

# Twisted Alexander polynomials detect fibered 3-manifolds

By STEFAN FRIEDL and STEFANO VIDUSSI

## Abstract

A classical result in knot theory says that for a fibered knot the Alexander polynomial is monic and that the degree equals twice the genus of the knot. This result has been generalized by various authors to twisted Alexander polynomials and fibered 3-manifolds. In this paper we show that the conditions on twisted Alexander polynomials are not only necessary but also sufficient for a 3-manifold to be fibered. By previous work of the authors this result implies that if a manifold of the form  $S^1 \times N^3$  admits a symplectic structure, then  $N$  fibers over  $S^1$ . In fact we will completely determine the symplectic cone of  $S^1 \times N$  in terms of the fibered faces of the Thurston norm ball of  $N$ .

## 1. Introduction

1.1. *Twisted Alexander polynomials and fibered 3-manifolds.* Let  $N$  be a compact, connected, oriented 3-manifold with empty or toroidal boundary. Given a nontrivial class  $\phi \in H^1(N; \mathbb{Z}) = \text{Hom}(\pi_1(N), \mathbb{Z})$  we say that  $(N, \phi)$  *fibers over*  $S^1$  if there exists a fibration  $f : N \rightarrow S^1$  such that the induced map  $f_* : \pi_1(N) \rightarrow \pi_1(S^1) = \mathbb{Z}$  agrees with  $\phi$ . Stated otherwise, the homotopy class in  $[N, S^1] = H^1(N; \mathbb{Z})$  identified by  $\phi$  can be represented by a fibration.

It is a classical result in knot theory that if a knot  $K \subset S^3$  is fibered, then the Alexander polynomial is monic (i.e. the top coefficient equals  $\pm 1$ ), and the degree of the Alexander polynomial equals twice the genus of the knot. This result has been generalized in various directions by several authors (e.g. [McM02], [Cha03], [GKM05], [FK06], [Kit07]) to show that twisted Alexander polynomials give necessary conditions for  $(N, \phi)$  to fiber.

To formulate this kind of result more precisely we have to introduce some definitions. Let  $N$  be a 3-manifold with empty or toroidal boundary and let

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$\phi \in H^1(N; \mathbb{Z})$ . Given  $(N, \phi)$  the *Thurston norm* of  $\phi$  (cf. [Thu86]) is defined as

$$\|\phi\|_T = \min\{\chi_-(S) \mid S \subset N \text{ properly embedded surface dual to } \phi\}.$$

Here, given a surface  $S$  with connected components  $S_1 \cup \dots \cup S_k$ , we define  $\chi_-(S) = \sum_{i=1}^k \max\{-\chi(S_i), 0\}$ .

In the following we assume that  $\phi \in H^1(N; \mathbb{Z})$  is nontrivial. Let  $\alpha : \pi_1(N) \rightarrow G$  be a homomorphism to a finite group. We have the permutation representation  $\pi_1(N) \rightarrow \text{Aut}(\mathbb{Z}[G])$  given by left multiplication, which we also denote by  $\alpha$ . We can therefore consider the twisted Alexander polynomial  $\Delta_{N,\phi}^\alpha \in \mathbb{Z}[t^{\pm 1}]$ , whose definition is detailed in Section 2.3. We denote by  $\phi_\alpha$  the restriction of  $\phi \in H^1(N; \mathbb{Z}) = \text{Hom}(\pi_1(N), \mathbb{Z})$  to  $\text{Ker}(\alpha)$ . Note that  $\phi_\alpha$  is necessarily nontrivial. We denote by  $\text{div}\phi_\alpha \in \mathbb{N}$  the divisibility of  $\phi_\alpha$ , i.e.

$$\text{div}\phi_\alpha = \max\{n \in \mathbb{N} \mid \phi_\alpha = n\psi \text{ for some } \psi : \text{Ker}(\alpha) \rightarrow \mathbb{Z}\}.$$

We can now formulate the following theorem which appears as [FK06, Th. 1.3 and Remark, p. 938].

**THEOREM 1.1.** *Let  $N \neq S^1 \times S^2, S^1 \times D^2$  be a 3-manifold with empty or toroidal boundary. Let  $\phi \in H^1(N; \mathbb{Z})$  be a nontrivial class. If  $(N, \phi)$  fibers over  $S^1$ , then for any homomorphism  $\alpha : \pi_1(N) \rightarrow G$  to a finite group the twisted Alexander polynomial  $\Delta_{N,\phi}^\alpha \in \mathbb{Z}[t^{\pm 1}]$  is monic and*

$$\text{deg}(\Delta_{N,\phi}^\alpha) = |G| \|\phi\|_T + (1 + b_3(N))\text{div}\phi_\alpha.$$

It is well-known that in general the constraint of monicness and degree for the ordinary Alexander polynomial falls short from characterizing fibered 3-manifolds. The main result of this paper is to show that on the other hand the collection of all twisted Alexander polynomials does detect fiberedness, i.e. the converse of Theorem 1.1 holds true:

**THEOREM 1.2.** *Let  $N$  be a 3-manifold with empty or toroidal boundary. Let  $\phi \in H^1(N; \mathbb{Z})$  be a nontrivial class. If, for any homomorphism  $\alpha : \pi_1(N) \rightarrow G$  to a finite group, the twisted Alexander polynomial  $\Delta_{N,\phi}^\alpha \in \mathbb{Z}[t^{\pm 1}]$  is monic and*

$$\text{deg}(\Delta_{N,\phi}^\alpha) = |G| \|\phi\|_T + (1 + b_3(N))\text{div}\phi_\alpha$$

*holds, then  $(N, \phi)$  fibers over  $S^1$ .*

Note that alternatively it is possible to rephrase this statement in terms of Alexander polynomials of the finite regular covers of  $N$ , by use of the fact that  $\Delta_{N,\phi}^\alpha = \Delta_{\tilde{N}, p^*(\phi)}$  (cf. [FV08b]), where  $p : \tilde{N} \rightarrow N$  is the cover of  $N$  determined by  $\text{Ker}(\alpha)$ . Also, note that this theorem asserts that twisted Alexander polynomials detect whether  $(N, \phi)$  fibers under the assumption that  $\|\phi\|_T$  is known; while it is known that twisted Alexander polynomials give lower bounds

(cf. [FK06, Th. 1.1]), it is still an open question whether twisted Alexander polynomials determine the Thurston norm.

In the case where  $\phi$  has trivial Thurston norm, this result is proven in [FV08a], by use of subgroup separability. Here, following a different route (see §1.3 for a summary of the proof), we prove the general case.

1.2. *Symplectic 4-manifolds and twisted Alexander polynomials.* In 1976 Thurston [Thu76] showed that if a closed 3-manifold  $N$  admits a fibration over  $S^1$ , then  $S^1 \times N$  admits a symplectic structure, i.e. a closed, nondegenerate 2-form  $\omega$ . It is natural to ask whether the converse to this statement holds true. In its simplest form, we can state this problem in the following way:

CONJECTURE 1.3. *Let  $N$  be a closed 3-manifold. If  $S^1 \times N$  is symplectic, then there exists a  $\phi \in H^1(N; \mathbb{Z})$  such that  $(N, \phi)$  fibers over  $S^1$ .*

Interest in this question was motivated by Taubes’ results in the study of Seiberg-Witten invariants of symplectic 4-manifolds (see [Tau94], [Tau95]) that gave initial evidence to an affirmative solution of this conjecture. In the special case where  $N$  is obtained via 0-surgery along a knot in  $S^3$ , this question appears also in [Kro98, Question 7.11]. Over the last ten years evidence for this conjecture was given by various authors [Kro98], [CM00], [Etg01], [McC01], [Vid03].

In [FV08b] the authors initiated a project relating Conjecture 1.3 to the study of twisted Alexander polynomials. The outcome of that investigation is that if  $S^1 \times N$  is symplectic, then the twisted Alexander polynomials of  $N$  behave like twisted Alexander polynomials of a fibered 3-manifold. More precisely, the following holds (cf. [FV08b, Th. 4.4]):

THEOREM 1.4. *Let  $N$  be an irreducible closed 3-manifold and  $\omega$  a symplectic structure on  $S^1 \times N$  such that  $\omega$  represents an integral cohomology class. Let  $\phi \in H^1(N; \mathbb{Z})$  be the Künneth component of  $[\omega] \in H^2(S^1 \times N; \mathbb{Z})$ . Then for any homomorphism  $\alpha : \pi_1(N) \rightarrow G$  to a finite group, the twisted Alexander polynomial  $\Delta_{N,\phi}^\alpha \in \mathbb{Z}[t^{\pm 1}]$  is monic and*

$$\deg(\Delta_{N,\phi}^\alpha) = |G| \|\phi\|_T + 2\text{div}\phi_\alpha.$$

Note that it follows from McCarthy’s work [McC01] (see also Lemma 7.1) and Perelman’s proof of the geometrization conjecture (cf. e.g. [MT07]) that if  $S^1 \times N$  is symplectic, then  $N$  is prime, i.e. either irreducible or  $S^1 \times S^2$ . The proof of Theorem 1.4 relies heavily on the results of [Kro98] and [Vid03], which in turn build on results of Taubes [Tau94], [Tau95] and Donaldson [Don96].

As the symplectic condition is open, the assumption that a symplectic manifold admits an integral symplectic form is not restrictive. Therefore, combining Theorem 1.2 with Theorem 1.4, we deduce that Conjecture 1.3 holds

true. In fact, in light of [FV11, Th. 1.2], we have the following more refined statement:

**THEOREM 1.5.** *Let  $N$  be a closed oriented 3-manifold. Then given  $\Omega \in H^2(S^1 \times N; \mathbb{R})$  the following are equivalent:*

- (1)  $\Omega$  can be represented by a symplectic structure;
- (2)  $\Omega$  can be represented by a symplectic structure which is  $S^1$ -invariant;
- (3)  $\Omega^2 > 0$  and the Künneth component  $\phi \in H^1(N; \mathbb{R})$  of  $\Omega$  lies in the open cone on a fibered face of the Thurston norm ball of  $N$ .

Note that the theorem allows us in particular to completely determine the symplectic cone of a manifold of the form  $S^1 \times N$  in terms of the fibered cones of  $N$ .

Combined with the results of [FV08b], [Theorem 1.2](#) shows in particular that the collection of the Seiberg-Witten invariants of all finite covers of  $S^1 \times N$  determines whether  $S^1 \times N$  is symplectic or not. In particular, we have the following corollary (we refer to [Vid99], [Vid03] for the notation and the formulation in the case that  $b_1(N) = 1$ ).

**COROLLARY 1.6.** *Let  $N$  be a closed 3-manifold with  $b_1(N) > 1$ . Then given a spin<sup>c</sup> structure  $K \in H^2(S^1 \times N; \mathbb{Z})$  there exists a symplectic structure representing a cohomology class  $\Omega \in H^2(S^1 \times N; \mathbb{R})$  with canonical class  $K$  if and only if the following conditions hold:*

- (1)  $K \cdot \phi = \|\phi\|_T$ , where  $\phi \in H^1(N; \mathbb{R})$  is the Künneth component of  $\Omega$ , and for any regular finite cover  $p : \tilde{N} \rightarrow N$ ;
- (2)  $SW_{S^1 \times \tilde{N}}(p^*(K)) = 1$ ;
- (3) for any Seiberg-Witten basic class  $\kappa \in H^2(S^1 \times \tilde{N}; \mathbb{Z})$  we have

$$|p_*(\kappa) \cdot \phi| \leq \deg(p) K \cdot \phi$$

(where  $p_*$  is the transfer map), and the latter equality holds if and only if  $\kappa = \pm p^*K$ .

(Note that, under the hypotheses of the corollary, all basic classes of  $S^1 \times \tilde{N}$  are the pull-back of elements of  $H^2(\tilde{N}; \mathbb{Z})$ ).

*Remark.* A different approach to [Conjecture 1.3](#) involves a deeper investigation of the consequence of the symplectic condition on  $S^1 \times N$ , that goes beyond the information encoded in [Theorem 1.4](#). A major breakthrough in this direction has recently been obtained by Kutluhan and Taubes ([KT09]). They show that if  $N$  is a 3-manifold such that  $S^1 \times N$  is symplectic, under some cohomological assumption on the symplectic form, then the Monopole Floer homology of  $N$  behaves like the Monopole Floer homology of a fibered

3-manifold. On the other hand it is known, due to the work of Ghiggini, Kronheimer and Mrowka, and Ni that Monopole Floer homology detects fibered 3-manifolds ([Ghi08], [Ni09], [KM10], [Ni08].) The combination of the above results proves in particular [Conjecture 1.3](#) in the case that  $b_1(N) = 1$ .

1.3. *Fibered 3-manifolds and finite solvable groups: outline of the proof.* In this subsection we will outline the strategy of the proof of [Theorem 1.2](#). It is useful to introduce the following definition.

*Definition.* Let  $N$  be a 3-manifold with empty or toroidal boundary, and let  $\phi \in H^1(N; \mathbb{Z})$  be a nontrivial class. We say that  $(N, \phi)$  satisfies Condition (\*) if for any homomorphism  $\alpha : \pi_1(N) \rightarrow G$  to a finite group the twisted Alexander polynomial  $\Delta_{N, \phi}^\alpha \in \mathbb{Z}[t^{\pm 1}]$  is monic and

$$\deg(\Delta_{N, \phi}^\alpha) = |G| \|\phi\|_T + (1 + b_3(N)) \operatorname{div} \phi_\alpha.$$

It is well-known (see [[McC01](#)] for the closed case, and [7.1](#) for the general case) that Condition (\*) implies, by geometrization, that  $N$  is prime; so we can restrict ourself to the case where  $N$  is irreducible.

Note that McMullen [[McM02](#)] showed that when the class  $\phi$  is primitive, the condition  $\Delta_{N, \phi} \neq 0$  implies that there exists a connected Thurston norm-minimizing surface  $\Sigma$  dual to  $\phi$ . It is well-known that to prove [Theorem 1.2](#) it is sufficient to consider a primitive  $\phi$ , and we will assume that in the following. Denote  $M = N \setminus \nu\Sigma$ ; the boundary of  $M$  contains two copies  $\Sigma^\pm$  of  $\Sigma$  and throughout the paper we denote the inclusion induced maps  $\Sigma \rightarrow \Sigma^\pm \rightarrow M$  by  $\iota_\pm$ .

By Stallings' theorem [[Sta62](#)] the surface  $\Sigma$  is a fiber of a fibration  $N \rightarrow S^1$  if and only if  $\iota_\pm : \pi_1(\Sigma) \rightarrow \pi_1(M)$  are isomorphisms. Hence to prove [Theorem 1.2](#) we need to show that if  $(N, \phi)$  satisfies Condition (\*), then the monomorphisms  $\iota_\pm : \pi_1(\Sigma) \rightarrow \pi_1(M)$  are in fact isomorphisms. Using purely group theoretic arguments we are not able to show directly that Condition (\*) implies the desired isomorphism; however, we have the following result:

**PROPOSITION 1.7.** *Assume that  $(N, \phi)$  satisfies Condition (\*) and that  $\phi$  is primitive. Let  $\Sigma \subset N$  be a connected Thurston norm-minimizing surface dual to  $\phi$  and let  $\iota$  be either of the two inclusion maps of  $\Sigma$  into  $M = N \setminus \nu\Sigma$ . Then  $\iota : \pi_1(\Sigma) \rightarrow \pi_1(M)$  induces an isomorphism of the prosolvable completions.*

We refer to [Section 2.4](#) for information regarding group completions. [Proposition 1.7](#) translates the information from Condition (\*) into information regarding the maps  $\iota_\pm : \pi_1(\Sigma) \rightarrow \pi_1(M)$ . From a purely group theoretic point of view it is a difficult problem to decide whether a homomorphism which gives rise to an isomorphism of prosolvable completions has to be an isomorphism itself (cf. [[Gro70](#)], [[BG04](#)], [[AdlHKŠ07](#)] and also [Lemma 4.7](#)). However in our

3-dimensional setting we can use a recent result of Agol [Ago08] to prove the following theorem.

**THEOREM 1.8.** *Let  $N$  be an irreducible 3-manifold with empty or toroidal boundary. Let  $\Sigma \subset N$  be a connected Thurston norm-minimizing surface. We write  $M = N \setminus \nu\Sigma$ . Assume the following hold:*

- (1) *The inclusion induced maps  $\iota_{\pm} : \pi_1(\Sigma) \rightarrow \pi_1(M)$  give rise to isomorphisms of the respective prosolvable completions, and*
- (2)  *$\pi_1(M)$  is residually finite solvable,*

*then  $\iota_{\pm} : \pi_1(\Sigma) \rightarrow \pi_1(M)$  are isomorphisms; hence  $M = \Sigma \times I$ .*

In light of Proposition 1.7, the remaining obstacle for the proof of Theorem 1.2 is the condition in Theorem 1.8 that  $\pi_1(M)$  has to be residually finite solvable. It is well-known that linear groups (and hence in particular hyperbolic 3-manifolds groups) are virtually residually  $p$  for all but finitely many primes  $p$  (cf. e.g. [Weh73, Th. 4.7] or [LS03, Window 7, Prop. 9]), in particular they are residually finite solvable. Thurston conjectured that 3-manifold groups in general are linear (cf. [Kir97, Problem 3.33]), but this is still an open problem. Using the recent proof of the geometrization conjecture (cf. e.g. [MT07]) we will prove the following result, which will be enough for our purposes.

**THEOREM 1.9.** *Let  $N$  be a closed prime 3-manifold. Then for all but finitely many primes  $p$  there exists a finite cover  $N'$  of  $N$  such that the fundamental group of any component of the JSJ decomposition of  $N'$  is residually a  $p$ -group.*

We can now deduce Theorem 1.2 as follows: We first show in Lemmas 7.1 and 7.2 that it suffices to show Theorem 1.2 for closed prime 3-manifolds. Theorem 1.2 in that situation now follows from combination of Theorems 1.7, 1.8 and 1.9 with a more technical theorem which allows us to treat the various JSJ pieces separately (cf. Theorem 6.4).

*Added in proof.* In a very recent paper ([AF10]) Matthias Aschenbrenner and the first author showed that any 3-manifold group is virtually residually  $p$ . This simplifies the proof of Theorem 1.2 as outlined in [FV09].

This paper is structured as follows. In Section 2 we recall the definition of twisted Alexander polynomials and some basics regarding completions of groups. In Section 3 we will prove Proposition 1.7 and in Section 4 we give the proof of Theorem 1.8. In Section 5 we prove Theorem 1.9 and in Section 6 we provide the proof for Theorem 6.4. Finally in Section 7 we complete the proof of Theorem 1.2.

*Conventions and notation.* Throughout the paper, unless otherwise stated, we will assume that all manifolds are oriented and connected, and all homology

and cohomology groups have integer coefficients. Furthermore all surfaces are assumed to be properly embedded and all spaces are compact and connected, unless stated explicitly otherwise. The derived series of a group  $G$  is defined inductively by  $G^{(0)} = G$  and  $G^{(n+1)} = [G^{(n)}, G^{(n)}]$ .

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**2. Preliminaries: Twisted invariants and completions of groups**

2.1. *Twisted homology.* Let  $X$  be a CW-complex with base point  $x_0$ . Let  $R$  be a commutative ring,  $V$  a module over  $R$  and  $\alpha : \pi_1(X, x_0) \rightarrow \text{Aut}_R(V)$  a representation. Let  $\tilde{X}$  be the universal cover of  $X$ . Note that  $\pi_1(X, x_0)$  acts on the left on  $\tilde{X}$  as a group of deck transformations. The cellular chain groups  $C_*(\tilde{X})$  are in a natural way right  $\pi_1(X)$ -modules, with the right action on  $C_*(\tilde{X})$  defined via  $\sigma \cdot g := g^{-1}\sigma$ , for  $\sigma \in C_*(\tilde{X})$ . By tensoring, we can form the chain complex  $C_*(\tilde{X}) \otimes_{\mathbb{Z}[\pi_1(X, x_0)]} V$ , which is a complex of  $R$ -modules. Now define  $H_i(X; V) := H_i(C_*(\tilde{X}) \otimes_{\mathbb{Z}[\pi_1(X, x_0)]} V)$ . The isomorphism type of the  $R$ -module  $H_i(X; V)$  does not depend on the choice of the base point; in fact it only depends on the homotopy type of  $X$  and the isomorphism type of the representation.

In this paper we will also frequently consider twisted homology for a finitely generated group  $\Gamma$ ; its definition can be reduced to the one above by looking at the twisted homology of the Eilenberg-MacLane space  $K(\Gamma, 1)$ .

The most common type of presentation we consider in this paper is as follows: Let  $X$  be a topological space,  $\alpha : \pi_1(X) \rightarrow G$  a homomorphism to a group  $G$  and  $H \subset G$  a subgroup of finite index. Then we get a natural action of  $\pi_1(X)$  on  $\text{Aut}_{\mathbb{Z}}(\mathbb{Z}[G/H])$  by left-multiplication, which gives rise to the homology groups  $H_i(X; \mathbb{Z}[G/H])$ .

We will now study the  $\mathbb{Z}[\pi_1(X)]$ -module  $\mathbb{Z}[G/H]$  in more detail. We write  $C := \alpha(\pi_1(X))$ . Consider the set of double cosets  $C \backslash G/H$ . By definition  $g, g' \in G$  represent the same equivalence class if and only if there exist  $c, c' \in C$  and  $h, h' \in H$  such that  $cgh = c'g'h'$ . Note that  $g_1, \dots, g_k \in G$  are a complete set of representatives of  $C \backslash G/H$  if and only if  $G$  is the disjoint union of  $Cg_1H, \dots, Cg_kH$ . The first part of the following lemma is an immediate consequence of [Bro94, II.5.2.], the second part follows either from Shapiro’s lemma or a straightforward calculation.

LEMMA 2.1. *Let  $g_1, \dots, g_k \in G$  be a set of representatives for the equivalence classes  $C \backslash G/H$ . For  $i = 1, \dots, k$  write  $\tilde{C}_i = C \cap g_i H g_i^{-1}$ . We then have*

the following isomorphisms of left  $\mathbb{Z}[C]$ -modules:

$$\mathbb{Z}[G/H] \cong \bigoplus_{i=1}^k \mathbb{Z}[C/\tilde{C}_i].$$

In particular  $H_0(X; \mathbb{Z}[G/H])$  is a free abelian group of rank  $k = |C \backslash G/H|$ .

**2.2. Induced maps on low dimensional homology groups.** In this section we will give criteria when maps between groups give rise to isomorphisms between low dimensional twisted homology groups. We start out with a study of the induced maps on 0-th twisted homology groups.

**LEMMA 2.2.** *Let  $\varphi : A \rightarrow B$  be a monomorphism of finitely generated groups. Suppose that  $B$  is a subgroup of a group  $\pi$  and let  $\tilde{\pi} \subset \pi$  be a subgroup of finite index. Let  $g_1, \dots, g_k \in \pi$  be a set of representatives for the equivalence classes  $B \backslash \pi / \tilde{\pi}$ . For  $i = 1, \dots, k$  we write  $\tilde{B}_i = B \cap g_i \tilde{\pi} g_i^{-1}$  and  $\tilde{A}_i = \varphi^{-1}(\tilde{B}_i)$ . Then*

$$\varphi_* : H_0(A; \mathbb{Z}[\pi/\tilde{\pi}]) \rightarrow H_0(B; \mathbb{Z}[\pi/\tilde{\pi}])$$

is an epimorphism of free abelian groups and it is an isomorphism if and only if  $\varphi : A/\tilde{A}_i \rightarrow B/\tilde{B}_i$  is a bijection for any  $i$ .

*Proof.* It is well-known that the induced map on 0-th twisted homology groups is always surjective (cf. e.g. [HS97, §6]) and by Lemma 2.1 both groups are free abelian groups. Now note that without loss of generality we can assume that  $A \subset B$  and that  $\varphi$  is the inclusion map. It follows from Lemma 2.1 that  $H_0(B; \mathbb{Z}[\pi/\tilde{\pi}])$  is a free abelian group of rank  $k = |B \backslash \pi / \tilde{\pi}|$ . By the same lemma we also have

$$\mathbb{Z}[\pi/\tilde{\pi}] \cong \bigoplus_{i=1}^k \mathbb{Z}[B/\tilde{B}_i]$$

as left  $\mathbb{Z}[B]$ -modules and hence also as left  $\mathbb{Z}[A]$ -modules. Applying Lemma 2.1 to the  $\mathbb{Z}[A]$ -modules  $\mathbb{Z}[B/\tilde{B}_i]$ , we see that  $H_0(A; \mathbb{Z}[\pi/\tilde{\pi}])$  is a free abelian group of rank  $k$  if and only if  $|A \backslash B/\tilde{B}_i| = 1$  for any  $i$ . It is straightforward to see that this is equivalent to  $A/\tilde{A}_i \rightarrow B/\tilde{B}_i$  being a bijection for any  $i$ . □

We will several times make use of the following corollary.

**COROLLARY 2.3.** *Let  $\varphi : A \rightarrow B$  be a monomorphism of finitely generated groups. Let  $\beta : B \rightarrow G$  be a homomorphism to a finite group. Then*

$$\varphi_* : H_0(A; \mathbb{Z}[G]) \rightarrow H_0(B; \mathbb{Z}[G])$$

is an epimorphism of free abelian groups and it is an isomorphism if and only if

$$\text{Im}\{A \rightarrow B \rightarrow G\} = \text{Im}\{B \rightarrow G\}.$$

*Proof.* Let  $\pi' = B \times G$  and  $\tilde{\pi}' = B$ . We can then apply Lemma 2.2 to  $A' = A, B' = \{(g, \beta(g)) \mid g \in B\} \subset \pi'$  and  $\varphi'(a) = (\varphi(a), \beta(\varphi(a)), a \in A'$ . It is straightforward to verify that the desired equivalence of statements follows. □



We now turn to the question when group homomorphisms induce isomorphisms of the 0-th and of the first twisted homology groups at the same time.

