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On the structure of the Selberg class, VII: $1 < d < 2$

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Abstract

The Selberg class $\mathcal{S}$ is a rather general class of Dirichlet series with functional equation and Euler product and can be regarded as an axiomatic model for the global L-functions arising from number theory and automorphic representations. One of the main problems of the Selberg class theory is to classify the elements of $\mathcal{S}$. Such a classification is based on a real-valued invariant d called degree, and the degree conjecture asserts that $d \in \mathbb{N}$ for every L-function in $\mathcal{S}$. The degree conjecture has been proved for $d < 5/3$, and in this paper we extend its validity to $d < 2$. The proof requires several new ingredients, in particular a rather precise description of the properties of certain nonlinear twists associated with the L-functions in $\mathcal{S}$.

1. **Introduction**

The Selberg class, introduced by Selberg [17] and denoted by $\mathcal{S}$, is a rather general class of Dirichlet series with functional equation and Euler product (see below for definitions) and contains, at least conjecturally, the global L-functions arising from number theory and automorphic representations. In fact, the Selberg class may be regarded as an axiomatic model of the L-functions, and the main problem, apart from classical open problems such as the Riemann Hypothesis, is to classify its elements. The classification is based on the degree d_F of the functions $F \in \mathcal{S}$, a real-valued invariant which somehow measures the analytic complexity of an L-function (see Bombieri [3]), and according to a rather widely accepted expectation can be formulated in two parts as follows. The first part, called the degree conjecture, states that d_F is a nonnegative integer for every $F \in \mathcal{S}$, while the second part, a kind of general analytic version of the Langlands program, predicts that the functions in $\mathcal{S}$ with integer degree d coincide with the automorphic L-functions of degree d.

We recall that the class $\mathcal{S}$ consists of the ordinary Dirichlet series $F(s)$ such that

i) $F(s)$ is absolutely convergent for $\sigma > 1$;
ii) $(s-1)^m F(s)$ is an entire function of finite order for some integer $m \geq 0$;

iii) $F(s)$ satisfies a functional equation of type $\Phi(s) = \omega\bar{\Phi}(1-s)$, where $|\omega| = 1$ and

$$\Phi(s) = Q^s \prod_{j=1}^{r} \Gamma(\lambda_j s + \mu_j) F(s)$$

with $r \geq 0$, $Q > 0$, $\lambda_j > 0$, $\Re\mu_j \geq 0$ (here and in the sequel we write $\bar{f}(s) = \overline{f(\bar{s})}$, and an empty product equals 1);

iv) the Dirichlet coefficients $a_F(n)$ of $F(s)$ satisfy $a_F(n) \ll n^\varepsilon$ for every $\varepsilon > 0$;

v) $\log F(s)$ is a Dirichlet series with coefficients $b_F(n)$ satisfying $b_F(n) = 0$ unless $n = p^m$, $m \geq 1$, and $b_F(n) \ll n^\vartheta$ for some $\vartheta < 1/2$.

We also recall that $\mathcal{S}^\sharp$ denotes the extended Selberg class, consisting of the nonzero functions satisfying only axioms i), ii) and iii). Further, the degree d_F, the conductor q_F and the shift θ_F of $F \in \mathcal{S}^\sharp$ are invariants defined by

$$d_F = 2\sum_{j=1}^{r} \lambda_j, \qquad q_F = (2\pi)^{d_F} Q^2 \prod_{j=1}^{r} \lambda_j^{2\lambda_j}, \qquad \theta_F = 2\Im\left(\sum_{j=1}^{r}\left(\mu_j - \frac{1}{2}\right)\right),$$

and $\mathcal{S}_d$ (resp. $\mathcal{S}_d^\sharp$) denotes the subclass of $\mathcal{S}$ (resp. $\mathcal{S}^\sharp$) consisting of the functions with given degree d. Given an entire $F \in \mathcal{S}_d^\sharp$ and $\tau \in \mathbb{R}$, the shifted function $F_\tau(s) = F(s + i\tau)$ belongs to $\mathcal{S}_d^\sharp$ as well, and clearly there exists a τ such that $\theta_{F_\tau} = 0$. Hence, in the proof of nonexistence results for entire $F \in \mathcal{S}_d^\sharp$ we may assume that $\theta_F = 0$. For further information about the Selberg class we refer to our survey papers [8], [6], [15] and [14]. Finally, we write $e(x) = e^{2\pi i x}$ and $f(x) \asymp g(x)$ if $f(x) \ll g(x) \ll f(x)$, we denote by $|\mathcal{A}|$ the cardinality of the set $\mathcal{A}$ and let $\delta_{a,b} = 1$ if $a = b$ and $\delta_{a,b} = 0$ otherwise.

Not much is known about the structure of the Selberg class. Richert [16], Bochner [2] and Conrey-Ghosh [4] independently proved that $\mathcal{S}_d = \emptyset$ for $0 < d < 1$, and Conrey-Ghosh [4] proved that $\mathcal{S}_0 = \{1\}$. Moreover, in [7] we showed that $\mathcal{S}_1$ consists of the Riemann zeta function $\zeta(s)$ and the shifted Dirichlet L-functions $L(s + i\tau, \chi)$ with $\tau \in \mathbb{R}$ and χ primitive, and in [9] we proved that $\mathcal{S}_d = \emptyset$ for $1 < d < 5/3$. These results confirm the above described expectation when the degree lies in the range $[0, 5/3)$. We remark that the degree conjecture is expected to hold in the more general framework of $\mathcal{S}^\sharp$, and in fact the nonexistence results for $d \in (0,1) \cup (1, 5/3)$ are proved in such a framework. However, the degree conjecture badly fails if the ordinary Dirichlet series in axiom i) are replaced by general Dirichlet series of type $\sum_{n=1}^\infty a_n k_n^{-s}$ with positive increasing $k_n \to \infty$. Indeed, in this case every functional equation in axiom iii) has uncountably many linearly independent such solutions; see [11]. This shows that the degree conjecture is a rather delicate problem, highly sensitive on the arithmetic structure of the k_n. In fact, integer k_n are fundamental in all proofs of nonexistence results, and are usually exploited via the periodicity of $e(\alpha k_n)$. In this paper we push further the validity of the degree conjecture by proving the following

THEOREM. *$\mathcal{S}_d^\sharp = \emptyset$ for $1 < d < 2$.*

Further proofs that $\mathcal{S}_d^\sharp = \emptyset$ for $0 < d < 1$ have been given by Molteni [13] and in our papers [9] and [12]. In particular, the proof in [12] is based on the *standard nonlinear twist*, defined for $\alpha > 0$ and $F \in \mathcal{S}_d^\sharp$ with $d > 0$ by

$$F_d(s,\alpha) = \sum_{n=1}^{\infty} \frac{a_F(n)}{n^s} e(-n^{1/d}\alpha), \qquad \sigma > 1$$

and satisfying the following properties: $F_d(s,\alpha)$ is meromorphic on $\mathbb{C}$ and, writing

$$n_\alpha = q_F d^{-d}\alpha^d, \qquad a_F(n_\alpha) = \begin{cases} a_F(n_\alpha) & \text{if } n_\alpha \in \mathbb{N} \\ 0 & \text{if } n_\alpha \notin \mathbb{N}, \end{cases} \tag{1.1}$$

it is entire if $a_F(n_\alpha) = 0$, while if $a_F(n_\alpha) \neq 0$ it has a simple pole at $s_0 = \frac{d+1}{2d} - i\frac{\theta_F}{d}$; see [12] for a fuller account. The set

$$\operatorname{Spec}(F) = \{\alpha > 0 : a_F(n_\alpha) \neq 0\} \tag{1.2}$$

is called the *spectrum* of $F(s)$ and is clearly an unbounded subset of $\mathbb{R}^+$. The nonexistence result for $0 < d < 1$ follows then by choosing $\alpha \in \operatorname{Spec}(F)$ and observing that $\Re s_0 = \frac{d+1}{2d} > 1$ in this case. The properties of the standard nonlinear twist, in the case $d = 1$ where it becomes the linear twist

$$F(s,\alpha) = \sum_{n=1}^{\infty} \frac{a_F(n)}{n^s} e(-n\alpha), \qquad \sigma > 1,$$

were exploited also in [7], in order to classify the functions of $\mathcal{S}_1$ and $\mathcal{S}_1^\sharp$; see Soundararajan [18] for a shorter proof. In [9] we used a transformation formula, valid for $1 < d < 2$, relating $F(s,\alpha)$ to the nonlinear twist

$$\sum_{n=1}^{\infty} \frac{\overline{a_F(n)}}{n^{s^*}} e(An^\kappa), \quad \kappa = \frac{1}{d-1}, \ A = \frac{d-1}{(\alpha q_F)^\kappa}, \ s^* = \frac{s + \frac{d}{2} - 1 + i\theta_F}{d-1} \tag{1.3}$$

to prove that $\mathcal{S}_d^\sharp = \emptyset$ for $1 < d < 5/3$. Moreover, since $\Re s^* > \sigma$ for $\sigma > 1/2$, choosing $\alpha = 1$ the periodicity of the linear twist implies the nonexistence of polar $F \in \mathcal{S}_d^\sharp$ for every $1 < d < 2$; see again [9]. In this paper we combine the above ideas to deal with the more difficult case of entire $F \in \mathcal{S}_d^\sharp$ with $1 < d < 2$. This process is rather complicated and requires several new ingredients; hence for sake of clarity we first outline the main steps.

Our main tool is a general transformation formula for nonlinear twists of functions $F \in \mathcal{S}_d^\sharp$, of which the above mentioned transformation formula for the linear twist (see [9]) is a special case. Let $d \geq 1$, $N \geq 0$ be an integer, $\boldsymbol{\alpha} = (\alpha_0, \dots, \alpha_N) \in \mathbb{R}^{N+1}$ and for $\xi > 0$

(1.4)

$$f(\xi, \boldsymbol{\alpha}) = \xi^{\kappa_0} \sum_{\nu=0}^{N} \alpha_\nu \xi^{-\omega_\nu} \quad \text{with} \;\; d\kappa_0 > 1,\; 0 = \omega_0 < \cdots < \omega_N < \kappa_0,\; \alpha_0 > 0.$$

The *multidimensional nonlinear twist* of $F \in \mathcal{S}_d^\sharp$ associated with $f(\xi, \boldsymbol{\alpha})$ is defined by

$$F(s; f) = \sum_{n=1}^{\infty} \frac{a_F(n)}{n^s} e(-f(n, \boldsymbol{\alpha})), \qquad \sigma > 1. \tag{1.5}$$

Let further

$$\mathcal{D}_f = \{\omega = \sum_{\nu=1}^{N} m_\nu \omega_\nu : m_\nu \in \mathbb{Z}, m_\nu \geq 0\}$$

(1.6)

$$\kappa_0^* = \frac{\kappa_0}{d\kappa_0 - 1}, \qquad \omega^* = \frac{\omega}{d\kappa_0 - 1}, \qquad s^* = \frac{s + \frac{d\kappa_0}{2} - 1}{d\kappa_0 - 1}, \qquad \sigma^* = \Re s^*$$

and

$$f^*(\xi, \boldsymbol{\alpha}) = \xi^{\kappa_0^*} \sum_{\omega \in \mathcal{D}_f,\, \omega < \kappa_0} A_\omega(\boldsymbol{\alpha}) \xi^{-\omega^*}$$

with certain coefficients $A_\omega(\boldsymbol{\alpha})$ described in Theorem 1.2 below. We have

THEOREM 1.1. *Let $d \geq 1$, $F \in \mathcal{S}_d^\sharp$ be entire with $\theta_F = 0$ and let $f(\xi, \boldsymbol{\alpha})$ be as in* (1.4). *Then there exist an integer $J \geq 0$, constants $0 = \eta_0 < \cdots < \eta_J$ and functions $W_0(s), \dots, W_J(s)$ holomorphic for $\sigma > 0$, with $W_0(s) \neq 0$ for $\sigma > 0$, such that*

$$F(s; f) = \sum_{j=0}^{J} W_j(s) \bar{F}(s^* + \eta_j; f^*) + G(s), \tag{1.7}$$

where $G(s)$ is holomorphic for $\sigma > 0$.

Note that the integer J, the constants η_j and the functions $W_j(s)$ and $G(s)$ depend on both $F(s)$ and $f(\xi, \boldsymbol{\alpha})$. Clearly, identity (1.7) means that the difference of the two terms involving the nonlinear twists, which make sense at least for $\min(\sigma, \sigma^*) > 1$, is holomorphic for $\sigma > 0$.

The function $f^*(\xi, \boldsymbol{\alpha})$ comes into play by a suitable application of the saddle point method. In fact, denoting by $g^\flat(\xi)$ the truncation of a series of the form $g(\xi) = \sum_{n=0}^\infty \alpha_n \xi^{\kappa_n}$, $\kappa_0 > \kappa_1 > \cdots \to -\infty$, obtained by dropping the terms with $\kappa_n \leq 0$, it is clear from Lemma 2.8 below that for ξ sufficiently large

$$f^*(\xi, \boldsymbol{\alpha}) = \frac{1}{2\pi} \Phi^\flat(x_0, \xi, \boldsymbol{\alpha}),$$

where $x_0 = x_0(\xi, \boldsymbol{\alpha})$ is the critical point of the function

$$\Phi(z, \xi, \boldsymbol{\alpha}) = z^{1/d} - 2\pi f\left(\frac{qz}{\xi}, \boldsymbol{\alpha}\right), \qquad q = q_F(2\pi d)^{-d}. \tag{1.8}$$

Moreover, $\Phi(z, \xi, \boldsymbol{\alpha})$ is holomorphic for $z = x + iy \in \mathbb{C} \setminus [0, -\infty)$ and x_0 is the unique z-solution of

$$\frac{\partial}{\partial z}\Phi(z, \xi, \boldsymbol{\alpha}) = 0$$

in the region $\{z \in \mathbb{C} : \Re z \geq 1, \ |\arg z| \leq \theta\}$ with a suitably small $\theta > 0$, and is real and simple; see Lemma 2.3 below. Writing

$$T(f)(\xi, \boldsymbol{\alpha}) = \frac{1}{4\pi^2 i}\int_{\mathcal{C}} \frac{\Phi(z, \xi, \boldsymbol{\alpha})\frac{\partial^2}{\partial z^2}\Phi(z, \xi, \boldsymbol{\alpha})}{\frac{\partial}{\partial z}\Phi(z, \xi, \boldsymbol{\alpha})}\mathrm{d}z, \tag{1.9}$$

where $\mathcal{C}$ is the circle $|z - \tilde{x}_0| = \delta\tilde{x}_0$ (contained in the region defined above),

$$\tilde{x}_0 = C_0\xi^{\frac{d\kappa_0}{d\kappa_0 - 1}} \qquad C_0 = (2\pi d\kappa_0 q^{\kappa_0}\alpha_0)^{-\frac{d}{d\kappa_0 - 1}} \tag{1.10}$$

is the approximation to x_0 in Lemma 2.3 below and $\delta > 0$ is sufficiently small, again thanks to Lemma 2.3 we have that x_0 lies inside $\mathcal{C}$, and hence

$$\frac{1}{2\pi}\Phi(x_0, \xi, \boldsymbol{\alpha}) = T(f)(\xi, \boldsymbol{\alpha}).$$

In particular, for every function $f(\xi, \boldsymbol{\alpha})$ defined by (1.4) we have

$$f^*(\xi, \boldsymbol{\alpha}) = T(f)^\flat(\xi, \boldsymbol{\alpha}). \tag{1.11}$$

In order to exploit the transformation formula in Theorem 1.1 we need to enter the finer structure of the operator T. A simple computation shows that (1.12)

$$(\kappa_0^*)^* = \frac{\kappa_0^*}{d\kappa_0^* - 1} = \kappa_0, \quad (\omega^*)^* = \frac{\omega^*}{d\kappa_0^* - 1} = \omega, \quad (s^*)^* = \frac{s^* + \frac{d\kappa_0^*}{2} - 1}{d\kappa_0^* - 1} = s,$$

where κ_0^*, ω^* and s^* are given by (1.6), and this suggests that the transformation in Theorem 1.1 is self-reciprocal. To this end we consider the following slightly more general situation. For $d \geq 1$ we denote by $\mathfrak{X}_d$ the set of all functions of type

$$f(\xi) = f(\xi, \boldsymbol{\alpha}) = \xi^{\kappa_0}\sum_{\omega \in \mathcal{D}_f} \alpha_\omega \xi^{-\omega} \quad \text{with } d\kappa_0 > 1 \text{ and } \alpha_0 > 0,$$

where D_f is the additive semigroup defined above with positive $\omega_1, \ldots, \omega_N$, and α_ω are real and satisfy $\alpha_\omega \ll c^\omega$ for some $c = c(f) > 0$. Sometimes, in the proofs, the α_ω's are treated as real variables in such ranges. We call κ_0 the *leading exponent* of $f(\xi, \boldsymbol{\alpha})$ and write

$$\kappa_0 = \text{lexp}(f), \qquad \alpha_\omega = \text{coeff}(f, \xi^{\kappa_0 - \omega}).$$

Clearly, every function defined by (1.4) belongs to $\mathfrak{X}_d$, every $f \in \mathfrak{X}_d$ is defined for ξ sufficiently large and the associated function $\Phi(z, \xi, \boldsymbol{\alpha})$, defined as in (1.8), is holomorphic for $\Re z$ sufficiently large. Note that, due to the presence of the invariant q, the definition of $\Phi(z,\xi,\boldsymbol{\alpha})$ involves also a function $F\in\mathcal{S}_d^\sharp$; although not strictly necessary, for simplicity we tacitly assume that the positive number q always comes from a function of $\mathcal{S}_d^\sharp$. Further, for $f\in\mathfrak{X}_d$, $T(f)(\xi,\boldsymbol{\alpha})$ is defined as in (1.9) and (1.10). We have

THEOREM 1.2. *Let $d\geq 1$ and ξ be sufficiently large.*

(i) *If $f\in\mathfrak{X}_d$, then*

$$T(f)(\xi,\boldsymbol{\alpha})=\xi^{\kappa_0^*}\sum_{\omega\in\mathcal{D}_f}A_\omega(\boldsymbol{\alpha})\xi^{-\omega^*},$$

where $A_0(\boldsymbol{\alpha})=A_0\alpha_0^{-\frac{1}{d\kappa_0-1}}$ with a certain constant $A_0>0$, and for $\omega>0$,

$$A_\omega(\boldsymbol{\alpha})=\alpha_0^{-\frac{1}{d\kappa_0-1}}P_\omega\Big((\alpha_0^{\frac{d\omega'}{d\kappa_0-1}}\alpha_{\omega'})_{0<\omega'\leq\omega}\Big)$$

with certain $P_\omega\in\mathbb{R}[(x_{\omega'})_{0<\omega'\leq\omega}]$ without constant term and

$$A_\omega(\boldsymbol{\alpha})=A'_\omega((\alpha_{\omega'})_{0\leq\omega'<\omega})+A''_\omega(\alpha_0)\alpha_\omega,\qquad A''_\omega(\alpha_0)\neq 0.$$

(ii) *If $f\in\mathfrak{X}_d$, then $T(f)\in\mathfrak{X}_d$, and the operator $T:\mathfrak{X}_d\to\mathfrak{X}_d$ satisfies $T^2=id_{\mathfrak{X}_d}$.*

For $f,g\in\mathfrak{X}_d$ and $\lambda\in\mathbb{R}$ we write $f\equiv g\ (\mathrm{mod}\ \xi^\lambda)$ to mean that $f(\xi)-g(\xi)=O(\xi^\lambda)$ as $\xi\to+\infty$; in such a case, the terms with exponents $>\lambda$ are equal. Note that if $f,g\in\mathfrak{X}_d$, $\lambda<\kappa_0=\mathrm{lexp}(f)$ and $f\equiv g\ (\mathrm{mod}\ \xi^\lambda)$, then (for a fixed $F\in\mathcal{S}_d^\sharp$)

$$T(f)\equiv T(g)\ (\mathrm{mod}\ \xi^{\lambda^*}),\qquad \lambda^*=\frac{\lambda}{d\kappa_0-1}. \tag{1.13}$$

Indeed, from the hypotheses of (1.13) we have that $\mathrm{lexp}(g)=\kappa_0$ and hence we can write

$$f(\xi)=\xi^{\kappa_0}\sum_{\omega\in\mathcal{D}_f}\alpha_\omega\xi^{-\omega},\qquad g(\xi)=\xi^{\kappa_0}\sum_{\omega\in\mathcal{D}_g}\beta_\omega\xi^{-\omega}$$

with $\alpha_\omega=\beta_\omega$ for $0\leq\omega<\kappa_0-\lambda$. Thus from part (i) of Theorem 1.2 we get $A_\omega(\boldsymbol{\alpha})=A_\omega(\boldsymbol{\beta})$ for $0\leq\omega<\kappa_0-\lambda$ (note that also the *shape* of the above polynomials $P_\omega((x_{\omega'})_{0<\omega'\leq\omega})$ depends only on the part of $f(\xi)$ with $0<\omega'\leq\omega$) and (1.13) follows. Choosing $\lambda=0$ in (1.13) we see that for $f\in\mathfrak{X}_d$

$$T(f^\flat)^\flat(\xi,\boldsymbol{\alpha})=T(f)^\flat(\xi,\boldsymbol{\alpha}). \tag{1.14}$$

Thus, (1.11) and (1.14) allow to transfer into the framework of nonlinear twists the properties of the operator T on $\mathfrak{X}_d$.

