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Absence of mixing in area-preserving flows on surfaces

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Abstract

We prove that minimal area-preserving flows locally given by a smooth Hamiltonian on a closed surface of genus $g \geq 2$ are typically (in the measure-theoretical sense) not mixing. The result is obtained by considering special flows over interval exchange transformations under roof functions with symmetric logarithmic singularities and proving absence of mixing for a full measure set of interval exchange transformations.

1. Definitions and main results

1.1. *Flows given by multi-valued Hamiltonians.* Let us consider the following natural construction of area-preserving flows on surfaces. On a closed connected orientable surface S of genus $g \geq 1$ with a fixed smooth area form ω, consider a smooth closed real-valued differential 1-form η. Let X be the vector field determined by $\eta = i_X \omega = \omega(\eta, \cdot)$ and consider the flow $\{\varphi_t\}_{t\in\mathbb{R}}$ on S associated to X. Since η is closed, the transformations φ_t, $t \in \mathbb{R}$, are area-preserving. The flow $\{\varphi_t\}_{t\in\mathbb{R}}$ is known as the *multi-valued Hamiltonian* flow associated to η. Indeed, the flow $\{\varphi_t\}_{t\in\mathbb{R}}$ is locally Hamiltonian; i.e., *locally* one can find coordinates (x, y) on S in which it is given by the solution to the equations $\dot{x} = \partial H/\partial y$, $\dot{y} = -\partial H/\partial x$ for some smooth real-valued Hamiltonian function H. A *global* Hamiltonian H cannot be in general be defined (see [NZ99, §1.3.4]), but one can think of $\{\varphi_t\}_{t\in\mathbb{R}}$ as globally given by a *multi-valued* Hamiltonian function.

The study of flows given by multi-valued Hamiltonians was initiated by S. P. Novikov [Nov82] in connection with problems arising in solid-state physics i.e., the motion of an electron in a metal under the action of a magnetic field. The orbits of such flows arise also in pseudo-periodic topology, as hyperplane sections of periodic surfaces in $\mathbb{T}^n$ (see e.g. Zorich [Zor99]).

From the point of view of topological dynamics, a decomposition into *minimal components* (i.e., subsurfaces on which the flow is minimal) and *periodic components* on which all orbits are periodic (elliptic islands around a center and cylinders filled by periodic orbits) was proved independently by Maier [May43], Levitt [Lev82] (in the context of foliations on surfaces) and Zorich

[Zor99] for multi-valued Hamiltonian flows. We consider the case in which the flow is *minimal*; i.e., all semi-infinite trajectories are dense. This excludes the presence of periodic components.

From the point of view of ergodic theory, one is naturally led to ask whether the flow on each minimal component is *ergodic* and, in this case, whether it is *mixing*. Ergodicity is equivalent to ergodicity of the Poincaré first return map on a cross section, which is isomorphic to a minimal interval exchange transformation (see §1.2 for definitions). A well-known and celebrated result asserts that *typical*[1] IETs are uniquely ergodic ([Vee82], [Mas82]).

In this paper we address the question of mixing. Let μ be the area associated to ω, renormalized so that $\mu(S) = 1$. Let us recall that $\{\varphi_t\}_{t\in\mathbb{R}}$ is said to be *mixing* if for each pair A, B of Borel-measurable sets one has

$$\lim_{t\to\infty} \mu(\varphi_t(A) \cap B) = \mu(A)\mu(B). \tag{1}$$

We assume that the 1-form η is *Morse*; i.e., it is locally the differential of a Morse function. This condition is generic in the space of perturbations of closed smooth 1-forms by closed smooth 1-forms. Thus, all zeros of η correspond to either centers or simple saddles, and, in particular, if the the associated flow $\{\varphi_t\}_{t\in\mathbb{R}}$ is minimal, then all zeros are simple saddles. Let us hence consider the space of multi-valued Hamiltonian flows with only simple saddles. A *measure-theoretical notion of typical* is defined as follows by using the *Katok fundamental class* (introduced by Katok in [Kat73]; see also [NZ99]). Let Σ be the set of fixed points of η (in our case simple saddles) and let k be the cardinality of Σ. Let $\gamma_1, \dots, \gamma_n$ be a base of the relative homology $H_1(S, \Sigma, \mathbb{R})$, where $n = 2g + k - 1$. The image of η by the period map Per is $\mathrm{Per}(\eta) = (\int_{\gamma_1} \eta, \dots, \int_{\gamma_n} \eta) \in \mathbb{R}^n$. The pull-back $\mathrm{Per}_* \mathrm{Leb}$ of the Lebesgue measure class by the period map gives the desired measure class on closed 1-forms. When we use the expression *typical* below, we mean full measure with respect to this measure class.

The main result is the following. Let us recall that a *saddle connection* is a flow trajectory from a saddle to a saddle and a *saddle loop* is a saddle connection from a saddle to the same saddle.

THEOREM 1.1. *Let $\{\varphi_t\}_{t\in\mathbb{R}}$ be the flow given by a multi-valued Hamiltonian associated to a smooth closed differential 1-form η on a closed surface of genus $g \geq 2$. Assume that $\{\varphi_t\}_{t\in\mathbb{R}}$ has only simple saddles and no saddle loops homologous to zero. For a* typical *such form η, the flow $\{\varphi_t\}_{t\in\mathbb{R}}$ is* not mixing.

[1]The notion of typical here is measure-theoretical; i.e., it refers to *almost every* IET in the sense defined before the statement of Theorem 1.2.

The assumption that there are no saddle loops homologous to zero excludes for typical flows the presence of periodic components and implies minimality by a result of Maier [May43]. In fact, a typical flow has no saddle connections other than saddle loops homologous to zero and periodic components are bounded by saddle connections.

Thus, Theorem 1.1 settles the open question (which appears, for example, in Forni [For02] and in the survey [KT06, §6.3.2] by Katok and Thouvenot) of whether a typical *minimal* multi-valued Hamiltonian flow with only simple saddles is mixing. Even if nonmixing, such flows are nevertheless typically *weakly mixing*[2] ([Ulc09]; see also §1.3). The asymptotic behavior of Birkhoff sums and its deviation spectrum for this class of flows was described by Forni in [For02].

Let us remark that both assumptions of Theorem 1.1 (i.e., simple saddles and no saddle loops homologous to zero) are crucial for the absence of mixing. Indeed, if a minimal flow has multi-saddles, corresponding to higher-order zeros of η, then φ_t is mixing, as proved by Kochergin [Koč75]. On the other hand, flows with saddle loops homologous to zero form an open set in the space of multi-valued Hamiltonians, and if there are such saddle loops, then one can typically produce mixing in each minimal component using the mechanism developed in [SK92] for genus one and in [Ulc07b] for higher genus.

In the next section we recall the definitions of interval exchange transformations and special flows and formulate the main theorem in the setting of special flows. (Theorem 1.2, from which Theorem 1.1 will be deduced.) Previous results on ergodic properties of special flows over IETs are recalled in Section 1.3.

1.2. *Special flows with logarithmic singularities.* Special flows give a useful tool to describe area-preserving flows on surfaces. When representing a flow on a surface (or one of its minimal components) as a special flow, it is enough to consider a transversal to the flow: the first return, or Poincaré map, to the transversal determines the base transformation T, while the function f gives the first return time of the flow to the transversal. Different functions f describe different time-reparametrizations of the same flow; hence they give rise to flows which, topologically, have the same orbits. Interval exchange transformations arise naturally as first return maps (up to smooth reparametrization); see Section 5. flows over IETs with this type of singularities, we get as a Corollary the following.

[2]Let us recall that a flow $\{\varphi_t\}_{t\in\mathbb{R}}$ preserving a probability measure μ is *weakly mixing* if for each pair A, B of measurable sets, $\frac{1}{T}\int_0^T |\mu(\varphi_t(A)\cap B)-\mu(A)\mu(B)|\,\mathrm{d}t$ converges to zero as T tends to infinity.

Interval exchange transformations. Let $I^{(0)} = [0, 1)$, let $\pi \in \mathcal{S}_d$, $d \geq 2$, be a permutation[3] and let Δ_{d-1} denote the simplex of vectors $\underline{\lambda} \in \mathbb{R}^d_+$ such that $\sum_{i=1}^d \lambda_i = 1$. The *interval exchange transformation* (IET) of d subintervals given by $(\underline{\lambda}, \pi)$ with $\underline{\lambda} \in \Delta_{d-1}$ is the map $T \colon I^{(0)} \to I^{(0)}$ given by[4]

$$T(x) = x - \sum_{i=1}^{j-1} \lambda_i + \sum_{i=1}^{j-1} \lambda_{\pi^{-1}(i)} \quad \text{for} \quad x \in I^{(0)}_j = [\sum_{i=1}^{j-1} \lambda_i, \sum_{i=1}^{j} \lambda_i), \quad j = 1, \dots, d.$$

In other words T is a piecewise isometry which rearranges the subintervals of lengths given by $\underline{\lambda}$ in the order determined by π. We shall often use the notation $T = (\underline{\lambda}, \pi)$. Let $\Sigma_{\underline{\lambda},\pi} = \{\sum_{i=1}^j \lambda_i, j = 1, \dots, d\} \cup \{0\}$ be the set of *discontinuities* of T together with the endpoints of $I^{(0)}$. We say that T is *minimal* if the orbit of all points are dense. We say that the permutation $\pi \in \mathcal{S}_d$ is *irreducible* if, whenever the subset $\{1, 2, \dots, k\}$ is π-invariant, then $k = d$. Irreducibility is a necessary condition for minimality. Recall that T satisfies the *Keane condition* if the orbits of all discontinuities in $\Sigma_{\underline{\lambda},\pi} \backslash \{0, 1\}$ are infinite and disjoint. If T satisfies this condition, then T is minimal [Kea75].

Special flows. Let $f \in L^1(I^{(0)}, dx)$ be a strictly positive function and assume that $\int_{I^{(0)}} f(x)dx = 1$. Let $X_f \doteqdot \{(x, y) \in \mathbb{R}^2 \mid x \in I^{(0)}, 0 \leq y < f(x)\}$ be the set of points below the graph of the roof function f and μ be the restriction to X_f of the Lebesgue measure $\mathrm{d}x\,\mathrm{d}y$. Given $x \in I^{(0)}$ and $r \in \mathbb{N}^+$ we denote by $S_r(f)(x) \doteqdot \sum_{i=0}^{r-1} f(T^i(x))$ the r^{th} nonrenormalized *Birkhoff sum* of f along the trajectory of x under T. By convention, $S_0(f)(x) \doteqdot 0$. Let $t > 0$. Given $x \in I^{(0)}$ denote by $r(x, t)$ the integer uniquely defined by $r(x, t) \doteqdot \max\{r \in \mathbb{N} \mid \quad S_r(f)(x) < t\}$.

The *special flow built over*[5] *T under the roof function f* is a one-parameter group $\{\varphi_t\}_{t \in \mathbb{R}}$ of μ-measure-preserving transformations of X_f whose action is given, for $t > 0$, by

$$\varphi_t(x, 0) = \left(T^{r(x,t)}(x), t - S_{r(x,t)}(f)(x)\right). \tag{2}$$

For $t < 0$, the action of the flow is defined as the inverse map and φ_0 is the identity. Under the action of the flow a point $(x, y) \in X_f$ moves with unit velocity along the vertical line up to the point $(x, f(x))$, then jumps instantly to the

[3] We are using here the notation for IETs classically introduced by Keane [Kea75] and Veech [Vee78], [Vee82]. We remark that recently Marmi-Moussa and Yoccoz introduced a new labeling of IETs (see the lecture notes by Yoccoz [Yoc06] or Viana [Via]), which considerably facilitates the analysis of Rauzy-Veech induction. We do not recall it here, since it does not bring any simplification to our proofs.

[4] The sums in the definition are by convention zero if over the empty set, e.g., for $j = 0$.

[5] One can define in the same way special flows over any measure-preserving transformation T of a probability space $(M, \mathscr{M}, \mu)$; see e.g., [CFS82].

point $(T(x), 0)$, according to the base transformation. Afterward it continues its motion along the vertical line until the next jump and so on. The integer $r(x,t)$ gives the number of *discrete iterations* of the base transformation T, which the point $(x, 0)$ undergoes when flowing up to time $t > 0$.

Logarithmic singularities. We consider the following class of roof functions with logarithmic symmetric singularities. The motivation for considering special flows over IETs under such roofs is explained in Section 5.

Let $0 \leq \overline{z}_0^+ < \overline{z}_1^+ < \cdots < \overline{z}_{s_1-1}^+ < 1$ be the s_1 points where the roof function is right-singular (i.e., the right limit is infinite) and $0 < \overline{z}_0^- < \overline{z}_1^- < \cdots < \overline{z}_{s_2-1}^- \leq 1$ the s_2 points where the roof function is left-singular (i.e., the left limit is infinite). Let us denote by $\{\{x\}\}$ the fractional part of x, that is a periodic function of period 1 such that $\{\{x\}\} = x$ if $x \in [0,1)$.

Definition 1.1. The function f has *logarithmic singularities* at the set of points $\{\overline{z}_0^+, \ldots, \overline{z}_{s_1-1}^+, \overline{z}_0^-, \ldots, \overline{z}_{s_2-1}^-\}$, where $0 \leq \overline{z}_0^+, \ldots, \overline{z}_{s_1-1}^+ < 1$ are right singularities and $0 < \overline{z}_1^-, \ldots, \overline{z}_{s_2-1}^- \leq 1$ are left singularities, if $f \in \mathscr{C}^2$ on

$$[0,1] \backslash \{\overline{z}_0^+, \ldots, \overline{z}_{s_1-1}^+, \overline{z}_0^-, \ldots, \overline{z}_{s_2-1}^-\}$$

and there exist constants C_i^+ $i = 0, \ldots, s_1 - 1$ and C_i^- for $i = 0, \ldots, s_2 - 1$ and a function w of bounded variation on $[0,1]$ such that

$$f = f_0 + w, \quad f_0(x) = \sum_{i=0}^{s_1-1} C_i^+ \left| \ln\{\{x - \overline{z}_i^+\}\} \right| + \sum_{i=0}^{s_2-1} C_i^- \left| \ln\{\{\overline{z}_i^- - x\}\} \right|.$$

The logarithmic singularities are called *symmetric* if, moreover, $\sum_{i=0}^{s_2-1} C_i^- = \sum_{i=0}^{s_1-1} C_i^+$.

We remark that the derivative f' of a function with symmetric logarithmic singularities is not integrable. Indeed, to express the derivative, let us introduce two auxiliary functions u, v defined on $(0,1)$ as follows:

$$u(x) := \frac{1}{x}, \quad v(x) := \frac{1}{1-x}$$

and extended to the whole real line so that they are periodic of period 1, i.e., for $x \in \mathbb{R}$, $u(x) = u(\{\{x\}\})$ and $v(x) = v(\{\{x\}\})$. Let us denote $u_i(x) = u(x - \overline{z}_i^+)$ for $i = 0, \ldots, s_1 - 1$ and $v_i(x) = v(x - \overline{z}_i^-)$, $i = 0, \ldots, s_2 - 1$. Then, we can write $f' = f_0 + w'$ where w' is integrable since w has bounded variation, while

$$f_0' = \sum_{i=0}^{s_2-1} C_i^- v_i - \sum_{i=0}^{s_1-1} C_i^+ u_i \tag{3}$$

is not integrable since it has singularities of type $1/x$.