LEMMA 2.4. *Let  $\varphi : A \rightarrow B$  be a monomorphism of finitely generated groups. Suppose that  $B$  is a subgroup of a group  $\pi$  and let  $\tilde{\pi} \subset \pi$  be a subgroup of finite index. Let  $g_1, \dots, g_k \in \pi$  be a set of representatives for the equivalence classes  $B \backslash \pi / \tilde{\pi}$ . For  $i = 1, \dots, k$  we write  $\tilde{B}_i = B \cap g_i \tilde{\pi} g_i^{-1}$  and  $\tilde{A}_i = \varphi^{-1}(\tilde{B}_i)$ . Then*

$$\varphi_* : H_i(A; \mathbb{Z}[\pi/\tilde{\pi}]) \rightarrow H_i(B; \mathbb{Z}[\pi/\tilde{\pi}])$$

is an isomorphism for  $i = 0$  and  $i = 1$  if and only if the following two conditions are satisfied:

- (1)  $\varphi : A/\tilde{A}_i \rightarrow B/\tilde{B}_i$  is a bijection for any  $i$ ,
- (2)  $\varphi : A/[\tilde{A}_i, \tilde{A}_i] \rightarrow B/[\tilde{B}_i, \tilde{B}_i]$  is a bijection for any  $i$ .

*Proof.* Without loss of generality we can assume that  $A \subset B$  and that  $\varphi$  is the inclusion map. By Lemmas 2.1 and 2.2 it suffices to show for any  $i$  the following: If  $A/\tilde{A}_i \rightarrow B/\tilde{B}_i$  is a bijection, then the map  $H_1(A; \mathbb{Z}[B/\tilde{B}_i]) \rightarrow H_1(B; \mathbb{Z}[B/\tilde{B}_i])$  is an isomorphism if and only if  $\varphi : A/[\tilde{A}_i, \tilde{A}_i] \rightarrow B/[\tilde{B}_i, \tilde{B}_i]$  is a bijection.

Using the above and using Shapiro’s Lemma we can identify

$$\begin{aligned}
 H_1(A; \mathbb{Z}[B/\tilde{B}_i]) &= H_1(A; \mathbb{Z}[A/\tilde{A}_i]) = \tilde{A}_i/[\tilde{A}_i, \tilde{A}_i] && \text{and} \\
 H_1(B; \mathbb{Z}[B/\tilde{B}_i]) &= \tilde{B}_i/[\tilde{B}_i, \tilde{B}_i].
 \end{aligned}$$

Note that  $A/\tilde{A}_i, B/\tilde{B}_i, A/[\tilde{A}_i, \tilde{A}_i]$  and  $B/[\tilde{B}_i, \tilde{B}_i]$  are in general not groups, but we can view them as pointed sets. We now consider the following commutative diagram of exact sequences of pointed sets:

$$\begin{array}{ccccccc}
 0 & \longrightarrow & H_1(A; \mathbb{Z}[B/\tilde{B}_i]) & \longrightarrow & A/[\tilde{A}_i, \tilde{A}_i] & \longrightarrow & A/\tilde{A}_i \longrightarrow 1 \\
 & & \downarrow \varphi & & \downarrow \varphi & & \downarrow \varphi \\
 0 & \longrightarrow & H_1(B; \mathbb{Z}[B/\tilde{B}_i]) & \longrightarrow & B/[\tilde{B}_i, \tilde{B}_i] & \longrightarrow & B/\tilde{B}_i \longrightarrow 1.
 \end{array}$$

Recall that the map on the right is a bijection. It now follows from the 5-Lemma for exact sequences of pointed sets that the middle map is a bijection if and only if the left-hand map is a bijection. □

We will several times make use of the following corollary which can be deduced from Lemma 2.4 the same way that Corollary 2.3 is deduced from Lemma 2.2.

COROLLARY 2.5. *Let  $\varphi : A \rightarrow B$  be a monomorphism of finitely generated groups, and assume there is given a homomorphism  $\beta : B \rightarrow G$  to a finite*

group  $G$ . Then

$$\varphi_* : H_i(A; \mathbb{Z}[G]) \rightarrow H_i(B; \mathbb{Z}[G]), \quad i = 0, 1$$

is an isomorphism if and only if the following two conditions hold:

- (1)  $\text{Im}\{A \rightarrow B \rightarrow G\} = \text{Im}\{B \rightarrow G\}$ ,
- (2)  $\varphi$  induces an isomorphism

$$A/[\text{Ker}(\beta \circ \varphi), \text{Ker}(\beta \circ \varphi)] \rightarrow B/[\text{Ker}(\beta), \text{Ker}(\beta)].$$

Under extra conditions we can also give a criterion for a map between groups to induce an isomorphism of second homology groups.

LEMMA 2.6. *Let  $\varphi : A \rightarrow B$  be a homomorphism between two groups such that  $X = K(A, 1)$  and  $Y = K(B, 1)$  are finite 2-complexes with vanishing Euler characteristic. Let  $\beta : B \rightarrow G$  be a homomorphism to a finite group such that*

$$\varphi_* : H_i(A; \mathbb{Z}[G]) \rightarrow H_i(B; \mathbb{Z}[G]), \quad i = 0, 1$$

is an isomorphism; then

$$\varphi_* : H_2(A; \mathbb{Z}[G]) \rightarrow H_2(B; \mathbb{Z}[G])$$

is also an isomorphism.

*Proof.* We can and will view  $X$  as a subcomplex of  $Y$ . It suffices to show that  $H_2(Y, X; \mathbb{Z}[G]) = 0$ . Note that our assumption implies that  $H_i(Y, X; \mathbb{Z}[G]) = 0$  for  $i = 0, 1$ . Now note that  $H_2(Y, X; \mathbb{Z}[G])$  is a submodule of  $C_2(Y, X; \mathbb{Z}[G])$ , in particular  $H_2(Y, X; \mathbb{Z}[G])$  is a free  $\mathbb{Z}$ -module. We therefore only have to show that  $\text{rank}H_2(Y, X; \mathbb{Z}[G]) = 0$ . Now note that

$$\begin{aligned} \text{rank}H_2(Y, X; \mathbb{Z}[G]) &= \text{rank}H_2(Y, X; \mathbb{Z}[G]) - \text{rank}H_1(Y, X; \mathbb{Z}[G]) \\ &\quad + \text{rank}H_0(Y, X; \mathbb{Z}[G]) \\ &= |G|\chi(Y, X) \\ &= |G|(\chi(Y) - \chi(X)) \\ &= 0. \end{aligned}$$

□

We conclude this section with the following lemma.

LEMMA 2.7. *Let  $\varphi : A \rightarrow B$  be a homomorphism. Let  $\widehat{B} \subset \widetilde{B} \subset B$  be two subgroups. Suppose that  $\widehat{B} \subset B$  is normal. We write  $\widehat{A} := \varphi^{-1}(\widehat{B})$  and  $\widetilde{A} := \varphi^{-1}(\widetilde{B})$ . Assume that*

$$\varphi : A/\widehat{A} \rightarrow B/\widehat{B} \text{ and } \varphi : A/[\widehat{A}, \widehat{A}] \rightarrow B/[\widehat{B}, \widehat{B}]$$

are bijections, then

$$\varphi : A/\widetilde{A} \rightarrow B/\widetilde{B} \text{ and } \varphi : A/[\widetilde{A}, \widetilde{A}] \rightarrow B/[\widetilde{B}, \widetilde{B}]$$

are also bijections.

*Proof.* In the following let  $n = 0$  or  $n = 1$ . Suppose that  $\varphi : A/\hat{A}^{(n)} \rightarrow B/\hat{B}^{(n)}$  is a bijection. Note that  $\hat{A}^{(n)} \subset A$  and  $\hat{B}^{(n)} \subset B$  are normal, in particular  $\varphi : A/\hat{A}^{(n)} \rightarrow B/\hat{B}^{(n)}$  is in fact an isomorphism. We have to show that  $\varphi : A/\tilde{A}^{(n)} \rightarrow B/\tilde{B}^{(n)}$  is a bijection.

*Claim.* The map  $\varphi$  induces a bijection  $\tilde{A}^{(n)}/\hat{A}^{(n)} \rightarrow \tilde{B}^{(n)}/\hat{B}^{(n)}$ .

We write  $\bar{A} := A/\hat{A}^{(n)}$ ,  $\bar{B} := B/\hat{B}^{(n)}$  and we denote by  $\bar{\varphi} : \bar{A} \rightarrow \bar{B}$  the induced map which by assumption is an isomorphism. We denote by  $\bar{H}$  the subgroup  $\tilde{B}/\hat{B}^{(n)} \subset \bar{B}$ . Note that  $\bar{\varphi}$  restricts to isomorphisms  $\bar{\varphi}^{-1}(\bar{H}) \rightarrow \bar{H}$  and  $\bar{\varphi}^{-1}(\bar{H}^{(n)}) \rightarrow \bar{H}^{(n)}$ . Since  $\bar{\varphi}^{-1}$  is an isomorphism it follows that  $(\bar{\varphi}^{-1}(\bar{H}))^{(n)} = \bar{\varphi}^{-1}(\bar{H}^{(n)})$ . Now recall that  $\bar{H} = \tilde{B}/\hat{B}^{(n)}$ , hence  $\bar{H}^{(n)} = \tilde{B}^{(n)}/\hat{B}^{(n)}$ . We clearly have  $\bar{\varphi}^{-1}(\bar{H}) = \tilde{A}/\hat{A}^{(n)}$  and therefore  $\bar{\varphi}^{-1}(\bar{H})^{(n)} = \tilde{A}^{(n)}/\hat{A}^{(n)}$ . This shows that the isomorphism  $\bar{\varphi} : \bar{\varphi}^{-1}(\bar{H})^{(n)} \rightarrow \bar{H}^{(n)}$  is precisely the desired isomorphism  $\tilde{A}^{(n)}/\hat{A}^{(n)} \rightarrow \tilde{B}^{(n)}/\hat{B}^{(n)}$ . This concludes the proof of the claim.

Now consider the following commutative diagram of short exact sequences of pointed sets:

$$\begin{array}{ccccccc}
 1 & \longrightarrow & \tilde{A}^{(n)}/\hat{A}^{(n)} & \longrightarrow & A/\hat{A}^{(n)} & \longrightarrow & A/\tilde{A}^{(n)} \longrightarrow 1 \\
 & & \downarrow \varphi & & \downarrow \varphi & & \downarrow \varphi \\
 1 & \longrightarrow & \tilde{B}^{(n)}/\hat{B}^{(n)} & \longrightarrow & B/\hat{B}^{(n)} & \longrightarrow & B/\tilde{B}^{(n)} \longrightarrow 1.
 \end{array}$$

The middle vertical map is a bijection by assumption and we just verified that the vertical map on the left is a bijection. It now follows from the 5-Lemma for exact sequences of pointed sets that the vertical map on the right is also a bijection. □

**2.3. Twisted Alexander polynomials.** In this section we are going to recall the definition of twisted Alexander polynomials. These were introduced, for the case of knots, by Xiao-Song Lin in 1990 (published in [Lin01]), and his definition was later generalized to 3-manifolds by Wada [Wad94], Kirk-Livingston [KL99] and Cha [Cha03].

Let  $N$  be a compact manifold. Let  $R$  be a commutative, Noetherian unique factorization domain (in our applications  $R = \mathbb{Z}$  or  $R = \mathbb{F}_p$ , the finite field with  $p$  elements) and  $V$  a finite free  $R$ -module. Let  $\alpha : \pi_1(N) \rightarrow \text{Aut}_R(V)$  be a representation and let  $\phi \in H^1(N; \mathbb{Z}) = \text{Hom}(\pi_1(N), \mathbb{Z})$  be a nontrivial element. We write  $V \otimes_R R[t^{\pm 1}] =: V[t^{\pm 1}]$ . Then  $\alpha$  and  $\phi$  give rise to a representation  $\alpha \otimes \phi : \pi_1(N) \rightarrow \text{Aut}_{R[t^{\pm 1}]}(V[t^{\pm 1}])$  as follows:

$$((\alpha \otimes \phi)(g))(v \otimes p) := (\alpha(g) \cdot v) \otimes (\phi(g) \cdot p) = (\alpha(g) \cdot v) \otimes (t^{\phi(g)} p),$$

where  $g \in \pi_1(N), v \otimes p \in V \otimes_R R[t^{\pm 1}] = V[t^{\pm 1}]$ .

Note that  $N$  is homotopy equivalent to a finite CW-complex, which, by abuse of notation, we also denote by  $N$ . Then we consider  $C_*(\tilde{N}) \otimes_{\mathbb{Z}[\pi_1(N)]} V[t^{\pm 1}]$  which is a complex of finitely generated  $R[t^{\pm 1}]$ -modules. Since  $R[t^{\pm 1}]$  is Noetherian it follows that for any  $i$  the  $R[t^{\pm 1}]$ -module  $H_i(N; V[t^{\pm 1}])$  is a finitely presented  $R[t^{\pm 1}]$ -module. This means  $H_i(N; V[t^{\pm 1}])$  has a free  $R[t^{\pm 1}]$ -resolution

$$R[t^{\pm 1}]^{r_i} \xrightarrow{S_i} R[t^{\pm 1}]^{s_i} \rightarrow H_i(N; V[t^{\pm 1}]) \rightarrow 0.$$

Without loss of generality we can assume that  $r_i \geq s_i$ .

*Definition.* The  $i$ -th twisted Alexander polynomial of  $(N, \alpha, \phi)$  is defined to be the order of the  $R[t^{\pm 1}]$ -module  $H_i(N; V[t^{\pm 1}])$ , i.e. the greatest common divisor (which exists since  $R[t^{\pm 1}]$  is a UFD as well) of the  $s_i \times s_i$  minors of the  $s_i \times r_i$ -matrix  $S_i$ . It is denoted by  $\Delta_{N, \phi, i}^\alpha \in R[t^{\pm 1}]$ .

Note that  $\Delta_{N, \phi, i}^\alpha \in R[t^{\pm 1}]$  is well-defined up to a unit in  $R[t^{\pm 1}]$ , i.e. up to an element of the form  $rt^i$  where  $r$  is a unit in  $R$  and  $i \in \mathbb{Z}$ . We say that  $f \in R[t^{\pm 1}]$  is *monic* if its top coefficient is a unit in  $R$ . Given a nontrivial  $f = \sum_{i=r}^s a_s t^i$  with  $a_r \neq 0, a_s \neq 0$  we write  $\deg f = s - r$ . For  $f = 0$  we write  $\deg(f) = -\infty$ . Note that  $\deg \Delta_{N, \phi, i}^\alpha$  is well-defined.

We now write  $\pi = \pi_1(N)$ . If we are given a homomorphism  $\alpha : \pi \rightarrow G$  to a finite group, then this gives rise to a finite dimensional representation of  $\pi$ , that we will denote by  $\alpha : \pi \rightarrow \text{Aut}_R(R[G])$  as well. In the case that we have a finite index subgroup  $\tilde{\pi} \subset \pi$  we get a finite dimensional representation  $\pi \rightarrow \text{Aut}_R(R[\pi/\tilde{\pi}])$  given by left-multiplication. When  $R = \mathbb{Z}$ , the resulting twisted Alexander polynomials will be denoted by  $\Delta_{N, \phi, i}^{\pi/\tilde{\pi}} \in \mathbb{Z}[t^{\pm 1}]$ , while for  $R = \mathbb{F}_p$  we will use the notation  $\Delta_{N, \phi, i}^{\pi/\tilde{\pi}, p} \in \mathbb{F}_p[t^{\pm 1}]$ . See [FV08b] for the relation between these polynomials.

Finally, in the case that  $\alpha : \pi \rightarrow \text{GL}(1, \mathbb{Z})$  is the trivial representation we drop the  $\alpha$  from the notation, and in the case that  $i = 1$  we drop the subscript “1” from the notation.

We summarize some of the main properties of twisted Alexander polynomials in the following lemma. It is a consequence of [FV08b, Lemmas 3.3 and 3.4] and [FK06, Prop. 2.5].

LEMMA 2.8. *Let  $N$  be a 3-manifold with empty or toroidal boundary. Let  $\phi \in H^1(N; \mathbb{Z})$  be nontrivial and  $\tilde{\pi} \subset \pi := \pi_1(N)$  a finite index subgroup. Denote by  $\phi_{\tilde{\pi}}$  the restriction of  $\phi$  to  $\tilde{\pi}$ ; then the following hold:*

- (1)  $\Delta_{N, \phi, 0}^{\pi/\tilde{\pi}} = (1 - t^{\text{div } \phi_{\tilde{\pi}}})$ ;
- (2) if  $\Delta_{N, \phi, 1}^{\pi/\tilde{\pi}} \neq 0$ , then  $\Delta_{N, \phi, 2}^{\pi/\tilde{\pi}} = (1 - t^{\text{div } \phi_{\tilde{\pi}}})^{b_3(N)}$ ;
- (3)  $\Delta_{N, \phi, i}^{\pi/\tilde{\pi}} = 1$  for any  $i \geq 3$ .

Assume we also have a subgroup  $\pi'$  with  $\tilde{\pi} \subset \pi' \subset \pi$ . Denote the covering of  $N$  corresponding to  $\pi'$  by  $N'$  and denote by  $\phi'$  the restriction of  $\phi$  to  $\pi'$ , then

$$\Delta_{N,\phi,i}^{\pi/\tilde{\pi}} = \Delta_{N',\phi',i}^{\pi'/\tilde{\pi}}$$

for any  $i$ . Finally note that the statements of the lemma also hold for the polynomial  $\Delta_{N,\phi,i}^{\pi/\tilde{\pi},p} \in \mathbb{F}_p[t^{\pm 1}]$ .

We also recall the following well-known result (cf. e.g. [Tur01]).

LEMMA 2.9. *Let  $N$  be a 3-manifold with empty or toroidal boundary. Let  $\phi \in H^1(N; \mathbb{Z})$  be nontrivial and  $\tilde{\pi} \subset \pi := \pi_1(N)$  a finite index subgroup. Then given  $i$  the following are equivalent:*

- (1)  $\Delta_{N,\phi,i}^{\pi/\tilde{\pi}} \neq 0$ ,
- (2)  $H_i(N; \mathbb{Z}[\pi/\tilde{\pi}][t^{\pm 1}])$  is  $\mathbb{Z}[t^{\pm 1}]$ -torsion,
- (3)  $H_i(N; \mathbb{Q}[\pi/\tilde{\pi}][t^{\pm 1}])$  is  $\mathbb{Q}[t^{\pm 1}]$ -torsion,
- (4) the rank of the abelian group  $H_i(N; \mathbb{Z}[\pi/\tilde{\pi}][t^{\pm 1}])$  is finite.

In fact if any of the four conditions holds, then

$$\deg \Delta_{N,\phi,i}^{\pi/\tilde{\pi}} = \text{rank } H_i(N; \mathbb{Z}[\pi/\tilde{\pi}][t^{\pm 1}]) = \dim H_i(N; \mathbb{Q}[\pi/\tilde{\pi}][t^{\pm 1}]).$$

2.4. *Completions of groups.* Throughout the paper it is convenient to use the language of completions of groups. Although the proof of Theorem 1.2 does not explicitly require this terminology, group completions are the natural framework for these results. We recall here the definitions and some basic facts, referring to [LS03, Window 4] and [Wil98], [RZ00] for proofs and for more information.

Let  $\mathcal{C}$  be a variety of groups (cf. [RZ00, p. 20] for the definition). Examples of varieties of pertinence to this paper are given by any one of the following:

- (1) finite groups,
- (2)  $p$ -groups for a prime  $p$ ,
- (3) the variety  $\mathcal{FS}(n)$  of finite solvable groups of derived length at most  $n$ ,
- (4) the variety  $\mathcal{FS}$  of finite solvable groups.

In the following we equip a finitely generated group  $A$  with its *pro- $\mathcal{C}$  topology*, this topology is the translation invariant topology uniquely defined by taking as a fundamental system of neighborhoods of the identity the collection of all normal subgroups of  $A$  such that the quotient lies in  $\mathcal{C}$ . Note that in particular all groups in  $\mathcal{C}$  are endowed with the discrete topology.

Given a group  $A$  denote by  $\hat{A}_{\mathcal{C}}$  its pro- $\mathcal{C}$  completion, i.e. the inverse limit

$$\hat{A}_{\mathcal{C}} = \varprojlim A/A_i$$

where  $A_i$  runs through the inverse system determined by the collection of all normal subgroups of  $A$  such that  $A/A_i \in \mathcal{C}$ . Then  $\hat{A}_{\mathcal{C}}$ , which we can view as a subgroup of the direct product of all  $A/A_i$ , inherits a natural topology.

Henceforth by homomorphisms between groups we will mean a homomorphism which is continuous with respect to the above topologies. Using the standard convention we refer to the pro- $\mathcal{FS}$  completion of a group as the prosolvable completion.

Note that by the assumption that  $\mathcal{C}$  is a variety, the pro- $\mathcal{C}$  completion is a covariant functor, i.e. given  $\varphi : A \rightarrow B$  we get an induced homomorphism  $\hat{\varphi} : \hat{A}_{\mathcal{C}} \rightarrow \hat{B}_{\mathcal{C}}$ .

A group  $A$  is called *residually  $\mathcal{C}$*  if for any nontrivial  $g \in A$  there exists a homomorphism  $\alpha : A \rightarrow G$  where  $G \in \mathcal{C}$  such that  $\alpha(g) \neq e$ . It is easily seen that  $A$  is residually  $\mathcal{C}$  if and only if the map  $A \rightarrow \hat{A}_{\mathcal{C}}$  is injective. In particular, if we are given a homomorphism  $\varphi : A \rightarrow B$  between residually  $\mathcal{C}$  groups  $A, B$  such that  $\hat{\varphi} : \hat{A}_{\mathcal{C}} \rightarrow \hat{B}_{\mathcal{C}}$  is an injection, then it follows from the following commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{\varphi} & B \\ \downarrow & & \downarrow \\ \hat{A}_{\mathcal{C}} & \xrightarrow{\hat{\varphi}} & \hat{B}_{\mathcal{C}} \end{array}$$

that  $\varphi$  is injective as well.

The following well-known lemma gives sufficient and necessary conditions for a homomorphism  $\varphi : A \rightarrow B$  to induce an isomorphism of pro- $\mathcal{C}$  completions.

LEMMA 2.10. *Let  $\mathcal{C}$  be a variety of groups and assume that there is a homomorphism  $\varphi : A \rightarrow B$ . Then the following are equivalent:*

- (1)  $\hat{\varphi} : \hat{A}_{\mathcal{C}} \rightarrow \hat{B}_{\mathcal{C}}$  is an isomorphism,
- (2) for any  $G \in \mathcal{C}$  the induced map

$$\varphi^* : \text{Hom}(B, G) \rightarrow \text{Hom}(A, G)$$

is a bijection.

We also note the following well-known lemma.

LEMMA 2.11. *Let  $\mathcal{C}$  be an extension-closed variety and let  $\varphi : A \rightarrow B$  be a homomorphism of finitely generated groups which induces an isomorphism of pro- $\mathcal{C}$  completions. Then for any homomorphism  $\beta : B \rightarrow G$  to a  $\mathcal{C}$ -group, the restriction of  $\varphi$  to  $\text{Ker}(\beta \circ \varphi) \rightarrow \text{Ker}(\beta)$  induces an isomorphism of pro- $\mathcal{C}$  completions.*

When a homomorphism  $\varphi : A \rightarrow B$  of finitely generated groups induces an isomorphism of their pro- $\mathcal{C}$  completions, then we have a relation of the twisted homology with coefficients determined by  $\mathcal{C}$ -groups. More precisely, we have the following.

LEMMA 2.12. *Let  $\mathcal{C}$  be a variety of groups and let  $\varphi : A \rightarrow B$  be a homomorphism of finitely generated groups which induces an isomorphism of pro- $\mathcal{C}$  completions. Then for any homomorphism  $\beta : B \rightarrow G$  to a  $\mathcal{C}$ -group the map  $\varphi_* : H_0(A; \mathbb{Z}[G]) \rightarrow H_0(B; \mathbb{Z}[G])$  is an isomorphism. Furthermore, if  $\mathcal{C}$  is an extension-closed variety containing all finite abelian groups, the map  $\varphi_* : H_1(A; \mathbb{Z}[G]) \rightarrow H_1(B; \mathbb{Z}[G])$  is an isomorphism.*

*Proof.* Observe that, by Corollary 2.3, the first part of the statement is equivalent to the assertion that, for any element  $\beta \in \text{Hom}(B, G)$ ,

$$\text{Im}\{\beta \circ \varphi : A \rightarrow G\} = \text{Im}\{\beta : B \rightarrow G\}.$$

Without loss of generality, we can reduce the proof of this isomorphism to the case where  $\beta$  is surjective. Denote  $\alpha = \beta \circ \varphi \in \text{Hom}(A, G)$ . Assume to the contrary that  $\alpha(A) \subsetneq G$ ; then  $\alpha \in \text{Hom}(A, \alpha(A)) \subset \text{Hom}(A, G)$  and as  $\alpha(A) \in \mathcal{C}$  there exists by hypothesis a map  $\beta' \in \text{Hom}(B, \alpha(A)) \subset \text{Hom}(B, G)$  such that  $\alpha = \beta' \circ \varphi$ . Now the two maps  $\beta, \beta' \in \text{Hom}(B, G)$  (that must differ as they have different images) induce the same map  $\alpha \in \text{Hom}(A, G)$ , contradicting the bijectivity of  $\text{Hom}(B, G)$  and  $\text{Hom}(A, G)$ .

