbb{R}$, $\mu(s, \pi \boxtimes \sigma, \psi)$ is holomorphic and ≥ 0.*
(ii) *If $\sigma^\vee \neq \sigma \otimes \omega_\pi$, then $\mu(0, \pi \boxtimes \sigma, \psi) \neq 0$ and $I_{P_r}(s, \pi \boxtimes \sigma)$ is irreducible for all $s \in \mathbb{R}$.*
(iii) *If $\sigma^\vee = \sigma \otimes \omega_\pi$, then there is a unique real $s_0 \geq 0$ such that $I_{P_r}(s_0, \pi \boxtimes \sigma)$ is reducible. Moreover, $s_0 > 0$ if and only if $\mu(0, \pi \boxtimes \sigma, \psi) = 0$, in which case s_0 is the unique pole of $\mu(s, \pi \boxtimes \sigma, \psi)$ on the positive real axis.*

When $\pi \boxtimes \sigma$ is a generic (not necessarily supercuspidal) representation, Shahidi showed that the meromorphic function $\mu(s, \pi \boxtimes \sigma, \psi)$ can be expressed as a product of gamma factors. As a result, he was able to determine with great precision the unique point of reducibility in (iii) of the above proposition when $\mu(0, \pi \boxtimes \sigma, \psi) = 0$. Let us recall his results for the case at hand.

The parabolic subgroup $P_r \subset G_r$ gives rise to a dual parabolic subgroup $P_r^\vee = M_r^\vee \cdot N_r^\vee$ in the dual group $G_r^\vee = \mathrm{GSp}_{2r+4}(\mathbb{C})$ and we have:

$$M_r^\vee = \mathrm{GSp}_4(\mathbb{C}) \times \mathrm{GL}_r(\mathbb{C}).$$

Under the adjoint action of $M_r^\vee$, the Lie algebra $\mathfrak{n}_r^\vee$ of the unipotent radical $N_r^\vee$ decomposes as $\mathfrak{n}_r^\vee = r_1 \oplus r_2$ with

$$r_1 = \mathrm{std}^\vee \boxtimes \mathrm{std} \quad \text{and} \quad r_2 = \mathrm{sim}^{-1} \otimes \mathrm{Sym}^2$$

where std denotes the relevant standard representation and sim is the similitude character of $\mathrm{GSp}_4(\mathbb{C})$. On the opposite nilpotent radical $\bar{\mathfrak{n}}_r^\vee$, the adjoint action of $M_r^\vee$ is the dual representation $r_1^\vee \oplus r_2^\vee$. Now we have:

PROPOSITION 9.2. *Suppose that $\pi \boxtimes \sigma$ is a generic representation of $M_r(F)$ $= \mathrm{GSp}_4(F) \times \mathrm{GL}_r(F)$. Then*

$$\mu(s, \pi \boxtimes \sigma, \psi) = \gamma(s, \pi \boxtimes \sigma, r_1, \psi) \cdot \gamma(s, \pi \boxtimes \sigma, r_1^\vee, \overline{\psi}) \cdot \gamma(2s, \pi \boxtimes \sigma, r_2, \psi) \cdot \gamma(2s, \pi \boxtimes \sigma, r_2^\vee, \overline{\psi}),$$

which is in turn equal to

$$\gamma(s, \phi_\pi^\vee \otimes \phi_\sigma, \psi) \cdot \gamma(-s, \phi_\pi \otimes \phi_\sigma^\vee, \overline{\psi}) \cdot \gamma(2s, \mathrm{Sym}^2 \phi_\sigma \otimes \mathrm{sim}\, \phi_\pi^{-1}, \psi) \cdot \gamma(-2s, \mathrm{Sym}^2 \phi_\sigma^\vee \otimes \mathrm{sim}\, \phi_\pi, \overline{\psi}).$$

Proof. The first equality is a result of Shahidi [Sha90, Th. 3.5]. The second equality follows from Theorem 8.1 and a result of Henniart [Hen10, Th. 1.4]. Indeed, Henniart showed that the local Langlands correspondence for GL_r respect the twisted symmetric square epsilon-factors up to multiplication by a root of unity α. Hence

$$\begin{aligned}\gamma(2s, \pi \boxtimes \sigma, r_2, \psi) \cdot \gamma(-2s, \pi \boxtimes \sigma, r_2^\vee, \overline{\psi}) &= \frac{\gamma(2s, \pi \boxtimes \sigma, r_2, \psi)}{\gamma(1+2s, \pi \boxtimes \sigma, r_2, \psi)} \\ &= \frac{\alpha \cdot \gamma(2s, \mathrm{Sym}^2 \phi_\sigma \otimes \mathrm{sim}\, \phi_\pi^{-1}, \psi)}{\alpha \cdot \gamma(1+2s, \mathrm{Sym}^2 \phi_\sigma \otimes \mathrm{sim}\, \phi_\pi^{-1}, \psi)},\end{aligned}$$

which is in turn equal to

$$\gamma(2s, \mathrm{Sym}^2 \phi_\sigma \otimes \mathrm{sim}\, \phi_\pi^{-1}, \psi) \cdot \gamma(-2s, \mathrm{Sym}^2 \phi_\sigma^\vee \otimes \mathrm{sim}\, \phi_\pi, \overline{\psi}).$$

This explains why the root of unity α disappears. We thank Henniart for explaining this point to us. □

After these preliminaries, the main result of this section is:

THEOREM 9.3. *Let $\{\pi, \pi'\}$ be an L-packet of $\mathrm{GSp}_4(F)$ such that π' is non-generic supercuspidal and π is a generic discrete series representation. Then for any supercuspidal representation σ of $\mathrm{GL}_r(F)$,*

$$\mu(s, \pi \boxtimes \sigma, \psi) = \mu(s, \pi' \boxtimes \sigma, \psi).$$

In particular, if π is also supercuspidal, then $I_{P_r}(s, \pi \boxtimes \sigma)$ is reducible if and only if $I_{P_r}(s, \pi' \boxtimes \sigma)$ is reducible.

COROLLARY 9.4. *Property* (vi) *of the Main Theorem holds.*

The rest of the section is devoted to the proof of Theorem 9.3, which is similar to the proof of [MS00a, Prop. 2.1]. We know by the results of Section 7 that the representations π and π' can be obtained as theta lifts from $\mathrm{GSO}_{2,2}(F)$ and $\mathrm{GSO}(D)$ respectively, where D is the quaternion division algebra over F. Thus, we have

$$\pi = \theta(\tau_1 \boxtimes \tau_2) \quad \text{and} \quad \pi' = \theta(\tau_1^D \boxtimes \tau_2^D),$$

where τ_i^D is a representation of $D^\times$ with the Jacquet-Langlands lift τ_i on $\mathrm{GL}_2(F)$. Moreover, we know that $\tau_1 \neq \tau_2$.

