Convex solutions to the mean curvature flow

By Xu-Jia Wang

Abstract

In this paper we study the classification of ancient convex solutions to the mean curvature flow in $\mathbf{R}^{n+1}$. An open problem related to the classification of type II singularities is whether a convex translating solution is k-rotationally symmetric for some integer $2 \le k \le n$, namely whether its level set is a sphere or cylinder $S^{k-1} \times \mathbf{R}^{n-k}$. In this paper we give an affirmative answer for entire solutions in dimension 2. In high dimensions we prove that there exist nonrotationally symmetric, entire convex translating solutions, but the blow-down in space of any entire convex translating solution is k-rotationally symmetric. We also prove that the blow-down in space-time of an ancient convex solution which sweeps the whole space $\mathbf{R}^{n+1}$ is a shrinking sphere or cylinder.

1. Introduction

Convex solutions arise in the study of singularities of the mean curvature flow. To study the geometric behavior at singularities one needs to classify such solutions. In this paper[1] we study the classification, or more precisely the geometric asymptotic behavior, of general ancient convex solutions, including the convex translating solutions arising at type II singularities [12], [11] and the ancient convex solutions arising at general singularities [27].

It was proved by Huisken-Sinestrari [12], [11] that if $\mathcal{M}$ is a mean convex flow, namely a mean curvature flow with mean convex solution, in the Euclidean space $\mathbf{R}^{n+1}$, then the limit flow obtained by a proper blow-up procedure near type II singular points is a convex translating solution (also called soliton), that is, in an appropriate coordinate system, a mean curvature flow of

---

[1]This is a revised version of the paper arXiv:math.DG/0404326. All results and the ideas of proofs are the same. The main change is the proof of the growth estimate (1.5) in Section 2. In this new version we divide it into two parts. We first prove it for dimension 2, then prove it for high dimensions.

the form $\mathcal{M}' = \{(x, u(x) + t) \in \mathbf{R}^{n+1} : x \in \mathbf{R}^n, t \in \mathbf{R}\}$, where u is a complete convex solution to the mean curvature equation

$$(1.1) \qquad \mathrm{div}\left(\frac{Du}{\sqrt{1 + |Du|^2}}\right) = \frac{1}{\sqrt{1 + |Du|^2}}.$$

Translating solutions play a similar role for the investigation of asymptotic behavior of type II singularities as self-similar solutions for type I singularities. It is known that a convex self-similar solution must be a shrinking sphere or cylinder [10]. For convex translating solutions there is a well-known conjecture among researchers in this area, which is explicitly formulated, for example in [27], which asserts that if u is a complete convex solution of (1.1), then the level sets $\{u = \mathrm{const}\}$ are spheres or cylinders. This Bernstein type problem attracted attention in recent years, as it is crucial for a classification of type II singularities of the mean convex flow. In this paper we prove the conjecture is true for entire solutions in dimension 2 (Theorem 1.1) and false in higher dimensions (Theorem 1.2).

In this paper we also study the classification of general ancient convex solutions to the mean curvature flow. In [27] White proved that any limit flow to the mean convex flow in $\mathbf{R}^{n+1}$ for $n < 7$, or any special limit flow, namely a blow-up solution before first time singularity for $n \geq 7$, is an ancient convex solution, namely at any time the solution is a convex hypersurface. We prove that an ancient convex solution is convex in space-time (Proposition 4.1), and that the parabolic blow-down in space-time of any entire, ancient convex solution, and the blow-down in space of any entire convex translating solution, is a shrinking sphere or cylinder (Theorem 1.3). This result corresponds to Perelman's classification of ancient κ-noncollapsing solutions with nonnegative sectional curvature to the three-dimensional Ricci flow [20]; see Section 6.

To study the above two problems, we will consider the following more general equation

$$(1.2) \qquad \mathcal{L}_\sigma[u] =: \sum_{i,j=1}^n \left(\delta_{ij} - \frac{u_i u_j}{\sigma + |Du|^2}\right) u_{ij} = 1,$$

where $\sigma \in [0, 1]$ is a constant. If u is a convex solution of (1.2), then $u + t$, as a function of $(x, t) \in \mathbf{R}^n \times \mathbf{R}$, is a translating solution to the flow

$$(1.3) \qquad u_t = \sqrt{\sigma + |Du|^2}\,\mathrm{div}\left(\frac{Du}{\sqrt{\sigma + |Du|^2}}\right).$$

When $\sigma = 1$, equation (1.2) is exactly the mean curvature equation (1.1), and (1.3) is the nonparametrized mean curvature flow. When $\sigma = 0$, (1.3) is the level set flow. That is, if u is a solution of (1.2) with $\sigma = 0$, the level set $\{u = -t\}$, where $-\infty < t < -\inf u$, evolves by mean curvature.

Conversely, if a family of convex hypersurfaces $\mathcal{M} = \{\mathcal{M}_t\}$, with time slice $\mathcal{M}_t$, evolves by mean curvature, then $\mathcal{M}$ can be represented as a graph of u in the space-time $\mathbf{R}^{n+1} \times \mathbf{R}^1$ with $x_{n+2} = -t$, and the function u satisfies (1.2) with $\sigma = 0$. We will show that the function u itself is convex (Proposition 4.1). Therefore for both problems it suffices to study the classification of convex solutions to equation (1.2).

We say a solution to the mean curvature flow is *ancient* if it exists from time $-\infty$. We say a solution u of (1.2) is *complete* if its graph is a complete hypersurface in $\mathbf{R}^{n+1}$, and u is an *entire* solution if it is defined in the whole space $\mathbf{R}^n$. Accordingly, an ancient convex solution $\mathcal{M}$ to the mean curvature flow in $\mathbf{R}^{n+1}$ is an entire solution if $\mathcal{M}$ is an entire graph in space-time $\mathbf{R}^{n+1} \times \mathbf{R}^1$, which is equivalent to saying that the flow $\mathcal{M}$ sweeps the whole space $\mathbf{R}^{n+1}$. We say u is *k-rotationally symmetric* if there exists an integer $1 \leq k \leq n$ such that in an appropriate coordinate system, u is rotationally symmetric with respect to $x_1, \ldots, x_k$ and is independent of $x_{k+1}, \ldots, x_n$. Therefore a function u is k-rotationally symmetric if and only if its level sets are spheres ($k = n$) or cylinders ($k < n$). For other related terminologies we refer the reader to [11], [27]. For any $1 \leq k \leq n$, there is a k-rotationally symmetric convex solution of (1.1), which is unique up to orthogonal transformations. When $n = 1$, the unique complete convex solution of (1.1) is the "grim reaper", given by $u(x) = \log \sec x_1$. To exclude hyperplanes in this paper we always consider convex solutions with positive mean curvature.

The results in this paper can be summarized in the following theorems.

THEOREM 1.1. *If $n = 2$, then any entire convex solution to (1.2) must be rotationally symmetric in an appropriate coordinate system.*

From Theorem 1.1 we obtain

COROLLARY 1.1. *A convex translating solution to the mean curvature flow must be rotationally symmetric if it is a limit flow to a mean convex flow in $\mathbf{R}^3$.*

THEOREM 1.2. *For any dimension $n \geq 2$ and $1 \leq k \leq n$, there exist complete convex solutions, defined in strip regions, to equation (1.2) which are not k-rotationally symmetric. If $n \geq 3$, there exist entire convex solutions to (1.2) which are not k-rotationally symmetric.*

Theorems 1.1 and 1.2 reflect a typical phenomenon; namely, the Bernstein theorem is in general true in low dimensions and false in higher dimensions. See [25] for a brief discussion.

THEOREM 1.3. *Let u be an entire convex solution of (1.2). Let $u_h(x) = h^{-1}u\sqrt{h}\,x)$. Then there is an integer $2 \leq k \leq n$ such that after a rotation of*

the coordinate system for each h, u_h converges to

$$(1.4) \qquad\qquad \eta_k(x) = \frac{1}{2(k-1)} \sum_{i=1}^{k} x_i^2.$$

The case $\sigma = 0$ of Theorems 1.1–1.3 describes the geometry of ancient convex solutions to the mean curvature flow, while the case $\sigma = 1$ of Theorems 1.1–1.3 resolves the problem on convex translating solutions. Note that if u is a convex solution which is not defined in the whole space, then u is defined in a convex strip region (Corollaries 2.1 and 2.2), and it cannot be a blow-up solution to the mean convex flow in general (Corollary 6.1). We also remark that in Theorem 1.3 we did not rule out the possibility that the axis of the cylinder-like level set $\{u_h = 1\}$ may rotate slowly as $h \to \infty$, as the convergence $u_h \to \eta_k$ is uniform only on any compact sets.

As a limit flow at the first time singularity is convex, by Brakke's regularity theorem, a blow-up sequence converges smoothly on any compact sets to an ancient convex solution [27]. Therefore by the above classifications one may infer that if $\mathcal{M} = \{\mathcal{M}_t\}$ is a mean convex flow in $\mathbf{R}^{n+1}$, $n \geq 2$, then $\mathcal{M}_t$ satisfies a canonical neighborhood condition, similar to the assertion in [21] for the Ricci flow, at any point $x_t \in \mathcal{M}_t$ with large mean curvature before the first time singularity. In particular if the mean curvature at x_t converges to infinity, then a proper scaling of $\mathcal{M}$ at x_t converges along subsequences to shrinking spheres or cylinders. See the discussion in Section 6.

Our proofs of the above theorems rely heavily on the convexity of solutions. To prove these theorems it suffices to consider the cases $\sigma = 0$ and $\sigma = 1$, as for any $\sigma > 0$, one can make the transform $\hat{u}(x) = \frac{1}{\sigma} u(\sqrt{\sigma} x)$ to change equation (1.2) to the case $\sigma = 1$. A key estimate for the proof of Theorem 1.3 is that for any entire convex solution u of (1.2), there exists a positive constant C such that

$$(1.5) \qquad\qquad u(x) \leq C(1 + |x|^2) \quad \forall\, x \in \mathbf{R}^n.$$

The constant C depends only on n and the upper bound of $u(0)$ and $|Du(0)|$. Note that (1.5) implies the compactness of the set of entire convex solutions to (1.2); see Corollary 2.3.

By Theorem 1.3 and estimate (1.5) we have, if $n = 2$,

$$C_1 |x|^2 \leq u(x) \leq C_2 |x|^2$$

for large $|x|$. Hence the case $\sigma = 0$ of Theorem 1.1 follows immediately from the asymptotic estimates in [7]. For the case $\sigma = 1$ we will prove furthermore, by an iteration argument, that

$$(1.6) \qquad\qquad |u(x) - u_0(x)| = o(|x|) \quad \text{as} \ \ |x| \to \infty,$$

where u_0 is the radial solution of (1.1). We then conclude $u = u_0$ by a Liouville type theorem of Bernstein [1], [2], which asserts that an entire solution w to an elliptic equation in $\mathbf{R}^2$ must be a constant if $|w(x)| = o(|x|)$ at infinity [23].

The proof of Theorem 1.2 is different for the cases $\sigma = 0$ and $\sigma = 1$. For the case $\sigma = 0$, we consider the Dirichlet problem

$$(1.7) \qquad \begin{cases} \mathcal{L}_0[u] = 1 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where Ω is a bounded convex domain in $\mathbf{R}^n$ ($n \geq 2$). The existence and uniqueness of viscosity solutions to (1.7) can be found in [3] and [6]. We prove that there exists a sequence of bounded convex domains $\{\Omega_k\}$ such that $u_k + |\inf_{\Omega_k} u_k|$, where u_k is the solution of (1.7) with $\Omega = \Omega_k$, converges to a complete convex solution u of $\mathcal{L}_0[u] = 1$ of which the level set $\{x \in \mathbf{R}^n : u(x) = 1\}$ is not a sphere. To prove that u is a complete convex solution we need the concavity of the function $\log(-u)$ (Lemma 4.1).

The concavity of $\log(-u)$ is still an open problem for the mean curvature equation (1.1). To construct a similar sequence of solutions (u_k) for equation (1.1), we use the Legendre transform to convert the mean curvature equation (1.1) to a fully nonlinear equation for which the convexity is a natural condition for the ellipticity of the equation. Let u be a smooth, uniformly convex function defined in a convex domain $\Omega \subset \mathbf{R}^n$. The Legendre transform of u, u^*, is a smooth, uniformly convex function defined in $\Omega^* = Du(\Omega)$, given by

$$(1.8) \qquad u^*(y) = \sup\{x \cdot y - u(x) : x \in \Omega\}.$$

The supremum is attained at the unique point x such that $y = Du(x)$, and u can be recovered from u^* by the same Legendre transform. If u is a convex solution of (1.1), u^* satisfies the fully nonlinear equation

$$(1.9) \qquad \det D^2 u^* = \sum \left(\delta_{ij} - \frac{y_i y_j}{1 + |y|^2} \right) F^{ij}[u^*],$$

where $\delta_{ij} = 1$ if $i = j$ and $\delta_{ij} = 0$ otherwise, and

$$F^{ij}[u^*] = \frac{\partial}{\partial r_{ij}} \det r \quad \text{at } r = D^2 u^*.$$

It is known that for any uniformly convex domain Ω and any smooth function φ on $\partial\Omega$, (1.9) has a unique convex solution u^* in Ω satisfying $u^* = \varphi$ on $\partial\Omega$; see Theorem 5.2. By Theorem 5.2 we can construct a sequence of convex solutions (u_k^*) to (1.9), such that (u_k), the Legendre transform of (u_k^*), converges to a complete convex solution u of (1.1) and the level set $\{x \in \mathbf{R}^n : u(x) = 1 + \inf u\}$ is not a sphere.

This paper is arranged as follows. In Section 2 we prove estimate (1.5) and Theorem 1.3. In Section 3 we prove Theorem 1.1. In Section 4 we prove

the case $\sigma = 0$ of Theorem 1.2. The case $\sigma = 1$ of Theorem 1.2 will be proved in Section 5. The final Section 6 contains some applications. We first prove Corollary 1.1, then discuss implications of Theorem 1.3, and finally mention a few unsolved problems related to our Theorems 1.1–1.3.

Recent developments. A major advance, after the paper was finished in early 2003, has been made by Huisken and Sinestrari [13], in which they studied the mean curvature flow with surgeries of 2-convex hypersurfaces in $\mathbf{R}^{n+1}$ for $n \geq 3$. They proved that at any point with large curvature, the hypersurface after normalization must be very close to the cylinder $S^{n-1} \times \mathbf{R}^1$ or a convex cap. Very recently, in another development, the author, together with Weimin Sheng [22], found a new proof for the following result of White [26], [27]; that is, for the mean convex flow up to the first time singularity, a blow-up sequence converges along a subsequence to a convex mean curvature flow, and the grim reaper is not a blow-up solution. This proof is based on the curvature pinching of Huisken and Sinestrari [12], [11].

2. **Level set estimates and proof of Theorem 1.3**

In this section we prove estimate (1.5) and Theorem 1.3. Our proof of (1.5) involves elementary, but delicate analysis. To illustrate our idea, we first prove it in the dimension-two case, then prove it for higher dimensions. For clarity we divide this section into three subsections. In Section 2.1 we prove (1.5) for $n = 2$. In Section 2.2 we prove (1.5) for $n > 2$. In Section 2.3 we prove Theorem 1.3.

Let u be a complete convex solution of (1.2). For any constant $h > 0$ we denote

$$(2.1) \qquad \Gamma_h = \Gamma_{h,u} = \{x \in \mathbf{R}^n : u(x) = h\},$$
$$\Omega_h = \Omega_{h,u} = \{x \in \mathbf{R}^n : u(x) < h\}.$$

Then $\Omega_h \subset \Omega_{h+\varepsilon}$ for any $\varepsilon > 0$. Let κ denote the mean curvature of the level set Γ_h. We have

$$(2.2) \qquad \mathcal{L}_\sigma[u] = \kappa u_\gamma + \frac{\sigma u_{\gamma\gamma}}{\sigma + u_\gamma^2} \geq \kappa u_\gamma = \mathcal{L}_0[u],$$

where γ is the unit outward normal to $\Omega_{h,u}$, and $u_{\gamma\gamma} = \gamma_i \gamma_j u_{ij}$.

We may assume that Ω_h does not contain a straight line. For if Ω_{h_0} contains a straight line, then for all $h \geq h_0$, by convexity we have the splitting

$\Omega_h = \Omega'_h \times \mathbf{R}^1$ for some convex set $\Omega'_h \subset \mathbf{R}^{n-1}$. The convexity implies that u is a function of $x_1, \dots, x_{n-1}$ if the straight line is parallel to the x_n-axis. Therefore the problem reduces to a lower dimension case. Furthermore, the graph of u, $\mathcal{M}_u$, does not contain any line segment. For if it does, the analyticity of u when $\sigma > 0$, or the constant ranking of $(D^2 u)$ when $\sigma = 0$ ([12], [11]), implies that $\mathcal{M}_u$ contains a straight line.

We also remark that our proof of (1.5) works for general convex solutions of (1.2). When $\sigma = 0$ and the solution is a limit flow (blow-up solution) to a mean convex flow, one may also use the noncollapsing result in [22], [26], [27] to give an alternative proof; see Remark 2.1 below.

2.1. *Proof of* (1.5) *for* $n = 2$. Let u be a complete convex solution of (1.2) satisfying $u(0) = 0$. We first prove that if $\Omega_1 \cap \{x_1 = 0\}$ is contained in $\{|x_2| \leq \beta\}$ for some small $\beta > 0$, then Ω_h is contained in a strip region for any $h > 0$. We prove the result in three lemmas. In the first one we assume that $\sigma = 0$ and u is symmetric in x_2. In the second one we remove the symmetry assumption. In the third one we remove the condition $\sigma = 0$.

LEMMA 2.1. *Let u be a complete convex solution of* (1.2). *Suppose $n = 2$, $\sigma = 0$, $u(0) = 0$, and u is symmetric in x_2, namely $u(x_1, x_2) = u(x_1, -x_2)$. Suppose there is a sufficiently small $\beta > 0$ such that $u(0, \beta) \geq 1$. Then u is defined in a strip region $\{|x_2| < C\}$.*

Proof. Let $\mathcal{M}_u^+$ denote the graph of u in the half-space $\{x_2 \geq 0\}$, and D the projection of $\mathcal{M}_u^+$ on the plane $\{x_2 = 0\}$. Then $\mathcal{M}_u^+$ can be represented as the graph of a function g in the form $\mathcal{M}_u^+ = \{(x_1, x_2, x_3) : x_2 = g(x_1, x_3)\}$, and g is positive, concave, monotone increasing in x_3, defined in D, and vanishes on ∂D. In the following we also regard the height parameter h as a variable and use h instead of x_3.

For any $h > 0$, we denote $g_h(x_1) = g(x_1, h)$ and $D_h = \{x_1 : (x_1, h) \in D\}$. So g_h is a positive, concave function of one variable, and D_h is an interval in $\mathbf{R}^1$ containing the origin, $D_h = (-\underline{a}_h, \overline{a}_h)$ (here $\underline{a}_h$ or $\overline{a}_h$ might be equal to infinity). Denote $\overline{b}_h = g_h(0)$.

Claim 1. For any given $h > 0$, if $\overline{a}_h, \underline{a}_h \geq \overline{b}_h$, then $\overline{a}_h \overline{b}_h \geq \frac{\pi}{32} h$.

To prove the claim, we assume $\overline{a}_h \leq \underline{a}_h$, otherwise we make the change $x_1 \to -x_1$. Denote $U_h = \Omega_h \cap \{x_1 > 0\}$. When $\sigma = 0$, the level set Γ_h is evolving at the velocity equal to its curvature (with time $t = -h$). By the convexity of U_h and the assumption $\overline{a}_h, \underline{a}_h \geq \overline{b}_h$, we have $\overline{a}_s, \underline{a}_s \geq \frac{1}{2} \overline{b}_s$ for all $s \in (\frac{1}{2} h, h)$. Hence by the concavity of g we have the gradient estimate $|\frac{d}{dx_1} g_s(0)| \leq 2$ for $s \in (\frac{1}{2} h, h)$. Note that $\frac{d}{ds} |U_s|_{\mathcal{H}^2}$ is equal to the arc-length of the set of the unit normals to $\Gamma_s \cap \{x_1 > 0\}$. Hence $\frac{d}{ds} |U_s|_{\mathcal{H}^2} \geq \frac{\pi}{4}$ for

$s \in (\frac{1}{2}h, h)$, which implies $|U_h|_{\mathcal{H}^2} \geq \frac{\pi}{8}h$. Here and below we use $|E|_{\mathcal{H}^k}$ to denote the k-dim Hausdorff measure of the set E. By the convexity of Ω_h and the assumption $\underline{a}_h \geq \bar{a}_h$, one sees that U_h is contained in $(0, \bar{a}_h) \times (-2\bar{b}_h, 2\bar{b}_h)$. Hence $\bar{a}_h \bar{b}_h \geq \frac{1}{4}|U_h|_{\mathcal{H}^2}$. We obtain $\bar{a}_h \bar{b}_h \geq \frac{\pi}{32}h$.

Claim 1 is also true when $\sigma > 0$. Indeed, by equation (2.2) and the convexity we have $\mathcal{L}_0[u] \leq 1$, which means Γ_h is moving at a velocity greater than or equal to its curvature. Therefore we also have $\frac{d}{ds}|U_s|_{\mathcal{H}^2} \geq \frac{\pi}{4}$ for $s \in (\frac{1}{2}h, h)$, and so it also follows $\bar{a}_h \bar{b}_h \geq \frac{\pi}{32}h$.

In particular, the proof implies Claim 1 holds for convex functions u satisfying $\mathcal{L}_0[u] \leq C_1$ for some positive constant C_1. That is, if $\mathcal{L}_0[u] \leq C_1$, then $\bar{a}_h \bar{b}_h \geq C_2 h$, where C_2 depends on C_1.

Claim 2. Denote $h_k = 2^k$, $\bar{a}_k = \bar{a}_{h_k}$, $\bar{b}_k = \bar{b}_{h_k}$, $g_k = g_{h_k}$, and $D_k = D_{h_k}$. Then

$$(2.3) \qquad g_k(0) \leq g_{k-1}(0) + 2^{-k/8} \quad \forall \ k \geq 1.$$

Note that Lemma 2.1 follows from Claim 2 immediately. Indeed, let $\Omega_\infty = \bigcup_{h>0} \Omega_h$ be the domain of definition of u. By (2.3), $\bar{b}_k$ is uniformly bounded. Hence by Claim 1, Ω_∞ is a convex set containing the whole x_1-axis. Hence $\Omega_\infty = I \times \mathbf{R}^1$ for some interval I in the x_2 axis. Estimate (2.3) implies that $I \subset (-2, 2)$ (see (2.5) below). Hence Ω_∞ must be a strip region.