We now turn to the proof of the second part of the statement. Let  $\beta : B \rightarrow G$  be a homomorphism to a  $\mathcal{C}$ -group. Again, without loss of generality, we can assume that  $\beta : B \rightarrow G$  is surjective. Note that by the above the homomorphism  $\beta \circ \varphi : A \rightarrow G$  is surjective as well. We now write  $B' = \text{Ker}(\beta)$  and  $A' = \text{Ker}(\beta \circ \varphi)$ . By Shapiro’s lemma, we have the commutative diagram

$$\begin{array}{ccc} H_1(A'; \mathbb{Z}) & \xrightarrow{\cong} & H_1(A; \mathbb{Z}[G]) \\ \downarrow & & \downarrow \\ H_1(B'; \mathbb{Z}) & \xrightarrow{\cong} & H_1(B; \mathbb{Z}[G]). \end{array}$$

The claim amounts therefore to showing that the map  $\varphi_* : H_1(A'; \mathbb{Z}) \rightarrow H_1(B'; \mathbb{Z})$  is an isomorphism. As  $A$  and  $B$  are finitely generated,  $A'$  and  $B'$  are finitely generated as well. When  $\mathcal{C}$  is extension closed, and contains all finite abelian groups, Lemma 2.11 asserts that the map  $\varphi$  induces a bijection between  $\text{Hom}(B', \Gamma)$  and  $\text{Hom}(A', \Gamma)$  for any finite abelian group  $\Gamma$ ; the desired isomorphism easily follows. □

### 3. Monic twisted Alexander polynomials and solvable groups

The aim of this section is to prove Proposition 1.7.

3.1. *Preliminary results.* We will often make use of the following proposition (cf. [McM02, §4 and Prop. 6.1]).

**PROPOSITION 3.1.** *Let  $N$  be a 3-manifold with empty or toroidal boundary and let  $\phi \in H^1(N; \mathbb{Z})$  be a primitive class. If  $\Delta_{N,\phi} \neq 0$ , then there exists a connected Thurston norm-minimizing surface  $\Sigma$  dual to  $\phi$ .*

Given a connected oriented surface  $\Sigma \subset N$  we will adopt the following conventions for the rest of the paper. We choose a neighborhood  $\Sigma \times [-1, 1] \subset N$  and write  $\nu\Sigma = \Sigma \times (-1, 1)$ . Let  $M := N \setminus \nu\Sigma$ ; we will write  $\Sigma^\pm = \Sigma \times \{\pm 1\} \subset \partial M$ , and we will denote the inclusion induced maps  $\Sigma \rightarrow \Sigma^\pm \subset M$  by  $\iota_\pm$ .

We pick a base point in  $M$  and endow  $N$  with the same base point. Also, we pick a base point for  $\Sigma$  and endow  $\Sigma^\pm$  with the corresponding base points. With these choices made, we write  $A = \pi_1(\Sigma)$  and  $B = \pi_1(M)$ . We also pick paths in  $M$  connecting the base point of  $M$  with the base points of  $\Sigma^-$  and  $\Sigma^+$ . We now have inclusion induced maps  $\iota_\pm : A \rightarrow B$  for either inclusion of  $\Sigma$  in  $M$  and, using the constant path, a map  $\pi_1(M) \rightarrow \pi_1(N)$ . Under the assumption that  $\Sigma$  is incompressible (in particular, whenever  $\Sigma$  is Thurston norm minimizing) these maps are injective. Since  $M$  and  $N$  have the same base point we can view  $B$  canonically as a subgroup of  $\pi_1(N)$ .

Before we state the first proposition we have to introduce a few more definitions. Let  $N$  be a 3-manifold with empty or toroidal boundary and let  $\phi \in H^1(N; \mathbb{Z})$  be a nontrivial class. Let  $\tilde{\pi} \subset \pi$  be a finite index subgroup. We denote by  $\phi_{\tilde{\pi}}$  the restriction of  $\phi \in H^1(N; \mathbb{Z}) = \text{Hom}(\pi, \mathbb{Z})$  to  $\tilde{\pi}$ . Note that  $\phi_{\tilde{\pi}}$  is necessarily nontrivial. We say that  $\tilde{\pi} \subset \pi$  has Property (M) if the twisted Alexander polynomial  $\Delta_{N,\phi}^{\pi/\tilde{\pi}} \in \mathbb{Z}[t^{\pm 1}]$  is monic and if

$$\text{deg}(\Delta_{N,\phi}^{\pi/\tilde{\pi}}) = [\pi : \tilde{\pi}] \|\phi\|_T + (1 + b_3(N))\text{div}\phi_{\tilde{\pi}}$$

holds. Note that a pair  $(N, \phi)$  satisfies Condition (\*) if and only if Property (M) is satisfied by all normal subgroups of  $\pi_1(N)$ .

The following proposition is the key tool for translating information on twisted Alexander polynomials into information on the maps  $\iota_\pm : A \rightarrow B$ . The proposition is well known in the classical case. In the case of normal subgroups a proof for the ‘only if’ direction of the proposition is given by combining [FV08b, §8] with [FV08a, §4].

**PROPOSITION 3.2.** *Let  $N$  be a 3-manifold with empty or toroidal boundary with  $N \neq S^1 \times D^2, N \neq S^1 \times S^2$ . Let  $\phi \in H^1(N; \mathbb{Z})$  be a primitive class which is dual to a connected Thurston norm-minimizing surface  $\Sigma$ . Let  $\tilde{\pi} \subset \pi$  be a finite index subgroup. Then  $\tilde{\pi}$  has Property (M) if and only if the maps  $\iota_\pm : H_i(A; \mathbb{Z}[\pi/\tilde{\pi}]) \rightarrow H_i(B; \mathbb{Z}[\pi/\tilde{\pi}])$  are isomorphisms for  $i = 0, 1$ .*

*Proof.* Let  $R = \mathbb{Z}$  or  $R = \mathbb{F}_p$  with  $p$  a prime. We have canonical isomorphisms  $H_i(\Sigma; R[\pi/\tilde{\pi}]) \cong H_i(A; R[\pi/\tilde{\pi}])$  and  $H_i(M; R[\pi/\tilde{\pi}]) \cong H_i(B; R[\pi/\tilde{\pi}])$  for  $i = 0, 1$ . It follows from [FK06, Prop. 3.2] that splitting  $N$  along  $\Sigma$  gives



rise to the following Mayer-Vietoris type exact sequence

$$\begin{aligned} & \dots && \rightarrow H_2(N; R[\pi/\tilde{\pi}][t^{\pm 1}]) \\ \rightarrow H_1(A; R[\pi/\tilde{\pi}]) \otimes R[t^{\pm 1}] & \rightarrow H_1(B; R[\pi/\tilde{\pi}]) \otimes R[t^{\pm 1}] & \rightarrow H_1(N; R[\pi/\tilde{\pi}][t^{\pm 1}]) & \rightarrow \\ \rightarrow H_0(A; R[\pi/\tilde{\pi}]) \otimes R[t^{\pm 1}] & \rightarrow H_0(B; R[\pi/\tilde{\pi}]) \otimes R[t^{\pm 1}] & \rightarrow H_0(N; R[\pi/\tilde{\pi}][t^{\pm 1}]) & \rightarrow 0, \end{aligned}$$

where the map

$$H_i(A; R[\pi/\tilde{\pi}]) \otimes R[t^{\pm 1}] \rightarrow H_i(B; R[\pi/\tilde{\pi}]) \otimes R[t^{\pm 1}]$$

is given by  $(\iota_+ - \iota_-)$ . We refer to this sequence as the Mayer-Vietoris sequence of  $(N, \Sigma)$  with  $R[\pi/\tilde{\pi}][t^{\pm 1}]$ -coefficients. First note that by Shapiro’s lemma the groups  $H_i(A; R[\pi/\tilde{\pi}])$  are the  $i$ -th homology with  $R$ -coefficients of a (possibly) disconnected surface. It follows that  $H_i(A; R[\pi/\tilde{\pi}])$  is a free  $R$ -module, in particular the  $R[t^{\pm 1}]$ -modules  $H_i(A; R[\pi/\tilde{\pi}]) \otimes R[t^{\pm 1}]$  are free  $R[t^{\pm 1}]$ -modules. We will several times make use of the observation that if  $H_i(N; R[\pi/\tilde{\pi}][t^{\pm 1}])$  is  $R[t^{\pm 1}]$ -torsion, then the map  $H_i(N; R[\pi/\tilde{\pi}][t^{\pm 1}]) \rightarrow H_{i-1}(A; R[\pi/\tilde{\pi}]) \otimes R[t^{\pm 1}]$  is necessarily zero.

We first assume that  $\tilde{\pi}$  has Property (M). Since  $\Delta_{N,\phi}^{\pi/\tilde{\pi}} \neq 0$  we have that the module  $H_1(N; \mathbb{Z}[\pi/\tilde{\pi}][t^{\pm 1}] \otimes_{\mathbb{Z}[t^{\pm 1}]} \mathbb{Q}(t))$  is trivial. Note that by Lemma 2.9 we have that  $H_0(N; \mathbb{Z}[\pi/\tilde{\pi}][t^{\pm 1}])$  is also  $\mathbb{Z}[t^{\pm 1}]$ -torsion. We now consider the Mayer-Vietoris sequence of  $(N, \Sigma)$  with  $\mathbb{Z}[\pi/\tilde{\pi}][t^{\pm 1}]$ -coefficients. Tensoring the exact sequence with  $\mathbb{Q}(t)$  we see that

$$\begin{aligned} \text{rank}_{\mathbb{Z}}(H_0(A; \mathbb{Z}[\pi/\tilde{\pi}])) &= \text{rank}_{\mathbb{Q}(t)}(H_0(A; \mathbb{Z}[\pi/\tilde{\pi}]) \otimes_{\mathbb{Z}} \mathbb{Q}(t)) \\ &= \text{rank}_{\mathbb{Q}(t)}(H_0(B; \mathbb{Z}[\pi/\tilde{\pi}]) \otimes_{\mathbb{Z}} \mathbb{Q}(t)) \\ &= \text{rank}_{\mathbb{Z}}(H_0(B; \mathbb{Z}[\pi/\tilde{\pi}])). \end{aligned}$$

Using this observation and using Lemma 2.2 we see that the maps

$$\iota_{\pm} : H_0(A; \mathbb{Z}[\pi/\tilde{\pi}]) \rightarrow H_0(B; \mathbb{Z}[\pi/\tilde{\pi}])$$

are epimorphisms between free abelian groups of the same rank. Hence the maps are in fact isomorphisms.

In order to prove that the maps  $\iota_{\pm} : H_1(A; \mathbb{Z}[\pi/\tilde{\pi}]) \rightarrow H_1(B; \mathbb{Z}[\pi/\tilde{\pi}])$  are isomorphisms we first consider the following claim.

*Claim.*  $H_1(A; \mathbb{Z}[\pi/\tilde{\pi}])$  and  $H_1(B; \mathbb{Z}[\pi/\tilde{\pi}])$  are free abelian groups of the same rank.

Let  $p$  be a prime. We consider the Mayer-Vietoris sequence of  $(N, \Sigma)$  with  $\mathbb{F}_p[\pi/\tilde{\pi}][t^{\pm 1}]$ -coefficients. Denote by  $\Delta_{N,\phi}^{\pi/\tilde{\pi},p} \in \mathbb{F}_p[t^{\pm 1}]$  the twisted Alexander polynomial with coefficients in  $\mathbb{F}_p$ . It follows from  $\Delta_{N,\phi}^{\pi/\tilde{\pi}}$  monic and from [FV08b, Prop. 6.1] that  $\Delta_{N,\phi}^{\pi/\tilde{\pi},p} \neq 0 \in \mathbb{F}_p[t^{\pm 1}]$ . Furthermore by Lemma 2.8 we

have that  $\Delta_{N,\phi,2}^{\pi/\tilde{\pi},p} \neq 0 \in \mathbb{F}_p[t^{\pm 1}]$ . In particular  $H_i(N; \mathbb{F}_p[\pi/\tilde{\pi}][t^{\pm 1}])$  is  $\mathbb{F}_p[t^{\pm 1}]$ -torsion for  $i = 1, 2$ . It follows from the fact that  $H_i(A; \mathbb{F}_p[\pi/\tilde{\pi}]) \otimes_{\mathbb{F}_p} \mathbb{F}_p[t^{\pm 1}]$  is a free  $\mathbb{F}_p[t^{\pm 1}]$ -module and the above observation that  $H_i(N; \mathbb{F}_p[\pi/\tilde{\pi}][t^{\pm 1}])$  is  $\mathbb{F}_p[t^{\pm 1}]$ -torsion for  $i = 1, 2$  that the Mayer-Vietoris sequence gives rise to the following short exact sequence

$$0 \rightarrow H_1(A; \mathbb{F}_p[\pi/\tilde{\pi}]) \otimes \mathbb{F}_p[t^{\pm 1}] \xrightarrow{t\iota_+ - \iota_-} H_1(B; \mathbb{F}_p[\pi/\tilde{\pi}]) \otimes \mathbb{F}_p[t^{\pm 1}] \rightarrow H_1(N; \mathbb{F}_p[\pi/\tilde{\pi}][t^{\pm 1}]) \rightarrow 0.$$

Tensoring with  $\mathbb{F}_p(t)$  we see that in particular  $H_1(A; \mathbb{F}_p[\pi/\tilde{\pi}]) \cong H_1(B; \mathbb{F}_p[\pi/\tilde{\pi}])$  as  $\mathbb{F}_p$ -vector spaces. The homology group  $H_0(A; \mathbb{Z}[\pi/\tilde{\pi}])$  is  $\mathbb{Z}$ -torsion free. It follows from the universal coefficient theorem applied to the complex of  $\mathbb{Z}$ -modules  $C_*(\tilde{\Sigma}) \otimes_{\mathbb{Z}[A]} \mathbb{Z}[\pi/\tilde{\pi}]$  that

$$H_1(A; \mathbb{Z}[\pi/\tilde{\pi}]) \otimes_{\mathbb{Z}} \mathbb{F}_p \cong H_1(A; \mathbb{F}_p[\pi/\tilde{\pi}])$$

for every prime  $p$ . The same statement holds for  $B$ . Combining our results we see that  $H_1(A; \mathbb{Z}[\pi/\tilde{\pi}]) \otimes_{\mathbb{Z}} \mathbb{F}_p$  and  $H_1(B; \mathbb{Z}[\pi/\tilde{\pi}]) \otimes_{\mathbb{Z}} \mathbb{F}_p$  are isomorphic for any prime  $p$ . Since  $H_1(A; \mathbb{Z}[\pi/\tilde{\pi}])$  is free abelian it follows that  $H_1(A; \mathbb{Z}[\pi/\tilde{\pi}]) \cong H_1(B; \mathbb{Z}[\pi/\tilde{\pi}])$ . This completes the proof of the claim.

We now equip the free  $\mathbb{Z}$ -modules  $H_1(A; \mathbb{Z}[\pi/\tilde{\pi}])$  and  $H_1(B; \mathbb{Z}[\pi/\tilde{\pi}])$  with a choice of basis. We study the Mayer-Vietoris sequence for  $(N, \Sigma)$  with  $\mathbb{Z}[\pi/\tilde{\pi}][t^{\pm 1}]$ -coefficients. Using an argument similar to the above, we see that it gives rise to the following exact sequence

$$H_1(A; \mathbb{Z}[\pi/\tilde{\pi}]) \otimes \mathbb{Z}[t^{\pm 1}] \xrightarrow{t\iota_+ - \iota_-} H_1(B; \mathbb{Z}[\pi/\tilde{\pi}]) \otimes \mathbb{Z}[t^{\pm 1}] \rightarrow H_1(N; \mathbb{Z}[\pi/\tilde{\pi}][t^{\pm 1}]) \rightarrow 0.$$

Since  $H_1(A; \mathbb{Z}[\pi/\tilde{\pi}])$  and  $H_1(B; \mathbb{Z}[\pi/\tilde{\pi}])$  are free abelian groups of the same rank, it follows that the above exact sequence is a resolution of  $H_1(N; \mathbb{Z}[\pi/\tilde{\pi}][t^{\pm 1}])$  by free  $\mathbb{Z}[t^{\pm 1}]$ -modules and that  $\Delta_{N,\phi}^{\pi/\tilde{\pi}} = \det(t\iota_+ - \iota_-)$ . Recall that Property (M) states in particular that

$$(1) \quad \deg \Delta_{N,\phi}^{\pi/\tilde{\pi}} = |\pi/\tilde{\pi}| \|\phi\|_T + (1 + b_3(N)) \operatorname{div} \phi_{\tilde{\pi}}.$$

Recall that we assumed  $N \neq S^1 \times D^2$  and  $N \neq S^1 \times S^2$ , in particular  $\chi(\Sigma) \leq 0$  and therefore  $-\chi(\Sigma) = \|\phi\|_T$ . Writing  $b_i = \operatorname{rank}_{\mathbb{Z}}(H_i(\Sigma; \mathbb{Z}[\pi/\tilde{\pi}])) = \operatorname{rank}_{\mathbb{Z}}(H_i(A; \mathbb{Z}[\pi/\tilde{\pi}]))$  a standard Euler characteristic argument now shows that

$$-b_0 + b_1 - b_2 = -|\pi/\tilde{\pi}| \chi(\Sigma) = |\pi/\tilde{\pi}| \cdot \|\phi\|_T.$$

By [FK06, Lemma 2.2] we have  $b_i = \deg \Delta_{N,\phi,i}^{\pi/\tilde{\pi}}$  for  $i = 0$  and  $i = 2$ . We also have  $\deg \Delta_{N,\phi,0}^{\pi/\tilde{\pi}} = \operatorname{div} \phi_{\pi/\tilde{\pi}}$  and  $\deg \Delta_{N,\phi,2}^{\pi/\tilde{\pi}} = b_3(N) \operatorname{div} \phi_{\tilde{\pi}}$  by Lemma 2.8. Combining these facts with (1) we conclude that  $\deg \Delta_{N,\phi}^{\pi/\tilde{\pi}} = b_1$ . So we now have  $\deg(\det(t\iota_+ - \iota_-)) = b_1$ . Since  $\iota_+$  and  $\iota_-$  are  $b_1 \times b_1$  matrices over

$\mathbb{Z}$  it now follows that  $\det(\iota_+)$  equals the top coefficient of  $\Delta_{N,\phi}^{\pi/\tilde{\pi}}$ , which by Property (M) equals  $\pm 1$ . By the symmetry of twisted Alexander polynomials we have that the bottom coefficient of  $\Delta_{N,\phi}^{\pi/\tilde{\pi}}$  also equals  $\pm 1$ , we deduce that  $\det(\iota_-) = \pm 1$ . This shows that  $\iota_+, \iota_- : H_1(A; \mathbb{Z}[\pi/\tilde{\pi}]) \rightarrow H_1(B; \mathbb{Z}[\pi/\tilde{\pi}])$  are isomorphisms. We thus showed that if  $\tilde{\pi}$  has Property (M), then the maps  $\iota : H_i(A; \mathbb{Z}[\pi/\tilde{\pi}]) \rightarrow H_i(B; \mathbb{Z}[\pi/\tilde{\pi}])$  are isomorphisms for  $i = 0, 1$ .

Now assume that we are given a finite index subgroup  $\tilde{\pi} \subset \pi$  such that the maps  $\iota_{\pm} : H_i(A; \mathbb{Z}[\pi/\tilde{\pi}]) \rightarrow H_i(B; \mathbb{Z}[\pi/\tilde{\pi}])$  are isomorphisms for  $i = 0, 1$ . It follows from the assumption that  $\iota_{\pm} : H_0(A; \mathbb{Z}[\pi/\tilde{\pi}]) \rightarrow H_0(B; \mathbb{Z}[\pi/\tilde{\pi}])$  are isomorphisms that the map

$$H_0(A; \mathbb{Z}[\pi/\tilde{\pi}]) \otimes \mathbb{Z}[t^{\pm 1}] \xrightarrow{t\iota_+ - \iota_-} H_0(B; \mathbb{Z}[\pi/\tilde{\pi}])$$

is injective. Specifically, the Mayer-Vietoris sequence of  $(N, \Sigma)$  with  $\mathbb{Z}[\pi/\tilde{\pi}][t^{\pm 1}]$ -coefficients gives rise to the following exact sequence

$$H_1(A; \mathbb{Z}[\pi/\tilde{\pi}]) \otimes \mathbb{Z}[t^{\pm 1}] \xrightarrow{t\iota_+ - \iota_-} H_1(B; \mathbb{Z}[\pi/\tilde{\pi}]) \otimes \mathbb{Z}[t^{\pm 1}] \rightarrow H_1(N; \mathbb{Z}[\pi/\tilde{\pi}][t^{\pm 1}]) \rightarrow 0.$$

As above  $H_1(A; \mathbb{Z}[\pi/\tilde{\pi}])$  is a free abelian group. Therefore, by our assumption,  $H_1(B; \mathbb{Z}[\pi/\tilde{\pi}]) \cong H_1(A; \mathbb{Z}[\pi/\tilde{\pi}])$  is also free abelian. In particular the above exact sequence defines a presentation for  $H_1(N; \mathbb{Z}[\pi/\tilde{\pi}][t^{\pm 1}])$  and we deduce that  $\Delta_{N,\phi}^{\pi/\tilde{\pi}} = \det(t\iota_+ - \iota_-)$ . Since  $\iota_-$  and  $\iota_+$  are isomorphisms it follows that  $\Delta_{N,\phi}^{\pi/\tilde{\pi}}$  is monic of degree  $b_1$ . An argument similar to the above now shows that

$$\deg \Delta_{N,\phi}^{\pi/\tilde{\pi}} = |\pi/\tilde{\pi}| \|\phi\|_T + (1 + b_3(N)) \operatorname{div} \phi_{\tilde{\pi}}. \quad \square$$

**3.2. Finite solvable quotients.** Given a solvable group  $S$  we denote by  $\ell(S)$  its derived length, i.e. the length of the shortest decomposition into abelian groups. Put differently,  $\ell(S)$  is the minimal number such that  $S^{(\ell(S))} = \{e\}$ . Note that  $\ell(S) = 0$  if and only if  $S = \{e\}$ .

For the sake of comprehension, we briefly recall the notation. Consider a 3-manifold  $N$  with empty or toroidal boundary, and fix a primitive class  $\phi \in H^1(N; \mathbb{Z})$ . We denote by  $\Sigma$  a connected Thurston norm-minimizing surface dual to  $\phi$ , and write  $A = \pi_1(\Sigma)$  and  $B = \pi_1(M)$  (with  $M = N \setminus \nu\Sigma$ ) denoting the two inclusion induced maps  $A \rightarrow B$  by  $\iota_{\pm}$ . We also write  $\pi = \pi_1(N)$ . Note that  $\pi = \langle B, t|\iota_-(A) = t\iota_+(A)t^{-1} \rangle$ .

Given  $n \in \mathbb{N} \cup \{0\}$  we denote by  $\mathcal{S}(n)$  the statement that for any finite solvable group  $S$  with  $\ell(S) \leq n$ , for  $\iota = \iota_-, \iota_+$  the map

$$\iota^* : \operatorname{Hom}(B, S) \rightarrow \operatorname{Hom}(A, S)$$

is a bijection. This is equivalent by [Lemma 2.10](#) to asserting that  $\iota : A \rightarrow B$  induces an isomorphism of pro- $\mathcal{FS}(n)$  completions. Recall that by [Corollary 2.3](#) and [Lemma 2.12](#) statement  $\mathcal{S}(n)$  implies that for any homomorphism  $\beta : B \rightarrow S$  to a finite solvable group  $S$  with  $\ell(S) \leq n$  we have  $\text{Im}\{\beta \circ \iota : A \rightarrow B \rightarrow S\} = \text{Im}\{\beta : B \rightarrow S\}$ .

Our goal is to show that  $\mathcal{S}(n)$  holds for all  $n$ . We will show this by induction on  $n$ . For the induction argument we use the following auxiliary statement: Given  $n \in \mathbb{N} \cup \{0\}$  we denote by  $\mathcal{H}(n)$  the statement that for any homomorphism  $\beta : B \rightarrow S$  where  $S$  is finite solvable with  $\ell(S) \leq n$ , for  $\iota = \iota_-, \iota_+$  the homomorphism

$$\iota_* : H_1(A; \mathbb{Z}[S]) \rightarrow H_1(B; \mathbb{Z}[S])$$

is an isomorphism.

In the next two sections we will prove the following two propositions:

**PROPOSITION 3.3.** *If  $\mathcal{H}(n)$  and  $\mathcal{S}(n)$  hold, then  $\mathcal{S}(n + 1)$  holds as well.*

**PROPOSITION 3.4.** *Assume that  $(N, \phi)$  satisfies Condition (\*). If  $\mathcal{S}(n)$  holds, then  $\mathcal{H}(n)$  holds as well.*

We can now prove the following corollary, which amounts to [Proposition 1.7](#).

**COROLLARY 3.5.** *Assume that  $(N, \phi)$  satisfies Condition (\*) and that  $\phi$  is primitive. Let  $\Sigma \subset N$  be a connected Thurston norm-minimizing surface dual to  $\phi$  and let  $\iota : A \rightarrow B$  be one of the two injections. Then for any finite solvable group  $G$  the map*

$$\text{Hom}(B, G) \xrightarrow{\iota^*} \text{Hom}(A, G)$$

*is a bijection; i.e.,  $\iota : A \rightarrow B$  induces an isomorphism of prosolvable completions.*

*Proof.* The condition  $\mathcal{S}(0)$  holds by fiat. It follows from [Proposition 3.2](#) applied to the trivial group that if  $(N, \phi)$  satisfies Condition (\*), then  $\iota_{\pm} : H_1(A; \mathbb{Z}) \rightarrow H_1(B; \mathbb{Z})$  are isomorphisms, i.e.  $\mathcal{H}(0)$  holds. The combination of [Propositions 3.3](#) and [3.4](#) then shows that  $\mathcal{H}(n)$  and  $\mathcal{S}(n)$  hold for all  $n$ . The corollary is now immediate. □

**3.3. Proof of Proposition 3.3.** In this section we will prove [Proposition 3.3](#). Let  $\iota = \iota_-$  or  $\iota = \iota_+$ . Since  $\mathcal{S}(n)$  holds we only have to consider the case of  $G$  a finite solvable group with  $\ell(G) = n + 1$ . By definition  $G$  fits into a short exact sequence

$$1 \rightarrow I \rightarrow G \rightarrow S \rightarrow 1,$$

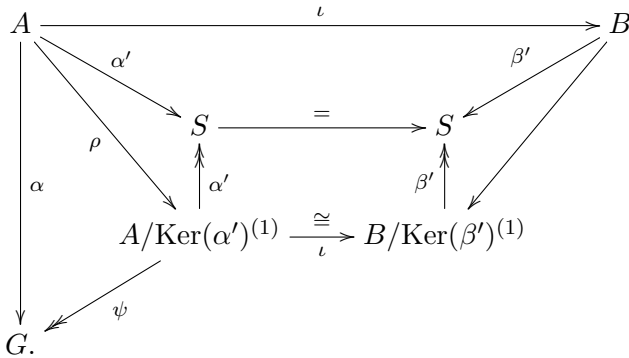
where  $I = G^{(n)}$  is finite abelian and  $S = G/G^{(n)}$  finite solvable with  $\ell(S) = n$ .