Absence of mixing. The main theorem that we prove in this context is the following. Here and in the rest of the paper we will say that a result holds for *almost every* IET if it holds for any *irreducible* permutation π on $d \geq 2$ symbols and almost every choice of the length vector $\underline{\lambda} \in \Delta_{d-1}$ with respect to the restriction of the d-Lebesgue measure to the simplex Δ_{d-1}.

THEOREM 1.2. *For almost every* IET $T = (\underline{\lambda}, \pi)$, *the special flow* $\{\varphi_t\}_{t\in\mathbb{R}}$, *built over* T *under a roof function* f *with symmetric logarithmic singularities at a subset of the discontinuities* $\Sigma_{\underline{\lambda},\pi}$ *of* T, *is* not mixing.

It is worth remarking that nevertheless special flows with logarithmic singularities over typical IETs are weakly mixing,[6] as proved by the author in [Ulc07a] and [Ulc09]. We show in Section 5 that flows on surfaces given by multi-valued Hamiltonians can be represented as flows over IETs with logarithmic singularities and that Theorem 1.1 can be deduced from Theorem 1.2.

1.3. *Ergodic properties of logarithmic special flows.*

Flows over rotations. Assume first that the base transformation is a rotation of the circle (i.e. the map $R_\alpha x = x + \alpha \,(\mathrm{mod}\, 1)$), which can be seen as an interval exchange of $d = 2$ intervals. Kochergin proved in [Koč76] that special flows with *symmetric* logarithmic singularities [Koč76] are not mixing for almost every α. Recently, in [Koč07], he shows that absence of mixing holds indeed for all α. An intermediate result for $s_1 = s_2 = 1$ and $C_0^- = C_0^+$ is a consequence of [Lem00]. In [FL03] Frączek and Lemańczyk consider the roof function $f(x) = |\ln x|/2 + |\ln(1-x)|/2$ and show that the corresponding special flow over R_α is *weakly mixing* for all α. They also push the investigation to more subtle spectral properties, showing in [FL05] that such flows are spectrally disjoint from all mixing flows.

On the other hand, if the roof has *asymmetric logarithmic singularities*, Khanin and Sinai proved in [SK92] that, under a diophantine condition on the rotation angle which holds for a full measure set of α, the corresponding special flow is *mixing*, answering affirmatively to a question asked by Arnold in [Arn91]. The diophantine condition of [SK92] was weakened by Kochergin in a series of works ([Koč03], [Koč04a], [Koč04b], [Koč04c]).

Flows over IET*s.* In [Ulc07a] and [Ulc07b] the author proved that special flows over typical IETs under a roof function f having a single *asymmetric* logarithmic singularity at the origin (i.e., as in Definition 1.1 with $s_1 = s_2 = 1$ and $C_0^+ \neq C_0^-$) are *mixing*. The same techniques can be applied to the situation

[6]The definition of weak mixing was recalled in footnote 2 (page 1745).

of several logarithmic singularities as long as the roof satisfies the asymmetry condition $\sum C_i^+ \neq \sum C_i^-$. Let us also recall that if the singularities are power-like (i.e., f blows up near singularities as $1/x^\alpha$ for $\alpha > 0$) rather than logarithmic, then mixing was proved by Kochergin in [Koč75].

If the singularities are *symmetric*, two results in special cases were recently proved. In [Ulc07a], the author showed the absence of mixing if the IET on the base satisfies a condition which is similar to α being bounded type for rotations (which in particular holds only for a measure zero set of IETs). Scheglov recently showed in [Sch09] that if $\pi = (54321)$, then for almost every $\underline{\lambda}$ the special flow over $(\underline{\lambda}, \pi)$ under a particular class of functions with symmetric logarithmic singularities[7] is not mixing. From his result it follows that Theorem 1.1 holds in the special case in which $g = 2$ and the flow has two isometric saddles. Unfortunately, his methods do not seem to extend to higher genus, for the reasons explained in the remark at the end of Section 4.2.

It is worth recalling also that IETs are never mixing and that special flows over IETs are never mixing if the function f is of bounded variation (both results were proved by Katok in [Kat80]). On the other side, Avila and Forni [AF07] showed that IETs which are not of rotation-type are typically weakly mixing and that special flows over IETs with piecewise constant roofs are also typically weakly mixing.

2. Background on cocycles and Rauzy-Veech induction

2.1. *Some properties of cocycles.* Let (X, μ, F) be a discrete dynamical system, where (X, μ) is a probability space and F is a μ-measure-preserving map on X. A measurable map $A : X \to \mathrm{SL}(d, \mathbb{Z})$ ($d \times d$ invertible matrices) determines a cocycle A on (X, μ, F). If we denote by $A_n(x) = A(F^n x)$ and by $A_F^n(x) = A_{n-1}(x) \cdots A_1(x) A_0(x)$, then the following *cocycle identity*

$$A_F^{m+n}(x) = A_F^m(F^n x) A_F^n(x) \tag{4}$$

holds for all $m, n \in \mathbb{N}$ and for all $x \in X$. If F is invertible, let us set $A_{-n}(x) = A(F^{-n}x)$. The map $A^{-1}(x) = A(x)^{-1}$ gives a cocycle over F^{-1} which we call *inverse cocycle*.

If $Y \subset X$ is a measurable subset with $\mu(Y) > 0$, then the *induced cocycle* A_Y on Y is a cocycle over (Y, μ_Y, F_Y) where F_Y is the induced map of F on Y, $\mu_Y = \mu/\mu(Y)$, and $A_Y(y)$ is defined for all $y \in Y$ which return to Y and is

[7]A function f with symmetric logarithmic singularities, as defined in [Sch09], is such that f' is a linear combination of the functions $f_i(x) = 1/(b_i - x) - 1/(x - a_i)$ defined on the interior of the IET subintervals $I_i^{(0)} = [a_i, b_i)$ for $i = 1, \ldots, d$ and the function $1/(1-x) - 1/x$. In particular, $s_1 = s_2$ and constants come in pairs $\{C_i^+, C_i^-\}$ such that $C_i^+ = C_i^-$. Thus, this class is more restrictive than the one given by Definition 1.1.

given by

$$A_Y(y) = A(F^{r_Y(y)-1}y) \cdots A\,(Fy)\,A\,(y)\,,$$

where $r_Y(y) = \min\{r \mid F^r y \in Y\}$ is the first return time. The induced cocycle is an acceleration of the original cocycle, i.e., if $\{n_k\}_{k\in\mathbb{N}}$ is the infinite sequence of return times of some $y \in Y$ to Y (i.e., $T^n y \in Y$ if and only if $n = n_k$ for some $k \in \mathbb{N}$ and $n_{k+1} > n_k$), then

$$(A_Y)_k(y) = A_{n_{k+1}-1}(y) \cdots A_{n_k+1}(y) A_{n_k}(y). \tag{5}$$

We say that $x \in X$ is *recurrent* to Y if there exists an infinite increasing sequence $\{n_k\}_{k\in\mathbb{N}}$ such that $T^{n_k}x \in Y$. Let us extend the definition of the induced cocycle A_Y to all $x \in X$ recurrent to Y. If the sequence $\{n_k\}_{k\in\mathbb{N}}$ is increasing and contains all $n \in \mathbb{N}^+$ such that $T^n x \in Y$, let us say that *x recurs to Y along* $\{n_k\}_{k\in\mathbb{N}}$. In this case, let us set

$$\begin{aligned} A_Y(x) &:= A_Y(y) A_F^{n_0}(x), && \text{where } y := F^{n_0}x \in Y, \\ (A_Y)_n(x) &:= (A_Y)_n(y), && \text{for } n \in \mathbb{N}^+. \end{aligned}$$

If F is ergodic, then μ-almost every $x \in X$ is recurrent to Y, and hence A_Y is defined on a full measure subset of X.

In the rest of the paper, we will use the norm $\|A\| = \sum_{ij} |A_{ij}|$ on matrices (more in general, the same results on cocycles hold for any norm on $\mathrm{SL}(d, \mathbb{Z})$). With this choice one has $\|A\| = \|A^T\|$. A cocycle over (X, F, μ) is called *integrable* if $\int_X \ln \|A(x)\| \mathrm{d}\mu(x) < \infty$. Integrability is the assumption which allows us to apply Oseledets Theorem. Let us recall the following properties of integrable cocycles.[8]

Remark 2.1. If A is an integrable cocycle over (X, F, μ) assuming values in $\mathrm{SL}(d, \mathbb{Z})$, then

(i) the dual cocycle $(A^{-1})^T$ and, if F is invertible, the inverse cocycle A^{-1} over (X, F^{-1}, μ) are integrable;
(ii) any induced cocycle A_Y of A on a positive-measure subset $Y \subset X$ is integrable.

[8] The integrability of the dual cocycle stated in Remark 2.1(i) is proved in Zorich [Zor97], and since $\|A^{-1}\| = \|(A^{-1})^T\|$, the integrability of the inverse cocycle follows. The proof of Remark 2.1(ii) if F is invertible follows from Kac's lemma representation of the space as towers and the noninvertible case can be reduced to the invertible one by considering the natural extension of F.

In Section 2.2 we will consider the Rauzy-Veech Zorich cocycle for IETs, and in Section 4 we will use various accelerations constructed using the following two lemmas. For $m < n$, let us denote by[9]

$$A^{(m,n)} = A_m A_{m+1} \cdots A_{n-1}.$$

LEMMA 2.1. *Let A^{-1} be an integrable cocycle over an ergodic and invertible (X, μ, F). There exist a measurable $E_1 \subset X$ with positive measure*[10] *and a constant $\overline{C}_1 > 0$ such that for all $x \in X$ recurrent to E_1 along the sequence $\{n_k\}_{k\in\mathbb{N}}$ we have*

$$\frac{\ln \|A^{(n,n_k)}(x)\|}{n_k - n} \leq \overline{C}_1, \quad \forall\, 0 \leq n < n_k. \tag{6}$$

Proof. Since A^{-1} is integrable, by Remark 2.1(i), also the inverse cocycle A over (X, μ, F^{-1}) is integrable. Hence, by Oseledets Theorem, the functions $\ln \|A^m_{F^{-1}}\|/m$ converge pointwise. There exists a set E_1 of positive measure such that by Egorov's theorem the convergence is uniform, so that $\ln \|A^m_{F^{-1}}(x)\| \leq cm$ for some $c > 0$ and all $x \in E_1$ and all $m \geq \overline{m} > 0$, and at the same time $\|A_{-m}(x)\|$ for $0 \leq m < \overline{m}$ are uniformly bounded. Thus, if $F^{n_k}x \in E_1$, we have $\ln \|A^m_{F^{-1}}(F^{n_k}x)\| \leq Cm$ for some $C > 0$ and all $m \geq 0$. Hence, since $A^m_{F^{-1}}(F^{n_k}x) = A^{(n_k-m+1, n_k+1)}(x)$, changing indexes by $n = n_k - m + 1$, we get (6). □

LEMMA 2.2. *Under the same assumptions of Lemma 2.1, for each $\varepsilon > 0$ there exist a measurable $E_2 \subset X$ with positive measure and a constant $\overline{C}_2 > 0$ such that if $x \in X$ is recurrent to E_2 along the sequence $\{m_k\}_{k\in\mathbb{N}}$, we have*

$$\|A_{m_k-n}(x)\| \leq \overline{C}_2 e^{\varepsilon n}, \quad \forall\, 0 \leq n \leq m_k. \tag{7}$$

Proof. Recall that since F^{-1} is ergodic, if f is integrable, then the functions $\{f \circ F^{-m}/m\}_{m\in\mathbb{N}}$ converge to zero for almost every $x \in X$ and hence, by Egorov's theorem, are eventually uniformly less than ε on some positive measure set for $m > \overline{m}$. Since A^{-1} is integrable, also A is integrable (Remark 2.1(i)) and applying this observation to $f = \ln \|A\|$, we can find a smaller positive measure set E_2 and $\overline{C}_2 > 0$ (in order to bound also $\|A(F^{-m}x)\|$ for $x \in E_2$, $0 \leq m \leq \overline{m}$) such that if $y \in E_2$, we have $\|A(F^{-n}y)\| \leq \overline{C}_2 e^{\varepsilon n}$ for all $n \geq 0$. When $y = F^{m_k}x \in E_2$, this gives (7). □

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[9]The reader should remark that here the order of the matrices in the product is the inverse order than the one used in (4). This notation is convenient since we will apply it to matrices Z where Z^{-1} is the Rauzy cocycle.

[10]The same proof gives that for each $\varepsilon > 0$ there exists E_1 with $\mu(E_1) > 1 - \varepsilon$.

2.2. *Rauzy-Veech-Zorich cocycle.*

Rauzy-Veech and Zorich algorithms. The Rauzy-Veech algorithm and the associated cocycle were originally introduced and developed in the works of Rauzy and Veech ([Rau79], [Vee78], [Vee82]) and proved since then to be a powerful tool to study interval exchange transformations. If $T = (\underline{\lambda}, \pi)$ satisfies the Keane's condition recalled in Section 1.2, which holds for almost every IET by [Kea75], then the Rauzy-Veech algorithm produces a sequence of IETs which are induced maps of T onto a sequence of nested subintervals contained in $I^{(0)}$. The intervals are chosen so that the induced maps are again IETs of the same number d of exchanged intervals. For the precise definition of the algorithm, we refer, e.g., to the recent lecture notes by Yoccoz [Yoc06] or Viana [Via]. We recall here only some basic definitions and properties needed in the proofs.

Let us use, here and in the rest of the paper, the vector norm $|\underline{\lambda}| = \sum_{i=1}^{d} \lambda_i$. If $I' \subset I^{(0)}$ is the subinterval associated to one step of the algorithm and T' is the corresponding induced IET, then the *Rauzy-Veech map* $\mathcal{R}$ associates to T the IET $\mathcal{R}(T)$ obtained by renormalizing T' by $\mathrm{Leb}(I')$ so that the renormalized IET is again defined on an unit interval. The natural domain of definition of the map $\mathcal{R}$ is a full Lebesgue measure subset of the space $X := \Delta_{d-1} \times \mathcal{R}(\pi)$, where $\mathcal{R}(\pi)$ is the Rauzy class[11] of the permutation π.

Veech proved in [Vee82] that $\mathcal{R}$ admits an invariant measure $\mu_{\mathcal{V}}$ which is absolutely continuous with respect to Lebesgue measure, but this measure is infinite. Zorich showed in [Zor96] that one can accelerate[12] the map $\mathcal{R}$ in order to obtain a map $\mathcal{Z}$, which we call *Zorich map*, that admits a *finite* invariant measure $\mu_{\mathcal{Z}}$. Let us also recall that both $\mathcal{R}$ and its acceleration $\mathcal{Z}$ are *ergodic* with respect to $\mu_{\mathcal{R}}$ and $\mu_{\mathcal{Z}}$ respectively [Vee82]. Let us recall the definition of the cocycle associated by the algorithm to the map $\mathcal{Z}$.