To prove (2.3) we observe that, since g is positive and concave, $g_k(0) \leq h_k g_0(0) \leq 2^k \beta$. Hence we may assume that $g_{k_0}(0) \leq 1$ for some sufficiently large k_0. By Claim 1, we have

$$(2.4) \qquad \bar{a}_k \geq C_0 h_k$$

for $k \leq k_0$ with $C_0 = \frac{\pi}{32}$. We prove (2.3) by induction, starting at $k = k_0$.

Suppose by induction that (2.3) holds up to k. Then by induction,

$$(2.5) \qquad g_k(0) \leq g_{k_0}(0) + \sum_{j=k_0}^{k} 2^{-j/8} \leq 2.$$

By the concavity of g and since $g \geq 0$, we have $g_{k+1}(0) \leq 2g_k(0) \leq 4$. By Claim 1, $\bar{a}_{k+1} \geq \frac{\pi}{128}h_k$. Hence (2.4) holds at $k+1$ with $C_0 = \frac{\pi}{128}$.

To prove (2.3) at $k+1$, denote

$$L_k = \left\{ x_1 \in \mathbf{R}^1 : -\frac{C_0}{4}h_k < x_1 < \frac{C_0}{4}h_k \right\}, \qquad Q_k = L_k \times [h_k, h_{k+1}] \subset D,$$

where C_0 is given as in (2.4). Since $g > 0$, g is concave, and $g(x_1, h)$ is defined in $2L_k$ for $h \geq \frac{1}{2}h_k$, we have

$$(2.6) \qquad \sup\{g(x_1, h) : (x_1, h) \in Q_k\} \leq 2\sup\{g(x_1, h_k) : x_1 \in L_k\}$$
$$\leq 4g(0, h_k) = 4g_k(0) \leq 8.$$

Moreover, for any $(x_1, h) \in Q_k$,

$$(2.7) \qquad |\partial_h g(x_1, h)| \leq \frac{g(x_1, h) - g(x_1, h_{k-1})}{h - h_{k-1}} < \frac{g(x_1, h)}{h_k - h_{k-1}} \leq \frac{16}{h_k}.$$

Similarly,

$$(2.8) \qquad |\partial_{x_1} g(x_1, h)| \leq \frac{g(x_1, h)}{\bar{a}_h - |x_1|} \leq \frac{2g(x_1, h)}{h_k} \leq \frac{16}{h_k} \quad \forall \, (x_1, h) \in Q_k.$$

From the above gradient estimates and the concavity of g, the average in Q_k of the second derivative satisfies

$$(*) \qquad |\partial_{x_1}^2 g(x_1, h)| \approx \sup_{x_1 \in L_k} |\partial_{x_1} g(x_1, h)| / |L_k|_{\mathcal{H}^1} \approx h_k^{-2}.$$

Here $a \approx b$ means $a \leq Cb$ and $b \leq Ca$ for some constant C. This simple observation is *critical* for the proof of (2.3), and actually the following weaker version is sufficient,

$$(2.9) \qquad |\partial_{x_1}^2 g(x_1, h)| \leq C h_k^{-5/4} \quad (x_1, h) \in Q_k$$

for some fixed constant C. Indeed, by equation (2.2), we have $\kappa u_\gamma = 1$ (when $\sigma = 0$). Note that $u_\gamma \approx (\partial_h g)^{-1}$ and $\kappa \approx \partial_{x_1 x_1} g$ when $(x_1, h) \in Q_k$. Hence if (2.9) holds, we have

$$(2.10) \qquad |\partial_h g(0, h)| \leq C h_k^{-5/4} \quad \text{for} \quad h \in (h_k, h_{k+1}).$$

It follows that

$$(2.11) \qquad g_{h_{k+1}}(0) - g_{h_k}(0) = g(0, h_{k+1}) - g(0, h_k) \leq C h_k^{-1/4} \leq h_k^{-1/8}$$

when k is large. We obtain (2.3).

However we have not proved the estimate (2.9) pointwise, even in the special case $\sigma = 0$. But we observe that the set where g does not satisfy (2.9) is very small, and the concavity of g ensures that this small set does not harm the estimate (2.3).

Denote

$$\chi = \{(x_1, h) \in Q_k : |\partial_{x_1}^2 g(x_1, h)| \geq h_k^{-5/4}\}.$$

If χ is empty, (2.3) is proved in (2.11) above. If $\chi \neq \emptyset$, we proceed as follows (the argument also applies to the case where χ is empty). For any $h \in (h_k, h_{k+1})$, by the gradient estimates and the concavity of g, an integration by parts gives

$$|\{x_1 \in L_k : (x_1, h) \in \chi\}|_{\mathcal{H}^1} h_k^{-5/4} \leq \left| \int_{L_k} \partial_{x_1 x_1} g \right| \leq 2 \sup_{L_k} |\partial_{x_1} g| \leq C h_k^{-1}.$$

Taking integration from $h = h_k$ to $h = h_{k+1}$ we obtain $|\chi|_{\mathcal{H}^2} h_k^{-5/4} \leq C$, namely

$$(2.12) \qquad |\chi|_{\mathcal{H}^2} \leq C h_k^{5/4}.$$

We say χ is a small set as the ratio $|\chi|_{\mathcal{H}^2} / |Q_k|_{\mathcal{H}^2} = O(h_k^{-3/4})$ is small.

For any given $y_1 \in L_k$ we denote $\chi_{y_1} = \chi \cap \{x_1 = y_1\}$. From (2.12) and by the Fubini Theorem, there is a set $\widetilde{L} \subset L_k$ with measure $|\widetilde{L}|_{\mathcal{H}^1} < h_k^{1/2}$ such that for any $y_1 \in L_k - \widetilde{L}$,

$$(2.13) \qquad\qquad |\chi_{y_1}|_{\mathcal{H}^1} \leq Ch_k^{3/4}.$$

For any $y_1 \in L_k - \widetilde{L}$, we have

$$g(y_1, h_{k+1}) - g(y_1, h_k) = \int_{\chi_{y_1}} \partial_h g(y_1, h)dh + \left(\int_{h_k}^{h_{k+1}} - \int_{\chi_{y_1}}\right) \partial_h g(y_1, h)dh.$$

By (2.13) and the gradient estimate (2.7), the first integral is bounded by $Ch_k^{-1/4}$. For the second one, similarly, as in (2.10), we have $\partial_h g \leq Ch_k^{-5/4}$ for any point $(x_1, h) \notin \chi$. Hence the second integral is also bounded by $Ch_k^{-1/4}$. Therefore we obtain

$$(2.14) \qquad\qquad g(y_1, h_{k+1}) - g(y_1, h_k) \leq Ch_k^{-1/4}.$$

From (2.14) we get (2.3) immediately. Indeed, replacing x_1 by $-x_1$ if necessary, we assume that $\partial_{x_1} g(0, h_k) \leq 0$, so that $g(x_1, h_k) \leq g(0, h_k)$ for all $x_1 \geq 0$. Since $|\widetilde{L}| < h_k^{1/2}$, the set $[0, h_k^{1/2}] - \widetilde{L}$ is not empty. Let $y_1 \in [0, h_k^{1/2}] - \widetilde{L}$. By (2.14) we obtain

$$(2.15) \qquad g(y_1, h_{k+1}) \leq g(y_1, h_k) + Ch_k^{-1/4} \leq g(0, h_k) + Ch_k^{-1/4}.$$

Note that g is positive and concave, and that g is defined on the interval $[0, \overline{a}_{k+1}]$ with $\overline{a}_{k+1} \geq Ch_{k+1}$ ($C = \frac{\pi}{128}$ as established before). We have

$$\frac{g_{k+1}(0)}{g_{k+1}(y_1)} \leq \frac{\overline{a}_{k+1}}{\overline{a}_{k+1} - |y_1|} = 1 + Ch_{k+1}^{-1/2}.$$

By (2.15) it follows (recall our notation, $g(y_1, h_{k+1}) = g_{k+1}(y_1)$) that

$$g_{k+1}(0) \leq g_k(0) + Ch_k^{-1/4}.$$

We obtain (2.3). Lemma 2.1 is proved. $\qquad\qquad\qquad\qquad\qquad\square$

In Lemma 2.1, we do not need to assume that the sublevel set Ω_h is compact, nor need we assume the whole sublevel set $\Omega_{h\,|\,h=1}$ is contained in a strip region. The assumption that $u(0, \beta) \geq 1$ for sufficiently small $\beta > 0$ implies that $\Omega_1 \cap \{x_1 > 0\}$ if $u_{x_1}(0, \beta) > 0$, or $\Omega_1 \cap \{x_1 < 0\}$ if $u_{x_1}(0, \beta) < 0$, is contained in a strip $\{|x_2| \leq \beta\}$. Next we remove the assumption that u is symmetric in x_2.

LEMMA 2.2. *Let u be a complete convex solution of (1.2). Suppose $n = 2$, $\sigma = 0$, $u(0) = 0$, and there is a sufficiently small $\beta > 0$ such that $u(0, \beta) \geq 1$ and $u(0, -\beta) \geq 1$. Then u is defined in a strip region.*

Note that the assumption $u(0, \beta) \geq 1$ and $u(0, -\beta) \geq 1$ is equivalent to the fact that $\Omega_1 \cap \{x_1 = 0\} \subset \{|x_2| < \beta\}$. Before proving Lemma 2.2, we state a property of convex domains, due to F. John [19], which is frequently used in the study of convex bodies and Monge-Ampère equations.

PROPOSITION 2.1. *Let Ω be a bounded, convex domain in $\mathbf{R}^n$, $n \geq 2$. Then among all (solid) ellipsoids containing Ω, there is a unique ellipsoid E of smallest volume such that*

$$(2.16) \qquad \frac{1}{n}E \subset \Omega \subset E,$$

where αE is the α-dilation of E with respect to its center.

We call E the *minimum ellipsoid* of Ω (it is a (solid) ellipse when $n = 2$). By a rotation of the coordinates, we may assume that E is given by $E = \left\{\sum_{i=1}^{n}\left(\frac{x_i - x_{0,i}}{r_i}\right)^2 < 1\right\}$, where $x_0 = (x_{0,1}, \dots, x_{0,n})$ is the center of E. We can make the linear transform $y_i = (x_i - x_{0,i})/r_i + x_{0,i}$, $i = 1, \dots, n$, such that E becomes the unit ball $B_1(x_0)$ and $B_{1/n}(x_0) \subset T(\Omega) \subset B_1(x_0)$.

Proof of Lemma 2.2. Let $R = 10^3$ and let E be the minimum ellipsoid of $\Omega_1 \cap B_R(0)$. By a rotation of coordinates we assume the axial directions of E coincide with those of the coordinate system (we do not need to assume the center of E is at the origin).

The proof is similar to that of Lemma 2.1. We indicate the necessary changes. Let $\mathcal{M}_u$ be the graph of u, which consists of two parts, $\mathcal{M}_u = \mathcal{M}^+ \cup \mathcal{M}^-$, where $\mathcal{M}^+ = \{(x, u(x)) \in \mathbf{R}^3 : \partial_{x_2}u(x) \geq 0\}$ and $\mathcal{M}^- = \{(x, u(x)) \in \mathbf{R}^3 : \partial_{x_2}u(x) \leq 0\}$. Then $\mathcal{M}^{\pm}$ can be represented as graphs of functions $g^{\pm}$ in the form $x_2 = g^{\pm}(x_1, x_3)$, $(x_1, x_3) \in D$ and D is the projection of $\mathcal{M}_u$ on the plane $\{x_2 = 0\}$. The functions g^+ and g^- are respectively concave and convex, and we have $x_3 = u(x_1, g^{\pm}(x_1, x_3))$. Denote

$$(2.17) \qquad g = g^+ - g^-.$$

Then g is a positive, concave function in D, vanishing on ∂D. For any $h > 0$ we also denote $g_h(x_1) = g(x_1, h)$, $g_h^{\pm}(x_1) = g^{\pm}(x_1, h)$, and $D_h = \{x_1 \in \mathbf{R}^1 : (x_1, h) \in D\}$. Then g_h is a positive, concave function in D_h, vanishing on ∂D_h, and $D_h = (-\underline{a}_h, \bar{a}_h)$ is an interval containing the origin. As before we denote $\bar{b}_h = g_h(0)$.

Claim 1. Suppose $\bar{a}_h, \underline{a}_h \geq \bar{b}_h$. Then $\bar{a}_h \bar{b}_h \geq \frac{\pi}{32}h$.

The claim can be proved in the same way as in Lemma 2.1, by observing that the gradient estimate $|\frac{d}{dx_1}g_s(0)| \leq 2$ also implies that the arc-length of the set of unit normals to $\Gamma_s \cap \{x_1 > 0\}$ is greater than $\frac{\pi}{4}$. Hence we also have $\frac{d}{ds}|U_s|_{\mathcal{H}^2} \geq \frac{\pi}{4}$ for $s \in (\frac{1}{2}h, h)$.

Claim 2. Denote $h_k = 2^k$, $\bar{a}_k = \bar{a}_{h_k}$, $\bar{b}_k = \bar{b}_{h_k}$, $g_k = g_{h_k}$, and $D_k = D_{h_k}$. We have

$$(2.18) \qquad\qquad g_k(0) \leq g_{k-1}(0) + 2^{-k/8} \quad \forall \ k \geq 1.$$

Lemma 2.2 follows from Claim 2 immediately. Indeed, let P be the projection of the graph $\mathcal{M}_g$ on the plane $\{x_3 = 0\}$. Then P is a convex set containing the x_1-axis. Hence $P = I \times \mathbf{R}^1$ for some interval I. Estimate (2.18) implies that $g(0, h) \leq 2$ for all h (due to (2.5)), so we have $I \subset [0, 2]$. Hence $\mathcal{M}_g$ is contained in the strip $\{(x_1, x_2, x_3) \in \mathbf{R}^3 : 0 \leq x_2 \leq 2\}$. By (2.17), $\mathcal{M}_u$ is also contained in a strip region $\{|a_1 x_1 + a_2 x_2| < 2\}$, where $(a_1, a_2, 0)$ is a unit vector in $\mathbf{R}^3$ with a_1 small and a_2 close to 1. We can make a_1 as small as we want, provided the constant R at the beginning of the proof is sufficiently large.

The proof of (2.18) is similar to that of (2.3). But for the argument from (2.9) to (2.10), one needs to use the equation (2.2). Therefore we need to consider g^+ and g^- instead of g.

First we establish (2.4)–(2.8) in the same way as in Lemma 2.1. Let L_k, Q_k and χ be as in Lemma 2.1. We also denote

$$\chi^{\pm} = \{(x_1, h) \in Q_k : |\partial_{x_1}^2 g^{\pm}(x_1, h)| \geq h_k^{-5/4}\}.$$

Then both χ^+ and χ^- are subsets of χ. For any $h \in (h_k, h_{k+1})$, by (2.8) and recalling that $L_k = (-\frac{1}{4}C_0 h_k, \frac{1}{4}C_0 h_k)$, we have

$$|\{x_1 \in L_k : (x_1, h) \in \chi^+\}|_{\mathcal{H}^1} h_k^{-5/4} \leq \left| \int_{L_k} \partial_{x_1 x_1} g^+ \right|$$

$$= \left| \partial_{x_1} g^+ \left(\frac{1}{4}C_0 h_k, h \right) - \partial_{x_1} g^+ \left(-\frac{1}{4}C_0 h_k, h \right) \right| \leq 2 \sup_{L_k} |\partial_{x_1} g| \leq C h_k^{-1},$$

where the second inequality is due to the fact that $g = g^+ - g^-$, g^+ is concave and g^- is convex. Hence $|\chi^+|_{\mathcal{H}^2} \leq C h_k^{5/4}$. Similarly we have $|\chi^-|_{\mathcal{H}^2} \leq C h_k^{5/4}$.

For any given $y_1 \in L_k$, denote $\chi_{y_1}^{\pm} = \chi^{\pm} \cap \{x_1 = y_1\}$. Then there is a set $\widetilde{L}^{\pm} \subset L_k$ with measure $|\widetilde{L}^{\pm}|_{\mathcal{H}^1} < h_k^{1/2}$ such that for any $y_1 \in L_k - \widetilde{L}^{\pm}$, we have $|\chi_{y_1}^{\pm}|_{\mathcal{H}^1} \leq C h_k^{3/4}$.

For any given $y_1 \in L_k - (\widetilde{L}^+ \cup \widetilde{L}^-)$, we have

$$g(y_1, h_{k+1}) - g(y_1, h_k) = g^+(y_1, h_{k+1}) - g^+(y_1, h_k) + |g^-(y_1, h_{k+1}) - g^-(y_1, h_k)|.$$

In the following we estimate $g^+(y_1, h_{k+1}) - g^+(y_1, h_k)$. The estimate also applies to $|g^-(y_1, h_{k+1}) - g^-(y_1, h_k)|$. We have

$$(2.19) \qquad g^+(y_1, h_{k+1}) - g^+(y_1, h_k) = \int_{h_k}^{h_{k+1}} \partial_h g^+(y_1, h) dh$$

$$= \int_{\chi_{y_1}^+} \partial_h g^+(y_1, h) dh + \left(\int_{h_k}^{h_{k+1}} - \int_{\chi_{y_1}^+} \right) \partial_h g^+(y_1, h) dh.$$

For the first integral on the right-hand side, note that $g = g^+ - g^-$, $g^+(y_1, h)$ is concave and increasing in h, and g^- is convex and decreasing in h. Hence $\partial_h g^+ \leq \partial_h g$. By the gradient estimate (2.7) and recalling that $|\chi^+_{y_1}|_{\mathcal{H}^1} \leq Ch_k^{3/4}$, we have

$$\int_{\chi^+_{y_1}} \partial_h g^+(y_1, h)dh \leq Ch_k^{-1/4}.$$

To estimate the second integration, we first introduce a mapping $\mathcal{T} : p \to q$ as follows. For a point $p = (x_1, h) \in D$, there is a corresponding point $P = (x_1, x_2, h)$ on the level set Γ_h, where $x_2 = g^+(x_1, h)$, such that p is the projection of P on the plane $\{x_2 = 0\}$. Let $q = (x_1, x_2)$ be the projection of P on the plane $\{x_3 = 0\}$.

By equation (2.2), we have $\kappa u_\gamma = 1$. Note that when $p = (x_1, h) \in Q_k$, the normal γ of the level set $\Gamma_h \subset \mathbf{R}^2$ at the point $q = \mathcal{T}(p) \in \Gamma_h$ satisfies

(2.20) $$|\gamma - e_2| < \varepsilon$$

for some small constant $\varepsilon > 0$, where $e_2 = (0, 1)$. This is because by induction, $\Omega_{h_k} \cap B_R$ is contained in a strip region (see discussion after (2.18)), and the axial directions of the minimum ellipsoid of $\Omega_{h_k} \cap B_R$ are a small perturbation of the axial directions of the coordinates, where R is the constant introduced at the beginning of the proof. Therefore we have

(2.21) $$\begin{cases} (\partial_h g^+)^{-1} = (1 + \varepsilon_1)u_\gamma, \\ \partial_{x_1 x_1} g^+ = (1 + \varepsilon_2)\kappa, \end{cases}$$

where ε_1, ε_2 are small constants provided R is sufficiently large. Hence

(2.22) $$|\partial_h g^+(y_1, h)| \leq C|\partial_{x_1 x_1} g^+| \leq Ch_k^{-5/4} \ \forall \ (y_1, h) \notin \chi^+.$$

It follows that

$$\left(\int_{h_k}^{h_{k+1}} - \int_{\chi^+_{y_1}}\right) \partial_h g^+(y_1, h)dh \leq Ch_k^{-1/4}.$$

Combining the above two estimates we obtain

$$g^+(y_1, h_{k+1}) - g^+(y_1, h_k) \leq Ch_k^{-1/4}.$$

Similarly we have $|g^-(y_1, h_{k+1}) - g^-(y_1, h_k)| \leq Ch_k^{-1/4}$. Therefore we obtain (2.18) just as we prove (2.3) from (2.14). $\qquad \square$

Next we remove the condition $\sigma = 0$ in Lemma 2.2.

LEMMA 2.3. *Let u be a complete convex solution of (1.2). Suppose $n = 2$, $u(0) = 0$, and there is a sufficiently small $\beta > 0$ such that $u(0, \beta) \geq 1$ and $u(0, -\beta) \geq 1$. Then u is defined in a strip region.*

Proof. We can follow the proof of Lemma 2.2 until (2.19) without any change. The estimate for the second integral on the right-hand side of (2.19) used the equation $\kappa u_\gamma = 1$. But when $\sigma \neq 0$, equation (2.2) contains an extra term $\frac{\sigma u_{\gamma\gamma}}{\sigma + u_\gamma^2}$. To handle this extra term, we need to divide the integral (2.19) into three parts,

$$(2.23) \qquad g^+(y_1, h_{k+1}) - g^+(y_1, h_k) = \int_I \partial_h g^+(y_1, h) dh$$

$$= \left(\int_{I_1} + \int_{I_2} + \int_{I_3} \right) \partial_h g^+(y_1, h) dh,$$

where $I = (h_k, h_{k+1})$,

$$I_1 = \chi_{y_1}^+,$$

$$I_2 = \left\{ h \in I - I_1 : \ \frac{\sigma u_{\gamma\gamma}(q)}{\sigma + u_\gamma^2(q)} \leq \frac{1}{2} \right\},$$

$$I_3 = I - (I_1 \cup I_2),$$

where $q = \mathcal{T}(p)$, $p = (y_1, h)$, and $\mathcal{T}$ is the mapping introduced after (2.19).

Similarly as in Lemma 2.2, we have $\partial_h g^+ \leq \partial_h g \leq C/h$ for $h \in (h_k, h_{k+1})$ and the first integral $\int_{I_1} \partial_h g^+(y_1, h) dh \leq Ch_k^{-1/4}$.

For the second one, noting that when $\frac{\sigma u_{\gamma\gamma}}{\sigma + u_\gamma^2} < \frac{1}{2}$, we have, by (2.21),

$$(\partial_h g^+)^{-1} \partial_{x_1 x_1} g^+ \approx \kappa u_\gamma \geq \frac{1}{2}.$$

Hence $\partial_h g^+ \leq C \partial_{x_1 x_1} g^+$ and we obtain $\int_{I_2} \partial_h g^+(y_1, h) dh \leq Ch_k^{-1/4}$.