Now choose a number field $\mathbb{F}$ such that for two places v_1 and v_2, one has $\mathbb{F}_{v_1} = \mathbb{F}_{v_2} = F$. Let $\mathbb{D}$ be the quaternion division algebra over $\mathbb{F}$ which is ramified precisely at v_1 and v_2. One can find a cuspidal representation Ξ of $\mathrm{GSO}_{2,2}(\mathbb{A}_\mathbb{F})$ such that $\Xi_{v_i} = \tau_1 \boxtimes \tau_2$ for $i = 1$ or 2. If $\Xi^\mathbb{D}$ denotes the Jacquet-Langlands lift of Ξ to $\mathrm{GSO}(\mathbb{D})(\mathbb{A}_\mathbb{F})$, then we have $\Xi^\mathbb{D}_{v_i} = \tau_1^D \boxtimes \tau_2^D$ for $i = 1$ or 2. We may now consider the global theta lifts $\Pi = \Theta(\Xi)$ and $\Pi' = \Theta(\Xi^\mathbb{D})$ of Ξ and $\Xi^\mathbb{D}$ to $\mathrm{GSp}_4(\mathbb{A}_\mathbb{F})$. We have:

LEMMA 9.5. *The global theta lifts* $\Pi = \Theta(\Xi)$ *and* $\Pi' = \Theta(\Xi^{\mathbb{D}})$ *are nonzero irreducible cuspidal representations of* $\mathrm{GSp}_4(\mathbb{A}_{\mathbb{F}})$. *Moreover, for* $i = 1$ *or* 2,

$$\Pi_{v_i} = \pi \quad \textit{and} \quad \Pi'_{v_i} = \pi',$$

and for all $v \neq v_1$ *or* v_2, $\Pi_v = \Pi'_v$.

Proof. Since Ξ is globally generic and $\tau_1 \neq \tau_2$, the nonvanishing and cuspidality of $\Pi = \Theta(\Xi)$ follows from [GRS97]. On the other hand, the nonvanishing and cuspidality of $\Pi' = \Theta(\Xi^{\mathbb{D}})$ follows from [GT3]. □

Hence, we have two irreducible cuspidal representations Π and Π' on $\mathrm{GSp}_4(\mathbb{A}_{\mathbb{F}})$ which are locally isomorphic at all $v \neq v_1$ or v_2, and whose local components at v_i ($i = 1$ or 2) are π and π' respectively.

Similarly, let Σ be a cuspidal representation of $\mathrm{GL}_r(\mathbb{A}_{\mathbb{F}})$ such that $\Sigma_{v_i} = \sigma$ for $i = 1$ and 2. Now consider the global induced representations on $G_r(\mathbb{A}_{\mathbb{F}})$:

$$I_{P_r}(s, \Pi \boxtimes \Sigma) \quad \text{and} \quad I_{P_r}(s, \Pi' \boxtimes \Sigma).$$

Fix an additive character Ψ of $\mathbb{F}\backslash\mathbb{A}_{\mathbb{F}}$ such that $\Psi_{v_1} = \Psi_{v_2} = \psi$. Then there are global standard intertwining operators

$$A_\Psi(s, \Pi \boxtimes \Sigma, N_r, \bar{N}_r) : I_{P_r}(s, \Pi \boxtimes \Sigma) \longrightarrow I_{\bar{P}_r}(s, \Pi \boxtimes \Sigma)$$

and

$$A_\Psi(s, \Pi' \boxtimes \Sigma, N_r, \bar{N}_r) : I_{P_r}(s, \Pi' \boxtimes \Sigma) \longrightarrow I_{\bar{P}_r}(s, \Pi' \boxtimes \Sigma),$$

which satisfy the functional equations

$$A_\Psi(s, \Pi \boxtimes \Sigma, \bar{N}_r, N_r) \circ A_\Psi(s, \Pi \boxtimes \Sigma, N_r, \bar{N}_r) = \mathrm{Id}$$

and

$$A_\Psi(s, \Pi' \boxtimes \Sigma, \bar{N}_r, N_r) \circ A_\Psi(s, \Pi' \boxtimes \Sigma, N_r, \bar{N}_r) = \mathrm{Id}\,.$$

These global intertwining operators are actually independent of the choice of Ψ, but their decompositions into local intertwining operators (for $\mathrm{Re}(s)$ large) depend on Ψ. By comparing the two global functional equations and using the fact that $\Pi_v = \Pi'_v$ for all $v \neq v_1$ or v_2, we deduce that

$$\mu(s, \pi \boxtimes \sigma, \psi)^2 = \mu(s, \pi' \boxtimes \sigma, \psi)^2 \quad \text{for all } s \in \mathbb{C}.$$

However, the two Plancherel measures are ≥ 0 when s is purely imaginary. So we have the desired equality:

$$\mu(s, \pi \boxtimes \sigma, \psi) = \mu(s, \pi' \boxtimes \sigma, \psi).$$

This completes the proof of Theorem 9.3.

In a similar fashion, the group $M_r = \mathrm{GSp}_4 \times \mathrm{GL}_r$ may be considered as a Levi subgroup of a maximal parabolic subgroup P'_r of $G'_r = \mathrm{GSp}_{2r+4}$. Thus, given a representation $\pi \boxtimes \sigma$ of $M_r(F)$, one may consider the generalized principal series representation $I_{P'_r}(s, \pi \boxtimes \sigma)$ and the associated Plancherel measure $\mu'(s, \pi \boxtimes \sigma, \psi)$. The same argument as above gives:

THEOREM 9.6. *Let $\{\pi, \pi'\}$ be an L-packet of $\mathrm{GSp}_4(F)$ such that π' is non-generic supercuspidal and π is a generic discrete series representation. Then for any supercuspidal representation σ of $\mathrm{GL}_r(F)$,*

$$\mu'(s, \pi \boxtimes \sigma, \psi) = \mu'(s, \pi' \boxtimes \sigma, \psi),$$

and these are in turn given by:

$$\gamma(s, \mathrm{std}\phi_\pi^\vee \otimes \phi_\sigma, \psi) \cdot \gamma(-s, \mathrm{std}\phi_\pi \otimes \phi_\sigma^\vee, \overline{\psi}) \cdot \gamma\Big(2s, \bigwedge^2 \phi_\sigma, \psi\Big) \cdot \gamma\Big(-2s, \bigwedge^2 \phi_\sigma^\vee, \overline{\psi}\Big).$$