Lengths-cocycle. Let us consider the Zorich map $\mathcal{Z}$ on $X = \Delta_{d-1} \times \mathcal{R}(\pi)$. We denote by $\{I^{(n)}\}_{n\in\mathbb{N}}$ the sequence of inducing intervals for T corresponding to the Zorich acceleration of the Rauzy-Veech algorithm (well defined if T satisfies the Keane's condition). Let $T^{(n)} = \mathcal{Z}^n(T)$ be the renormalized induced IET, which is given by $T^{(n)} := (\pi^{(n)}, \underline{\lambda}^{(n)}/\lambda^{(n)})$, where $\lambda^{(n)} = |\underline{\lambda}^{(n)}| = \mathrm{Leb}(I^{(n)})$. For each $T = T^{(0)}$ for which $\mathcal{Z}(T) = (\pi^{(1)}, \underline{\lambda}^{(1)}/\lambda^{(1)})$ is defined, let us associate to T the matrix $Z = Z(T)$ in $\mathrm{SL}(d, \mathbb{Z})$ such that $\underline{\lambda}^{(0)} = Z \cdot \underline{\lambda}^{(1)}$.

[11]Let us recall that the Rauzy class of π is the subset of all permutations π' of d symbols which appear as permutations of an IET $T' = (\underline{\lambda}', \pi')$ in the orbit under $\mathcal{R}$ of some IET $(\underline{\lambda}', \pi)$ with initial permutation π.

[12]The acceleration of a map is obtained by defining almost everywhere an integer valued function $z(T)$ which gives the return time to an appropriate section. The accelerated map is then given by $\mathcal{Z}(T) := \mathcal{R}^{z(T)}(T)$.

The map $Z^{-1}\colon X \to \mathrm{SL}(d,\mathbb{Z})$ is a cocycle over $(X, \mu_{\mathcal{Z}}, \mathcal{Z})$, which we call the *Zorich lengths-cocycle*. Zorich proved in [Zor96] that Z^{-1} is *integrable*.

Defining $Z_n = Z_n(T) \doteqdot Z(\mathcal{Z}^n(T))$ and $Z^{(n)} \doteqdot Z_0 \cdots Z_{n-1}$ and iterating the lengths relation, we get

$$\underline{\lambda}^{(n)} = \left(Z^{(n)}\right)^{-1} \underline{\lambda}, \quad \text{where} \quad \mathcal{Z}^n(T) := \left(\frac{\underline{\lambda}^{(n)}}{\lambda^{(n)}}, \pi^{(n)}\right). \tag{8}$$

For more general products with $m < n$ we use the notation

$$Z^{(m,n)} \doteqdot Z_m Z_{m+1} \cdots Z_{n-1}.$$

By our choice of the norm $|\underline{\lambda}| = \sum_i |\lambda_i|$ on vectors and $\|A\| = \sum_{i,j} |A_{ij}|$ on matrices, from (8),

$$|\underline{\lambda}^{(m)}| = \lambda^{(m)} \leq \|Z^{(m,n)}\| \lambda^{(n)}. \tag{9}$$

Moreover, if $Z^{(n,m)} = A_1 \cdots A_N$ where each of the matrices A_i has strictly positive entries, then

$$\lambda^{(m)} \geq d^N \lambda^{(n)}. \tag{10}$$

The *natural extension* $\widehat{\mathcal{Z}}$ of the map $\mathcal{Z}$ is an invertible map defined on a domain $\widehat{X}$ such that there exists a projection $p : \widehat{X} \to X$ for which $p\widehat{\mathcal{Z}} = \mathcal{Z}p$ (see [Yoc06] and [Via] for the explicit definition of $\widehat{X}$, which admits a geometric interpretation in terms of the space of zippered rectangles). The natural extension $\widehat{\mathcal{Z}}$ preserves a natural invariant measure $\mu_{\widehat{\mathcal{Z}}}$, which gives $\mu_{\mathcal{Z}}$ as pull back by p. The cocycle Z^{-1} can be extended to a cocycle over $(\widehat{X}, \mu_{\widehat{\mathcal{Z}}}, \widehat{\mathcal{Z}})$ by defining the extended cocycle, for which we will use the same notation Z^{-1}, to be constant on the fibers of p.

Towers and induced partitions. The action of the initial interval exchange T can be seen in terms of Rohlin towers over $T^{(n)} := \mathcal{Z}^n(T)$ as follows. Let $\underline{h}^{(n)} \in \mathbb{N}^d$ be the vector such that $h_j^{(n)}$ gives the return time of any $x \in I_j^{(n)}$ to $I^{(n)}$. Define the sets

$$Z_j^{(n)} \doteqdot \bigcup_{l=0}^{h_j^{(n)}-1} T^l I_j^{(n)}.$$

Each $Z_j^{(n)}$ can be visualized as a tower over $I_j^{(n)} \subset I^{(n)}$, of height $h_j^{(n)}$, whose floors are $T^l I_j^{(n)}$. Under the action of T every floor but the top one (i.e., every $T^e I_j^{(n)}$ with $0 \leq l < h_j^{(n)}-1$) moves one step up, while the image by T of the last one (corresponding to $l = h_j^{(n)}-1$) is $T^{(n)} I_j^{(n)}$.

Let us denote by $\phi^{(n)}$ the partition of $I^{(0)}$ into floors of step n, i.e., intervals of the form $T^l I_j^{(n)}$. We say that $F \in \phi^{(n)}$ is of *type* j, where $1 \leq j \leq d$, if it is a floor of $Z_j^{(n)}$. The following well-known fact is proved for example in [Yoc06], [Via].

Remark 2.2. If T satisfies the Keane's condition, then the partitions $\phi^{(n)}$ converge as n tends to infinity to the trivial partitions into points.

We recall also that the entry $Z_{ij}^{(n)}$ of the matrix $Z^{(n)}$ equals the number of visits of the orbit of any point $x \in I_j^{(n)}$ to the interval $I_i^{(0)}$ of the original partition before its first return to $I^{(n)}$. Moreover, the height vectors $\underline{h}^{(n)}$ can be obtained by applying the dual cocycle to the column vector $\underline{h}^{(0)}$ with all entries equal to 1, i.e.,

$$\underline{h}^{(n)} = (Z^T)^{(n)} \underline{h}^{(0)}. \tag{11}$$

Balanced return times. Consider an orbit $\{\mathcal{Z}^n(T)\}_{n\in\mathbb{N}}$ of a T satisfying Keane's condition. Let us say that a sequence $\{n_l\}_{l\in\mathbb{N}}$ is a sequence of *balanced times* for T if there exists $\nu > 1$ such that the following hold for all $l \in \mathbb{N}$:

$$\frac{1}{\nu} \le \frac{\lambda_i^{(n_l)}}{\lambda_j^{(n_l)}} \le \nu, \quad \frac{1}{\nu} \le \frac{h_i^{(n_l)}}{h_j^{(n_l)}} \le \nu, \quad \forall\, 1 \le i,\, j \le d. \tag{12}$$

If n is such that the tower representation over $\mathcal{Z}^n(T)$ satisfies (12), then we call n a *balanced return time*. Lengths and heights of the induction towers are approximately of the same size if n is a balanced return time or, more precisely:

$$\frac{1}{d\nu}\lambda^{(n)} \le \lambda_j^{(n)} \le \lambda^{(n)}, \quad \frac{1}{\nu\lambda^{(n)}} \le h_j^{(n)} \le \frac{\nu}{\lambda^{(n)}}, \quad \forall j = 0, \dots, d. \tag{13}$$

Hilbert metric and projective contractions. Consider on the simplex $\Delta_{d-1}a \subset \mathbb{R}_+^d$ the *Hilbert distance* d_H, defined as follows:

$$d_H(\underline{\lambda}, \underline{\lambda}') \doteqdot \log \left(\frac{\max_{i=1,\dots,d} \frac{\lambda_i}{\lambda'_i}}{\min_{i=1,\dots,d} \frac{\lambda_i}{\lambda'_i}} \right).$$

Let us write $A \ge 0$ if A has nonnegative entries and $A > 0$ if A has strictly positive entries. Recall that to each $A \in \mathrm{SL}(d, \mathbb{Z})$, $A \ge 0$, one can associate a projective transformation $\widetilde{A} : \Delta_{d-1} \to \Delta_{d-1}$ given by

$$\widetilde{A}\underline{\lambda} = \frac{A\underline{\lambda}}{|A\underline{\lambda}|}.$$

When $A \ge 0$, $d_H(\widetilde{A}\underline{\lambda}, \widetilde{A}\underline{\lambda}') \le d_H(\underline{\lambda}, \underline{\lambda}')$. Furthermore, if $A > 0$, then we get a contraction. More precisely, if $A > 0$, since the closure $\widetilde{A}\,(\Delta_{d-1})$ is contained in Δ_{d-1}, we have

$$d_H(\widetilde{A}\underline{\lambda}, \widetilde{A}\underline{\lambda}') \le (1 - e^{-D(A)}) d_H(\underline{\lambda}, \underline{\lambda}'), \text{ where } D(A) \doteqdot \sup_{\underline{\lambda}, \underline{\lambda}' \in \Delta_{d-1}} d_H(\widetilde{A}\underline{\lambda}, \widetilde{A}\underline{\lambda}') < \infty. \tag{14}$$

3. **Rigidity sets and the Kochergin criterion**

3.1. *A condition for absence of mixing.*

Rigidity sets. Interval exchange transformations present some type of rigidity, which was used by Katok in [Kat80] to show that they are never mixing. Let us formalize it in the following definition.

Definition 3.1 (Rigidity sets and times). The sequence $\{E_k\}_{k\in\mathbb{N}}$ of measurable subsets $E_k \subset I$ forms a sequence of *rigidity sets* if there exist a corresponding increasing sequence of *rigidity times* $\{r_k\}_{k\in\mathbb{N}}$, $r_k \in \mathbb{N}_+$, a sequence of finite partitions $\{\xi_k\}_{k\in\mathbb{N}}$ converging to the trivial partition into points and a constant $\alpha > 0$ such that

(i) $\mathrm{Leb}(E_k) \geq \alpha$ for all $k \in \mathbb{N}$;
(ii) for any $F \in \xi_k$, $T^{r_k}(F \cap E_k) \subset F$.

Condition (ii) is a way to express that T^{r_k} is close to identity on E_k.

In order to show absence of mixing for a special flow whose base presents this type of rigidity, it is enough to verify the following criterion, which was proved in [Koč76] by Kochergin and there applied to flows over rotations.

Lemma 3.1 (Absence of mixing criterion). *If there exist a sequence $\{E_k\}_{k\in\mathbb{N}}$ of rigidity sets $E_k \subset I$ with corresponding rigidity times $\{r_k\}_{k\in\mathbb{N}}$ and a constant $M > 0$ such that*

(iii) *for all $k \in \mathbb{N}$, for all $y_1, y_2 \in E_k$, $|S_{r_k}(f)(y_1) - S_{r_k}(f)(y_2)| < M$,*

then the special flow $\{\varphi_t\}_{t\in\mathbb{R}}$ is not mixing.

Condition (iii) is described sometimes by saying that Birkhoff sums $S_{r_k}(f)$ "do not stretch". Stretching of Birkhoff sums is the main mechanism which produces mixing in special flows over rotations or over interval exchange transformations when the roof function has logarithmic asymmetric singularities (see e.g. [SK92], [Ulc07b]). Lemma 3.1 shows that stretching of the Birkhoff sums is also a necessary condition to produce mixing, when there is rigidity in the base.

We use Lemma 3.1 to prove Theorem 1.2. In Section 3.2 we describe the construction of a class of sequences of rigidity sets E_k and times r_k for typical IETs, which are used in the proof of Theorem 1.2. The sets that we construct are analogous to the type of sets used by Katok in [Kat80] to show that IETs are never mixing, but are constructed with the help of Rohlin towers for Rauzy-Veech induction. A variation of this construction is used by the author also in [Ulc09], for the proof of weak mixing for this class of flows. The heart of the proof of absence of mixing is the proof that (iii) holds, given in Section 4.

3.2. *Construction of rigidity sets.* Assume that T satisfies Keane's condition. Let $\{n_k\}_{k\in\mathbb{N}}$ be a sequence of balanced times for T. Consider the corresponding towers $Z_j^{(n_k)}$ for $j = 1, \dots, d$. By the pigeon hole principle, since $\sum_j h_j^{(n_k)} \lambda_j^{(n_k)} = 1$, we can choose j_0 such that

$$h_{j_0}^{(n_k)} \lambda_{j_0}^{(n_k)} \geq \frac{1}{d}. \tag{15}$$

The map T' which is obtained inducing on T on $I_{j_0}^{(n_k)}$ is an IET of at most $d+2$ intervals (see for example [CFS82]), which we denote $(I_{j_0}^{(n_k)})_l$, where $0 \leq l \leq d+2$. Let $h_{j_0,l}^{(n_k)}$ be the first return time of $(I_{j_0}^{(n_k)})_l$ to $I_{j_0}^{(n_k)}$ under T. Choose l_0 so that

$$\mathrm{Leb}\,(I_{j_0}^{(n_k)})_{l_0} \geq \frac{1}{d+2}\,\mathrm{Leb}\, I_{j_0}^{(n_k)}. \tag{16}$$

Let $J_k \subset (I_{j_0}^{(n_k)})_{l_0}$ be any subinterval such that $\mathrm{Leb}\, J_k \geq \beta\, \mathrm{Leb}\,(I_{j_0}^{(n_k)})_{l_0}$ for some $0 \leq \beta \leq 1$. Define

$$E_k := \bigcup_{i=0}^{h_{j_0}^{(n_k)}-1} T^i J_k; \quad r_k := h_{j_0,l_0}^{(n_k)}; \tag{17}$$

i.e., E_k is the part of the tower $Z_{j_0}^{(n_k)}$ which *lies above* J_k. Let $\xi_k = \phi^{(n_k)}$ be the sequence of partitions into floors corresponding to the considered balanced steps.

LEMMA 3.2. *The sequences* $\{\xi_k\}_{k\in\mathbb{N}}$, $\{r_k\}_{k\in\mathbb{N}}$ *and* $\{E_k\}_{k\in\mathbb{N}}$ *defined above satisfy the assumptions* (i) *and* (ii) *of Definition* 3.1.

Remark 3.1. For any $0 \leq j < h_{j_0}^{(n_k)}$, all $T^i(T^j(I_{j_0}^{(n_k)})_{l_0})$ with $0 \leq i < r_k$ are disjoint intervals, which are rigid translates of $(I_{j_0}^{(n_k)})_{l_0}$. The same is true for $T^i J_k$, $0 \leq i < r_k$.

The proof of Lemma 3.2 and of Remark 3.1 can be found in [Ulc09].