To estimate the third integral in (2.23), note that for any point $p = (y_1, h)$ with $h \in I_3$, we have

$$(2.24) \qquad \begin{cases} u_\gamma(q) = u_{x_2}(q)(1 + \varepsilon_1), \\ u_{\gamma\gamma}(q) = u_{x_2 x_2}(q)(1 + \varepsilon_2) + o(u_{x_2}) \end{cases}$$

at the point $q = \mathcal{T}(p)$, for some small constants ε_1 and ε_2. The first formula is due to (2.20). To verify the second one in (2.24), one chooses a coordinate system (z_1, z_2) such that q is the origin, γ is in the z_2-axis and Γ_h is locally given by $z_2 = \eta(z_1)$. One then differentiates $u(z_1, \eta(z_1)) = h$ twice to obtain $u_{z_1 z_1} + u_{z_2}\kappa = 0$. Recall that when $p = (y_1, h)$ with $h \in I_3$, $\kappa \leq Ch_k^{-5/4}$. Hence $u_{z_1 z_1} = o(u_{z_2})$, from which one easily obtains (2.24).

Since for any $p \in \{y_1\} \times I_3$, $\frac{\sigma u_{\gamma\gamma}(q)}{\sigma + u_\gamma^2(q)} \geq \frac{1}{2}$. Therefore by (2.24) we have

$$\frac{\sigma u_{x_2 x_2}(q)}{\sigma + u_{x_2}^2(q)} \geq \frac{1}{3}.$$

Notice that $0 \leq \sigma \leq 1$. Hence

$$(2.25) \qquad u_{x_2 x_2} \geq \frac{1}{4}(\sigma + u_{x_2}^2) \geq \frac{1}{4} u_{x_2}^2.$$

Now by the relation $h = u(y_1, x_2)$ and $x_2 = g^+(y_1, h)$, we have $h = u(y_1, g^+(y_1, h))$. As y_1 is fixed, we can regard u and g^+ as functions of one variable. Differentiating in h gives $1 = u'(g^+)'$; differentiating twice we get $0 = u''(g^+)'^2 + u'(g^+)''$. Hence

$$(g^+)'' = -\frac{u''}{u'}(g^+)'^2 = -u''(g^+)'^3.$$

By (2.25) we then obtain

$$(2.26) \qquad (g^+)'' \leq -\varphi(h)(g^+)',$$

where $\varphi(h) = \frac{1}{4}$ if $h \in I_3$ and $\varphi(h) = 0$ otherwise. Observing that $(g^+)' > 0$, we obtain

$$\int_{h_k}^h \frac{(g^+)''}{(g^+)'} \leq -\int_{h_k}^h \varphi = -\frac{1}{4}|I_{3,h}| \quad \forall\, h \in (h_k, h_{k+1});$$

namely

$$\log(g^+)'(h) \leq \log(g^+)'(h_k) - \frac{1}{4}|I_{3,h}|,$$

or equivalently

$$(2.27) \qquad (g^+)'(h) \leq (g^+)'(h_k)e^{-|I_{3,h}|/4},$$

where $I_{3,h} = I_3 \cap [h_k, h]$. Since $I_3 = \cup_k(a_k, b_k)$ is the union of intervals, and g^+ is increasing, the third integral in (2.23) is equal to $\mathrm{osc}_{I_3}g^+ = \sum_k g^+(b_k) - g^+(a_k)$. We have

$$(2.28) \qquad \mathrm{osc}_{I_3}g^+ \leq (g^+)'(h_k)\int_{I_3} e^{-|I_{3,h}|/4}$$

$$\leq (g^+)'(h_k)\int_{h_k}^{h_{k+1}} e^{-(h-h_k)/4}$$

$$\leq 2(g^+)'(h_k) \leq \frac{C}{h_k}.$$

This completes the proof. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Next we prove an auxiliary lemma.

LEMMA 2.4. *Let u be a complete convex solution of* (1.2). *Suppose $n = 2$, $u(0) = 0$, $\delta := \inf\{|x| : x \in \Gamma_1\}$ is attained at $x_0 = (0, -\delta) \in \Gamma_1$, and $\delta > 0$ is sufficiently small. Then D_1 contains the interval $(-R, R)$ with*

$$(2.29) \qquad R \geq (-\log\delta - C)^{1/2},$$

where $C > 0$ is independent of δ and D_h is the set introduced in the proof of Lemma 2.2.

Proof. Suppose near x_0, Γ_1 is given by

$$x_2 = g(x_1).$$

Then g is a convex function, $g(0) = -\delta$, and $g'(0) = 0$. Let $a, b > 0$ be two constants such that $g(a) = 0$ and $g'(b) = 1$. To prove (2.29) it suffices to prove

$$(2.30) \qquad\qquad b \geq (-\log \delta - C)^{1/2}.$$

For any $y = (y_1, y_2) \in \Gamma_1$, where $y_1 \in [0, b]$, let $\xi = y/|y|$. By the convexity of u,

$$u_\xi(y) \geq \frac{u(y) - u(0)}{|y|} = \frac{1}{|y|}.$$

Let θ denote the angle between ξ and the tangential vector $\frac{1}{\sqrt{1+g'^2}}(1, g')$ of Γ_1 at y. Then

$$\cos \theta = \frac{\xi_1 + \xi_2 g'(y_1)}{\sqrt{1 + g'^2}}, \qquad \sin \theta = \sqrt{1 - \cos^2 \theta} = \frac{\xi_1 g' - \xi_2}{\sqrt{1 + g'^2}}.$$

Hence

$$u_\gamma(y) = u_\xi(y)/\sin \theta \geq \frac{\sqrt{1 + g'^2}}{y_1 g' - y_2},$$

where γ is the normal of the sublevel set $\Omega_1 = \{u < 1\}$. By $\mathcal{L}_0[u] \leq 1$, we obtain,

$$\frac{g''}{(1 + g'^2)^{3/2}} \frac{\sqrt{1 + g'^2}}{y_1 g' - y_2} \leq \kappa u_\gamma(y) \leq 1,$$

where κ is the curvature of the level set $\Gamma_1 = \{u = 1\}$. Hence

$$(2.31) \qquad g''(y_1) \leq (1 + g'^2)(y_1 g' - y_2)$$

$$\leq \begin{cases} 2(y_1 g' + \delta) & \text{if } y_2 \leq 0, \\ 2y_1 g' & \text{if } y_2 \geq 0, \end{cases}$$

where $y_2 = g(y_1)$ and $g'(y_1) \leq 1$ for $y_1 \in (0, b)$. We consider the equation

$$\rho''(t) = \begin{cases} 2(t\rho'(t) + \delta) & \text{if } \rho(t) \leq 0, \\ 2t\rho'(t) & \text{if } \rho(t) \geq 0 \end{cases}$$

with the initial condition $\rho(0) = -\delta$ and $\rho'(0) = 0$. Let $\alpha > 0$ be such that $\rho(\alpha) = 0$. Then for $t \in (0, \alpha)$ we have

$$\rho'(t) = 2\delta e^{t^2} \int_0^t e^{-s^2} ds.$$

Hence we have $C_1 \leq \alpha \leq C_2$ and $\rho'(\alpha) \leq C_2 \delta$ for some constants C_1, C_2. Let $\beta > \alpha$ be such that $\rho'(\beta) = 1$. Consider the equation

$$\rho'' = 2t\rho'$$

in the interval (α, β). Then from $\log \rho' \big|_\alpha^\beta = t^2 \big|_\alpha^\beta$, we obtain

$$\beta^2 \geq |\log \delta| - C.$$

By the comparison principle we have $g \leq \rho$. Hence (2.30) holds.   □

THEOREM 2.1. *Let u be an entire convex solution of* (1.2) *in* $\mathbf{R}^2$. *Then*

$$(2.32) \qquad\qquad u(x) \leq C(1 + |x|^2),$$

where the constant C depends only on the upper bound for $u(0)$ and $|Du(0)|$.

Proof. By adding a constant to u we may suppose $u(0) = 0$. To prove (2.32) it suffices to prove that $\operatorname{dist}(0, \Gamma_h) \geq Ch^{1/2}$ for all large h. By the rescaling $u_h(x) = \frac{1}{h} u(h^{1/2} x)$ it suffices to prove $\operatorname{dist}(0, \Gamma_{1, u_h}) \geq C$. Note that $|Du_h(0)| = h^{-1/2} |Du(0)| \leq |Du(0)|$. Hence by convexity, $\inf_{B_1(0)} u_h$ is uniformly bounded from below. Note also that u_h satisfies equation (1.2) with $\sigma \to 0$ as $h \to \infty$.

Denote $\delta =: \inf\{|x| : x \in \Gamma_{1, u_h}\}$. Suppose the infimum is attained at $x_0 = (0, -\delta)$. If $\delta > 0$ is sufficiently small, by Lemma 2.4, $D_1 = D_{1, u_h}$ contains the interval $(-R, R)$, where $R = (-\log \delta - C)^{1/2}$. Let $\delta^* > 0$ be such that $u_h(0, \delta^*) = 1$. Then δ^* must also be very small, for otherwise by convexity the ellipse

$$E = \left\{ (x_1, x_2) \in \mathbf{R}^2 : \frac{x_1^2}{(R/4)^2} + \frac{|x_2 - (\delta^* - \delta)/2|^2}{[(\delta^* + \delta)/8]^2} < 1 \right\}$$

is contained in sublevel set Ω_{1, u_h}.

When $\sigma = 0$, the level set Γ_{-t, u_h} is a solution to the curve-shortening flow (for time t starting at -1). Let E_{-t} be the solution to the curve-shortening flow with initial condition $E_{-1} = E$, where E is the ellipse given above. Therefore we have the inclusion $E_{-t} \subset \Omega_{-t, u_h}$ for all $t > -1$. It takes the time $T = |E|_{\mathcal{H}^2}$ for the solution E_{-t} to shrink to a point. Hence we have $\inf_{B_1(0)} u_h \leq 1 - T$. But when δ is small and δ^* has a positive lower bound (independent of δ), $T = |E|_{\mathcal{H}^2}$ becomes sufficiently large, which contradicts the assertion that $\inf_{B_1(0)} u_h$ is uniformly bounded from below.

When $\sigma \in (0, 1]$, u_h is a solution of (1.2) with $\sigma \leq 1/h$. If there is a sequence $h_k \to \infty$ and $\delta_k^* \geq \delta^*$ for some $\delta^* > 0$ such that $u_{h_k}(0, \delta_k^*) = 1$, we define E as above. Now let v_σ be the solution of $\mathcal{L}_\sigma(v) = 1$ in E and $v = 1$ on ∂E. Then for any given $\delta, \delta^* > 0$ and $R > 1$, the solution v_σ converges to v_0, the solution to $\mathcal{L}_0(v_0) = 1$ in E and $v_0 = 1$ on ∂E. The level set of v_0 is a solution to the curve-shortening flow. Hence $\inf v_\sigma \to -\infty$ as $\delta, \sigma \to 0$. We also reach a contradiction.   □

COROLLARY 2.1. *Let u be a complete convex solution of* (1.2). *Then u is either an entire solution, or is defined in a strip region. In particular, there is no complete convex solution of* (1.2) *defined in a half-space.*

Proof. Let us assume $u(0) = 0$. If u is not an entire solution, then for any $M > 1$, there exists $x_0 \in \mathbf{R}^n$ such that $u(x_0) > M|x_0|^2$. Let $u_h(x) = h^{-1}u(h^{1/2}x)$, where $h = u(x_0)$. Then the distance from the origin to the level set $\Gamma_1 = \{u_h = 1\}$ is less than M^{-1}. The proof of Theorem 2.1 then implies that u is defined in a strip region. $\square$

Note that in the above proof we have used the following lemma.

LEMMA 2.5. *Let u_k be a sequence of convex solutions of* (1.2) *with $\sigma = \sigma_k \in [0,1]$. Suppose $\sigma_k \to \sigma$ and $u_k \to u$. Then u is a convex solution of* (1.2).

Proof. Lemma 2.5 is well known if $\sigma_k \equiv 0$ or $\sigma_k \equiv 1$. If $\sigma_k \to \sigma > 0$, replacing u_k by $\frac{1}{\sigma_k}u(\sqrt{\sigma_k}x)$ we may suppose $\sigma_k \equiv 1$. We need only to consider the case when $\sigma_k \to 0$.

In this case we show that u is a viscosity solution of $\mathcal{L}_0[u] = 0$. Indeed, since $\mathcal{L}_{\sigma_k}[u_k] = 1$, by convexity we have $\mathcal{L}_0[u_k] \le 1$ and so $\mathcal{L}_0[u] \le 1$. On the other hand, for any fixed $\widehat{\sigma} > 0$, by convexity we have $\mathcal{L}_{\widehat{\sigma}}[u_k] \ge \mathcal{L}_{\sigma_k}[u_k] = 1$ if k is sufficiently large such that $\sigma_k < \widehat{\sigma}$. Hence $\mathcal{L}_{\widehat{\sigma}}[u] \ge 1$. As $\widehat{\sigma} > 0$ is arbitrary, we have $\mathcal{L}_0[u] \ge 1$. $\square$

Remark 2.1. When $\sigma = 0$ and u is a blow-up solution (limit flow) to a given mean convex flow, by a compactness argument, together with Lemma 2.4 and the proof of Theorem 2.1, one sees that (1.5) also follows from the noncollapsing in [22], [26], [27]. That is, if (1.5) is not true, there exists a sequence of blow-up solutions u_k to a given mean convex flow such that $w_k(x) := k^{-1}u_k(k^{1/2}x)$ converges to a multiplicity-two plane. But a multiplicity-two plane does not occur as a blow-up solution [22], [26], [27].

2.2. *Proof of* (1.5) *for $n > 2$*. In this subsection we extend the results in Section 2.1 to high dimensions.

Let u be a complete convex solution of (1.2). Let $\mathcal{M}_u$ denote the graph of u, and D the projection of $\mathcal{M}_u$ on the plane $\{x_n = 0\}$. We divide $\mathcal{M}_u$ into two parts, $\mathcal{M}_u = \mathcal{M}^+ \cup \mathcal{M}^-$, where $\mathcal{M}^\pm = \{(x, u(x)) \in \mathbf{R}^{n+1} : \partial_{x_n}u(x) \gtrless 0\}$. Then $\mathcal{M}^+$ and $\mathcal{M}^-$ can be represented respectively as graphs of the form $x_n = g^+(x', x_{n+1})$ and $x_n = g^-(x', x_{n+1})$, where $x' = (x_1, \dots, x_{n-1})$, $(x', x_{n+1}) \in D$. The functions g^+ and g^- are respectively concave and convex, and satisfy the relation $x_{n+1} = u(x', g^\pm(x', x_{n+1}))$. As before we denote $g = g^+ - g^-$. Then g is a positive, concave function in D, vanishing on ∂D.

For any $h > 0$ we also denote $g_h(x') = g(x', h)$, $g_h^\pm(x') = g^\pm(x', h)$, and $D_h = \{x' \in \mathbf{R}^{n-1} : (x', h) \in D\}$. Then g_h is a positive, concave function in D_h, vanishing on ∂D_h, and D_h is a convex domain in $\mathbf{R}^{n-1}$ containing the origin. Hence ∂D_h can be represented as a radial graph of a positive function a_h on S^{n-2}, $\partial D_h = \{p \cdot a_h(p) : p \in S^{n-2}\}$, where

$$a_h(p) = \sup\{t : tp \in D_h\}, \quad p \in S^{n-2}.$$

Denote

$$\bar{a}_h = \inf\{a_h(p) : p \in S^{n-2}\}, \qquad \bar{b}_h = g_h(0).$$

We want to extend Lemma 2.3 to high dimensions, that is if $\bar{b}_1(0)$ is small, then u is defined in a strip region. First we prove a lemma which corresponds to Claim 1 in the proof of Lemmas 2.1 and 2.2.

LEMMA 2.6. *Let u be a complete convex solution of* (1.2) *satisfying $u(0)$ $= 0$. Suppose $\bar{a}_h \geq \bar{b}_h$. Then there is a positive constant C_n, depending only on n, such that*

(2.33)
$$\bar{a}_h\bar{b}_h \geq C_n h.$$

Proof. When $n = 2$, (2.33) was proved in Claim 1 in Lemmas 2.1 and 2.2. When $n \geq 3$, we reduce (2.33) to the case $n = 2$.

Assume that $\bar{a}_h = a_h(p)$ for $p = (1, 0, \ldots, 0)$. Observing that $\bar{a}_h\bar{b}_h$ is proportional to the area of the section $\{x \in \Omega_h : x_1 > 0, x_2 = \cdots = x_{n-1} = 0\}$, we can prove (2.33) by making a rotation of coordinates. For a given $h > 0$, by a rotation of the coordinates we assume that $\inf\{|x| : x \in \Gamma_{h,u}\}$ is attained at b^*e_n, where $b^* \in (0, \bar{b}_h]$ and e_k is the unit vector in the x_k-axis, $k = 1, \ldots, n$. Then it suffices to prove (2.33) for $\bar{a}_h, \bar{b}_h$ defined in this new coordinate system. Since $\bar{a}_h \geq \bar{b}_h$, by the convexity of $\Gamma_{h,u}$ we have

(2.34)
$$\Gamma_{h,u} \cap \{|x| < \bar{a}_h\} \subset \{|x_n| \leq 2\bar{b}_h\}.$$

Let $\hat{u}$ be the restriction of u on the 2-plane spanned by the x_1 and x_n axes. From the proof for the case $n = 2$ in Lemma 2.1, we see that (2.33) holds if one can verify that $\mathcal{L}_0[\hat{u}] \leq C$ for some constant C depending only on n, where $\mathcal{L}_0$ is the operator in (2.2).

For any given point $y = (y_1, 0, \ldots, 0, y_n) \in \Gamma_{h,\hat{u}}$, let κ be the mean curvature of $\Gamma_{h,u}$, and $\hat{\kappa}$ be the curvature of $\Gamma_{h,\hat{u}}$ at y. Let $\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_n)$ be the unit normal of $\Gamma_{h,u}$ at y, and $\hat{\gamma} = (\hat{\gamma}_1, 0, \ldots, 0, \hat{\gamma}_n)$ be the unit normal of $\Gamma_{h,\hat{u}}$ at y in the 2-plane spanned by the x_1 and x_n axes. Suppose for a moment that

(2.35)
$$\gamma \cdot \hat{\gamma} = \gamma_1\hat{\gamma}_1 + \gamma_n\hat{\gamma}_n \geq C_1$$

for some positive constant C_1. Then by the convexity of $\Gamma_{h,u}$ we have $\hat{\kappa} \leq C_2\kappa$ and $\hat{u}_{\hat{\gamma}} \leq C_2 u_\gamma$. By (2.2), we have $\kappa u_\gamma \leq 1$. Hence $\mathcal{L}_0[\hat{u}] = \hat{\kappa}\hat{u}_{\hat{\gamma}} \leq C$ and so (2.33) holds.

Let $P = \{x \in \mathbf{R}^n : \gamma \cdot (x - y) = 0\}$ be the tangent plane of $\Gamma_{h,u}$ at the point y. Let $p_k = z_k e_k$, $k = 1, \ldots, n$, be the intersection of P with the x_k-axis. Then by $\gamma \cdot (x - y) = 0$ at $x = p_k$, we have

$$\gamma_k z_k = \gamma_1 y_1 + \gamma_n y_n \quad \forall \ k = 2, \ldots, n-1.$$

Hence if for all $k = 2, \ldots, n-1$, $|z_k| \geq C\sqrt{y_1^2 + y_n^2}$, we have $|\gamma_k| \leq \sqrt{\gamma_1^2 + \gamma_n^2}$, which implies $\sqrt{\gamma_1^2 + \gamma_n^2} \geq C_1 > 0$ as γ is a unit vector. Observe that the vector

$(\gamma_1, 0, \ldots, 0, \gamma_n)$ is parallel to the unit vector $\hat{\gamma} = (\hat{\gamma}_1, 0, \ldots, 0, \hat{\gamma}_n)$. Hence we obtain (2.35).

To prove $|z_k| \geq C\sqrt{y_1^2 + y_n^2}$, notice that γ and $\hat{\gamma}$ are invariant if we translate the level set $\Gamma_{h,u}$. Without loss of generality let us assume that $\gamma_n > 0$. The case $\gamma_n < 0$ can be treated similarly. We translate $\Gamma_{h,u}$ in the x_n-direction by a distance $2\bar{b}_h$, so that $\Gamma_{h,u} \cap \{|x| < \bar{a}_h\}$ is contained in $\{x_n > 0\}$. Since P is a tangent plane of $\Gamma_{h,u}$ lying above the set $\Gamma_{h,u}$, we must have $|z_k| \geq \bar{a}_h$. On the other hand, $|y_1| \leq \bar{a}_h$ and $|y_n| \leq 4\bar{b}_h$ (after the translation). Hence by the assumption $\bar{b}_h \leq \bar{a}_h$, we have $|z_k| \geq \frac{1}{5}\sqrt{y_1^2 + y_n^2}$. $\square$

LEMMA 2.7. *Let u be a complete convex solution of* (1.2). *Suppose $u(0){=}0$ and $u(\beta e_n) \geq 1$, $u(-\beta e_n) \geq 1$ for some sufficiently small $\beta > 0$, where $e_n = (0, \ldots, 0, 1)$. Then u is defined in a strip region.*

Note that the level set $\Gamma_{h,u} = \{u = h\}$ may not be compact. Note also that the strip region in Lemma 2.7 may not take the form $\{x \in \mathbf{R}^n : -C_1 \leq x_n \leq C_2\}$, except in some special cases such as when u is symmetric in x_n. But as in Lemmas 2.2 and 2.3, the axes of the minimum ellipsoid of $\Omega_h \cap B_R(0)$ is a small perturbation of axes of the coordinates.

To prove Lemma 2.7 we will prove that the graph of g, $\mathcal{M}_g = \{(x, x_{n+1}) : x_n = g(x', x_{n+1}), (x', x_{n+1}) \in D\}$, is contained in a strip $\{(x, x_{n+1}) \in \mathbf{R}^{n+1} : 0 \leq x_n \leq C\}$. By convexity it suffices to prove $\bar{b}_h = g_h(0)$ is uniformly bounded. The idea of our proof is very similar to the two-dimensional case given in Section 2.1, where we divided the proof into three lemmas. Here we present it in a single lemma.

Proof of Lemma 2.7. Let $R = 10^3$ and let E be the minimum ellipsoid of $\Omega_1 \cap B_R(0)$. By a rotation of coordinates we assume the axial directions of E coincide with those of the coordinate system.