Next we consider the *shift operator*

$$S(f)(\xi,\boldsymbol{\alpha})=f(\xi,\boldsymbol{\alpha})+\xi$$

which acts trivially on the nonlinear twists,

$$F(s; S(f)) = F(s; f), \tag{1.15}$$

but is nontrivial on $\mathfrak{X}_d$. Note that, due to condition $d\kappa_0 > 1$ in the definition of $\mathfrak{X}_d$, $S^{-1}(\mathfrak{X}_d)$ is not contained in $\mathfrak{X}_d$ (e.g. $S^{-1}(-\xi) = 0$ identically); on the other hand, $S(f)$ is well defined for any function $f : \mathbb{R} \to \mathbb{R}$. Note also that T and S do not commute, and this is important for our purposes. In fact, the proof of our theorem is based on a suitable combination of the operators T and S, which we now describe. For $1 < d < 2$ let $\mathfrak{S}_d$ be the group generated by T and S. Since T has order 2, every element of $\mathfrak{S}_d$ is a (formal) product of elements of type T and S^m, $m \in \mathbb{Z}$, and the inverse is the same product in reverse order with $-m$ in place of m. Since $S^{-1}(\mathfrak{X}_d) \not\subset \mathfrak{X}_d$, the elements of $\mathfrak{S}_d$ in general are not well defined on $\mathfrak{X}_d$; thus some care is needed when dealing with them. Given a sequence $m_k \in \mathbb{Z}$, $k \geq 1$, we consider the sequence $\mathcal{L}_k \in \mathfrak{S}_d$, $k \geq 0$, defined recursively by $\mathcal{L}_0 = S$ and

$$\mathcal{L}_{k+1} = \mathcal{L}_k^{-1} T S^{m_{k+1}} T \mathcal{L}_k.$$

The number of factors of $\mathcal{L}_k$ grows exponentially as a function of k; for example,

$$\mathcal{L}_0 = S, \quad \mathcal{L}_1 = S^{-1} T S^{m_1} T S,$$
$$\mathcal{L}_2 = S^{-1} T S^{-m_1} T S T S^{m_2} T S^{-1} T S^{m_1} T S, \quad \dots .$$

We write $e_0 = 1$ and for $k \geq 1$

$$e_k = d^k - d^{k-1} - \cdots - d - 1 = (1-\kappa) d^k + \kappa,$$

where κ is defined in (1.3). The sequence e_k is strictly decreasing to $-\infty$ since $1 < d < 2$, hence there exists an integer k_d such that

$$e_{k_d} \geq \frac{1}{d} > e_{k_d+1} = d e_{k_d} - 1. \tag{1.16}$$

We want $k_d \geq 1$, and this implies that $e_1 = d - 1 \geq 1/d$, hence $\frac{1+\sqrt{5}}{2} \leq d < 2$. Note that the range $1 < d < \frac{1+\sqrt{5}}{2}$ is already covered by the theorem in [9] and, moreover, that it is easy to deal with such a range by our present method; see after (1.17) below. We write

$$g_0(\xi) = \alpha \xi^{1/d}, \qquad \alpha > 0$$

(corresponding to the standard nonlinear twist; $g_0 \notin \mathfrak{X}_d$) and study the action of $\mathcal{L}_{k_d}$ on $g_0(\xi)$. To this end, for $f \in \mathfrak{X}_d$ with $\operatorname{lexp}(f) = \kappa_0$ we further write

$$s^*(s, T, f) = \frac{s + \frac{d\kappa_0}{2} - 1}{d\kappa_0 - 1},$$

and for any $f : \mathbb{R} \to \mathbb{R}$ and $m \in \mathbb{Z}$

$$s^*(s, S^m, f) = s.$$

In view of Theorem 1.1, this notation reflects the behaviour of s under the two generators of $\mathfrak{S}_d$. Moreover, the behaviour of s under any element of $\mathfrak{S}_d$ is inductively defined as

$$s^*(s,\mathcal{H}\mathcal{K},f)=s^*\big(s^*(s,\mathcal{K},f),\mathcal{H},\mathcal{K}(f)\big),$$

provided $\mathcal{H},\mathcal{K}\in\mathfrak{S}_d$ and $\mathcal{H}\mathcal{K}(f)$ is well defined. With the above notation, thanks to Theorems 1.1 and 1.2 we can prove

THEOREM 1.3. *Let $\frac{1+\sqrt{5}}{2}\leq d<2$. There exist positive integers $m_1,\dots,$ m_{k_d-1} and $\overline{m}_{k_d}$ such that for every integer $m_{k_d}>\overline{m}_{k_d}$ the operator $\mathcal{L}_{k_d}$ is well defined on $g_0(\xi)$ and*

$$\operatorname{lexp}(\mathcal{L}_{k_d}(g_0))=e_{k_d},\qquad \operatorname{coeff}(\mathcal{L}_{k_d}(g_0),\xi^{e_{k_d}})=a_0+b_0m_{k_d}$$

with certain real numbers a_0 and $b_0\neq 0$ depending on $\alpha,m_1,\dots,m_{k_d-1}$. Moreover,

$$s^*(s,\mathcal{L}_{k_d},g_0)=s.$$

Now we are ready for proof of the Theorem. We first remark that $k_d\to\infty$ as $d\to 2$; hence the complexity of the argument increases as d approaches 2. With a slight abuse of notation we will denote by T and S the transformations of nonlinear twists corresponding to the operators T and S, respectively (i.e. Theorem on page 1399 and (1.15)). We recall that the Theorem in [9] excludes the existence of polar functions $F\in\mathcal{S}_d^\sharp$ with $1<d<2$; therefore we may restrict our attention to entire $F\in\mathcal{S}_d^\sharp$ with $\theta_F=0$. Given any such function, we start with the associated standard nonlinear twist $F_d(s,\alpha)$, apply S and get

$$F_d(s,\alpha)=\sum_{n=1}^\infty\frac{a_F(n)}{n^s}e(-n-\alpha n^{1/d}).$$

Then we apply T thus getting, thanks to Theorem 1.1, (1.11) and (1.14), that

$$F_d(s,\alpha)=\sum_{j=0}^{J}W_j(s)\bar{F}(s^*+\eta_j;TS(g_0)^\flat)+G(s),\tag{1.17}$$

where the quantities on the right-hand side are as in Theorem 1.1. By the way, note that choosing $\alpha\in\operatorname{Spec}(F)$, the left-hand side of (1.17) has a pole at $s_0=\frac{d+1}{2d}$; thus (1.17) already gives a contradiction if $1<d<\frac{1+\sqrt{5}}{2}$, since in this case $\sigma_0^*=(s_0+d/2-1)/(d-1)>1$. If $d\geq\frac{1+\sqrt{5}}{2}$, then we choose $\mathcal{L}_{k_d}$ as in Theorem 1.3 and apply recursively the remaining factors of $\mathcal{L}_{k_d}$ to all nonlinear twists on the right-hand side of (1.17). Observing that $\Re s^*(s,\mathcal{H},f)>1/2$ for $\mathcal{H}=S$ and $\mathcal{H}=T$ if $\sigma>1/2$, we deduce that

$$\Re s^*(s,\mathcal{H},f)>1/2\qquad\text{for every }\mathcal{H}\in\mathfrak{S}_d\text{ and }\sigma>1/2$$

(of course provided that $\mathcal{H}(f)$ is well defined). Hence by Theorem 1.1, (1.11), (1.14) and Theorem 1.3 we get an expression of type (note that the operator T is applied an even number of times)

$$F_d(s,\alpha) = \sum_{j=0}^{J'} W'_j(s) F(s+\eta'_j; \mathcal{L}_{k_d}(g_0)^\flat) + G'(s), \tag{1.18}$$

where the involved quantities satisfy the properties stated in Theorem 1.1. To conclude we have to consider two cases.

Case $de_{k_d} > 1$. By Theorem 1.3, in this case $\mathcal{L}_{k_d}(g_0) \in \mathfrak{X}_d$; hence a further application of Theorem 1.1 to the right-hand side of (1.18) leads to an expression of type

$$F_d(s,\alpha) = \sum_{j=0}^{\tilde{J}} \widetilde{W}_j(s) \bar{F}(s^* + \tilde{\eta}_j; T\mathcal{L}_{k_d}(g_0)^\flat) + \widetilde{G}(s), \tag{1.19}$$

where $s^* = s^*(s, T, \mathcal{L}_{k_d}(g_0))$ and, once again, the involved quantities are as in Theorem 1.1. Choosing $\alpha \in \mathrm{Spec}(F)$ and observing that by (1.16)

$$s^*\left(\frac{d+1}{2d}, T, \mathcal{L}_{k_d}(g_0)\right) = \frac{\frac{d+1}{2d} + \frac{de_{k_d}}{2} - 1}{de_{k_d} - 1} = \frac{1}{2} + \frac{1}{2de_{k_d+1}} > 1,$$

we get a contradiction, since the right-hand side of (1.19) is holomorphic at $s_0 = \frac{d+1}{2d}$.

Case $de_{k_d} = 1$. This case is more involved and requires the full force of Theorem 1.3 as well as the following partial extension of Theorem 1 in [12] on the analytic properties of the standard nonlinear twist. Let $d \geq 1$, $N \geq 0$ be an integer, $(\beta, \beta_1, \ldots, \beta_N) \in \mathbb{R}^{N+1}$ and for $\xi > 0$

$$f(\xi, \boldsymbol{\beta}) = \beta\xi^{1/d} + \xi^{1/d} \sum_{\nu=1}^{N} \beta_\nu \xi^{-\omega_\nu}, \qquad 0 < \omega_1 < \cdots < \omega_N < 1/d, \ \beta > 0. \tag{1.20}$$

For any such function $f(\xi, \boldsymbol{\beta})$ and $F \in \mathcal{S}_d^\sharp$, the nonlinear twist $F(s; f)$ and the spectrum $\mathrm{Spec}(F)$ are defined as in (1.5) and (1.2) (with β in place of α), respectively. We have

THEOREM 1.4. *Let* $d \geq 1$, $F \in \mathcal{S}_d^\sharp$ *and* $f(\xi, \boldsymbol{\beta})$ *be as in* (1.20) *with* $\beta \notin \mathrm{Spec}(F)$. *Then* $F(s; f)$ *is holomorphic for* $\sigma > 0$.

We remark that the full extension of Theorem 1 in [12] can be obtained by suitably adapting the arguments in the proof of Theorem 1.1 and of Theorem 1 in [12]; however, such an extension is not needed here. We shall deal with it, as well as with other results of similar nature and their applications, in a future paper.

If $de_{k_d} = 1$, then by Theorem 1.3 we have that for all sufficiently large integers $m_{k_d} = m$ the function $\mathcal{L}_{k_d}(g_0)^\flat(\xi)$ is of the form (1.20) with

$$\beta = a_0 + b_0 m, \qquad\qquad b_0 \neq 0.$$

Choosing $\alpha \in \mathrm{Spec}(F)$ in (1.18) and recalling that $W_0'(s) \neq 0$ for $\sigma > 0$, from (1.18) we deduce that $\bar{F}(s;\mathcal{L}_{k_d}(g_0)^\flat)$ has a pole at $s_0 = \frac{d+1}{2d}$; hence Theorem 1.4 implies that $a_0 + b_0 m \in \mathrm{Spec}(F)$ for every sufficiently large integer m. In particular, in view of (1.1) and writing n_m in place of $n_{a_0+b_0m}$, we have

$$n_m = q_F d^{-d}(a_0 + b_0 m)^d \in \mathbb{N} \quad \text{for } m \text{ large.} \tag{1.21}$$

But the second difference $\Delta^2 n_m = n_{m+2} - 2n_{m+1} + n_m$ of n_m is an integer and satisfies

$$\Delta^2 n_m \ll m^{d-2} \to 0;$$

hence $\Delta^2 n_m = 0$ for large m. Therefore, there exist $a_1, b_1 \in \mathbb{R}$ such that $n_m = a_1 + b_1 m$ (see e.g. [1, Thm. 4.1]), a contradiction in view of (1.21) since $1 < d < 2$.

Our main theorem is therefore proved modulo Theorems 1.1–1.4, and the rest of the paper is devoted to the proof of such results.

We conclude with the following two simple consequences of the Theorem. The first consequence concerns primitivity in $\mathcal{S}$ and $\mathcal{S}^\sharp$. We refer to Selberg [17] for the notion of primitivity in $\mathcal{S}$ and for the Selberg orthonormality conjecture (SOC in short), and to our paper [10] for the notion of almost-primitivity in $\mathcal{S}^\sharp$. Moreover, we recall that SOC implies that the only polar primitive function in $\mathcal{S}$ is $\zeta(s)$; see Conrey-Ghosh [4]. The nonexistence of functions in $\mathcal{S}^\sharp$ with degree in $(0,1) \cup (1,2)$ and the additivity of the degree immediately imply

Corollary 1. *Every function $F \in \mathcal{S}_d^\sharp$ with $2 < d < 3$ is almost-primitive, and every function $F \in \mathcal{S}_d$ with $2 < d < 3$ is primitive. In particular, assuming* SOC *every $F \in \mathcal{S}_d$ with $2 < d < 3$ is entire.*

The second consequence is a sharpening of Corollary 4 of [12], and its proof follows by the same argument.

Corollary 2. *Let $F \in \mathcal{S}_d$ with $d \geq 1$. If the series*

$$\sum_{n=1}^{\infty} \frac{a_F(n) - 1}{n^s} \qquad\qquad \Big(\textit{resp. } \sum_{n=1}^{\infty} \frac{a_F(n)}{n^s}\Big)$$

converges for $\sigma > 1/4 - \delta$ with some $\delta > 0$, then $F(s) = \zeta(s)$ (resp. $F(s) = L(s + i\theta, \chi)$ with some $\theta \in \mathbb{R}$ and a primitive Dirichlet character χ).

Note that $1/4$ is probably the (upper) limit of this type of characterizations of $\zeta(s)$ and $L(s,\chi)$, due to the conjectural bound $O(x^{1/4+\varepsilon})$ for the coefficient sums of classical L-functions of degree 2.

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2. **Proof of Theorem 1.1**

1. *Set up.* Let $F(s)$ and $f(\xi,\boldsymbol{\alpha})$ be as in Theorem 1.1 and $d \geq 1$. For simplicity we write $a(n) = a_F(n)$ and $\kappa_\nu = \kappa_0 - \omega_\nu$, and denote by $c, c', \dots$ (with or without suffix) some constants whose value will not necessarily be the same at each occurrence and by $\varepsilon, \delta, \eta > 0$ sufficiently small constants, again not necessarily having the same value at each occurrence. Let $w_0, \dots, w_N \in \mathbb{C}$, $w_\nu = u_\nu + iv_\nu$ and $X > 1$. Writing (with a slight abuse of notation)

$$z_\nu = \frac{1}{X} + 2\pi i\alpha_\nu, \qquad \boldsymbol{w} = \sum_{\nu=0}^{N} \kappa_\nu w_\nu, \qquad \mathrm{d}\boldsymbol{w} = \mathrm{d}w_0 \dots \mathrm{d}w_N,$$

$$F_X(s;f) = \sum_{n=1}^{\infty} \frac{a(n)}{n^s} \exp\left(-n^{\kappa_0} \sum_{\nu=0}^{N} z_\nu n^{-\omega_\nu}\right), \qquad G(\boldsymbol{w}) = \prod_{\nu=0}^{N} \Gamma(w_\nu) z_\nu^{-w_\nu},$$

by Mellin's transform we have

$$F_X(s;f) = \frac{1}{(2\pi i)^{N+1}} \int_{(2)} \cdots \int_{(2)} F(s+\boldsymbol{w}) G(\boldsymbol{w}) \mathrm{d}\boldsymbol{w}, \qquad \sigma > 0. \tag{2.1}$$

In order to deal with the multiple integral in (2.1) without entering the theory of several complex variables we prove an ad hoc result. Let (again with a slight abuse of notation) $[N] = \{0, 1, \dots, N\}$ and for $\emptyset \neq \mathcal{A} \subset [N]$ write

$$\boldsymbol{w}_{|\mathcal{A}} = \sum_{\nu\in\mathcal{A}} \kappa_\nu w_\nu, \qquad G(w_\nu) = \Gamma(w_\nu) z_\nu^{-w_\nu}, \qquad G(\boldsymbol{w}_{|\mathcal{A}}) = \prod_{\nu\in\mathcal{A}} G(w_\nu),$$

$$\mathrm{d}\boldsymbol{w}_{|\mathcal{A}} = \prod_{\nu\in\mathcal{A}} \mathrm{d}w_\nu, \qquad \int_{\boldsymbol{L}_{|\mathcal{A}}} = \int_{(-\frac{1}{2}-\eta)} \cdots \int_{(-\frac{1}{2}-\eta)},$$

integrating with respect to the variables w_ν with $\nu \in \mathcal{A}$.

Lemma 2.1. *With the above notation, for* $\sigma > 0$ *we have*

$$F_X(s;f) = F(s) + \sum_{\emptyset\neq\mathcal{A}\subset[N]} \frac{1}{(2\pi i)^{|\mathcal{A}|}} \int_{\boldsymbol{L}_{|\mathcal{A}}} F(s+\boldsymbol{w}_{|\mathcal{A}}) G(\boldsymbol{w}_{|\mathcal{A}}) d\boldsymbol{w}_{|\mathcal{A}}.$$

Proof. By induction over N, starting from (2.1). For $N = 0$ it is a standard application of the residue theorem. Assuming $N \geq 1$ and the lemma true up to $N - 1$ we have

$$\frac{1}{(2\pi i)^{N+1}} \int_{(2)} \cdots \int_{(2)} F(s+\boldsymbol{w}) G(\boldsymbol{w}) \mathrm{d}\boldsymbol{w} = \frac{1}{2\pi i} \int_{(2)} \Big\{ F(s+\kappa_N w_N)$$

$$+ \sum_{\emptyset\neq\mathcal{A}\subset[N-1]} \frac{1}{(2\pi i)^{|\mathcal{A}|}} \int_{\boldsymbol{L}_{|\mathcal{A}}} F(s+\boldsymbol{w}_{|\mathcal{A}}+\kappa_N w_N) G(\boldsymbol{w}_{|\mathcal{A}}) \mathrm{d}\boldsymbol{w}_{|\mathcal{A}} \Big\} G(w_N) \mathrm{d}w_N$$

$$= \frac{1}{2\pi i}\int_{(2)} F(s+\kappa_N w_N)G(w_N)\mathrm{d}w_N + \sum_{\emptyset\neq\mathcal{A}\subset[N-1]} \frac{1}{(2\pi i)^{|\mathcal{A}|}}$$

$$\times \int_{\boldsymbol{L}_{|\mathcal{A}}} \left\{ \frac{1}{2\pi i}\int_{(2)} F(s+\boldsymbol{w}_{|\mathcal{A}}+\kappa_N w_N)G(w_N)\mathrm{d}w_N \right\} G(\boldsymbol{w}_{|\mathcal{A}})\mathrm{d}\boldsymbol{w}_{|\mathcal{A}}$$

$$= F(s) + \frac{1}{2\pi i}\int_{\boldsymbol{L}_{|\{N\}}} F(s+\kappa_N w_N)G(w_N)\mathrm{d}w_N$$

$$+ \sum_{\emptyset\neq\mathcal{A}\subset[N-1]} \frac{1}{(2\pi i)^{|\mathcal{A}|}} \int_{\boldsymbol{L}_{|\mathcal{A}}} F(s+\boldsymbol{w}_{|\mathcal{A}})G(\boldsymbol{w}_{|\mathcal{A}})\mathrm{d}\boldsymbol{w}_{|\mathcal{A}} + \sum_{\emptyset\neq\mathcal{A}\subset[N-1]} \frac{1}{(2\pi i)^{|\mathcal{A}|+1}}$$

$$\times \int_{\boldsymbol{L}_{|\mathcal{A}}} \int_{\boldsymbol{L}_{|\{N\}}} F(s+\boldsymbol{w}_{|\mathcal{A}}+\kappa_N w_N)G(\boldsymbol{w}_{|\mathcal{A}})G(w_N)\mathrm{d}\boldsymbol{w}_{|\mathcal{A}}\mathrm{d}w_N$$

$$= F(s) + \sum_{\substack{\emptyset\neq\mathcal{A}\subset[N]\\ N\notin\mathcal{A}}} \frac{1}{(2\pi i)^{|\mathcal{A}|}} \int_{\boldsymbol{L}_{|\mathcal{A}}} F(s+\boldsymbol{w}_{|\mathcal{A}})G(\boldsymbol{w}_{|\mathcal{A}})\mathrm{d}\boldsymbol{w}_{|\mathcal{A}}$$

$$+ \sum_{\substack{\emptyset\neq\mathcal{A}\subset[N]\\ N\in\mathcal{A}}} \frac{1}{(2\pi i)^{|\mathcal{A}|}} \int_{\boldsymbol{L}_{|\mathcal{A}}} F(s+\boldsymbol{w}_{|\mathcal{A}})G(\boldsymbol{w}_{|\mathcal{A}})\mathrm{d}\boldsymbol{w}_{|\mathcal{A}}.$$

The lemma follows. □

In view of Lemma 2.1, for $\mathcal{A}\subset[N]$, $\mathcal{A}\neq\emptyset$, we have to study integrals of type

$$I_X(s,\mathcal{A}) = \frac{1}{(2\pi i)^{|\mathcal{A}|}} \int_{\boldsymbol{L}_{|\mathcal{A}}} F(s+\boldsymbol{w}_{|\mathcal{A}})G(\boldsymbol{w}_{|\mathcal{A}})\mathrm{d}\boldsymbol{w}_{|\mathcal{A}}. \tag{2.2}$$