10. **Characterization of the map L**

In this section, we shall show that the map L is characterized by the properties (i), (iii), (v) and (vi) in the Main Theorem. In particular, this will complete the proof of the Main Theorem. More precisely, we show:

THEOREM 10.1. *There is at most one map*

$$L : \Pi(\mathrm{GSp}_4) \longrightarrow \Phi(\mathrm{GSp}_4)$$

satisfying:

(a) *the central character ω_π of π corresponds to the similitude character $\mathrm{sim}(\phi_\pi)$ of $\phi_\pi := L(\pi)$ under local class field theory*;

(b) *π is an essentially discrete series representation if and only if ϕ_π does not factor through any proper Levi subgroup of $\mathrm{GSp}_4(\mathbb{C})$*;

(c) *if π is generic or nonsupercuspidal, then for any irreducible representation σ of $\mathrm{GL}_r(F)$ with $r \leq 2$,*

$$\begin{cases} L(s, \pi \times \sigma) = L(s, \phi_\pi \otimes \phi_\sigma), \\ \varepsilon(s, \pi \times \sigma, \psi) = \varepsilon(s, \phi_\pi \otimes \phi_\sigma, \psi). \end{cases}$$

(d) *if π is nongeneric supercuspidal, then for any supercuspidal representation σ of $\mathrm{GL}_r(F)$ with $r \leq 2$, the Plancherel measure $\mu(s, \pi \boxtimes \sigma)$ is equal to*

$$\gamma(s, \phi_\pi^\vee \otimes \phi_\sigma, \psi) \cdot \gamma(-s, \phi_\pi \otimes \phi_\sigma^\vee, \overline{\psi}) \cdot \gamma(2s, \mathrm{Sym}^2\phi_\sigma \otimes \mathrm{sim}\,\phi_\pi^{-1}, \psi) \cdot \gamma(-2s, \mathrm{Sym}^2\phi_\sigma^\vee \otimes \mathrm{sim}\,\phi_\pi, \overline{\psi}).$$

The rest of the section is devoted to the proof of this theorem. Recall from Lemma 6.1 that the natural inclusion

$$\iota : \mathrm{GSp}_4(\mathbb{C}) \longrightarrow \mathrm{GL}_4(\mathbb{C}) \times \mathrm{GL}_1(\mathbb{C})$$

gives rise to an injection

$$\iota_* : \Phi(\mathrm{GSp}_4) \hookrightarrow \Phi(\mathrm{GL}_4) \times \Phi(\mathrm{GL}_1).$$

where the projection to $\Phi(\mathrm{GL}_1)$ is simply given by taking similitude character. Hence, if L_1 and L_2 are two maps satisfying the requirements of the theorem,

it suffices to show that $\iota_* \circ L_1 = \iota_* \circ L_2$. In view of the requirement (a) in the theorem, it suffices to show that for each $\pi \in \Pi(\mathrm{GSp}_4)$ with $\phi_i := L_i(\pi)$ ($i = 1$ or 2), ϕ_1 and ϕ_2 are equivalent as 4-dimensional representations of the Weil-Deligne group WD_F. We treat two different cases separately.

Case 1: *π is generic or nonsupercuspidal.* In this case, the requirement (c) implies that

$$\begin{cases} L(s, \phi_1 \otimes \phi_\sigma) = L(s, \phi_2 \otimes \phi_\sigma), \\ \varepsilon(s, \phi_1 \otimes \phi_\sigma, \psi) = \varepsilon(s, \phi_2 \otimes \phi_\sigma, \psi) \end{cases} \tag{10.2}$$

for any representation σ of $\mathrm{GL}_r(F)$ with $r \leq 2$. At this point, we remark that if the above equalities are assumed to hold for $r \leq 3$, then the results of Henniart in [Hen02, Cor. 1.4 and Th. 1.7] would immediately imply that $\phi_1 \cong \phi_2$, as desired. However, since we are only requiring the above equalities to hold for $r \leq 2$, we need a refinement of these results of Henniart.

The first refinement (which refines [Hen02, Cors. 1.4 and 1.9]) is the so-called $n \times (n-2)$ local converse theorem for GL_n which was shown in the thesis of J.-P. Chen and has recently appeared in print [Che06]. We are of course interested in the case $n = 4$. Applying [Che06, Th. 1.1], we see that if ϕ_1 and ϕ_2 are both irreducible representations of the Weil-Deligne group WD_F, then equation (10.2) implies that $\phi_1 \cong \phi_2$.

We thus need to consider the case when, without loss of generality, ϕ_1 is reducible. Our argument below is nothing more than an elaboration of the proof of [Hen02, Th. 1.7], taking into account the key observation that since ϕ_1 and ϕ_2 factor through $\mathrm{GSp}_4(\mathbb{C})$, they do not contain any irreducible 3-dimensional constituent.

To be more precise, any irreducible representation of WD_F is of the form $\rho \boxtimes S_r$ where ρ is an irreducible representation of the Weil group W_F and S_r is the irreducible r-dimensional representation of $\mathrm{SL}_2(\mathbb{C})$. If ϕ is a semisimple representation of WD_F, then for each $\rho \boxtimes S_r$, let $m_\phi(\rho, r) \geq 0$ be the multiplicity of $\rho \boxtimes S_r$ in ϕ. Moreover, let $\nu_\phi(\rho, r) \geq 0$ denote the order of poles at $s = 0$ of the local L-factor $L(s, \phi \otimes (\rho \boxtimes S_r)^\vee)$. We make a few simple observations:

(i) From the definitions, one has:

$$L(s, \rho \boxtimes S_r) = L\left(s + \frac{r-1}{2}, \rho\right).$$

(ii) With $t = \min(r, k)$,

$$L(s, (\rho \boxtimes S_r) \otimes (\varphi \boxtimes S_k)^\vee) = \prod_{j=0}^{t-1} L\left(s + \frac{r+k-2-2j}{2}, \rho \otimes \varphi^\vee\right).$$

(iii) For any semisimple representation ϕ of WD_F,

$$\nu_\phi(\rho, r) = \sum_{k \geq 1} \sum_{j=1}^{\min(r,k)} m_\phi\left(\rho| - |^{-(\frac{r+k}{2}-j)}, k\right).$$

(iv) If the ϕ_1 and ϕ_2 above have a common constituent, then $\phi_1 \cong \phi_2$. Indeed, after cancelling this common constituent, we have two representations ϕ_1' and ϕ_2' of dimension < 4. Moreover, L- and ε-factors behave multiplicatively with respect to direct sums of representations. Hence, the analog of equation (10.2) holds for ϕ_i' with $r \leq 2$. Thus, one can apply [Hen02, Cor. 1.4 and Th. 1.7] to conclude that $\phi_1' \cong \phi_2'$.