We will construct a map  $\Phi : \text{Hom}(A, G) \rightarrow \text{Hom}(B, G)$  which is an inverse to  $\iota^* : \text{Hom}(B, G) \rightarrow \text{Hom}(A, G)$ . Let  $\alpha : A \rightarrow G$  be a homomorphism. Without loss of generality we can assume that  $\alpha$  is an epimorphism. Denote  $A \xrightarrow{\alpha} G \rightarrow S$  by  $\alpha'$  and denote the map  $A \rightarrow A/\text{Ker}(\alpha')^{(1)}$  by  $\rho$ . Note that  $\alpha$  sends  $\text{Ker}(\alpha')$  to the abelian group  $I$ ; hence  $\alpha$  vanishes on  $\text{Ker}(\alpha')^{(1)}$ . This shows that  $\alpha$  factors through  $\rho$ , in particular  $\alpha = \psi \circ \rho$  for some  $\psi : A/\text{Ker}(\alpha')^{(1)} \rightarrow G$ .

Recall that  $\ell(S) = n$ , therefore by  $\mathcal{S}(n)$  we have that  $\alpha' : A \rightarrow S$  equals  $\iota^*(\beta')$  for some  $\beta' : B \rightarrow S$ . By Lemma 2.12,  $\mathcal{S}(n)$  guarantees that  $i_* : H_0(A; \mathbb{Z}[S]) \rightarrow H_0(B; \mathbb{Z}[S])$  is an isomorphism; on the other hand  $\mathcal{H}(n)$  asserts that  $i_* : H_1(A; \mathbb{Z}[S]) \rightarrow H_1(B; \mathbb{Z}[S])$  is an isomorphism as well. By Corollary 2.5 this implies that  $\iota$  induces an isomorphism

$$\iota : A/\text{Ker}(\alpha')^{(1)} \xrightarrow{\cong} B/\text{Ker}(\beta')^{(1)}.$$

The various homomorphisms can be summarized in the following commutative diagram:



Now we define  $\Phi(\alpha) \in \text{Hom}(B, G)$  to be the homomorphism

$$B \rightarrow B/\text{Ker}(\beta')^{(1)} \xrightarrow{\iota^{-1}} A/\text{Ker}(\alpha')^{(1)} \xrightarrow{\psi} G.$$

It is now straightforward to check that  $\Phi$  and  $\iota^*$  are inverses to each other.

3.4. Proof of Proposition 3.4. In this section we will prove Proposition 3.4.

Let  $\beta : B \rightarrow S$  be a homomorphism to a finite solvable group  $S$  with  $\ell(S) \leq n$ . If  $\beta$  extends to  $\pi_1(N)$ ,  $\mathcal{H}(n)$  will follow immediately from Proposition 3.2. In general  $\beta$  though will not extend; however, using  $\mathcal{S}(n)$  we will construct a homomorphism  $\pi = \langle B, t|_{t_-(A)} = t|_{t_+(A)}t^{-1} \rangle \rightarrow G$  to a finite group  $G$  which contains  $\beta : B \rightarrow S$  to get the required isomorphism.

We first need some notation. Given groups  $C$  and  $H$  we define

$$C(H) = \bigcap_{\gamma \in \text{Hom}(C, H)} \text{Ker}(\gamma).$$

We summarize a few properties of  $C(H) \subset C$  in the following lemma.

LEMMA 3.6. *Let  $C$  be a finitely generated group. Then the subgroup  $C(H) \subset C$  has the following properties:*

- (1)  $C(H) \subset C$  is normal and characteristic.
- (2) If  $H$  is finite and solvable, then  $C/C(H)$  is finite and solvable with  $\ell(C/C(H)) \leq \ell(H)$ .

*Proof.* Statement (1) is immediate. To prove the rest, consider the injection

$$C/C(H) = C / \bigcap_{\gamma \in \text{Hom}(C,H)} \text{Ker}(\gamma) \rightarrow \prod_{\gamma \in \text{Hom}(C,H)} C/\text{Ker}(\gamma).$$

If  $H$  is finite, then  $\text{Hom}(C, H)$  is a finite set (since  $C$  is finitely generated); hence  $C/C(H)$  is finite. If  $H$  is furthermore solvable, then for any  $\gamma \in \text{Hom}(C, H)$  the groups  $C/\text{Ker}(\gamma)$  are solvable, hence  $C/C(H)$  is solvable as well. Moreover for any  $\gamma \in \text{Hom}(C, H)$  we have  $\ell(C/\text{Ker}(\gamma)) \leq \ell(H)$ . We therefore get

$$\ell(C/C(H)) \leq \max_{\gamma \in \text{Hom}(C,H)} \ell(C/\text{Ker}(\gamma)) \leq \ell(H). \quad \square$$

We will also need the following group homomorphism extension lemma.

LEMMA 3.7. *Assume that  $\mathcal{S}(n)$  holds and that  $S$  is a finite solvable group with  $\ell(S) \leq n$ . Let  $\beta : B \rightarrow S$  be a homomorphism.*

*Then there exists a  $k \in \mathbb{N}$ , a semidirect product  $\mathbb{Z}/k \rtimes B/B(S)$  and a homomorphism*

$$\pi = \langle B, t | \iota_-(A) = t\iota_+(A)t^{-1} \rangle \rightarrow \mathbb{Z}/k \rtimes B/B(S)$$

*which extends  $B \rightarrow B/B(S)$ , i.e. we have the following commutative diagram:*

$$\begin{array}{ccccccc} 1 & \longrightarrow & B/B(S) & \longrightarrow & \mathbb{Z}/k \rtimes B/B(S) & \longrightarrow & \mathbb{Z}/k \longrightarrow 1 \\ & & \uparrow & & \uparrow & & \\ & & B & \longrightarrow & \pi & & \end{array}$$

*Proof.* Assume that  $\mathcal{S}(n)$  holds and that  $S$  is a finite solvable group with  $\ell(S) \leq n$ . Let  $\beta : B \rightarrow S$  be a homomorphism. We denote the projection map  $B \rightarrow B/B(S)$  by  $\rho$ .

*Claim.* There exists an automorphism  $\gamma : B/B(S) \rightarrow B/B(S)$  such that  $\rho(\iota_+(a)) = \gamma(\rho(\iota_-(a)))$  for all  $a \in A$ .

Let  $\iota = \iota_-$  or  $\iota = \iota_+$ . By Lemma 3.6 we know that  $B/B(S)$  is finite solvable with  $\ell(B/B(S)) \leq n$ . It follows from  $\mathcal{S}(n)$  that

$$\iota_* : A/\text{Ker}\{A \xrightarrow{\iota} B \xrightarrow{\rho} B/B(S)\} \rightarrow B/B(S)$$

is an isomorphism. On the other hand it is also a straightforward consequence of  $\mathcal{S}(n)$  that

$$\text{Ker}\{A \xrightarrow{\iota} B \xrightarrow{\rho} B/B(S)\} = A(S).$$

Combining these two observations we see that  $\iota$  gives rise to an isomorphism  $\iota_* : A/A(S) \rightarrow B/B(S)$ . We now take  $\gamma := \iota_{+*} \circ (\iota_{-*})^{-1}$ . This concludes the proof of the claim.

We now write  $H = B/B(S)$ . It is now straightforward to verify that

$$\begin{aligned} \pi = \langle B, t | \iota_{-}(A) = t \iota_{+}(A) t^{-1} \rangle &\rightarrow \mathbb{Z} \rtimes H = \langle H, t | H = t \gamma(H) t^{-1} \rangle \\ b &\mapsto \rho(b), \quad b \in B, \\ t &\mapsto t \end{aligned}$$

defines a homomorphism. Since  $H = B/B(S)$  is a finite group it follows that the automorphism  $\gamma$  has finite order  $k$ ; in particular the projection map  $\mathbb{Z} \rtimes B/B(S) \rightarrow \mathbb{Z}/k \rtimes B/B(S)$  is a homomorphism. Clearly the resulting homomorphism  $\pi \rightarrow \mathbb{Z}/k \rtimes B/B(S)$  has all the required properties.  $\square$

We are in position now to prove [Proposition 3.4](#).

*Proof of Proposition 3.4.* Recall that we assume that  $(N, \phi)$  satisfies Condition  $(*)$  and that  $\mathcal{S}(n)$  holds. We have to show that  $\mathcal{H}(n)$  holds as well. So let  $\beta : B \rightarrow S$  be a homomorphism to a finite solvable group  $S$  with  $\ell(S) \leq n$ . We have to show that for  $\iota = \iota_{-}, \iota_{+}$  the homomorphism

$$\iota_* : H_1(A; \mathbb{Z}[S]) \rightarrow H_1(B; \mathbb{Z}[S])$$

is an isomorphism. Without loss of generality we can assume that  $\beta$  is surjective. Recall that  $\mathcal{S}(n)$  implies that  $\beta \circ \iota : A \rightarrow S$  is surjective as well.

We now apply [Lemma 3.7](#) to find a homomorphism

$$\pi = \langle B, t | \iota_{-}(A) = t \iota_{+}(A) t^{-1} \rangle \rightarrow \mathbb{Z}/k \rtimes B/B(S)$$

which extends  $B \rightarrow B/B(S)$ . Note that

$$\begin{aligned} (2) \quad \text{Ker}\{\gamma : B \rightarrow \pi \rightarrow \mathbb{Z}/k \rtimes B/B(S)\} &= \text{Ker}\{B \rightarrow B/B(S)\}, \\ \text{Ker}\{\gamma \circ \iota : A \rightarrow B \rightarrow \pi \rightarrow \mathbb{Z}/k \rtimes B/B(S)\} &= \text{Ker}\{\iota : A \rightarrow B \rightarrow B/B(S)\}. \end{aligned}$$

We let

$$\begin{aligned} \widehat{B} &= \text{Ker}\{B \rightarrow B/B(S)\}, \\ \widetilde{B} &= \text{Ker}(\beta). \end{aligned}$$

Clearly  $\widehat{B} \subset \widetilde{B}$  by the definition of  $B/B(S)$ . We also write  $\widehat{A} = \iota^{-1}(\widehat{B})$  and  $\widetilde{A} = \iota^{-1}(\widetilde{B})$ . We now consider the epimorphism  $\pi_1(N) = \pi \rightarrow \mathbb{Z}/k \rtimes B/B(S)$ . By Condition  $(*)$ , [equation \(2\)](#), [Proposition 3.2](#) and [Corollaries 2.5 and 2.5](#) it follows that the maps

$$\iota : A/\widehat{A} \rightarrow B/\widehat{B} \quad \text{and} \quad \iota : A/[\widehat{A}, \widehat{A}] \rightarrow B/[\widehat{B}, \widehat{B}]$$

are isomorphisms. It now follows from [Lemma 2.7](#) and [Corollary 2.5](#) that the maps

$$\iota : A/\tilde{A} \rightarrow B/\tilde{B} \text{ and } \iota : A/[\tilde{A}, \tilde{A}] \rightarrow B/[\tilde{B}, \tilde{B}]$$

are also isomorphisms. □

#### 4. A product criterion

In this section we will apply a theorem of Agol to prove a criterion for a manifold to be a product, which complements [Proposition 1.7](#).

In order to state our result, we first recall the definition of a sutured manifold (cf. [\[Gab83, Def. 2.6\]](#) or [\[CC03, p. 364\]](#)). A *sutured manifold*  $(M, \gamma)$  is a compact oriented 3-manifold  $M$  together with a set  $\gamma \subset \partial M$  of pairwise disjoint annuli  $A(\gamma)$  and tori  $T(\gamma)$ . Furthermore, the structure of a sutured manifold consists of the following choices of orientations:

- (1) For each  $A \in A(\gamma)$  a choice of a simple closed, oriented curve in  $A$  (called *suture*) such that  $A$  is the tubular neighborhood of the curve, and
- (2) the choice of an orientation for each component of  $\partial M \setminus A(\gamma)$ .

The orientations must be compatible, i.e. the orientation of the components of  $\partial M \setminus A(\gamma)$  must be coherent with the orientations of the sutures.

Given a sutured manifold  $(M, \gamma)$  we define  $R_+(\gamma)$  as the components of  $\overline{\partial M \setminus \gamma}$  where the orientation agrees with the orientation induced by  $M$  on  $\partial M$ , and  $R_-(\gamma)$  as the components of  $\overline{\partial M \setminus \gamma}$  where the two orientations disagree. We define also  $R(\gamma) = R_+(\gamma) \cup R_-(\gamma)$ .

A sutured manifold  $(M, \gamma)$  is called *taut* if  $M$  is irreducible and if each component of  $R(\gamma)$  is incompressible and Thurston norm-minimizing in  $H_2(M, \gamma; \mathbb{Z})$  (we refer to [\[Sch89\]](#) for information regarding the Thurston norm on sutured manifolds).

An example of a taut sutured manifold is given by taking an oriented surface  $\Sigma$  and considering  $\Sigma \times I$  with sutures given by the annuli  $\partial \Sigma \times I$ . The sutures are oriented by the orientation of  $\partial \Sigma$ . We can pick orientations such that  $R_-(\gamma) = \Sigma \times 0$  and  $R_+(\gamma) = \Sigma \times 1$ . If a sutured manifold  $(M, \gamma)$  is diffeomorphic (as a sutured manifold) to such a product, then we say that  $(M, \gamma)$  is a *product sutured manifold*.

Another example of a taut sutured manifold comes from consideration of an oriented incompressible Thurston norm-minimizing surface  $\Sigma \subset N$  in an irreducible 3-manifold with empty or toroidal boundary. We let  $(M, \gamma) = (N \setminus \nu \Sigma, \partial N \cap (N \setminus \nu \Sigma))$ . With appropriate orientations  $(M, \gamma)$  is a taut sutured manifold such that  $R_-(\gamma) = \Sigma^-$  and  $R_+(\gamma) = \Sigma^+$ .

The following theorem immediately implies [Theorem 1.8](#).



**THEOREM 4.1.** *Assume there is a taut sutured manifold  $(M, \gamma)$  which has the following properties:*

- (1)  $R_{\pm}(\gamma)$  consist of one component  $\Sigma^{\pm}$  each, and the inclusion induced maps  $\pi_1(\Sigma^{\pm}) \rightarrow \pi_1(M)$  give rise to isomorphisms of the respective prosolvable completions;
- (2)  $\pi_1(M)$  is residually finite solvable.

*Then  $(M, \gamma)$  is a product sutured manifold.*

The key ingredient to the proof of [Theorem 4.1](#) is a result of Agol’s [\[Ago08\]](#) which we recall in [Section 4.1](#). We will then provide the proof for [Theorem 4.1](#) in [Sections 4.2 and 4.3](#).

*Remarks.* (1) It is an immediate consequence of ‘peripheral subgroup separability’ [\[LN91\]](#) that the theorem holds under the assumption that the inclusion induced maps  $\pi_1(\Sigma^{\pm}) \rightarrow \pi_1(M)$  give rise to isomorphisms of the respective profinite completions. It is not clear how the approach of [\[LN91\]](#) can be adapted to prove [Theorem 4.1](#).

(2) It is also interesting to compare [Theorem 4.1](#) with a result of Grothendieck. In [\[Gro70, §3.1\]](#) Grothendieck proved that if  $\varphi : A \rightarrow B$  is a homomorphism between finitely presented, residually finite groups which induces an isomorphism of the profinite completions, and if  $A$  is arithmetic (e.g. a surface group), then  $\varphi$  is an isomorphism. It is an interesting question whether [Theorem 4.1](#) can be proved using purely group theoretic arguments. We refer to [\[AdlHKŠ07\]](#) for more information regarding this question.

**4.1. Agol’s theorem.** Before we can state Agol’s result we have to introduce more definitions. A group  $G$  is called *residually finite  $\mathbb{Q}$ -solvable* or RFRS if there exists a filtration of groups  $G = G_0 \supset G_1 \supset G_2 \cdots$  such that the following hold:

- (1)  $\cap_i G_i = \{1\}$ ,
- (2)  $G_i$  is a normal, finite index subgroup of  $G$  for any  $i$ ,
- (3) for any  $i$  the map  $G_i \rightarrow G_i/G_{i+1}$  factors through  $G_i \rightarrow H_1(G_i; \mathbb{Z})/\text{torsion}$ .

Note that RFRS groups are in particular residually finite solvable, but the RFRS condition is considerably stronger than being residually finite solvable. The notion of an RFRS group was introduced by Agol [\[Ago08\]](#) whose paper we refer to for more information on RFRS groups.

Given a sutured manifold  $(M, \gamma)$ , the double  $DM_{\gamma}$  is defined to be the double of  $M$  along  $R(\gamma)$ , i.e.  $DM_{\gamma} = M \cup_{R(\gamma)} M$ . Note that the annuli  $A(\gamma)$  give rise to toroidal boundary components of  $DM_{\gamma}$ . By  $r : DM_{\gamma} \rightarrow M$ , we denote the retraction map given by ‘folding’ the two copies of  $M$  along  $R(\gamma)$ .

We are now in a position to state Agol’s result. The theorem as stated here is clearly implicit in the proof of [\[Ago08, Th. 6.1\]](#).

**THEOREM 4.2.** *Let  $(M, \gamma)$  be a connected, taut sutured manifold such that  $\pi_1(M)$  satisfies property RFRS. Write  $W = DM_\gamma$ . Then there exists an epimorphism  $\alpha : \pi_1(M) \rightarrow S$  to a finite solvable group, such that in the covering  $p : \widetilde{W} \rightarrow W$  corresponding to  $\alpha \circ r_* : \pi_1(W) \rightarrow S$  the pull back of the class  $[R_-(\gamma)] \in H_2(W, \partial W; \mathbb{Z})$  lies on the closure of the cone over a fibered face of  $\widetilde{W}$ .*

Note that  $[R_+(\gamma)] = \pm[R_-(\gamma)]$  in  $H_2(W, \partial W; \mathbb{Z})$ ; i.e.  $[R_-(\gamma)]$  is a fibered class if and only if  $[R_+(\gamma)]$  is a fibered class. In case  $\widetilde{W}$  has vanishing Thurston norm, we adopt the usual convention that by the fibered face we actually mean  $H^1(\widetilde{W}, \mathbb{R}) \setminus \{0\}$ .

4.2. *Proof of Theorem 4.1.* From now on assume we have a taut sutured manifold  $(M, \gamma)$  with the following properties:

- (1)  $R_\pm(\gamma)$  consist of one component  $\Sigma^\pm$  each and the inclusion induced maps  $\pi_1(\Sigma^\pm) \rightarrow \pi_1(M)$  give rise to isomorphisms of the respective prosolvable completions.
- (2)  $\pi_1(M)$  is residually finite solvable.

Since Theorem 4.1 is obvious in the case  $M = S^2 \times [0, 1]$  we will henceforth assume that  $M \neq S^2 \times [0, 1]$ .

Our main tool in proving Theorem 4.1 is Theorem 4.2. In order to apply it we need the following claim.

*Claim.* The group  $\pi_1(M)$  is RFRS.

*Proof.* By assumption the group  $\pi_1(M)$  is residually finite solvable. This means that we can find a sequence  $\pi_1(M) = B_0 \supset B_1 \supset B_2 \dots$  with the following properties:

- (1)  $\cap_i B_i = \{1\}$ ;
- (2)  $B_i$  is a normal, finite index subgroup of  $\pi_1(M)$  for any  $i$ ;
- (3) for any  $i$ , the map  $B_i \rightarrow B_i/B_{i+1}$  factors through  $B_i \rightarrow H_1(B_i; \mathbb{Z})$ .

It remains to show that  $B_i \rightarrow B_i/B_{i+1}$  factors through  $H_1(B_i; \mathbb{Z})/\text{torsion}$ . In fact we claim that  $H_1(B_i; \mathbb{Z})$  is torsion-free. Indeed, first note that by Shapiro’s lemma  $H_1(B_i; \mathbb{Z}) \cong H_1(B; \mathbb{Z}[B/B_i]) \cong H_1(M; \mathbb{Z}[B/B_i])$ . Furthermore, by Lemma 2.12 we have

$$H_1(\Sigma^-; \mathbb{Z}[B/B_i]) \xrightarrow{\cong} H_1(M; \mathbb{Z}[B/B_i]),$$

but the first group is clearly torsion-free as it is the homology of a finite cover of a surface. □

In the following,  $W = DM_\gamma$ . By the above claim we can apply Theorem 4.2 which says that there exists an epimorphism  $\alpha : \pi_1(M) \rightarrow S$  to a finite solvable group, such that in the covering  $p : \widetilde{W} \rightarrow W$  corresponding to

$\alpha \circ r_* : \pi_1(W) \rightarrow S$  the pull back of the class  $[R_-(\gamma)] = [\Sigma^-] \in H_2(W, \partial W; \mathbb{Z})$  lies on the closure of the cone over of a fibered face of  $\widetilde{W}$ .

Note that we can view  $\widetilde{W}$  as the double of the cover  $(\widetilde{M}, \tilde{\gamma})$  of  $(M, \gamma)$  induced by  $\alpha : \pi_1(M) \rightarrow S$ . We summarize the main properties of  $\widetilde{\Sigma}^\pm$  and  $\widetilde{W}$  in the following lemma.

- LEMMA 4.3. (1)  $\widetilde{\Sigma}^\pm := p^{-1}(\Sigma^\pm)$  are connected surfaces;  
 (2) The inclusion induced maps  $\pi_1(\widetilde{\Sigma}^\pm) \rightarrow \pi_1(\widetilde{M})$  give rise to isomorphisms of prosolvable completions;  
 (3) If  $\widetilde{\Sigma}^-$  is the fiber of a fibration  $\widetilde{W} = D\widetilde{M}_{\tilde{\gamma}} \rightarrow S^1$ , then  $\widetilde{M}$  is a product over  $\widetilde{\Sigma}^-$ ;  
 (4)  $M$  is a product over  $\Sigma^-$  if and only if  $\widetilde{M}$  is a product over  $\widetilde{\Sigma}^-$ .

*Proof.* First note that it follows from Lemma 2.12 and Corollary 2.3 and the assumption that  $\pi_1(\Sigma^\pm) \rightarrow \pi_1(M)$  give rise to isomorphisms of the respective prosolvable completions that  $\pi_1(\Sigma^\pm) \rightarrow \pi_1(M) \rightarrow S$  is surjective, i.e. the preimages  $\widetilde{\Sigma}^\pm := p^{-1}(\Sigma^\pm)$  are connected. The second claim follows from Lemma 2.11 since the maps  $\pi_1(\Sigma^\pm) \rightarrow \pi_1(M)$  give rise to isomorphisms of their prosolvable completions.

For the third claim consider the following commutative diagram

$$\begin{array}{ccc}
 \pi_1(\widetilde{\Sigma}^-) & \xrightarrow{\quad\quad\quad} & \pi_1(\widetilde{W} \setminus \nu\widetilde{\Sigma}^-) \\
 & \searrow & \nearrow \\
 & \pi_1(\widetilde{M}). &
 \end{array}$$

If  $\widetilde{\Sigma}^-$  is the fiber of a fibration  $D\widetilde{M}_{\tilde{\gamma}} \rightarrow S^1$ , then the top map in the above commutative diagram is an isomorphism. We can think of  $\widetilde{W} \setminus \nu\widetilde{\Sigma}^-$  as  $\widetilde{M} \cup_{\widetilde{\Sigma}^+} \widetilde{M}$ . It is now clear that the lower two maps are injective. But then the lower left map also has to be an isomorphism; i.e.,  $\widetilde{M}$  is a product over  $\widetilde{\Sigma}^-$ . The last claim is well-known; it is for example a consequence of [Hem76, Th. 10.5].  $\square$

By the above lemma it is now clear that the following lemma implies Theorem 4.1.

LEMMA 4.4. Let  $(M, \gamma)$  be a taut sutured manifold such that  $R_\pm(\gamma)$  consist of one component  $\Sigma^\pm$  each. Assume the following hold:

- (A) The inclusion induced maps  $\pi_1(\Sigma^\pm) \rightarrow \pi_1(M)$  give rise to isomorphisms of the respective prosolvable completions.  
 (B) The class in  $H^1(DM_\gamma; \mathbb{Z})$  represented by the surface  $\Sigma^-$  lies on the closure of the cone over a fibered face of  $DM_\gamma$ .

Then  $\Sigma^-$  is the fiber of a fibration  $DM_\gamma \rightarrow S^1$ .

In the following we write  $W = DM_\gamma$ . Note that we have a canonical involution  $\tau$  on  $W$  with fixed point set  $R(\gamma)$ . From now on we think of  $W = DM_\gamma$  as  $M \cup_{R(\gamma)} \tau(M)$ .