Assuming that $0<\sigma<\delta$, by the functional equation of $F(s)$ and the reflection formula $\Gamma(z)\Gamma(1-z)=\pi/\sin(\pi z)$ we get

$$I_X(s,\mathcal{A}) = \omega\pi^{-r}Q^{1-2s}\sum_{n=1}^{\infty}\frac{\overline{a(n)}}{n^{1-s}}\frac{1}{(2\pi i)^{|\mathcal{A}|}} \times \int_{\boldsymbol{L}_{|\mathcal{A}}} \widetilde{G}(s+\boldsymbol{w}_{|\mathcal{A}})S(s+\boldsymbol{w}_{|\mathcal{A}})G(\boldsymbol{w}_{|\mathcal{A}})\left(\frac{n}{Q^2}\right)^{\boldsymbol{w}_{|\mathcal{A}}}\mathrm{d}\boldsymbol{w}_{|\mathcal{A}}, \tag{2.3}$$

where, writing $\boldsymbol{x}_{|\mathcal{A}} = s+\boldsymbol{w}_{|\mathcal{A}}$,

$$\widetilde{G}(\boldsymbol{x}_{|\mathcal{A}}) = \prod_{j=1}^{r}\Gamma(\lambda_j(1-\boldsymbol{x}_{|\mathcal{A}})+\overline{\mu}_j)\Gamma(1-\lambda_j\boldsymbol{x}_{|\mathcal{A}}-\mu_j),$$

$$S(\boldsymbol{x}_{|\mathcal{A}}) = \prod_{j=1}^{r}\sin\pi(\lambda_j\boldsymbol{x}_{|\mathcal{A}}+\mu_j).$$

As in [9, Lemma 2.1], by means of Stirling's formula we transform $\widetilde{G}(\boldsymbol{x}_{|\mathcal{A}})$ to (roughly) a single Γ-factor. In fact, given a sufficiently large integer L and recalling that $\theta_F = 0$, similar computations as in such a lemma, applied to the variable $\boldsymbol{x}_{|\mathcal{A}}$, give

$$\widetilde{G}(\boldsymbol{x}_{|\mathcal{A}}) = (d^d/\beta)^{\boldsymbol{x}_{|\mathcal{A}}} \sum_{\ell=0}^{L} c_\ell \Gamma\left(\frac{d+1}{2} - d\boldsymbol{x}_{|\mathcal{A}} - \ell\right) + R_1(\boldsymbol{x}_{|\mathcal{A}}), \tag{2.4}$$

where c_ℓ are constants with $c_0 \neq 0$, $\beta = \prod_{j=1}^{r} \lambda_j^{2\lambda_j}$ and $R_1(\boldsymbol{x}_{|\mathcal{A}})$ is meromorphic on $\mathbb{C}$, holomorphic for $0 < \sigma < \delta$ and in such a strip satisfies the uniform bound

$$R_1(\boldsymbol{x}_{|\mathcal{A}}) \ll e^{-\frac{\pi}{2}d|\Im \boldsymbol{x}_{|\mathcal{A}}|}(1 + |\Im \boldsymbol{x}_{|\mathcal{A}}|)^{-L+c} \tag{2.5}$$

with some $c > 0$. Analogously, by means of $\sin(z) = \frac{1}{2i}(e^{iz} - e^{-iz})$ we transform $S(\boldsymbol{x}_{|\mathcal{A}})$ to the more convenient form

$$S(\boldsymbol{x}_{|\mathcal{A}}) = c_1 e^{-i\frac{\pi}{2}d\boldsymbol{x}_{|\mathcal{A}}} + c_2 e^{i\frac{\pi}{2}d\boldsymbol{x}_{|\mathcal{A}}} + R_2(\boldsymbol{x}_{|\mathcal{A}}) \tag{2.6}$$

with constants $c_1, c_2 \neq 0$ and $R_2(\boldsymbol{x}_{|\mathcal{A}})$ entire and satisfying

$$R_2(\boldsymbol{x}_{|\mathcal{A}}) \ll e^{\frac{\pi}{2}(d-\eta)|\Im \boldsymbol{x}_{|\mathcal{A}}|} \tag{2.7}$$

with some $\eta > 0$. From (2.3)–(2.7), for $0 < \sigma < \delta$ we have

$$I_X(s,\mathcal{A}) = e^{a_2 s + b_2} \sum_{n=1}^{\infty} \frac{\overline{a(n)}}{n^{1-s}} \sum_{\ell=0}^{L} \Big(c_\ell I_X(s,\mathcal{A},n,\ell) + c'_\ell J_X(s,\mathcal{A},n,\ell) + h_X(s,\mathcal{A},n,\ell)\Big), \tag{2.8}$$

where $a_2 \in \mathbb{R}$ and b_2 are certain constants, $c_0 c'_0 \neq 0$,

$$I_X(s,\mathcal{A},n,\ell) = \frac{1}{(2\pi i)^{|\mathcal{A}|}} \int_{\boldsymbol{L}_{|\mathcal{A}}} \Gamma\left(\frac{d+1}{2} - d\boldsymbol{x}_{|\mathcal{A}} - \ell\right) (e^{-i\frac{\pi}{2}d} d^d/\beta)^{\boldsymbol{x}_{|\mathcal{A}}} G(\boldsymbol{w}_{|\mathcal{A}}) \left(\frac{n}{Q^2}\right)^{\boldsymbol{w}_{|\mathcal{A}}} \mathrm{d}\boldsymbol{w}_{|\mathcal{A}},$$

$$J_X(s,\mathcal{A},n,\ell) = \frac{1}{(2\pi i)^{|\mathcal{A}|}} \int_{\boldsymbol{L}_{|\mathcal{A}}} \Gamma\left(\frac{d+1}{2} - d\boldsymbol{x}_{|\mathcal{A}} - \ell\right) (e^{i\frac{\pi}{2}d} d^d/\beta)^{\boldsymbol{x}_{|\mathcal{A}}} G(\boldsymbol{w}_{|\mathcal{A}}) \left(\frac{n}{Q^2}\right)^{\boldsymbol{w}_{|\mathcal{A}}} \mathrm{d}\boldsymbol{w}_{|\mathcal{A}},$$

and since

$$h_X(s,\mathcal{A},n,\ell) \ll e^{c|t|} n^{-\eta} \int_{\boldsymbol{L}_{|\mathcal{A}}} (1 + |\Im \boldsymbol{w}_{|\mathcal{A}}|)^{-L+c} |\mathrm{d}\boldsymbol{w}_{|\mathcal{A}}|,$$

$h_X(s,\mathcal{A},n,\ell)$ is meromorphic on $\mathbb{C}$, holomorphic for $0 < \sigma < \delta$ and satisfies

$$h_X(s,\mathcal{A},n,\ell) \ll e^{c|t|} n^{-\eta} \tag{2.9}$$

for some $c, \eta > 0$, uniformly for $\mathcal{A} \subset [N]$, $0 \le \ell \le L$, $0 < \sigma < \delta$ and $X \to \infty$.

2. *Mellin's transform.* For a given $0 \le \ell \le L$ assume that

$$\frac{d+1}{2d} - \frac{\ell}{d} < \sigma < \frac{d+1}{2d} - \frac{\ell}{d} + \eta \tag{2.10}$$

with some $\eta > 0$. The inverse Mellin transform $\tilde{I}_X(y)$ of $I_X(s, \mathcal{A}, n, \ell)$ is well defined in the range (2.10), and we have

$$\begin{aligned}\tilde{I}_X(y) &= \frac{1}{2\pi i}\int_{(\sigma)} I_X(s,\mathcal{A},n,\ell) y^{-s}\mathrm{d}s \\ &= \frac{1}{(2\pi i)^{|\mathcal{A}|}}\int_{\boldsymbol{L}_{|\mathcal{A}}}\biggl\{\frac{1}{2\pi i}\int_{(\sigma)}\Gamma\Bigl(\frac{d+1}{2} - d\boldsymbol{x}_{|\mathcal{A}} - \ell\Bigr) \\ &\qquad \times (e^{-i\frac{\pi}{2}d}d^d/\beta)^{\boldsymbol{x}_{|\mathcal{A}}} y^{-s}\mathrm{d}s\biggr\} G(\boldsymbol{w}_{|\mathcal{A}})\Bigl(\frac{n}{Q^2}\Bigr)^{\boldsymbol{w}_{|\mathcal{A}}}\mathrm{d}\boldsymbol{w}_{|\mathcal{A}}.\end{aligned} \tag{2.11}$$

In the inner integral we make the substitution

$$\frac{d+1}{2} - ds - d\boldsymbol{w}_{|\mathcal{A}} - \ell = \theta, \qquad s = \frac{d+1}{2d} - \boldsymbol{w}_{|\mathcal{A}} - \frac{\ell}{d} - \frac{\theta}{d}$$

so that, thanks to (2.10),

$$\theta_\sigma = \Re\theta = \frac{d+1}{2} - d\sigma + d\Bigl(\frac{1}{2}+\eta\Bigr)\sum_{\nu\in\mathcal{A}}\kappa_\nu - \ell > 0. \tag{2.12}$$

Therefore, in view of (2.12), the inner integral in (2.11) becomes

$$\begin{aligned}&\frac{1}{d}(e^{-i\frac{\pi}{2}d}d^d/\beta)^{\frac{d+1}{2d}-\frac{\ell}{d}} y^{-\frac{d+1}{2d}+\frac{\ell}{d}+\boldsymbol{w}_{|\mathcal{A}}}\frac{1}{2\pi i}\int_{(\theta_\sigma)}\Gamma(\theta)(e^{-i\frac{\pi}{2}}(d^d/\beta y)^{1/d})^{-\theta}\mathrm{d}\theta \\ &\qquad = \frac{1}{d}(e^{-i\frac{\pi}{2}d}d^d/\beta)^{\frac{d+1}{2d}-\frac{\ell}{d}} y^{-\frac{d+1}{2d}+\frac{\ell}{d}+\boldsymbol{w}_{|\mathcal{A}}} e^{i(d^d/\beta y)^{1/d}},\end{aligned}$$

and hence

$$\begin{aligned}\tilde{I}_X(y) &= \frac{1}{d}(e^{-i\frac{\pi}{2}d}d^d/\beta)^{\frac{d+1}{2d}-\frac{\ell}{d}} e^{i(d^d/\beta y)^{1/d}} y^{-\frac{d+1}{2d}+\frac{\ell}{d}} \\ &\qquad \times \prod_{\nu\in\mathcal{A}}\biggl\{\frac{1}{2\pi i}\int_{(-\frac{1}{2}-\eta)}\Gamma(w_\nu)\Bigl(\frac{z_\nu Q^{2\kappa_\nu}}{n^{\kappa_\nu}y^{\kappa_\nu}}\Bigr)^{-w_\nu}\mathrm{d}w_\nu\biggr\} \\ &= c_\ell e^{i(d^d/\beta y)^{1/d}} y^{-\frac{d+1}{2d}+\frac{\ell}{d}}\prod_{\nu\in\mathcal{A}}\Bigl(e^{-\frac{z_\nu Q^{2\kappa_\nu}}{n^{\kappa_\nu}y^{\kappa_\nu}}} - 1\Bigr)\end{aligned} \tag{2.13}$$

with $c_\ell \neq 0$. Applying the Mellin transform, from (2.11), (2.13) and the change of variable $\frac{d^d}{\beta y} = x$ we obtain that for s in the range (2.10)

$$\begin{aligned}I_X(s,\mathcal{A},n,\ell) &= \int_0^\infty \tilde{I}_X(y)y^{s-1}\mathrm{d}y \\ &= c_\ell(d^d/\beta)^s\int_0^\infty e^{ix^{1/d}}\prod_{\nu\in\mathcal{A}}\Bigl(e^{-z_\nu(\frac{qx}{n})^{\kappa_\nu}} - 1\Bigr)x^{\frac{d+1}{2d}-\frac{\ell}{d}-s-1}\mathrm{d}x\end{aligned} \tag{2.14}$$

with $c_\ell \neq 0$, where q is defined in (1.8). Exactly in the same way we also obtain (2.15)

$$J_X(s,\mathcal{A},n,\ell) = c'_\ell(d^d/\beta)^s \int_0^\infty e^{-ix^{1/d}} \prod_{\nu\in\mathcal{A}} \big(e^{-z_\nu(\frac{qx}{n})^{\kappa_\nu}} - 1\big) x^{\frac{d+1}{2d}-\frac{\ell}{d}-s-1}\mathrm{d}x,$$

again with s in the range (2.10) and $c'_\ell \neq 0$.

In order to sum over the subsets $\mathcal{A}$ we use the identity

$$\sum_{\emptyset\neq\mathcal{A}\subset[N]} \prod_{\nu\in\mathcal{A}} (X_\nu - 1) = \prod_{\nu=0}^{N} X_\nu - 1, \tag{2.16}$$

which holds for arbitrary complex numbers X_ν, $\nu = 0,\dots,N$, and can be proved by a simple induction over N. From (2.14) and (2.16) we obtain that for s in the range (2.10)

$$\begin{aligned} &\sum_{\emptyset\neq\mathcal{A}\subset[N]} I_X(s,\mathcal{A},n,\ell) \\ &\quad= c_\ell(d^d/\beta)^s \int_0^\infty e^{ix^{1/d}} \Big(e^{-\sum_{\nu=0}^N z_\nu(\frac{qx}{n})^{\kappa_\nu}} - 1\Big)\, x^{\frac{d+1}{2d}-\frac{\ell}{d}-s-1}\mathrm{d}x \\ &\quad= c_\ell(d^d/\beta)^s \int_0^\infty e^{ix^{1/d}} \Big(e^{-\Psi_X(x,n)} e\Big(-f\Big(\frac{qx}{n},\boldsymbol{\alpha}\Big)\Big) - 1\Big)\, x^{\frac{d+1}{2d}-\frac{\ell}{d}-s-1}\mathrm{d}x \\ &\quad= c_\ell(d^d/\beta)^s I_X(s,n,\ell), \end{aligned} \tag{2.17}$$

say, with $c_\ell \neq 0$, $f(\xi,\boldsymbol{\alpha})$ given by (1.4) and

$$\Psi_X(x,n) = \frac{1}{X}\sum_{\nu=0}^{N}\Big(\frac{qx}{n}\Big)^{\kappa_\nu}. \tag{2.18}$$

In the same range of s, from (2.15) and (2.16) we have

$$\begin{aligned} &\sum_{\emptyset\neq\mathcal{A}\subset[N]} J_X(s,\mathcal{A},n,\ell) \\ &\quad= c'_\ell(d^d/\beta)^s \int_0^\infty e^{-ix^{1/d}} \Big(e^{-\sum_{\nu=0}^N z_\nu(\frac{qx}{n})^{\kappa_\nu}} - 1\Big)\, x^{\frac{d+1}{2d}-\frac{\ell}{d}-s-1}\mathrm{d}x \\ &\quad= c'_\ell(d^d/\beta)^s \int_0^\infty e^{-ix^{1/d}} \Big(e^{-\Psi_X(x,n)} e\Big(-f\Big(\frac{qx}{n},\boldsymbol{\alpha}\Big)\Big) - 1\Big)\, x^{\frac{d+1}{2d}-\frac{\ell}{d}-s-1}\mathrm{d}x \\ &\quad= c'_\ell(d^d/\beta)^s J_X(s,n,\ell), \end{aligned} \tag{2.19}$$

say, with $c'_\ell \neq 0$. Note that the integrals in (2.17) and (2.19) are clearly convergent at ∞ for every s thanks to the middle exponential term, and at 0 in the range (2.10). The integral in (2.19) is simpler to deal with since there is no critical point, and we have

Lemma 2.2. *The function $J_X(s,n,\ell)$ is meromorphic over $\mathbb{C}$ and is holomorphic for $0<\sigma<\delta$ for some $\delta>0$. Moreover, $J_X(s,n,\ell)$ satisfies*

$$J_X(s,n,\ell) \ll e^{c|t|}n^{-\eta}$$

for some $c,\eta>0$, uniformly for $0\le\ell\le L$, $0<\sigma<\delta$ and $X\to\infty$.

Proof. Since the integrand in $J_X(s,n,\ell)$ is holomorphic as a function of the complex variable x for $\Re x > 0$, we adhere to the notation in the introduction and use instead of x the complex variable z. Consider the half-line $z = \rho e^{-i\phi}$ with $\rho \geq 0$ and $\phi > 0$ sufficiently small. Recalling that $\alpha_0 > 0$ and $\kappa_0 > \kappa_1 > \ldots$, on such a half-line we have

$$|e^{-iz^{1/d}}| = e^{-\rho^{1/d}\sin(\phi/d)}, \qquad |e^{-\Psi_X(z,n)}| \leq 1,$$
$$\left|e\left(-f\left(\frac{qz}{n},\boldsymbol{\alpha}\right)\right)\right| = e^{-2\pi\sum_{\nu=0}^{N}\alpha_\nu(\frac{q\rho}{n})^{\kappa_\nu}\sin(\phi\kappa_\nu)} \ll 1;$$

hence we switch the integration in $J_X(s,n,\ell)$ from $(0,\infty)$ to $(0,\infty e^{-i\phi})$ and split the new integral as

$$J_X(s,n,\ell) = \int_0^{\infty e^{-i\phi}} \cdots = \int_0^{n^\varepsilon e^{-i\phi}} \cdots + \int_{n^\varepsilon e^{-i\phi}}^{\infty e^{-i\phi}} \cdots = J_1(s) + J_2(s), \tag{2.20}$$

say, where $\varepsilon > 0$ is sufficiently small. In view of the above estimates we have

$$J_2(s) \ll e^{c|t|}\int_{n^\varepsilon}^{\infty} e^{-\rho^{1/d}\sin(\phi/d)}\rho^{\frac{d+1}{2d}-\frac{\ell}{d}-\sigma-1}\mathrm{d}\rho \ll e^{c|t|}n^{-c'} \tag{2.21}$$

with some $c > 0$ and any $c' > 0$, uniformly for $0 \leq \ell \leq L$, σ in any fixed interval and $X \to \infty$; moreover, $J_2(s)$ is an entire function.

In order to treat $J_1(s)$, for s in the range (2.10) we use the Taylor expansion of the two exponential terms in the first expression for $J_X(s,n,\ell)$ in (2.19). In such a way, given any large integer K, the integrand in $J_1(s)$ is expressed as a finite sum of terms of type

$$n^{-\eta}P(z_0,\ldots,z_N)z^{\gamma-s-1},$$

where $\eta > 0$, $P(z_0,\ldots,z_N)$ is a polynomial and γ is of the form

$$\frac{d+1}{2d} + \frac{k}{d} + \sum_{\nu=0}^{N} h_\nu\kappa_\nu, \qquad k \geq \ell,\ h_\nu \geq 0,\ k, h_\nu \in \mathbb{Z},$$

plus an error term $R_K(s,z)$ which is an entire function of s and satisfies

$$R_K(s,z) \ll n^{-\eta}e^{c|t|}\rho^{K-\sigma-1}.$$

Hence $J_1(s)$ is the sum of a finite number of terms of type

$$n^{-\eta}P(z_0,\ldots,z_N)\frac{(n^\varepsilon e^{-i\phi})^{(\gamma-s)}}{\gamma-s},$$

which are meromorphic over $\mathbb{C}$, plus the term

$$\int_0^{n^\varepsilon e^{-i\phi}} R_K(s,z)\mathrm{d}z$$

which is absolutely convergent, and hence holomorphic, for $\sigma < K$. Since K is arbitrarily large we have that $J_1(s)$ is meromorphic over $\mathbb{C}$, holomorphic for $0 < \sigma < \delta$ and, choosing $\varepsilon > 0$ sufficiently small, satisfies

$$J_1 \ll n^{-\eta} e^{c|t|} n^{\varepsilon c K} \ll n^{-\eta} e^{c|t|} \tag{2.22}$$

for some $c, \eta > 0$, uniformly for $0 \le \ell \le L$, $0 < \sigma < \delta$ and $X \to \infty$. The lemma follows at once from (2.20)–(2.22). □

From (2.17), (2.19) and Lemma 2.2 we have that for s in the range (2.10)

$$\sum_{\emptyset \neq \mathcal{A} \subset [N]} \Big(c_\ell I_X(s,\mathcal{A},n,\ell) + c'_\ell J_X(s,\mathcal{A},n,\ell) \Big) = c''_\ell (d^d/\beta)^s I_X(s,n,\ell) + h_X(s,n,\ell), \tag{2.23}$$

say, where $c''_\ell \neq 0$ and $h_X(s,n,\ell)$ is meromorphic over $\mathbb{C}$, holomorphic for $0 < \sigma < \delta$ and satisfies

$$h_X(s,n,\ell) \ll e^{c|t|} n^{-\eta}$$

for some $c, \eta > 0$, uniformly for $0 \le \ell \le L$, $0 < \sigma < \delta$ and $X \to \infty$.