For $i = 1$ or 2, let us write $m_i(\rho, r)$ and $\nu_i(\rho, r)$ for $m_{\phi_i}(\rho, r)$ and $\nu_{\phi_i}(\rho, r)$ respectively. Now using the above observations (see also [Hen02, §4.2]), one sees that for any 1-dimensional character χ of W_F, one has the following system of equations:

$$\begin{cases} \nu_i(\chi, 2) = m_i(\chi| - |^{-1/2}, 1) + m_i(\chi, 2) \\ \qquad\qquad + m_i(\chi| - |^{-1}, 2) + m_i(\chi| - |^{-1}, 4) + m_i(\chi| - |^{-2}, 4), \\ \nu_i(\chi| - |^{-1/2}, 1) = m_i(\chi| - |^{-1/2}, 1) + m_i(\chi| - |^{-1}, 2) + m_i(\chi| - |^{-2}, 4), \\ \nu_i(\chi| - |^{1/2}, 1) = m_i(\chi| - |^{1/2}, 1) + m_i(\chi, 2) + m_i(\chi| - |^{-1}, 4). \end{cases}$$

In particular, this implies that

$$m_i(\chi, 1) = \nu_i(\chi, 1) + \nu_i(\chi| - |^{-1}, 1) - \nu_i(\chi| - |^{-1/2}, 2).$$

Thus, using the requirement (c), we conclude that any 1-dimensional irreducible constituent of ϕ_1 occurs with the same multiplicity in ϕ_2. In particular, if ϕ_1 (and thus ϕ_2) contains a 1-dimensional irreducible constituent, we are done by observation (iv) above.

We are thus reduced to the case where ϕ_1 is the sum of two (possibly equivalent) irreducible 2-dimensional representations. If ρ is an irreducible 2-dimensional representation of W_F, then

$$\nu_i(\rho, 1) = m_i(\rho, 1) + m_i(\rho| - |^{-1/2}, 2).$$

Thus, if ϕ_1 contains ρ and ϕ_2 does not, we must have $\phi_2 = \rho| - |^{-1/2} \boxtimes S_2$ and $\nu_i(\rho, 1) = m_1(\rho, 1) = 1$. This would imply that $\phi_1 = \rho \oplus \rho_0$ with $\rho_0 \neq \rho$, and thus $L(s, \phi_1 \otimes \rho_0^\vee)$ has a pole at $s = 0$ whereas $L(s, \phi_2 \otimes \rho_0^\vee)$ is holomorphic at $s = 0$. With this contradiction, we see that if ϕ_1 contains ρ, so must ϕ_2. Then by observation (iv) above, we have $\phi_1 \cong \phi_2$.

Finally, we are reduced to the case where

$$\phi_1 = (\chi \boxtimes S_2) \oplus (\mu \boxtimes S_2),$$

and ϕ_2 is either of the same form as ϕ_1 or is irreducible. In this case, one has

$$\nu_i(\chi| - |^{1/2}, 1) = m_i(\chi, 2) + m_i(\chi| - |^{-1}, 4).$$

Thus, if ϕ_2 does not contain $\chi \boxtimes S_2$, then we must have $\phi_2 = \chi|-|^{-1} \boxtimes S_4$ and $m_1(\chi, 2) = 1$. This implies that $\mu \neq \chi$ and one has

$$\nu_1(\mu, 2) = 1 \quad \text{and} \quad \nu_2(\mu, 2) = 0$$

which is a contradiction. Hence, ϕ_2 must also contain $\chi \boxtimes S_2$ and by observation (iv) again, we deduce that $\phi_1 \cong \phi_2$.

Case 2: *π is nongeneric supercuspidal.* In this case, we know that $\pi = \theta(\tau_1^D \boxtimes \tau_2^D)$ for a representation $\tau_1^D \boxtimes \tau_2^D$ of $\mathrm{GSO}(D)$. If τ_i is the Jacquet-Langlands lift of τ_i^D, then set

$$\Phi = \phi_{\tau_1} \oplus \phi_{\tau_2}.$$

We have shown in the previous section that $\mu(s, \pi \boxtimes \sigma)$ is equal to

$$\gamma(s, \Phi \otimes \phi_\sigma, r_1, \psi) \cdot \gamma(s, \Phi \otimes \phi_\sigma, r_1^\vee, \psi) \cdot \gamma(2s, \pi \boxtimes \sigma, r_2, \psi) \cdot \gamma(2s, \pi \boxtimes \sigma, r_2^\vee, \psi).$$

Hence, if L is a map satisfying the conditions of the theorem with $\phi_\pi := L(\pi)$, then the requirement (d) implies that

(10.3)

$$\gamma(s, \phi_\pi \otimes \phi_\sigma, r_1, \psi) \cdot \gamma(s, \phi_\pi \otimes \phi_\sigma, r_1^\vee, \psi) = \gamma(s, \Phi \otimes \phi_\sigma, r_1, \psi) \cdot \gamma(s, \Phi \otimes \phi_\sigma, r_1^\vee, \psi).$$

We need to show that this forces ϕ_π to be equal to Φ.

Suppose first that τ_1 is supercuspidal. Taking $\sigma = \tau_1$, the right-hand side of equation (10.3) has a zero at $s = 0$, which implies that $L(-s, \phi_\pi \otimes \phi_{\tau_1}^\vee) \cdot L(s, \phi_\pi^\vee \otimes \phi_{\tau_1})$ has a pole at $s = 0$. Thus, for some irreducible constituent $\phi = \rho \boxtimes S_r$ of ϕ_π, the function

$$L(-s, \phi \otimes \phi_{\tau_1}^\vee) \cdot L(s, \phi^\vee \otimes \phi_{\tau_1})$$

has a pole at $s = 0$. Since τ_1 is supercuspidal, this occurs if and only if

$$\rho = \phi_{\tau_1} \otimes |-|^{\pm(r-1)/2}.$$

Hence, we conclude that ϕ_π must contain $\phi_{\tau_1}^\vee \cdot |-|^{\pm(r-1)/2} \boxtimes S_r$ as a constituent and further $r = 1$ or 2. In other words, either ϕ_π contains ϕ_{τ_1} or else ϕ_π is equal to the irreducible representation $\phi_{\tau_1}|-|^{\pm 1/2} \boxtimes S_2$.