Denote $h_k = 2^k$, $\bar{a}_k = \bar{a}_{h_k}$, $\bar{b}_k = \bar{b}_{h_k}$, $g_k = g_{h_k}$, and $D_k = D_{h_k}$. As in Section 2.1 we use an induction argument to prove

$$(2.36) \qquad\qquad g_k(0) \leq g_{k-1}(0) + 2^{-k/4n} \quad \forall \ k \geq 1.$$

As shown in Section 2.1, (2.36) implies that u is defined in a strip region.

The proof of (2.36) is similar to (2.3), we point out the difference here. As in Section 2.1, when β is sufficiently small, by convexity we have $\bar{b}_k \leq h_k \bar{b}_0 \leq 2^k \beta \leq 1$ when $k \leq k_0$ and our induction argument starts at $k = k_0$.

Suppose by induction that (2.36) holds up to k. By the induction assumption, $g_k(0) \leq g_{k_0}(0) + \sum_{j=k_0}^{k} 2^{-j/4n} \leq 2$. By the concavity, $\bar{b}_{k+1} = g_{k+1}(0) \leq 2g_k(0) \leq 4$. Hence by Lemma 2.6 we have

$$(2.37) \qquad\qquad\qquad \bar{a}_{k+1} \geq C_0 h_{k+1}.$$

Next we prove (2.36) at $k + 1$. Rotate the axes such that $\partial_i g_k(0) \leq 0$ for all $i = 1, \ldots, n - 1$. By the concavity of g we have

(2.38) $$g_k(0) = \sup\{g_k(x') : x_1 > 0, \ldots, x_{n-1} > 0\}.$$

Denote

$$L_k = \left\{x' \in \mathbf{R}^{n-1} : -\frac{C_0}{2n}h_k < x_i < \frac{C_0}{2n}h_k, i = 1, \ldots, n - 1\right\},$$

and $Q_k = L_k \times [h_k, h_{k+1}] \subset D$, where C_0 is the constant in (2.37). Then similarly to (2.6),

$$\sup\{g(x', h) : (x', h) \in Q_k\} \leq 2\sup\{g(x', h_k) : x' \in L_k\} \leq 4g(0, h_k) \leq 8.$$

Observing that $2L_k \subset D_k$, by the convexity of u we have $L_k \subset D_{k-1}$. Hence by the concavity of g and (2.38) we have, for any $(x', h) \in Q_k$,

(2.39) $$|\partial_h g(x', h)| \leq \frac{g(x', h) - g(x', h_{k-1})}{h - h_{k-1}}$$
$$\leq \frac{g(x', h)}{h_k - h_{k-1}} \leq \frac{2g(x', h_k)}{h_k - h_{k-1}} \leq \frac{2g(0, h_k)}{h_k - h_{k-1}} \leq \frac{8}{h_k}.$$

By (2.37) and (2.38), the concavity of g, and since $g \geq 0$, we also have,

(2.40) $$|D_{x'}g(x', h)| \leq C/h_k \quad \forall \, (x', h) \in Q_k.$$

From the above gradient estimates and the concavity of g, the average in Q_k of the second order derivatives $|\partial^2 g| \leq Ch_k^{-2} \leq h_k^{-5/4}$. But we have not proved this estimate pointwise. We need to treat the sets $\{|\partial^2 g| < h_k^{-5/4}\}$ and $\{|\partial^2 g| \geq h_k^{-5/4}\}$ separately. Denote

(2.41) $$\chi = \{(x', h) \in Q_k : |\Sigma_{i=1}^{n-1}\partial_i^2 g_h(x')| \geq h_k^{-5/4}\},$$
$$\chi^+ = \{(x', h) \in Q_k : |\Sigma_{i=1}^{n-1}\partial_i^2 g_h^+(x')| \geq h_k^{-5/4}\}.$$

Obviously $\chi^+ \subset \chi$. By the gradient estimates, we have

$$|\chi|_{\mathcal{H}^n} h_k^{-5/4} \leq \left|\int_{Q_k} \Delta_{x'}g\right| \leq \int_{\partial L_k \times [h_k, h_{k+1}]} |D_{x'}g| \leq Ch_k^{n-2}.$$

In the above formula g is a function of (x', h). Hence we obtain

(2.42) $$|\chi^+|_{\mathcal{H}^n} \leq |\chi|_{\mathcal{H}^n} \leq Ch_k^{n-3/4}.$$

From (2.42) and by the Fubini Theorem, there is a set $\widetilde{L} \subset L_k$ with measure $|\widetilde{L}|_{\mathcal{H}^{n-1}} < h_k^{n-3/2}$ such that for any $y' \in L_k - \widetilde{L}$,

(2.43) $$|\chi_{y'}^+|_{\mathcal{H}^1} \leq Ch_k^{3/4},$$

where $\chi_{y'}^+ = \chi^+ \cap \{x' = y'\}$.

For any given $y' \in L_k - \widetilde{L}$, we want to prove

(2.44) $$g_{k+1}^+(y') - g_k^+(y') \leq Ch_k^{-1/4}.$$

Similarly we can estimate $|g_{k+1}^-(y') - g_k^-(y')|$. Hence if (2.44) is proved, we have

$$g_{k+1}(y') - g_k(y') \leq Ch_k^{-1/4},$$

which corresponds to (2.14). As in the argument after (2.14), we can choose a point $y_1 \in L_k - \widetilde{L}$ with $|y_1| \leq Ch_k^{1-1/2(n-1)}$ such that $g_{k+1}(y_1) \leq g_k(0) + Ch_k^{-1/4}$. But now

$$\frac{g_{k+1}(0)}{g_{k+1}(y_1)} \leq \frac{\overline{a}_{k+1}}{\overline{a}_{k+1} - |y_1|} \leq 1 + Ch_k^{-1/2(n-1)}.$$

Therefore we obtain (2.36).

To prove (2.44), we have

(2.45)
$$g_{k+1}^+(y') - g_k^+(y') = \int_I \partial_h g^+(y', h) dh = \left(\int_{I_1} + \int_{I_2} + \int_{I_3} \right) \partial_h g^+(y', h) dh,$$

where as in (2.23), $I = (h_k, h_{k+1})$, $I_1 = \chi^+ \cap \{x' = y'\}$, $I_2 = \{h \in I - I_1 : \frac{\sigma u_{\gamma\gamma}(q)}{\sigma + u_\gamma^2(q)} \leq \frac{1}{2}\}$, and $I_3 = I - (I_1 \cup I_2)$, where $q = \mathcal{T}(p)$ with $p = (y', h)$, and $\mathcal{T} : p \to q$ is the mapping introduced before (2.20).

For the first integral in (2.45), by (2.39) and (2.43) we have

$$\int_{I_1} \partial_h g^+(y', h) dh \leq Ch_k^{-1/4}.$$

Note that in I_2, similarly to (2.22), we have

$$|\partial_h g^+(y_1, h)| \leq C|\partial_{x_1 x_1} g^+| \leq Ch_k^{-5/4} \quad \forall (y_1, h) \notin \chi^+.$$

Hence we have the estimate for the second integral in (2.45),

$$\int_{I_2} \partial_h g^+(y', h) dh \leq Ch_k^{-1/4}.$$

For the third one, the argument between (2.24) and (2.28) applies and we also have the estimate $\operatorname{osc}_{I_3} g^+ \leq C/h_k$. Hence (2.44) holds. $\qquad \square$

The next lemma corresponds to Lemma 2.4 in Section 2.1.

LEMMA 2.8. *Let u be a complete convex solution of* (1.2). *Suppose $u(0)=0$ and the infimum $\inf\{|x| : x \in \Gamma_1\}$ is attained at $x_0 = (0, \ldots, 0, -\delta) \in \Gamma_1$ for some $\delta > 0$ sufficiently small. As above let D_1 be the projection of Γ_1 on the plane $\mathbf{R}^n \cap \{x_n = 0\}$. Then $D_1 \supset \{x' \in \mathbf{R}^{n-1} : |x'| < R\}$ with*

(2.46) $$R \geq \frac{1}{C_n}(-\log \delta - C)^{1/2},$$

where C_n is a constant depending only on n, and $C > 0$ is a constant independent of δ.

Proof. Estimate (2.46) is equivalent to $a_1(p) \geq \frac{1}{C_n}(-\log \delta - C)^{1/2}$ for any $p \in S^{n-2}$. Suppose $\inf a_1(p)$ is attained at $p = (1, 0, \ldots, 0)$. By restricting u to the 2-plane $\{x_2 = \cdots = x_{n-1} = 0\}$, we reduce the proof to the two-dimensional case in Lemma 2.4, as we have shown, in the proof of Lemma 2.6, that $\mathcal{L}_0[u] \leq C_n$. $\qquad\square$

With Lemmas 2.7 and 2.8, we extend Theorem 2.1 to high dimensions.

THEOREM 2.2. *Let u be an entire convex solution of* (1.2) *in* $\mathbf{R}^n$. *Then there exists a positive constant C such that for any $x \in \mathbf{R}^n$,*

(2.47)
$$u(x) \leq C(1 + |x|^2),$$

where C depends on n and the upper bound of $u(0)$ and $|Du(0)|$.

Proof. The proof is very similar to that of Theorem 2.1. Let δ, δ^* and u_h be as in the proof of Theorem 2.1. Instead of an ellipse, here we use the ellipsoid

$$E = \left\{ x \in \mathbf{R}^2 : \sum\nolimits_{i=1}^{n-1} \frac{x_i^2}{(R/2n)^2} + \frac{|x_2 - (\delta^* - \delta)/2|^2}{[(\delta^* + \delta)/8]^2} < 1 \right\}.$$

When $\sigma = 0$, the level set Γ_{-t,u_h} is a solution to the mean curvature flow. Let E_{-t} be the solution to the mean curvature flow with initial condition $E_{-1} = E$, so that $E_{-t} \subset \Omega_{-t,u_h}$ for all $t > -1$. Suppose it takes time T for E_{-t} to shrink to a point. Then we have $\inf_{B_1(0)} u_h \leq 1 - T$. Observe that for any fixed δ^*, E_{-t} converges to a pair of parallel planes, and so $T \to \infty$ as $R \to \infty$ (or $\delta \to 0$). Hence when δ is small, we reach a contradiction with the assertion that $\inf_{B_1(0)} u_h$ is uniformly bounded from below. The case $\sigma > 0$ can be proved in the same way as in Theorem 2.1. $\qquad\square$

From Theorem 2.2, we have accordingly

COROLLARY 2.2. *Let u be a complete convex solution of* (1.2). *Then u is either an entire solution, or is defined in a strip region. There is no complete convex solution of* (1.2) *defined in a half-space.*

Note that estimate (2.47) also implies the follows compactness result. This compactness result is not just for the set of blow-up solutions to mean convex flow but for all entire convex solutions of (1.2). We do not know whether an entire convex solution to (1.2) must be a blow-up solution to mean convex flow.

COROLLARY 2.3. *For any constant $C > 0$, the set of all entire convex solutions u to* (1.2) *satisfying $u(0) = 0$ and $|Du(0)| \leq C$ is compact.*

2.3. *Proof of Theorem 1.3.* First we prove a lemma.

LEMMA 2.9. *Let u be an entire convex solution of (1.2). Suppose $u \geq 0$ and $u(0) = 0$. Then the convex set $\{u = 0\}$ is either a single point or it is a linear subspace of $\mathbf{R}^n$.*

Proof. If $\sigma > 0$, u is analytic. As the set $\{u = 0\}$ is convex, it must be a single point or a linear subspace of $\mathbf{R}^n$. In the following we consider the case $\sigma = 0$.

If the set $\{u = 0\}$ is bounded, then $\Gamma_{h,u}$ is a closed, bounded convex hypersurface. As $\Gamma_{h,u}$ evolves by mean curvature (with time $t = -h$), from [7], [9] it follows that $\{u = 0\}$ is a single point.

If the set $\{u = 0\}$ contains a straight line, say the line $\ell = (t, 0, \ldots, 0)$ $(t \in \mathbf{R})$, then by convexity u is independent of x_1. Hence to prove Lemma 2.9, we need only to rule out the possibility that $\{u = 0\}$ contains a ray but no straight line lies in it.

Suppose the ray $r = (t, 0, \ldots, 0)$ $(t > 0)$ is contained in $\{u = 0\}$. We may also suppose that $\{u = 0\}$ contains no straight lines and the asymptotical cone of $\{u = 0\}$ is contained in $\{x_1 > 0\}$. Then u is decreasing in x_1. Denote $u_m(x_1, x_2, \ldots, x_n) = u(x_1 + m, x_2, \ldots, x_n)$, where $m > 0$ is a constant. Then u_m is nonnegative and decreasing in m. By choosing a subsequence we suppose $u_m \to \hat{u}$ as $m \to \infty$. Then the straight line $\ell = (t, 0, \ldots, 0)$ $(t \in \mathbf{R})$ is contained in the graph of $\hat{u}$. By convexity, $\hat{u}$ is independent of x_1. Since $\mathcal{L}_0[\hat{u}] = 1$, $\hat{u}$ does not vanish completely, and so we must have $n \geq 3$. Moreover, we have $\hat{u} < u$ except on the set $\{u = \hat{u} = 0\}$.

Since u and $\hat{u}$ are both solutions to $\mathcal{L}_0[u] = 1$, the level sets $\{u = -t\}$ and $\{\hat{u} = -t\}$ evolve by mean curvature (with time t). Denote $\mathcal{M}_t = \{u = -t\}$ $\cap\{x_1 = 0\}$ and $\widehat{\mathcal{M}}_t = \{\hat{u} = -t\} \cap \{x_1 = 0\}$. Then $\hat{\mathcal{M}}_t$ evolves by mean curvature as $\hat{u}$ is independent of x_1. We assert that $\mathcal{M}_t$ evolves at a velocity greater than its mean curvature. Indeed, for any given point $p \in \mathcal{M}_t$, we assume the hypersurface $\{u = -t\}$ is locally given by $x_n = \psi(x_1, \ldots, x_{n-1})$, and locally $\mathcal{M}_t$ is given by $x_n = \psi(0, x_2, \ldots, x_{n-1})$. By choosing the coordinate system properly we also assume that $\partial_{x_i}\psi = 0$ for $i = 2, \ldots, n-1$ at p. Then $\mathcal{M}_t$ evolves at the velocity $\sqrt{1 + |D\psi|^2} \operatorname{div} \frac{D\psi}{\sqrt{1+|D\psi|^2}}$, by convexity which is greater than $\sum_{i=2}^{n-1} \partial_{x_i}^2 \psi$, the mean curvature of $\mathcal{M}_t$ at p.

On the other hand, since $\hat{u} < u$, $\mathcal{M}_t$ is strictly contained in the interior of $\widehat{\mathcal{M}}_t$ for any $t < 0$. Moreover $\mathcal{M}_t$ is a bounded, closed convex hypersurface, as the asymptotical cone of $\{u = 0\}$ is contained in $\{x_1 > 0\}$. By the comparison principle, $\mathcal{M}_t$ is strictly contained in the interior of $\widehat{\mathcal{M}}_t$ for all $t \leq 0$. We reach a contradiction as $\hat{u} = u = 0$ at the origin. $\square$

Therefore [9, Lemma 2.9] implies that the singularity set of a mean curvature flow of convex, noncompact hypersurfaces in $\mathbf{R}^{n+1}$ must be a subspace $\mathbf{R}^{n-k}$ for some $1 \leq k \leq n$. But then by convexity, u is a function of k variables.

Proof of Theorem 1.3. *Step* 1. First we prove that there is a subsequence of u_h, where $u_h(x) = h^{-1}u(h^{1/2}x)$, which converges to η_k for some $2 \le k \le n$, where η_k is the function given in (1.4).

By adding a constant we may suppose $u(0) = 0$. Let $T = \{x_{n+1} = a(x)\}$ be the tangent plane of u at the origin. By Theorems 2.1 and 2.2 and the convexity of u we have

$$a(x) \le u(x) \le C(1 + |x|^2).$$

Hence

$$\frac{1}{\sqrt{h}}a(x) \le u_h(x) \le C\left(\frac{1}{\sqrt{h}} + |x|^2\right).$$

By convexity it follows that Du_h is locally uniformly bounded. Hence u_h subconverges to a convex function u_0 which satisfies $u_0(0) = 0$,

$$(2.48) \qquad\qquad 0 \le u_0(x) \le C|x|^2.$$

By Lemma 2.5, u_0 is an entire convex solution of $\mathcal{L}_0[u] = 1$.

Case 1. the set $\{x \in \mathbf{R}^n : u_0(x) = 0\}$ is bounded. Then by convexity the level set $\Gamma_{1,u_0} = \{x \in \mathbf{R}^n : u_0(x) = 1\}$ is a bounded convex hypersurface. Since the level set $\{u_0 = -t\}$, with time $t \in (-\infty, 0)$, evolves by mean curvature, by the asymptotic estimates in [7] and [9],

$$(2.49) \qquad\qquad u_0(x) = \frac{1}{2(n-1)}|x|^2 + \varphi(x),$$

where $\varphi(x) = o(|x|^2)$ for $x \ne 0$ near the origin. Hence for any $\varepsilon > 0$, there is a sufficiently small $h' > 0$, such that

$$B_{(1-\varepsilon)r}(0) \subset \Omega_{h',u_0} \subset B_{(1+\varepsilon)r}(0),$$

where $r = \sqrt{2(n-1)h'}$. Hence there is a sequence $h_m \to \infty$ such that

$$(2.50) \qquad\qquad B_{(1-\frac{1}{m})r_m}(0) \subset \Omega_{h_m,u} \subset B_{(1+\frac{1}{m})r_m}(0),$$

where $r_m = \sqrt{2(n-1)h_m}$. Let $u_{h_m}(x) = \frac{1}{h_m}u(\sqrt{h_m}\,x)$. Then u_{h_m} subconverges to $\hat{u}_0$ which satisfies $\mathcal{L}_0[\hat{u}_0] = 1$. From (2.50), the level set $\Gamma_{1,\hat{u}_0}$ is a sphere. Hence $\hat{u}_0(x) = \frac{1}{2(n-1)}|x|^2$.

Case 2. The set $\{u_0 = 0\}$ is unbounded. Then by Lemma 2.9, the set $\{u_0 = 0\}$ is a linear subspace of $\mathbf{R}^n$. Suppose

$$\{u_0 = 0\} = \{x \in \mathbf{R}^n : x_{k+1} = \cdots = x_n = 0\}.$$

We must have $k \ge 2$, as the level set $\{u_0 = -t\}$ evolves by its mean curvature. It follows that u_0 is a convex function depending only on $\hat{x} = (x_1, \ldots, x_k)$. Similarly, as above,

$$(2.51) \qquad\qquad u_0(x) = \frac{1}{2(k-1)}|\hat{x}|^2 + o(|\hat{x}|^2)$$

near the origin. Hence for any $\varepsilon > 0$,

$$\widehat{B}_{(1-\varepsilon)r}(0) \subset \widehat{\Omega}_{h',u_0} \subset \widehat{B}_{(1+\varepsilon)r}(0)$$

provided h' is sufficiently small, where $r = \sqrt{2(k-1)h'}$, $\widehat{B}_r(0) = B_r(0) \cap \{\widetilde{x} = 0\}$ and $\widehat{\Omega}_{h',u} = \Omega_{h',u} \cap \{\widetilde{x} = 0\}$, $\widetilde{x} = (x_{k+1}, \ldots, x_n)$. It follows that for any $R > 0$,

$$\{x \in \mathbf{R}^n : |\hat{x}| < (1-\varepsilon)r\} \cap \{|\widetilde{x}| < R\} \subset \Omega_{h',u_{h_m}} \cap \{|\widetilde{x}| < R\}$$
$$\subset \{x \in \mathbf{R}^n : |\hat{x}| < (1+\varepsilon)r\} \cap \{|\widetilde{x}| < R\}$$

if h_m is sufficiently large. Hence there exist $\tau_m \to \infty$ and (a different sequence) $h_m \to \infty$ such that

(2.52) $$\left\{x \in \mathbf{R}^n : |\hat{x}| < \left(1 - \frac{1}{m}\right)r_m\right\}$$
$$\cap \{|\widetilde{x}| < \tau_m r_m\} \subset \Omega_{h_m,u} \cap \{|\widetilde{x}| < \tau_m r_m\}$$
$$\subset \left\{x \in \mathbf{R}^n : |\hat{x}| < \left(1 + \frac{1}{m}\right)r_m\right\} \cap \{|\widetilde{x}| < \tau_m r_m\},$$

where $r_m = \sqrt{2(k-1)h_m}$. Hence $u_{h_m} \to \frac{1}{2(k-1)}|\hat{x}|^2$.

Step 2. Now we prove that u_h itself, after a rotation of axes, converges to the function η_k.

In Step 1 we proved that u_{h_m} converges to η_k for some $2 \leq k \leq n$. Let us choose the sequence $\{h_m\}$ properly such that k is the largest such integer, namely if $u_{h'_m}$ converges to $\eta_{k'}$, then $k' \leq k$. From the above proof we can also choose h_m such that (2.50) or (2.52) holds.

Case 1: $k = n$. We prove that for any constant $\varepsilon > 0$,

(2.53) $$B_{(1-\varepsilon)r}(0) \subset \Omega_{h,u} \subset B_{(1+\varepsilon)r}(0)$$

if $h > 0$ is sufficiently large, where $r = \sqrt{2(n-1)h}$. Suppose (2.53) is not true. Let $h_m \to \infty$ be a sequence such that (2.50) holds. Let

(2.54) $$\hat{h}_m = \inf\{h' \leq h_m : \ (2.53) \text{ holds for any } h \in (h', h_m)\}.$$

Since $u_{h_m} \to \frac{1}{2(n-1)}|x|^2$, we have $h_m/\hat{h}_m \to \infty$ as $m \to \infty$. Let $\alpha > 1$ be a fixed constant which will be determined below. Then the sequence $u_{\alpha \hat{h}_m}$ sub-converges to a convex function u_0 satisfying $u_0(0) = 0$, $u_0 \geq 0$ and $\mathcal{L}_0[u] = 1$. By our choice of $\hat{h}_m$, the level set Ω_{1,u_0} satisfies

$$B_{(1-\varepsilon)r}(0) \subset \Omega_{1,u_0} \subset B_{(1+\varepsilon)r}(0)$$

with $r = \sqrt{2(n-1)}$. Hence from [7], [9], the level set Ω_{h,u_0} satisfies

$$B_{(1-\delta)r}(0) \subset \Omega_{h,u_0} \subset B_{(1+\delta)r}(0)$$

with $\delta \to 0$ as $h \to 0$, where $r = \sqrt{2(n-1)h}$. Hence we have

$$B_{(1-2\delta)r}(0) \subset \Omega_{h,u_{\alpha\hat{h}_m}} \subset B_{(1+2\delta)r}(0)$$

if m is sufficiently large. Choose h sufficiently small such that $\delta \leq \frac{1}{3}\varepsilon$ and let $\alpha = h^{-1}$. Then scaling back we find that $\Omega_{\hat{h}_m,u}$ satisfies

$$B_{(1-2\delta)r_m}(0) \subset \Omega_{\hat{h}_m,u} \subset B_{(1+2\delta)r_m}(0)$$

with $r = \sqrt{2(n-1)\hat{h}_m}$. When $\delta < \frac{1}{2}\varepsilon$, this is in contradiction with our choice of $\hat{h}_m$.