3. *Saddle point.* For future reference we switch from n to the real variable ξ. Given $f(\xi, \boldsymbol{\alpha})$ as in (1.4), recall that $d\kappa_0 > 1$ and

$$\Phi(z,\xi,\boldsymbol{\alpha}) = z^{1/d} - 2\pi f\left(\frac{qz}{\xi}, \boldsymbol{\alpha}\right);$$

see (1.8). Then $\Phi(z,\xi,\boldsymbol{\alpha})$ is holomorphic for $z = x + iy \in \mathbb{C} \setminus [0, -\infty)$, and we write

$$\Phi'(z,\xi,\boldsymbol{\alpha}) = \frac{\partial}{\partial z}\Phi(z,\xi,\boldsymbol{\alpha}).$$

With this notation we have

Lemma 2.3. *Let ξ be sufficiently large. Then $\Phi'(z,\xi,\boldsymbol{\alpha})$ has exactly one zero $x_0 = x_0(\xi,\boldsymbol{\alpha})$ in the region $R = \{z \in \mathbb{C} : \Re z \ge 1, |\arg z| \le \theta\}$ with a suitably small $\theta > 0$. Moreover, x_0 is real and simple, and for some $\eta > 0$ satisfies*

$$x_0 = \tilde{x}_0 + O\left(\xi^{\frac{d\kappa_0}{d\kappa_0 - 1} - \eta}\right) \text{ with } \tilde{x}_0 = C_0 \xi^{\frac{d\kappa_0}{d\kappa_0 - 1}} \text{ and } C_0 = (2\pi d \alpha_0 \kappa_0 q^{\kappa_0})^{-\frac{d}{d\kappa_0 - 1}}. \tag{2.24}$$

Proof. We first consider the problem for the real variable x. Writing for simplicity $\Phi'(z) = \Phi'(z,\xi,\boldsymbol{\alpha})$, for $x \ge 1$ we have

$$\Phi'(x) = \frac{1}{d} x^{\frac{1}{d}-1} - 2\pi \sum_{\nu=0}^{N} c_\nu \left(\frac{x}{\xi}\right)^{\kappa_\nu} \frac{1}{x}, \qquad c_\nu = \alpha_\nu \kappa_\nu q^{\kappa_\nu}.$$

Hence, since $\kappa_0 > 1/d$, for ξ sufficiently large

$$\Phi'(1) = \frac{1}{d} + O(\xi^{-\eta}) > 0, \quad \Phi'(x) \sim -2\pi c_0 \left(\frac{x}{\xi}\right)^{\kappa_0} \frac{1}{x} < 0 \quad \text{as } x \to \infty.$$

Therefore, there exists at least one solution of $\Phi'(x) = 0$ in $[1,\infty)$, and we denote by x_0 one such solution. Assuming that $x_0 \le c\xi$ for a sufficiently small constant $c > 0$ we get

$$x_0^{1/d} = 2\pi d \sum_{\nu=0}^{N} c_\nu \left(\frac{x_0}{\xi}\right)^{\kappa_\nu} \ll c^{\kappa_N} < 1,$$

a contradiction since $x_0 > 1$. Hence $x_0 > c\xi$; thus, in particular, $x_0/\xi \gg 1$ and therefore

$$x_0^{1/d} = 2\pi d \sum_{\nu=0}^{N} c_\nu \left(\frac{x_0}{\xi}\right)^{\kappa_\nu} \ll \left(\frac{x_0}{\xi}\right)^{\kappa_0}.$$

From this inequality we obtain that $x_0/\xi \gg \xi^{\frac{1}{d\kappa_0 - 1}}$; hence

$$x_0^{1/d} = 2\pi d \sum_{\nu=0}^{N} c_\nu \left(\frac{x_0}{\xi}\right)^{\kappa_\nu} = 2\pi d c_0 \left(\frac{x_0}{\xi}\right)^{\kappa_0} \left(1 + O(\xi^{-\eta})\right)$$

for some $\eta > 0$ and consequently x_0 satisfies (2.24). Next we show that x_0 is unique and simple. With $\tilde{x}_0$ as in (2.24) we have that $(\tilde{x}_0/\xi)^{\kappa_0} \asymp \tilde{x}_0^{1/d}$, hence for $|x - \tilde{x}_0| < \varepsilon \tilde{x}_0$, $\varepsilon > 0$ sufficiently small, we obtain

(2.25)

$$\Phi''(x) = \Phi''(\tilde{x}_0) + O\left(\varepsilon \tilde{x}_0 \max_{(1-\varepsilon)\tilde{x}_0 \le x \le (1+\varepsilon)\tilde{x}_0} \Phi'''(x)\right) = \Phi''(\tilde{x}_0) + O\left(\varepsilon \tilde{x}_0^{\frac{1}{d}-2}\right).$$

On the other hand, recalling that $c_\nu = \alpha_\nu \kappa_\nu q^{\kappa_\nu}$ we have

(2.26)

$$\begin{aligned}
\Phi''(\tilde{x}_0) &= \frac{1}{d}\left(\frac{1}{d} - 1\right) \tilde{x}_0^{\frac{1}{d}-2} - 2\pi \sum_{\nu=0}^{N} c_\nu \left(\frac{\tilde{x}_0}{\xi}\right)^{\kappa_\nu} \frac{1}{\tilde{x}_0^2} \\
&= -\frac{d-1}{d^2} \tilde{x}_0^{\frac{1}{d}-2} \left\{1 + 2\pi c_0 \frac{\kappa_0(\kappa_0 - 1)d^2}{d-1} \left(\frac{\tilde{x}_0}{\xi}\right)^{\kappa_0} \tilde{x}_0^{-\frac{1}{d}} \left(1 + O(\xi^{-\eta})\right)\right\} \\
&= -\frac{d-1}{d^2} \tilde{x}_0^{\frac{1}{d}-2} \left\{\frac{d\kappa_0 - 1}{d-1} + O(\xi^{-\eta})\right\}
\end{aligned}$$

and, since $d\kappa_0 > 1$, the expression inside brackets is positive for ξ sufficiently large. Hence from (2.25) and (2.26) we have

$$\Phi''(x) < 0, \qquad |x - \tilde{x}_0| < \varepsilon \tilde{x}_0; \tag{2.27}$$

thus, in view of (2.24), the solution x_0 is unique and simple.

Now we write

$$h(z) = \frac{1}{d} z^{1/d} - 2\pi c_0 \left(\frac{z}{\xi}\right)^{\kappa_0}, \qquad k(z) = -2\pi \sum_{\nu=1}^{N} c_\nu \left(\frac{z}{\xi}\right)^{\kappa_\nu}$$

and observe that $h(z) + k(z) = \Phi'(z)/z$ and $z = \tilde{x}_0$ is the only zero of $h(z)$ in the region R. Moreover, recalling that $\tilde{x}_0 = C_0 \xi^{\frac{d\kappa_0}{d\kappa_0 - 1}}$, for $|z - \tilde{x}_0| = \delta \tilde{x}_0$ we have

$$|k(z)| \ll \left|\frac{z}{\xi}\right|^{\kappa_1} \leq c\xi^{\frac{\kappa_1}{d\kappa_1 - 1}},$$

$$|h(z)| = \frac{1}{d}|z|^{1/d}\left|1 - 2\pi c_0 \frac{z^{\kappa_0 - 1/d}}{\xi^{\kappa_0}}\right| \gg |z|^{1/d} \geq c'\xi^{\frac{\kappa_0}{d\kappa_0 - 1}}$$

for some constants $c, c' > 0$. Since $\kappa_0 > \kappa_1$, for $|z - \tilde{x}_0| = \delta\tilde{x}_0$ we therefore have $|k(z)| < |h(z)|$; hence by Rouché's theorem (see §3.42 of Titchmarsh [19]) x_0 is the only zero of $\Phi'(z)$ inside $|z - \tilde{x}_0| < \delta\tilde{x}_0$. Since the other zeros of $h(z)$ are bounded away from the region R, a similar argument applies to any closed contour in R containing the circle $|z - \tilde{x}_0| = \delta\tilde{x}_0$, and hence the lemma follows. □

By Lemma 2.3, $x_0 = x_0(\xi, \boldsymbol{\alpha})$ is the critical point of the integral $I_X(s, n, \ell)$ in (2.23) with n replaced by ξ (see also (2.17)). For the reasons explained in the introduction, we continue the treatment of such an integral with n replaced by ξ, and only in the end we will choose $\xi = n$. Moreover, as in Lemma 2.2, since the integrand in $I_X(s, \xi, \ell)$ is holomorphic in the complex variable x for $\Re x > 0$, we use instead of x the complex variable z. By the saddle point method, for ξ sufficiently large the main contribution to $I_X(s, \xi, \ell)$ is expected to come from $K_X(s + \frac{\ell}{d}, \xi)$, where

$$K_X(s, \xi) = \gamma x_0^{\frac{d+1}{2d} - s} \int_{-r}^{r} e^{-\Psi_X(z,\xi) + i\Phi(z,\xi,\boldsymbol{\alpha})}(1 + \gamma\lambda)^{\frac{d+1}{2d} - s - 1}\mathrm{d}\lambda, \tag{2.28}$$

$\Psi_X(z, \xi)$ and $\Phi(z, \xi, \boldsymbol{\alpha})$ are defined by (2.18) and (1.8), respectively, and

$$\gamma = 1 - i, \qquad z = x_0(1 + \gamma\lambda), \qquad r = \frac{\log \xi}{\sqrt{|R|}} \quad \text{and} \quad R = x_0^2\Phi''(x_0, \xi, \boldsymbol{\alpha}). \tag{2.29}$$

Note that $K_X(s, \xi)$ is entire, and by (2.24) and (2.26) we have

$$|\Phi''(x_0, \xi, \boldsymbol{\alpha})| \asymp x_0^{\frac{1}{d} - 2}, \qquad |R| \asymp x_0^{1/d} \asymp \xi^{\frac{\kappa_0}{d\kappa_0 - 1}}, \qquad r < 1. \tag{2.30}$$

Inspired by the saddle point techniques in Jutila's book [5] we prove the following

Lemma 2.4. *Let ξ_0 be sufficiently large. Then for $\xi \geq \xi_0$ we have*

$$I_X(s, \xi, \ell) = K_X(s + \frac{\ell}{d}, \xi) \ + \ k_X(s, \xi, \ell),$$

where $K_X(s+\frac{\ell}{d}, \xi)$ is entire and $k_X(s, \xi, \ell)$ is meromorphic over $\mathbb{C}$, holomorphic for $0 < \sigma < \delta$ and satisfies

$$k_X(s, \xi, \ell) \ll e^{c|t|}\xi^{-\eta}$$

for some $c, \eta > 0$*, uniformly for* $0 \le \ell \le L$*,* $0 < \sigma < \delta$ *and* $X \to \infty$*. Moreover, for* $\xi < \xi_0$ *the integral* $I_X(s,\xi,\ell)$ *has the same properties of* $k_X(s,\xi,\ell)$*, with* $\eta = 0$ *and uniformly for* $\xi < \xi_0$*.*

Proof. The function $K_X(s+\frac{\ell}{d},\xi)$ is clearly entire. Given $\phi > 0$ sufficiently small and recalling (2.29), we consider the points

$$x_0' = \rho_0' e^{i\phi}, \qquad x_0'' = \rho_0'' e^{-i\phi}, \qquad x_0^- = x_0(1-\gamma r), \qquad x_0^+ = x_0(1+\gamma r)$$

with

$$\rho_0' = \frac{x_0}{\cos\phi + \sin\phi}, \qquad \rho_0'' = \frac{x_0}{\cos\phi - \sin\phi}.$$

Then, given $\varepsilon > 0$ sufficiently small and using the simpler notation $\Psi_X(z) = \Psi_X(z,\xi)$ and $\Phi(z) = \Phi(z,\xi,\boldsymbol{\alpha})$, for s in the range (2.10) we have

(2.31)

$$\begin{aligned} I_X(s,\xi,\ell) = &\int_0^{\xi^\varepsilon e^{i\phi}} e^{iz^{1/d}} \left(e^{-\Psi_X(z)} e\left(-f\left(\frac{qz}{\xi},\boldsymbol{\alpha}\right)\right) - 1\right) z^{\frac{d+1}{2d}-\frac{\ell}{d}-s-1} \mathrm{d}z \\ &- \int_{\xi^\varepsilon e^{i\phi}}^{\infty e^{i\phi}} e^{iz^{1/d}} z^{\frac{d+1}{2d}-\frac{\ell}{d}-s-1} \mathrm{d}z \\ &+ \left\{ \int_{\xi^\varepsilon e^{i\phi}}^{x_0'} + \int_{x_0'}^{x_0^-} + \int_{x_0^-}^{x_0^+} + \int_{x_0^+}^{x_0''} + \int_{x_0''}^{\infty e^{-i\phi}} \right\} e^{-\Psi_X(z)+i\Phi(z)} z^{\frac{d+1}{2d}-\frac{\ell}{d}-s-1} \mathrm{d}z \\ = &A - B + C + D + E + F + G, \end{aligned}$$

say. We write $z = \rho e^{i\theta}$ with $\theta = \pm\phi$ or $\theta = -\pi/4$. Note that, after a change of variable,

$$E = K_X\left(s + \frac{\ell}{d}, \xi\right). \tag{2.32}$$

Thus we have to prove that the remaining terms in (2.31) satisfy the properties of $k_X(s,\xi,\ell)$ in the lemma. In the remaining part of the proof we implicitly assume that the bounds we get satisfy the uniformity requirements in the lemma.

A: we treat the term A as in the proof of Lemma 2.2, thus getting that A satisfies the properties of $k_X(s,\xi,\ell)$.

B: since $|e^{iz^{1/d}}| = e^{-\sin(\phi/d)\rho^{1/d}}$, the integral is absolutely convergent for every s, and hence B is an entire function. Moreover, with the same notation as in (2.21) we have

$$B \ll e^{c|t|} \int_{\xi^\varepsilon}^\infty e^{-\sin(\phi/d)\rho^{1/d}} \rho^{\frac{d+1}{2d}-\frac{\ell}{d}-\sigma-1} \mathrm{d}\rho \ll e^{c|t|} \xi^{-c'}.$$

C: for ε and ϕ sufficiently small, $\theta = \phi$ and $\xi^\varepsilon \le \rho \le \rho_0'$ we have $|e^{-\Psi_X(z)}| \le 1$ and $|e^{i\Phi(z)}| = e^{-\Im\Phi(z)}$. Writing $c_\nu = \alpha_\nu q^{\kappa_\nu}$ and using (2.24) we obtain

$$\begin{aligned}\Im\Phi(z) &= \rho^{1/d}\sin(\phi/d) - 2\pi\sum_{\nu=0}^{N} c_\nu \sin(\kappa_\nu\phi)\left(\frac{\rho}{\xi}\right)^{\kappa_\nu} \\ &\geq \rho^{1/d}\sin(\phi/d)\left\{1 - 2\pi c_0 \frac{\sin(\kappa_0\phi)}{\sin(\phi/d)}\frac{{\rho_0'}^{(d\kappa_0-1)/d}}{\xi^{\kappa_0}}\Big(1+O(\xi^{-\eta})\Big)\right\} \\ &= \rho^{1/d}\sin(\phi/d)\left\{1 - \frac{1}{d\kappa_0}\frac{\sin(\kappa_0\phi)}{\sin(\phi/d)}\frac{1}{(\cos\phi+\sin\phi)^{(d\kappa_0-1)/d}} + O(\xi^{-\eta})\right\} \\ &\geq c_1(\phi)\rho^{1/d}\end{aligned}$$

with $c_1(\phi) > 0$, since for ϕ sufficiently small

$$\frac{1}{d\kappa_0}\frac{\sin(\kappa_0\phi)}{\sin(\phi/d)}\frac{1}{(\cos\phi+\sin\phi)^{(d\kappa_0-1)/d}} < 1.$$

Hence, again with the notation in (2.21), we have

$$C \ll e^{c|t|}\int_{n^\varepsilon}^{\rho_0'} e^{-c_1(\phi)\rho^{1/d}}\rho^{\frac{d+1}{2d}-\frac{\ell}{d}-\sigma-1}\mathrm{d}\rho \ll e^{c|t|}\xi^{-c'},$$

and C is clearly an entire function.

D: again, D is an entire function. We make the change of variable $z = x_0(1+\gamma\lambda)$ as in (2.29), thus getting for some $c_1, c_2 > 0$ that

$$D \ll \xi^{c_1} e^{c_2|t|}\int_{-r'}^{-r} e^{-\Im\Phi(z)}\mathrm{d}\lambda,$$

where r is as in (2.29) and $r' = \sin\phi/(\cos\phi+\sin\phi)$; here we used the fact that $\Re\Psi_X(x_0(1+\gamma\lambda)) > 0$ for $-r' \leq \lambda \leq -r$. Since $\Phi(x_0) \in \mathbb{R}$, $\Phi'(x_0) = 0$, $\Phi''(x_0) < 0$ (see (2.27)), $\Im\gamma^2 = -2$ and $|z^3\Phi'''(z)| \asymp x_0^{1/d} \asymp |x_0^2\Phi''(x_0)| = |R|$ for $-r' \leq \lambda \leq r$, for some $c > 0$ we have

$$\begin{aligned}\Im\Phi(z) &= \Im\left\{\frac{1}{2}\Phi''(x_0)x_0^2(\gamma\lambda)^2 + O\left(\max_{-r'\leq\lambda\leq r}|z^3\Phi'''(z)\lambda^3|\right)\right\} \\ &= |R|\lambda^2(1+O(|\lambda|)) \;\geq\; c|R|\lambda^2.\end{aligned} \tag{2.33}$$

Hence a computation shows that, with the notation in (2.21),

$$D \ll \xi^{c_1}e^{c_2|t|}\int_r^\infty e^{-c|R|\lambda^2}\mathrm{d}\lambda \ll e^{c|t|}\xi^{-c'}.$$

F: we treat F exactly in the same way as D and obtain the same result.

G: in this case we have $\theta = -\phi$ and $\rho \geq \rho_0''$. We proceed as in the case of C, with the difference that this time the term corresponding to κ_0 dominates.

Writing again $c_\nu = \alpha_\nu q^{\kappa_\nu}$ we have

$$
\begin{aligned}
&\Im\Phi(z) = -\rho^{1/d}\sin(\phi/d) + 2\pi\sum_{\nu=0}^{N} c_\nu \sin(\kappa_\nu\phi)\left(\frac{\rho}{\xi}\right)^{\kappa_\nu}\\
&\geq 2\pi c_0\sin(\kappa_0\phi)\left(\frac{\rho}{\xi}\right)^{\kappa_0}\left\{1-\frac{\sin(\phi/d)\rho_0''^{(1-d\kappa_0)/d}\xi^{\kappa_0}}{2\pi c_0\sin(\kappa_0\phi)}+O(\xi^{-\eta})\right\}\\
&= 2\pi c_0\sin(\kappa_0\phi)\left(\frac{\rho}{\xi}\right)^{\kappa_0}\left\{1-\frac{d\kappa_0\sin(\phi/d)(\cos\phi-\sin\phi)^{(d\kappa_0-1)/d}}{\sin(\kappa_0\phi)}+O(\xi^{-\eta})\right\}\\
&\geq c_2(\phi)(\frac{\rho}{\xi})^{\kappa_0}
\end{aligned}
$$

with $c_2(\phi) > 0$, since for $\phi > 0$ sufficiently small

$$
\frac{d\kappa_0\sin(\phi/d)(\cos\phi-\sin\phi)^{(d\kappa_0-1)/d}}{\sin(\kappa_0\phi)} < 1.
$$

Hence, once again with the notation in (2.21), since $\rho_0''/\xi \asymp \xi^{1/(d\kappa_0-1)}$ we have

$$
G \ll e^{c_1|t|}\int_{\rho_0''}^{\infty} e^{-c_2(\phi)(\frac{\rho}{\xi})^{\kappa_0}}\rho^{c_2}\mathrm{d}\rho \ll e^{c_1|t|}\xi^{c_2}\int_{\rho_0''/\xi}^{\infty} e^{-c_2(\phi)y^{\kappa_0}}y^{c_3}\mathrm{d}y \ll e^{c|t|}\xi^{-c'},
$$

and G is clearly an entire function.

The first part of the lemma follows now from (2.31), (2.32) and the bounds and properties of A, B, C, E, F and G. When $\xi < \xi_0$ the proof is simpler. For simplicity, we still split $I_X(s,\xi,\ell)$ as in (2.31) and proceed as above in the cases of A, B, C and G (in this case $\eta = 0$). Then, the integral from x_0' to x_0'', i.e. $D+E+F$, is trivially bounded by $e^{c|t|}$, and the second part of the lemma follows as well. □

From Lemma 2.4 we see that $I_X(s,\xi,\ell)$ is meromorphic on $\mathbb{C}$; hence (2.23) holds for $s \in \mathbb{C}$. Therefore, from Lemma 2.1, (2.2), (2.8), (2.9), (2.23) and applying Lemma 2.4 with $\xi = n$, summing over n we get

$$
F_X(s;f) = e^{a_3 s+b_3}\sum_{\ell=0}^{L} c_\ell \sum_{n=n_0}^{\infty}\frac{\overline{a(n)}}{n^{1-s}}K_X\left(s+\frac{\ell}{d},n\right)+\sum_{n=1}^{\infty}\frac{\overline{a(n)}}{n^{1-s}}H_X(s,n) \tag{2.34}
$$

provided the two series converge, where n_0 is a sufficiently large integer, $a_3 \in \mathbb{R}$ and b_3 are certain constants, $c_0 \neq 0$, $K_X(s,\xi)$ is defined by (2.28) and is an entire function, and $H_X(s,n)$ is holomorphic for $0 < \sigma < \delta$ and satisfies

$$
H_X(s,n) \ll e^{c|t|}n^{-\eta} \tag{2.35}
$$

uniformly for $0 < \sigma < \delta$ and $X \to \infty$.