Our main tool in the proof of [Lemma 4.4](#) will be the interplay between the Thurston norm and McMullen’s Alexander norm [[McM02](#)]. Recall that given a 3-manifold  $V$  with  $b_1(V) \geq 2$  the Alexander norm  $\| - \|_A : H^1(V; \mathbb{R}) \rightarrow \mathbb{R}_{\geq 0}$  has the following properties:

- (a) The Alexander norm ball is dual to the Newton polyhedron defined by the symmetrized Alexander polynomial  $\Delta_V \in \mathbb{Z}[H_1(V; \mathbb{Z})/\text{torsion}]$ .
- (b) The Alexander norm ball is a (possibly noncompact) polyhedron with finitely many faces.
- (c) For any  $\phi \in H^1(V; \mathbb{R})$  we have  $\|\phi\|_A \leq \|\phi\|_T$ , and equality holds for fibered classes.
- (d) Let  $C \subset H^1(V; \mathbb{R})$  be a fibered cone, i.e. the cone on a fibered face of the Thurston norm ball; then  $C$  is contained in the cone on the interior of a top-dimensional face of the Alexander norm ball.
- (e) When  $C_1, C_2 \subset H^1(V; \mathbb{R})$  are fibered cones contained in the same cone on the interior of a top-dimensional face of the Alexander norm ball, then  $C_1 = C_2$ .

Our assumption that the induced maps  $\pi_1(\Sigma^\pm) \rightarrow \pi_1(M)$  give rise to isomorphisms of the respective prosolvable completions implies that  $W = DM_\gamma$  ‘looks algebraically’ the same as  $\Sigma^- \times S^1$ . More precisely, we have the following lemma which to be proved in [Section 4.3](#).

**LEMMA 4.5.** *Let  $(M, \gamma)$  be a taut, sutured manifold with the property that  $R_\pm(\gamma)$  consist of one component  $\Sigma^\pm$  each. Assume that (A) holds. Then the following hold:*

- (1) *There exists an isomorphism*

$$f : \mathbb{R} \oplus H_1(\Sigma^-, \partial\Sigma^-; \mathbb{R}) \rightarrow H_2(W, \partial W; \mathbb{R})$$

*such that  $f(1, 0) = [\Sigma^-]$  and such that  $\tau(f(r, h)) = f(r, -h)$ .*

- (2) *The class  $\phi = PD(\Sigma^-) \in H^1(W; \mathbb{Z})$  lies in the cone  $D$  on the interior of a top-dimensional face of the Alexander norm ball.*

Note that (1) implies in particular that  $b_1(W) \geq 2$ . Assuming this lemma we are now in a position to prove [Lemma 4.4](#).

*Proof of Lemma 4.4.* Let  $(M, \gamma)$  be a taut sutured manifold with the property that  $R_\pm(\gamma)$  consist of one component  $\Sigma^\pm$  each. Assume that (A) and (B) hold.

By [Lemma 4.5](#) there exists a cone  $D \subset H^1(W; \mathbb{R})$  on the interior of a top-dimensional face of the Alexander norm ball which contains  $\phi = PD([\Sigma^-])$ .

We denote the map

$$\mathbb{R} \oplus H_1(\Sigma^-, \partial\Sigma^-; \mathbb{R}) \xrightarrow{f} H_2(W, \partial W; \mathbb{R}) \xrightarrow{PD} H^1(W; \mathbb{R})$$

by  $\Phi$ .

By (B) we can find  $h \in H_1(\Sigma^-, \partial\Sigma^-; \mathbb{R})$  such that  $\Phi(1, h)$  and  $\Phi(1, -h)$  lie in  $D$  and such that  $\Phi(1, h)$  lies in the cone  $C$  on a fibered face  $F$  of the Thurston norm ball. Note that  $\tau_* : H^1(W; \mathbb{R}) \rightarrow H^1(W; \mathbb{R})$  sends fibered classes to fibered classes and preserves the Thurston norm. In particular  $\tau(\Phi(1, h)) = \Phi(1, -h)$  is fibered as well and it lies in  $\tau(C)$  which is the cone on the fibered face  $\tau(F)$  of the Thurston norm ball. Recall that  $\tau(\Phi(1, h)) = \Phi(1, -h)$  lies in  $D$ ; it follows from Property (d) of the Alexander norm that  $\tau(C) \subset D$ . We then use (e) to conclude that  $C = \tau(C)$ . In particular  $\Phi(1, h)$  and  $\Phi(1, -h)$  lie in  $C$ . Since  $C$  is convex it follows that  $\phi = \Phi(1, 0) \in C$ , i.e.,  $\phi$  is a fiber class. □

4.3. *Proof of Lemma 4.5.* By Lemma 2.12 the following lemma is just the first statement of Lemma 4.5.

LEMMA 4.6. *Let  $(M, \gamma)$  be a taut sutured manifold with the property that  $R_{\pm}(\gamma)$  consist of one component  $\Sigma^{\pm}$  each. Assume that  $\iota_{\pm} : H_1(\Sigma^{\pm}; \mathbb{R}) \rightarrow H_1(M; \mathbb{R})$  are isomorphisms. Then there exists an isomorphism*

$$f : \mathbb{R} \oplus H_1(\Sigma^-, \partial\Sigma^-; \mathbb{R}) \rightarrow H_2(W, \partial W; \mathbb{R})$$

such that  $f(1, 0) = [\Sigma^-]$  and such that  $\tau(f(b, c)) = f(b, -c)$ .

*Proof.* We start out with the following two claims.

*Claim.*  $M$  has no toroidal sutures.

*Proof.* Denote the toroidal sutures by  $T_1, \dots, T_n$ . Recall that for any compact 3-manifold  $X$  we have  $b_1(\partial X) \leq 2b_1(X)$ . In our case it is easy to see that we have  $b_1(\partial M) = b_1(\Sigma^-) + b_1(\Sigma^+) + \sum_{i=1}^n b_1(T_i) = 2b_1(\Sigma) + 2n$ . On the other hand, since  $H_1(\Sigma^{\pm}; \mathbb{R}) \rightarrow H_1(M; \mathbb{R})$  are isomorphisms we have  $b_1(M) = b_1(\Sigma)$ . It now follows from  $b_1(\partial M) \leq 2b_1(M)$  that  $n = 0$ . □

*Claim.* The inclusion induced maps  $H_1(\Sigma^{\pm}, \partial\Sigma^{\pm}; \mathbb{R}) \rightarrow H_1(M, A(\gamma); \mathbb{R})$  are isomorphisms.

Now consider the following commutative diagram:

$$\begin{CD} H_1(\partial\Sigma^-; \mathbb{R}) @>>> H_1(\Sigma^-; \mathbb{R}) @>>> H_1(\Sigma^-, \partial\Sigma^-; \mathbb{R}) @>>> H_0(\partial\Sigma^-; \mathbb{R}) @>>> H_0(\Sigma^-; \mathbb{R}) \\ @VVV @VV\cong V @VVV @VVV @VV\cong V \\ H_1(A(\gamma); \mathbb{R}) @>>> H_1(M; \mathbb{R}) @>>> H_1(M, A(\gamma); \mathbb{R}) @>>> H_0(A(\gamma); \mathbb{R}) @>>> H_0(M; \mathbb{R}). \end{CD}$$

Note that by the compatibility condition in the definition of sutured manifolds we have that for each component  $A$  of  $A(\gamma)$  the subset  $\partial A \cap \Sigma^- = A \cap \partial\Sigma^- \subset \partial A$  consists of exactly one boundary component of  $A$ . This implies that the

maps  $H_i(\partial\Sigma^- \cap A; \mathbb{R}) \rightarrow H_i(A; \mathbb{R})$  are isomorphisms. The claim now follows immediately from the above commutative diagram and from the assumption that  $H_1(\Sigma^\pm; \mathbb{R}) \rightarrow H_1(M; \mathbb{R})$  are isomorphisms.

We now define

$$g : H_1(\Sigma^-, \partial\Sigma^-; \mathbb{R}) \rightarrow H_2(W, \partial W; \mathbb{R})$$

as follows: given an element  $c \in H_1(\Sigma^-, \partial\Sigma^-; \mathbb{R})$  represent it by a chain  $c^-$ . Since the maps  $H_1(\Sigma^\pm, \partial\Sigma^\pm; \mathbb{R}) \rightarrow H_1(M, A(\gamma); \mathbb{R})$  are isomorphisms we can find a chain  $c^+$  in  $\Sigma^+$  such that  $[c^-] = [c^+] \in H_2(M, A(\gamma); \mathbb{R})$ . Now let  $d$  be a 2-chain in  $M$  such that  $\partial d = c^- \cup -c^+$ . Then define  $g(c)$  to be the element in  $H_2(W, \partial W; \mathbb{R})$  represented by the closed 2-chain  $d \cup -\tau(d)$ . It is easy to verify that  $g$  is a well-defined homomorphism. Note that  $\partial W = A(\gamma) \cup \tau(A(\gamma))$  since  $W$  has no toroidal sutures. It is now straightforward to check, using a Mayer-Vietoris sequence, that the map

$$\begin{aligned} f : \mathbb{R} \oplus H_1(\Sigma^-, \partial\Sigma^-; \mathbb{R}) &\rightarrow H_2(W, \partial W; \mathbb{R}) \\ (b, c) &\mapsto b[\Sigma^-] + g(c) \end{aligned}$$

is an isomorphism. Clearly  $f(1, 0) = [\Sigma^-]$ . It is also easy to verify that  $\tau(f(b, c)) = f(b, -c)$ . This shows that  $f$  has all the required properties.  $\square$

The second statement of [Lemma 4.5](#) is more intricate. We start with the following lemma which in light of [[Gro70](#)], [[BG04](#)] and [[AdlHKŠ07](#)] has perhaps some independent interest.

**LEMMA 4.7.** *Let  $\varphi : A \rightarrow B$  be a homomorphism of finitely generated metabelian groups which induces an isomorphism of prosolvable completions. Then  $\varphi$  is also an isomorphism.*

*Proof.* We first show that  $A \rightarrow B$  is an injection. We consider the following commutative diagram:

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \downarrow \\ \widehat{A}_{\mathcal{FS}} & \longrightarrow & \widehat{B}_{\mathcal{FS}}. \end{array}$$

The vertical maps are injections since metabelian groups are residually finite (cf. [[Hal59](#)]). The bottom map is an isomorphism by assumption. It now follows that the top map is an injection.

Now suppose that the homomorphism  $A \rightarrow B$  is not surjective. We identify  $H_1(A; \mathbb{Z}) = H_1(A; \mathbb{Z}) \xrightarrow{\cong} H_1(B; \mathbb{Z}) = H_1(B; \mathbb{Z})$  via  $\varphi$  and refer to the group as  $H$ . Let  $g' \in B \setminus \varphi(A)$ . We can pick an  $a \in A$  such that  $\varphi(a)$  and  $g'$  represent the same element in  $H$ . Let  $g = \varphi(a)^{-1}g'$ . Then  $g$  represents the trivial element in  $H$  but  $g$  is also an element in  $B \setminus \varphi(A)$ .

We will show that there exists a homomorphism  $\alpha : B \rightarrow G$  to a finite metabelian group such that  $\alpha$  separates  $g$  from  $\varphi(A)$ , i.e. such that  $\alpha(g) \notin \alpha(\varphi(A))$ . This then immediately contradicts, via [Lemma 2.10](#), our assumption

that  $\varphi : A \rightarrow B$  induces an isomorphism of prosolvable completions. Our construction of finding such  $\alpha$  builds on some ideas in the proof of [LN91, Th. 1].

We write  $B_1 = B_2 = B$  and denote the inclusion maps  $A \rightarrow B_i = B$  by  $\varphi_i$ . We let  $C = B_1 *_A B_2$ . It is then easy to see that the homomorphisms  $B_i \rightarrow C$  give rise to an isomorphism  $H_1(B_i; \mathbb{Z}) = H \rightarrow H_1(C; \mathbb{Z})$ . Denote by

$$s : B_1 *_A B_2 \rightarrow B_1 *_A B_2$$

the switching map, i.e. the map induced by  $s(b) = b \in B_2$  for  $b \in B_1$  and  $s(b) = b \in B_1$  for  $b \in B_2$ . Note that  $s$  acts as the identity on  $A \subset C$ . Also note that  $s$  descends to a map  $C/C^{(2)} \rightarrow C/C^{(2)}$  which we also denote by  $s$ . We now view  $g$  as an element in  $B_1$  and hence as an element in  $C$ . Note that the fact that  $g$  represents the trivial element in  $H$  implies that  $g$  represents an element in  $C^{(1)}/C^{(2)}$ . We will first show that  $s(g) \neq g \in C/C^{(2)}$ . Consider the following commutative diagram of exact sequences

$$\begin{array}{ccccccc}
 H_1(A; \mathbb{Z}[H]) & \rightarrow & H_1(B_1; \mathbb{Z}[H]) \oplus H_1(B_2; \mathbb{Z}[H]) & \rightarrow & H_1(C; \mathbb{Z}[H]) & \rightarrow & 0 \\
 \downarrow \cong & & \downarrow \cong & & \downarrow \cong & & \downarrow \\
 A^{(1)} & \rightarrow & B_1^{(1)} \times B_2^{(1)} & \rightarrow & C^{(1)}/C^{(2)} & \rightarrow & 0. \\
 h & \mapsto & (\varphi_1(h), \varphi_2(h)^{-1}) & & & & 
 \end{array}$$

Since  $g \in B_1^{(1)} \setminus \varphi(A^{(1)})$  it follows that  $(g, g^{-1})$  does not lie in the image of  $A^{(1)}$  in  $B_1^{(1)} \times B_2^{(1)}$ . It therefore follows from the above diagram that  $gs(g)^{-1} \neq e \in C^{(1)}/C^{(2)}$ .

Note that  $C/C^{(2)}$  is metabelian, and hence by [Hal59] residually finite. We can therefore find an epimorphism  $\alpha : C/C^{(2)} \rightarrow G$  onto a finite group  $G$  (which is necessarily metabelian) such that  $\alpha(gs(g)^{-1}) \neq e$ . Now consider  $\beta : C/C^{(2)} \rightarrow G \times G$  given by  $\beta(h) = (\alpha(h), \alpha(s(h)))$ . Then clearly  $\beta(g) \notin \beta(A) \subset \{(g, g) \mid g \in G\}$ . The restriction of  $\beta$  to  $B = B_1$  now clearly separates  $g$  from  $A$ . □

**COROLLARY 4.8.** *Let  $\varphi : A \rightarrow B$  be a homomorphism of finitely generated groups which induces an isomorphism of prosolvable completions. Then the induced map  $A/A^{(2)} \rightarrow B/B^{(2)}$  is an isomorphism.*

*Proof.* It follows immediately from Lemma 2.10 that  $\varphi$  induces an isomorphism of the prosolvable completions of the metabelian groups  $A/A^{(2)}$  and  $B/B^{(2)}$ . It now follows from Lemma 4.7 that the induced map  $A/A^{(2)} \rightarrow B/B^{(2)}$  is an isomorphism. □

We now turn to the proof of the second claim of Lemma 4.5. For the remainder of this section let  $(M, \gamma)$  be a taut sutured manifold with the property that  $R_{\pm}(\gamma)$  consist of one component  $\Sigma^{\pm}$  each. We assume that (A) holds,

i.e. the inclusion induced maps  $\pi_1(\Sigma^\pm) \rightarrow \pi_1(M)$  give rise to isomorphisms of the respective prosolvable completions. We have to show that the class  $\phi = PD(\Sigma^-) \in H^1(W; \mathbb{Z})$  lies in the cone on the interior of a top-dimensional face of the Alexander norm ball.

For the remainder of the section we pick a base point  $x^- \in \Sigma^-$  and a base point  $x^+ \in \Sigma^+$ . We endow  $W, M$  and  $\tau(M)$  with the base point  $x^-$ . Furthermore we pick a path  $\gamma$  in  $M$  connecting  $x^- \in \Sigma^-$  to  $x^+ \in \Sigma^+$ .

Our goal is to understand the Alexander norm ball of  $W$ . In order to do this we first have to study  $H = H_1(W; \mathbb{Z})$ . Let  $t$  denote the element in  $H$  represented by the closed path  $\gamma \cup -\tau(\gamma)$ . It follows from a straightforward Mayer-Vietoris sequence argument that we have an isomorphism

$$\begin{aligned} f : H_1(\Sigma^-; \mathbb{Z}) \oplus \langle t \rangle &\rightarrow H_1(W; \mathbb{Z}) \\ (b, t^k) &\mapsto \iota(b) + kt. \end{aligned}$$

In particular  $H$  is torsion-free. We write  $F = H_1(\Sigma^-; \mathbb{Z})$  and use  $f$  to identify  $H$  with  $F \times \langle t \rangle$  and to identify  $\mathbb{Z}[H]$  with  $\mathbb{Z}[F][t^{\pm 1}]$ .

Recall that the module  $H_1(W; \mathbb{Z}[H])$ . Recall that  $H_1(W; \mathbb{Z}[H])$  is the homology of the covering of  $W$  corresponding to  $\pi_1(W) \rightarrow H_1(W; \mathbb{Z}) = H$  together with the  $\mathbb{Z}[H]$ -module structure given by deck transformations.

In the following claim we compare  $W$  with  $\Sigma \times S^1$ . We also write  $F = H_1(\Sigma; \mathbb{Z})$  and we can identify  $H_1(\Sigma \times S^1; \mathbb{Z})$  with  $H = F \times \langle t \rangle$ . In particular we identify  $H_1(\Sigma \times S^1; \mathbb{Z})$  with  $H_1(W; \mathbb{Z})$ . With these identifications we can now state the following lemma.

LEMMA 4.9. *The  $\mathbb{Z}[H]$ -module  $H_1(W; \mathbb{Z}[H])$  is isomorphic to the  $\mathbb{Z}[H]$ -module  $H_1(\Sigma \times S^1; \mathbb{Z}[H])$ .*

*Proof.* In the following we identify  $\Sigma$  with  $\Sigma^- \subset W$ . We denote by  $X$  the result of gluing  $M$  and  $\tau(M)$  along  $\Sigma = \Sigma^-$ . Note that we have two canonical maps  $r : \Sigma^+ \rightarrow M \rightarrow X$  and  $s : \Sigma^+ \rightarrow \tau(M) \rightarrow X$ . We furthermore denote the canonical inclusion maps  $\Sigma \rightarrow M, \Sigma \rightarrow \tau(M)$  and  $\Sigma = \Sigma^- \rightarrow X$  by  $i$ . Throughout this proof we denote by  $i, r, s$  the induced maps on solvable quotients as well.

*Claim A.* The map  $i : \pi_1(\Sigma) \rightarrow \pi_1(X)$  gives rise to an isomorphism  $\pi_1(\Sigma)/\pi_1(\Sigma)^{(2)} \rightarrow \pi_1(X)/\pi_1(X)^{(2)}$ .

In the following let  $M'$  be either  $M$  or  $\tau(M)$ . Recall that we assume that  $\pi_1(\Sigma) \rightarrow \pi_1(M')$  gives rise to isomorphisms of the prosolvable completions. It now follows from Corollary 4.8 that  $\pi_1(\Sigma)/\pi_1(\Sigma)^{(2)} \rightarrow \pi_1(M'/\pi_1(M')^{(2)})$  is an isomorphism. Now let  $g : \pi_1(X) = \pi_1(M \cup_\Sigma \tau(M)) \rightarrow \pi_1(M)$  be the ‘folding map’. Note that

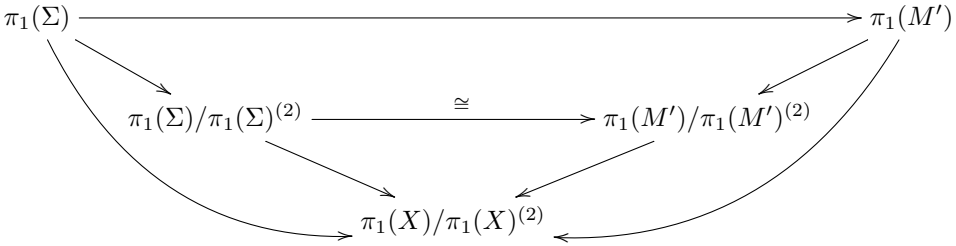
$$\pi_1(\Sigma)/\pi_1(\Sigma)^{(2)} \xrightarrow{i} \pi_1(X)/\pi_1(X)^{(2)} \xrightarrow{g} \pi_1(M)/\pi_1(M)^{(2)} \xleftarrow{\cong} \pi_1(\Sigma)/\pi_1(\Sigma)^{(2)}$$



is the identity map. In particular  $\pi_1(\Sigma)/\pi_1(\Sigma)^{(2)} \xrightarrow{i} \pi_1(X)/\pi_1(X)^{(2)}$  is injective. On the other hand it follows from the van Kampen theorem that

$$\pi_1(X) = \pi_1(M) *_{\pi_1(\Sigma)} \pi_1(M').$$

In particular  $\pi_1(X)/\pi_1(X)^{(2)}$  is generated by the images of  $\pi_1(M)$  and  $\pi_1(\tau(M))$  in  $\pi_1(X)/\pi_1(X)^{(2)}$ . But it results immediately from the following commutative diagram



that the image of  $\pi_1(\Sigma)/\pi_1(\Sigma)^{(2)}$  in  $\pi_1(X)/\pi_1(X)^{(2)}$  also generates the group. This concludes the proof of [Claim A](#).

*Claim B.* For any  $g \in \pi_1(\Sigma^+)/\pi_1(\Sigma^+)$  we have

$$r(g) = s(g) \in \pi_1(X)/\pi_1(X)^{(2)}.$$

Denote by  $\tau : X = M \cup_{\Sigma} \tau(M) \rightarrow X = M \cup_{\Sigma} \tau(M)$  the map given by switching the two copies of  $M$ . Clearly  $r(g) = \tau_*(s(g))$ . But  $\tau_*$  acts trivially on the image of  $\pi_1(\Sigma)/\pi_1(\Sigma)^{(2)}$  in  $\pi_1(X)/\pi_1(X)^{(2)}$ . By the above claim this means that  $\tau_*$  acts trivially on  $\pi_1(X)/\pi_1(X)^{(2)}$ . This concludes the proof of [Claim B](#).

We now view  $W$  as the result of gluing the two copies of  $\Sigma^+$  in  $\partial X$  by the identity map. First note that by the van Kampen theorem we have

$$\pi_1(W) = \langle t, \pi_1(X) \mid ts(g)t^{-1} = r(g), g \in \pi_1(\Sigma^+) \rangle.$$

Note that by [Claim B](#) the obvious assignments give rise to a well-defined map

$$\pi_1(W) = \langle t, \pi_1(X) \mid ts(g)t^{-1} = r(g), g \in \pi_1(\Sigma^+) \rangle \rightarrow \langle t \rangle \times \pi_1(X)/\pi_1(X)^{(2)}.$$

Since  $\pi_1(X)/\pi_1(X)^{(2)}$  is metabelian this map descends to a map

$$\Phi : \langle t, \pi_1(X) \mid ts(g)t^{-1} = r(g), g \in \pi_1(\Sigma^+) \rangle / (\dots)^{(2)} \rightarrow \langle t \rangle \times \pi_1(X)/\pi_1(X)^{(2)}.$$

*Claim C.* The map  $\Phi : \pi_1(W)/\pi_1(W)^{(2)} \rightarrow \langle t \rangle \times \pi_1(X)/\pi_1(X)^{(2)}$  is an isomorphism.

We denote by  $\Psi$  the following map:

$$\begin{aligned} \langle t \rangle \times \pi_1(X)/\pi_1(X)^{(2)} &\rightarrow \langle t, \pi_1(X)/\pi_1(X)^{(2)} \mid ts(g)t^{-1} = r(g), g \in \pi_1(\Sigma^+) \rangle \\ &= \langle t, \pi_1(X) \mid ts(g)t^{-1} = r(g), g \in \pi_1(\Sigma^+), \pi_1(X)^{(2)} \rangle \\ &\rightarrow \langle t, \pi_1(X) \mid ts(g)t^{-1} = r(g), g \in \pi_1(\Sigma^+) \rangle / (\dots)^{(2)}. \end{aligned}$$

Clearly  $\Psi$  is surjective and we have  $\Phi \circ \Psi = \text{id}$ . It follows that  $\Phi$  is an isomorphism. This concludes the proof of the claim.

Finally note that we have a canonical isomorphism

$$\pi_1(\Sigma \times S^1)/\pi_1(\Sigma \times S^1)^{(2)} = \langle t \rangle \times \pi_1(\Sigma)/\pi_1(\Sigma)^{(2)}.$$

It now follows from the above discussion that there is an isomorphism

$$\begin{aligned} \pi_1(W)/\pi_1(W)^{(2)} &\xrightarrow{\Phi} \langle t \rangle \times \pi_1(X)/\pi_1(X)^{(2)} \\ &\cong \langle t \rangle \times \pi_1(\Sigma)/\pi_1(\Sigma)^{(2)} \\ &= \pi_1(\Sigma \times S^1)/\pi_1(\Sigma \times S^1)^{(2)} \end{aligned}$$

which is again denoted by  $\Phi$ . Under the abelianization the map  $\Phi$  descends to the above identification  $H_1(\Sigma \times S^1; \mathbb{Z}) = H = H_1(W; \mathbb{Z})$ . We now get the following commutative diagram of exact sequences

$$\begin{array}{ccccccc} 0 & \longrightarrow & H_1(W; \mathbb{Z}[H]) & \longrightarrow & \pi_1(W)/\pi_1(W)^{(2)} & \longrightarrow & H := H_1(W; \mathbb{Z}) \longrightarrow 0 \\ & & \downarrow & & \downarrow \Phi & & \downarrow = \\ 0 & \longrightarrow & H_1(\Sigma \times S^1; \mathbb{Z}[H]) & \longrightarrow & \pi_1(\Sigma \times S^1)/\pi_1(\Sigma \times S^1)^{(2)} & \longrightarrow & H_1(\Sigma \times S^1; \mathbb{Z}) \longrightarrow 0. \end{array}$$

The lemma is now immediate. □

We are now ready to prove the second statement of [Lemma 4.5](#). The isomorphism of Alexander modules implies that the Alexander polynomials  $\Delta_W$  and  $\Delta_{\Sigma \times S^1}$  agree in  $\mathbb{Z}[H]$ . It is well-known that  $\Delta_{\Sigma \times S^1} = (t - 1)^{\|\phi\|_T} \in \mathbb{Z}[H] = \mathbb{Z}[F][t^{\pm 1}]$ . Recall that we are interested in  $\phi = PD([\Sigma])$ , and that  $\phi$  as an element in  $\text{Hom}(H, \mathbb{Z}) = \text{Hom}(F \times \langle t \rangle, \mathbb{Z})$  is given by  $\phi(t) = 1, \phi|_F = 0$ . It is now obvious from  $\Delta_W = \Delta_{\Sigma \times S^1} = (t - 1)^{\|\phi\|_T}$  that  $\phi$  lies in the interior of a top-dimensional face of the Alexander norm ball of  $W$ . This concludes the proof of the second statement of [Lemma 4.5](#) modulo the proof of the claim.