Case 2. $k < n$. For any given small $\varepsilon > 0$, by (2.52), $\Gamma_{h_m,u}$ is ε-close to the cylinder $S^{k-1} \times R^{n-k}$ if m is sufficiently large, namely

$$(2.55) \quad \{x \in \mathbf{R}^n : |\hat{x}| < (1-\varepsilon)r\} \cap \{|\tilde{x}| < \varepsilon^{-1}r\} \subset \Omega_{h_m,u} \cap \{|\tilde{x}| < \varepsilon^{-1}r\}$$

$$\subset \{x \in \mathbf{R}^n : |\hat{x}| < (1+\varepsilon)r\} \cap \{|\tilde{x}| < \varepsilon^{-1}r\},$$

where $r = \sqrt{2(k-1)h_m}$. Let $\hat{h}_m < h_m$ be the least number such that $\Gamma_{h,u}$ is ε-close to the cylinder $S^{k-1} \times R^{n-k}$ (the axes of the cylinder may vary as h varies) for any $h \in [\hat{h}_m, h_m]$. Then by our assumption that k is the largest possible integer, we have, due to (2.51), that $u_{\alpha\hat{h}_m} = \frac{1}{2(k-1)}|\hat{x}|^2 + o(|\hat{x}|^2)$ for any given $\alpha > 1$. Here we regard $u_{\alpha\hat{h}_m}$ as a function of $\hat{x} = (x_1, \ldots, x_k)$ by letting $x_{k+1} = \cdots = x_n = 0$. Similar to Case 1, we can choose $\alpha > 1$ such that $\Gamma_{\hat{h}_m,u}$ is $\frac{1}{2}\varepsilon$-close to the cylinder $S^{k-1} \times R^{n-k}$, which is a contradiction to our choice of $\hat{h}_m$. Hence Theorem 1.3 is proved. $\qquad\square$

Note that Case 2 in Step 2 follows readily from Step 1 and the fact that k is an integer. In Step 1 it is shown that u_h converges along a subsequence to the function η_k but k is an integer so it must be the same integer for all subsequences.

Remark 2.2. Theorem 1.3 asserts that u_h, which is the blow-down of u with respect to the origin in space-time, subconverges to a self-similar solution. We point out that for ancient convex solutions w to the level set flow (1.2) (with $\sigma = 0$), under some very mild conditions the corresponding level set $\{w = h\}$, after proper translation, subconverges as $h \to \infty$ to a translating solution. In particular, if w is a complete convex solution of (1.2) defined in a strip region in $\mathbf{R}^2$, then after proper translation, the level set must converge along a subsequence to the grim reaper.

3. **Proof of Theorem 1.1**

In this section we prove Theorem 1.1 and throughout we suppose the dimension $n = 2$.

Let u be an entire convex solution of (1.2). By Theorem 1.3 we have

$$(3.1) \qquad u(x) = \frac{1}{2}|x|^2 + \varphi(x)$$

with $|\varphi(x)| = o(|x|^2)$ as $|x| \to \infty$. To prove Theorem 1.1, we first consider the case $\sigma = 0$.

THEOREM 3.1. *Let u be an entire convex solution of (1.2) with $\sigma = 0$. Then $u(x) = \frac{1}{2}|x|^2$ in a proper coordinate system.*

Proof. By a translation of the graph of u, we may suppose $u \geq 0$, $u(0) = 0$, and (3.1) holds. For any constant $h > 1$, denote $u_h(y) = u(h^{1/2}y)/h$. Then u_h is also an entire convex solution of (1.2) and by (3.1), the sublevel set $\Omega_{1/2,u_h}$ satisfies

$$(3.2) \qquad B_{1-\varepsilon}(0) \subset \Omega_{1/2,u_h} \subset B_{1+\varepsilon}(0)$$

with $\varepsilon \to 0$ as $h \to \infty$. By Gage-Hamilton [7], we have

$$(3.3) \qquad u_h(y) = \frac{1}{2}|y|^2 + \varphi(y)$$

with

$$|\varphi(y)| \leq C|y|^{2+\alpha}$$

for some $\alpha \in (0,1)$, and C is a constant independent of h. Rescaling back to the x-coordinate we obtain

$$u(x) = \frac{1}{2}|x|^2 + h^2\varphi(x/h),$$

where for any fixed x, $h^2\varphi(x/h) \to 0$ as $h \to \infty$. Hence $u(x) \equiv \frac{1}{2}|x|^2$. $\square$

Remark 3.1. . By the asymptotic estimates in [9], Theorem 3.1 also holds in high dimensions if the solution u satisfies

$$(3.4) \qquad C_1|x|^2 \leq u(x) \leq C_2|x|^2.$$

Indeed, if u satisfies (3.4), we have $u(x) = \frac{1}{2(n-1)}|x|^2 + o(|x|^2)$ by Theorem 1.3. Next we consider the case $\sigma = 1$ of Theorem 1.1.

THEOREM 3.2. *Let u be an entire convex solution of the mean curvature equation (1.1). Then u is rotationally symmetric in an appropriate coordinate system.*

To prove Theorem 3.2 we need a few lemmas.

LEMMA 3.1. *Let Ω be a bounded convex domain in $\mathbf{R}^2$. Let u_0 and u_σ be respectively solutions of $\mathcal{L}_0[u] = 1$ and $\mathcal{L}_\sigma[u] = 1$ in Ω, vanishing on $\partial\Omega$, where $\sigma \in (0,1]$. Suppose u_σ is convex. Then for any constant $a > 0$, there exists a*

constant $C > 0$, depending on a and the upper and lower bounds of $|Du_\sigma|$ on the set $\{x \in \Omega : \inf u_\sigma + a \le u_\sigma(x) < 0\}$, such that for any $0 > h > a + \inf_\Omega u_\sigma$,

$$(3.5) \qquad 0 \le |\Omega_{h,u_0}| - |\Omega_{h,u_\sigma}| \le C\sigma.$$

Note that the constant C in (3.5) is large when the lower bound of $|Du_\sigma|$ is small. We do not impose conditions on Ω but it is convex and its shape is controlled by the lower bound of $|Du_\sigma|$.

For the proof of Theorem 1.1, the solution u satisfies (3.1) and Ω is a small perturbation of the unit disc. In this case (3.5) can be proved easily. In fact, the difference $|\Omega_{h,u_0}| - |\Omega_{h,u_\sigma}|$ is controlled by $\frac{\sigma u_{\gamma\gamma}}{\sigma + u_\gamma^2} = O(\sigma)$; see (3.6) below.

Proof. Denote $u = u_\sigma$ and suppose without loss of generality that $u(0) = \inf_\Omega u = -1$. By convexity we have $\mathcal{L}_0[u] \le 1$. By the comparison principle we have $u \ge u_0$. Hence $\Omega_{h,u} \subset \Omega_{h,u_0}$ and $|\Omega_{h,u_0}| \ge |\Omega_{h,u}|$.

Write the equation $\mathcal{L}_\sigma[u] = 1$ in the form

$$(3.6) \qquad \kappa u_\gamma = 1 - \frac{\sigma u_{\gamma\gamma}}{\sigma + u_\gamma^2},$$

where κ is the curvature of the level set $\Gamma_{h,u}$, and γ is the unit outward normal to $\Omega_{h,u}$. Now, (3.6) implies that the level set $\Gamma_{h,u}$ is moving with the velocity (regard $t = -h$ as the time)

$$v = u_\gamma^{-1} = \frac{\kappa}{1 - \frac{\sigma u_{\gamma\gamma}}{\sigma + u_\gamma^2}}.$$

Let $w = w(\cdot, h) \in C(S^1)$ denote the supporting function of $\Gamma_{h,u}$. That is,

$$w(p) = w(p, h) = \sup\{\langle p, x\rangle : x \in \Gamma_{h,u}\}, \quad p \in S^1.$$

The supremum is attained at the point x at which the unit outer normal $\gamma(x) = p$, and the curvature κ at x is given by

$$\kappa(x) = \frac{1}{(w'' + w)(p)},$$

where S^1 is parametrized by $p = (\cos\theta, \sin\theta)$ and $w' = \frac{d}{d\theta}w$. The area of the domain $\Omega_{h,u}$ is given by

$$|\Omega_{h,u}| = \frac{1}{2}\int_{S^1} w(w'' + w).$$

Observing that $\partial_h w = v = u_\gamma^{-1}$, we have

$$(3.7) \qquad \frac{d}{dh}|\Omega_{h,u}| = \frac{d}{dh}\int_{S^1} \frac{1}{2}w(w'' + w) = \int_{S^1} \partial_h w(w'' + w)$$
$$= \int_{S^1} \frac{\kappa}{1 - \frac{\sigma u_{\gamma\gamma}}{\sigma + u_\gamma^2}}(w'' + w).$$

For any $h \in (a + \inf_\Omega u, 0)$, denote $D = S^1 \times (h, 0)$. Let G denote the diffeomorphism from $\mathcal{M}_{u,h} =: \mathcal{M}_u \cap \{h < u < 0\}$ to D, where $\mathcal{M}_u$ is the graph of u, such that for any point $(x, t) \in \mathcal{M}_{u,h}$, $G(x, t) = (G_t(x), t) \in D$, where G_t is the Gauss mapping from the level set $\Gamma_{t,u}$ to S^1.

We divide D into two parts, $D = D_1 \cup D_2$, such that

$$D_1 = \left\{ 1 - \frac{\sigma u_{\gamma\gamma}}{\sigma + u_\gamma^2} \geq \frac{1}{2} \right\}$$

and $D_2 = D - D_1$. Observing that $\kappa(w'' + w) = 1$, from (3.7) we have

$$|\Omega_{0,u}| - |\Omega_{h,u}| = \int_{D_1} \frac{1}{1 - \frac{\sigma u_{\gamma\gamma}}{\sigma + u_\gamma^2}} + \int_{D_2} \frac{\kappa}{1 - \frac{\sigma u_{\gamma\gamma}}{\sigma + u_\gamma^2}} (w'' + w).$$

On D_1 we have

$$\left(1 - \frac{\sigma u_{\gamma\gamma}}{\sigma + u_\gamma^2} \right)^{-1} \leq 1 + \frac{2\sigma u_{\gamma\gamma}}{\sigma + u_\gamma^2} \leq 1 + C_1 \sigma u_{\gamma\gamma}$$

and on D_2 we have

(3.8)
$$\frac{\kappa}{1 - \frac{\sigma u_{\gamma\gamma}}{\sigma + u_\gamma^2}} = u_\gamma^{-1} \leq C_2^{-1},$$

where both constants C_1 and C_2 depend on the lower bound of $|Du|$ on the set $\{u > a + \inf u\}$. Hence

(3.9)
$$|\Omega_{0,u}| - |\Omega_{h,u}| \leq \int_h^0 \int_{S^1} (1 + C\sigma u_{\gamma\gamma}) + C \int_{D_2} (w'' + w)$$

$$= 2\pi|h| + C\sigma \int_h^0 \int_{S^1} u_{\gamma\gamma} + C|G^{-1}(D_2)|$$

$$\leq 2\pi|h| + C\sigma + C|G^{-1}(D_2)|.$$

To estimate $|G^{-1}(D_2)|$ we suppose $\inf u$ is attained at the origin. For any unit vector τ in the plane $\{x_3 = 0\}$ starting at the origin, let P_τ be the plane in $\mathbf{R}^3$ containing τ and the x_3-axis and let E_τ denote the intersection of P_τ with $G^{-1}(D_2)$. On E_τ we have, by our definition of D_2, $u_{\gamma\gamma} \geq \frac{C}{\sigma}$. Noting that by equation, $u_{\xi\xi} \leq C$ in $G^{-1}(D_2)$ for any unit vector tangential to $\Gamma_{h,u}$ and that the inner product $\langle \gamma, \tau \rangle \geq C'$ for some constants $C, C' > 0$ depending on the upper and lower bounds of $|Du|$ on the set $\{x \in \Omega : \inf u + a \leq u(x) < 0\}$ (which also determine the geometric shape of Ω), we have $u_{\tau\tau} \geq \frac{C}{\sigma}$ for a different C (for small $\sigma > 0$). It follows that the one-dimensional Lebesgue measure $|E_\tau|_{\mathcal{H}^1} \leq C\sigma$ for some C depending on the upper bound of $|Du|$. Hence the two-dimensional Lebesgue measure $|G^{-1}(D_2)|_{\mathcal{H}^2} \leq C\sigma$.

Observing that the level set Γ_{h,u_0} is moving by its curvature (with time $t = -h$), we have

$$|\Omega_{0,u_0}| - |\Omega_{h,u_0}| = 2\pi|h|.$$

Hence by (3.9),

$$|\Omega_{h,u_0}| - |\Omega_{h,u}| \leq C\sigma + C|G^{-1}(D_2)| \leq C\sigma.$$

We obtain (3.5). □

LEMMA 3.2. *Let $\{\ell_t\}$ be a convex solution to the curve-shortening flow. Suppose ℓ_0 is in the δ_0-neighborhood of a unit circle S^1 and $\{\ell_t\}$ shrinks to a point (the origin) at $t = \frac{1}{2}$. Let $\hat{\ell}_t = \frac{1}{\sqrt{1-2t}}\ell_t$ be the normalization of ℓ_t. Then $\hat{\ell}_t$ is in the δ_t-neighborhood of the unit circle centered at the origin,*

$$(3.10) \qquad\qquad \hat{\ell}_t \subset N_{\delta_t}(S^1),$$

with

$$\delta_t \leq C\delta_0 \left(\frac{1}{2} - t\right)^\alpha,$$

where $\alpha \in (0,1)$ is a positive constant.

Proof. First observe, by the comparison principle, that when $t \leq \frac{1}{4}$, ℓ_t is pinched between two concentrated circles with Hausdorff distance $C\delta_0$. By the Schauder estimate, for $t \in (\frac{1}{8}, \frac{1}{4})$ the C^k norm of $\hat{\ell}_t$ is in the $C\delta_0$-neighborhood of the unit circle; that is,

$$(3.11) \qquad\qquad \|\hat{\ell}_t - S^1\|_{C^k} \leq C\delta_0.$$

With estimate (3.11) we obtain (3.10) from [7, §§5.7.10–5.7.15]. □

Remark 3.2. (i) By the Schauder estimate one can simplify some estimates in [7, §§5.1–5.6]. In [7, §§5.7.10–5.7.15], it was proved that for any $\alpha > 0$ small, there exists $\delta_0 > 0$ such that if (3.11) holds at $t = 0$, then

$$(3.12) \qquad\qquad \int_{S^1} [\kappa'(\tau)]^2 \leq e^{-\alpha\tau} \int_{S^1} [\kappa'(0)]^2,$$

where $\tau = \frac{1}{2}\log(\frac{1}{2} - t)$, and κ' denotes the derivative of the curvature κ with respect to the are-length parameter. Similar inequalities for high order derivatives of κ were also proved there.

(ii) Let u be a convex solution of $L_0[u] = 0$ which attains its minimum 0 at y_1 (namely $u(y_1) = \inf u = 0$). Suppose the level set $\Gamma_{1/2} \subset N_{\delta_0}(S^1)$ for some small $\delta_0 > 0$. Then $|y_1| < C\delta_0$ for some $C > 0$ independent of δ_0. Therefore by a translation we may assume that u attains its minimum at 0 and $\Gamma_{1/2,u} \subset N_{C^*\delta_0}(S^1)$ for a different constant C^*.

To prove $|y_1| < C\delta_0$, let $\hat{u} = \frac{1}{2}|x|^2$ be the rotationally symmetric solution to $L_0[u] = 0$. As in the proof of Lemma 3.2, let $w(p, h)$ and $\hat{w}(p, h)$ be respectively the support functions of $\Gamma_{h,u}$ and $\Gamma_{h,\hat{u}}$, where $p = (\cos\theta, \sin\theta)$.

Denote $t = -h$ (regard $t \in (-\frac{1}{2}, 0)$ as the time). Then $w_t(w'' + w) = -1$, $\hat{w}_t(\hat{w}'' + \hat{w}) = -1$. Denote $\varphi = w - \hat{w}$. Direct computation shows that

$$(w'' + w + \hat{w}'' + \hat{w})\varphi_t = -(w_t + \hat{w}_t)(\varphi'' + \varphi).$$

Hence φ satisfies the equation

$$\varphi_t = (w_t \hat{w}_t)(\varphi'' + \varphi).$$

We have $\hat{w}_t = -\frac{1}{\sqrt{|t|}}$ and by estimate (3.12) (for higher order derivatives), $w_t = -\frac{1+o(1)}{\sqrt{|t|}}$, as $t \to 0$. We obtain $w_t \hat{w}_t = \frac{1+o(1)}{|t|}$. The estimate (3.12) also implies that the curvature of $\Gamma_{u,h}$ is equal to that of $\Gamma_{\hat{u},h}$ up to a lower order perturbation, namely $\varphi'' + \varphi \leq C\delta_0 |t|^\alpha$ for some $\alpha > 0$. We obtain $|\varphi_t| \leq C\delta_0 |t|^{\alpha-1}$ and so $|y_1| \leq \sup |\varphi| \leq C\delta_0$.

Next we need a refinement of (3.1).

LEMMA 3.3. *Let u be an entire convex solution of* (1.1) *with* $\inf u = 0$. *Then in an appropriate coordinate system, we have* (3.1) *with*

(3.13) $|\varphi(x)| = O(|x|^{2/3})$ *as* $|x| \to \infty$.

Proof. Let $u_h(x) = h^{-1} u(h^{1/2} x)$. Then u_h satisfies the equation $\mathcal{L}_\sigma[u] = 1$ in $\mathbf{R}^2$ with $\sigma = h^{-1}$. By Theorem 1.3, u_h converges to the function $u^* = \frac{1}{2}|x|^2$, and the level set $\Gamma_{1/2, u_h}$ converges to the unit circle S^1 as $h \to \infty$.

For any given sufficiently small constant $\delta_0 > 0$, let $h > 0$ sufficiently large such that

(3.14) $\Gamma_{1/2, u_h} \subset N_{\delta_0}(S^1)$

for some unit circle S^1. We claim that for any $\tau > \tau_0$, where $\tau_0 > 3\max(\delta_0, \sigma)$,

(3.15) $\Gamma_{\tau, u_h} \subset \sqrt{2\tau}(N_{\delta_\tau}(S^1))$

with

$$\delta_\tau \leq C_1(\tau)\sigma^{2/3} + C_2 \delta_0 \tau^\alpha,$$

where $\alpha(N_\delta(S^1)) = N_{\alpha\delta}(\alpha S^1)$, and αS^1 is the α-dilation of S^1 with the same center, the constants C_1 and C_2 are independent of δ_0 and h, and C_2 is also independent of τ. The center of the S^1 in (3.15) is the minimum point of u_0, the solution of $\mathcal{L}_0[u] = 1$ in $\Omega_{\frac{1}{2}, u_h}$ satisfying $u_0 = u_h = \frac{1}{2}$ on $\partial\Omega_{\frac{1}{2}, u_h}$.

To prove (3.15), by Lemma 3.2 we have, for any $\tau > 0$,

(3.16) $\Gamma_{\tau, u_0} \subset \sqrt{2(\tau + a_0)}(N_{\delta_1}(S^1))$

with $\delta_1 \leq C\delta_0(\tau + a_0)^\alpha$, where $a_0 = -\inf u_0 \geq 0$. By the comparison principle we have $u_0 \leq u_h$ in $\Omega_{\frac{1}{2}, u_h}$. By (3.14) we also have

$$u_0 \geq \frac{1}{2}(|x|^2 - (1 + \delta_0)^2) + \frac{1}{2}   \text{ in } \Omega_{1/2, u_h}.$$

Hence $a_0 \leq 3\delta_0$.

We will use the following simple result: Let Ω be a convex domain contained in B_R. If the area $|B_R - \Omega| \leq \varepsilon$, then

$$\text{(3.17)} \qquad \text{dist}(\partial B_R, \partial \Omega) \leq C\varepsilon^{2/3} R^{-1/3},$$

where $\text{dist}(A, B)$ denotes the least constant $\delta > 0$ such that $A \subset N_\delta(B)$ and $B \subset N_\delta(A)$.

We use (3.17) to prove (3.15). Let ℓ be the largest circle, with center at the minimum point of u_0, contained in Ω_{τ, u_0}. Let $\widetilde{\Omega}_{\tau, u}$ be the common area enclosed by $\Omega_{\tau, u}$ and ℓ, and denote $\widetilde{\Gamma}_{\tau, u} = \partial\widetilde{\Omega}_{\tau, u}$. Since $\Omega_{\tau, u} \subset \Omega_{\tau, u_0}$, we have

$$\text{(3.18)} \qquad \text{dist}(\Gamma_{\tau, u}, \Gamma_{\tau, u_0}) \leq \text{dist}(\widetilde{\Gamma}_{\tau, u}, \ell) + \text{dist}(\ell, \Gamma_{\tau, u_0}).$$

Since $\mathcal{L}_0[u_0] = 1$, we have $\frac{d}{dt}|\Omega_{t, u_0}| = -2\pi$. Hence $|\Omega_{\tau, u_0}| = 2\pi(\tau + a_0)$. By (3.16), Γ_{τ, u_0} is in the $\sqrt{2(\tau + a_0)}\delta_1 = C\delta_0(\tau + a_0)^{1/2+\alpha}$ neighborhood of $\sqrt{2(\tau + a_0)}S^1$. Hence

$$\text{dist}(\ell, \Gamma_{\tau, u_0}) \leq C\delta_0(\tau + a_0)^{1/2+\alpha},$$

where $C > 0$ is independent of δ_0, h, and τ. Recall that $\Omega_{\tau, u} \subset \Omega_{\tau, u_0}$ and $|\Omega_{\tau, u_0} - \Omega_{\tau, u}| \leq C\sigma$ by (3.5). Hence by (3.17) we have

$$\text{dist}(\widetilde{\Gamma}_{\tau, u}, \ell) \leq C\sigma^{2/3}(\tau + a_0)^{-1/6}.$$

Combining (3.16) and (3.18), and noting that $a_0 \leq 3\delta_0 < \tau$, we obtain (3.15).