4. *Limit as* $X \to \infty$. Writing EBV(X) for “entire and bounded on every vertical strip, depending on X and on the strip”, we have

LEMMA 2.5. *Under the hypotheses of Theorem* 1.1 *and with the notation of* (2.34) *we have*

$$F_X(s;f) = e^{a_3 s+b_3}\sum_{\ell=0}^{L} c_\ell \sum_{n=n_0}^{\infty} \frac{\overline{a(n)}}{n^{1-s}} K_X\left(s+\frac{\ell}{d},n\right) + H_X(s) = M_X(s) + H_X(s),$$

say, where $c_0 \neq 0$, $M_X(s)$ *is* EBV(X) *and* $H_X(s)$ *satisfies the following properties*:

(1) $H_X(s)$ *is* EBV(X);
(2) $H_X(s) \ll e^{c|t|}$ *uniformly for* $0 < \sigma < \delta$ *and* $X \to \infty$ *with some* $c, \delta > 0$;
(3) $H_X(s) \ll 1$ *uniformly as* $X \to \infty$ *for* s *in every strip* $\sigma_0 < \sigma < \sigma_0 + 1$ *with* $\sigma_0 > d\kappa_0$, *and*

$$\lim_{X\to\infty} H_X(s)$$

exists, convergence as $X \to \infty$ *being uniform on compact subsets of such strips.*

Proof. Thanks to (2.35) and to the absolute convergence of $F(s)$ for $\sigma > 1$, we deduce that the second series in (2.34) is absolutely convergent for $0 < \sigma < \delta$ for a sufficiently small $\delta > 0$; hence

$$H_X(s) = \sum_{n=1}^{\infty} \frac{\overline{a(n)}}{n^{1-s}} H_X(s,n) \ll e^{c|t|} \tag{2.36}$$

uniformly for $0 < \sigma < \delta$ and $X \to \infty$, and $H_X(s)$ is holomorphic for $0 < \sigma < \delta$. Concerning the first series in (2.34), recalling the notation in (2.29) we observe that

$$K_X\left(s+\frac{\ell}{d},n\right) \ll x_0^{\frac{d+1}{2d}-\frac{\ell}{d}-\sigma} \int_{-r}^{r} e^{-\Re\Psi_X(z,n)-\Im\Phi(z,n,\boldsymbol{\alpha})} \mathrm{d}\lambda$$

and, thanks to (2.33), for some $c, c' > 0$

$$\Re\Psi_X(z,n) = \frac{1}{X}\sum_{\nu=0}^{N}\left(\frac{q|z|}{n}\right)^{\kappa_\nu}\cos(\kappa_\nu \arg z) \geq \frac{c}{X} n^{\frac{\kappa_0}{d\kappa_0-1}}$$
$$\Im\Phi(z,n,\boldsymbol{\alpha}) \geq c'|R|\lambda^2.$$

Therefore, by (2.29) and (2.30) we have

$$\begin{aligned} K_X\left(s+\frac{\ell}{d},n\right) &\ll n^{\frac{d\kappa_0}{d\kappa_0-1}(\frac{d+1}{2d}-\frac{\ell}{d}-\sigma)} e^{-\frac{c}{X}n^{\kappa_0/(d\kappa_0-1)}} \int_{-r}^{r} \mathrm{d}\lambda \\ &\ll e^{-\frac{c}{X}n^{\kappa_0/(d\kappa_0-1)}} n^{\frac{d\kappa_0}{d\kappa_0-1}(\frac{d}{2d}-\frac{\ell}{d}-\sigma)} \log n. \end{aligned} \tag{2.37}$$

Summing over n, from (2.37) we obtain that $M_X(s)$ is EBV(X). Now, (1) follows since both $F_X(s;f)$ and $M_X(s)$ are EBV(X), and (2) follows from

(2.36). Moreover, by (2.37) we have

$$\sum_{n\geq n_0} \frac{\overline{a(n)}}{n^{1-s}} K_X\left(s+\frac{\ell}{d}, n\right) \ll \sum_{n\geq n_0} |a(n)| n^{\sigma-1+\frac{d\kappa_0}{d\kappa_0-1}(\frac{1}{2}-\sigma)} \log n$$

uniformly as $X\to\infty$ and $0\leq\ell\leq L$. Hence, as $X\to\infty$, $M_X(s)$ is uniformly bounded in every vertical strip contained in the half-plane $\sigma > d\kappa_0/2$. Moreover,

$$\lim_{X\to\infty} M_X(s)$$

exists, convergence as $X\to\infty$ being uniform on compact subsets of such strips. The same properties hold trivially for $F_X(s;f)$ in every vertical strip contained in the half-plane $\sigma>1$. Therefore (3) follows since $d\kappa_0>1$, and the lemma is proved. □

In order to take the limit as $X\to\infty$ of $F_X(s;f)$ we need the following general result.

Lemma 2.6. *If a function* $H_X(s)$, *depending on a parameter* $X>1$, *satisfies properties* (1), (2) *and* (3) *in Lemma* 2.5, *then*

$$H(s)=\lim_{X\to\infty} H_X(s)$$

exists and is holomorphic for $\sigma>0$.

Proof. For a suitable constant $c>0$ consider the function

$$G_X(s)=\Gamma(cs+1)H_X(s),$$

holomorphic for $\sigma>-\delta$ for some $\delta>0$. By (2), (3) and Stirling's formula, for $\sigma\in(0,\delta)\cup(\sigma_0,\sigma_0+1)$ we have that $G_X(s)$ is uniformly bounded as $X\to\infty$. Moreover, by (1), $G_X(s)$ is bounded, depending on X, on the vertical strip $-\delta<\sigma<\sigma_0+1/2$. Hence, by the Phragmén-Lindelöf theorem, $G_X(s)$ is uniformly bounded as $X\to\infty$ for $0<\sigma<\sigma_0$. In view of (3), by Vitali's convergence theorem (see §5.21 of Titchmarsh [19])

$$G(s)=\lim_{X\to\infty} G_X(s)$$

exists and is holomorphic for $0<\sigma<\sigma_0$; thus the same holds for $H(s)$ as well. The lemma follows since σ_0 can be chosen arbitrarily large. □

Writing

$$K(s,\xi)=\lim_{X\to\infty} K_X(s,\xi),$$

with the notation in (2.29) we have

$$K(s,\xi)=\gamma x_0^{\frac{d+1}{2d}-s}\int_{-r}^{r} e^{i\Phi(z,\xi,\boldsymbol{\alpha})}(1+\gamma\lambda)^{\frac{d+1}{2d}-s-1}\mathrm{d}\lambda. \tag{2.38}$$

Therefore, from Lemmas 2.5 and 2.6 we obtain

$$F(s;f) = e^{a_3 s + b_3} \sum_{\ell=0}^{L} c_\ell \sum_{n=n_0}^{\infty} \frac{\overline{a(n)}}{n^{1-s}} K\left(s+\frac{\ell}{d}, n\right) + H(s) = M(s) + H(s), \tag{2.39}$$

say, where $a_3 \in \mathbb{R}$ and b_3 are certain constants, $c_0 \neq 0$, n_0 is sufficiently large and $H(s)$ is holomorphic for $\sigma > 0$. Note that $F(s;f)$ and $M(s)$ are defined, respectively, for $\sigma > 1$ and σ sufficiently large.

5. *Computing* $f^*(\xi, \boldsymbol{\alpha})$. In order to prove Theorem 1.1 we have to extract the main contribution from $K(s+\frac{\ell}{d}, n)$, thus transforming (2.39) into (1.7). Again we work with the real variable ξ in place of n.

LEMMA 2.7. *Let ξ_0 be sufficiently large, $\xi \geq \xi_0$, x_0 be as in* (2.24) *and $K(s,\xi)$ be as in* (2.38). *Then with the notation in* (2.29) *we have*

$$K\left(s+\frac{\ell}{d}, \xi\right) = \frac{\gamma\sqrt{\pi}}{\sqrt{|R|}} e\left(\frac{1}{2\pi}\Phi(x_0,\xi,\boldsymbol{\alpha})\right) x_0^{\frac{d+1}{2d}-\frac{\ell}{d}-s} + f_1(s,\xi,\ell),$$

where $f_1(s,\xi,\ell)$ is entire and for $0 \leq \ell \leq L$ satisfies

$$f_1(s,\xi,\ell) \ll \xi^{-\frac{2-d}{2}\frac{\kappa_0}{d\kappa_0-1}-\sigma} \log^7 \xi.$$

Proof. Write $\Phi(z) = \Phi(z,\xi,\boldsymbol{\alpha})$. With the notation in (2.29) we have

$$K\left(s+\frac{\ell}{d}, \xi\right) = \gamma x_0^{\frac{d+1}{2d}-\frac{\ell}{d}-s} e^{i\Phi(x_0)} I(s), \tag{2.40}$$

where

$$I(s) = \int_{-r}^{r} e^{i(\Phi(z)-\Phi(x_0))}(1+\gamma\lambda)^{\frac{d+1}{2d}-\frac{\ell}{d}-s-1} \mathrm{d}\lambda.$$

For every integer $m \geq 2$ let $R_m = x_0^m \Phi^{(m)}(x_0)$, so that $R = R_2$ and, writing $c_\nu = \alpha_\nu q^{\kappa_\nu}$,

$$R_m = \left(\frac{1}{d}\right)_m x_0^{1/d} - 2\pi \sum_{\nu=0}^{N} c_\nu (\kappa_\nu)_m \left(\frac{x_0}{\xi}\right)^{\kappa_\nu}$$

where, as usual, for $x \in \mathbb{R}$ we write $(x)_m = x(x-1)\cdots(x-m+1)$. Note that if $k \in \mathbb{N}$ is such that $k-1 < |x| \leq k$, then

$$\begin{aligned} |(x)_m| &\leq |x|(|x|+1)\cdots(|x|+m-1) \leq k(k+1)\cdots(k+m-1) \\ &= \binom{k+m-1}{k-1} m! \leq 2^{k+m-1} m! \leq 2^{|x|+m} m!; \end{aligned}$$

hence by (2.30)

$$\frac{|R_m|}{m!} \ll 2^m \left(x_0^{1/d} + \left(\frac{x_0}{\xi}\right)^{\kappa_0}\right) \ll 2^m \xi^{\frac{\kappa_0}{d\kappa_0-1}}, \qquad |R| \asymp \xi^{\frac{\kappa_0}{d\kappa_0-1}}. \tag{2.41}$$

Therefore, recalling (2.29), (2.30) and that $\Phi'(x_0)=0$, for $-r\le\lambda\le r$ we have

$$\Phi(z)-\Phi(x_0)=\sum_{m=2}^{\infty}\frac{R_m}{m!}(\gamma\lambda)^m=-iR\lambda^2+\frac{1}{6}\gamma^3R_3\lambda^3+O\left(\xi^{\frac{\kappa_0}{d\kappa_0-1}}\lambda^4\right)$$

and hence

$$I(s)=\tilde{I}(s)+O\left(\xi^{\frac{\kappa_0}{d\kappa_0-1}}\int_{-r}^{r}\lambda^4\mathrm{d}\lambda\right)=\tilde{I}(s)+O\left(\xi^{-\frac{3}{2}\frac{\kappa_0}{d\kappa_0-1}}\log^5\xi\right), \tag{2.42}$$

where, since $R<0$ by (2.27) and (2.29),

$$\tilde{I}(s)=\int_{-r}^{r}e^{-|R|\lambda^2+\frac{i}{6}\gamma^3R_3\lambda^3}(1+\gamma\lambda)^{\frac{d+1}{2d}-\frac{\ell}{d}-s-1}\mathrm{d}\lambda.$$

But

$$e^{\frac{i}{6}\gamma^3R_3\lambda^3}=1+\frac{i}{6}\gamma^3R_3\lambda^3+O(|R_3|^2\lambda^6),\qquad (1+\gamma\lambda)^{\frac{d+1}{2d}-\frac{\ell}{d}-s-1}=1+O(|\lambda|);$$

hence

$$\begin{aligned}\tilde{I}(s)&=\int_{-r}^{r}e^{-|R|\lambda^2}(1+\gamma\lambda)^{\frac{d+1}{2d}-\frac{\ell}{d}-s-1}\mathrm{d}\lambda+\frac{i}{6}\gamma^3R_3\int_{-r}^{r}e^{-|R|\lambda^2}\lambda^3\mathrm{d}\lambda\\&\quad+O\left(|R_3|\int_{-r}^{r}e^{-|R|\lambda^2}\lambda^4\mathrm{d}\lambda\right)\\&\quad+O\left(|R_3|^2\int_{-r}^{r}e^{-|R|\lambda^2}\lambda^6\mathrm{d}\lambda\right)=A+B+C+D,\end{aligned} \tag{2.43}$$

say. Using the expansion $(1+\gamma\lambda)^{\frac{d+1}{2d}-\frac{\ell}{d}-s-1}=1+c(s)\lambda+O(\lambda^2)$, in view of (2.29) we get

$$\begin{aligned}A&=\int_{-r}^{r}e^{-|R|\lambda^2}\mathrm{d}\lambda+c(s)\int_{-r}^{r}e^{-|R|\lambda^2}\lambda\mathrm{d}\lambda+O\left(\int_{-r}^{r}e^{-|R|\lambda^2}\lambda^2\mathrm{d}\lambda\right)\\&=\frac{2}{|R|}\int_0^{\log\xi}e^{-u^2}\mathrm{d}u+O(|R|^{-3/2}\log^3\xi)=\frac{\sqrt{\pi}}{|R|}+O\left(\xi^{-\frac{3}{2}\frac{\kappa_0}{d\kappa_0-1}}\log^3\xi\right).\end{aligned}$$

Clearly $B=0$, and by (2.41)

$$C\ll|R_3||R|^{-5/2}\log^5\xi\ll\xi^{-\frac{3}{2}\frac{\kappa_0}{d\kappa_0-1}}\log^5\xi,$$

$$D\ll|R_3|^2|R|^{-7/2}\log^7\xi\ll\xi^{-\frac{3}{2}\frac{\kappa_0}{d\kappa_0-1}}\log^7\xi.$$

From (2.42), (2.43) and the evaluations of A, B, C and D we obtain

$$I(s)=\frac{\sqrt{\pi}}{\sqrt{|R|}}+O\left(\xi^{-\frac{3}{2}\frac{\kappa_0}{d\kappa_0-1}}\log^7\xi\right), \tag{2.44}$$

and hence, since by (2.24) we have

$$\begin{aligned}\frac{d\kappa_0}{d\kappa_0-1}\left(\sigma+\frac{\ell}{d}-\frac{d+1}{2d}\right)+\frac{3}{2}\frac{\kappa_0}{d\kappa_0-1}&\ge\sigma+\left(\frac{3}{2}-\frac{d+1}{2}\right)\frac{\kappa_0}{d\kappa_0-1}\\&=\sigma+\frac{2-d}{2}\frac{\kappa_0}{d\kappa_0-1};\end{aligned}$$

the lemma follows from (2.40) and (2.44). □

To conclude the proof of Theorem 1.1 we need suitable expansions for the critical point x_0 and the associated critical value $\Phi(x_0, \xi, \boldsymbol{\alpha})$. Such expansions, as well as the one in (2.47) below, are absolutely convergent; we will tacitly use this fact in what follows. Recalling the definition of $\mathcal{D}_f$, κ_0^* and ω^* in the introduction we have

LEMMA 2.8. *Let ξ_0 be sufficiently large, $\xi \geq \xi_0$ and C_0 be as in* (2.24). *There exist coefficients $a_\omega(\boldsymbol{\alpha})$ and $A_\omega(\boldsymbol{\alpha})$, $\omega \in \mathcal{D}_f$ and $a_0(\boldsymbol{\alpha}) = 1$, such that*

$$x_0 = C_0 \xi^{d\kappa_0^*} \sum_{\omega \in \mathcal{D}_f} a_\omega(\boldsymbol{\alpha}) \xi^{-\omega^*},$$

$$\frac{1}{2\pi} \Phi(x_0, \xi, \boldsymbol{\alpha}) = \xi^{\kappa_0^*} \sum_{\omega \in \mathcal{D}_f} A_\omega(\boldsymbol{\alpha}) \xi^{-\omega^*}.$$

Proof. From the definition of x_0 and writing $c_\nu = \alpha_\nu \kappa_\nu q^{\kappa_\nu}$ we have

$$x_0^{1/d} - 2\pi d c_0 \left(\frac{x_0}{\xi}\right)^{\kappa_0} = 2\pi d \sum_{\nu=1}^{N} c_\nu \left(\frac{x_0}{\xi}\right)^{\kappa_\nu}. \tag{2.45}$$

Writing $x_0 = \tilde{x}_0 + r$ with $|r| < 1$ (see Lemma 2.3), by (2.24) the left-hand side of (2.45) becomes

$$\tilde{x}_0^{1/d}\left(1 + \frac{r}{\tilde{x}_0}\right)^{1/d} - \frac{2\pi d c_0}{\xi^{\kappa_0}} \tilde{x}_0^{\kappa_0}\left(1 + \frac{r}{\tilde{x}_0}\right)^{\kappa_0}$$
$$= \tilde{x}_0^{1/d}\left\{\left(1 + \frac{r}{\tilde{x}_0}\right)^{1/d} - \left(1 + \frac{r}{\tilde{x}_0}\right)^{\kappa_0}\right\} = \tilde{x}_0^{1/d} h\left(\frac{r}{\tilde{x}_0}\right),$$

say, where for $|z| < 1$

$$h(z) = \sum_{m=1}^{\infty} \left\{\binom{1/d}{m} - \binom{\kappa_0}{m}\right\} z^m, \qquad h(0) = 0,\ h'(0) = 1/d - \kappa_0 < 0.$$

Therefore, the local inverse of $h(z)$ exists, and we write $h^{-1}(z) = \sum_{m=1}^{\infty} c_m z^m$. Since (2.45) can be written as (see (2.24))

$$h\left(\frac{r}{\tilde{x}_0}\right) = \tilde{x}_0^{-1/d} 2\pi d \sum_{\nu=1}^{N} c_\nu \xi^{-\kappa_\nu} \tilde{x}_0^{\kappa_\nu}\left(1 + \frac{r}{\tilde{x}_0}\right)^{\kappa_\nu}$$
$$= 2\pi d \sum_{\nu=1}^{N} c_\nu C_0^{\kappa_\nu - \frac{1}{d}} \xi^{-\frac{\kappa_0 - \kappa_\nu}{d\kappa_0 - 1}} \sum_{k=0}^{\infty} \binom{\kappa_\nu}{k}\left(\frac{r}{\tilde{x}_0}\right)^k,$$

recalling that $\kappa_\nu = \kappa_0 - \omega_\nu$ and the definition of ω^* we have

$$\frac{r}{\tilde{x}_0} = h^{-1}\left(2\pi d \sum_{\nu=1}^{N} c_\nu C_0^{\kappa_\nu - \frac{1}{d}} \xi^{-\omega_\nu^*} \sum_{k=0}^{\infty} \binom{\kappa_\nu}{k}\left(\frac{r}{\tilde{x}_0}\right)^k\right).$$

Hence expanding and interchanging summation we get

$$\frac{r}{\tilde{x}_0} = \sum_{k=0}^{\infty} \sum_{\omega\in\mathcal{D}_f,\, \omega>0} c_{\omega,k}(\boldsymbol{\alpha})\xi^{-\omega^*}\left(\frac{r}{\tilde{x}_0}\right)^k, \tag{2.46}$$

with certain coefficients $c_{\omega,k}(\boldsymbol{\alpha})$; the manipulations of the series are justified by absolute convergence. By an iterative application of (2.46) we obtain that

$$\frac{r}{\tilde{x}_0} = \sum_{\omega\in\mathcal{D}_f,\, \omega>0} a_\omega(\boldsymbol{\alpha})\xi^{-\omega^*}$$

with certain coefficients $a_\omega(\boldsymbol{\alpha})$, and hence in view of (2.24)

$$\begin{aligned} x_0 = \tilde{x}_0\left(1+\frac{r}{\tilde{x}_0}\right) &= C_0\xi^{\frac{d\kappa_0}{d\kappa_0-1}}\left(1+\sum_{\omega\in\mathcal{D}_f,\, \omega>0} a_\omega(\boldsymbol{\alpha})\xi^{-\omega^*}\right) \\ &= C_0\xi^{d\kappa_0^*}\sum_{\omega\in\mathcal{D}_f} a_\omega(\boldsymbol{\alpha})\xi^{-\omega^*} \end{aligned}$$

with $a_0(\boldsymbol{\alpha})=1$, and the first assertion of the lemma follows.