4. **Upper bounds on Birkhoff sums of derivatives.**

The key ingredient to show condition (iii) of the absence of mixing criterion (Lemma 3.1) are upper bounds on the Birkhoff sums $|S_{r_k}(f_0')|$ on some rigidity set E_k, where f_0 is the pure logarithmic part of $f = f_0 + w$ (see Definition 1.1) and r_k is the rigidity time corresponding to E_k (see definitions in §3.1). Let us first consider Birkhoff sums of the form $S_{r_k}(f_0')(z_0)$ where $z_0 \in I_j^{(n_k)}$ and $r_k = h_j^{(n_k)}$ is exactly the return time. We call this type of sums *Birkhoff sums along a tower*, since the orbit segment $\{T_i z_0\}_{i=0}^{r_k-1}$ has exactly one point in each floor of the tower $Z_j^{(n_k)}$.

Let us denote by $x_i^{\min}$, for $i = 0, \ldots, s_1 - 1$, the minimum distance from the singularity $\overline{z}_i^+$ of the orbit points to the right of $\overline{z}_i^+$ and by $y_i^{\min}$, for $i = 0, \ldots, s_2 - 1$, the minimum distance from the singularity $\overline{z}_i^-$ of the points to the left of $\overline{z}_i^-$. In formulas, denoting by $(x)^{\mathrm{pos}}$ the positive part of x defined by $(x)^{\mathrm{pos}} = x$ if $x \geq 0$ and $(x)^{\mathrm{pos}} = +\infty$ if $x < 0$, these minimum distances are given by

$$x_i^{\min} := \min\{(T^j z_0 - \overline{z}_i)^{\mathrm{pos}}, \quad 0 \leq j < r_k\}, \qquad i = 0, \ldots, s_1 - 1, \tag{18}$$

$$y_i^{\min} := \min\{(\overline{z}_i - T^j z_0)^{\mathrm{pos}}, \quad 0 \leq j < r_k\}, \qquad i = 0, \ldots, s_2 - 1. \tag{19}$$

We adopt the convention that $1/\infty = 0$.

PROPOSITION 4.1. *For almost every* IET T *there exist a constant* M *and a sequence of balanced induction times* $\{c_l\}_{l \in \mathbb{N}}$ *such that, if* $z_0 \in I_j^{(c_l)}$ *and* $r_l = h_j^{(c_l)}$,

$$|S_{r_l}(f_0')(z_0)| \leq M r_l + \sum_{i=0}^{s_1 - 1} \frac{C_i^+}{x_i^{\min}} + \sum_{i=0}^{s_2 - 1} \frac{C_i^-}{y_i^{\min}}. \tag{20}$$

Remark that if $x_i^{\min} = +\infty$ (and similarly if $y_i^{\min} = +\infty$), since by convention $\frac{1}{\infty} = 0$, the term $\frac{1}{x_i^{\min}}$ does not contribute to the sum (in other words, only the closest visits from the side of the singularity contribute to the sum).

The proof of Proposition 4.1 is given in Section 4.2, using the lemmas proved in Section 4.1. The estimate of Proposition 4.1 for Birkhoff sums along towers is then used in Section 4.3 to give bounds on more general Birkhoff sums.

Let us remark that the linear growth in (20) is essentially due to a principal value phenomenon of cancellations between symmetric sides of the singularities, which is peculiar to the symmetric case. A similar principal value phenomenon was used, in the case of rotations, in [SU08]. In presence of an asymmetric singularity, as shown in [Ulc07b], $S_{r_k}(f_0')$ grows as $r_k \log r_k$ on a set of measure tending to 1 as k tends to infinity.

4.1. *Deviations estimates.* In order to estimate deviations of ergodic averages, it is standard to first consider deviations for the number of elements of $\phi^{(n)}$ of type j inside $I_i^{(m)}$, i.e. for the quantities

$$N_{ij}^{(m,n)} \doteqdot \#\{h \mid T^h I_j^{(n)} \subset I_i^{(m)}, 0 \leq h < h_j^{(n)}\}.$$

In terms of the cocycle matrices, $N_{ij}^{(m,n)} = Z_{ij}^{(m,n)}$ also gives the cardinality of elements of $\phi^{(n)}$ of type j inside each element of $\phi^{(m)}$ of type i. Let us recall that in [Zor97] Zorich proved an asymptotic result on deviations of ergodic averages for characteristic functions of intervals of $\phi^{(0)}$ (hence on the asymptotic growth of $N_{ij}^{(m,n)}$).

4.1.1. *Balanced acceleration.* Let Z be the Zorich cocycle over the natural extension $\widehat{\mathcal{Z}}$ (see §2.2). Let $\widehat{K}$ be a compact subset of $\widehat{X}$ and denote by $A := Z_{\widehat{K}}$ the induced cocycle of Z on $\widehat{K}$. If $\widehat{T}$ is recurrent to $\widehat{K}$, denote by $\{a_n\}_{n\in\mathbb{N}}$ the sequence of visits of $\widehat{T}$ to $\widehat{K}$. One can choose the compact set[13] $\widehat{K}$ so that, considering the acceleration corresponding to return times to $\widehat{K}$, the following properties hold (the notation is the one introduced in Section 2.2 and more details can be found in [Ulc07a] and [AGY06]).

LEMMA 4.1. *There exists $\overline{D} > 0$ and $\nu > 1$ depending only on $\widehat{K}$ such that*

(i) $A_n = A(\mathcal{Z}^{a_n} T) > 0$ *for each* $n \in \mathbb{N}$;
(ii) $D(A_n) \doteqdot \sup_{\underline{\lambda},\underline{\lambda}' \in \Delta_{d-1}} d_H(\widetilde{A}_n\underline{\lambda}, \widetilde{A}_n\underline{\lambda}') \leq \overline{D}$;
(iii) *the return times* $\{a_n\}_{n\in\mathbb{N}}$ *to* $\widehat{K}$ *are ν-balanced times.*

4.1.2. *Deviations estimates for partition intervals.* Using the balanced acceleration A we can control quantitatively the convergence of $N_{ij}^{(m,n)}$ corresponding to m, n which belong to the sequence $\{a_k\}_{k\in\mathbb{N}}$ of visits to $\widehat{K}$.

LEMMA 4.2. *Let* $(N_A)_{ij}^{(m,n)} := N_{ij}^{(a_m,a_n)}$. *There exists $C_{\overline{D}} > 0$ such that for each recurrent $\widehat{T} \in \widehat{K}$, for each pair $a_m < a_n$ of return times, we have*

$$(21)\quad (N_A)_{ij}^{(m,n)} = \delta_j^{(a_n)} \frac{\lambda_i^{(a_m)}}{\lambda_j^{(a_n)}} \left(1 + \varepsilon_{ij}^{(a_m,a_n)}\right), \quad \left|\varepsilon_{ij}^{(a_m,a_n)}\right| \leq C_{\overline{D}} (1 - e^{-\overline{D}})^{n-m},$$

for all $1 \leq i, j \leq d$, *where* $\delta_j^{(a_n)} = h_j^{(a_n)} \lambda_j^{(a_n)}$.

Let us remark that the leading term in (21) is $h_j^{(a_n)} \operatorname{Leb}(I_i^{(a_m)})$, as expected by ergodicity. The form of the leading term in (21) shows that it is proportional to the ratio of lengths of the intervals where the density $\delta_j^{(a_n)}$ depends only on j. The error, i.e., the deviations from this leading behavior, decreases exponentially in the number of visits to $\widehat{K}$.

Proof. Let us denote $\varepsilon_n := (1 - e^{-\overline{D}})^{n-1}\overline{D}$ where $\overline{D} > 0$ is as in Property (ii) in Lemma 4.1. Let us prove first that for each $1 \leq i, j \leq d$ and $m < n$ we have

$$(22)\qquad e^{-2\varepsilon_{n-m}} \lambda_i^{(a_m)} \leq \frac{A_{ij}^{(m,n)}}{h_j^{(a_n)}} \leq e^{2\varepsilon_{n-m}} \lambda_i^{(a_m)}.$$

[13] Using the notation in [AGY06], we can, for example, choose $\widehat{K}$ to be a subset of the Zorich cross-section contained in $\Delta_\gamma \times \Theta_\gamma$, where γ is a path in the Rauzy class of π starting and ending at π, chosen to be *positive* and *neat* (see §§3.2.1 and 4.1.3 in [AGY06] for the corresponding definitions).

Let us consider the sets $\widetilde{A^{(m,l)}}\Delta_{d-1} \subset \Delta_{d-1}$ for $l > m$ and let us remark that give a sequence of nested sets. Since by (8) we have $\underline{\lambda}^{(a_m)} = A^{(m,l)}\underline{\lambda}^{(a_l)}$, we also have

$$\frac{\underline{\lambda}^{(a_m)}}{\lambda^{(a_m)}} \in \bigcap_{l>m} \widetilde{A^{(m,l)}}\Delta_{d-1}.$$

When $l = n$, since $D(A_i) \leq \overline{D}$ for each $i \in \mathbb{N}$ by in Property (ii) in Lemma 4.1, applying $n - m - 1$ times the contraction estimate (14), we get

$$D(A^{(m,n)}) \leq (1 - e^{-\overline{D}})^{n-m-1}\overline{D} \leq \varepsilon_{n-m}. \tag{23}$$

Let us denote by $\underline{e}_j$ the unit vector $(\underline{e}_j)_i = \delta_{ij}$ (δ is here the Kronecker symbol). Since both vectors $\widetilde{A^{(m,n)}}\underline{e}_j$ and $\frac{\underline{\lambda}^{(m)}}{\lambda^{(m)}}$ belong to the closure of $\widetilde{A^{(m,n)}}\Delta_{d-1}$, it follows by (23), using compactness, that

$$d_H\left(\frac{\underline{\lambda}^{(a_m)}}{\lambda^{(a_m)}}, \widetilde{A^{(m,n)}\underline{e}_j}\right) = \log \frac{\max_{i=1,\dots,d} \frac{A_{ij}^{(m,n)}}{\lambda_i^{(a_m)}}}{\min_{i=1,\dots,d} \frac{A_{ij}^{(m,n)}}{\lambda_i^{(a_m)}}} \leq \varepsilon_{n-m},$$

where we also used the invariance of the distance expression by multiplication of the arguments by a scalar. Equivalently, for each $1 \leq i,\, k \leq d$,

$$e^{-\varepsilon_{n-m}}\left(A_{kj}^{(m,n)}\lambda_i^{(a_m)}\right) \leq A_{ij}^{(m,n)}\lambda_k^{(a_m)} \leq e^{\varepsilon_{n-m}}\left(A_{kj}^{(m,n)}\lambda_i^{(a_m)}\right) \tag{24}$$

and summing over k or respectively multiplying (24) by $h_i^{(a_m)}$ and then summing over both k and i and using that $\sum_i h_i^{(a_m)}\lambda_i^{(a_m)} = 1$ and (11), we get respectively

$$e^{-\varepsilon_{n-m}} \leq \frac{A_{ij}^{(m,n)}\lambda^{(a_m)}}{\sum_k A_{kj}^{(m,n)}\lambda_i^{(a_m)}} \leq e^{\varepsilon_{n-m}}, \quad e^{-\varepsilon_{n-m}} \leq \frac{h_j^{(a_n)}\lambda^{(a_m)}}{\sum_k A_{kj}^{(m,n)}} \leq e^{\varepsilon_{n-m}}. \tag{25}$$

Producing the estimates in (25) gives (22). Since, for $n - m$ sufficiently large, $\varepsilon_{n-m} \leq 1/2$ and $|1 - e^{\pm 2\varepsilon_{n-m}}| \leq 4\varepsilon_{n-m}$, the lemma follows from (22) by remarking that $(N_A)_{ij}^{(m,n)} = A_{ij}^{(m,n)}$ and setting $\delta_j^{(a_n)} := h_j^{(a_n)}\lambda_j^{(a_n)}$. $\square$

4.1.3. *Power form of the deviation.* Let us show that, for times corresponding to a further appropriate acceleration of the cocycle A, the deviations can be expressed as a small power of the main order.

Lemma 4.3. *For almost every T there exist a subsequence $\{b_k := a_{n_k}\}_{k\in\mathbb{N}} \subset \{a_n\}_{n\in\mathbb{N}}$ and $0<\gamma<1$ such that for all $k\in\mathbb{N}$, for all $0\le k'<k$, we have*

$$(N_B)_{ij}^{(k',k)} := (N_A)_{ij}^{(n_{k'},n_k)} = \delta_j^{(b_k)}\left(\frac{\lambda_i^{(b_{k'})}}{\lambda_j^{(b_k)}} + \mathscr{E}_{ij}^{(b_{k'},b_k)}\right),$$
$$\left|\mathscr{E}_{ij}^{(b_{k'},b_k)}\right| \le \mathrm{const}\left(\frac{\lambda_i^{(b_{k'})}}{\lambda_j^{(b_k)}}\right)^{\gamma}.$$

Proof. By Lemma 2.1, there exist a measurable set $E_B\subset\widehat{K}$ with positive measure and $\overline{C}_1>0$ such that if $\widehat{T}\in E_B$ is recurrent to E_B along $\{a_{n_k}\}_{k\in\mathbb{N}}$ (which is a subsequence of the visits to $\widehat{K}$ since $E_B\subset\widehat{K}$), then (6) holds. By ergodicity of $\widehat{\mathcal{Z}}$, almost every $\widehat{T}\in\widehat{X}$ is recurrent to E_B. Thus for almost every T, there exists $\widehat{T}\in p^{-1}(T)$ recurrent to E_B (indeed a full measure set of $\widehat{T}$ in the fiber is recurrent), and we can define $\{a_{n_k}\}_{k\in\mathbb{N}}$ to be the sequence along which $\widehat{T}$ is recurrent.

Since $(N_A)_{ij}^{(n_{k'},n_k)}$ satisfies, by Lemma 4.2, the estimate (21) and $\delta_j^{(b_k)}\le 1$, it is enough to prove that, for some[14] $\gamma<1$ and const >0,

$$\left(\frac{\lambda_i^{(b_{k'})}}{\lambda_j^{(b_k)}}\right)\varepsilon_{ij}^{(b_{k'},b_k)} \le \mathrm{const}\left(\frac{\lambda_i^{(b_{k'})}}{\lambda_j^{(b_k)}}\right)^{\gamma}. \tag{26}$$

By (9) and (13), $\lambda_i^{(b_{k'})}\le d\nu\|A^{(n_{k'},n_k)}\|\lambda_j^{(b_k)}$. Let $1-\gamma:=-\log(1-e^{-D})/\overline{C}_1>0$, so that $\gamma<1$ and, recalling the estimate of $\varepsilon_{ij}^{(a_{n_{k'}},a_{n_k})}=\varepsilon_{ij}^{(b_{k'},b_k)}$ and using (6), we have

$$\left(\frac{\lambda_j^{(b_k)}}{\lambda_i^{(b_{k'})}}\right)^{1-\gamma} \ge \left(\frac{1}{d\nu\|A^{(n_{k'},n_k)}\|}\right)^{1-\gamma} \ge \frac{(d\nu)^{\gamma-1}}{\left(e^{\overline{C}_1(n_k-n_{k'})}\right)^{1-\gamma}}$$
$$= (d\nu)^{\gamma-1}(1-e^{-D})^{n_k-n_{k'}} \ge c\left|\varepsilon_{ij}^{(b_{k'},b_k)}\right|$$

for some constant $c>0$, which is exactly (26). □

We will denote by B the induced cocycle of A corresponding to this acceleration. In particular, for $k\in\mathbb{N}$, let $B_k(T)=B_k(\widehat{T})=A^{(b_k,b_{k+1})}(T)$, where $\{b_k\}_{k\in\mathbb{N}}$ is the sequence of visits of a chosen lift $\widehat{T}\in p^{-1}(T)$, which is recurrent to E_B.