Now we fix a $\tau_0 > 0$ small such that $C_2\tau_0^\alpha < 1/4$. From (3.15) we obtain

$$\text{(3.19)} \qquad \Gamma_{\tau_0, u_h} \subset \sqrt{2\tau_0}(N_\delta(S^1))$$

with $\delta \leq C\sigma^{2/3} + \delta_0/4$, where C is independent of δ_0 and h.

Now Lemma 3.3 follows from (3.19) by iteration. We start at the level τ_0^{-k} for some sufficiently large k. Denote $\Omega_k = \sqrt{2\tau_0^k}\Omega_{\tau_0^{-k}, u}$ and $\Gamma_k = \partial\Omega_k$. By (3.1), Γ_k converges to the unit circle as $k \to \infty$. Suppose Γ_k is in the δ_k-neighborhood of S^1, where $\delta_k \to 0$ as $k \to \infty$. Let y_k denote the minimum point of the solution of $\mathcal{L}_0[u] = 1$ in Ω_{k+1} and $u = \frac{1}{2}$ on Γ_{k+1}. By (3.19), Γ_{k-1} is in the δ_{k-1}-neighborhood of a unit circle S^1 centered at y_{k-1} with

$$\delta_{k-1} \leq C\tau_0^{2(k-1)/3} + \delta_k/4.$$

By induction, we obtain

$$\delta_{k-2} \leq C\tau_0^{2(k-2)/3} + \delta_{k-1}/4.$$

Hence we have

$$\delta_j \leq 2C\tau_0^{2j/3} + \delta_k \quad \forall\, j < k.$$

Letting $k \to \infty$ we obtain

$$\text{(3.20)} \qquad \Gamma_j \subset N_{\delta_j}(S^1)$$

with $\delta_j \leq 2C\tau_0^{2j/3}$, where S^1 is centered at y_j. It follows that for $h = \tau_0^{-j}$ sufficiently large,

$$(3.20') \qquad\qquad\qquad \Gamma_{h,u} \subset N_\delta(\sqrt{2h}S^1)$$

with $\delta \leq 2Ch^{-1/6}$, where S^1 is centered at $z_j = h^{1/2}y_j$.

Next we estimate $|z_j - z_{j-1}|$. Let $u_j(x) = \tau_0^j u(\tau_0^{-j/2}x)$. Let v_1 and v_0 be the solutions of $\mathcal{L}_0[v] = 1$ which satisfy respectively $v_1 = 1$ on $\{u_j = 1\}$ and $v_0 = \tau_0$ on $\{u_j = \tau_0\}$. By (3.20) we have $\{v_0 = \tau_0\} \subset N_\delta(\sqrt{2\tau_0}S^1_{p_0})$ and $\{v_1 = \tau_0\} \subset N_\delta(\sqrt{2\tau_0}S^1_{p_1})$ with $\delta < C\tau_0^{2j/3}$ for some points p_0 and p_1. As remarked before Lemma 3.3, we may assume p_0 and p_1 are the minimum points of v_0 and v_1, so that $z_j = \tau_0^{-j/2}p_1$ and $z_{j-1} = \tau_0^{-j/2}p_0$. By Lemma 3.1 and (3.17) we also have $\{v_0 = \tau_0\} \subset N_\delta(\{v_1 = \tau_0\})$. Hence $|p_0 - p_1| \leq C\delta$. We obtain $|z_j - z_{j-1}| \leq C\tau_0^{j/6}$.

From the above estimate, the sequence $\{z_j\}$ is convergent. Assume that $z_j \to 0$. Then the above estimate implies that $|z_j| \leq C\tau_0^{j/6}$ for any large j. Hence for $h = \tau_0^{-j}$,

$$\Gamma_{h,u} \subset N_\delta(\sqrt{2h}S^1),$$

where $\delta \leq Ch^{-1/6}$ and S^1 is centered at the origin. It is easy to see the estimate also holds for all $h > 1$. Hence Lemma 3.3 is proved. $\qquad\square$

To finish the proof we need the following fundamental Liouville Theorem by Bernstein [1], [2]; see also [23, p. 245].

PROPOSITION 3.1. *Let u be an entire solution to the elliptic equation*

$$(3.21) \qquad\qquad \sum_{i,j=1}^2 a_{ij}(x)u_{ij} = 0 \quad in \ \ \mathbf{R}^2.$$

If u satisfies the asymptotic estimate

$$(3.22) \qquad\qquad |u(x)| = o(|x|) \quad as \ \ |x| \to \infty,$$

then u is a constant.

We remark that the operator in the above proposition need not be uniformly elliptic. Condition (3.21) can be replaced by a weaker condition that $u_{11}u_{22} - u_{12}^2 \leq 0$ and $\not\equiv 0$.

Proof of Theorem 3.2. Assume u is locally uniformly convex, namely the Hessian matrix $(D^2u) > 0$ pointwise, which will be proved below. Let u^* be the Legendre transform of u. Then u^* satisfies equation (1.9). First we have

$$(3.23) \qquad\qquad u^*(x) = \frac{1}{2}|x|^2 + O(|x|^{2/3}).$$

Indeed, for any $h > 1$, let $u_h(x) = h^{-1}u(h^{1/2}x)$. Then by Lemma 3.3,

$$u_h(x) = \frac{1}{2}|x|^2 + O(h^{-2/3})$$

in $B_1(0)$. Denote u_h^* the Legendre transforms of u_h. Then

$$u_h^*(x) = \frac{1}{2}|x|^2 + O(h^{-2/3})$$

in $B_1(0)$. Observing that $u_h^*(x) = h^{-1}u^*(h^{1/2}x)$, we obtain (3.23).

Let u_0 be the unique radial solution of (1.1) satisfying $u(0) = 0$, and let u_0^* denote the Legendre transform of u_0. Similar to (3.23) we have

(3.24) $$u_0^*(x) = \frac{1}{2}|x|^2 + O(|x|^{2/3}).$$

Write equation (1.9) in the form

(3.25) $$G[x, D^2u^*] =: \frac{\det D^2u^*}{\sum \left(\delta_{ij} - \frac{x_i x_j}{1+|x|^2}\right) F^{ij}[u^*]} = 1.$$

Since both u^* and u_0^* satisfy equation (1.9), $v = u^* - u_0^*$ satisfies equation (3.21) in the entire $\mathbf{R}^2$ with coefficients

$$a_{ij} = \int_0^1 G^{ij}[x, D^2u_0^* + t(D^2u^* - D^2u_0^*)]dt,$$

where $G^{ij}[x, r] = \frac{\partial}{\partial r_{ij}}G[x, r]$ for any symmetric matrix r. By (3.23) and (3.24), $|v(x)| = O(|x|^{2/3})$ as $|x| \to \infty$. By the above proposition we conclude that v is a constant. □

Remark 3.3. When using the Legendre transform we have implicitly used the local uniform convexity of u, namely the Hessian matrix $\{D^2u\} > 0$. In dimension 2, this was proved in [12] by Hamilton's maximum principle; for high dimensions, see [11]. We also note that the reason for using the Legendre transform in the above proof is that equation (3.21) does not involve the first order derivatives.

4. Translating solutions to the level set flow

In this section we prove the case $\sigma = 0$ of Theorem 1.2 and that an ancient convex (in space) solution to the mean curvature flow is convex in space-time. We point out that when $n \geq 4$, the proof is simpler; see Remark 4.1.

THEOREM 4.1. *For any $n \geq 2$ and $1 \leq k \leq n$, there exist complete convex solutions, defined in strip regions, to the equation*

(4.1) $$\sum_{i,j=1}^n \left(\delta_{ij} - \frac{u_i u_j}{|Du|^2}\right) u_{ij} = 1$$

which are not k-rotationally symmetric. If $n \geq 3$, there exist entire convex solutions to (4.1) *which are not k-rotationally symmetric.*

By our definition, a function u is k-rotationally symmetric if $u(x) = \varphi(|\hat{x}|)$ in an appropriate coordinate system, where $\hat{x} = (x_1, \ldots, x_k)$. To prove Theorem 4.1 we will need the following logarithm concavity of solutions to (4.1).

LEMMA 4.1. *Let Ω be a smooth, bounded, convex domain in $\mathbf{R}^n$. Let u be the solution of* (4.1) *in Ω, vanishing on $\partial\Omega$. Then for any constant h satisfying $\inf_\Omega u < h < 0$, the level set $\Gamma_{h,u} = \{u = h\}$ is convex. Moreover, $\log(-u)$ is a concave function.*

Proof. Since u is a solution of (4.1), $\psi = -\log(-u)$ satisfies

$$(4.2) \qquad \left(\delta_{ij} - \frac{\psi_i \psi_j}{|D\psi|^2}\right)\psi_{ij} = e^\psi.$$

Since $\psi(x) \to +\infty$ as $x \to \partial\Omega$, the results in [15] (see Section III.12) implies ψ is convex. $\square$

Denote

$$(4.3) \qquad \Omega_{r,t} = \left\{x \in \mathbf{R}^n : \frac{|x'|^2}{r^2} + \frac{x_n^2}{t^2} < 1\right\},$$

where r, t are positive constants, $x' = (x_1, \ldots, x_{n-1})$. Let $u_{r,t}$ denote the solution of (4.1) in $\Omega_{r,t}$, vanishing on $\partial\Omega_{r,t}$. Denote $M_{r,t} = -\inf u_{r,t}$ and $\Gamma_{r,t} = \{u_{r,t} = -M_{r,t} + 1\}$. Obviously $M_{r,t} \to \infty$ as $r, t \to \infty$.

The following lemma plays a key role for our construction of nonradial convex solutions. A similar idea was used in [4], where we proved that for any ellipsoid E, there exists an entire convex solution u to the Monge-Ampère equation $\det D^2 u = f$ such that $u(0) = 0$, $u \geq 0$, and the minimum ellipsoid of the sublevel set $\{u < 1\}$ is similar to E.

LEMMA 4.2. *For any $\theta > 0$ and $K > 1$, there exist $r = r(\theta, K)$ and $t = t(\theta, K)$ such that $M_{r,t} = K$ and*

$$(4.4) \qquad \sup\{|x'| : x \in \Gamma_{r,t}\} = \theta \sup\{x_n : x \in \Gamma_{r,t}\}.$$

Proof. The solution $u_{r,t}$ depends continuously on r and t, and $M_{r,t}$ is monotone increasing in r and t. For any $K > 1$, we have $M_{r,t} = K$ when $r = t = \sqrt{2(n-1)K}$.

It is easy to see that for any fixed $r > 0$, $M_{r,t} \to 0$ as $t \to 0$. Hence for any given $r > \sqrt{2(n-1)K}$, there exists a unique $t = t_r < \sqrt{2(n-1)K}$ such that $M_{r,t} = K$. Moreover we have $t_r \to 0$ as $r \to \infty$. Similarly for any fixed $t > 0$, we have $M_{r,t} \to 0$ as $r \to 0$. Hence for any given $t > \sqrt{2(n-1)K}$, there exists a unique $r = r_t < \sqrt{2(n-1)K}$ such that $M_{r,t} = K$.

Observe that for any fixed K, $\sup\{x_n : x \in \Gamma_{r,t}\} \to 0$ as $t \to 0$ and by convexity $\sup\{x_n : x \in \Gamma_{r,t}\} \to \infty$ as $t \to \infty$, where $r = r_t$ is such that $M_{r,t} = K$. By the continuity of $u_{r,t}$ in r and t, there exist $r > 0$ and $t = t_r$ such that (4.4) holds.   □

For any fixed $\theta \neq 1$, by Lemma 4.2 there exist $r = r_k$ and $t = t_k$ such that $M_{r_k,t_k} = k$ and (4.4) holds. From the proof of Lemma 4.2 we also have

(4.5) $t_k > r_k$ if $\theta < 1$.

Hence as $k \to \infty$ we have $t_k \to \infty$ if $\theta < 1$ or $r_k \to \infty$ if $\theta > 1$. Denote $w_k = u_{r_k,t_k} + k$. Then $w_k \geq w_k(0) = 0$. We want to prove w_k converges to a complete solution of (4.1).

We say that a sequence of (embedded) convex hypersurfaces $\{\mathcal{M}_k\}$ locally converges to $\mathcal{M}$ if for any $R > 1$ and $\delta > 0$, there exists $k_0 > 1$ such that when $k \geq k_0$, $\mathcal{M}_k \cap B_R(0) \subset N_\delta(\mathcal{M} \cap B_R(0))$ and $\mathcal{M} \cap B_R(0) \subset N_\delta(\mathcal{M}_k \cap B_R(0))$, where B_R denotes the ball of radius R and N_δ denotes the δ-neighborhood.

For any fixed integer j, by Lemma 4.1, $\varphi_{j,k} = \log j - \log(j - w_k)$ $(k \geq j)$ is an even, convex function. Let $\mathcal{M}_{j,k}$ denote the graph of $\varphi_{j,k}$. Observe that for any fixed j and h, by (4.4) the sets $\mathcal{M}_{j,k} \cap \{x_{n+1} < h\}$ are uniformly bounded in k, and can be represented as radial graphs with center at the point $(0, \ldots, 0, \frac{1}{2})$. Hence we may suppose by choosing subsequences that $\mathcal{M}_{j,k}$ converges locally to a complete, convex hypersurface $\mathcal{M}_j$. Let $\overline{D}_j$ denote the projection of $\mathcal{M}_j$ on $\{x_{n+1} = 0\}$ and D_j denote the interior of $\overline{D}_j$. Then D_j is a convex domain and as $k \to \infty$, $\varphi_{j,k}$ converges locally in D_j to a function φ_j. It follows that w_k converges locally in D_j to a function w. Obviously w is a viscosity solution of (4.1) in D_j. Repeating the procedure for $j = 1, 2, \ldots$, by the Arzela-Ascoli lemma we obtain a sequence of domains $D_1 \subset D_2 \subset \cdots$ such that w_k subconverges locally to w in all D_j, $j = 1, 2, \ldots$. Let $D = \cup D_j$. Then D is a convex domain and w_k converges locally to w in D.

By (4.4), w is not rotationally symmetric. To prove Theorem 4.1, we will prove w is convex and $w(x) \to \infty$ as $x \to \partial D$.

LEMMA 4.3. *For any $x_0 \in \partial D$, we have*

(4.6) $\lim_{x \to x_0} w(x) = \infty.$

Proof. For any fixed k, the level set $\{w_k = -t\}$ is a convex solution to the mean curvature flow (with time $t \in (-k, 0)$). For any fixed t, by the discussion above we see that $\{w_k = -t\}$ converges to the level set $\{w = -t\}$ as $k \to \infty$. Hence, when $-\infty < t < 0$, $\{w = -t\}$ is also a convex solution to the mean curvature flow. It follows that for any $t \in (-\infty, 0)$, $\{w = -t\}$ is smooth and locally uniformly convex. Hence at any time $t \in (-\infty, 0)$, the hypersurface $\{w = -t\}$ is moving at positive velocity. Hence $w(x) \to \infty$ as $x \to \partial D$.   □

We can also write the graph of w_k locally in the form $x_n = v_k(x', t)$, where $t = -x_{n+1}$. Then v_k satisfies the nonparametric mean curvature flow equation

$$(4.7) \qquad\qquad v_t = \sqrt{1 + |Dv|^2} \, \mathrm{div} \frac{Dv}{\sqrt{1 + |Dv|^2}},$$

where $Dv = (v_{x_1}, \dots, v_{x_{n-1}})$. Hence if v is convex in x' and if $v_t > 0$ at some point, then $v_t > 0$ everywhere by the Harnack inequality. Using this property one also easily obtain (4.6). We remark that Lemmas 4.2 and 4.3 were also observed by White; see [27].

LEMMA 4.4. *The solution w is convex.*

Proof. Since the level set of w_k is convex, so is the level set of w. For any point $y \in D$ and any positive constant $\delta < \min(1, \frac{1}{2}d_y)$, where $d_y = \mathrm{dist}(y, \partial D)$, there exists a constant $M_0 > 0$ depending on δ such that $\sup\{w(x) : x \in B_\delta(y)\} \le M_0$ and $\sup\{|Dw(x)| : x \in B_\delta(y)\} \le M_0$. Denote $v_k = \log(k - w)$. By the concavity of v_k we have

$$|Dv_k(y)| \le \sup_{x \in \partial B_\delta(y)} \frac{1}{\delta} (|v_k(x) - v_k(y)|)$$
$$\le C(\log(k + M_0) - \log(k - M_0)) \le C/k,$$

where $C > 0$ depends on M_0 and δ, but is independent of k.

By the concavity of v_k, we have, furthermore,

$$\{\partial_i \partial_j w_k(y)\} = -e^{v_k} \{\partial_i \partial_j v_k + \partial_i v_k \partial_j v_k\}$$
$$\ge -e^{v_k} \{\partial_i v_k \partial_j v_k\} \ge -\frac{C}{k} I,$$

where I is the unit matrix. Sending $k \to \infty$ we obtain $\{\partial_i \partial_j w(y)\} \ge 0$. Hence w is convex. $\qquad\square$

From [9] we know that $\{w = -t\}$ shrinks to a round point as $t \to 0$. Hence $w > 0$ for any $x \neq 0$ and so w is not k-rotationally symmetric for any $1 \le k \le n$. If we choose $\theta > 1$ sufficiently large, then w must be defined in a strip region by Lemma 2.7. We have thus proved the first part of Theorem 4.1.

We would like to point out that, from the proof of Theorems 1.1 and 1.3, the function w is defined in a strip region for any $\theta > 1$. If $n = 2$, then by Theorem 1.1, w is defined in a strip for any $\theta \neq 1$.

Next we prove the second part of Theorem 4.1. We will prove the solution w obtained above is an entire solution if $n \ge 3$ and $\theta < 1$. Denote

$$r_h = r_{h,w} = \sup\{|x'| : (x', x_n) \in \Omega_{h,w}\},$$
$$t_h = t_{h,w} = \sup\{x_n : (x', x_n) \in \Omega_{h,w}\}.$$

LEMMA 4.5. *Suppose $t_h \geq \delta r_h$ for some positive constant $\delta > 0$. Then*

$$(4.8) \qquad \frac{(\delta r_h)^2}{4(n-1)} \leq h \leq \frac{r_h^2}{2(n-2)}.$$

Proof. Let

$$\varphi = \frac{1}{2(n-1)} \left(|x|^2 - \frac{1}{2} (\delta r_h)^2 \right).$$

Then $\mathcal{L}_0[\varphi] = 1$ in $\Omega_{h,w} = \{w < h\}$ and $\varphi \geq 0$ on $\partial\Omega_{h,w}$. By the comparison principle it follows that $w - h \leq \varphi$ in $\Omega_{h,w}$. Hence

$$h \geq -\inf \varphi = \frac{(\delta r_h)^2}{4(n-1)}.$$

To prove the second inequality of (4.8), let

$$\varphi = \frac{1}{2(n-2)} (|x'|^2 - r_h^2).$$

Then $\mathcal{L}_0[\varphi] = 1$ in $\Omega_{h,w}$ and $\varphi \leq 0$ on $\partial\Omega_{h,w}$. It follows $w - h \geq \varphi$ in $\Omega_{h,w}$. Hence

$$h \leq -\varphi(0) = \frac{r_h^2}{2(n-2)}.$$

This completes the proof.   □

Therefore to prove the second part of Theorem 4.1 it suffices to prove that there exists $\delta > 0$ such that

$$(4.9) \qquad \sup\{x_n : x \in \Gamma_{h,w}\} \geq \delta \sup\{|x'| : x \in \Gamma_{h,w}\}$$

for any $h > 0$, where $\Gamma_{h,w} = \{w = h\}$. Denote

$$r_{h,k} = \sup\{|x'| : x \in \Gamma_{h,w_k}\},$$
$$t_{h,k} = \sup\{x_n : x \in \Gamma_{h,w_k}\}.$$

Then $r_{h,k \mid h=k} = r_k$ and $t_{h,k \mid h=k} = t_k$, where t_k and r_k satisfy (4.5). If there is a subsequence of $\{k\}$ such that

$$(4.10) \qquad t_{h,k} \geq r_{h,k} \quad \forall\, h \in (0, k),$$

then (4.9) holds with $\delta = 1$ for all $h > 0$. Hence w is defined in the entire space $\mathbf{R}^n$.

If (4.10) is not true, let

$$h_k = \sup\{h > 0 : t_{h,k} < r_{h,k}\}.$$

By (4.5) we have $h_k < k$. If the sequence $\{h_k\}$ is uniformly bounded, w is defined in the entire space $\mathbf{R}^n$.

If $h_k \to \infty$, we denote $\widetilde{w}_k(x) = h_k^{-1} w_k(h_k^{1/2} x)$. Then $\mathcal{L}_0[\widetilde{w}_k] = 1$ in $\{\widetilde{w}_k < 1\}$ and

$$(4.11) \qquad \sup\{|x'| : x \in \{\widetilde{w}_k < 1\}\} = \sup\{x_n : x \in \{\widetilde{w}_k < 1\}\}.$$

Observe that the level set $\{\widetilde{w}_k = -t\}$ is a convex solution to the mean curvature flow (with time $t \in (-1, 0)$). From [9] we see that $\{\widetilde{w}_k = h\}$ shrinks to a round point at $h \to 0$. Hence by (4.11) we have, for any $h \in (0, 1)$,

$$(4.12) \qquad \sup\{x_n : x \in \{\widetilde{w}_k = h\}\} \geq \delta \sup\{|x'| : x \in \{\widetilde{w}_k = h\}\}$$

for some $\delta > 0$ independent of h and k. Rescaling back we obtain (4.9). This completes the proof of Theorem 4.1.

Remark 4.1. When $n \geq 4$, we can also construct entire convex solutions of (4.1) as follows. Denote $\hat{x} = (x_1, \ldots, x_{n-2})$, $\widetilde{x} = (x_{n-1}, x_n)$. Let $\Omega_{r,t} = \{x \in \mathbf{R}^n : \frac{|\hat{x}|^2}{r^2} + \frac{|\widetilde{x}|^2}{t^2} = 1\}$. Let $u_{r,t}$ be the solution of (4.1) with $\Omega = \Omega_{r,t}$, which vanishes on $\partial\Omega$. As before we choose r_k and t_k such that $\inf u_{r_k,t_k} = -k$ and

$$(4.13) \qquad \sup\{|\hat{x}| : x \in \Gamma_{r,t}\} = \theta \sup\{|\widetilde{x}| : x \in \Gamma_{r,t}\},$$

where $\Gamma_{r,t} = \{u_{r,t} = -k + 1\}$, and $\theta \neq 1$ is any given positive constant. Denote $w_k = u_{r_k,t_k} + k$. Then w_k is nonnegative and $w_k(0) = 0$. Observe that the function $\varphi = \frac{1}{2}|\widetilde{x}|^2$ satisfies the equation $\mathcal{L}_0[\varphi] = 1$. By the comparison principle we have

$$(4.14) \qquad\qquad\qquad w_k(x) \leq \frac{1}{2}|x|^2.$$

Let $w = \lim_{k \to \infty} w_k$. Then w satisfies (4.14). Hence by Lemma 4.4, w is an entire convex solution of (4.1). Obviously w is not k-rotationally symmetric for any $1 \leq k \leq n$.