On the other hand, suppose that τ_1 is the twisted Steinberg representation St_χ so that $\phi_{\tau_1} = \chi \boxtimes S_2$. Taking $\sigma = \chi$, we see that the right-hand side of equation (10.3) has a zero at $s = 1/2$. Thus, for some constituent $\phi = \rho \boxtimes S_r$ of ϕ_π, the function

$$\gamma(s, \rho^\vee \cdot \chi \boxtimes S_r) \cdot \gamma(-s, \rho \cdot \chi^{-1} \boxtimes S_r)$$
$$= (\varepsilon\text{-factors}) \cdot \frac{L(\frac{r-1}{2} + 1 - s, \rho \cdot \chi^{-1}) \cdot L(\frac{r-1}{2} + 1 + s, \rho^\vee \cdot \chi)}{L(\frac{r-1}{2} + s, \rho^\vee \cdot \chi) \cdot L(\frac{r-1}{2} - s, \rho \cdot \chi^{-1})}$$

has a zero at $s = 1/2$. Thus, the denominator must have a pole at $s = 1/2$. This occurs if and only if

$$\rho = \chi \cdot |-|^{r/2} \quad \text{or} \quad \rho = \chi \cdot |-|^{-(r-2)/2}.$$

Now by requirement (b) of the theorem, we know that ϕ_π is either irreducible or is the sum of two inequivalent 2-dimensional irreducible representations. Thus, $r = 2$ or 4 above. In other words, ϕ_π is either the irreducible 4-dimensional representation S_4 up to twisting, or else ϕ_π contains $\chi|-| \boxtimes S_2$ or $\chi \boxtimes S_2$.

Taking note of the fact that $\tau_1 \neq \tau_2$ but $\omega_{\tau_1} = \omega_{\tau_2}$, the above discussion shows that ϕ_π must be the sum of two irreducible 2-dimensional representations of WD_F, with each τ_i contributing a constituent equal to ϕ_{τ_i} or $\phi_{\tau_i} \cdot |-|$. Moreover, the latter can only occur if τ_i is nonsupercuspidal. However, it is not difficult to check that equation (10.3) cannot hold for all σ if ϕ_π contains $\phi_{\tau_i} \cdot |-|$ as a constituent. The proof of Theorem 10.1 is complete.

Remarks. Observe that the requirement (b) is only used in the analysis of Case 2 above. If the requirement (c) in the theorem is known to hold for all representations π, then the proof given above in Case 1 shows that the map L is characterized by (a) and (c) alone.

11. **Comparison with construction of Vigneras**

When $p \neq 2$, Vigneras [Vig86] and Roberts [Rob01] have given an alternative definition of L-packets. In this section, we verify that their definition agrees with ours. For that, we shall begin by recalling their construction.

We shall focus on an L-parameter ϕ for GSp_4 which is trivial on $\mathrm{SL}_2(\mathbb{C})$ and such that ϕ is irreducible of the form

$$\phi = \mathrm{ind}_{W_K}^{W_F} \phi_\rho.$$

Here K/F is a quadratic field extension and ϕ_ρ is an irreducible 2-dimensional representation of W_K which does not extend to W_F but such that $\det \phi_\rho$ does extend. In this case, there are two extensions of $\det \phi_\rho$ to W_F, differing from each other by twisting by $\omega_{K/F}$. One of these extensions of $\det \phi_\rho$ is precisely the similitude character μ of ϕ. This is the crucial case treated in [Vig86].

To construct the L-packet associated to ϕ following [Vig86], consider the rank 4 quadratic spaces

$$V_K^+ = (K, \mathbb{N}_K) \oplus \mathbb{H} \quad \text{and} \quad V_K^- = (K, \delta \cdot \mathbb{N}_K) \oplus \mathbb{H},$$

where $\delta \in F^\times \smallsetminus \mathbb{N}_K(K^\times)$. The associated orthogonal similitude groups of these two quadratic spaces are isomorphic:

$$\mathrm{GSO}(V_K^\varepsilon) \cong (\mathrm{GL}_2(K) \times F^\times)/\Delta K^\times,$$

where $\Delta(a) = (a, \mathbb{N}_K(a)^{-1})$ for $a \in K^\times$.

If ρ is the supercuspidal representation of $\mathrm{GL}_2(K)$ associated to ϕ_ρ, then ρ is not obtained via base change from $\mathrm{GL}_2(F)$. One may consider the representation $\rho \boxtimes \mu$ of $\mathrm{GSO}(V_K^\varepsilon)$. Considering the theta correspondence for $\mathrm{GSO}(V_K^\varepsilon) \times \mathrm{GSp}_4(F)^+$, one gets two irreducible representations $\theta^\varepsilon(\rho \boxtimes \mu)$ of $\mathrm{GSp}_4(F)^+$, which are exchanged under the conjugation action of an element of $\mathrm{GSp}_4(F) \setminus \mathrm{GSp}_4(F)^+$. Thus we have an irreducible representation

$$V(\rho \boxtimes \mu) = \mathrm{ind}_{\mathrm{GSp}_4^+}^{\mathrm{GSp}_4} \theta^\varepsilon(\rho \boxtimes \mu).$$

The packet associated to ϕ by Vigneras consists of the single supercuspidal representation $V(\rho \boxtimes \mu)$.

On the other hand, if Π_ρ is the supercuspidal representation of $\mathrm{GL}_4(F)$ attached to ϕ, then Π_ρ is the automorphic induction of ρ from $\mathrm{GL}_2(K)$ to $\mathrm{GL}_4(F)$. According to our definition of L-packets, the packet associated to ϕ is the theta lift of $\Pi_\rho \boxtimes \mu$ to $\mathrm{GSp}_4(F)$. Thus, to show that our definition is consistent with that of Vigneras and Roberts, it suffices to show:

Proposition 11.1. *Under the theta lifting from* $\mathrm{GSp}_4(F)$ *to* $\mathrm{GSO}(V)$, *we have*

$$\theta(V(\rho \boxtimes \mu)) = \Pi_\rho \boxtimes \mu.$$

Proof. We shall show the proposition using global means. Pick a quadratic extension $\mathbb{K}/\mathbb{F}$ of number fields such that for some place v of $\mathbb{F}$, we have $\mathbb{K}_v/\mathbb{F}_v = K/F$. Consider the quadratic space $V_\mathbb{K} = (\mathbb{K}, \mathbb{N}_\mathbb{K}) \oplus \mathbb{H}$ over $\mathbb{F}$ so that

$$\mathrm{GSO}(V_\mathbb{K}) \cong (\mathrm{GL}_2(\mathbb{K}) \times \mathbb{F}^\times)/\Delta\mathbb{K}^\times.$$

Let $\Xi \boxtimes \Upsilon$ be a cuspidal representation of $\mathrm{GSO}(V_\mathbb{K})(\mathbb{A}_\mathbb{K})$ whose local component at the place v is isomorphic to $\rho \boxtimes \mu$. Consider the global theta lift $V(\Xi \boxtimes \Upsilon)$ of $\Xi \boxtimes \Upsilon$ to $\mathrm{GSp}_4(\mathbb{A}_\mathbb{K})$. Then $V(\Xi \boxtimes \Upsilon)$ is a nonzero globally generic cuspidal representation with central character Υ and the local component at v of an irreducible constituent of $V(\Xi \boxtimes \Upsilon)$ is the representation $V(\rho \boxtimes \mu)$.