### 5. Residual properties of 3-manifold groups

[Proposition 1.7](#) and [Theorem 4.1](#) are almost enough to deduce [Theorem 1.2](#), but we still have to deal with the assumption in [Theorem 4.1](#) that  $\pi_1(W)$  has to be residually finite solvable.

Using well-known arguments (see [§7](#) for details) one can easily see that [Proposition 1.7](#) and [Theorem 4.1](#) imply [Theorem 1.2](#) for 3-manifolds  $N$  which have virtually residually finite solvable fundamental groups. Here we say that

a group  $\pi$  has virtually a property if a finite index subgroup of  $\pi$  has this property. It seems reasonable to conjecture that all 3-manifold groups are virtually residually finite solvable. For example, linear groups (and hence fundamental groups of hyperbolic 3-manifolds and Seifert fibered spaces) are virtually residually finite solvable and (virtually) fibered 3-manifold groups are easily seen to be (virtually) residually finite solvable.

It is not known though whether all 3-manifold groups are linear. In the case of 3-manifolds with nontrivial JSJ decomposition we therefore use a slightly different route to deduce [Theorem 1.2](#) from [Proposition 1.7](#) and [Theorem 4.1](#). In [Lemmas 7.1 and 7.2](#) we first show that it suffices in the proof of [Theorem 1.2](#) to consider closed prime 3-manifolds. In this section we will show that given a closed prime 3-manifold  $N$ , there exists a finite cover  $N'$  of  $N$  such that all pieces of the JSJ decomposition of  $N'$  have residually finite solvable fundamental groups ([Theorem 5.1](#)). Finally in [Section 6](#) we will prove a result which allows us in the proof of [Theorem 1.2](#) to work with each JSJ piece separately ([Theorem 6.4](#)).

**5.1. Statement of the theorem.** We first recall some definitions. Let  $p$  be a prime. A  $p$ -group is a group such that the order of the group is a power of  $p$ . Note that any  $p$ -group is in particular finite solvable. A group  $\pi$  is called *residually  $p$*  if for any nontrivial  $g \in \pi$  there exists a homomorphism  $\alpha : \pi \rightarrow P$  to a  $p$ -group such that  $\alpha(g) \neq e$ . A residually  $p$  group is evidently also residually finite solvable.

For the reader's convenience we recall the statement of [Theorem 1.9](#) which we will prove in this section.

**THEOREM 5.1.** *Let  $N$  be a closed irreducible 3-manifold. Then for all but finitely many primes  $p$  there exists a finite cover  $N'$  of  $N$  such that the fundamental group of any JSJ component of  $N'$  is residually  $p$ .*

*Remarks.* (1) Note that this theorem relies on the geometrization results of Thurston and Perelman.

(2) A slight modification of our proof shows that the statement of the theorem also holds for irreducible 3-manifolds with toroidal boundary.

**5.2. Proof of [Theorem 5.1](#).** The proof of the theorem combines in a straightforward way ideas from the proof that finitely generated subgroups of  $\mathrm{GL}(n, \mathbb{C})$  are virtually residually  $p$  for all but finitely many primes  $p$  (cf. e.g. [[Weh73](#), Th. 4.7] or [[LS03](#), Window 7, Prop. 9]) with ideas from the proof that 3-manifold groups are residually finite (cf. [[Hem87](#)]). Since all technical results can be found in either [[Weh73](#)] or [[Hem87](#)], and in order to save space, we only give an outline of the proof by referring heavily to [[Weh73](#)] and [[Hem87](#)].

In the following recall that given a positive integer  $n$  there exists a unique characteristic subgroup of  $\mathbb{Z} \oplus \mathbb{Z} = \pi_1(\text{torus})$  of index  $n^2$ , namely  $n(\mathbb{Z} \oplus \mathbb{Z})$ .

*Definition 5.2.* Let  $N$  be a 3-manifold which is either closed or has toroidal boundary. Given a prime  $p$  we say that a subgroup  $\Gamma \subset \pi_1(N)$  has Property  $(p)$  if it satisfies the following two conditions:

- (1)  $\Gamma$  is residually  $p$ , and
- (2) for any torus  $T \subset \partial N$  the group  $\Gamma \cap \pi_1(T)$  is the unique characteristic subgroup of  $\pi_1(T)$  of index  $p^2$ .

**PROPOSITION 5.3.** *Let  $N$  be a compact orientable 3-manifold with empty or toroidal boundary such that the interior has a complete hyperbolic structure of finite volume. Then for all but finitely many primes  $p$  there exists a finite index subgroup of  $\pi_1(N)$  which has Property  $(p)$ .*

*Proof.* [Proposition 5.3](#) is essentially a combination of [[Hem87](#), Lemma 4.1] with the proof that finitely generated linear groups are virtually residually  $p$ . We will use the notation of the proof of [[Hem87](#), Lemma 4.1]. First we pick a finitely generated subring  $A \subset \mathbb{C}$  as in [[Hem87](#), Proof of Lemma 4.1]. In particular we can assume that  $\pi_1(N) \subset \text{SL}(2, A)$  where  $A \subset \mathbb{C}$ . We pick a prime  $p$  and a maximal ideal  $\mathfrak{m} \subset A$  as in [[Hem87](#), p. 391]. We then have in particular that  $\text{char}(A/\mathfrak{m}) = p$ . For  $i \geq 1$  we now let  $\Gamma_i = \text{Ker}\{\pi_1(N) \rightarrow \text{SL}(2, A/\mathfrak{m}^i) \times H/p^i H\}$  where  $H = H_1(N; \mathbb{Z})/\text{torsion}$ . We claim that  $\Gamma_1 \subset \pi_1(N)$  is a finite index subgroup which has Property  $(p)$ . Clearly  $\Gamma_1$  is of finite index in  $\pi_1(N)$  and by [[Hem87](#), p. 391] the subgroup  $\Gamma_1$  also satisfies condition (2). The proof that finitely generated linear groups are virtually residually  $p$  (cf. [[Weh73](#), Th. 4.7] or [[LS03](#), Window 7, Prop. 9]) then shows immediately that all the groups  $\Gamma_1/\Gamma_i, i \geq 1$  are  $p$ -groups and that  $\bigcap_{i=1}^{\infty} \Gamma_i = \{1\}$ . In particular  $\Gamma_1$  is residually  $p$ .  $\square$

**PROPOSITION 5.4.** *Let  $N$  be a Seifert fibered space. Then for all but finitely many primes  $p$ , there exists a finite index subgroup of  $\pi_1(N)$  which has Property  $(p)$ .*

*Proof.* If  $N$  is a closed Seifert fibered space, then it is well-known that  $\pi_1(N)$  is linear, and the proposition immediately follows from the fact that linear groups are virtually residually  $p$  for almost all primes  $p$ .

Now consider the case that  $N$  has a boundary. It is well-known (cf. for example [[Ham01](#), Lemma 6] and see also [[Hem87](#), p. 391]) that there exists a finite cover  $q : N' \rightarrow N$  with the following two properties:

- (1)  $N' = S^1 \times F$  for some surface  $F$ ,
- (2) for any torus  $T \subset \partial N$  the group  $\pi_1(N') \cap \pi_1(T)$  is the unique characteristic subgroup of  $\pi_1(T)$  of index  $p^2$ .

We now write  $\Gamma := \pi_1(N') \subset \pi_1(N)$ . The group  $\Gamma$  is residually  $p$  since free groups are residually  $p$ . It now follows from (2) that  $\Gamma$  has the required properties. □

We are now in a position to prove [Theorem 5.1](#).

*Proof of Theorem 5.1.* Let  $N$  be a closed irreducible 3-manifold. Let  $N_1, \dots, N_r$  be the JSJ components. For all but finitely many primes  $p$  we can by [Propositions 5.3 and 5.4](#) find finite index subgroups  $\Gamma_i \subset \pi_1(N_i)$  for  $i = 1, \dots, r$  which have Property (p). We denote by  $N'_i$  the cover of  $N_i$  corresponding to  $\Gamma_i$ .

By the second condition of Property (p) the intersections of the subgroups  $\Gamma_i, i = 1, \dots, r$  with the fundamental group of any torus of the JSJ decomposition coincide. We can therefore appeal to [[Hem87](#), Th. 2.2] to find a finite cover  $N'$  of  $N$  such that any component in the JSJ decomposition of  $N'$  is homeomorphic to some  $N'_i, i \in \{1, \dots, r\}$ . Recall that  $\pi_1(N'_i) = \Gamma_i$  is residually  $p$  for any  $i$ ; hence the cover  $N'$  of  $N$  has the desired properties. □

### 6. The JSJ decomposition and prosolvable completions

Let  $N$  be a closed 3-manifold and let  $\phi \in H^1(N; \mathbb{Z})$  primitive with  $\|\phi\|_T > 0$ . If  $(N, \phi)$  fibers, and if  $\Sigma \subset N$  is a surface dual to  $\phi$  which is the fiber of the fibration, then it is well-known (cf. e.g. [[EN85](#)]) that the JSJ tori of  $N$  cut the product  $N \setminus \nu\Sigma \cong \Sigma \times [0, 1]$  into smaller products.

If  $(N, \phi)$  satisfies Condition (\*), and if  $\Sigma \subset N$  is a connected Thurston norm-minimizing surface dual to  $\phi$ , then we will see in [Lemma 6.3](#) and [Theorem 6.4](#) that the JSJ tori of  $N$  cut the manifold  $N \setminus \nu\Sigma$  into smaller pieces which look like products ‘on the level of prosolvable completions’. This result will play an important role in the proof of [Theorem 1.2](#) as it allows us to work with each JSJ piece separately.

6.1. *The statement of the theorem.* Throughout this section let  $N$  be a closed irreducible 3-manifold. Furthermore let  $\phi \in H^1(N; \mathbb{Z})$  be a primitive class which is dual to a *connected* Thurston norm-minimizing surface. (Recall that by [Proposition 3.1](#) this is in particular the case if  $(N, \phi)$  satisfies Condition (\*).) Finally we assume that  $\|\phi\|_T > 0$ .

We now fix once and for all embedded tori  $T_1, \dots, T_r \subset N$  which give the JSJ decomposition of  $N$ . (Recall that the  $T_1, \dots, T_r$  are unique up to reordering and isotopy.)

We will make use several times of the following well-known observations:

LEMMA 6.1. *Let  $\Sigma \subset N$  be an incompressible surface in general position with the JSJ torus  $T_i, i \in 1, \dots, r$ . Let  $c$  be a component of  $\Sigma \cap T_i$ . Then  $c$  represents a nontrivial element in  $\pi_1(T_i)$  if and only if  $c$  represents a nontrivial element in  $\pi_1(\Sigma)$ .*

LEMMA 6.2. *There exists an embedded Thurston norm-minimizing surface  $\Sigma \subset N$  dual to  $\phi$  with the following three properties:*

- (1)  $\Sigma$  is connected,
- (2) the tori  $T_i, i = 1, \dots, r$  and the surface  $\Sigma$  are in general position,
- (3) any component of  $\Sigma \cap T_i, i = 1, \dots, r$ , represents a nontrivial element in  $\pi_1(T_i)$ .

Now, among all surfaces dual to  $\phi$  satisfying the properties of the lemma we pick a surface  $\Sigma$  which minimizes the number  $\sum_{i=1}^r b_0(\Sigma \cap T_i)$ .

Given  $\Sigma$  we can and will fix a tubular neighborhood  $\Sigma \times [-1, 1] \subset N$  such that the tori  $T_i, i = 1, \dots, r$ , and the surface  $\Sigma \times t$  are in general position for any  $t \in [-1, 1]$ . We from now on write  $M = N \setminus \Sigma \times (-1, 1)$  and  $\Sigma^\pm = \Sigma \times \pm 1$ .

We denote the components of  $N$  cut along  $T_1, \dots, T_r$  by  $N_1, \dots, N_s$ . Let  $\{A_1, \dots, A_m\}$  be the set of components of the intersection of the tori  $T_1 \cup \dots \cup T_r$  with  $M$ . Note that the surfaces  $A_i \subset M, i = 1, \dots, m$  are properly embedded since we assumed that the tori  $T_i$  and the surfaces  $\Sigma^\pm = \Sigma \times \pm 1$  are in general position. We also let  $\{M_1, \dots, M_n\}$  be the set of components of the intersection of  $N_i$  with  $M$  for  $i = 1, \dots, s$ . Put differently,  $M_1, \dots, M_n$  are the components of  $M$  cut along  $A_1, \dots, A_m$ . For  $i = 1, \dots, n$  we furthermore write  $\Sigma_i^\pm = M_i \cap \Sigma^\pm$ .

Let  $i \in \{1, \dots, m\}$ . If the surface  $A_i$  is an annulus, then we say that  $A_i$  connects  $\Sigma^-$  and  $\Sigma^+$  if one boundary component of  $A_i$  lies on  $\Sigma^-$  and the other boundary component lies on  $\Sigma^+$ . The following lemma will be proved in Section 6.2

LEMMA 6.3. *When  $(N, \phi)$  satisfies Condition  $(*)$ , then for  $i = 1, \dots, m$  the surface  $A_i$  is an annulus which connects  $\Sigma^-$  and  $\Sigma^+$ .*

We can now formulate the main theorem of this section. The proof will be given in Sections 6.2, 6.3 and 6.4.

THEOREM 6.4. *Assume that for  $i = 1, \dots, m$  the surface  $A_i$  is an annulus which connects  $\Sigma^-$  and  $\Sigma^+$ . Furthermore assume that the inclusion induced maps  $\pi_1(\Sigma^\pm) \rightarrow \pi_1(M)$  give rise to an isomorphism of prosolvable completions. Then for  $i = 1, \dots, n$  the following hold:*

- (1) The surfaces  $\Sigma_i^\pm$  are connected.
- (2) Given  $j \in \{1, \dots, n\}$  with  $M_j \subset N_j$  the inclusion induced map  $\pi_1(M_j) \rightarrow \pi_1(N_j)$  is injective.
- (3) The inclusion induced maps  $\pi_1(\Sigma_i^\pm) \rightarrow \pi_1(M_i)$  give rise to isomorphisms of the respective prosolvable completions.

We remind the reader that at the beginning of the section we assumed that  $\|\phi\|_T > 0$ .

6.2. *Proof of Lemma 6.3.* We first recall the following theorem from an earlier paper (cf. [FV08a, Th. 5.2]).

**THEOREM 6.5.** *Let  $Y$  be a closed irreducible 3-manifold. Let  $\psi \in H^1(Y; \mathbb{Z})$  be a primitive class. Assume that for any epimorphism  $\alpha : \pi_1(Y) \rightarrow G$  onto a finite group  $G$  the twisted Alexander polynomial  $\Delta_{Y,\psi}^\alpha \in \mathbb{Z}[t^{\pm 1}]$  is nonzero. Let  $T \subset Y$  be an incompressible embedded torus. Then either  $\psi|_T \in H^1(T; \mathbb{Z})$  is nonzero, or  $(Y, \psi)$  fibers over  $S^1$  with fiber  $T$ .*

With this theorem we are now able to prove Lemma 6.3. We use the notation from the previous section and assume that  $(N, \phi)$  is a pair which satisfies Condition (\*). In particular,  $\Delta_{N,\phi}^\alpha \neq 0$  for any epimorphism  $\alpha : \pi_1(N) \rightarrow G$  onto a finite group  $G$ . We can therefore apply Theorem 6.5 to the tori  $T_1, \dots, T_r \subset N$  to conclude that either  $(N, \phi)$  fibers over  $S^1$  with toroidal fiber, or  $\phi|_{T_i} \in H^1(T_i; \mathbb{Z})$  is nonzero for  $i = 1, \dots, r$ . Recall that we assumed that  $\|\phi\|_T > 0$ , we therefore only have to deal with the latter case. From  $\phi|_{T_i} \in H^1(T_i; \mathbb{Z})$  nonzero we obtain that  $\Sigma$  (which is dual to  $\phi$ ) necessarily intersects  $T_i$  in at least one curve which is homologically essential on  $T_i$ . In fact by our assumption on  $\Sigma$  and  $T_1, \dots, T_r$  any intersection curve  $\Sigma \cap T_i \subset T_i$  is essential; in particular the components of  $T_i$  cut along  $\Sigma$  are indeed annuli.

In order to prove Lemma 6.3 it now remains to show that each of the annuli  $A_i$  connects  $\Sigma^-$  and  $\Sigma^+$ . So assume there exists an  $i \in \{1, \dots, m\}$  such that the annulus  $A_i$  does not connect  $\Sigma^-$  and  $\Sigma^+$ . Without loss of generality we can assume that  $\Sigma^+ \cap A_i = \emptyset$ . We equip  $A_i$  with an orientation. Denote the two oriented components of  $\partial A_i$  by  $c$  and  $-d$ . By our assumption  $c$  and  $d$  lie in  $\Sigma^-$ , and they cobound the annulus  $A_i \subset M$ .

Now recall that by Proposition 3.2 our assumption that  $(N, \phi)$  satisfies Condition (\*) implies in particular that  $H_1(\Sigma^-; \mathbb{Z}) \rightarrow H_1(M; \mathbb{Z})$  is an isomorphism. Note that  $c, d$  are homologous in  $M$  via the annulus  $A := A_i$ , and since  $H_1(\Sigma^-; \mathbb{Z}) \rightarrow H_1(M; \mathbb{Z})$  is an isomorphism we deduce that  $c$  and  $d$  are homologous in  $\Sigma^-$  as well. Since  $\Sigma^-$  is closed we can now find two subsurfaces  $\Sigma_1, \Sigma_2 \subset \Sigma^-$  such that  $\partial \Sigma_1 = -c \cup d$ , and such that (with the orientations induced from  $\Sigma^-$ ) the following hold:  $\Sigma_1 \cup \Sigma_2 = \Sigma$ ,  $\partial \Sigma_2 = c \cup -d$  and  $\Sigma_1 \cap \Sigma_2 = c \cup d$ . Note that possibly one of  $\Sigma_1$  or  $\Sigma_2$  is disconnected.

*Claim.* The surfaces  $\bar{\Sigma}_1 = \Sigma_1 \cup A$  and  $\bar{\Sigma}_2 = \Sigma_2 \cup -A$  are closed, orientable and connected. Furthermore, there exists a  $j \in \{1, 2\}$  such that  $\text{genus}(\bar{\Sigma}_j) = \text{genus}(\Sigma)$  and such that  $\bar{\Sigma}_j$  is homologous to  $\Sigma$  in  $N$ .

*Proof.* It is clear that  $\bar{\Sigma}_1$  and  $\bar{\Sigma}_2$  are closed, orientable and connected. We give  $\bar{\Sigma}_k, k = 1, 2$  the orientation which restricts to the orientation of  $\Sigma_k$ . We therefore only have to show the second claim.

Recall that Condition (\*) implies that the inclusion induced maps

$$H_j(\Sigma^-; \mathbb{Z}) \rightarrow H_j(M; \mathbb{Z}), \quad j = 0, 1$$

are isomorphisms. It follows from Lemma 2.6 that we also have an isomorphism  $H_2(\Sigma^-; \mathbb{Z}) \rightarrow H_2(M; \mathbb{Z})$ . In particular  $H_2(M; \mathbb{Z})$  is generated by  $[\Sigma^-]$ . Now note that  $\bar{\Sigma}_1$  and  $\bar{\Sigma}_2$  represent elements in  $H_2(M; \mathbb{Z})$ . We can write  $[\bar{\Sigma}_k] = l_k[\Sigma^-]$ ,  $k = 1, 2$ , for some  $l_k \in \mathbb{Z}$ . Note that  $[\bar{\Sigma}_1] + [\bar{\Sigma}_2] = [\Sigma^-]$ ; i.e.,  $l_1 + l_2 = 1$ .

Now let  $k \in \{1, \dots, r\}$  such that  $A_i \subset T_k$ . Recall that any component of  $\Sigma \cap T_k$  represents a nontrivial element in  $\pi_1(T_k)$ . By Lemma 6.1 any component of  $\Sigma \cap T_k$  therefore also represents a nontrivial element in  $\pi_1(\Sigma)$ . In particular  $c$  and  $d$  do not bound disks on  $\Sigma$ , which in turn implies that  $\chi(\Sigma_k) \leq 0$ ,  $k = 1, 2$ . It follows that

$$(3) \quad -\chi(\bar{\Sigma}_k) = -\chi((\Sigma^- \setminus \Sigma_{3-k}) \cup A) = -\chi(\Sigma) + \chi(\Sigma_{3-k}) \leq -\chi(\Sigma), \quad k = 1, 2.$$

On the other hand, by the linearity of the Thurston norm and the genus minimality of  $\Sigma$  we have

$$(4) \quad -\chi(\bar{\Sigma}_k) \geq -|l_k|\chi(\Sigma), \quad k = 1, 2.$$

Now recall our assumption that  $\chi(\Sigma) = \|\phi\|_T > 0$ . It follows that  $l_1 + l_2 = 1$  and the inequalities (3) and (4) can only be satisfied if there exists a  $j$  with  $l_j = 1$  and  $\chi(\bar{\Sigma}_j) = \chi(\Sigma)$ . (Necessarily,  $l_{3-j} = 0$  and  $\bar{\Sigma}_{3-j}$  is a torus.)  $\square$

Now, there exists a proper isotopy of  $A \subset M$  to an annulus  $A' \subset M$  such that  $\partial A'$  lies entirely in  $\Sigma_j$  and such that  $A'$  is disjoint from all the other  $A_j, j = 1, \dots, r$ . We then let  $\Sigma'_j \subset \Sigma_j$  be the subsurface of  $\Sigma_j$  such that  $\partial \Sigma'_j = \partial A'$ . Clearly  $\Sigma' := \Sigma'_j \cup -A'$  is isotopic to  $\Sigma_j \cup -A$ , in particular by the claim,  $\Sigma'$  is a closed connected surface homologous to  $\Sigma$  in  $N$  with  $\text{genus}(\Sigma') = \text{genus}(\Sigma)$  which satisfies all the properties listed in Lemma 6.2. On the other hand we evidently have  $b_0(\Sigma' \cap T_j) \leq b_0(\Sigma) - 2$ . Since we did not create any new intersections we in fact have  $\sum_{i=1}^r b_0(\Sigma' \cap T_i) < \sum_{i=1}^r b_0(\Sigma \cap T_i)$ . But this contradicts the minimality of  $\sum_{i=1}^r b_0(\Sigma \cap T_i)$  in our choice of the surface  $\Sigma$ . We therefore showed that the assumption that  $A_i$  does not connect  $\Sigma^-$  and  $\Sigma^+$  leads to a contradiction. This concludes the proof of Lemma 6.3.

6.3. *Preliminaries on the components  $M_1, \dots, M_n$ .* We continue with the notation from the previous sections. Using Lemma 6.3 we can now prove the following lemma, which in particular implies the first statement of Theorem 6.4.

LEMMA 6.6. *Assume that the inclusion induced maps  $\pi_1(\Sigma^\pm) \rightarrow \pi_1(M)$  give rise to isomorphisms of the prosolvable completions. Let  $i \in \{1, \dots, n\}$ . Then the following hold:*

- (1) *The surfaces  $\Sigma_i^\pm$  are connected.*



(2) For any homomorphism  $\alpha : \pi_1(M) \rightarrow S$  to a finite solvable group the inclusion maps induce isomorphisms

$$H_j(\Sigma_i^\pm; \mathbb{Z}[S]) \rightarrow H_j(M_i; \mathbb{Z}[S])$$

for  $j = 0, 1$ .

*Proof.* We first consider statement (1). Recall that  $M_1, \dots, M_n$  are the components of  $M$  split along  $A_1, \dots, A_m$ . We therefore get the following commutative diagram of Mayer-Vietoris sequences

$$\begin{CD} \dots @>>> \bigoplus_{k=1}^m H_j(A_k \cap \Sigma^\pm; \mathbb{Z}) @>>> \bigoplus_{l=1}^n H_j(M_l \cap \Sigma^\pm; \mathbb{Z}) @>>> H_j(\Sigma^\pm; \mathbb{Z}) @>>> \dots \\ @. @VVV @VVV @VVV \\ \dots @>>> \bigoplus_{k=1}^m H_j(A_k; \mathbb{Z}) @>>> \bigoplus_{l=1}^n H_j(M_l; \mathbb{Z}) @>>> H_j(M; \mathbb{Z}) @>>> \dots \end{CD}$$

Note that the vertical homomorphisms on the left are isomorphisms since by Lemma 6.3 for any  $i = 1, \dots, m$  the  $A_i$  is an annulus which connects  $\Sigma^-$  and  $\Sigma^+$ , i.e.  $A_i$  is a product on  $A_i \cap \Sigma^\pm$ . Also note that the vertical homomorphisms on the right are isomorphisms for  $j = 0, 1$  by Proposition 3.2 and for  $j = 2$  by Lemma 2.6. We can now appeal to the 5-Lemma to deduce that the middle homomorphisms are isomorphisms as well. But for any  $j$  the middle homomorphism is a direct sum of homomorphisms. It therefore follows in particular that the maps  $H_j(\Sigma_i^\pm; \mathbb{Z}) \rightarrow H_j(M_i; \mathbb{Z}), j = 0, 1$  are isomorphisms for any  $i \in \{1, \dots, n\}$ . In particular  $b_0(\Sigma_i^\pm) = b_0(M_i) = 1$ ; i.e., the surfaces  $\Sigma_i^\pm$  are connected.