[14]In the statement of Lemma 4.3 we require $0<\gamma<1$, but if (26) holds for some $\gamma\le 0$, since $\lambda_i^{(b_{k'})}/\lambda_j^{(b_k)}\ge 1$, then, by positivity, it also holds for $0<\gamma'<1$.

4.1.4. *Deviations for any interval.* Let T, $\{b_k\}_{k\in\mathbb{N}}$ and $0<\gamma<1$ be as in Lemma 4.3. Let us now consider an interval $I\subset I^{(0)}$ and let us denote the number of intervals of type j of $\phi^{(b_k)}$ contained in I by

$$(N_B)_j^{(k)}(I) \doteqdot \#\{l \mid T^l I_j^{(b_k)} \subset I, 0\le l< h_j^{(b_k)}\}.$$

In order to describe the deviations from ergodic averages, set by convention $B_{-1}(T):=B_0(T)$ and introduce, for any $0\le k'<k$, the quantity

$$\Theta_{k'}^k = \Theta_{k'}^k(T) := \sum_{n=0}^{k-k'} \frac{\|B_{k'+n-1}(T)\|}{d^{\gamma n}}. \tag{27}$$

The following lemma shows that the deviations of $(N_B)_j^{(k)}(I)$ can be estimated in terms of $\Theta_{k(I)}^k$, where

$$k(I) := \min\{k \mid \text{there exists}\, F\in\phi^{(b_k)}\, \text{such that}\, F\subset I\}. \tag{28}$$

LEMMA 4.4. *For almost every T and for all $k\in\mathbb{N}$ and all $1\le j\le d$, given any interval I of length* $\mathrm{Leb}(I)\ge\lambda_j^{(b_k)}$, *we have*

$$(N_B)_j^{(k)}(I)=\delta_j^{(b_k)}\left(\frac{\mathrm{Leb}(I)}{\lambda_j^{(b_k)}}+\mathscr{E}_j^{(k)}(I)\right),\quad \left|\mathscr{E}_j^{(k)}(I)\right|\le \mathrm{const}\;\Theta_{k(I)}^k(T)\left(\frac{\mathrm{Leb}(I)}{\lambda_j^{(b_k)}}\right)^{\gamma},$$

where $\{b_k\}_{k\in\mathbb{N}}$, $0<\gamma<1$ and $\delta_j^{(b_k)}$ are the same as in Lemma 4.3.

Proof. Let us decompose I into elements of the partitions $\phi^{(b_k)}$, $k\ge k(I)$, as follows. Consider all intervals of $\phi^{(b_{k(I)})}$ which are completely contained in I. By definition of $k(I)$, this set is not empty and, moreover, I is contained in at most two intervals of $\phi^{(b_{k(I)-1})}$. Hence, if we denote by $\#_i^{k(I)}$ the number of intervals of $\phi^{(b_{k(I)})}$ of type i contained in I, if $k(I)>0$, then we have $\#_i^{k(I)}\le 2\max_{1\le l\le d}(B_{k(I)-1})_{li}\le 2\|B_{k(I)-1}\|$. If $k(I)=0$, since $I^{(0)}$ contains $\sum_{1\le l\le d}(B_0)_{li}$ elements of $\phi^{(b_0)}$ of type i, we have $\#_i^0\le\|B_0\|$. Thus, if by convention we set $B_{-1}:=B_0$, we have

$$\begin{aligned}\mathrm{Leb}(I) &= \sum_{i=1}^d \#_i^{k(I)}\lambda_i^{(b_{k(I)})}+\delta(I,k(I)),\\ \#_i^{k(I)} &\le 2\|B_{k(I)-1}\|,\quad \delta(I,k(I))\le 2\max_{1\le i\le d}\lambda_i^{(b_{k(I)})},\end{aligned}$$

where $\delta(I,k(I))$ is the length of the remainder (possibly empty), given by the two intervals (at the two ends) left after subtracting from I all interval of $\phi^{(b_{k(I)})}$ completely contained in it. Decompose in the same way the two remainders into intervals of $\phi^{(b_{k(I)+1})}$ (if any) completely contained in it and two new remainder intervals and so on by induction, until decomposing into elements of $\phi^{(b_k)}$. As before, if $\#_i^{k'}$ is the number of intervals of $\phi^{(b_{k'})}$ of type

i involved in the decomposition, then $\#_i^{k'} \leq 2\|B_{k'-1}\|$ since by construction each of the two remainders is contained in an interval of $\phi^{(b_{k'-1})}$. Thus, we get

(29)
$$\mathrm{Leb}(I)=\sum_{k'=k(I)}^{k}\sum_{i=1}^{d}\#_i^{k'}\lambda_i^{(b_{k'})}+\delta(I,k),\ \#_i^{k'}\leq 2\|B_{k'-1}\|,\quad \delta(I,k)\leq 2\max_{1\leq i\leq d}\lambda_i^{(b_k)}.$$

Using this decomposition to estimate $(N_B)_j^{(k)}(I)$ in terms of $(N_B)_{ij}^{(k',k)}$ and then applying Lemma 4.3, we get

$$(N_B)_j^{(k)}(I)=\sum_{k'=k(I)}^{k}\sum_{i=1}^{d}\#_i^{k'}(N_B)_{ij}^{(k',k)}=\delta_j^{(b_k)}\sum_{k'=k(I)}^{k}\sum_{i=1}^{d}\#_i^{k'}\left(\frac{\lambda_i^{(b_{k'})}}{\lambda_j^{(b_k)}}+\mathscr{E}_{ij}^{(b_{k'},b_k)}\right).$$

Recalling (29), we have

$$(N_B)_j^{(k)}(I)=\delta_j^{(b_k)}\frac{\mathrm{Leb}(I)}{\lambda_j^{(b_k)}}+\delta_j^{(b_k)}\sum_{k'=k(I)}^{k}\sum_{i=1}^{d}\#_i^{k'}\mathscr{E}_{ij}^{(b_{k'},b_k)}-\delta_j^{(b_k)}\frac{\delta(I,k)}{\lambda_j^{(b_k)}}. \tag{30}$$

In order to conclude, let us show that the contribution to the error coming from the last two terms in (30) is of the desired form. The very last term in (30) is less then 2ν by (29) and balance (13) and hence, since $\mathrm{Leb}(I)/\lambda_j^{(b_k)}\geq 1$ by assumption, it is controlled by choosing the constant appropriately. For the other term, applying Lemma 4.3,

$$\sum_{k'=k(I)}^{k}\sum_{i=1}^{d}\#_i^{k'}\mathscr{E}_{ij}^{(b_{k'},b_k)}\leq \mathrm{const}\left(\frac{\mathrm{Leb}(I)}{\lambda_j^{(b_k)}}\right)^{\gamma}\sum_{k'=k(I)}^{k}\|B_{k'-1}\|\left(\frac{\lambda_i^{(b_{k'})}}{\mathrm{Leb}(I)}\right)^{\gamma}.$$

Since by definition of $k(I)$, (10) and balance (13) we have $\mathrm{Leb}(I)\geq \frac{1}{d\nu}\lambda^{(b_{k(I)})}\geq \frac{1}{d\nu}d^{k'-k(I)}\lambda_i^{(b_{k'})}$, the sum is controlled as desired by

$$\sum_{k'=k(I)}^{k}\|B_{k'-1}\|\left(\frac{\lambda_i^{(b_{k'})}}{\mathrm{Leb}(I)}\right)^{\gamma}\leq (d\nu)^{\gamma}\sum_{k'=k(I)}^{k}\frac{\|B_{k'-1}\|}{d^{\gamma(k'-k(I))}}=(d\nu)^{\gamma}\,\Theta_{k(I)}^{k}. \qquad \square$$

In Section 4.2.1 we consider more in general intervals $I=(a,b)\subset\mathbb{R}$, such that $\mathrm{Leb}(I)\leq 1$ and either a or b belong to the set of singularities $\{\overline{z}_0^+,\dots,\overline{z}_{s_1-1}^+,\overline{z}_0^-,\dots,\overline{z}_{s_2-1}^-\}$. We consider I as a subset of $I^{(0)}$ *modulo one*; i.e., if $a<1<b$ we consider the union $(a,1)\cup(0,b-1)$, and if $a<0<b$ we consider $(0,b)\cup(a+1,1)$. One can also decompose this type of intervals so that (29) holds. Indeed, if I modulo one is a disjoint union, then one can decompose any interval of this type so that $k(I)=0$ since one of the two intervals ($(a,1)$ or $(0,b)$ respectively) is a union of elements of $\phi^{(b_0)}$, whose total number is bounded by $\|B_0\|$, and the other interval can be decomposed as before. Thus, the same proof that shows Lemma 4.4 gives also the following remark.

Remark 4.1. Lemma 4.4 also holds for intervals I modulo one such that $\mathrm{Leb}(I) \leq 1$ and one of the endpoints of I is a singularity of f.

4.2. *Cancellations.* Let b_k be an induction time of the sequence $\{b_k\}_{k\in\mathbb{N}}$ of Lemma 4.4. Let $x_0 \in I_{j_0}^{(b_k)}$ and let $r_k := h_{j_0}^{(b_k)}$. Consider the distances of the points in the orbit segment $\{T^i x_0\}_{i=0}^{r_k-1}$ from the right singularity $\overline{z}_i^+$, for $i = 0, \ldots, s_1 - 1$, taken modulo one[15] (i.e. $T^j z_0 - \overline{z}_i^+ (\bmod 1), 0 \leq j < r_k$), and, respectively, consider the distances from the left singularity $\overline{z}_i^-$, for $i = 0, \ldots, s_2 - 1$, taken modulo one (i.e. $\overline{z}_i^- - T^j z_0 \ (\bmod 1), 0 \leq j < r_k$). Rearrange each group in increasing order, renaming by $x_i(j)$ (or respectively $y_i(j)$) the j^{th} distance from the right (respectively from the left), so that the following equalities of sets hold:[16]

(31)
$$\bigcup_{j=0}^{r_k-1} \{x_i(j)\} = \bigcup_{j=0}^{r_k-1} \left\{T^j z_0 - \overline{z}_i^+ \ (\bmod 1)\right\}, x_i(j_1) < x_i(j_2) \ \ \forall j_1 < j_2 (0 \leq i < s_1),$$

(32)
$$\bigcup_{j=0}^{r_k-1} \{y_i(j)\} = \bigcup_{j=0}^{r_k-1} \left\{\overline{z}_i^- - T^j z_0 \ (\bmod 1)\right\}, y_i(j_1) < y_i(j_2) \ \ \forall j_1 < j_2 (0 \leq i < s_2).$$

4.2.1. *Deviations from an arithmetic progression.* As a consequence of Lemma 4.4, we have the following. For $j = 0, \ldots, r_k - 1$, let $I_i^+(j)$, for $i = 0, \ldots, s_1 - 1$, be the interval $(\overline{z}_i^+, \overline{z}_i^+ + x_i(j))$ considered modulo one and let $I_i^-(j)$, for $i = 0, \ldots, s_2 - 1$ be the interval $(\overline{z}_i^- - y_i(j), \overline{z}_i^-)$ considered modulo one. For brevity, let $k_i^\pm(j) := k\left(I_i^\pm(j)\right)$ (see the definition given in (28)).

COROLLARY 4.1. *For all $1 \leq j < r_k$, we have*

$$x_i(j) = \frac{\lambda_{j_0}^{(b_k)}}{\delta_{j_0}^{(b_k)}} \left(j + O\left(\Theta^k_{k_i^+(j)} \ (\|B_k\|\, j)^\gamma\right)\right), \quad i = 0, \ldots, s_1 - 1, \tag{33}$$

$$y_i(j) = \frac{\lambda_{j_0}^{(b_k)}}{\delta_{j_0}^{(b_k)}} \left(j + O\left(\Theta^k_{k_i^-(j)} \ (\|B_k\|\, j)^\gamma\right)\right), \quad i = 0, \ldots, s_2 - 1. \tag{34}$$

Proof. Consider the interval $I_i^+(j)$. Since by definition $x_i(j)$ is the distance of the j^{th} closest point to the right of $\overline{z}_i^+$, there are j points of the orbit in $I_i^+(j)$,

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[15]For example, if $T^j z_0 < \overline{z}_i^+$, then $T^j z_0 - \overline{z}_i^+ = 1 + T^j z_0 - \overline{z}_i^+ \ (\bmod 1)$. In this way, since $u_i(x)$ and $v_i(s)$ are 1-periodic, the quantity $T^j z_0 - \overline{z}_i^+ \ (\bmod 1)$ (respectively $\overline{z}_i^- - T^j z_0 \ (\bmod 1)$) gives the value of $1/u_i(T^j z_0)$ (respectively $1/v_i(T^j z_0)$).