To prove the second assertion we substitute the above series expansion of x_0 into the definition of $\Phi(x_0,\xi,\boldsymbol{\alpha})$ and observe that $\kappa_0(d\kappa_0^*-1)=\kappa_0^*$, thus getting

$$\begin{aligned} \frac{\Phi(x_0,\xi,\boldsymbol{\alpha})}{2\pi} = \xi^{\kappa_0^*}\Bigg\{ &\frac{C_0^{1/d}}{2\pi}\left(\sum_{\omega\in\mathcal{D}_f} a_\omega(\boldsymbol{\alpha})\xi^{-\omega^*}\right)^{1/d} \\ &- \sum_{\nu=0}^{N}\alpha_\nu q^{\kappa_\nu} C_0^{\kappa_\nu}\xi^{-\omega_\nu^*}\left(\sum_{\omega\in\mathcal{D}_f} a_\omega(\boldsymbol{\alpha})\xi^{-\omega^*}\right)^{\kappa_\nu}\Bigg\}. \end{aligned}$$

The assertion follows recalling that $a_0(\boldsymbol{\alpha})=1$ and applying the expansions of $(1+x)^{1/d}$ and $(1+x)^{\kappa_\nu}$, and then rearranging the terms. □

By Lemma 2.8 and (2.26) with x_0 in place of $\tilde{x}_0$ we also obtain that there exist coefficients $b_\omega(\boldsymbol{\alpha})$ with $b_0(\boldsymbol{\alpha})\neq 0$ such that for ξ sufficiently large

$$\frac{1}{\sqrt{|R|}} = \big(-x_0^2\Phi''(x_0,\xi,\boldsymbol{\alpha})\big)^{-1/2} = \xi^{-\frac{\kappa_0}{2(d\kappa_0-1)}}\sum_{\omega\in\mathcal{D}_f} b_\omega(\boldsymbol{\alpha})\xi^{-\omega^*}. \tag{2.47}$$

Theorem 1.1 follows now from the expansions in Lemma 2.8 and (2.47). Indeed, by Lemma 2.8 we have

$$x_0^{\frac{d+1}{2d}-\frac{\ell}{d}-s} = \xi^{\frac{\kappa_0}{d\kappa_0-1}(\frac{d+1}{2}-\ell-ds)}\sum_{\omega\in\mathcal{D}_f} g_\omega(s,\boldsymbol{\alpha},\ell)\xi^{-\omega^*}$$

with $g_\omega(s,\boldsymbol{\alpha},\ell)$ holomorphic and $g_0(s,\boldsymbol{\alpha},\ell)\neq 0$ for $\sigma>0$; hence by (2.47)

$$\frac{\gamma\sqrt{\pi}}{\sqrt{|R|}}x_0^{\frac{d+1}{2d}-\frac{\ell}{d}-s} = \xi^{\frac{\kappa_0}{d\kappa_0-1}(\frac{d}{2}-\ell-ds)}\sum_{\omega\in\mathcal{D}_f} h_\omega(s,\boldsymbol{\alpha},\ell)\xi^{-\omega^*} \tag{2.48}$$

with $h_\omega(s,\boldsymbol{\alpha},\ell)$ holomorphic and $h_0(s,\boldsymbol{\alpha},\ell) \neq 0$ for $\sigma > 0$. Moreover, from Lemma 2.8 we also get

$$e\Big(\frac{1}{2\pi}\Phi(x_0,\xi,\boldsymbol{\alpha})\Big) = e\Bigg(\xi^{\kappa_0^*}\sum_{\omega\in\mathcal{D}_f,\,\omega<\kappa_0} A_\omega(\boldsymbol{\alpha})\xi^{-\omega^*}\Bigg)\sum_{m=0}^\infty \frac{c_m(\boldsymbol{\alpha})}{\xi^{\delta_m}} = e\Big(f^*(\xi,\boldsymbol{\alpha})\Big)\sum_{m=0}^\infty \frac{c_m(\boldsymbol{\alpha})}{\xi^{\delta_m}} \tag{2.49}$$

with $c_0(\boldsymbol{\alpha}) \neq 0$ and $0 = \delta_0 < \delta_1 < \cdots \to \infty$. Let now $\sigma_0 > 2d\kappa_0$, M be a sufficiently large integer and ξ be sufficiently large. Recalling the definition of s^* in the introduction, for $0 < \sigma < \sigma_0$ and $0 \le \ell \le L$ from (2.48) and (2.49) we have

$$\frac{\gamma\sqrt{\pi}}{\sqrt{|R|}}e\Big(\frac{1}{2\pi}\Phi(x_0,\xi,\boldsymbol{\alpha})\Big)x_0^{\frac{d+1}{2d}-\frac{\ell}{d}-s} = \frac{e(f^*(\xi,\boldsymbol{\alpha}))}{\xi^{s^*+s-1}}\sum_{m=0}^M \frac{c_m(s,\boldsymbol{\alpha},\ell)}{\xi^{\delta'_m+\frac{\ell\kappa_0}{d\kappa_0-1}}} + f_2(s,\xi,\ell) \tag{2.50}$$

with $c_m(s,\boldsymbol{\alpha},\ell)$, $f_2(s,\xi,\ell)$ holomorphic and $c_0(s,\boldsymbol{\alpha},\ell) \neq 0$ for $\sigma > 0$, $0 = \delta'_0 < \cdots < \delta'_M$, and

$$f_2(s,\xi,\ell) = O(\xi^{-\sigma-1}). \tag{2.51}$$

Thanks to Lemma 2.7, (2.51) and the absolute convergence of $F(s)$ for $\sigma > 1$, the function

$$\widetilde{H}(s) = e^{a_3s+b_3}\sum_{\ell=0}^L c_\ell \sum_{n=n_0}^\infty \frac{\overline{a(n)}}{n^{1-s}}\big\{f_1(s,n,\ell) + f_2(s,n,\ell)\big\} \tag{2.52}$$

is holomorphic for $0 < \sigma < 2d\kappa_0$. Therefore, from (2.39), Lemma 2.7, (2.50) and (2.52) we get

$$F(s;f) = e^{a_3s+b_3}\sum_{\ell=0}^L c_\ell \sum_{n=n_0}^\infty \frac{\overline{a(n)}}{n^{1-s}}\frac{e(f^*(n,\boldsymbol{\alpha}))}{n^{s^*+s-1}}\Bigg\{\sum_{m=0}^M \frac{c_m(s,\boldsymbol{\alpha},\ell)}{n^{\delta'_m+\frac{\ell\kappa_0}{d\kappa_0-1}}}\Bigg\} + H(s) + \widetilde{H}(s). \tag{2.53}$$

However, $H(s)$, $F(s;f)$ and the first term on the right-hand side of (2.53) are holomorphic for $\sigma > d\kappa_0$, thus $\widetilde{H}(s)$ is holomorphic for $\sigma > 0$. Hence from (2.53) we finally obtain

$$\begin{aligned} F(s;f) &= e^{a_3s+b_3}\sum_{\ell=0}^L c_\ell \sum_{m=0}^M c_m(s,\boldsymbol{\alpha},\ell)\sum_{n=1}^\infty \frac{\overline{a(n)}}{n^{s^*+\delta'_m+\frac{\ell\kappa_0}{d\kappa_0-1}}}e(f^*(n,\boldsymbol{\alpha})) \\ &\quad - e^{a_3s+b_3}\sum_{\ell=0}^L c_\ell \sum_{n<n_0} \frac{\overline{a(n)}}{n^{1-s}}\frac{e(f^*(n,\boldsymbol{\alpha}))}{n^{s^*+s-1}}\Bigg\{\sum_{m=0}^M \frac{c_m(s,\boldsymbol{\alpha},\ell)}{n^{\delta'_m+\frac{\ell\kappa_0}{d\kappa_0-1}}}\Bigg\} + H(s) + \widetilde{H}(s) \\ &= \sum_{j=0}^J W_j(s)\bar{F}(s^*+\eta_j;f^*) + G(s), \end{aligned}$$

say, where the quantities in the last equation are as in Theorem 1.1. This concludes the proof.

3. **Proof of Theorem 1.2**

1. *Computing* $T(f)(\xi,\boldsymbol{\alpha})$. We first remark that the arguments in Lemmas 2.4 and 2.8 carry over to the case of $f\in\mathfrak{X}_d$; hence whenever needed we will freely use such lemmas for any $f\in\mathfrak{X}_d$. Moreover, in what follows we will tacitly use the absolute convergence of the involved expansions to justify formal manipulations.

Let $f\in\mathfrak{X}_d$ and ξ be sufficiently large. Recalling the definitions of $f(\xi,\boldsymbol{\alpha})$ and $\Phi(z,\xi,\boldsymbol{\alpha})$ in the introduction and writing $f(\xi,\boldsymbol{\alpha})=f(\xi,\boldsymbol{\alpha}_\omega)$ (a slightly incorrect but very convenient notation in what follows), we have

$$\begin{aligned}&\frac{\frac{\partial^2}{\partial z^2}\Phi(z,\xi,\boldsymbol{\alpha})}{\frac{\partial}{\partial z}\Phi(z,\xi,\boldsymbol{\alpha})}\\ &\quad=\frac{\frac{-1}{z^2}\big(\frac{z^{1/d}}{d}-2\pi f(\frac{z}{\xi},q^{\kappa_0-\omega}(\kappa_0-\omega)\boldsymbol{\alpha}_\omega)\big)+\frac{1}{z^2}\big(\frac{z^{1/d}}{d^2}-2\pi f(\frac{z}{\xi},q^{\kappa_0-\omega}(\kappa_0-\omega)^2\boldsymbol{\alpha}_\omega)\big)}{\frac{1}{z}\big(\frac{z^{1/d}}{d}-2\pi f(\frac{z}{\xi},q^{\kappa_0-\omega}(\kappa_0-\omega)\boldsymbol{\alpha}_\omega)\big)}.\end{aligned}$$

Hence, recalling also the definition of $T(f)(\xi,\boldsymbol{\alpha})$ and observing that the point $z=0$ is outside the circle $\mathcal{C}$, by Cauchy's theorem we get

$$\begin{aligned}&T(f)(\xi,\boldsymbol{\alpha})\\ &\quad=\frac{1}{4\pi^2 di}\int_{\mathcal{C}}\frac{\big(z^{1/d}-2\pi f(\frac{z}{\xi},q^{\kappa_0-\omega}\boldsymbol{\alpha}_\omega)\big)\big(z^{1/d}-2\pi f(\frac{z}{\xi},q^{\kappa_0-\omega}d^2(\kappa_0-\omega)^2\boldsymbol{\alpha}_\omega)\big)}{z^{1/d}-2\pi f(\frac{z}{\xi},q^{\kappa_0-\omega}d(\kappa_0-\omega)\boldsymbol{\alpha}_\omega)}\frac{\mathrm{d}z}{z};\end{aligned}$$

thus by the change of variable $z=\tilde{x}_0 w$ and denoting by $\mathcal{C}'$ the circle $|w-1|=\delta$ we obtain

$$\begin{aligned}&T(f)(\xi,\boldsymbol{\alpha})=\frac{\tilde{x}_0^{1/d}}{2\pi d}\frac{1}{2\pi i}\\ &\times\int_{\mathcal{C}'}\frac{\big(w^{1/d}-2\pi\tilde{x}_0^{-1/d}f(\frac{\tilde{x}_0 w}{\xi},q^{\kappa_0-\omega}\boldsymbol{\alpha}_\omega)\big)\big(w^{1/d}-2\pi\tilde{x}_0^{-1/d}f(\frac{\tilde{x}_0 w}{\xi},q^{\kappa_0-\omega}d^2(\kappa_0-\omega)^2\boldsymbol{\alpha}_\omega)\big)}{w^{1/d}-2\pi\tilde{x}_0^{-1/d}f(\frac{\tilde{x}_0 w}{\xi},q^{\kappa_0-\omega}d(\kappa_0-\omega)\boldsymbol{\alpha}_\omega)}\frac{\mathrm{d}w}{w}.\end{aligned}$$

Writing for simplicity

$$B_\omega=(2\pi d\kappa_0\alpha_0)^{\frac{d\omega}{d\kappa_0-1}-1},\qquad \boldsymbol{\beta}_\omega=B_\omega\left(\frac{q}{\xi}\right)^{\frac{\omega}{d\kappa_0-1}}\boldsymbol{\alpha}_\omega \tag{3.1}$$

and recalling (2.24), a computation shows that

$$\tilde{x}_0^{-1/d}f\left(\frac{\tilde{x}_0 w}{\xi},q^{\kappa_0-\omega}\boldsymbol{\alpha}_\omega\right)=f\left(w,\left(\frac{\tilde{x}_0 q}{\xi}\right)^{\kappa_0-\omega}\tilde{x}_0^{-1/d}\boldsymbol{\alpha}_\omega\right)=f(w,\boldsymbol{\beta}_\omega);$$

hence

(3.2)
$$T(f)(\xi,\boldsymbol{\alpha}) = \frac{\tilde{x}_0^{1/d}}{2\pi d}\frac{1}{2\pi i} \times \int_{\mathcal{C}'} \frac{\big(w^{1/d}-2\pi f(w,\boldsymbol{\beta}_\omega)\big)\big(w^{1/d}-2\pi f(w,d^2(\kappa_0-\omega)^2\boldsymbol{\beta}_\omega)\big)}{w^{1/d}-2\pi f(w,d(\kappa_0-\omega)\boldsymbol{\beta}_\omega)}\frac{\mathrm{d}w}{w}.$$

But
$$w^{1/d}-2\pi f(w,d(\kappa_0-\omega)\boldsymbol{\beta}_\omega) = w^{1/d}-w^{\kappa_0}-2\pi f_1(w,d(\kappa_0-\omega)\boldsymbol{\beta}_\omega)$$

with
$$f_1(w,\boldsymbol{\alpha}_\omega) = w^{\kappa_0}\sum_{\omega\in\mathcal{D}_f,\,\omega>0}\alpha_\omega w^{-\omega};$$

hence the denominator of the main fraction of the integrand in (3.2) equals
$$\frac{1}{w^{1/d}-w^{\kappa_0}}\frac{1}{1-2\pi\frac{f_1(w,d(\kappa_0-\omega)\boldsymbol{\beta}_\omega)}{w^{1/d}-w^{\kappa_0}}} = \sum_{m=0}^{\infty}\frac{(2\pi)^m f_1(w,d(\kappa_0-\omega)\boldsymbol{\beta}_\omega)^m}{(w^{1/d}-w^{\kappa_0})^{m+1}}.$$

Thus, writing $X = d(\kappa_0-\omega)$, (3.2) becomes

(3.3)
$$T(f)(\xi,\boldsymbol{\alpha}) = \frac{\tilde{x}_0^{1/d}}{2\pi d}\sum_{m=0}^{\infty}(2\pi)^m \times \frac{1}{2\pi i}\int_{\mathcal{C}'}\frac{\big(w^{1/d}-2\pi f(w,\boldsymbol{\beta}_\omega)\big)\big(w^{1/d}-2\pi f(w,X^2\boldsymbol{\beta}_\omega)\big)f_1(w,X\boldsymbol{\beta}_\omega)^m}{(w^{1/d}-w^{\kappa_0})^{m+1}}\frac{\mathrm{d}w}{w}.$$

Recalling the definition of ω^* in the introduction, by (3.1) we write the ω-component β_ω of $\boldsymbol{\beta}_\omega$ as
$$\beta_\omega = \xi^{-\omega^*}\alpha_0^{\frac{d\omega}{d\kappa_0-1}-1}\alpha_\omega c_\omega,$$
where c_ω are certain constants not depending on ξ and on the α_ω's. Note also that $\beta_0 = (2\pi d\kappa_0)^{-1}$ does not depend on α_0. Hence, the numerator of the integrand in (3.3) can be rearranged in the form

$$\delta_{0,m}k_{0,m}(w) + \sum_{\omega\in\mathcal{D}_f,\,\omega>0}\xi^{-\omega^*}Q_{\omega,m}\left(\left(\alpha_0^{\frac{d\omega'}{d\kappa_0-1}-1}\alpha_{\omega'}\right)_{0<\omega'\leq\omega}\right)k_{\omega,m}(w), \tag{3.4}$$

where $Q_{\omega,m}\in\mathbb{R}[(x_{\omega'})_{0<\omega'\leq\omega}]$ are without constant term and the functions $k_{\omega,m}(w)$ are holomorphic inside and on the circle $\mathcal{C}'$, and do not depend on ξ and on the α_ω's. Substituting (3.4) into (3.3) we obtain

$$T(f)(\xi,\boldsymbol{\alpha}) = \frac{\tilde{x}_0^{1/d}}{2\pi d}\sum_{m=0}^{\infty}(2\pi)^m\Big\{\delta_{0,m}c_{0,m} + \sum_{\omega\in\mathcal{D}_f,\,\omega>0}\xi^{-\omega^*}Q_{\omega,m}\Big(\Big(\alpha_0^{\frac{d\omega'}{d\kappa_0-1}-1}\alpha_{\omega'}\Big)_{0<\omega'\leq\omega}\Big)c_{\omega,m}\Big\}$$

with certain constants $c_{\omega,m}$ not depending on ξ and on the α_ω's. By a further rearrangement of terms, and recalling the definition of $\tilde{x}_0$ in (2.24), we finally get

$$T(f)(\xi,\boldsymbol{\alpha}) = \xi^{\kappa_0^*} \sum_{\omega\in\mathcal{D}_f} A_\omega(\boldsymbol{\alpha})\xi^{-\omega^*}, \tag{3.5}$$

where $A_0(\boldsymbol{\alpha}) = A_0\alpha_0^{-\frac{1}{d\kappa_0-1}}$ with a constant $A_0\in\mathbb{R}$, and for $\omega>0$

$$A_\omega(\boldsymbol{\alpha}) = \alpha_0^{-\frac{1}{d\kappa_0-1}} P_\omega\left(\left(\alpha_0^{\frac{d\omega'}{d\kappa_0-1}}\alpha_{\omega'}\right)_{0<\omega'\le\omega}\right)$$

with $P_\omega\in\mathbb{R}[(x_{\omega'})_{0<\omega'\le\omega}]$ without constant term. In order to deal with A_0 we note that

$$\begin{aligned}
A_0 &= \frac{(2\pi d\kappa_0 q^{\kappa_0})^{-\frac{1}{d\kappa_0-1}}}{2\pi d}\frac{1}{2\pi i}\int_{\mathcal{C}'}\frac{(w^{1/d}-(d\kappa_0)^{-1}w^{\kappa_0})(w^{1/d}-d\kappa_0 w^{\kappa_0})}{w^{1/d}-w^{\kappa_0}}\frac{\mathrm{d}w}{w}\\
&= \frac{(2\pi d\kappa_0 q^{\kappa_0})^{-\frac{1}{d\kappa_0-1}}}{2\pi d}\lim_{w\to1}(w-1)\left\{\frac{(w^{1/d}-(d\kappa_0)^{-1}w^{\kappa_0})(w^{1/d}-d\kappa_0 w^{\kappa_0})}{w^{1/d}-w^{\kappa_0}}\frac{1}{w}\right\}\\
&= \frac{(2\pi d\kappa_0 q^{\kappa_0})^{-\frac{1}{d\kappa_0-1}}}{2\pi d}\frac{(1-(d\kappa_0)^{-1})(1-d\kappa_0)}{1/d-\kappa_0},
\end{aligned}$$

and a computation shows that

$$A_0 = (d\kappa_0-1)q_F^{-\kappa_0^*} > 0. \tag{3.6}$$

Hence the first assertions in part (i) of Theorem 1.2 follow from (3.5) and (3.6).

Now we turn to the proof of the last statement of part (i). To this end we first note that the component β_ω of $\boldsymbol{\beta}_\omega$ in (3.1) is

$$\beta_\omega = q^{1/d}(2\pi d q^{1/d}\kappa_0)^{d\omega^*-1}\alpha_0^{d\omega^*-1}\alpha_\omega\xi^{-\omega^*}. \tag{3.7}$$

Moreover, recalling the notation in (3.3), with obvious notation we write the numerator of the integrand in (3.3) as

$$\begin{aligned}
&\left(w^{1/d}-\frac{w^{\kappa_0}}{d\kappa_0}-2\pi\sum_{\omega>0}\beta_\omega w^{\kappa_0-\omega}\right)\\
&\qquad\times\left(w^{1/d}-d\kappa_0 w^{\kappa_0}-2\pi\sum_{\omega>0}X^2\beta_\omega w^{\kappa_0-\omega}\right)\left(\sum_{\omega>0}X\beta_\omega w^{\kappa_0-\omega}\right)^m\\
&\qquad= (H(w)-2\pi\Sigma_1(w))(K(w)-2\pi\Sigma_2(w))(\Sigma_3(w))^m,
\end{aligned} \tag{3.8}$$

say. Since $\tilde{x}_0^{1/d} = C_0^{1/d}\xi^{\kappa_0^*}$ (see Lemma 2.3), it is clear from (3.3), (3.7) and (3.8) that the part containing α_ω of the coefficient $A_\omega(\boldsymbol{\alpha})$ comes only from the terms with $m = 0$ and $m = 1$ in (3.3). More precisely, such a part equals

(3.9)
$$\begin{aligned}
&\frac{C_0^{1/d}}{2\pi d} q^{1/d}(2\pi d q^{1/d}\kappa_0)^{d\omega^*-1}\alpha_0^{d\omega^*-1}\alpha_\omega(-2\pi) \\
&\qquad \times \frac{1}{2\pi i}\int_{\mathcal{C}'} \frac{H(w)X^2 w^{\kappa_0-\omega} + K(w)w^{\kappa_0-\omega}}{w^{1/d}-w^{\kappa_0}}\frac{\mathrm{d}w}{w} \\
&\qquad + \frac{C_0^{1/d}}{2\pi d} 2\pi q^{1/d}(2\pi d q^{1/d}\kappa_0)^{d\omega^*-1}\alpha_0^{d\omega^*-1}\alpha_\omega \frac{1}{2\pi i}\int_{\mathcal{C}'} \frac{H(w)K(w)Xw^{\kappa_0-\omega}}{(w^{1/d}-w^{\kappa_0})^2}\frac{\mathrm{d}w}{w} \\
&\quad = \frac{C_0^{1/d}}{d} q^{1/d}(2\pi d q^{1/d}\kappa_0)^{d\omega^*-1}\alpha_0^{d\omega^*-1}\alpha_\omega(-A+B),
\end{aligned}$$

say. A computation similar to the one leading to the value of A_0 in (3.6) gives

$$-A = \frac{X^2}{\kappa_0} - d. \tag{3.10}$$

To compute B we observe that

$$B = X \lim_{w\to 1} \frac{\mathrm{d}}{\mathrm{d}w}\left\{\frac{(w-1)^2 H(w)K(w)w^{\kappa_0-\omega-1}}{(w^{1/d}-w^{\kappa_0})^2}\right\}$$

and

$$\frac{(w-1)^2}{(w^{1/d}-w^{\kappa_0})^2} = \frac{1}{e_1^2 + e_1e_2(w-1) + g(w)},$$

where $g(w) = O((w-1)^2)$ as $w \to 1$ and

$$e_1 = \frac{1}{d} - \kappa_0, \qquad e_2 = \frac{1}{d}\left(\frac{1}{d} - 1\right) - \kappa_0(\kappa_0 - 1).$$

Hence, writing $h(w) = H(w)K(w)w^{\kappa_0-\omega-1}$, we obtain

$$B = X\frac{h'(1)e_1^2 - h(1)e_1e_2}{e_1^4},$$

and a computation shows that

$$\begin{aligned}
h(1) &= -\frac{(d\kappa_0-1)^2}{d\kappa_0}, \\
h'(1) &= \frac{d\kappa_0 - 1}{d^2\kappa_0} - \kappa_0(d\kappa_0-1) + h(1)\left(\frac{X}{d} - 1\right), \\
e_1 &= -\frac{d\kappa_0-1}{d}, \\
e_2 &= -e_1 - \frac{(d\kappa_0-1)(d\kappa_0+1)}{d^2}.
\end{aligned}$$

Therefore, writing $Y = d\kappa_0 - 1$, we have

(3.11)
$$B = \frac{X}{e_1^4}\left\{\frac{Y^3}{d^4\kappa_0} - \frac{\kappa_0 Y^3}{d^2} - \frac{Y^4 X}{d^4\kappa_0} - h(1)e_1^2 + h(1)e_1^2 + h(1)\frac{Y(d\kappa_0+1)}{d^2}e_1\right\}$$
$$= \frac{X}{e_1^4}\left\{-\frac{Y^4X}{d^4\kappa_0} + \frac{Y^3}{d^4\kappa_0} - \frac{(d\kappa_0)^2Y^3}{d^4\kappa_0} + \frac{Y^4(d\kappa_0+1)}{d^4\kappa_0}\right\} = -\frac{X^2}{\kappa_0},$$

and the last statement of part (i) of Theorem 1.2 follows from (3.9), (3.10) and (3.11).