We now prove statement (2). Let  $\alpha : \pi_1(M) \rightarrow S$  be a homomorphism to a finite solvable group. Recall that by Lemmas 2.12 and 2.6, the inclusion induced maps  $H_j(\Sigma^\pm; \mathbb{Z}[S]) \rightarrow H_j(M; \mathbb{Z}[S])$  are isomorphisms for  $j = 0, 1, 2$ . It now follows from the commutative diagram of Mayer-Vietoris sequences as above, but with  $\mathbb{Z}[S]$ -coefficients (cf. [FK06] for details) that

$$H_j(\Sigma_i^\pm; \mathbb{Z}[S]) \rightarrow H_j(M_i; \mathbb{Z}[S])$$

is an isomorphism for any  $i \in \{1, \dots, n\}$  and  $j = 0, 1$ . □

The following lemma in particular implies the second statement of Theorem 6.4.

LEMMA 6.7. For any pair  $(i, j)$  such that  $M_i \subset N_j$  we have a commutative diagram of injective maps as follows:

$$\begin{CD} \pi_1(\Sigma_i^\pm) @>>> \pi_1(M_i) @>>> \pi_1(N_j) \\ @VVV @VVV @VVV \\ \pi_1(\Sigma^\pm) @>>> \pi_1(M) @>>> \pi_1(N). \end{CD}$$

*Proof.* First note that since  $\Sigma$  is incompressible we know that the two bottom maps are injective. Furthermore recall that  $N_j$  is a JSJ component of  $N$ , i.e. a component of the result of cutting  $N$  along incompressible tori, hence  $\pi_1(N_j) \rightarrow \pi_1(N)$  is injective.

*Claim.* For any  $k \in \{1, \dots, n\}$  the maps  $\pi_1(\Sigma_k^\pm) \rightarrow \pi_1(\Sigma)$  are injective.

Let  $c$  be a component of  $\Sigma \cap T_l$  for some  $l \in \{1, \dots, r\}$ . Recall that by our choice of tori  $T_1, \dots, T_r$  the curve  $c$  represents a nontrivial element in  $\pi_1(T_l)$ . By Lemma 6.1 the curve  $c$  also represents a nontrivial element in  $\pi_1(\Sigma)$ . In particular none of the components of  $\Sigma^\pm \setminus \Sigma_k^\pm$  are disks and therefore the maps  $\pi_1(\Sigma_k^\pm) \rightarrow \pi_1(\Sigma^\pm)$  are injective. This concludes the proof of the claim.

Now let  $K = \{k \in \{1, \dots, n\} \mid M_k \subset N_j\}$ . It follows from the claim and the above commutative diagram that for any  $k \in K$  the inclusion induced map  $\pi_1(\Sigma_k) \rightarrow \pi_1(N_j)$  is injective; i.e., for any  $k \in K$  the surface  $\Sigma_k \subset N_j$  is incompressible. Since  $M_i$  is a component of cutting  $N_j$  along the incompressible surfaces  $\Sigma_k^- \subset N_j, k \in K$  we have that  $\pi_1(M_i) \rightarrow \pi_1(N_j)$  is injective.

By commutativity of the above diagram we now obtain that all other maps are injective as well. □

6.4. *The conclusion of the proof of Theorem 6.4.* In this section we will finally prove the third statement of Theorem 6.4. The main ingredient in the proof is the following result.

PROPOSITION 6.8. *Let  $\Sigma$  be a closed surface and  $\Sigma' \subset \Sigma$  a connected subsurface such that  $\pi_1(\Sigma') \rightarrow \pi_1(\Sigma)$  is injective. Let  $\alpha : \pi_1(\Sigma') \rightarrow S$  be a homomorphism to a finite solvable group. Then there exists a homomorphism to a finite solvable group  $\beta : \pi_1(\Sigma) \rightarrow T$  and a homomorphism  $\pi : T' := \text{Im}\{\pi_1(\Sigma') \rightarrow T\} \rightarrow S$  such that the following diagram commutes:*

$$\begin{array}{ccc}
 \pi_1(\Sigma') & \longrightarrow & \pi_1(\Sigma) \\
 \alpha \downarrow & \searrow & \downarrow \beta \\
 S & \xleftarrow{\pi} T' \longrightarrow & T.
 \end{array}$$

Put differently, the prosolvable topology on  $\pi_1(\Sigma')$  agrees with the topology on  $\pi_1(\Sigma')$  induced from the prosolvable topology on  $\pi_1(\Sigma)$ .

*Remark.* Note that in general  $H_1(\Sigma'; \mathbb{Z}) \rightarrow H_1(\Sigma; \mathbb{Z})$  is not injective, even if  $\pi_1(\Sigma') \rightarrow \pi_1(\Sigma)$  is an injection. In particular in general a homomorphism  $\pi_1(\Sigma') \rightarrow S$  to an abelian group will not extend to a homomorphism from  $\pi_1(\Sigma)$  to an abelian group. This shows that in general we cannot take  $T = S$  or  $T$  of the same solvability length as  $S$  in the above proposition.

*Proof.* The statement of the proposition is trivial if  $\Sigma' = \Sigma$ ; we will therefore henceforth only consider the case that  $\Sigma' \neq \Sigma$ . Let  $\alpha : \pi_1(\Sigma') \rightarrow S$  be

a homomorphism to a finite solvable group. It suffices to show that there exists a homomorphism  $\beta : \pi_1(\Sigma) \rightarrow T$  to a finite solvable group such that  $\text{Ker}(\beta) \cap \pi_1(\Sigma') \subset \text{Ker}(\alpha)$ .

Denote by  $\Sigma_1, \dots, \Sigma_l$  the components of  $\overline{\Sigma \setminus \Sigma'}$ . Note that the condition that  $\pi_1(\Sigma') \rightarrow \pi_1(\Sigma)$  is injective is equivalent to saying that none of the sub-surfaces  $\Sigma_1, \dots, \Sigma_l$  is a disk.

It is straightforward to see that for each  $j = 1, \dots, l$  we can find an annulus  $A_j \in \text{int}(\Sigma_j)$  such that  $(\Sigma' \cup \Sigma_j) \setminus A_j$  is still connected.

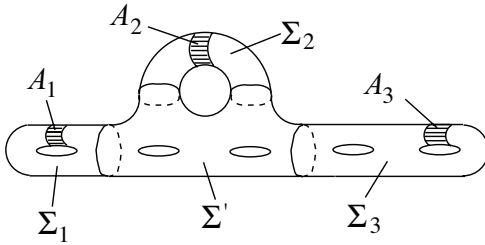


Figure 1. Surface  $\Sigma' \subset \Sigma$  with annuli  $A_i \subset \Sigma_i, i = 1, 2, 3$ .

Now let  $\Sigma'' = \overline{\Sigma \setminus \cup_{j \in J} A_j}$ . Clearly  $\Sigma''$  is a connected surface with boundary. By assumption  $\pi_1(\Sigma') \rightarrow \pi_1(\Sigma)$  is injective. Since  $\pi_1(\Sigma') \rightarrow \pi_1(\Sigma)$  factors through  $\pi_1(\Sigma'')$  we see that  $\Sigma'$  is a subsurface of  $\Sigma''$  such that  $\pi_1(\Sigma') \rightarrow \pi_1(\Sigma'')$  is injective. Since  $\Sigma''$  is a surface with boundary (contrary to  $\Sigma$ ) this implies that  $\pi_1(\Sigma')$  is in fact a free factor of  $\pi_1(\Sigma'')$ ; i.e., we have an isomorphism  $\gamma : \pi_1(\Sigma'') \xrightarrow{\cong} \pi_1(\Sigma') * F$  where  $F$  is a free group such that the map  $\pi_1(\Sigma'') \xrightarrow{\gamma} \pi_1(\Sigma') * F \rightarrow \pi_1(\Sigma')$  splits the inclusion induced map  $\pi_1(\Sigma') \rightarrow \pi_1(\Sigma'')$ .

We now write  $\pi := \pi_1(\Sigma'')$  and we denote by  $\alpha''$  the projection map  $\pi \rightarrow \pi/\pi(S)$  (referring to §3.4 for the definition and the properties of the characteristic subgroup  $\pi(S)$  of  $\pi$ ). We can extend  $\alpha : \pi_1(\Sigma') \rightarrow S$  to  $\pi_1(\Sigma'') \xrightarrow{\gamma} \pi_1(\Sigma') * F \rightarrow \pi_1(\Sigma') \xrightarrow{\alpha} S$ . It follows immediately that  $\text{Ker}(\alpha'') \cap \pi_1(\Sigma') \subset \text{Ker}(\alpha)$ .

We now extend  $\alpha'' : \pi_1(\Sigma'') \rightarrow \pi/\pi(S)$  to a homomorphism  $\beta : \pi_1(\Sigma) \rightarrow \mathbb{Z}/n \times \pi/\pi(S)$  where  $1 \in \mathbb{Z}/n$  acts in an appropriate way on  $\pi/\pi(S)$ . In order to do this we will first study the relationship between  $\pi_1(\Sigma'')$  and  $\pi_1(\Sigma)$ .

Evidently  $\Sigma = \Sigma'' \cup \cup_{i=1}^k A_i$ . We pick an orientation for  $\Sigma$  and give  $A_1, \dots, A_k$  the induced orientations. We write  $\partial A_i = -a_i \cup b_i, i = 1, \dots, k$  (see Figure 2). We now pick a base point for  $\Sigma''$ . We can find based curves  $c_1, \dots, c_l, d_1, \dots, d_l$  and paths from the base point to the curves  $a_1, \dots, a_k, b_1, \dots, b_k$  (and from now on we do not distinguish in the notation between

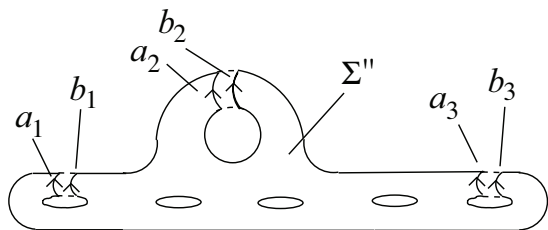


Figure 2. Surface  $\Sigma'' \subset \Sigma$  with oriented boundary curves  $a_i, b_i$ .

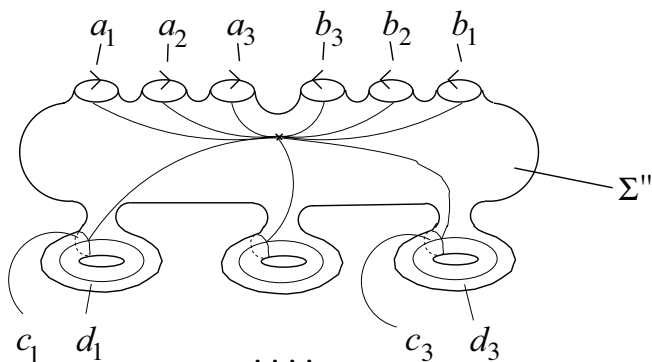


Figure 3. Surface  $\Sigma'' \subset \Sigma$  with oriented based curves  $a_i, b_i, c_i, d_i$ .

curves and based curves) such that

$$\pi = \langle a_1, \dots, a_k, b_1, \dots, b_k, c_1, \dots, c_l, d_1, \dots, d_l \mid a_1 \dots a_k b_k^{-1} \dots b_1^{-1} = [c_l, d_l] \dots [c_1, d_1] \rangle.$$

(See Figure 3 for an illustration.) By the van Kampen theorem we then have

$$\pi_1(\Sigma) = \langle \pi_1(\Sigma''), t_1, \dots, t_k \mid t_i a_i t_i^{-1} = b_i, i = 1, \dots, k \rangle.$$

*Claim.* There exists an automorphism  $\varphi : \pi \rightarrow \pi$  such that  $\varphi(a_i) = b_i$  and  $\varphi(b_i) = a_i$  for any  $i \in \{1, \dots, k\}$ .

Let  $\Gamma$  be the free group generated by  $a_i, b_i, i = 1, \dots, k$  and  $c_i, d_i, i = 1, \dots, l$  and consider the isomorphism  $\varphi : \Gamma \rightarrow \Gamma$  defined by  $\varphi(a_i) = b_i, \varphi(b_i) = a_i, i = 1, \dots, k$  and  $\varphi(c_i) = d_{l+1-i}, \varphi(d_i) = c_{l+1-i}, i = 1, \dots, l$ . In the following we write  $w = [c_l, d_l] \dots [c_1, d_1]$  and we write  $r = a_1 \dots a_k b_k^{-1} \dots b_1^{-1} \cdot [c_1, d_1]^{-1} \dots [c_l, d_l]^{-1}$  for the relator. Note that we have a canonical isomorphism  $\pi \cong \Gamma / \langle\langle r \rangle\rangle$ . We calculate

$$\begin{aligned}
 \varphi(r) &= \varphi(a_1 \dots a_k b_k^{-1} \dots b_1^{-1} \cdot [c_1, d_1]^{-1} \dots [c_l, d_l]^{-1}) \\
 &= b_1 \dots b_k a_k^{-1} \dots a_1^{-1} \cdot [d_l, c_l]^{-1} \dots [d_1, c_1]^{-1} \\
 &= b_1 \dots b_k a_k^{-1} \dots a_1^{-1} \cdot [c_l, d_l] \dots [c_1, d_1] \\
 &= w^{-1} [c_l, d_l] \dots [c_1, d_1] b_1 \dots b_k a_k^{-1} \dots a_1^{-1} w \\
 &= w^{-1} r^{-1} w.
 \end{aligned}$$

This shows that  $\varphi$  restricts to an automorphism of the subgroup of  $\Gamma$  normally generated by the relator  $r$ . In particular  $\varphi$  descends to an automorphism of  $\pi$ . This concludes the proof of the claim.

Recall that  $\pi(S)$  is a characteristic subgroup of  $\pi$ , hence  $\varphi : \pi \rightarrow \pi$  descends to an automorphism  $\pi/\pi(S) \rightarrow \pi/\pi(S)$  which we again denote by  $\varphi$ . Furthermore recall that  $\pi/\pi(S)$  is a finite solvable group. Since  $\pi/\pi(S)$  is finite there exists  $n > 0$  such that  $\varphi^n : \pi/\pi(S) \rightarrow \pi/\pi(S)$  acts as the identity. We can therefore consider the semidirect product  $\mathbb{Z}/n \ltimes \pi/\pi(S)$  where  $1 \in \mathbb{Z}/n$  acts on  $\pi/\pi(S)$  via  $\varphi$ .

It is now straightforward to check that the assignment

$$\begin{aligned}
 g &\mapsto (0, \alpha''(g)), \quad g \in \pi_1(\Sigma''), \\
 t_i &\mapsto (1, 0)
 \end{aligned}$$

defines a homomorphism

$$\pi_1(\Sigma) = \langle \pi_1(\Sigma''), t_1, \dots, t_k \mid t_i a_i t_i^{-1} = b_i, i = 1, \dots, k \rangle \rightarrow \mathbb{Z}/n \ltimes \pi/\pi(S)$$

which we denote by  $\beta$ . Clearly  $\beta : \pi_1(\Sigma) \rightarrow \mathbb{Z}/n \ltimes \pi/\pi(S)$  restricts to  $\pi_1(\Sigma'') \rightarrow \pi/\pi(S)$  and hence has the required properties. □

We can now prove the third statement of [Theorem 6.4](#).

*Proof of Theorem 6.4(3).* In light of [Lemma 6.6](#) (together with [Corollary 2.3](#)) and [Lemma 6.7](#) it suffices to show the following claim:

*Claim.* Let  $M$  be a 3-manifold and  $\Sigma \subset \partial M$  such that  $\pi_1(\Sigma) \rightarrow \pi_1(M)$  induces an isomorphism of prosolvable completions. Furthermore let  $M' \subset M$  be a submanifold with the following properties:

- (A)  $\Sigma' := \Sigma \cap M'$  is a connected subsurface of  $\Sigma'$ ,
- (B)  $\pi_1(\Sigma') \rightarrow \pi_1(\Sigma)$  is injective, and
- (C) for any homomorphism  $\alpha : \pi_1(M) \rightarrow S$  to a finite solvable group the inclusion map induces isomorphisms

$$H_j(\Sigma'; \mathbb{Z}[S]) \rightarrow H_j(M'; \mathbb{Z}[S])$$

for  $j = 0, 1$  and we have

$$\text{Im}\{\pi_1(\Sigma') \rightarrow \pi_1(M) \xrightarrow{\alpha} S\} = \text{Im}\{\pi_1(M') \rightarrow \pi_1(M) \xrightarrow{\alpha} S\}.$$

Then  $\pi_1(\Sigma') \rightarrow \pi_1(M')$  induces an isomorphism of prosolvable completions.

By [Lemma 2.10](#) we have to show that for any finite solvable group  $S$  the map

$$\iota^* : \text{Hom}(\pi_1(M'), S) \rightarrow \text{Hom}(\pi_1(\Sigma'), S)$$

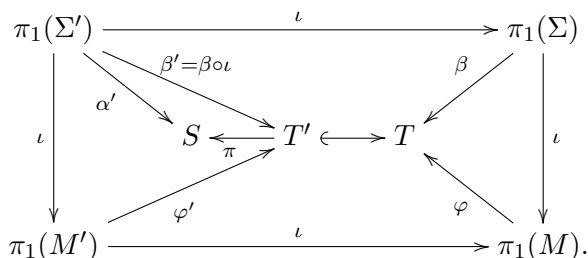
is a bijection.

So let  $S$  be a finite solvable group. We first show that  $\iota^* : \text{Hom}(\pi_1(M'), S) \rightarrow \text{Hom}(\pi_1(\Sigma'), S)$  is surjective. The various groups and maps in the proof are summarized in the diagram below. Assume we are given a homomorphism  $\alpha' : \pi_1(\Sigma') \rightarrow S$ . By (B) and [Proposition 6.8](#) there exists a homomorphism  $\beta : \pi_1(\Sigma) \rightarrow T$  to a finite solvable group and a homomorphism  $\pi : \text{Im}\{\pi_1(\Sigma') \rightarrow T\} \rightarrow S$  such that  $\pi \circ (\beta \circ \iota) = \alpha'$ . We write  $T' = \text{Im}\{\pi_1(\Sigma') \rightarrow T\}$  and  $\beta' = \beta \circ \iota : \pi_1(\Sigma') \rightarrow T'$ .

By our assumption  $\pi_1(\Sigma) \rightarrow \pi_1(M)$  induces isomorphisms of prosolvable completions and by [Lemma 2.10](#) there exists a homomorphism  $\varphi : \pi_1(M) \rightarrow T$  such that  $\beta = \varphi \circ \iota$ . By (C) we have

$$\begin{aligned} \text{Im}\{\pi_1(M') \rightarrow \pi_1(M) \xrightarrow{\varphi} T\} &= \text{Im}\{\pi_1(\Sigma') \xrightarrow{\iota} \pi_1(M) \xrightarrow{\varphi} T\} \\ &= \text{Im}\{\pi_1(\Sigma') \xrightarrow{\beta'} T\} = T'. \end{aligned}$$

Now denote the induced homomorphism  $\pi_1(M') \rightarrow T'$  by  $\varphi'$ . Clearly  $\varphi' \circ \iota = \beta'$ . Hence  $\alpha' = \pi \circ \beta' = (\pi \circ \varphi') \circ \iota$ . This shows that  $\iota^* : \text{Hom}(\pi_1(M'), S) \rightarrow \text{Hom}(\pi_1(\Sigma'), S)$  is surjective. The following diagram summarizes the homomorphisms in the proof of the previous claim:



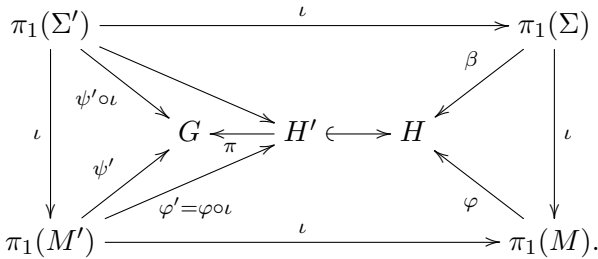
We now show that  $\iota^* : \text{Hom}(\pi_1(M'), S) \rightarrow \text{Hom}(\pi_1(\Sigma'), S)$  is injective. Let  $\alpha_1, \alpha_2 : \pi_1(M') \rightarrow S$  be two different homomorphisms. Let  $n$  be the maximal integer such that the homomorphisms  $\pi_1(M') \rightarrow S \rightarrow S/S^{(n)}$  induced by  $\alpha_1$  and  $\alpha_2$  agree. We will show that the restriction to  $\pi_1(\Sigma')$  of the maps  $\pi_1(M') \rightarrow S \rightarrow S/S^{(n+1)}$  induced by  $\alpha_1$  and  $\alpha_2$  are different. Without loss of generality we can therefore assume that  $S = S/S^{(n+1)}$ .

We denote by  $\psi'$  the homomorphism  $\pi_1(M') \rightarrow S \rightarrow S/S^{(n)} =: G$ , induced by  $\alpha_1$  and  $\alpha_2$ .

*Claim.* There exists a homomorphism  $\varphi : \pi_1(M) \rightarrow H$  to a finite solvable group and a homomorphism  $\pi : \text{Im}\{\pi_1(M') \rightarrow \pi_1(M) \rightarrow H\} \rightarrow G$  such that  $\psi' = \pi \circ (\varphi \circ \iota)$ .

By (B) and Proposition 6.8 there exists a homomorphism  $\beta : \pi_1(\Sigma) \rightarrow H$  to a finite solvable group  $H$  and a homomorphism  $\pi : \text{Im}\{\pi_1(\Sigma') \rightarrow H\} \rightarrow G$  such that  $\pi' \circ (\beta \circ \iota) = \psi' \circ \iota$ . By our assumption and by Lemma 2.10 there exists a homomorphism  $\varphi : \pi_1(M) \rightarrow H$  such that  $\beta = \varphi \circ \iota$ . By (C) we have  $\text{Im}\{\pi_1(\Sigma') \rightarrow H\} = \text{Im}\{\pi_1(M') \rightarrow H\} =: H'$ . It is now clear that  $\varphi$  and  $\pi$  have the required properties. This concludes the proof of the claim.

The following diagram summarizes the homomorphisms in the proof of the previous claim:



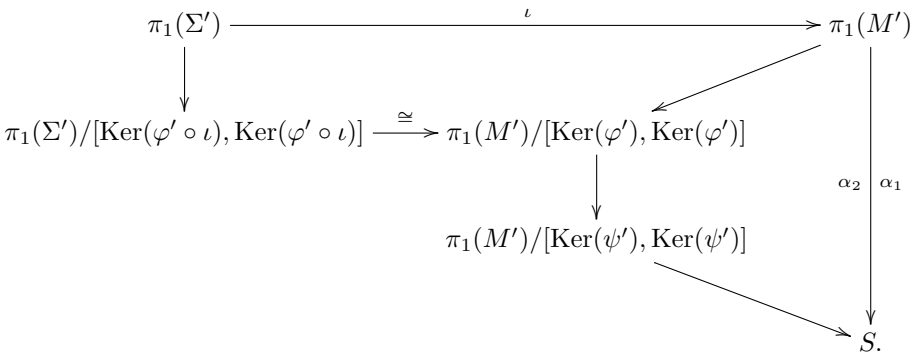
We now apply (C) and Corollary 2.5 to the case  $A = \pi_1(\Sigma'), B = \pi_1(M')$  and  $\varphi' : B \rightarrow H'$  to conclude that the inclusion map induces an isomorphism

$$\pi_1(\Sigma') / [\text{Ker}(\varphi' \circ \iota), \text{Ker}(\varphi' \circ \iota)] \rightarrow \pi_1(M') / [\text{Ker}(\varphi'), \text{Ker}(\varphi')].$$

We now consider the homomorphisms  $\alpha_1, \alpha_2 : \pi_1(M') \rightarrow S = S/S^{(n+1)}$ . First note that they factor through  $\pi_1(M') / [\text{Ker}(\psi'), \text{Ker}(\psi')]$ . Now note that  $\text{Ker}(\varphi') \subset \text{Ker}(\psi') \subset \pi_1(M')$ ; in particular we have a surjection  $\pi_1(M') / \text{Ker}(\varphi') \rightarrow \pi_1(M') / \text{Ker}(\psi')$  which gives rise to a surjection

$$\pi_1(M') / [\text{Ker}(\varphi'), \text{Ker}(\varphi')] \rightarrow \pi_1(M') / [\text{Ker}(\psi'), \text{Ker}(\psi')].$$

In particular  $\alpha_1, \alpha_2$  factor through  $\pi_1(M') / [\text{Ker}(\varphi'), \text{Ker}(\varphi')]$ . We therefore obtain the following commutative diagram



It is now clear that  $\alpha_1 \circ \iota$  and  $\alpha_2 \circ \iota$  are different. This concludes the proof that  $\iota^* : \text{Hom}(\pi_1(M'), S) \rightarrow \text{Hom}(\pi_1(\Sigma'), S)$  is injective. As we pointed out before, it now follows from [Lemma 2.10](#) that  $\iota : \pi_1(\Sigma') \rightarrow \pi_1(M')$  induces an isomorphism of prosolvable completions. □

### 7. The proof of [Theorem 1.2](#)

We start out with the following two results which allow us to reduce the proof of [Theorem 1.2](#) to the case of closed prime 3-manifolds.

**LEMMA 7.1.** *Let  $N$  be a 3-manifold with empty or toroidal boundary and let  $\phi \in H^1(N; \mathbb{Z})$  be nontrivial. If  $\Delta_{N, \phi}^\alpha$  is nonzero for any homomorphism  $\alpha : \pi_1(N) \rightarrow G$  to a finite group  $G$ , then  $N$  is prime.*

Note that the main idea for the proof of this lemma can already be found in [\[McC01\]](#).