[16]Here the notation $\{x\}$ denotes the singleton set containing $x \in \mathbb{R}$ as its element and should not be confused with the fractional part which we denote by $\{\{x\}\}$.

so that $(N_B)^{(k)}_{j_0}(I_i^+(j)) = j$. Hence, Lemma 4.4 together with Remark 4.1 give

$$j = \delta^{(b_k)}_{j_0}\left(\frac{x_i(j)}{\lambda^{(b_k)}_{j_0}} + \mathscr{E}^{(k)}_{j_0}(I_i^+(j))\right), \quad \left|\mathscr{E}^{(k)}_{j_0}(I_i^+(j))\right| = O\left(\Theta^k_{k_i^+(j)}\left(\frac{x_i(j)}{\lambda^{(b_k)}_{j_0}}\right)^{\gamma}\right),$$

for some $0 < \gamma < 1$, or, rearranging the terms,

$$x_i(j) = \lambda^{(b_k)}_{j_0}\left(\frac{j}{\delta^{(b_k)}_{j_0}} - \mathscr{E}^{(k)}_{j_0}(I_i^+(j))\right).$$

This gives (33) if we show that $x_i(j)/\lambda^{(b_k)}_{j_0} \le c\|B_k\| j$ for some constant c. Since by definition $\{b_k\}_{k\in\mathbb{N}}$ is a subsequence of a balanced sequence, we have $B_k > 0$. Thus, inside each element of $\phi^{(b_{k-1})}$ there is at least one element of $\phi^{(b_k)}$ of type j_0, or, equivalently, one point of the orbit $\{T^i x_0\}_{i=0}^{r_k-1}$. Since a lower bound for the number of elements of $\phi^{(b_{k-1})}$ in $I_i^+(j)$ is given by $[x_i(j)/\lambda^{(b_{k-1})}]$, where $[\cdot]$ denotes the integer part, we get $j \ge x_i(j)/\lambda^{(b_{k-1})} - 1$. Using that $\lambda^{(b_{k-1})} \le \|B_k\|\lambda^{(b_k)}$, we get $x_i(j) \le \|B_k\|(j+1)\lambda^{(b_k)}$. Together with the elementary inequality $\delta^{(b_k)}_{j_0} \le 1$ (recall that $\delta^{(b_k)}_{j_0} = h^{(b_k)}_{j_0}\lambda^{(b_k)}_{j_0}$; see Lemma 4.2), this concludes the proof of (33). In the same way, the proof of (34) follows by applying Lemma 4.4 together with Remark 4.1 to the interval $I_i^-(j)$. $\square$

Let us remark that, in the special case in which the permutation π is (54321) and $\overline{z}_i^{\pm}$ are the endpoints of a subinterval $I_i^{(0)}$ of T (see footnote 7, p. 1749), Scheglov [Sch09] shows that for almost every $\underline{\lambda}$ one can find a subsequence of times $\{b_k\}_{k\in\mathbb{N}}$ and a constant $K > 0$ such that the $|x_i(j) - y_i(j)| \le K\lambda^{(b_k)}_{j_0}$. This stronger form of control of the deviations, which presumably holds for all combinatorics of the form $(n\, n-1 \cdots 21)$, crucially exploits the symmetry of the permutation and hence can be used to prove Theorem 1.1 only for $g = 2$ (see footnote 21, p. 1775).

4.2.2. *Acceleration for cancellations.* Let us now accelerate one more time in order to prove Proposition 4.1.

Proof of Proposition 4.1. By Lemma 2.2 applied to[17] $\varepsilon = \overline{\gamma}\ln(2d/3) \ge 0$ where $\overline{\gamma} := \min\{\gamma, 1-\gamma\} \ge 0$, we can find $E_C \subset E_B$ such that if $\widehat{T} \in p^{-1}(T)$ is recurrent to E_C along $\{b_{k_l}\}_{l\in\mathbb{N}}$ (which is a subsequence of the return times $\{b_k\}_{k\in\mathbb{N}}$ to E_B), we have

$$\|B_{k_l - m}\| \le \overline{C}_2(2d/3)^{\overline{\gamma}m}, \quad \forall 0 \le m \le k_l. \tag{35}$$

[17] One can set $\varepsilon = \overline{\gamma}\ln(d/t)$, where t is any number $1 < t < 2$, so that $d \ge 2 > t$ gives $\varepsilon > 0$. Later we need $t > 1$. Here, for concreteness, we choose $t = 3/2$.

We remark that almost every T has a lift $\widehat{T}$ recurrent to E_C, by ergodicity of $\widehat{\mathcal{Z}}$. Let us set $c_l := b_{k_l}$ and let $x_0 \in I_{j_0}^{(c_l)}$, $r_l = h_{j_0}^{(c_l)}$. Recall that f_0' is of the form (3).

For clarity, let us first give the proof in the special case $s_2 = s_1 = 1$ and $f_0'(x) = -C_0^+/x + C_0^-/(1-x)$ with $C_0^+ = C_0^- = C$, since the notation is heavier in the general case. Using the relabeling of (31) and (32) for $s = 1$, let $x_0(j)$ and $y_0(j)$ denote the distances of the j^{th} orbit point from 0 and 1 respectively, so that

$$S_{r_l}(f_0')(z_0) = \sum_{j=0}^{r_l-1} \left(\frac{C}{1 - T^j z_0} - \frac{C}{T^j z_0} \right)$$
$$= \sum_{j=0}^{r_l-1} \left(\frac{C}{y_0(j)} - \frac{C}{x_0(j)} \right) = C \sum_{j=0}^{r_l-1} \frac{x_0(j) - y_0(j)}{x_0(j) y_0(j)}.$$

We remark that the points in $\{T^i z_0\}_{i=0}^{r_l-1}$ belong to distinct floors of $Z_{j_0}^{(c_l)}$ and have the same relative position within the floors. Hence, $\min_{i \neq j} |T^i z_0 - T^j z_0| \geq \lambda_{j_0}^{(c_l)}$ and we can estimate the denominator by using that $x_0(j), y_0(j) \geq j \lambda_{j_0}^{(b_{k_l})}$ when $j \geq 1$. To estimate the numerator, let us apply Corollary 4.1. By (33) and (34) the leading term in j in $x_0(j)$ and $y_0(j)$ for $j \geq 1$ are the same and cancel out. Moreover, $1/\delta_{j_0}^{(c_l)} \leq d\nu^2$ by balance (12). So, setting aside the contribution of the two closest points $x_0(0) = x_0^{\min}$ and $y_0(0) = y_0^{\min}$, we get

$$\left| S_{r_l}(f_0')(z_0) - \frac{C}{y_0^{\min}} + \frac{C}{x_0^{\min}} \right| \leq \sum_{j=1}^{r_l-1} \frac{j^\gamma \lambda_{j_0}^{(b_{k_l})} \|B_{k_l}\|^\gamma \left(O\left(\Theta_{k_0^+(j)}^{k_l}\right) + O\left(\Theta_{k_0^-(j)}^{k_l}\right)\right)}{j^2 (\lambda_{j_0}^{(b_{k_l})})^2}.$$

Let us first bound the part of the above sum which involves $\Theta_{k_0^-}^{k_l}(j)$. Let $I_{1^-}^k$ be the interval of $\phi^{(b_k)}$ which contains 1 as a right endpoint. Set by convention $I_{1^-}^{(-1)} := I^{(0)}$. The intervals $\{I_{1^-}^k\}_{k \in \mathbb{N}}$ are nested and we can use them to rearrange the sum over j as follows. Let us remark that if $y_0(j) \in I_{1^-}^{k'-1} \backslash I_{1^-}^{k'}$, then, by definition of $k_0^-(j)$, we have $k_0^-(j) = k'$. Moreover, since we kept aside the closest point to 1, there are no orbit points in $I_{1^-}^{k_l}$. Thus, recalling also that $\|B_{k_l}\| \leq \overline{C}_2$ by (35), we obtain

$$\sum_{j=1}^{r_l-1} \frac{\|B_{k_l}\|^\gamma O\left(\Theta_{k_0^-(j)}^{k_l}\right)}{j^{2-\gamma} \lambda_{j_0}^{(b_{k_l})}} \leq \text{const} \sum_{k'=0}^{k_l} \sum_{y_0(j) \in I_{1^-}^{k'-1} \backslash I_{1^-}^{k'}} \frac{\Theta_{k'}^{k_l}}{j^{2-\gamma}\, \lambda_{j_0}^{(b_{k_l})}}$$
$$\leq \text{const } r_l \sum_{k'=0}^{k_l} \frac{\Theta_{k'}^{k_l}}{d^{(1-\gamma)(k_l - k')}},$$

where in the last inequality we used that $\lambda_{j_0}^{(b_{k_l})} \geq \text{const}(r_l)^{-1}$, which follows by balance, and that, if $j_{k'}$ denotes the minimum j such that $y_0(j) \notin I_{1-}^{k'}$, using balance and positivity as in the proof of Corollary 4.1, we get

(36)
$$\sum_{y_0(j)\in I_{1-}^{k'-1}\setminus I_{1-}^{k'}} \frac{1}{j^{2-\gamma}} = O\left(\frac{1}{j_{k'}^{1-\gamma}}\right) \quad \text{and} \quad j_{k'} \geq \left[\frac{\min_j \lambda_j^{(b_{k'})}}{\lambda^{(b_{k_l}-1)}}\right] \geq \text{const } d^{k_l-k'}.$$

Recall that $\overline{\gamma} = \min\{\gamma, 1-\gamma\}$ and recall the definition of $\Theta_{k'}^{k_l}$ given in (27). Changing indexes by $k = k_l - k'$ first and $m = k - n$ later in order to rearrange the sums, since k_l by (35) is such that $\|B_{k_l-m-1}\| \leq \text{const}\left(\frac{2d}{3}\right)^{\overline{\gamma}m}$, we have

$$\sum_{k'=0}^{k_l} \frac{\Theta_{k'}^{k_l}}{d^{(1-\gamma)(k_l-k')}} \leq \sum_{k=0}^{k_l} \frac{\Theta_{k_l-k}^{k_l}}{d^{(\overline{\gamma})k}} \leq \sum_{k=0}^{k_l}\sum_{n=0}^{k} \frac{\|B_{k_l-k+n-1}\|}{d^{\overline{\gamma}(n+k)}} = \sum_{k=0}^{k_l}\sum_{m=0}^{k} \frac{\|B_{k_l-m-1}\|}{d^{\overline{\gamma}(2k-m)}}$$
$$\leq \sum_{k=0}^{k_l} \frac{\text{const}}{d^{\overline{\gamma}(2k)}} \sum_{m=0}^{k} \left(\frac{2^{\overline{\gamma}}(d^2)^{\overline{\gamma}}}{3^{\overline{\gamma}}}\right)^m.$$

Since $(2d^2/3)^{\overline{\gamma}} > 1$ and thus $\sum_{m=0}^{k}(2d^2/3)^{\overline{\gamma}m} = O(d^{2\overline{\gamma}k}(2/3)^{\overline{\gamma}k})$, the latter expression is bounded by const $\sum_{k=0}^{k_l}(2/3)^{\overline{\gamma}k}$, which is uniformly bounded independently on l.

The proof that also the sum involving $\Theta_{k_0^+}^{k_l}(j)$ is uniformly bounded in l is analogous and gives also an upper bound by a fixed constant. This concludes the proof in this case.

In the general case, when $f_0' = \sum_{i=0}^{s_2-1} C_i^- v_i - \sum_{i=0}^{s_1-1} C_i^+ u_i$, reducing to a common denominator, and denoting by $X(j) := \prod_{l=0}^{s_1-1} x_l(j)$, $Y(j) := \prod_{l=0}^{s_2-1} y_l(j)$ and by $X_i(j) := \prod_{\substack{1\leq 1\leq s_1-1\\ l\neq i}} x_l(j)$, $Y_i(j) := \prod_{\substack{1\leq 1\leq s_2-1\\ l\neq i}} y_l(j)$, we get

$$S_{r_l}(f_0')(z_0) = \sum_{j=0}^{r_l-1} \frac{\sum_{i=0}^{s_2-1} C_i^- Y_i(j)X(j) - \sum_{i=0}^{s_1-1} C_i^+ Y(j)X_i(j)}{X(j)Y(j)}. \tag{37}$$

By Corollary 4.1, recalling that $\|B_{k_l}\| \leq \text{const}$, we have

$$X(j) = \left(\frac{\lambda_{j_0}^{(c_l)}}{\delta_{j_0}^{(c_l)}}\right)^{s_1}\left(j^{s_1} + \sum_{m_1=1}^{s_1} j^{s_1-m_1+\gamma m_1} \sum_{1\leq i_1<\cdots<i_{m_1}\leq s_1} \Theta_{k_{i_1}^+(j)}^{k_l}\cdots\Theta_{k_{i_{m_1}}^+(j)}^{k_l}\right),$$
$$X_i(j) = \left(\frac{\lambda_{j_0}^{(c_l)}}{\delta_{j_0}^{(c_l)}}\right)^{s_1-1}\left(j^{s_1-1} + \sum_{m_2=1}^{s_1-1} j^{s_1-1-m_2+\gamma m_2} \sum_{\substack{i_1<\cdots<i_{m_2}\in\\ \{1,\ldots,s_1\}\setminus\{i\}}} \Theta_{k_{i_1}^+(j)}^{k_l}\cdots\Theta_{k_{i_{m_2}}^+(j)}^{k_l}\right)$$

and analogous expressions hold for $Y(j)$ and $Y_i(j)$ with $\Theta_{k_i^-(j)}^{i}$ and s_2 instead of $\Theta_{k_i^+(j)}^{i}$ and s_1. Thus, since the coefficients of $j^{s_1+s_2-1}$ in $Y(j)X_i(j)$

and $Y_i(j)X(j)$ are the same, by symmetry of the constants, the main order in j of the numerator of the right-hand side of (37) cancels out. Moreover, using as before the minimum distance between points, we have $X(j) \geq (j\lambda_{j_0}^{(c_l)})^{s_1}, Y(j) \geq (j\lambda_{j_0}^{(c_l)})^{s_2}$ when $j \geq 1$. Thus, from balance (13) we can estimate $(\lambda_{j_0}^{(c_l)})^{s_1+s_2-1}/(\lambda_{j_0}^{(c_l)})^{s_1+s_2}$ by const r_l. Producing the lower-order terms, for each $1 \leq m_1 \leq s_1$ and $1 \leq m_2 \leq s_2 - 1$ (or $1 \leq m_1 \leq s_1 - 1$ and $1 \leq m_2 \leq s_2$), we are left with a bounded number of terms to estimate. Each one, after simplifying the power of j which is $(s_1 - m_1 + \gamma m_1) + (s_2 - 1 - m_2 + \gamma m_2)$ at numerator and $s_1 + s_2$ at denominator, is of the form

$$\sum_{j=1}^{r_l-1} \frac{\Theta^{k_l}_{k_1^+}(j) \cdots \Theta^{k_l}_{k_{m_1}^+}(j) \Theta^{k_l}_{k_1^-}(j) \cdots \Theta^{k_l}_{k_{m_2}^-}(j)}{j^{1+m_1+m_2-(m_1+m_2)\gamma}}$$
$$\leq \sum_{j=1}^{r_l-1} \frac{\Theta^{k_l}_{k_{i_0}^\pm}(j)}{j^{2-\gamma}} \left(\max_{1 \leq j < r_l} \max_{\substack{0 \leq i^+ \leq s_1-1 \\ 0 \leq i^- \leq s_2-1}} \left(\frac{\Theta^{k_l}_{k_{i^\pm}^\pm(j)}}{j^{1-\gamma}} \right) \right)^{m_1+m_2-1},$$

where $k_{i_0}^\pm$ is any index among $k_1^+, \ldots, k_{m_1}^+$, $k_1^-, \ldots, k_{m_2}^-$. Let us conclude the proof by showing that each of these terms is bounded (uniformly in l). Let $I^k_{i^+}$ (respectively $I^k_{i^-}$) be the interval of the partition $\phi^{(b_k)}$ which has $\overline{z}_i^+$ as left endpoint (respectively $\overline{z}_i^-$ as right endpoint). The sum over j is estimated exactly as before, decomposing the sum using the nested intervals $I^{k-1}_{i_0^\pm} \backslash I^k_{i_0^\pm}$. To estimate the maximum, we remark that if $x_i(j)$ (respectively $y_i(j)$) belongs to $I^{k-1}_{i^+} \backslash I^k_{i^+}$ (respectively $I^{k-1}_{i^-} \backslash I^k_{i^-}$), then $k_i^+(j) = k$ (or respectively $k_i^-(j) = k$) and $j \geq$ const $d^{k_l - k}$ (see (36)), so that

$$\frac{\Theta^{k_l}_{k_i^\pm(j)}}{j^{1-\gamma}} \leq \frac{\Theta^{k_l}_k}{d^{\overline{\gamma}(k_l-k)}} \leq \sum_{n=0}^{k_l-k} \frac{\|B_{k+n-1}\|}{d^{\overline{\gamma}(n+k_l-k)}}$$
$$= \sum_{m=0}^{k_l-k} \frac{\|B_{k_l-m-1}\|}{d^{\overline{\gamma}(2(k_l-k)-m)}} \leq \frac{\text{const}}{d^{2\overline{\gamma}(k_l-k)}} \sum_{m=0}^{k_l-k} \left(\frac{2d^2}{3}\right)^{\overline{\gamma}m},$$

where, reasoning as before, we changed the indexes by $m = k_l - k - n$ and used (35). Since the sum in the last expression is $O\left(d^{2\overline{\gamma}(k_l-k)}\,(2/3)^{\overline{\gamma}(k_l-k)}\right)$, we get a uniform bound for all $0 \leq k \leq k_l$, which concludes the proof. □

4.3. *Decomposition into Birkhoff sums along towers.* From the estimate of Birkhoff sums along a tower given by Proposition 4.1, let us derive an estimate for more general Birkhoff sums.