2. *Self-reciprocity.* Let ξ be sufficiently large. From the properties of the coefficients $A_\omega(\boldsymbol{\alpha})$ of $T(f)(\xi,\boldsymbol{\alpha})$ in part (i), it is clear that $T(f) \in \mathfrak{X}_d$ if $f \in \mathfrak{X}_d$. In order to prove that $T^2 = \mathrm{id}_{\mathfrak{X}_d}$ we recall that, with the notation in Lemma 2.3, if $\Re z$ is sufficiently large and $|\arg z| \le \theta$, then there exists exactly one zero of $\Phi'(z,\xi,\boldsymbol{\alpha})$, and it is real and simple. Given $f \in \mathfrak{X}_d$, we denote by

$$x_0 = x_0(\xi,\boldsymbol{\alpha}) \quad \text{and} \quad x_0^* = x_0^*(\xi,\boldsymbol{\alpha}) \tag{3.12}$$

such a zero, referred to $f(\xi,\boldsymbol{\alpha})$ and $T(f)(\xi,\boldsymbol{\alpha})$, respectively. We also write

$$W(\xi,\boldsymbol{\alpha}) = \frac{x_0\left(\frac{qx_0^*(\xi,\boldsymbol{\alpha})}{\xi},\boldsymbol{\alpha}\right)}{x_0^*(\xi,\boldsymbol{\alpha})}.$$

Note that $W(\xi,\boldsymbol{\alpha})$ is well defined since $x_0^*/\xi \to \infty$ as $\xi \to \infty$ by Lemma 2.3. Note also that $f(\xi,\boldsymbol{\alpha})$ and $T^m(f)(\xi,\boldsymbol{\alpha})$, $m \ge 1$ are differentiable in each component α_ω of $\boldsymbol{\alpha}$.

LEMMA 3.1. *Let $f \in \mathfrak{X}_d$ and ξ be sufficiently large. Then for every $\omega \in \mathcal{D}_f$ we have*

$$\frac{\partial}{\partial\alpha_\omega}T^2(f)(\xi,\boldsymbol{\alpha}) = \big(W(\xi,\boldsymbol{\alpha})\xi\big)^{\kappa_0-\omega}, \qquad \kappa_0 = \mathrm{lexp}(f).$$

Proof. Denoting by $\mathcal{C}^*$ the circle $|z - \tilde{x}_0^*| = \delta\tilde{x}_0^*$ with $\delta > 0$ sufficiently small and

$$\tilde{x}_0^* = C_0^*\xi^{\frac{d\kappa_0^*}{d\kappa_0^*-1}}, \qquad C_0^* = \left(2\pi d\kappa_0^* q^{\kappa_0^*}A_0(\boldsymbol{\alpha})\right)^{-\frac{d}{d\kappa_o^*-1}}$$

and writing $\Phi^*(z) = \Phi^*(z,\xi,\boldsymbol{\alpha}) = z^{1/d} - 2\pi T(f)(\frac{qz}{\xi},\boldsymbol{\alpha})$, we have

$$T^2(f)(\xi,\boldsymbol{\alpha}) = \frac{1}{4\pi^2 i}\int_{\mathcal{C}^*}\frac{\Phi^*(z)\frac{\partial^2}{\partial z^2}\Phi^*(z)}{\frac{\partial}{\partial z}\Phi^*(z)}\mathrm{d}z.$$

Hence for every $\omega \in \mathcal{D}_f$ we get

$$\frac{\partial}{\partial\alpha_\omega}T^2(f)(\xi,\boldsymbol{\alpha}) = \frac{1}{4\pi^2 i}\int_{\mathcal{C}^*}\frac{\frac{\partial}{\partial\alpha_\omega}\Phi^*(z)\frac{\partial^2}{\partial z^2}\Phi^*(z)}{\frac{\partial}{\partial z}\Phi^*(z)}\mathrm{d}z + I^*, \tag{3.13}$$

where

$$I^* = \frac{1}{4\pi^2 i}\int_{\mathcal{C}^*} \Phi^*(z)\frac{\partial}{\partial\alpha_\omega}\left\{\frac{\frac{\partial^2}{\partial z^2}\Phi^*(z)}{\frac{\partial}{\partial z}\Phi^*(z)}\right\}\mathrm{d}z.$$

Thanks to the above reported properties of the zero x_0^* in (3.12) we have

$$\frac{\frac{\partial^2}{\partial z^2}\Phi^*(z)}{\frac{\partial}{\partial z}\Phi^*(z)} = \frac{1}{z-x_0^*} + \Psi_1^*(z)$$

with $\Psi_1^*(z)$ holomorphic inside and on the circle $\mathcal{C}^*$, thus with obvious notation

$$\frac{\partial}{\partial\alpha_\omega}\left\{\frac{\frac{\partial^2}{\partial z^2}\Phi^*(z)}{\frac{\partial}{\partial z}\Phi^*(z)}\right\} = \frac{1}{(z-x_0^*)^2}\frac{\partial}{\partial\alpha_\omega}x_0^*(\xi,\boldsymbol{\alpha}) + \Psi_2^*(z),$$

again with $\Psi_2^*(z)$ holomorphic inside and on the circle $\mathcal{C}^*$. Therefore by Cauchy's theorem and the definition of x_0^* we obtain

$$I^* = \frac{\frac{\partial}{\partial\alpha_\omega}x_0^*(\xi,\boldsymbol{\alpha})}{4\pi^2 i}\int_{\mathcal{C}^*}\frac{\Phi^*(z)}{(z-x_0^*)^2}\mathrm{d}z = \frac{\frac{\partial}{\partial\alpha_\omega}x_0^*(\xi,\boldsymbol{\alpha})}{2\pi}\frac{\partial}{\partial z}\Phi^*(x_0^*) = 0. \tag{3.14}$$

To treat the integral in (3.13) we observe that the definition of $\Phi^*(z)$ and (1.9) imply that

$$\begin{aligned}\frac{\partial}{\partial\alpha_\omega}\Phi^*(z) &= -2\pi\frac{\partial}{\partial\alpha_\omega}T(f)(\frac{qz}{\xi},\boldsymbol{\alpha})\\ &= -\frac{\partial}{\partial\alpha_\omega}\frac{1}{2\pi i}\int_{\mathcal{C}}\frac{\Phi(w,\frac{qz}{\xi},\boldsymbol{\alpha})\frac{\partial^2}{\partial w^2}\Phi(w,\frac{qz}{\xi},\boldsymbol{\alpha})}{\frac{\partial}{\partial w}\Phi(w,\frac{qz}{\xi},\boldsymbol{\alpha})}\mathrm{d}w\end{aligned}$$

(here $\mathcal{C}$ is the usual circle, referred to $x_0(\frac{qz}{\xi},\boldsymbol{\alpha})$). Arguing as before, thanks to the definition and the properties of $x_0(\frac{qz}{\xi},\boldsymbol{\alpha})$, we see that

$$I = \frac{1}{2\pi i}\int_{\mathcal{C}}\Phi(w,\frac{qz}{\xi},\boldsymbol{\alpha})\frac{\partial}{\partial\alpha_\omega}\left\{\frac{\frac{\partial^2}{\partial w^2}\Phi(w,\frac{qz}{\xi},\boldsymbol{\alpha})}{\frac{\partial}{\partial w}\Phi(w,\frac{qz}{\xi},\boldsymbol{\alpha})}\right\}\mathrm{d}w = 0.$$

Thus, writing for simplicity $\Phi(w) = \Phi(w,\frac{qz}{\xi},\boldsymbol{\alpha})$, we obtain that

(3.15)

$$\begin{aligned}\frac{\partial}{\partial\alpha_\omega}\Phi^*(z) &= -\frac{1}{2\pi i}\int_{\mathcal{C}}\frac{\frac{\partial}{\partial\alpha_\omega}\Phi(w)\frac{\partial^2}{\partial w^2}\Phi(w)}{\frac{\partial}{\partial w}\Phi(w)}\mathrm{d}w\\ &= \frac{1}{2\pi i}\int_{\mathcal{C}}\frac{\frac{\partial}{\partial\alpha_\omega}2\pi f(\frac{w\xi}{z},\boldsymbol{\alpha})\frac{\partial^2}{\partial w^2}\Phi(w)}{\frac{\partial}{\partial w}\Phi(w)}\mathrm{d}w\\ &= 2\pi(\frac{\xi}{z})^{\kappa_0-\omega}\frac{1}{2\pi i}\int_{\mathcal{C}}\frac{w^{\kappa_0-\omega}\frac{\partial^2}{\partial w^2}\Phi(w)}{\frac{\partial}{\partial w}\Phi(w)}\mathrm{d}w = 2\pi\left(\frac{x_0(\frac{qz}{\xi},\boldsymbol{\alpha})\xi}{z}\right)^{\kappa_0-\omega}.\end{aligned}$$

Hence (3.13), (3.14) and (3.15) finally give

$$\frac{\partial}{\partial \alpha_\omega} T^2(f)(\xi,\boldsymbol{\alpha}) = \xi^{\kappa_0-\omega}\frac{1}{2\pi i}\int_{\mathcal{C}^*}\left(\frac{x_0(\frac{qz}{\xi},\boldsymbol{\alpha})}{z}\right)^{\kappa_0-\omega}\frac{\frac{\partial^2}{\partial z^2}\Phi^*(z)}{\frac{\partial}{\partial z}\Phi^*(z)}\mathrm{d}z$$
$$= \left(\frac{x_0(\frac{qx_0^*(\xi,\boldsymbol{\alpha})}{\xi},\boldsymbol{\alpha})}{x_0^*(\xi,\boldsymbol{\alpha})}\right)^{\kappa_0-\omega}\xi^{\kappa_0-\omega}$$

and the lemma follows. □

Suppose now that $W(\xi,\boldsymbol{\alpha}) = 1$ identically. Then from Lemma 3.1 we obtain that for every $\omega \in \mathcal{D}_f$

$$\frac{\partial}{\partial \alpha_\omega} T^2(f)(\xi,\boldsymbol{\alpha}) = \xi^{\kappa_0-\omega}.$$

On the other hand, from part (i) of Theorem 1.2 and (1.12) we get

$$T^2(f)(\xi,\boldsymbol{\alpha}) = \xi^{\kappa_0}\sum_{\omega\in\mathcal{D}_f} H_\omega(\boldsymbol{\alpha})\xi^{-\omega}$$

with some coefficients $H_\omega(\boldsymbol{\alpha})$, and hence such coefficients satisfy

$$\frac{\partial}{\partial \alpha_\omega} H_{\omega'}(\boldsymbol{\alpha}) = \delta_{\omega,\omega'}.$$

Therefore, since each $H_{\omega'}(\boldsymbol{\alpha})$ depends only on a finite number of α_ω's, we get

$$H_\omega(\boldsymbol{\alpha}) = \alpha_\omega + c_\omega$$

with some constant c_ω.

A simple direct computation based on $A_0(\boldsymbol{\alpha}) = A_0\alpha_0^{-\frac{1}{d\kappa_0-1}}$ and (3.6) shows that $H_0(\boldsymbol{\alpha}) = \alpha_0$. Moreover, writing $\bar{\boldsymbol{\alpha}} = (\alpha_0, 0, 0\dots)$, it is clear from part (i) that $A_\omega(\bar{\boldsymbol{\alpha}}) = 0$ for $\omega > 0$; hence also $H_\omega(\bar{\boldsymbol{\alpha}}) = 0$ for $\omega > 0$ and therefore $c_\omega = 0$ for every $\omega \in \mathcal{D}_f$. Thus $T^2 = \mathrm{id}_{\mathfrak{X}_d}$. Therefore, in view of the definition of $W(\xi,\boldsymbol{\alpha})$, part (ii) of Theorem 1.2 follows from

Lemma 3.2. *Let ξ be sufficiently large and x_0, x_0^* be as in* (3.12). *Then* $x_0(\frac{qx_0^*}{\xi},\boldsymbol{\alpha}) = x_0^*(\xi,\boldsymbol{\alpha})$.

Proof. Arguing as in the proof of (3.14) in Lemma 3.1 (using definition and properties of $x_0(\frac{qz}{\xi},\boldsymbol{\alpha})$ as well as Cauchy's theorem) and writing for simplicity $\Phi(w) = \Phi(w,\frac{qz}{\xi},\boldsymbol{\alpha})$ we obtain

$$2\pi\frac{\partial}{\partial z} T(f)\left(\frac{qz}{\xi},\boldsymbol{\alpha}\right) = \frac{1}{2\pi i}\int_{\mathcal{C}}\frac{\frac{\partial}{\partial z}\Phi(w)\frac{\partial^2}{\partial w^2}\Phi(w)}{\frac{\partial}{\partial w}\Phi(w)}\mathrm{d}w,$$

where $\mathcal{C}$ is the usual circle, referred to $x_0(\frac{qz}{\xi}, \boldsymbol{\alpha})$. Hence by the change of variable $\rho = \frac{w\xi}{z}$ we get

$$\begin{aligned}\text{(3.16)} \qquad \pi\frac{\partial}{\partial z}T(f)(\frac{qz}{\xi}, \boldsymbol{\alpha}) &= \frac{\partial}{\partial z}\Phi\left(w, \frac{qz}{\xi}, \boldsymbol{\alpha}\right)_{|w=x_0(\frac{qz}{\xi},\boldsymbol{\alpha})} \\ &= 2\pi\frac{x_0(\frac{qz}{\xi}, \boldsymbol{\alpha})\xi}{z^2}\frac{\partial}{\partial\rho}f(\rho, \boldsymbol{\alpha})\Big|_{\rho=\frac{x_0(\frac{qz}{\xi},\boldsymbol{\alpha})\xi}{z}}.\end{aligned}$$

Since $x_0^* = x_0^*(\xi, \boldsymbol{\alpha})$ is the solution of

$$\frac{\partial}{\partial z}\left(z^{1/d} - 2\pi T(f)\left(\frac{qz}{\xi}, \boldsymbol{\alpha}\right)\right) = 0,$$

from (3.16) we deduce that

$$\text{(3.17)} \qquad \frac{1}{d}x_0^{*\frac{1}{d}-1} - 2\pi\frac{x_0(\frac{qx_0^*}{\xi}, \boldsymbol{\alpha})\xi}{x_0^{*2}}\frac{\partial}{\partial\rho}f(\rho, \boldsymbol{\alpha})\Big|_{\rho=\frac{x_0(\frac{qx_0^*}{\xi},\boldsymbol{\alpha})\xi}{x_0^*}} = 0.$$

But $x_0(\frac{qx_0^*}{\xi}, \boldsymbol{\alpha})$ is the solution of

$$\frac{\partial}{\partial z}\left(z^{1/d} - 2\pi f\left(\frac{z\xi}{x_0^*}, \boldsymbol{\alpha}\right)\right) = 0$$

and hence, by the change of variable $\rho = \frac{z\xi}{x_0^*}$, it satisfies

$$\text{(3.18)} \qquad \frac{1}{d}x_0\left(\frac{qx_0^*}{\xi}, \boldsymbol{\alpha}\right)^{\frac{1}{d}-1} - 2\pi\frac{\xi}{x_0^*}\frac{\partial}{\partial\rho}f(\rho, \boldsymbol{\alpha})\Big|_{\rho=\frac{x_0(\frac{qx_0^*}{\xi},\boldsymbol{\alpha})\xi}{x_0^*}} = 0.$$

Multiplying (3.18) by $x_0(\frac{qx_0^*}{\xi}, \boldsymbol{\alpha})/x_0^*$ and then subtracting from (3.17) we finally obtain

$$\frac{1}{d}x_0^{*\frac{1}{d}-1} = \frac{1}{d}\frac{x_0(\frac{qx_0^*}{\xi}, \boldsymbol{\alpha})^{\frac{1}{d}}}{x_0^*};$$

i.e., $x_0(\frac{qx_0^*}{\xi}, \boldsymbol{\alpha}) = x_0^*$ and the lemma follows. □

The proof of Theorem 1.2 is now complete.

4. **Proof of Theorem 1.3**

In this section we assume that $1 < d < 2$. We use the notation

$$\prod_{j=k}^{1}\mathcal{H}_j = \mathcal{H}_k \cdots \mathcal{H}_1, \qquad \mathcal{H}_1, \ldots, \mathcal{H}_k \in \mathfrak{S}_d,$$

and an empty product is the identity; note that the order of factors is important since $\mathfrak{S}_d$ is noncommutative. Our first goal is to obtain a more explicit expression for the operators $\mathcal{L}_k$, defined recursively in the introduction (remember that the $\mathcal{L}_k$'s depend on a sequence of integers m_k),

LEMMA 4.1. *Let $1 < d < 2$. For $k \geq 0$ we have*

$$\mathcal{L}_k = \Big(\prod_{j=N_k}^{1} S^{(-1)^j} \mathcal{X}_j \Big) S$$

where $N_k = 2^k - 1$ and $\mathcal{X}_j = TS^{n_j}T$ with some $n_j \in \{\pm m_1, \ldots, \pm m_k\}$.

Proof. By induction on k. If $k = 0$ the assertion is trivial, so we assume the lemma true up to a $k > 0$. Then

$$\begin{aligned} \mathcal{L}_{k+1} &= \mathcal{L}_k^{-1} TS^{m_{k+1}} T \mathcal{L}_k \\ &= (S^{-1}\mathcal{X}_1^{-1} S \cdots \mathcal{X}_{N_k}^{-1} S)(TS^{m_{k+1}} T)(S^{-1}\mathcal{X}_{N_k} \cdots S^{-1}\mathcal{X}_1 S) \\ &= \Big(\prod_{j=2N_k+1}^{1} S^{(-1)^j} \mathcal{X}_j \Big) S, \end{aligned}$$

where $2N_k + 1 = N_{k+1}$ and for $N_k + 1 \leq j \leq N_{k+1}$

$$\mathcal{X}_j = \begin{cases} TS^{m_{k+1}} T & \text{if } j = N_k + 1 \\ \mathcal{X}_{N_{k+1}-j+1}^{-1} & \text{if } j = N_k + 2, \ldots, N_{k+1}. \end{cases} \tag{4.1}$$

Since for $j = 1, \ldots, N_k$ we have $\mathcal{X}_j = TS^{n_j}T$ with $n_j \in \{\pm m_1, \ldots, \pm m_k\}$, and clearly $\mathcal{X}_j^{-1} = TS^{-n_j}T$, from (4.1) we obtain that $\mathcal{X}_j = TS^{n_j}T$ with $n_j \in \{\pm m_1, \ldots, \pm m_{k+1}\}$ for $j = 1, \ldots, N_{k+1}$, and the lemma follows. □

We need the following auxiliary lemma; we refer to the introduction for the notation.