Now consider the theta lift of $V(\Xi \boxtimes \Upsilon)$ from $\mathrm{GSp}_4(\mathbb{A}_\mathbb{F})$ to $\mathrm{GSO}(V)(\mathbb{A}_\mathbb{F})$. It is not difficult to see that this theta lift is nonzero and cuspidal. Thus we obtain a cuspidal representation $\Sigma \boxtimes \Upsilon$ on $\mathrm{GL}_4(\mathbb{A}_\mathbb{F}) \times \mathrm{GL}_1(\mathbb{A}_\mathbb{F})$, whose local component at v is the representation $\theta(V(\rho \boxtimes \mu))$.

From the functoriality of theta correspondences for unramified representations (Proposition 3.4), we see that Σ is nearly equivalent to the automorphic induction of Ξ from $\mathrm{GL}_2(\mathbb{A}_\mathbb{K})$ to $\mathrm{GL}_4(\mathbb{A}_\mathbb{F})$. By the strong multiplicity one theorem for GL_4, these two representations are thus isomorphic. In particular, the proposition follows by extracting the local component at v. □

Corollary 11.2. *When* $p \neq 2$, *Vigneras' construction of L-packets in* [Vig86] *exhausts all irreducible representations of* $\mathrm{GSp}_4(F)$.

12. **Global generic lifting from** GSp_4 **to** GL_4

We conclude this paper with a global consequence of our local results. Hence, in this section, F will denote a number field and $\mathbb{A}$ its ring of adeles. Jacquet, Piatetski-Shapiro and Shalika announced some thirty years ago that they have used the global theta correspondence for $\mathrm{GSp}_4 \times \mathrm{GSO}(V)$ to show the weak lifting of globally generic cuspidal representations of $\mathrm{GSp}_4(\mathbb{A})$ to $\mathrm{GL}_4(\mathbb{A})$, but unfortunately their proof was never published. Recently, an alternative proof of this weak lifting, with some further refinements, was given by Asgari-Shahidi [AS06] using the converse theorem.

Though the results of Jacquet-Piatetski-Shapiro-Shalika were not published, essentially all the details of their proof can be found in the papers [Sou87] and [GRS97]. Namely, given a globally generic cuspidal representation π of $\mathrm{GSp}_4(\mathbb{A})$, one considers its global theta lift $\Theta(\pi)$ to $\mathrm{GSO}(V)$ and shows that the Whittaker-Fourier coefficients of $\Theta(\pi)$ can be expressed in terms of the Whittaker coefficient of π [GRS97, Prop. 2.2 and Cor. 2.5]. Thus one concludes that $\Theta(\pi)$ is globally generic (and in particular is nonzero). Moreover, by the functoriality of the local theta correspondence for unramified representations (Proposition 3.4), one sees that $\Theta(\pi)$ is a weak lift of π.

The purpose of this last section is to strengthen this weak lifting to a strong one:

THEOREM 12.1. *The global theta lifting $\pi \mapsto \Theta(\pi) = \Pi \boxtimes \mu$ defines an injection from the set of globally generic cuspidal representations of $\mathrm{GSp}_4(\mathbb{A})$ to the set of globally generic automorphic representations $\Pi \boxtimes \mu$ of $\mathrm{GL}_4(\mathbb{A}) \times \mathrm{GL}_1(\mathbb{A})$. Moreover, one has*:

(i) *$\mu = \omega_\pi$ and the central character of Π is ω_π^2;*
(ii) *$\Pi \cong \Pi^\vee \otimes \omega_\pi$;*
(iii) *The image of the lifting consists precisely of those automorphic representations $\Pi \boxtimes \mu$ satisfying one of the following*:
  (a) *Π is cuspidal and the (partial) twisted exterior square L-function $L^S(s, \Pi, \bigwedge^2 \otimes \mu^{-1})$ has a pole at $s = 1$;*
  (b) *Π is an isobaric sum $\tau_1 \boxplus \tau_2$, where $\tau_1 \neq \tau_2$ are cuspidal representations of $\mathrm{GL}_2(\mathbb{A})$ such that $\tau_i = \tau_i^\vee \otimes \mu$. In this case, π is the global theta lift of the cuspidal representation $\tau_1 \boxtimes \tau_2$ of $\mathrm{GSO}_{2,2}(\mathbb{A})$.*
(iv) *for each place v of F, one has an equality of L-parameters*

$$\phi_{\pi,v} = \phi_{\Pi,v} : WD_{F_v} \longrightarrow \mathrm{GL}_4(\mathbb{C}).$$

In other words, $\pi \mapsto \Theta(\pi)$ is a **strong** *lift and one has the following equalities of global L-functions and ε-factors*:

$$L(s, \pi \times \Sigma) = L(s, \Pi \times \Sigma) \quad \textit{and} \quad \varepsilon(s, \pi \times \Sigma) = \varepsilon(s, \Pi \times \Sigma)$$

for any cuspidal representation Σ of $\mathrm{GL}_r(\mathbb{A})$.

We remark that (i), (ii) and parts of (iii) constitute the main theorem of [AS06] and are proved by entirely different methods there. More precisely, [ASh] showed that the image of the global generic lifting is contained in the set of automorphic representations satisfying the conditions (a) and (b) of (iii). Thus the new results in our theorem are (iv) and the other half of (iii), namely that any automorphic representation satisfying (a) and (b) of (iii) is in the image.

The rest of the section is devoted to the proof of the theorem. By the local results of this paper, the local theta lift for $\mathrm{GSp}_4 \times \mathrm{GSO}(V)$ satisfies Howe's conjecture. The analogous result at archimedean places has been proved by Adams-Barbasch [AB95] over $\mathbb{C}$ and by A. Paul [Pa] over $\mathbb{R}$ (suitably extended to the similitude case). Thus the map $\pi \mapsto \Theta(\pi) = \Pi \boxtimes \mu$ is injective. Since the local theta correspondence preserves central characters, we immediately deduce (i). Moreover, for a finite place, (iv) follows directly from our definition of L-parameters for GSp_4 and Theorem 5.6(iii)(b). For an infinite place, (iv) follows by [AB95] and [Pau05]. Thus, the weak lifting is strong.