PROPOSITION 4.2. *For almost every T, there exist a constant M' and sequence of induction times $\{n_k\}_{k\in\mathbb{N}}$ such that, whenever $z_0 \in I^{(n_k)}_{j_0}$ for some*

k and $0 < r \leq h_{j_0}^{(n_k)}$, we have

$$|S_r(f_0')(z_0)| \leq M'r + \sum_{i=0}^{s_1-1} \frac{C_i^+}{x_i^{\min}} + \sum_{i=0}^{s_2-1} \frac{C_i^-}{y_i^{\min}}, \tag{38}$$

where $x_i^{\min}$ and $y_i^{\min}$ are the closest points to the singularities defined in (18) *and* (19).

Comparing Proposition 4.1 below with Proposition 4.2, the difference is that the time r considered here is any $0 \leq r \leq h_{j_0}^{(n_k)}$.

Proof. Let $\{c_l\}_{l\in\mathbb{N}}$ be the sequence associated to almost every T in Proposition 4.1 and let B be the induced cocycle defined at the end of Section 4.1.3. Let us denote by C the accelerated cocycle over the first return map of $\mathcal{Z}$ to E_C (defined at the beginning of Proposition 4.1) so that $C_l := B^{(c_l, c_{l+1})}$. Let $E_D \subset E_C$ be given by Lemma 2.2 for $\varepsilon = \ln(d/2)$. For almost every T we can assume that the chosen lift $\widehat{T}$ is recurrent to E_D along a subsequence $\{n_k := c_{l_k}\}_{k\in\mathbb{N}}$. Then, by Lemma 2.2, we have

$$\|C_{l_k - l}\| \leq \overline{C}_3 (d/2)^l, \quad \forall\, 0 \leq l \leq l_k. \tag{39}$$

Without loss of generality, we can assume that $h_{j_1}^{(c_{l_k-1})} \leq r \leq h_{j_0}^{(c_{l_k})}$ for some j_1 and $z_0 \in I_{j_1}^{(c_{l_k-1})}$. (Indeed, if not, since $I^{(c_{l_{k'}})}$ are nested, we can define $j_{k'}$ for $0 \leq k' < k$ such that $z_0 \in I_{j_{k'}}^{(c_{l_{k'}})}$. Since $h_{j_{k'}}^{(c_{l_{k'}})}$ are increasing in k', we can then substitute k with the unique k' for which $h_{j_{k'-1}}^{(c_{l_{k'}-1})} < r < h_{j_{k'}}^{(c_{l_{k'}})}$.)

Let us use the following notation: Let us denote the orbit segment

$$\{z_0, Tz_0, \ldots, T^{r-1}z_0\}$$

by $\mathcal{O}_r(z_0)$. On $\mathcal{O}_r(z_0)$ introduce an ordering $\prec$ and a distance $d_{\mathcal{O}}$ using the natural ordering given by T as follows. If $z_1, z_2 \in \mathcal{O}_r(z_0)$ and $z_1 = T^{i_1}z_0$, $z_2 = T^{i_2}z_0$ for $i_1, i_2 > 0$, let $z_1 \prec z_2$ if and only if $i_1 < i_2$ and let $d_{\mathcal{O}}(z_1, z_2) = k$ if and only if $|i_1 - i_2| = k$.

Let us decompose the orbit $\mathcal{O}_r(z_0)$ into Birkhoff sums along towers as follows. Consider first Birkhoff sums along the towers of $\phi^{(c_{l_k-1})}$. Let $z_j^{(l_k-1)}$, for $0 \leq j \leq a_{l_k-1}$, be the elements of $\mathcal{O}_r(z_0)$ which are contained in $I^{(c_{l_k-1})}$ in increasing order with respect to $\prec$. More precisely, define by induction $z_0^{(l_k-1)} = z_0$ and $z_{j+1}^{(l_k-1)} = T^{(c_{l_k-1})} z_j^{(l_k-1)}$, so that $z_{j+1}^{(l_k-1)}$ is the smallest $z \succ z_j^{(l_k-1)}$ such that $z \in \mathcal{O}_r(z_0) \cap I^{(c_{l_k-1})}$. The last step of the induction is a_{l_k-1} where $a_{l_k-1} = \max\{j \mid z_j^{(l_k-1)} \prec T^r z_0\}$. Let us also define $r_j^{(l_k-1)} = d_{\mathcal{O}}(z_0, z_j^{(l_k-1)})$. Since $r \geq h_{j_1}^{(c_{l_k-1})}$, $a_{l_k-1} \geq 1$. Moreover, since $d_{\mathcal{O}}(z_j^{(l_k-1)}, z_{j+1}^{(l_k-1)}) \geq \min_l h_l^{(c_{l_k-1})}$, using balance (12) and (11) and $r \leq h_{j_0}^{(c_{l_k})}$, we also have that $a_{l_k-1} \leq r/\min_l h_l^{(c_{l_k-1})} \leq \nu \|C_{l_k}\|$. So far we can write

$$S_r(f_0')(z_0) = \sum_{j=0}^{a_{l_k-1}-1} S_{r_{j+1}^{(l_k-1)}-r_j^{(l_k-1)}}(f_0')(z_j^{(l_k-1)}) + S_{r'}(f_0')(z_{a_{l_k-1}}^{(l_k)}), \qquad a_{l_k-1} \leq \nu \|C_{l_k}\|,$$

where $r' = r - r_{a_{l_k-1}}^{(l_k-1)}$ and each term in the sum is by construction a Birkhoff sum along a tower of $\phi^{(c_{l_k-1})}$ while the last term is a remainder that cannot be decomposed any more into Birkhoff sums along towers of the same order.

Let us continue by induction to decompose the remainder into Birkhoff sums along the towers of $\phi^{(c_l)}$ with $0 \leq l < l_k - 1$. To get from step $l+1$ to step l, let $z_0^{(l)} = z_{a_{l+1}}^{(l+1)}$ and $z_{j+1}^{(l)} = T^{(c_l)} z_j^{(l)} \in I^{(c_l)}$ for $j = 0, \ldots, a_l$ with $a_l = \max\{j \,|\, z_j^{(l)} \prec T^r z_0\}$. In this way again $z_1^{(l)} \prec \cdots \prec z_{a_l}^{(l)}$ are all elements $z \in \mathcal{O}_r(z_0)$ with $z \succ z_{a_{l+1}}^{(l+1)}$ for which $z \in I^{(c_l)}$. Letting $r_j^{(l)} = d_{\mathcal{O}}(z_0, z_j^{(l)})$, we have

$$S_r(f_0')(z_0) = \sum_{l=0}^{l_k-1} \sum_{j=0}^{a_l-1} S_{r_{j+1}^{(l)}-r_j^{(l)}}(f_0')(z_j^{(l)}) + S_{r'}(f_0')(z_{a_0}^{(0)}), \quad r' = r - r_{a_0}^{(0)} \leq \max_l h_l^{(c_0)}, \tag{40}$$

where, if $a_l = 0$ (it might happen for $l < l_k - 1$), the sum over j is taken by convention to be zero. Moreover, as before, by construction we have $a_l \leq \nu \|C_{l+1}\|$, $0 \leq l \leq l_k - 1$.

Let us apply Proposition 4.1 to each addend in the double sum in (40), denoting, in each Birkhoff sum along a tower, the points which are closest to right and left singularities (recalling that $(\,\cdot\,)^{\mathrm{pos}}$ denotes the positive part) by

$$(x_i^{\min})_j^{(l)} = \min_{0 \leq s < r_{j+1}^{(l)} - r_j^{(l)}} (T^s z_i^{(l)} - \overline{z}_i^+)^{\mathrm{pos}}, \quad i = 0, \ldots, s_1 - 1,$$

$$(y_i^{\min})_j^{(l)} = \min_{0 \leq s < r_{j+1}^{(l)} - r_j^{(l)}} (\overline{z}_i^- - T^s z_i^{(l)})^{\mathrm{pos}}, \quad i = 0, \ldots, s_2 - 1.$$

We get

$$\left|S_r(f_0')(z_0)\right| \leq \sum_{l=0}^{l_k-1} \sum_{j=0}^{a_l-1} \left(\sum_{i=0}^{s_1-1} \frac{C_i^+}{(x_i^{\min})_j^{(l)}} + \sum_{i=0}^{s_2-1} \frac{C_i^-}{(y_i^{\min})_j^{(l)}} + M(r_{j+1}^{(l)} - r_j^{(l)}) \right) + \left| S_{r'}(f_0')(z_{a_0}^{(0)}) \right|. \tag{41}$$

Since the sum $\sum_{l=0}^{l_k-1} \sum_{j=0}^{a_l-1} (r_{j+1}^{(l)} - r_j^{(l)})$ is telescopic, it reduces to $r_{a_0}^{(0)} \leq r$. Moreover, the last term in (41) can be estimated by $Mr' + \sum_{i=0}^{s_1-1} \frac{C_i^+}{x_i^{\min}} +$

$\sum_{i=0}^{s_2-1} \frac{C_i^-}{y_i^{\min}}$ with $r' \leq \max_j h_j^{(c_0)}$, which is a constant. Hence, to conclude the proof of (38), we are left to estimate the sums of contributions from closest points to the singularities in each cycle. Let us show that their contributions decrease exponentially in the order k of the towers, thanks to the choice (39) of the times $\{c_{l_k}\}_{k\in\mathbb{N}}$.

Given any $0 \leq i \leq s_1 - 1$, let us consider the contribution to (41) coming from the points

$$\left\{ (x_i^{\min})_j^{(l)}, \quad j = 0, \ldots, a_l - 1 \right\}. \tag{42}$$

Assume first that $0 \leq l < l_k - 1$. We remark that all these points belong by construction to a unique tower of order $l+1$, the tower $Z_{j(l+1)}^{(c_{l+1})}$ such that $z_0^{(l)} = z_{a_{l+1}+1}^{(l+1)} \in I_{j(l+1)}^{(c_{l+1})}$. Thus the minimum spacing between them is by balance (12) at least $\lambda^{(c_{l+1})}/d\nu$, and if we consider separately the minimum of (42), each of the other $a_l - 1$ points of (42) gives a contribution less than $C_i^+ d\nu/\lambda^{(c_{l+1})}$. Since the minimum of (42) is bigger than the minimum $(x_i^{\min})_{a_{l+1}+1}^{(l+1)}$ of the level $l+1$ orbit segment of length $h_{j(l+1)}^{(c_{l+1})}$, which contains all points in (42), it can be included in the analogous estimate corresponding to $l+1$, by considering a_{l+1} contributions equals to $C_i^+ d\nu/\lambda^{(c_{l+2})}$ rather than $a_{l+1}-1$. When $l = l_k - 1$, the minimum is simply given by $x_i^{\min}$ and the contributions of all the other points are again estimated by $C_i^+ d\nu/\lambda^{(c_{l_k})}$ since they are all contained in different floors of the tower $Z_{j_0}^{(c_{l_k})}$.

Hence, first recalling that $a_l \leq \nu\|C_{l+1}\|$, then using the fact that $\lambda^{(c_{l+1})} \geq d^{l_k-l-1}\lambda^{(c_{l_k})}$ and setting $l' := l_k - l - 1$, we get

$$\sum_{l=0}^{l_k-1}\sum_{j=0}^{a_l-1} \frac{C_i^+}{(x_i^{\min})_j^{(l)}} \leq \frac{C_i^+}{x_i^{\min}} + \sum_{l=0}^{l_k-1} \nu\|C_{l+1}\| \frac{C_i^+ d\nu}{\lambda^{(c_{l+1})}} \leq \frac{C_i^+}{x_i^{\min}} + \frac{C_i^+ d\nu^2}{\lambda^{(c_{l_k})}} \sum_{l'=0}^{l_k-1} \frac{\|C_{l_k-l'}\|}{d^{l'}}, \tag{43}$$

where the last series is uniformly bounded for all k by (39). Since by (12) and (13) we have $1/\lambda^{(c_{l_k})} \leq \nu^2 h_{j_0}^{(c_{l_k})}$ and $h_{j_0}^{(c_{l_k})} \leq \|C_{l_k}\|\nu^2 h_{j_1}^{(c_{l_k-1})} \leq \overline{C}_3\nu^2 r$ (where the last inequality uses again (39)), we get a bound of the desired form.

Since, for any $0 \leq i \leq s_2 - 1$, the contribution from $(y_i^{\min})_j^{(l)}$, $j = 0, \ldots, a_l - 1$ is estimated in the same way, this concludes the proof. $\square$

4.4. *Birkhoff sums variations and proof of absence of mixing.* In this section we complete the proof of Theorem 1.2.

Proof of Theorem 1.2. Let us verify the assumptions of the criterion for absence of mixing (Lemma 3.1). Given a typical T, consider the subsequence $\{n_k\}_{k\in\mathbb{N}}$ of balanced times for which Proposition 4.2 holds. In Section 3.2 we already defined, starting from $\{n_k\}_{k\in\mathbb{N}}$, a corresponding class of sets E_k and times r_k which verify conditions (i) and (ii) of Definition 3.1 of rigidity sets. Let us hence define the subintervals J_k introduced in Section 3.2 in such a way that condition (iii) is also satisfied. Referring to the notation in Section 3.2, if $(I^{(n_k)}_{j_0})_{l_0} = [a, b)$, and $\lambda = b - a$ is its length, define $J_k = [a + \lambda/4, b - \lambda/4)$.