Remark 4.2. In Lemma 4.4 we showed that a solution to (4.1) is convex if the level set $\Gamma_h = \{u = h\}$ is bounded and convex for all large $h > 0$. This assertion is true even if Γ_h is unbounded.

PROPOSITION 4.1. *Let u be a solution to* (4.1) *whose graph is a complete hypersurface. Suppose that the level set $\Gamma_{h,u}$ is convex for any h. Then u is convex.*

Proof. If the sublevel sets $\Omega_{h,u}$ are bounded, Proposition 4.1 is proved in Lemma 4.4. If $\Omega_{h,u}$ is unbounded, from the proof of Lemma 4.4 it suffices to show that $\log(h - u)$ is concave for any large constant h. By Lemma 4.1, it suffices to show that for any given $h > 0$, u can be approximated locally in $\Omega_{h,u}$ by a sequence of solutions to (4.1) whose level sets are bounded and convex.

Let $\{D_k\}$ be a sequence of convex domains in $\mathbf{R}^n$ satisfying $D_1 \subset D_2 \subset \cdots$, such that $\cup D_k = \Omega_{h,u}$ for some fixed h. Let w_k be the solution of (4.1) in D_k satisfying $w_k = h$ on ∂D_k. Then $w_k \geq u$ in D_k and w_k is decreasing in k. Hence w_k converges as $k \to \infty$ to a solution w of (4.1) in Ω_h, satisfying $w = h$ on $\partial\Omega_{h,u}$ and $w \geq u$ in $\Omega_{h,u}$.

To prove $w = u$, we need to prove that for any $h' < h$ (h fixed), the level set $\Gamma_{h',w}$ is sufficiently close to $\Gamma_{h',u}$. Note that both $\Gamma_{h',w}$ and $\Gamma_{h',u}$ evolves by

mean curvature (with time $t = -h'$), so it suffices to prove $\Gamma_{h',w}$ is sufficiently close to $\Gamma_{h',u}$ at infinity.

Let $\{x_k\}$ be a sequence on $\Gamma_{h,u}$ with $|x_k| \to \infty$. By the convexity of $\Gamma_{h,u}$, the normal of $\Gamma_{h,u}$ (regarded as a hypersurface in $\mathbf{R}^n = \{x_{n+1} = h\}$) at x_k converges along a subsequence to a boundary point of the Gauss mapping image of $\Gamma_{h,u}$. Hence after translation, the convex hypersurface $\Gamma_{h,u}^k = \{x - x_k : x \in \Gamma_{h,u}\}$ converges to a convex hypersurface which can be split as $\mathbf{R}^1 \times \Sigma_{h,u}$. Similarly for any $h' < h$, $\Gamma_{h',u}^k = \{x - x_k : x \in \Gamma_{h',u}\}$ converges to a convex hypersurface $\mathbf{R}^1 \times \Sigma_{h',u}$, and $\Gamma_{h',w}^k = \{x - x_k : x \in \Gamma_{h',w}\}$ converges to $\mathbf{R}^1 \times \Sigma_{h',w}$, and both $\Sigma_{h',u}$ and $\Sigma_{h',w}$ evolve by mean curvature (with $t = -h'$) with initial hypersurface $\Sigma_{h,u}$. By an induction argument on dimension we conclude that $\Sigma_{h',u} = \Sigma_{h',w}$ for any $h' < h$. Namely $\Gamma_{h',w}$ is sufficiently close to $\Gamma_{h',u}$ at infinity. $\qquad\square$

5. Translating solutions to the mean curvature flow

In this section we prove the case $\sigma = 1$ of Theorem 1.2. That is

THEOREM 5.1. *For any dimension $n \geq 2$ and $1 \leq k \leq n$, there exist complete convex solutions to equation* (1.1), *defined in strip regions, which are not k-rotationally symmetric. If $n \geq 3$, there are entire convex solutions to* (1.1) *which are not k-rotationally symmetric.*

The argument in Section 4 cannot be extended to the mean curvature equation (1.1), as the logarithm concavity in Lemma 4.1 is still an open problem for equation (1.1). To prove Theorem 5.1 we will use the Legendre transform. The purpose is to introduce the Legendre transform to obtain *convex* solutions to the mean curvature equation (1.1). As remarked at the end of Section 3, we can always assume that a convex solution is locally uniformly convex.

For clarity we divide this section into three subsections. But similarly as in Section 4, the argument is much simpler in the case $n \geq 4$; see discussions at the beginning of Section 5.3.

5.1. *The Legendre transform.* For a smooth, uniformly convex function u defined in a convex domain $\Omega \subset \mathbf{R}^n$. The Legendre transform of u, u^*, is a smooth, uniformly convex function defined in the domain $\Omega^* = Du(\Omega)$, given by

(5.1) $$u^*(x) = \sup\{x \cdot y - u(y) : y \in \Omega\}.$$

For example, if $u(x) = a|x|^2$, then $u^*(y) = \frac{1}{4a}|y|^2$; and if $u(x) = a|x|^{1+\beta}$, then $u^*(y) = c|y|^{1+1/\beta}$ with $c = a\beta/[a(1+\beta)]^{1+1/\beta}$. The function u can be recovered from u^* by the same Legendre transform, namely $u(y) = \sup\{x \cdot y - u^*(x) : x \in \Omega^*\}$. The supremum is attained at the unique point y which

satisfies

$$x = Du(y) \quad \text{and} \quad y = Du^*(x).$$

It follows that the Hessian matrix (D^2u) at y is the inverse of the Hessian matrix (D^2u^*) at x. That is

(5.2) $$(D^2u) = (D^2u^*)^{-1} = (F^{ij}[u^*])/\det D^2u^*,$$

where $F^{ij}[u^*]$ is the (i,j)-entry of the cofactor matrix of (D^2u^*),

$$F^{ij}[u^*] = \frac{\partial}{\partial r_{ij}} \det r \quad \text{at } r = D^2u^*.$$

Hence if u is a uniformly convex solution of (1.2), u^* is a solution of $\mathcal{L}_\sigma^*[u^*] = 1$, where

$$\mathcal{L}_\sigma^*[u^*] = \det D^2u^* \Big/ \sum \left(\delta_{ij} - \frac{x_i x_j}{\sigma + |x|^2} \right) F^{ij}[u^*]$$

is a fully nonlinear partial differential equation, which is elliptic at convex functions. In particular equation (1.1) is equivalent to the equation

(5.3) $$\mathcal{L}_1^*[u^*] = 1.$$

We have the following classical solvability for the Dirichlet problem of equation (5.3).

THEOREM 5.2. *Let Ω^* be a smooth, uniformly convex domain in $\mathbf{R}^n$ and φ be a smooth function defined on $\partial\Omega^*$. Then there is a unique, smooth, uniformly convex solution $u^* \in C^\infty(\overline{\Omega}^*)$ to (5.3) such that $u^* = \varphi$ on $\partial\Omega^*$.*

For the proof of Theorem 5.2, we observe that the uniqueness of convex solutions follows from the comparison principle. For the existence of smooth convex solutions, by the continuity method it suffices to establish the global regularity estimates. By Evans and Krylov's elliptic regularity theory (see, e.g., [5], [8], [16]), it suffices to establish the global second order derivative estimates.

Different proofs for the global second order derivative estimates are available [17], [18], [24]. In [17] Krylov provided a probabilistic proof for (degenerate) Bellman equations. An analytic proof was later given in [18]. Krylov's estimation covers equation (5.3) as it is equivalent to a concave equation (5.4) below and so can be expressed as a Bellman equation. For Hessian equations (such as (5.4)) the proof in [18] was simplified in [14].

To apply the *a priori* estimates in [24] we need to write equation (5.3) as a Hessian quotient equation, namely

(5.4) $$\frac{F_n[w]}{F_{n-1}[w]} = \frac{-1}{p_{n+1}}, \quad p \in S^*,$$

where $S^* = \{\frac{(x,-1)}{(1+|x|^2)^{1/2}} \in S^n : x \in \Omega^*\}$, S^n is the unit sphere,

$$w(p) = (1 + |x|^2)^{-1/2} u^*(x), \quad p = \frac{(x,-1)}{(1+|x|^2)^{1/2}}, \ x \in \Omega^*.$$

If u^* is the Legendre transform of u, the function w is indeed the support function of $\mathcal{M}_u$ (the graph of u), which can also be defined by

$$(5.5) \qquad\qquad w(p) = \sup\{p \cdot X : X \in \mathcal{M}_u\}.$$

Moreover, $\mathcal{M}_u$ can be recovered from w by $\mathcal{M}_u = \partial K$ with $K = \{X \in \mathbf{R}^{n+1} : p \cdot X \leq w(p) \ \forall \ p \in S^*\}$.

In (5.4) we denote by $F_k[w]$ the kth elementary symmetric polynomial of the eigenvalues $\lambda = (\lambda_1, \ldots, \lambda_n)$ of the matrix $\{\nabla^2 w + wI\}(p)$,

$$(5.6) \qquad\qquad F_k[w] = \sum_{1 \leq i_1 < \cdots < i_k \leq n} \lambda_{i_1} \cdots \lambda_{i_k}, \quad 1 \leq k \leq n,$$

where ∇ denotes the covariant derivative with respect to a local orthonormal basis on S^n, and I is the unit matrix. As the supremum in (5.5) is attained at the unique point $X_p \in \mathcal{M}_u$ with normal p, the principal radii of $\mathcal{M}_u$ are equal to the eigenvalues of the Hessian matrix $\{\nabla^2 w + wI\}$ (which also follows from (5.2)). Hence by equation (1.1), w satisfies equation (5.4).

The global second order derivative estimates for Hessian quotient equations in Euclidean domains were established by Trudinger [24]. It is not hard to extend the argument in [24] to equation (5.4) with domains on the unit sphere.

5.2. *Complete convex solutions.* With Theorem 5.2 we can now construct a sequence of convex solutions (w_k) of (1.1), such that w_k converges to a complete convex solution of (1.1) which is not k-rotationally symmetric for any $1 \leq k \leq n$.

For any positive constants r, t, denote

$$\Omega_{r,t}^* = \left\{ x \in \mathbf{R}^n : \ \frac{|x'|^2}{r^2} + \frac{x_n^2}{t^2} < 1 \right\},$$

where $n \geq 2$, $x' = (x_1, \ldots, x_{n-1})$. By Theorem 5.2, the Dirichlet problem

$$(5.7) \qquad\qquad \begin{cases} \mathcal{L}_1^*[v] = 1 & \text{in } \Omega_{r,t}^*, \\ v = 0 & \text{on } \partial\Omega_{r,t}^* \end{cases}$$

has a unique smooth convex solution $u_{r,t}^*$. Denote $M_{r,t}^* = -\inf u_{r,t}^*$ and $\Gamma_{r,t}^* = \{x \in \mathbf{R}^n : u_{r,t}^*(x) = -M_{r,t}^* + 1\}$. Then $M_{r,t}^* \to \infty$ as $r, t \to \infty$. Similarly to the proof of Lemma 4.2 we have

LEMMA 5.1. *For any constants $\theta > 0$ and $K > 1$, there exist $r = r(\theta, K)$ and $t = t(\theta, K)$ such that $M^*_{r,t} = K$ and*

$$(5.8) \qquad\qquad \sup\{|x'| : x \in \Gamma^*_{r,t}\} = \theta \sup\{x_n : x \in \Gamma^*_{r,t}\}.$$

Now we fix a positive constant $\theta \neq 1$. By Lemma 5.1 there exist positive constants $r = r_k$ and $t = t_k$ such that $M^*_{r_k,t_k} = k$ and (5.8) holds. Similar to (4.5) (after the Legendre transform the case $\theta > 1$ here corresponds to the case $\theta < 1$ in §4) we have

$$(5.9) \qquad\qquad r_k > t_k \quad \text{if } \theta > 1.$$

Denote $w^*_k = u^*_{r_k,t_k} + k$, $\Omega^*_k = \Omega^*_{r_k,t_k}$. Then $w^*_k \geq w^*_k(0) = 0$.

Now we use the Legendre transform to change back to equation (1.1). Let w_k be the Legendre transform of w^*_k. Then w_k is a convex function defined in the domain $\Omega_k =: Dw^*_k(\Omega^*_k)$ and satisfies the mean curvature equation (1.1) in Ω_k. By (5.8), there exists a constant $R_0 > 0$, depending only on n and θ, such that $w^*_k \geq 1$ on $\partial B_{R_0}(0)$. Hence $|Dw^*_k| \geq 1/R_0$ on $\partial B_{R_0}(0)$ and so $B_{R_0^{-1}}(0) \subset \Omega_k$ for any large k.

LEMMA 5.2. *Let $u \in C^2(\Omega)$ be a convex solution of* (1.1). *Suppose $u(0) = 0$, $u \geq 0$, and u is an even function. Then for any $M > 0$, there exists a constant $C > 0$ such that for any $y \in \Omega$, if $u(y) < M$, we have*

$$(5.10) \qquad\qquad |Du(y)| \leq C.$$

Proof. Write equation (1.1) in the form

$$(5.11) \qquad\qquad \kappa u_\gamma + \frac{u_{\gamma\gamma}}{1 + u_\gamma^2} = 1,$$

where κ is the mean curvature of the level set $\{u = \text{const}\}$ and γ is the unit outer normal to the level set. The normal $\gamma(x)$ is a smooth vector field in $\Omega - \{O\}$. Hence for any point $y \in \Omega$, there is a smooth curve ℓ_y connecting the origin O to y such that $\gamma(x)$ is tangential to the curve at any point $x \in \ell_y$. Since u is an even function, we may suppose y is in the positive cone $\{x = (x_1, \ldots, x_n) : x_i \geq 0\}$. It follows by the convexity of u that ℓ_y lies in the positive cone and for any $x \in \ell_y$, $\gamma(x)$ is also a point in the positive cone. Hence the arclength L of ℓ_y is less than $n|y|$.

Let ψ be the restriction of u on the curve ℓ_y, and let ℓ_y be parametrized by the arclength t. Then we have $\psi' = u_\gamma$ and $\psi'' = u_{\gamma\gamma}$. Hence

$$\frac{\psi''}{1 + \psi'^2} = g(t),$$

where $g(t) = 1 - \kappa\psi' \leq 1$. It follows that

$$\text{arctg}\, \psi'(t) = G(t) =: \int_0^t g(s)ds;$$

thus, $\psi'(t) = \mathrm{tg}G(t)$. Hence $G(L) < \frac{\pi}{2}$ and $G(L) \geq \frac{\pi}{4}$ if $\psi'(L) \geq 1$. Integrating, we have

$$\psi(L) = \int_0^L \mathrm{tg}G(t)dt.$$

Denote $L_0 = \inf\{t : G(t) \geq \frac{\pi}{4}\}$. Then

$$u(y) = \psi(L) \geq \frac{\sqrt{2}}{2} \int_{L_0}^L \frac{1}{\cos G(t)}.$$

If $|Du(y)| = \psi'(L)$ is sufficiently large, then $G(L)$ must be very close to $\frac{\pi}{2}$. This means $\psi(L)$ must be very large since $G'(t) < g(t) < 1$. Hence Lemma 5.2 is proved. $\qquad\square$

Note that for the relation $\psi'' = u_{\gamma\gamma}$ we have used the fact that γ is a normal to the level set $\{u = \text{const}\}$. If ℓ_y is replaced by an arbitrary curve ℓ, then we have

$$\psi'' = u_{\gamma\gamma} + \kappa u_\eta,$$

where γ is a unit vector tangential to ℓ, κ is the curvature of ℓ, and η is a unit normal to ℓ.

Note that by the convexity of u, to prove (5.10) it suffices to consider boundary points. The estimate (5.10) on the boundary of a convex domain can also be obtained by constructing proper subsolutions (more precisely, a solution of $\frac{u_{tt}}{1+u_t^2} = 1$ with one variable t).

LEMMA 5.3. *We have*

(5.12) $$m_k = \inf\{w_k(x) : x \in \partial\Omega_k\} \to \infty \quad \text{as } k \to \infty.$$

Proof. First observe that for any constant $h > 0$, there exists $R_h > 0$ (independent of k) such that for any $x \in \Omega_k$ with $|x| > R_h$, we have the estimate

(5.13) $$w_k(x) \geq h.$$

In fact this estimate results from the following geometric property of the Legendre transform: Let P_h denote the set of linear functions g such that $g < w_k^*$ and $g(0) = -h$. Let $\overline{g}(x) = \sup\{g(x) : g \in P_h\}$. Then the graph of $\overline{g}$ is a convex cone and $D\overline{g}(\mathbf{R}^n) = \{w_k \leq h\}$, where $D\overline{g}(\mathbf{R}^n)$ is the image of the subgradient mapping,

$$D\overline{g}(\mathbf{R}^n) = D\overline{g}(\{0\}) = \{p \in \mathbf{R}^n : \overline{g}(x) \geq p \cdot x + \overline{g}(0) \ \forall \ x \in \mathbf{R}^n\}.$$

By (5.8) we have $|D\overline{g}| < C$. Namely for any $h > 0$, the set $\{w_k < h\}$ is uniformly bounded. Hence (5.13) holds.

By convexity it follows that for any boundary point $x_k \in \partial\Omega_k$, we have $w_k(x_k) \to \infty$ if $|x_k| \to \infty$. Therefore to prove (5.12) we need only to consider

any bounded sequence $x_k \in \partial\Omega_k$. However, if $|x_k|$ and $w_k(x_k)$ are both uniformly bounded, then by Lemma 5.2, $y_k = Dw_k(x_k) \in \partial\Omega_k^*$ are also uniformly bounded. Hence by the Legendre transform,

$$k = w_k^*(y_k) = x_k \cdot y_k - w_k(x_k)$$

are also uniformly bounded, a contradiction.   □

LEMMA 5.4. *There is a subsequence of $\{w_k\}$ which converges to a complete convex solution w of* (1.1).

Proof. For any constant $h > 0$, by (5.13) the sets $\mathcal{M}_{w_k} \cap \{x_{n+1} < h\}$ are uniformly bounded in k, where $\mathcal{M}_{w_k}$ denotes the graph of w_k. Hence we may suppose by choosing subsequences that M_{w_k} converges locally to a convex hypersurface $\mathcal{M}$, which is complete by Lemmas 5.2 and 5.3. Let $\overline{D}$ be the projection of $\mathcal{M}$ on the plane $\{x_{n+1} = 0\}$ and D the interior of $\overline{D}$. Then D is a convex domain and w_k converges locally in D to a convex function w, and w is a convex solution of (1.1) in D. We claim that $w(x) \to \infty$ as $x \to \partial D$. Indeed, if this is not true, then there is a point $p \in \mathcal{M}$ at which the tangent plane of $\mathcal{M}$ is perpendicular to the plane $\{x_{n+1} = 0\}$. Hence there is a sequence $p_k = (x_k, w_k(x_k)) \in \mathcal{M}_{w_k}$, $p_k \to p$, such that $w_k(x_k)$ is uniformly bounded but $|Dw_k(x_k)| \to \infty$. This is in contradiction to Lemma 5.2. Consequently w is a complete convex solution of (1.1). By (5.8), w is not rotationally symmetric.   □

Now we can prove the first part of Theorem 5.1. Indeed, let P denote the set of linear functions g such that $g < w$ and $g(0) = -1$. Let $\overline{g}(x) = \sup\{g(x) : g \in P\}$. Then the graph of $\overline{g}$ is a convex cone and $D\overline{g}(\mathbf{R}^n) = \{w^* \leq 1\}$, where $w^* = \lim_{k\to\infty} w_k^*$ is the Legendre transform of w. Hence if the constant $\theta > 0$ in Lemma 5.1 is chosen sufficiently small, then the level set $\Gamma_{\overline{g}} = \{x \in \mathbf{R}^n : \overline{g}(x) = 1\}$ satisfies

$$\sup\{|x'| : x \in \Gamma_{\overline{g}}\} = \theta' \sup\{x_n : x \in \Gamma_{\overline{g}}\}$$

for some $\theta' > 1$ sufficiently large. Hence by Lemma 2.7, w is defined in a strip region.

5.3. *Entire convex solutions.* Next we prove the second part of Theorem 5.1. We prove that if $n \geq 3$ and $\theta > 1$, the solution w obtained in Lemma 5.4 is defined in the entire space $\mathbf{R}^n$. The following proof is necessary only when $n = 3$, since if $n \geq 4$, one can construct a sequence of functions w_k^* as above such that w_k, the Legendre transform of w_k^*, takes the form $w_k(x) = w_k(|\hat{x}|, |\widetilde{x}|)$ (where $\hat{x} = (x_1, \ldots, x_{n-2})$, $\widetilde{x} = (x_{n-1}, x_n)$; see Remark 4.1). Then $w_k(x) \leq \frac{1}{2}|x|^2$ and so $\{w_k\}$ subconverges to an entire convex solution of (1.1).

For any $h > 0$, denote

(5.14) $\qquad a_{h,k} = \sup\{|x'| : x \in \Gamma_{h,k}\}, \qquad b_{h,k} = \sup\{x_n : x \in \Gamma_{h,k}\},$

where $\Gamma_{h,k} = \{w_k = h\}$. Letting

$$\hat{w} = \frac{1}{2(n-2)}(|x'|^2 - a_{h,k}^2) + h,$$

we have $\mathcal{L}_1[\hat{w}] \geq 1$. By the comparison principle, $w_k \geq \hat{w}$ and so $\hat{w}(0) \leq 0$. We obtain $a_{h,k} \geq (2(n-2)h)^{1/2}$. When $k \to \infty$,

(5.15) $\qquad\qquad\qquad a_h \geq (2(n-2)h)^{1/2},$

where we denote

$$a_h = \sup\{|x'| : x \in \Gamma_h\}, \qquad b_h = \sup\{x_n : x \in \Gamma_h\},$$

where $\Gamma_h = \{w = h\}$. To prove that w is an entire solution, it suffices to prove

(5.16) $\qquad\qquad\qquad b_h \to \infty \ \text{ as } \ h \to \infty.$

To prove (5.16) we use Lemma 2.7. That is, if there exist $h_0 > 1$ and $\beta > 0$ sufficiently small, such that $b_{h_0,k} \leq \beta h_0^{1/2}$, then there exists a constant $C > 0$ independent of k, such that $b_{h,k} \leq C$ for all $h \in (1, m_k)$; that is, w_k is defined in the strip $\{|x_n| < C\}$, where m_k is the constant in (5.12).