By (iv) and the property of the local Langlands correspondence for GSp_4 proved in this paper, one knows that for each place v, $\phi_{\Pi,v}$ factors through $\mathrm{GSp}_4(\mathbb{C})$ with similitude character μ_v. So we have $\phi_{\Pi,v} \cong \phi_{\Pi,v}^\vee \otimes \mu_v$. By the properties of the local Langlands correspondence for GL_4, one deduces that

$$\Pi_v \cong \Pi_v^\vee \otimes \mu_v.$$

This proves (ii).

Finally we come to (iii). By the functoriality of the theta correspondence for unramified representations, we see that the degree 6 (partial) standard L-function of $\Theta(\pi) = \Pi \boxtimes \mu$ admits a factorization:

$$L^S(s, \Theta(\pi), \mathrm{std}) = \zeta^S(s) \cdot L^S(s, \pi, \mathrm{std}).$$

Since $L^S(s, \Theta(\pi), \mathrm{std}) = L^S(s, \Pi, \bigwedge^2 \otimes \mu^{-1})$ and $L^S(s, \pi, \mathrm{std})$ is nonzero at $s = 1$, we see that the twisted exterior square L-function $L^S(s, \Pi, \bigwedge^2 \otimes \mu^{-1})$ has a pole at $s = 1$ for any $\Pi \boxtimes \mu$ in the image of the global theta lift.

To complete the proof of (iii), suppose first that Π is a cuspidal representation of $\mathrm{GL}_4(\mathbb{A})$ such that $L^S(s, \Pi, \bigwedge^2 \otimes \mu^{-1})$ has a pole at $s = 1$. By a result of Jacquet-Shalika [JS90], this is equivalent to Π having nonzero Shalika period with respect to μ. Now let us consider the global theta lift of $\Pi \boxtimes \mu$ to $\mathrm{GSp}_4(\mathbb{A})$, which one can easily check to be cuspidal. If one computes the Whittaker-Fourier coefficient of $\Theta(\Pi \boxtimes \mu)$, one obtains an expression involving the Shalika period of Π with respect to μ. A proof of this can be found in [Sou87] and [GT10b, Prop. 3.1]. From this, one concludes that $\pi = \Theta(\Pi \boxtimes \mu)$ is a globally generic cuspidal representation of $\mathrm{GSp}_4(\mathbb{A})$. Thus, $\Pi \boxtimes \mu = \Theta(\pi)$ is in the image of the global theta lift.

On the other hand, if $\Theta(\pi) = \Pi \boxtimes \mu$ is noncuspidal, then the global theta lift of π to $\mathrm{GSO}_{2,2}(\mathbb{A})$ is nonzero (by the tower property of theta correspondence). It is easy to check that the theta lift of π to $\mathrm{GSO}_{1,1}(\mathbb{A})$ vanishes; indeed, the local theta lift of a representation of $\mathrm{GSO}_{1,1}$ to GSp_4 is never generic. Thus, $\pi = \Theta(\tau_1 \boxtimes \tau_2)$ for a cuspidal representation $\tau_1 \boxtimes \tau_2$ of $\mathrm{GSO}_{2,2}(\mathbb{A})$ with $\omega_\pi = \omega_{\tau_i}$ so that $\tau_i = \tau_i^\vee \otimes \omega_\pi$. Moreover, $\tau_1 \neq \tau_2$, for otherwise the theta lift of $\tau_1 \boxtimes \tau_2$ to $\mathrm{GSp}_4(\mathbb{A})$ would not be cuspidal. Conversely, if $\Pi \boxtimes \mu$ is of the type in (iii)(b), so that $\Pi = \tau_1 \boxplus \tau_2$, then we may consider the global theta lift of the cuspidal representation $\tau_1 \boxtimes \tau_2$ of $\mathrm{GSO}_{2,2}(\mathbb{A})$ to $\mathrm{GSp}_4(\mathbb{A})$. This gives us a globally generic cuspidal representation $\pi = \Theta(\tau_1 \boxtimes \tau_2)$ of $\mathrm{GSp}_4(\mathbb{A})$. The theta lift of π to $\mathrm{GSO}(V)(\mathbb{A})$ is then equal to $\Pi \boxtimes \mu$.

This completes the proof of the theorem. □

References

[AB95] J. Adams and D. Barbasch, Reductive dual pair correspondence for complex groups, *J. Funct. Anal.* **132** (1995), 1–42. MR 1346217. Zbl 0841.22014. doi: 10.1006/jfan.1995.1099.

[AP06] J. D. Adler and D. Prasad, On certain multiplicity one theorems, *Israel J. Math.* **153** (2006), 221–245. MR 2254643. Zbl 1137.22009. doi: 10.1007/BF02771784.

[AS08] M. Asgari and R. Schmidt, On the adjoint L-function of the p-adic GSp(4), *J. Number Theory* **128** (2008), 2340–2358. MR 2394824. Zbl 05565653. doi: 10.1016/j.jnt.2007.08.012.

[AS06] M. Asgari and F. Shahidi, Generic transfer from GSp(4) to GL(4), *Compos. Math.* **142** (2006), 541–550. MR 2231191. Zbl 1112.11027. doi: 10.1112/S0010437X06001904.

[CG] P. S. Chan and W. T. Gan, The local Langlands conjecture for GSp(4) III: stability and (twisted) endoscopy, in preparation.

[Che06] J.-P. J. Chen, The $n \times (n-2)$ local converse theorem for $\mathrm{GL}(n)$ over a p-adic field, *J. Number Theory* **120** (2006), 193–205. MR 2257542. Zbl 1193.11047. doi: 10.1016/j.jnt.2005.12.001.

[DR09] S. DeBacker and M. Reeder, Depth-zero supercuspidal L-packets and their stability, *Ann. of Math.* **169** (2009), 795–901. MR 2480618. Zbl 1193.11111. doi: 10.4007/annals.2009.169.795.

[GTa] W. T. Gan and S. Takeda, On the regularized Siegel-Weil formula (the second term identity) and non-vanishing of theta lifts from orthogonal groups, to appear in *J. Reine Angew. Math.* doi: 10.1515/CRELLE.2011.076.

[GTb] ———, Theta correspondences for GSp_4, to appear in *Represent. Theory.* arXiv 1006.3367.

[GT10a] ———, The local Langlands conjecture for Sp(4), *Int. Math. Res. Not.* **2010** (2010), 2987–3038. MR 2673717. Zbl 05786178.

[GT10b] W. T. Gan and S. Takeda, On Shalika periods and a theorem of Jacquet-Martin, *Amer. J. Math.* **132** (2010), 475–528. MR 2654780. Zbl 05706736. doi: 10.1353/ajm.0.0109.