LEMMA 4.2. *Let $1 < d < 2$, $f, g \in \mathfrak{X}_d$, κ be as in (1.3), $\kappa_0 = \text{lexp}(f) < \kappa$, $\lambda < \kappa_0$ and $\mathcal{X} = TS^nT$ with some $n \in \mathbb{Z}$ be such that $\mathcal{X}(f)$ and $\mathcal{X}(g)$ are well defined. Then*

$$\mathcal{X}(f) \equiv f \pmod{\xi^{d\kappa_0 - 1}} \quad \textit{and} \quad \text{lexp}(\mathcal{X}(f)) = \text{lexp}(f).$$

Moreover, if $f \equiv g \pmod{\xi^\lambda}$ then

$$\text{lexp}(\mathcal{X}(f)) = \text{lexp}(\mathcal{X}(g)) = \text{lexp}(f) = \text{lexp}(g)$$

and for every integer r

$$S^r\mathcal{X}(f) \equiv S^r\mathcal{X}(g) \pmod{\xi^\lambda}.$$

Proof. Since $\kappa_0 < \kappa$, we have that $\kappa_0^* > 1$. Hence from (i) of Theorem 1.2 we obtain that $S^n T(f)(\xi) \equiv T(f) \pmod{\xi}$, and therefore (1.13) and Theorem 1.2 give

$$\mathcal{X}(f) = TS^n T(f) \equiv T^2(f) = f \left(\bmod\, \xi^{\frac{1}{d\kappa_0^*-1}}\right). \tag{4.2}$$

But $\frac{1}{d\kappa_0^*-1} = d\kappa_0 - 1$; hence (4.2) proves the first claim of the lemma. Moreover, $d\kappa_0 - 1 < \kappa_0$ since $\kappa_0 < \kappa$; thus the second claim follows as well. The third claim follows in the same way, since $\lambda < \kappa_0$ implies that $\text{lexp}(g) = \kappa_0$. To prove the last claim we recall that $\kappa_0^* > 1$; hence $\text{lexp}(S^n T(f)) = \text{lexp}(S^n T(g)) = \kappa_0^*$. Moreover,

$$S^n T(f) \equiv S^n T(g) \pmod{\xi^{\lambda^*}}$$

by (1.13). Since $\lambda^* < \kappa_0^*$ and $(\lambda^*)^* = \lambda$, again by (1.13) we also have that

$$\mathcal{X}(f) \equiv \mathcal{X}(g) \pmod{\xi^{\lambda}},$$

and the lemma follows. □

Let $k_d \geq 1$ be as in (1.16), κ, e_k and $g_0(\xi) = \alpha\xi^{1/d}$ be as in the introduction, N_k and $\mathcal{X}_j$ be as in Lemma 4.1 and for $\ell = 0, \ldots, N_{k_d}$ let

$$f_\ell(\xi) = \left(\prod_{j=\ell}^{1} S^{(-1)^j} \mathcal{X}_j\right) S(g_0)(\xi); \tag{4.3}$$

clearly $f_0(\xi) = S(g_0)(\xi)$. Note that $f_{N_k}(\xi) = \mathcal{L}_k(g_0)(\xi)$, and that the functions $f_\ell(\xi)$ depend on the integers m_k in the definition of the $\mathcal{L}_k$'s, although this does not appear explicitly in our notation. Moreover, it is not *a priori* clear if such functions are well defined. However, this follows from

LEMMA 4.3. *Let $\frac{1+\sqrt{5}}{2} \leq d < 2$. There exist integers $m_1, \ldots, m_{k_d-1}$ and $\overline{m}_{k_d}$ such that for every integer $m_{k_d} > \overline{m}_{k_d}$ the functions $f_\ell(\xi)$ are well defined for $0 \leq \ell \leq N_{k_d}$. Moreover, for $0 \leq k \leq k_d$ and $1 \leq \ell \leq N_{k_d}$ we have*

$$e_k \leq \text{lexp}(f_\ell) \leq 1, \qquad N_k \leq \ell < N_{k+1}, \tag{4.4}$$

$$\text{coeff}(f_\ell, \xi^{e_{k+1}}) = a_\ell + b_\ell m_{k+1} \neq 0, \qquad N_k < \ell \leq N_{k+1} \tag{4.5}$$

with $a_\ell, b_\ell \in \mathbb{R}$ depending on α, $m_1, \ldots, m_k$ and $b_\ell \neq 0$, and

$$f_\ell \equiv S^{(-1)^\ell}(f_{N_{k+1}-\ell}) \pmod{\xi^{e_{k+1}}}, \qquad N_k < \ell \leq N_{k+1}. \tag{4.6}$$

Proof. Clearly $f_0(\xi)$ is well defined. We proceed by induction on ℓ. For $\ell = 1$ we have $k = 1$ in (4.4), and we have to deal with $\text{lexp}(S^{-1} TS^{m_1} TS(g_0))$. Recalling that $d \geq \frac{1+\sqrt{5}}{2}$ implies $d - 1 \geq 1/d$ and that $S(g_0)(\xi) = \xi + \alpha\xi^{1/d}$, a computation based on (i) of Theorem 1.2 shows

$$
\begin{aligned}
(4.7)\qquad TS(g_0)(\xi) &= c_0\xi^\kappa + c_1(\alpha)\xi^{\kappa/d} + \cdots \\
S^{m_1}TS(g_0)(\xi) &= c_0\xi^\kappa + m_1\xi + c_1(\alpha)\xi^{\kappa/d} + \cdots \\
TS^{m_1}TS(g_0)(\xi) &= \xi + c_2(m_1)\xi^{d-1} + c_3(\alpha, m_1)\xi^{1/d} + \cdots \\
f_1(\xi) = S^{-1}TS^{m_1}TS(g_0)(\xi) &= c_2(m_1)\xi^{d-1} + c_3(\alpha, m_1)\xi^{1/d} + \cdots
\end{aligned}
$$

with $c_2(m_1) \neq 0$ for m_1 sufficiently large if $d-1 > 1/d$, and $c_2(m_1) + c_3(\alpha, m_1) \neq 0$ for m_1 sufficiently large if $d-1 = 1/d$. Hence $\text{lexp}(f_1) = e_1$; thus (4.4) is verified in this case. Next we take $k = 0$ in (4.5) and (4.6). From the third equation in (4.7) and (i) of Theorem 1.2 we have that $c_2(m_1)$ is of the form $a + bm_1$ with $b \neq 0$ if $d-1 > 1/d$, and similarly for $c_2(m_1) + c_3(\alpha, m_1)$ if $d-1 = 1/d$. Since $e_1 = d-1$, (4.5) is therefore verified for $\ell = 1$ thanks to the last equation in (4.7). To verify (4.6) we apply Lemma 4.2 with $f(\xi) = S(g_0)(\xi)$ and $\mathcal{X} = \mathcal{X}_1$. Since $\kappa_0 = \text{lexp}(f) = 1 < \kappa$ and $d\kappa_0 - 1 = d-1$, from Lemma 4.2 we get

$$\mathcal{X}_1 S(g_0) \equiv S(g_0) \pmod{\xi^{d-1}} = f_0 \pmod{\xi^{e_1}}.$$

Hence applying S^{-1} to both sides we obtain that $f_1 \equiv S^{-1}(f_0) \pmod{\xi^{e_1}}$, and (4.6) follows for $\ell = 1$.

Let now $\ell \geq 2$ and suppose that the lemma holds up to $\ell - 1$. We consider the following two cases.

Case I: $\ell = N_k + 1$ for some $1 \leq k < k_d$. In this case ℓ is even; hence $f_\ell(\xi) = S\mathcal{X}_\ell(f_{\ell-1})(\xi)$ by (4.3), and from the inductive assumption we have

$$
(4.8)\qquad
\begin{cases}
e_k \leq \text{lexp}(f_{\ell-1}) \leq 1, \\
\text{coeff}(f_{\ell-1}, \xi^{e_k}) = a_{\ell-1} + b_{\ell-1}m_k \neq 0, \\
f_{\ell-1} \equiv S^{-1}(f_0) \pmod{\xi^{e_k}}.
\end{cases}
$$

From the third assumption we see that $f_{\ell-1} \equiv g_0 \equiv 0 \ (\text{mod}\xi^{e_k})$ since $e_k > 1/d$; hence from the first one we have that $\text{lexp}(f_{\ell-1}) = e_k$. Therefore, by an application of $S\mathcal{X}_\ell = STS^{m_{k+1}}T$ (see (4.1)) to $f_{\ell-1}(\xi)$, the leading exponent of $f_{\ell-1}(\xi)$ has the following evolution:

$$(4.9)\qquad e_k \xrightarrow{T} e_k/e_{k+1} \xrightarrow{S^{m_{k+1}}} e_k/e_{k+1} \xrightarrow{T} e_k \xrightarrow{S} 1,$$

since $e_k < 1$, the sequence e_k is strictly decreasing and $de_k - 1 = e_{k+1}$. Hence $f_\ell(\xi)$ is well defined for every m_{k+1}, and $\text{lexp}(f_\ell) = 1$ thus proving (4.4). Now we apply Lemma 4.2 with $f(\xi) = f_{\ell-1}(\xi)$, $\mathcal{X} = \mathcal{X}_\ell$ and $\kappa_0 = e_k$ thus getting

$$\mathcal{X}_\ell(f_{\ell-1}) \equiv f_{\ell-1} \pmod{\xi^{e_{k+1}}}.$$

A further application of S to both sides gives, since $N_k = N_{k+1} - \ell$, that

$$f_\ell = S\mathcal{X}_\ell(f_{\ell-1}) \equiv S(f_{N_k}) = S(f_{N_{k+1}-\ell}) \pmod{\xi^{e_{k+1}}},$$

and (4.6) is proved. To prove (4.5) we apply twice part (i) of Theorem 1.2 as in (4.7) and (4.9). Starting from the second assumption in (4.8) we get

$$\begin{aligned} T(f_{\ell-1})(\xi) &= c_0(\alpha, m_1, \dots, m_k)\xi^{e_k/e_{k+1}} + \cdots \qquad c_0(\alpha, m_1, \dots, m_k) \neq 0, \\ S^{m_{k+1}}T(f_{\ell-1})(\xi) &= c_0(\alpha, m_1, \dots, m_k)\xi^{e_k/e_{k+1}} + \cdots + m_{k+1}\xi + \cdots, \\ \text{coeff}(TS^{m_{k+1}}T(f_{\ell-1}), \xi^{e_{k+1}}) &= c_1(\alpha, m_1, \dots, m_k) + c_2(\alpha, m_1, \dots, m_k)m_{k+1} \end{aligned}$$

with $c_2(\alpha, m_1, \dots, m_k) \neq 0$, since the term in ξ in the second equation is transformed by T into the term in $\xi^{e_{k+1}}$. Hence (4.5) follows by taking m_{k+1} sufficiently large.

Case II: $N_k + 2 \le \ell \le N_{k+1}$ for some $1 \le k < k_d$. In this case we have $f_\ell(\xi) = S^{(-1)^\ell}\mathcal{X}_\ell(f_{\ell-1})(\xi)$ and the inductive assumptions

$$\text{(4.10)} \qquad \begin{cases} e_k \le \text{lexp}(f_{\ell-1}) \le 1, \\ \text{coeff}(f_{\ell-1}, \xi^{e_{k+1}}) = a_{\ell-1} + b_{\ell-1}m_{k+1} \neq 0, \qquad b_{\ell-1} \neq 0, \\ f_{\ell-1} \equiv S^{(-1)^{\ell-1}}(f_{N_{k+1}-\ell+1}) \pmod{\xi^{e_{k+1}}}. \end{cases}$$

Again $f_\ell(\xi)$ is well defined. Indeed, by (4.1) and $T^{-1} = T$ we have $\mathcal{X}_\ell = TS^{n_\ell}T$ with $n_\ell \in \{\pm m_1, \dots, \pm m_k\}$, and hence the leading exponent κ_0 of $f_{\ell-1}$ has the following evolution:

$$\kappa_0 \xrightarrow{T} \kappa_0^* \ (\ge \kappa > 1) \xrightarrow{S^{n_\ell}} \kappa_0^* \xrightarrow{T} \kappa_0,$$

and then $S^{(-1)^\ell}$ is certainly well defined. Applying Lemma 4.2 with $f(\xi) = f_{\ell-1}(\xi)$, $g(\xi) = S^{(-1)^{\ell-1}}(f_{N_{k+1}-\ell+1})(\xi)$, $\lambda = e_{k+1}$, $r = (-1)^\ell$ and $\mathcal{X} = \mathcal{X}_\ell$, and using the third assumption in (4.10) we obtain

$$\begin{aligned} f_\ell = S^{(-1)^\ell}\mathcal{X}_\ell(f_{\ell-1}) &\equiv S^{(-1)^\ell}\mathcal{X}_\ell S^{(-1)^{\ell-1}}(f_{N_{k+1}-\ell+1}) \pmod{\xi^{e_{k+1}}} \\ &= S^{(-1)^\ell}\mathcal{X}_\ell S^{(-1)^{\ell-1}}S^{(-1)^{N_{k+1}-\ell+1}}\mathcal{X}_{N_{k+1}-\ell+1}(f_{N_{k+1}-\ell}) \pmod{\xi^{e_{k+1}}} \\ &= S^{(-1)^\ell}\mathcal{X}_\ell\mathcal{X}_{N_{k+1}-\ell+1}(f_{N_{k+1}-\ell}) \pmod{\xi^{e_{k+1}}} \end{aligned}$$

since N_{k+1} is odd. By (4.1) we have $\mathcal{X}_\ell = \mathcal{X}^{-1}_{N_{k+1}-\ell+1}$; hence

$$f_\ell \equiv S^{(-1)^\ell}(f_{N_{k+1}-\ell}) \pmod{\xi^{e_{k+1}}}$$

and (4.6) follows.

To prove (4.5) we proceed as follows. Since $e_{k+1} < 1$ we have

$$\text{(4.11)} \quad \begin{aligned} \text{coeff}(f_\ell, \xi^{e_{k+1}}) &= \text{coeff}(S^{(-1)^\ell}\mathcal{X}_\ell(f_{\ell-1}), \xi^{e_{k+1}}) = \text{coeff}(\mathcal{X}_\ell(f_{\ell-1}), \xi^{e_{k+1}}) \\ &= \text{coeff}(TS^{n_\ell}T(f_{\ell-1}), \xi^{e_{k+1}}), \end{aligned}$$

and the inductive assumption gives $\text{lexp}(f_{\ell-1}) \ge e_k > e_{k+1}$. Hence from (i) of Theorem 1.2 we get

$$\text{coeff}(T(f_{\ell-1}), \xi^{e^*_{k+1}}) = A + B\,\text{coeff}(f_{\ell-1}, \xi^{e_{k+1}}),$$

where A and $B \neq 0$ depend on α and $m_1, \dots, m_k$. Therefore

$$\text{coeff}(S^{n_\ell} T(f_{\ell-1}), \xi^{e^*_{k+1}}) = A' + B \,\text{coeff}(f_{\ell-1}, \xi^{e_{k+1}}) \tag{4.12}$$

with A' again depending on α and $m_1, \dots, m_k$. Moreover, since $\kappa_0 = \text{lexp}(f_{\ell-1}) \leq 1$, $\frac{x}{dx-1}$ is decreasing and $1 < d < 2$, from (i) of Theorem 1.2 we obtain

$$\text{lexp}(T(f_{\ell-1})) = \kappa_0^* = \frac{\kappa_0}{d\kappa_0 - 1} \geq \frac{1}{d-1} > 1,$$

and hence

$$\text{lexp}(S^{n_\ell} T(f_{\ell-1})) = \text{lexp}(T(f_{\ell-1})). \tag{4.13}$$

Therefore, from (4.12), (4.13) and using again (i) of Theorem 1.2 we obtain

$$\text{coeff}(TS^{n_\ell} T(f_{\ell-1}), \xi^{e_{k+1}}) = A'' + B' \,\text{coeff}(f_{\ell-1}, \xi^{e_{k+1}}), \tag{4.14}$$

where A'' and B' have the same properties of A and B above. Thus (4.5) follows from (4.11) and (4.14), using the second assumption in (4.10) and choosing m_{k+1} sufficiently large.

Finally, to check (4.4) we note that (4.6) gives

$$\text{lexp}(f_\ell) \leq \max(1, \text{lexp}(f_{N_{k+1}-\ell})) = 1 \tag{4.15}$$

since $N_{k+1} - \ell < \ell$. Suppose that $N_{k+1} - \ell \geq 1$ and let $K \geq 0$ be such that $N_K < N_{k+1} - \ell \leq N_{K+1}$. Then $0 \leq K \leq k-1$ and by the inductive assumption we have

$$\text{coeff}(f_{N_{k+1}-\ell}, \xi^{e_{K+1}}) \neq 0;$$

hence also $\text{coeff}(S^{(-1)^\ell}(f_{N_{k+1}-\ell}), \xi^{e_{K+1}}) \neq 0$ since $e_{K+1} < 1$. Observing that $e_{K+1} \geq e_k > e_{k+1}$, from (4.6) we deduce that $\text{lexp}(f_\ell) \geq e_k$, and (4.4) follows thanks to (4.15) if $N_{k+1} - \ell \geq 1$. The case $\ell = N_{k+1}$ is easy, since (4.5) directly shows that $\text{lexp}(f_{N_{k+1}}) \geq e_{k+1}$; thus (4.4) follows thanks to (4.15) in this case as well, and the lemma is proved. □

All the statements, but the last one, of Theorem 1.3 follow easily from Lemma 4.3 by choosing $\ell = N_{k_d}$. Indeed, $f_{N_{k_d}}(\xi) = \mathcal{L}_{k_d}(g_0)(\xi)$ thanks to Lemma 4.1. Moreover, by (4.4) we have that $\text{lexp}(\mathcal{L}_{k_d}(g_0)) \geq e_{k_d}$, while by (4.6)

$$\mathcal{L}_{k_d}(g_0) \equiv S^{-1}S(g_0) \equiv 0 \pmod{\xi^{e_{k_d}}}$$

since $e_{k_d} \geq 1/d$, see (1.16). Therefore $\text{lexp}(\mathcal{L}_{k_d}(g_0)) \leq e_{k_d}$; hence $\text{lexp}(\mathcal{L}_{k_d}(g_0)) = e_{k_d}$. The claim concerning $\text{coeff}(\mathcal{L}_{k_d}(g_0), \xi^{e_{k_d}})$ follows at once from (4.5). Recalling the notation in the introduction, the last statement of Theorem 1.3 follows from

Lemma 4.4. *Let* $1 < d < 2$. *Then for* $k = 0, \dots, k_d$ *we have* $s^*(s, \mathcal{L}_k, g_0) = s$.

Proof. In view of Lemma 4.1 and (4.3) it is enough to show that

$$s^*(s, S^{(-1)^\ell}\mathcal{X}_\ell, f_{\ell-1}) = s$$

for $\ell = 1, \dots, N_k$ and $k \le k_d$. Since application of $S^{\pm 1}$ does not change s we have

$$s^*(s, S^{(-1)^\ell}\mathcal{X}_\ell, f_{\ell-1}) = s^*(s, \mathcal{X}_\ell, f_{\ell-1}),$$

and we write $\mathcal{X}_\ell = TS^{n_\ell}T$. By (4.4) we have that $\kappa_0 = \mathrm{lexp}(f_{\ell-1}) \le 1$ and hence from Theorem 1.2, since $\frac{x}{dx-1}$ is decreasing and $1 < d < 2$, we obtain

$$\mathrm{lexp}(T(f_{\ell-1})) = \kappa_0^* = \frac{\kappa_0}{d\kappa_0 - 1} \ge \frac{1}{d-1} > 1.$$

Therefore $\mathrm{lexp}(S^{n_\ell}T(f_{\ell-1})) = \kappa_0^*$, and hence again by Theorem 1.2 we get

$$\mathrm{lexp}(TS^{n_\ell}T(f_{\ell-1})) = (\kappa_0^*)^* = \kappa_0.$$

As a consequence, appying $\mathcal{X}_\ell$ the evolution of s is as follows:

$$s \xrightarrow{T} s^* \xrightarrow{S^{n_\ell}} s^* \xrightarrow{T} (s^*)^* = s,$$

and the lemma follows. □

The proof of Theorem 1.3 is therefore complete.

5. **Proof of Theorem 1.4**

The arguments leading to the proof of Theorem 1.1 can be modified to prove Theorem 1.4. We only give a very brief sketch of the required changes. An inspection of the proof of Theorem 1.1 shows that condition $d\kappa_0 > 1$ is not used in the first two subsections of Section 2, and hence the results up to equation (2.23) hold true with any function $f(n, \boldsymbol{\beta})$ as in (1.20). In the proof of Theorem 1.4 we do not need to switch from n to the real variable ξ, hence coming to the saddle point method (see subsection 3), in the present case we have to deal with the function

$$\Phi(x, n, \boldsymbol{\beta}) = x^{1/d} - 2\pi f\left(\frac{qx}{n}, \boldsymbol{\beta}\right),$$

where q is as in (1.8). Therefore, writing $c_\nu = \beta_\nu \kappa_\nu q^{\kappa_\nu}$ for $\nu = 1, \dots, N$, following the proof of Lemma 2.3 we have that $\Phi'(x, n, \boldsymbol{\beta}) = 0$ if and only if

$$0 = \frac{1}{d} x^{\frac{1}{d}-1} \left\{ 1 - 2\pi\beta \left(\frac{q}{n}\right)^{1/d} - 2\pi d \sum_{\nu=1}^{N} c_\nu n^{-\kappa_\nu} x^{-\omega_\nu} \right\}. \tag{5.1}$$

Hence, if

$$1 - 2\pi\beta \left(\frac{q}{n}\right)^{1/d} \ne 0, \tag{5.2}$$

then the nonzero term $1-2\pi\beta(\frac{q}{n})^{1/d}$ is dominating inside the brackets in (5.1), and hence $\Phi'(x,n,\boldsymbol{\beta}) \neq 0$ for every $x \geq 1$. As a consequence, the analogue of the integral $I_X(s,n,\ell)$ in Lemma 2.4 has the properties of the function $k_X(s,n,\ell)$ in such a lemma if (5.2) holds; i.e., the function $K_X(s+\frac{\ell}{d},n)$ arising from the critical point x_0 is not present under condition (5.2), since there is no critical point x_0 in this case. As a matter of fact, under (5.2) the analogue of Lemma 2.4 can be replaced by the analogue of Lemma 2.2.

Now observe that, in view of (1.8), condition $\beta \notin \mathrm{Spec}(F)$ means that for any given $n \in \mathbb{N}$, either (5.2) holds or $a_F(n) = 0$. Therefore, if $\beta \notin \mathrm{Spec}(F)$ then after summation over n to get the analogue of (2.34) we obtain

$$F_X(s;f) = \sum_{n=1}^{\infty} \frac{\overline{a_F(n)}}{n^{1-s}} H_X(s,n),$$

where $H_X(s,n)$ has the properties stated after (2.34). Hence the arguments in subsection 4 lead in the present case to (2.39) without the term $M(s)$; thus $F(s;f)$ is holomorphic for $\sigma > 0$ if $\beta \notin \mathrm{Spec}(F)$ and Theorem 1.4 is proved.

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