Let us consider any two points $y_1, y_2 \in E_k$. By definition of E_k, we can write $y_1 = T^{k_1} z_1$, $y_2 = T^{k_2} z_2$ where $z_1, z_2 \in J_k$ are two points in the base and $0 \le k_1, k_2 < h^{(n_k)}_{j_0}$. In order to prove (iii), let us decompose the Birkhoff sums of the roof function f as

$$\begin{aligned}(44)\quad S_{r_k}(f)(y_1) - S_{r_k}(f)(y_2) &= \left(S_{r_k}(f)(T^{k_1} z_1) - S_{r_k}(f)(z_1)\right)\\ &+ \left(S_{r_k}(f)(z_1) - S_{r_k}(f)(z_2)\right) + \left(S_{r_k}(f)(z_2) - S_{r_k}(f)(T^{k_2} z_2)\right).\end{aligned}$$

Remark, moreover, that the first and the last term in (44) can be written, for $\nu = 1, 2$, as

$$\begin{aligned}(45)\quad S_{r_k}(f)(T^{k_\nu} z_\nu) - S_{r_k}(f)(z_\nu) &= (S_{r_k - k_\nu}(f)(T^{k_\nu} z_\nu) + S_{k_\nu}(f)(T^{r_k} z_\nu))\\ &- (S_{k_\nu}(f)(z_\nu) + S_{r_k-k_\nu}(f)(T^{k_\nu} z_\nu)) = S_{k_\nu}(f)(T^{r_k} z_\nu) - S_{k_\nu}(f)(z_\nu).\end{aligned}$$

Recall that $f = f_0 + w$, where f_0 is the pure logarithmic part and w has bounded variation (see Definition 1.1). Thus, we can write $S_{r_k}(f) = S_{r_k}(f_0) + S_{r_k}(w)$. We will first estimate the terms in (44) coming from w, then the ones from f_0. Consider first the central term in (44) coming from w. Since $z_1, z_2 \in J_k$, the iterates T^i for $0 \le i < r_k$ of the interval between z_1 and z_2 are disjoint since r_k is the first return time of J_k to $(I^{(n_k)}_{j_0})_{l_0}$. Thus,

$$|S_{r_k}(w)(z_1) - S_{r_k}(w)(z_2)| \le \mathrm{Var}(w).$$

Similarly, since $z_\nu \in J_k$ which by construction is contained in $(I^{(n_k)}_{j_0})_{l_0}$ and since $T^{r_k} z_\nu \in T^{r_k} J_k \subset T^{r_k}(I^{(n_k)}_{j_0})_{l_0} \subset I^{(n_k)}_{j_0}$, $0 \le k_\nu < h^{(n_k)}_{j_0}$ and the iterates $T^i I^{(n_k)}_{j_0}$ for $0 \le i < h^{(n_k)}_{j_0}$ are disjoint, we can estimate

$$|S_{k_\nu}(w)(T^{r_k} z_\nu) - S_{k_\nu}(w)(z_\nu)| \le \mathrm{Var}(w), \quad \nu = 1, 2.$$

We know have to estimate the terms in (44) coming from f_0. Let us again consider first the central term in (44). Since J_k is by construction contained in a continuity interval of T^i for $0 \le i < r_k$, the mean value theorem gives that there exists some z_0 in J_k such that

$$(46)\quad |S_{r_k}(f_0)(z_1) - S_{r_k}(f_0)(z_2)| \le |S_{r_k}(f_0')(z_0)||z_1 - z_2| \le |S_{r_k}(f_0')(z_0)|\lambda/2.$$

The return time r_k is in particular a return time to $I^{(n_k)}_{j_0}$ and hence to $I^{(n_k)}$; assume that it is the n^{th} return to $I^{(n_k)}$, where $n = \sum_{j=1}^d a^k_j$ and a^k_j is the number of returns to $I^{(n_k)}_j$ (so $a^k_{j_0} \ge 1$). Hence we can write $r_k = \sum_{j=1}^d a^k_j h^{(n_k)}_j$.

Let us decompose the Birkhoff sum into Birkhoff sums along the towers of $\phi^{(n_k)}$ as

$$S_{r_k}(f_0')(z_0) = \sum_{j=1}^{d}\sum_{l=1}^{a_j^k} S_{h_j^{(n_k)}}(f_0')(z_l^j), \tag{47}$$

where the intermediate points[18] are $z_1^{j_0} = z_0$ and $z_l^j \in I_j^{(n_k)}$ for $l = 1, \ldots, a_j^k$, and for $j \neq j_0$ one can have $a_j^k = 0$, in which case the corresponding sum is empty.

To each of the Birkhoff sums in (47) let us apply Proposition 4.1. Let us remark that $T^i(I_{j_0}^{(n_k)})_{l_0}$ for $i = 0, \ldots, r_k - 1$ are all disjoint and rigidly translated by T and that the singularities $\overline{z}_j^{\pm}$ are all contained in the boundary of the floors of the towers, so that, since z_0 belongs to a central subinterval $J_k \subset (I_{j_0}^{(n_k)})_{l_0}$, the distance of each point in each orbit segment $\{T^i z_l^j\}_{i=0,\ldots,h_j^{(n_k)}-1}$ from the singularities $\overline{z}_j^+$, $0 \leq j < s_1$ and $\overline{z}_j^-$, $0 \leq j < s_2$ is at least $\lambda/4$. Moreover, we have $\sum_j a_j^k \leq 2d(d+2)\nu$, as it follows by combining that, by disjointness of $T^i(I_{j_0}^{(n_k)})_{l_0}$, we have $\sum_j a_j^k h_j^{(n_k)} \lambda/2 \leq 1$ and that, by balance (12) and construction (15), (16) of J_k, we have $h_j^{(n_k)} \geq 1/(\lambda d(d+2)\nu)$. Hence, setting $\overline{C} = \sum_{j=0}^{s_1-1} C_j^+ + \sum_{j=0}^{s_2-1} C_j^-$, we have

$$|S_{r_k}(f_0')(z_0)| \leq 2d(d+2)\nu\left(M \max_j h_j^{(n_k)} + \frac{4\overline{C}}{\lambda}\right). \tag{48}$$

The combination of equations (46) and (48), using again balance (13), gives the bound on the central term of (44) coming from f_0 by a constant. Each of the other two terms of (44) coming from f_0, as in (45), can be written, for $\nu = 1, 2$, as

$$\begin{aligned} S_{r_k}(f_0)(T^{k_\nu} z_\nu) - S_{r_k}(f_0)(z_\nu) &= S_{k_\nu}(f_0)(T^{r_k} z_\nu) - S_{k_\nu}(f_0)(z_\nu) \\ &= S_{k_\nu}(f_0')(u_\nu)(T^{r_k} z_\nu - z_\nu), \end{aligned} \tag{49}$$

where u_ν is a point in $I_{j_0}^{(n_k)}$ between z_ν and $T^{r_k} z_\nu$. Since $z_\nu \in J_k$ and, by construction of J_k, $J_k \subset (I_{j_0}^{(n_k)})_{l_0}$ and $T^{r_k} z_\nu \in T^{r_k} J_k \subset T^{r_k}(I_{j_0}^{(n_k)})_{l_0} \subset I_{j_0}^{(n_k)}$, u_ν has distance at least $\lambda/4$ from both endpoints of $I_{j_0}^{(n_k)}$. Moreover, $T^i I_{j_0}^{(n_k)}$

[18] To construct the intermediate points and show (47) one can use induction on n. For brevity, let $h := h_{j_0}^{(n_k)}$. If $n = 1$, then $T^h(z_0) \in J_k$ and there is nothing to prove, since $r_k = h$ and setting $a_{j_0}^k = 1$ and $a_j^k = 0$ for $j \neq j_0$, (47) becomes an identity. Otherwise, if $n > 1$, let j be such that $T^h z_0 \in I_j^{(n_k)}$ and define $z_j^1 = T^h(z_0)$ so that $S_{r_k}(f_0')(z_0) = S_h(f_0')(z_0) + S_{r'}(f_0')(z_j^1)$, with $r' = r_k - h$. Since for r' the number of return times is by construction $n-1$, the definition of the remaining intermediate points z_l^j follows by induction.

for $0 \le i < h_{j_0}^{(n_k)}$ are disjoint and are rigid translates of $I_{j_0}^{(n_k)}$, so that all the points $T^i u_\nu$ for $0 \le i \le k_\nu$ have distance at least $\lambda/4$ from both endpoints of $T^i I_{j_0}^{(n_k)}$ and hence, in particular, from the singularities $\overline{z}_j^+$, $j = 0, \dots, s_1 - 1$, $\overline{z}_j^-$, $j = 0, \dots, s_2 - 1$.

Hence, we can apply Proposition 4.2 with $x_i^{\min}, y_i^{\min} \ge \lambda/4$ to get

$$(50) \quad \left|S_{k_\nu}(f_0')(u_\nu)(T^{r_k} z_\nu - z_\nu)\right| \le \left(M' k_\nu + 4\overline{C}/\lambda\right) \lambda_{j_0}^{(n_k)} \le M' + 8\overline{C}(d+2),$$

where in the last inequality we used that $\lambda \ge \lambda_{j_0}^{(n_k)}/2(d+2)$, by (16) and definition of J_k and that $k_\nu \lambda_{j_0}^{(n_k)} \le h_{j_0}^{(n_k)} \lambda_{j_0}^{(n_k)} \le 1$. Thus, combining (49) and (50) we get the upper bound of the other two terms in (44) coming from f_0 by a constant. This concludes the proof that E_k and r_k satisfy also Property (iii) of Lemma 3.1 and hence, using Lemma 3.1, the proof of Theorem 1.2. $\square$

5. **Reduction to special flows.**

In this section we derive Theorem 1.1 from Theorem 1.2.

Proof of Theorem 1.1. Let us assume that the multi-valued Hamiltonian flow associated to η has only simple saddles and no saddle loops homologous to zero. In the set of multi-valued Hamiltonian flows without saddle loops homologous to zero, the flow associated to a typical η in the sense defined before Theorem 1.1 does not have saddle connections. Thus, from a result by Calabi [Cal69] or by Katok [Kat73], there exists a holomorphic 1-form (or Abelian differential) α whose associated vertical flow determines the same measured foliation.

For any γ cross-section transversal to the flow, the Poincaré first return map T on γ preserves the measure induced by the area form on the transversal. Up to reparametrization (using the smooth conjugacy that sends the induced invariant measure to the Lebesgue measure on an interval of unit length parametrizing γ) we can assume that T is an IET $(\underline{\lambda}, \pi)$ on $I^{(0)} = [0, 1)$. Since we assume that the flow is minimal, π is irreducible. The flow $\{\varphi_t\}_{t\in\mathbb{R}}$ is isomorphic (up to the smooth conjugacy above) to a special flow over T under a roof function f which is given by the first return time to the transversal. Using the representation of typical Abelian differentials as zippered rectangles (for which we refer for example to [Yoc06] or [Via]), one can see that a full measure set of IETs gives a set of full measure of Abelian differentials. Moreover, since the transverse measure of the multi-valued Hamiltonian flow $\{\varphi_t\}_{t\in\mathbb{R}}$ and of the vertical flow of the Abelian differential α are the same (in other words the horizontal components of the periods $\int_{\gamma_i} \alpha$ are the same then $\int_{\gamma_i} \eta$), a full measure set of IETs also gives a full measure set of multi-valued Hamiltonian flows. Thus, in order to deduce Theorem 1.1 from Theorem 1.2, it is enough to check that f satisfies the assumptions of Theorem 1.2.

Let us choose the transversal γ so that the backward flow orbits of the transversal endpoints both contain a saddle, but the endpoints are not at saddles, for example, by considering a standard cross-section chosen for the zippered rectangle representation of the corresponding Abelian differential and shifting it by a sufficiently small $t_0 < 0$ along the flow direction. Both discontinuities of T and singularities of f occur at points $\overline{z}_i$ which lie on a separatrix, hence whose forward orbit under φ_t limit toward a saddle. By our choice of the transversal endpoints, the IET exchanges $d = 2g + s - 1$ intervals, where g is the genus and s the number of saddles, and since the saddles are simple, by Gauss-Bonnet formula[19] $d = 4g - 3 = 2s + 1$.

Since the parametrization is locally Hamiltonian, trajectories are slowed down more and more the closer they come to a saddle. The one-sided limit $\lim_{x\to\overline{z}_i^+} f(x)$ (or $\lim_{x\to\overline{z}_i^-} f(x)$) of the return time $f(x)$ blows up near $\overline{z}_i$ if the forward trajectories of the nearby points $x > \overline{z}_i$ (or $x < \overline{z}_i$) under the vertical flow of the Abelian differential, considered up to their return time, come arbitrarily close to a saddle. From the canonical form of a simple saddle, one can show (see [Koč76]) that the singularities are in this case logarithmic, i.e., of the form $C_i|\log(x - \overline{z}_i)|$ for $x \geq \overline{z}_i$ (or $x \leq \overline{z}_i$) up to a function w of bounded variation, where the constant C_i depends on the saddle.[20]

One can see, for example, by using the zippered rectangle representation, that out of the $2d = 4s + 2$ (two for each interval) one-sided limits of the form $\lim_{x\to\overline{z}_i^\pm} f(x)$, where $\overline{z}_i$ is either a discontinuity of T or an endpoint of $I^{(0)}$, exactly $4s$ are infinite and give discontinuities of f, since the corresponding zippered rectangle boundary contains a saddle. The remaining two limits are finite and the corresponding return times for nearby $x > \overline{z}_i$ or $x < \overline{z}_i$ are bounded. Each of the s saddles has two incoming separatrices, each of which generates a left and a right logarithmic singularity of f with the same constant C_i depending on the saddle. Thus, the number of right and left singularities is $s_1 = s_2 = 2s$ and each constant C_i appears four times, twice in a right-side singularity and twice in a left-side one. In particular, the logarithmic singularities are symmetric. □

Remark 5.1. From the proof of Theorem 1.1 one can see that the class of special flows which are used to represent multi-valued Hamiltonian flows is less general than the class considered in Theorem 1.2. The permutations π that

[19] If k_i, for $i = i, \dots, s$, denote the orders of the zeros, the Gauss-Bonnet formula gives $\sum_{i=1}^{s} k_i = 2g - 2$ (we refer for example to [Yoc06] or [Via]) and since we are assuming that $k_i = 1$ for all $k = 1, \dots, s$, we have $s = 2g - 2$.

[20] The logarithmic nature of the singularities was first remarked by Arnold in [Arn91]. A detailed calculation which gives more information on the function w can be found in [FU].

arise are not all irreducible ones, but only the ones which correspond to Abelian differentials in the principal stratum $\mathscr{H}(1,\dots,1)$ of Abelian differentials with simple zeros,[21] and the roof functions have symmetric logarithmic singularities in which $s_1 = s_2$ and all constants appear in quadruples $C^+_{i_1}, C^+_{i_2}, C^-_{j_1}, C^-_{j_2}$ (for some $0 \le i_1 \ne i_2 < s_1 = s$, $0 \le j_1 \ne j_2 < s_2 = s$) such that $C^+_{i_1} = C^+_{i_2} = C^-_{j_1} = C^-_{j_2}$.

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[21] For example the permutations of the form $(n\, n-1 \cdots 21)$ correspond to a principal stratum only if $g = 2$ and $n = 5$.

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