[GT] W. T. Gan and W. Tantono, The local Langlands conjecture for GSp(4) II: the case of inner forms, preprint.

[GRS97] D. Ginzburg, S. Rallis, and D. Soudry, Periods, poles of L-functions and symplectic-orthogonal theta lifts, *J. Reine Angew. Math.* **487** (1997), 85–114. MR 1454260. Zbl 0928.11025.

[HT01] M. Harris and R. Taylor, *The geometry and cohomology of some simple Shimura varieties*, *Annals of Mathematics Studies* **151**, Princeton Univ. Press, Princeton, NJ, 2001, With an appendix by Vladimir G. Berkovich. MR 1876802. Zbl 1036.11027.

[Hen00] G. Henniart, Une preuve simple des conjectures de Langlands pour GL(n) sur un corps p-adique, *Invent. Math.* **139** (2000), 439–455. MR 1738446. Zbl 1048.11092. doi: 10.1007/s002220050012.

[Hen02] ———, Une caractérisation de la correspondance de Langlands locale pour GL(n), *Bull. Soc. Math. France* **130** (2002), 587–602. MR 1947454. Zbl 1029.22023.

[Hen10] ———, Correspondance de Langlands et fonctions L des carrés extérieur et symétrique, *Int. Math. Res. Not.* **2010** (2010), 633–673. MR 2595008. Zbl 1184.22009.

[JS90] H. Jacquet and J. Shalika, Exterior square L-functions, in *Automorphic Forms, Shimura Varieties, and L-Functions, Vol.* II (Ann Arbor, MI, 1988), *Perspect. Math.* **11**, Academic Press, Boston, MA, 1990, pp. 143–226. MR 1044830. Zbl 0695.10025.

[JS04] D. Jiang and D. Soudry, Generic representations and local Langlands reciprocity law for p-adic SO_{2n+1}, in *Contributions to Automorphic Forms, Geometry, and Number Theory*, Johns Hopkins Univ. Press, Baltimore, MD, 2004, pp. 457–519. MR 2058617. Zbl 1062.11077. doi: 10.1515/CRELLE.2011.076.

[Kud86] S. S. Kudla, On the local theta-correspondence, *Invent. Math.* **83** (1986), 229–255. MR 0818351. Zbl 0583.22010. doi: 10.1007/BF01388961.

[KR05] S. S. Kudla and S. Rallis, On first occurrence in the local theta correspondence, in *Automorphic Representations, L-Functions and Applications: Progress and Prospects*, *Ohio State Univ. Math. Res. Inst. Publ.* **11**, de Gruyter, Berlin, 2005, pp. 273–308. MR 2192827. Zbl 1109.22012.

[LR05] E. M. Lapid and S. Rallis, On the local factors of representations of classical groups, in *Automorphic Representations, L-Functions and Applications: Progress and Prospects*, *Ohio State Univ. Math. Res. Inst. Publ.* **11**, de Gruyter, Berlin, 2005, pp. 309–359. MR 2192828. Zbl 1188.11023.

[MS00a] G. Muić and G. Savin, Complementary series for Hermitian quaternionic groups, *Canad. Math. Bull.* **43** (2000), 90–99. MR 1749954. Zbl 0945.22005. doi: 10.4153/CMB-2000-014-5.

[MS00b] G. Muić and G. Savin, Symplectic-orthogonal theta lifts of generic discrete series, *Duke Math. J.* **101** (2000), 317–333. MR 1738175. Zbl 0955.22014. doi: 10.1215/S0012-7094-00-10128-7.

[Pau05] A. Paul, On the Howe correspondence for symplectic-orthogonal dual pairs, *J. Funct. Anal.* **228** (2005), 270–310. MR 2175409. Zbl 1084.22008. doi: 10.1016/j.jfa.2005.03.015.

[Pra93] D. Prasad, On the local Howe duality correspondence, *Internat. Math. Res. Notices* (1993), 279–287. MR 1248702. Zbl 0804.22009. doi: 10.1155/S1073792893000315.

[Ral84] S. Rallis, On the Howe duality conjecture, *Compositio Math.* **51** (1984), 333–399. MR 0743016. Zbl 0624.22011. Available at http://www.numdam.org/item?id=CM_1984__51_3_333_0.

[Rob96] B. Roberts, The theta correspondence for similitudes, *Israel J. Math.* **94** (1996), 285–317. MR 1394579. Zbl 0870.22011. doi: 10.1007/BF02762709.

[Rob01] ———, Global L-packets for GSp(2) and theta lifts, *Doc. Math.* **6** (2001), 247–314. MR 1871665. Zbl 1056.11029.

[RS07] B. Roberts and R. Schmidt, *Local Newforms for* GSp(4), *Lecture Notes in Math.* **1918**, Springer-Verlag, New York, 2007. MR 2344630. Zbl 1126.11027. doi: 10.1007/978-3-540-73324-9.

[ST93] P. J. Sally, Jr. and M. Tadić, Induced representations and classifications for $\mathrm{GSp}(2, F)$ and $\mathrm{Sp}(2, F)$, *Mém. Soc. Math. France* (1993), 75–133. MR 1212952. Zbl 0784.22008.

[Sha90] F. Shahidi, A proof of Langlands' conjecture on Plancherel measures; complementary series for p-adic groups, *Ann. of Math.* **132** (1990), 273–330. MR 1070599. doi: 10.2307/1971524.

[Sil79] A. J. Silberger, *Introduction to Harmonic Analysis on Reductive p-adic Groups*, *Math. Notes* **23**, Princeton Univ. Press, Princeton, N.J., 1979. MR 0544991. Zbl 0458.22006.

[Sou87] D. Soudry, A uniqueness theorem for representations of GSO(6) and the strong multiplicity one theorem for generic representations of GSp(4), *Israel J. Math.* **58** (1987), 257–287. MR 0917359. Zbl 0642.22003. doi: 10.1007/BF02771692.

[Vig86] M.-F. Vignéras, Correspondances entre representations automorphes de GL(2) sur une extension quadratique de GSp(4) sur $\mathbf{Q}$, conjecture locale de Langlands pour GSp(4), in *The Selberg Trace Formula and Related Topics* (Brunswick, Maine, 1984), *Contemp. Math.* **53**, Amer. Math. Soc., Providence, RI, 1986, pp. 463–527. MR 0853573. Zbl 0595.12009.

[Zin] W. Zink, letter to M.-F. Vigneras (1984).

(Received: December 4, 2009)

University of California San Diego, La Jolla, CA
E-mail: wgan@math.ucsd.edu

Purdue University, West Lafayette, IN
E-mail: stakeda@math.purdue.edu