*Proof.* Let  $N$  be a 3-manifold with empty or toroidal boundary which is not prime, i.e.  $N = N_1 \# N_2$  with  $N_1, N_2 \neq S^3$ . We have to show that there exists a homomorphism  $\alpha : \pi_1(N) \rightarrow G$  to a finite group such that  $\Delta_{N, \phi}^\alpha = 0$ . Recall that by [Lemma 2.9](#) we have  $\Delta_{N, \phi}^\alpha = 0$  if and only if  $H_1(N; \mathbb{Q}[G][t^{\pm 1}])$  is not  $\mathbb{Q}[t^{\pm 1}]$ -torsion. Note that we can write  $N = (N_1 \setminus \text{int}D^3) \cup_{S^2} (N_2 \setminus \text{int}D^3)$  and that  $H_j(N_i \setminus \text{int}D^3; \mathbb{Q}[t^{\pm 1}]) = H_j(N_i; \mathbb{Q}[t^{\pm 1}])$  for  $j = 0, 1$  and  $i = 1, 2$ . The Mayer-Vietoris sequence corresponding to  $N = (N_1 \setminus \text{int}D^3) \cup_{S^2} (N_2 \setminus \text{int}D^3)$  now gives rise to the following long exact sequence:

$$\begin{aligned} H_1(S^2; \mathbb{Q}[t^{\pm 1}]) &\rightarrow H_1(N_1; \mathbb{Q}[t^{\pm 1}]) \oplus H_1(N_2; \mathbb{Q}[t^{\pm 1}]) \rightarrow H_1(N; \mathbb{Q}[t^{\pm 1}]) \rightarrow \\ H_0(S^2; \mathbb{Q}[t^{\pm 1}]) &\rightarrow H_0(N_1; \mathbb{Q}[t^{\pm 1}]) \oplus H_0(N_2; \mathbb{Q}[t^{\pm 1}]) \rightarrow H_0(N; \mathbb{Q}[t^{\pm 1}]) \rightarrow 0. \end{aligned}$$

A straightforward computation shows that  $H_0(S^2; \mathbb{Q}[t^{\pm 1}]) = \mathbb{Q}[t^{\pm 1}]$  and that  $H_1(S^2; \mathbb{Q}[t^{\pm 1}]) = 0$ .

First assume that  $b_1(N_i) > 0$  for  $i = 1, 2$ . Denote by  $\phi_i$  the restriction of  $\phi : H_1(N; \mathbb{Q}) \rightarrow \mathbb{Q}$  to  $H_1(N_i; \mathbb{Q})$ . If  $\phi_i$  is nontrivial for  $i = 1$  and  $i = 2$ , then it follows from [Lemmas 2.2](#) and [2.9](#) that  $H_0(N_i; \mathbb{Q}[t^{\pm 1}])$  is  $\mathbb{Q}[t^{\pm 1}]$ -torsion for  $i = 1, 2$ . On the other hand we have  $H_0(S^2; \mathbb{Q}[t^{\pm 1}]) = \mathbb{Q}[t^{\pm 1}]$ . It follows from the above Mayer-Vietoris sequence that  $H_1(N; \mathbb{Q}[t^{\pm 1}])$  cannot be  $\mathbb{Q}[t^{\pm 1}]$ -torsion. On the other hand, if  $\phi_i$  is trivial for some  $i \in \{1, 2\}$ , then  $H_1(N_i; \mathbb{Q}[t^{\pm 1}])$  is isomorphic to  $H_1(N_i; \mathbb{Q}) \otimes \mathbb{Q}[t^{\pm 1}]$ . In particular  $H_1(N_i; \mathbb{Q}[t^{\pm 1}])$  is not  $\mathbb{Q}[t^{\pm 1}]$ -torsion, and by use of  $H_1(S^2; \mathbb{Q}[t^{\pm 1}]) = 0$  it follows again from the above Mayer-Vietoris sequence that  $H_1(N; \mathbb{Q}[t^{\pm 1}])$  is not  $\mathbb{Q}[t^{\pm 1}]$ -torsion.

Now assume that either  $b_1(N_1) = 0$  or  $b_1(N_2) = 0$ . Without loss of generality we can assume that  $b_1(N_2) = 0$ . Since  $b_1(N) = b_1(N_1) + b_1(N_2)$  we have  $b_1(N_1) > 0$ . By the Geometrization Conjecture  $\pi_1(N_2)$  is nontrivial and residually finite (cf. [\[Thu82\]](#) and [\[Hem87\]](#)). In particular there exists an



epimorphism  $\alpha : \pi_1(N_2) \rightarrow G$  onto a nontrivial finite group  $G$ . Denote the homomorphism  $\pi_1(N) = \pi_1(N_1) * \pi_1(N_2) \rightarrow \pi_1(N_2) \rightarrow G$  by  $\alpha$  as well. Then by [Lemma 2.8](#) we have

$$\Delta_{N,\phi}^\alpha = \Delta_{N_G,\phi_G}$$

where  $p : N_G \rightarrow N$  is the cover of  $N$  corresponding to  $\alpha$  and  $\phi_G = p^*(\phi)$ . But the prime decomposition of  $N_G$  has  $|G|$  copies of  $N_1$ . By the argument above we now have  $\Delta_{N_G,\phi_G} = 0$ , which implies that  $\Delta_{N,\phi}^\alpha = \Delta_{N_G,\phi_G} = 0$ . □

**LEMMA 7.2.** *Let  $N$  be an irreducible 3-manifold with nonempty toroidal boundary and let  $\phi \in H^1(N; \mathbb{Z})$  be nontrivial. Let  $W = N \cup_{\partial N} N$  be the double of  $N$  along the boundary of  $N$ . Let  $\Phi = p^*(\phi) \in H^1(W; \mathbb{Z})$  where  $p : W \rightarrow N$  denotes the folding map. Then the following hold:*

- (1)  $(W, \Phi)$  fibers over  $S^1$  if and only if  $(N, \phi)$  fibers over  $S^1$ ;
- (2) if  $(N, \phi)$  satisfies Condition (\*), then  $(W, \Phi)$  satisfies Condition (\*).

In the proof of [Lemma 7.2](#) we will make use of the following well-known lemma. We refer to [[EN85](#), Th. 4.2] and [[Rou74](#)] for the first statement, and to [[EN85](#), p. 33] for the second statement.

**LEMMA 7.3.** *Let  $Y$  be a closed 3-manifold. Let  $T \subset Y$  be a union of incompressible tori such that  $T$  separates  $Y$  into two connected components  $Y_1$  and  $Y_2$ . Let  $\psi \in H^1(Y; \mathbb{Z})$ . Then the following hold:*

- (1) If  $\|\phi\|_{T,Y} > 0$ , then  $(Y, \psi)$  fibers over  $S^1$  if and only if  $(Y_1, \psi|_{Y_1})$  and  $(Y_2, \psi|_{Y_2})$  fiber over  $S^1$ ;
- (2)  $\|\psi\|_{T,Y} = \|\psi|_{Y_1}\|_{T,Y_1} + \|\psi|_{Y_2}\|_{T,Y_2}$ .

*Proof of Lemma 7.2.* First note that an irreducible 3-manifold with boundary a union of tori has compressible boundary if and only if it is the solid torus. Since the lemma holds trivially in the case that  $N = S^1 \times D^2$  we will from now on assume that  $N$  has incompressible boundary. This implies in particular that  $\|\phi\|_T > 0$ . The first statement is now an immediate consequence of [Lemma 7.3](#) and the observation that  $\Phi|_N = \phi$ .

Now assume that  $(N, \phi)$  satisfies Condition (\*). In the following we write  $N_i = N, i = 1, 2$  and we think of  $W$  as  $W = N_1 \cup_{\partial N_1 = \partial N_2} N_2$ . Let  $\alpha : \pi_1(N) \rightarrow G$  be a homomorphism to a finite group  $G$ . We write  $n = |G|, V = \mathbb{Z}[G]$  and we slightly abuse notation by denoting by  $\alpha$  the representation  $\pi_1(W) \rightarrow \text{Aut}(V)$  given by left multiplication. We have to show that  $\Delta_{W,\Phi}^\alpha \in \mathbb{Z}[t^{\pm 1}]$  is monic and that

$$\deg(\Delta_{W,\Phi}^\alpha) - \deg(\Delta_{W,\Phi,0}^\alpha) - \deg(\Delta_{W,\Phi,2}^\alpha) = n \|\Phi\|_T$$

(here we used [Lemma 2.8](#) to rephrase the last condition). For any submanifold  $X \subset W$  we denote the restriction of  $\Phi$  and  $\alpha$  to  $\pi_1(X)$  by  $\Phi$  and  $\alpha$  as well. Evidently the restriction of  $\Phi$  to  $N = N_i, i = 1, 2$  just agrees with  $\phi$ .

In order to prove the claims on  $\Delta_{W,\Phi}^\alpha$  we will in the following express  $\Delta_{W,\Phi}^\alpha$  in terms of  $\Delta_{N_i,\phi_i}^\alpha$ ,  $i = 1, 2$ . The following statement combines the assumption that  $(N, \phi)$  satisfies Condition  $(*)$  with [Lemmas 2.8 and 2.9](#).

*Fact 1.* For  $i = 1, 2$ ,

$$\deg(\Delta_{N_i,\phi}^\alpha) - \deg(\Delta_{N_i,\Phi,0}^\alpha) - \deg(\Delta_{N_i,\Phi,2}^\alpha) = n \|\Phi\|_{T,N_i}.$$

Furthermore for all  $j$  we have that  $\Delta_{N_i,\Phi,j}^\alpha$  is monic.

We now turn to the twisted Alexander polynomials of the boundary tori of  $\partial N$ . The following is an immediate consequence of [Theorem 6.5](#).

*Fact 2.* If  $\Delta_{N,\phi} \neq 0$  (in particular if  $(N, \phi)$  satisfies Condition  $(*)$ ), then for any boundary component  $T \subset \partial N$  the restriction of  $\phi$  (and hence of  $\Phi$ ) to  $\pi_1(T)$  is nontrivial.

This fact and a straightforward computation now gives us the following fact (cf. e.g. [\[KL99\]](#)).

*Fact 3.* Let  $T \subset \partial N$  be any boundary component. Then

- (1)  $\Delta_{T,\Phi,i}^\alpha$  is monic for any  $i$ .
- (2)  $H_i(T; V[t^{\pm 1}]) = 0$  for  $i \geq 2$ ; in particular  $\Delta_{T,\Phi,i}^\alpha = 1$  for  $i \geq 2$ ,
- (3)  $\Delta_{T,\Phi,0}^\alpha = \Delta_{T,\Phi,1}^\alpha$ .

We now consider the following Mayer-Vietoris sequence:

$$\begin{array}{ccccccc} 0 & \rightarrow & H_2(N_1; V[t^{\pm 1}]) \oplus H_2(N_2; V[t^{\pm 1}]) & \rightarrow & H_2(W; V[t^{\pm 1}]) & \rightarrow & \\ H_1(\partial N; V[t^{\pm 1}]) & \rightarrow & H_1(N_1; V[t^{\pm 1}]) \oplus H_1(N_2; V[t^{\pm 1}]) & \rightarrow & H_1(W; V[t^{\pm 1}]) & \rightarrow & \\ H_0(\partial N; V[t^{\pm 1}]) & \rightarrow & H_0(N_1; V[t^{\pm 1}]) \oplus H_0(N_2; V[t^{\pm 1}]) & \rightarrow & H_0(W; V[t^{\pm 1}]) & \rightarrow & 0. \end{array}$$

Recall that we assume that  $(N, \phi)$  (and hence  $(N_i, \phi), i = 1, 2$ ) satisfy Condition  $(*)$ . By [Lemmas 2.8 and 2.9](#) and [Facts 1 and 3](#) it follows that all homology modules in the above long exact sequence but possibly  $H_1(W; V[t^{\pm 1}])$  and  $H_2(W; V[t^{\pm 1}])$  are  $\mathbb{Z}[t^{\pm 1}]$ -torsion. But then evidently  $H_1(W; V[t^{\pm 1}])$  and  $H_2(W; V[t^{\pm 1}])$  also have to be  $\mathbb{Z}[t^{\pm 1}]$ -torsion. Furthermore it follows from [Fact 3](#), [\[Tur01, Th. 3.4\]](#) and [\[Tur01, Th. 4.7\]](#) that

$$(5) \quad \frac{\Delta_{W,\Phi,1}^\alpha}{\Delta_{W,\Phi,0}^\alpha \Delta_{W,\Phi,2}^\alpha} = \frac{\Delta_{N_1,\Phi,1}^\alpha}{\Delta_{N_1,\Phi,0}^\alpha \Delta_{N_1,\Phi,2}^\alpha} \cdot \frac{\Delta_{N_2,\Phi,1}^\alpha}{\Delta_{N_2,\Phi,0}^\alpha \Delta_{N_2,\Phi,2}^\alpha}.$$

Note that  $\Delta_{W,\Phi,0}^\alpha$  and  $\Delta_{W,\Phi,2}^\alpha$  are monic by [Lemma 2.8](#), it now follows from [Fact 1](#) and [equality \(5\)](#) that  $\Delta_{W,\Phi,1}^\alpha$  is monic as desired.

Finally we can appeal to [Lemma 7.3](#) to conclude that  $\|\Phi\|_{T,W} = \|\Phi\|_{T,N_1} + \|\Phi\|_{T,N_2}$ . It therefore follows from [Fact 1](#) and [equation \(5\)](#) that

$$\deg(\Delta_{W,\Phi}^\alpha) - \deg(\Delta_{W,\Phi,0}^\alpha) - \deg(\Delta_{W,\Phi,2}^\alpha) = n \|\Phi\|_{T,W}$$

as required. □

Let  $N$  be a 3-manifold with empty or toroidal boundary. We write  $\pi = \pi_1(N)$ . Let  $\tilde{\pi} \subset \pi$  be a finite index subgroup and  $\phi \in H^1(N; \mathbb{Z})$  be nontrivial. We now say that the pair  $(\tilde{\pi}, \phi)$  has Property (M) if the twisted Alexander polynomial  $\Delta_{N, \phi}^{\pi/\tilde{\pi}} \in \mathbb{Z}[t^{\pm 1}]$  is monic and if

$$\deg(\Delta_{N, \phi}^{\pi/\tilde{\pi}}) = [\pi : \tilde{\pi}] \|\phi\|_T + (1 + b_3(N)) \operatorname{div} \phi_{\tilde{\pi}}$$

holds.

The first statement of the following lemma is well-known; the second one can be easily verified and the third is an immediate consequence of the second statement.

LEMMA 7.4. *Let  $N$  be a 3-manifold with empty or toroidal boundary and let  $\phi \in H^1(N; \mathbb{Z})$  be nontrivial. Let  $k \neq 0 \in \mathbb{Z}$ . Then the following hold:*

- (1)  $(N, \phi)$  fibers over  $S^1$  if and only if  $(N, k\phi)$  fibers over  $S^1$ .
- (2) Let  $\tilde{\pi} \subset \pi$  be a finite index subgroup. Then  $(\tilde{\pi}, \phi)$  has Property (M) if and only if  $(\tilde{\pi}, k\phi)$  has Property (M).
- (3)  $(N, \phi)$  satisfies Condition (\*) if and only if  $(N, k\phi)$  satisfies Condition (\*).

We will also need the following lemma.

LEMMA 7.5. *Let  $N$  be a 3-manifold with empty or toroidal boundary and let  $\phi \in H^1(N; \mathbb{Z})$  be nontrivial. Suppose that all finite index normal subgroups of  $\pi_1(N)$  have Property (M); then in fact all finite index subgroups of  $\pi_1(N)$  have Property (M).*

*Proof.* We write  $\pi := \pi_1(N)$ . Let  $\phi \in H^1(N; \mathbb{Z})$  be nontrivial. By Lemma 7.4(2) we can without loss of generality assume that  $\phi$  is primitive. Let  $\tilde{\pi} \subset \pi$  be a finite index subgroup. We denote by  $\hat{\pi} \subset \pi$  the core of  $\tilde{\pi}$ , i.e.  $\hat{\pi} = \bigcap_{g \in \pi} g\tilde{\pi}g^{-1}$ . Note that  $\hat{\pi}$  is normal in  $\pi$  and contained in  $\tilde{\pi}$ .

By Proposition 3.1 the class  $\phi$  is dual to a connected Thurston norm-minimizing surface  $\Sigma$ . We write  $A = \pi_1(\Sigma)$  and  $B = \pi_1(N \setminus \nu\Sigma)$  as before.

We write  $\widehat{B} := B \cap \hat{\pi}$  and  $\widehat{A}^{\pm} := (\iota_{\pm})^{-1}(\widehat{B})$ . We now pick representatives  $g_1, \dots, g_m$  for the equivalence classes of  $B \setminus \pi/\tilde{\pi}$ . For  $i = 1, \dots, m$  we write  $\widetilde{B}_i := B \cap g_i\tilde{\pi}g_i^{-1}$  and  $\widetilde{A}_i^{\pm} := (\iota_{\pm})^{-1}(\widetilde{B}_i)$ .

Since  $\hat{\pi} \subset \pi$  is normal and since we assume that normal finite index subgroups have Property (M) we can now apply Proposition 3.2 and Lemma 2.1 to conclude that

$$\iota_{\pm} : H_j(A; \mathbb{Z}[B/\widehat{B}]) \rightarrow H_j(B; \mathbb{Z}[B/\widehat{B}])$$

are isomorphisms for  $j = 0, 1$ . It now follows from Corollary 2.5 that the maps

$$\iota_{\pm} : A/\widehat{A}^{\pm} \rightarrow B/\widehat{B} \text{ and } \iota_{\pm} : A/[\widehat{A}^{\pm}, \widehat{A}^{\pm}] \rightarrow B/[\widehat{B}, \widehat{B}]$$

are isomorphisms. Recall that  $\hat{\pi}$  is normal in  $\pi$ , it follows that  $\widehat{B} \subset B$  is normal and for any  $i$  we have  $\widehat{B} = B \cap g_i\hat{\pi}g_i^{-1} \subset B \cap g_i\tilde{\pi}g_i^{-1} = \widetilde{B}_i$ . We now deduce

from Lemma 2.7 that

$$\iota_{\pm} : A/\tilde{A}_i^{\pm} \rightarrow B/\tilde{B}_i \text{ and } \iota_{\pm} : A/[\tilde{A}_i^{\pm}, \tilde{A}_i^{\pm}] \rightarrow B/[\tilde{B}_i, \tilde{B}_i]$$

are bijections for  $i = 1, \dots, m$ . It now follows from Lemma 2.4 that the maps

$$\iota_{\pm} : H_j(A; \mathbb{Z}[\pi/\tilde{\pi}]) \rightarrow H_j(B; \mathbb{Z}[\pi/\tilde{\pi}])$$

are isomorphisms. It now follows from Proposition 3.2 that  $\tilde{\pi}$  also has Property (M). □

We will now use the previous lemma to prove the following lemma.

LEMMA 7.6. *Let  $N$  be a 3-manifold with empty or toroidal boundary and let  $\phi \in H^1(N; \mathbb{Z})$  be nontrivial. Let  $p : N' \rightarrow N$  be a finite cover. We write  $\phi' = p^*(\phi) \in H^1(N'; \mathbb{Z})$ . Then the following hold:*

- (1)  $\phi'$  is nontrivial;
- (2)  $(N, \phi)$  fibers over  $S^1$  if and only if  $(N', \phi')$  fibers over  $S^1$ ;
- (3) if  $(N, \phi)$  satisfies Condition (\*), then  $(N', \phi')$  satisfies Condition (\*).

*Proof.* The first statement is well-known. The second statement is a consequence of [Hem76, Th. 10.5]. We now turn to the third statement. Assume that  $(N, \phi)$  satisfies Condition (\*).

Let  $\tilde{\pi}$  be a normal finite index subgroup of  $\pi' = \pi_1(N')$ . We have to show that  $(\tilde{\pi}, \phi')$  has Property (M). Note that  $\tilde{\pi}$  viewed as a subgroup of  $\pi = \pi_1(N)$  is not necessarily normal. It nonetheless follows from the assumption that  $(N, \phi)$  satisfies Condition (\*) and from Lemma 7.5 that the twisted Alexander polynomial  $\Delta_{N, \phi}^{\pi/\tilde{\pi}} \in \mathbb{Z}[t^{\pm 1}]$  is monic and that

$$\deg(\Delta_{N, \phi}^{\pi/\tilde{\pi}}) = [\pi : \tilde{\pi}] \|\phi\|_T + (1 + b_3(N))\text{div}\phi_{\tilde{\pi}}$$

holds. It now follows easily from Lemma 2.8,  $b_3(N) = b_3(N')$ , and the multiplicative property of the Thurston norm under finite covers (cf. [Gab83, Cor. 6.13]) that the twisted Alexander polynomial  $\Delta_{N', \phi'}^{\pi'/\tilde{\pi}} \in \mathbb{Z}[t^{\pm 1}]$  is monic and that the following equality holds:

$$\deg(\Delta_{N', \phi'}^{\pi'/\tilde{\pi}}) = [\pi' : \tilde{\pi}] \|\phi'\|_T + (1 + b_3(N'))\text{div}\phi'_{\tilde{\pi}}.$$

In particular  $(\tilde{\pi}, \phi')$  has Property (M). □

We are now finally in a position to prove Theorem 1.2.

*Proof of Theorem 1.2.* First note that the combination of Theorem 1.9 and Lemmas 7.1, 7.2, 7.6 and 7.4 shows that it suffices to observe the following:

*Claim.* Assume we are given a pair  $(N, \phi)$  where

- (1)  $N$  is a closed irreducible 3-manifold such that the fundamental group of each JSJ component is residually  $p$ , and

(2)  $\phi$  is primitive.

If  $(N, \phi)$  satisfies Condition  $(*)$ , then  $(N, \phi)$  fibers over  $S^1$ .

Let  $(N, \phi)$  be a pair (as in the claim) which satisfies Condition  $(*)$ . If  $\|\phi\|_T = 0$ , then it follows from [FV08a, Prop. 4.6] that  $(N, \phi)$  fibers over  $S^1$ .

We can and will henceforth assume that  $\|\phi\|_T > 0$ . We denote the tori of the JSJ decomposition of  $N$  by  $T_1, \dots, T_r$ . We pick a connected Thurston norm-minimizing surface  $\Sigma$  dual to  $\phi$  and a tubular neighborhood  $\nu\Sigma = \Sigma \times [-1, 1] \subset N$  as in Section 6.1. In particular we can and will throughout assume that  $\Sigma \times t$  and the tori  $T_1, \dots, T_r$  are in general position for any  $t \in [-1, 1]$  and that for any  $i \in \{1, \dots, r\}$  any component of  $\Sigma \cap T_i$  represents a nontrivial element in  $\pi_1(T_i)$ . Furthermore, as in Section 6, we assume that our choice of  $\Sigma$  minimizes the number  $\sum_{i=1}^r b_0(\Sigma \cap T_i)$ .

Let  $A_1, \dots, A_m$  be the components of the intersection of the tori  $T_1, \dots, T_r$  with  $M := N \setminus \Sigma \times (-1, 1)$ . Furthermore let  $M_1, \dots, M_n$  be the components of  $M$  cut along  $A_1 \cup \dots \cup A_m$ . Recall that any  $M_i$  is a submanifold of a JSJ component of  $N$ .

For  $i = 1, \dots, m$  write  $C_i = A_i \cap \Sigma^-$ . It follows from Lemma 6.3 that for  $i = 1, \dots, m$  the surface  $A_i$  is an annulus which is a product on  $C_i$ , i.e.  $C_i$  consists of one component and  $\pi_1(C_i) \rightarrow \pi_1(A_i)$  is an isomorphism.

In order to show that  $M$  is a product on  $\Sigma^-$  it suffices to show that  $\pi_1(\Sigma_i^-) \rightarrow \pi_1(M_i)$  is an isomorphism for any  $i \in \{1, \dots, n\}$ . So let  $i \in \{1, \dots, n\}$ . Since  $(N, \phi)$  satisfies Condition  $(*)$  it follows from Proposition 1.7 that the maps  $\pi_1(\Sigma^\pm) \rightarrow \pi_1(M)$  induce an isomorphism of prosolvable completions. By Theorem 6.4(1) the surfaces  $\Sigma_i^\pm$  are connected, and by Theorem 6.4(3) the inclusion induced maps  $\pi_1(\Sigma_i^\pm) \rightarrow \pi_1(M_i)$ ,  $i = 1, \dots, n$ , also induce isomorphisms of prosolvable completions. By Theorem 6.4(2) we have that the group  $\pi_1(M_i)$  is a subgroup of the fundamental group of a JSJ component of  $N$ . By our assumption this implies that  $\pi_1(M_i)$  is residually  $p$ , in particular residually finite solvable.

In the following we view  $M_i$  as a sutured manifold with sutures given by  $\gamma_i = \partial N \cap M_i$ . We can pick orientations such that  $R_-(\gamma_i) = \Sigma_i^-$  and  $R_+(\gamma_i) = \Sigma_i^+$ . Since  $\Sigma \subset N$  is Thurston norm minimizing it follows that  $(M_i, \gamma_i)$  is a taut sutured manifold. We can therefore now apply Theorem 4.1 to conclude that  $(M_i, \gamma_i)$  is a product sutured manifold, i.e.  $\pi_1(\Sigma_i^-) \rightarrow \pi_1(M_i)$  is an isomorphism. □

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MATHEMATISCHES INSTITUT, UNIVERSITÄT ZU KÖLN, GERMANY

*E-mail:* [sfriedl@gmail.com](mailto:sfriedl@gmail.com)

UNIVERSITY OF CALIFORNIA, RIVERSIDE, CA

*E-mail:* [svidussi@math.ucr.edu](mailto:svidussi@math.ucr.edu)