It follows that if $b_{\hat{h},k} \geq \beta \hat{h}^{1/2}$ for some large $\hat{h} > 1$, then there exists $\delta > 0$, independent of $\hat{h}$ and k, such that $b_{h,k} \geq \delta h^{1/2}$ for all $1 < h < \hat{h}$. Therefore to prove w is an entire solution, it suffices to prove that there exists a constant $\beta > 0$ independent of k, and a sequence τ_k, where $\tau_k \leq m_k$ and $\tau_k \to \infty$ as $k \to \infty$, such that

(5.17) $\qquad\qquad\qquad b_{\tau_k,k} \geq \beta \tau_k^{1/2}.$

Denote

(5.18) $\qquad r_{h,k} = \sup\{|x'| : x \in \Gamma_{h,k}^*\}, \qquad t_{h,k} = \sup\{x_n : x \in \Gamma_{h,k}^*\},$

where $\Gamma_{h,k}^* = \{w_k^* = h\}$.

LEMMA 5.5. *Suppose $r_{h,k} \geq t_{h,k}$. Then we have the estimates*

(5.19) $\qquad\qquad\qquad \sqrt{h/n} \leq r_{h,k} \leq \sqrt{2h}.$

Proof. The function

$$v = h + \frac{n}{2}(|x|^2 - 2r_{h,k}^2)$$

is a subsolution of (5.3), namely $\mathcal{L}_1^*[v] \geq 1$. Moreover, since $r_{h,k} \geq t_{h,k}$, we have $v \leq h$ on $\{w_k^* = h\}$. Hence by the comparison principle, $v \leq w_k^*$. In particular $0 = w_k^*(0) \geq v(0) = h - nr_{h,k}^2$. The first inequality of (5.19) is proved.

Next observe that when $n \geq 3$, the function

$$v = h + \frac{1}{2}(|x'|^2 - r_{h,k}^2) + Kx_n^2$$

is a super-solution of (5.3), namely $\mathcal{L}_1^*[v] \leq 1$, for any $K > 1$. For any $\varepsilon > 0$ we can choose K sufficiently large such that $v > w_k^* - \varepsilon$ on $\{w_k^* = h\}$. Hence $v(0) \geq w_k^*(0) = 0$ and we obtain the second inequality. $\square$

Note that $\mathcal{L}_1^*[u^*] \geq 1$ (≤ 1, resp.) is equivalent to $\mathcal{L}_1[u] \leq 1$ (≥ 1, resp.), where u^* is the Legendre transform of u.

Proof of Theorem 5.1. The first part of Theorem 5.1 has been proved in Section 5.2. We need only to prove that when $n \geq 3$ and $\theta > 1$, w is an entire solution. It suffices to prove (5.17).

By (5.9) we have $r_{h,k} \geq t_{h,k}$ when h is close to k. If $r_{h,k} \geq t_{h,k}$ for all $h < k$, then by (5.19),

$$(5.20) \qquad \frac{1}{2}|x'|^2 \leq w_k^*(x', 0) \leq n|x'|^2.$$

By the Legendre transform,

$$\frac{1}{4n}|x'|^2 \leq w_k(x', 0) \leq \frac{1}{2}|x'|^2.$$

Sending $k \to \infty$ we obtain

$$(5.21) \qquad \frac{1}{4n}|x'|^2 \leq w(x', 0) \leq \frac{1}{2}|x'|^2.$$

Hence

$$a_h \leq (4nh)^{1/2} \quad \forall \, h > 1.$$

By Lemma 2.6, $a_h b_h \geq Ch$ (a_h and b_h here correspond to $\bar{a}_h$ and $\bar{b}_h$ in Lemma 2.6). Hence $b_h \to \infty$ as $h \to \infty$, and w is defined in the whole space $\mathbf{R}^n$.

If there exists $h > 0$ such that $r_{h,k} < t_{h,k}$, we denote

$$(5.22) \qquad h_k = \inf\{h : r_{h',k} \geq t_{h',k} \ \forall \, h < h' < k\}.$$

If $\{h_k\}$ is uniformly bounded, or contains a uniformly bounded subsequence, then (5.20), and so also (5.21), holds for $|x'|$ large. Hence w is also an entire solution.

Finally we consider the case $h_k \to \infty$ as $k \to \infty$. Denote $G_k^* = \{w_k^* < h_k\}$ and $G_k = Dw_k^*(G_k^*)$ and consider the solution w_k in the domain G_k. Denote

$$(5.23) \qquad \tau_k =: \inf\{w_k(x) : x \in \partial G_k\}.$$

Since $h_k \to \infty$, similarly to Lemma 5.3 we have $\tau_k \to \infty$ as $k \to \infty$. Denote $\widehat{G}_k^* = \frac{1}{\sqrt{h_k}}G_k^*$, $\widehat{G}_k = \frac{1}{\sqrt{h_k}}G_k$, and

$$(5.24) \qquad \hat{w}_k^*(x) = \frac{1}{h_k}w_k^*(\sqrt{h_k}x), \qquad \hat{w}_k(x) = \frac{1}{h_k}w_k(\sqrt{h_k}x).$$

Then $\hat{w}_k$ is the Legendre transform of $\hat{w}_k^*$, and

$$(5.25) \qquad c_k =: \inf\{\hat{w}_k(x) : x \in \partial\widehat{G}_k\} = \tau_k/h_k.$$

We claim

$$(5.26) \qquad c_k \leq 4n \quad \forall\, k.$$

Indeed, by (5.22), $\widehat{G}_k^*$ has a good shape, namely

$$(5.27) \qquad \sup\{|x'| : x \in \widehat{G}_k^*\} = \sup\{x_n : x \in \widehat{G}_k^*\}.$$

As the domain $\widehat{G}_k^*$ is rotationally symmetric with respect to x', by Lemma 5.5 and (5.27) we have $\widehat{G}_k^* \subset B_2(0)$. Note that $\hat{w}_k^*$ is the Legendre transform of $\hat{w}_k$; now, $D\hat{w}_k(\widehat{G}_k) = \widehat{G}_k^*$. Hence

$$(5.28) \qquad |D\hat{w}_k| \leq 2 \quad \text{in } \widehat{G}_k.$$

Since $\widehat{G}_k^* \subset B_2(0)$, there exists $r \leq 2$ such that $\widehat{G}_k^* \subset B_r(0)$, and $\partial\widehat{G}_k^*$ and $\partial B_r(0)$ have a common boundary point x_0. Observe that $v = 1 + \frac{n}{2}(|x|^2 - r^2)$ is a subsolution of (5.3) and $v \leq \hat{w}_k^* = 1$ on $\partial\widehat{G}_k^*$. Also, $|D\hat{w}_k^*(x_0)| \leq |Dv(x_0)| \leq 2n$. By the Legendre transform, $y_0 = D\hat{w}_k^*(x_0)$ is a boundary point of $\widehat{G}_k$. Since $\hat{w}_k^*$ is constant on $\partial\widehat{G}_k^*$, one easily verifies that the domain $\widehat{G}_k = D\hat{w}_k^*(\widehat{G}_k^*)$ is star-shaped. By (5.28) we obtain

$$\hat{w}_k(y_0) \leq |y_0| \sup |D\hat{w}_k| \leq 4n;$$

that is, (5.26) holds.

Denote

$$\varepsilon_k = \inf\{x_n > 0 : \hat{w}_k(0,\ldots,0,x_n) \geq c_k\} = \sup\{x_n : \hat{w}_k(x',x_n) \leq c_k\}.$$

Then by (5.26) and the rescaling (5.24), to prove (5.17) it suffices to prove that ε_k has a uniform positive lower bound, namely

$$(5.29) \qquad \varepsilon_k \geq \varepsilon_0 > 0 \quad \forall\, k.$$

We prove (5.29) as follows. Observing that $\hat{w}_k^*(0) = 0$, $\hat{w}_k^* = 1$ on $\partial\widehat{G}_k^*$, and $\widehat{G}_k^* \subset B_2(0)$, we see that $|D\hat{w}_k^*| \geq 1/2$ on $\partial\widehat{G}_k^*$. That is, $\widehat{G}_k \supset B_{1/2}(0)$ for all k. By (5.28) we may suppose $\hat{w}_k \to \hat{w}$ in $B_{1/2}(0)$. Then $\hat{w} \geq 0$, $\hat{w}(0) = 0$ and $|D\hat{w}| \leq 2$ in $B_{1/2}(0)$ since $\hat{w}_k$ satisfies the equation $\mathcal{L}_\sigma[v] = 1$ in $\widehat{G}_k$, where $\sigma = 1/h_k$, and $\mathcal{L}_\sigma$ is the operator in (1.2). By Lemma 2.5, $\hat{w}$ is a solution of

$$(5.30) \qquad \mathcal{L}_0[v] = 1 \quad \text{in } B_{1/2}(0).$$

If $\hat{w}(x) > 0$ for any $x \neq 0$, then $c = \inf\{\hat{w}(x) : |x| = 1/2\} > 0$. Hence $c_k \geq \frac{1}{2}c$ for sufficiently large k. By (5.28) we then have $\varepsilon_k \geq c_k/2 \geq c/4$ for large k. Hence (5.29) holds.

If the convex set $\{\hat{w} = 0\}$ contains a line segment ℓ, then ℓ is either contained in the x_n-axis, or in the plane $\{x_n = 0\}$, as the function $\hat{w}$ is rotationally

symmetric with respect to x' and symmetric in x_n. Furthermore, by the comparison principle, the set $\{\hat{w} = 0\}$ contains no interior points, i.e. the Lebesgue measure $|\{\hat{w} = 0\}| = 0$. Note that the origin is the middle point of ℓ as $\hat{w}$ is an even function.

In the former case, namely if ℓ is contained in the x_n-axis, we have $\hat{w}(x) > 0$ for any point $x \in \partial B_{1/2}(0)$ not lying on the x_n-axis, for otherwise $\hat{w} = 0$ on a set with interior points. Hence

$$c_k \geq \inf_{\partial Q} \hat{w}_k(x) = \hat{w}_k \left(0, \ldots, 0, \frac{d}{2}\right),$$

where $Q = \{x \in \mathbf{R}^n : |x'| < \frac{d}{2}, |x_n| < \frac{d}{2}\}$, where d is the arclength of ℓ. Hence we have $\varepsilon_k \geq \frac{d}{2}$ and (5.29) holds.

We claim the latter case, namely the case when the line segment ℓ is contained in the plane $\{x_n = 0\}$, does not occur. Indeed, if $\ell \subset \{x_n = 0\}$, then $\{\hat{w} = 0\}$ is a disc type set $D = \{x \in B_{1/2}(0) : |x'| < d, x_n = 0\}$ for some $d > 0$. Since the set $\{\hat{w} = 0\}$ contains no interior points, the level set $\{x \in \mathbf{R}^n : \hat{w}(x) = h\}$ is contained in a strip $\{|x_n| < \delta\}$ with $\delta \to 0$ as $h \to 0$. Therefore if $h > 0$ is sufficiently small, there exist points on the level set $\{\hat{w} = h\}$ at which the mean curvature κ of $\{\hat{w} = h\}$ is sufficiently small. Writing equation (5.30) in the form $\kappa |D\hat{w}| = 1$ (see (2.2) with $\sigma = 0$), we find that $|D\hat{w}|$ is very large. On the other hand we have $|D\hat{w}| \leq 2$ by (5.28), reaching a contradiction. Hence the latter case does not occur. This completes the proof. $\square$

6. Applications to the mean curvature flow

First we consider applications of Theorem 1.1. For a mean convex flow $\mathcal{M} = \cup_{t \in [0,T)} \mathcal{M}_t$ in $\mathbf{R}^{n+1}$, let $\mathcal{F}$ denote the set of all limit flows (namely blow-up solutions) to $\mathcal{M}$ before first time singularity. A key result in [26] is that a limit flow in $\mathcal{F}$ cannot be a hyperplane of multiplicity-two, from which it follows that the "grim reaper" $x_{n+1} = \log \sec x_1$ cannot be a limit flow in $\mathcal{F}$; see [22] and [27]. As indicated in the introduction, we will consider solutions with positive mean curvature only.

COROLLARY 6.1. *A flow $\mathcal{M}' \in \mathcal{F}$ (with positive mean curvature at some point) must sweep the whole space $\mathbf{R}^{n+1}$.*

Proof. By Proposition 4.1, $\mathcal{M}'$ is a graph in space-time (with $x_{n+2} = -t$) of a convex function u on a convex domain in $\mathbf{R}^{n+1}$, and u is a complete convex solution of (1.2) with $\sigma = 0$. If u is not defined in the entire $\mathbf{R}^{n+1}$, by Corollaries 2.1 and 2.2, u is defined in a strip region of the form $\{x \in \mathbf{R}^{n+1} : |x_{n+1}| < C\}$ (in appropriate coordinates). By Lemma 2.6, the projection of $\{u = h\}$ on the $x' = (x_1, \ldots, x_n)$ plane contains the ball $\{|x'| < Ch\}$ for some

$C > 0$ independent of $h \geq 1$. It follows that the tangent flow at infinity to the solution u (more precisely, there is a limit flow to the original mean convex flow which) is a multiplicity-two plane; see Corollary 12.5 in [26] or Corollary 4 in [27]. But a multiplicity-two plane does not occur as a limit flow in $\mathcal{F}$. Hence u must be defined in the whole $\mathbf{R}^{n+1}$. □

If a convex translating solution $\mathcal{M}'$ is a limit flow to a mean convex flow in $\mathbf{R}^3$, then $\mathcal{M}'$ is the graph of a convex function u satisfying (1.1). By Corollary 6.1, u is defined in the whole $\mathbf{R}^2$. By Theorem 1.1, u is rotationally symmetric. We obtain Corollary 1.1.

From the case $\sigma = 0$ in Theorem 1.1, we also have the following:

COROLLARY 6.2. *A convex solution to the curve shortening flow which sweeps the whole space $\mathbf{R}^2$ must be a shrinking circle.*

From Theorem 1.2, there exist closed convex solutions to the curve shortening flow which are not the shrinking circle.

Next we consider applications of Theorem 1.3. First we have

COROLLARY 6.3. *Let $\mathcal{M} = \{\mathcal{M}_t\}$ be an ancient convex solution to the mean curvature flow. Let $\mathcal{M}'_t = \{x \in R^{n+1} : (-t)^{1/2}x \in \mathcal{M}_t\}$ be a dilation of $\mathcal{M}_t$. Then $\mathcal{M}'_t$, after a proper rotation of coordinates, converges as $t \to -\infty$ to one of the following*:
  (i) *an n-sphere of radius $\sqrt{2n}$;*
  (ii) *a cylinder $S^k \times \mathbf{R}^{n-k}$, where S^k is a k-sphere of radius $\sqrt{2k}$;*
  (iii) *the plane $\mathbf{R}^n$ of multiplicity-two.*

Proof. By Proposition 4.1, $\mathcal{M}$ is the graph in space-time of a convex function u in $\mathbf{R}^{n+1}$ satisfying equation (1.2) with $\sigma = 0$. If u is an entire solution, (iii) does not occur and Corollary 6.3 is equivalent to Theorem 1.3. If u is not an entire solution, by Corollaries 2.1 and 2.2, u is defined in a strip region. As in the proof of Corollary 6.1, the projection of $\mathcal{M}_t = \{u = -t\}$ on the x'-plane contains the ball $\{|x'| < C|t|\}$; by convexity we see that $\mathcal{M}'_t$ converges to a plane. □

From Theorem 1.3 (the case $\sigma = 0$) we also obtain

COROLLARY 6.4. *Let $\mathcal{M} = \bigcup_{t \in [0,T)} \mathcal{M}_t$ be a mean convex flow in $\mathbf{R}^{n+1}$. Suppose $h_i \to \infty$ and $p_i \in M_{t_i}$ such that the blow-up sequence*

$$\mathcal{M}_i = \{(h_i(p - p_i), h_i^2(t - t_i)) : (p, t) \in \mathcal{M}\}$$

converges to an ancient convex solution. Then there exist $h'_i \to \infty$ such that the corresponding blow-up sequence $\mathcal{M}'_i = \{(h'_i(p-p_i), h'^2_i(t-t_i)) : (p, t) \in \mathcal{M}\}$, converges along a subsequence to a shrinking sphere or cylinder.

At type II singularity, from Theorem 1.3 (the case $\sigma = 1$) we have

COROLLARY 6.5. *Let $\mathcal{M}_i$ be as in Corollary 6.4. If $\mathcal{M}_i$ converges to a convex translating solution, then there is a sequence of positive constants $\lambda_i \to 0$ such that, in a proper coordinate system, the blow-up sequence $\widetilde{\mathcal{M}}_i = \{(\lambda_i x_1, \ldots, \lambda_i x_n, \lambda_i^2 x_{n+1}, \lambda_i^2 t) : (x_1, \ldots, x_{n+1}, t) \in \mathcal{M}_i\}$ converges to the flow $\mathcal{M}' = \{(x, \eta_k(x) + t) \in \mathbf{R}^{n+1} : x \in \mathbf{R}^n, t \in \mathbf{R}\}$, where η_k is as given in (1.4).*

In Corollaries 6.4–6.5, we need not restrict ourselves to limit flows in $\mathcal{F}$.

As indicated in the introduction, our Theorem 1.3 corresponds to Perelman's classification of ancient κ-noncollapsing solutions with nonnegative sectional curvature to the three-dimensional Ricci flow [20]. Indeed, Theorems 2.1 and 2.2 imply that the set of entire, ancient convex solutions in $\mathbf{R}^{n+1}$, which includes all limit flows in $\mathcal{F}$ by Corollary 6.1, is compact if one normalizes the solutions such that their mean curvature equals one at some fixed point, say the origin. To see this, let $\mathcal{M} = \{\mathcal{M}_t\}$ be an ancient convex solution. Then $\mathcal{M}$ is the graph in space-time of an entire convex function u. If the mean curvature of $\mathcal{M}$ equals one at the origin, then $|Du(0)| = 1$, and so the compactness follows from Theorems 2.1 and 2.2.

For ancient convex solutions in $\mathbf{R}^3$, we have the following

COROLLARY 6.6. *Let $\mathcal{M} = \{\mathcal{M}_t\}$ be an ancient convex solutions which is a limit flow to a mean convex flow in $\mathbf{R}^3$. Suppose at time $t = 0$, the time slice $\mathcal{M}_0 = \{u = 0\}$ is noncompact. Then for all $\varepsilon > 0$, there is a compact set G such that any point in $\mathcal{M}_0 \backslash G$ has a neighborhood which is, after normalization, ε-close (see (2.55)) to the cylinder $S^1 \times \mathbf{R}^1$.*

Proof. As indicated above, $\mathcal{M}$ is the graph in space-time of a convex function u defined in the whole $\mathbf{R}^3$. If the level set $\{u = h\}$ is a cylinder, then u is a convex function of two variables and Corollary 6.6 follows from Theorem 1.1. In this case the set $G = \emptyset$. Otherwise by convexity we may assume by choosing an appropriate coordinate system that $u(0) = 0$, $u \geq 0$ in $\{x_1 \leq 0\}$, and $u \leq 0$ on the positive x_1-axis. For $a \in (0, \infty)$, let x_a be the point such that $u(x_a) = \inf\{u(x) : x_1 = a\}$. If $\inf u$ is bounded, let $u_a(x) = u(x + x_a) - u(x_a)$. By the convexity of u, $|Du(x_a)|$ is decreasing as $a \to \infty$, so it is uniformly bounded. By the above mentioned compactness result (Theorem 2.1), u_a subconverges to a convex function u_0. By our choice of coordinates, we have $u_0 \leq 0$ on the positive x_1-axis. But since $\inf u$ is bounded, we have $u_0 = 0$ on the positive x_1-axis. By Lemma 2.9 it follows that u_0 is independent of x_1. That is u_0 is a function of x_2 and x_3. By Theorem 1.1, u_0 is rotationally symmetric in x_2 and x_3.

If $\inf u = -\infty$, then $u(x_a) \to -\infty$ as $a \to \infty$. By the compactness Theorem 2.1, the sequence $u_a(x) = \frac{1}{h_a}[u(\sqrt{h_a}x + x_a) - u(x_a)]$, where $h_a = |u(x_a)|$ converges along a subsequence to a convex solution u_0 of (4.1). Since

$|Du(x_a)|$ is uniformly bounded, we see that $u_0 = 0$ on the positive x_1-axis; see (2.48). By Lemma 2.9 it follows that u_0 is independent of x_1. By Theorem 1.1, u_0 is rotationally symmetric in x_2 and x_3. $\qquad\square$

By a compactness argument, one easily sees that the diameter of G is uniformly bounded if the maximum of the mean curvature of $\mathcal{M}_0$ is equal to 1. We remark that by Theorem 1.2, Corollary 6.6 is not true for convex solutions in $\mathbf{R}^n$ for $n \geq 4$. But in high dimensions we have, accordingly, Corollary 6.3, which says that an ancient convex solution behaves asymptotically as $t \to -\infty$ like a sphere or cylinder. Note that for any dimension $n \geq 2$, the set $\mathcal{F}$ contains all blow-up solutions before the first time singularity, and that the blow-up sequence converges smoothly on any compact sets to an ancient convex solution [27]. Therefore, similar to [21], one may infer that if $\mathcal{M} = \{\mathcal{M}_t\}$ is a mean convex flow, then at any point $x_t \in \mathcal{M}_t$ with large mean curvature before the first time singularity, $\mathcal{M}_t$ satisfies a canonical neighborhood condition. If $n = 2$, the condition is very similar to that in [21]. In high dimensions, the condition is more complicated. We will not get into details in this direction. Concerning the geometry of the singularity set it would be more interesting if one could prove the following:

CONJECTURE. *Let K be a smooth and compact domain in $\mathbf{R}^{n+1}$. Suppose the boundary ∂K has positive mean curvature. Let $\mathcal{M} = \cup_t \mathcal{M}_t$ be the solution to the mean curvature flow with initial condition ∂K. Then*

(i) *singularity occurs only at finitely many times;*
(ii) *at each singular time the singularity set consists of finitely many connected components;*
(iii) *each connected component is contained in a C^1 smooth $(n-1)$-submanifold (with or without boundary). If $n = 2$, each component is either a single point or a C^1 curve.*

One may wish to assume $n < 7$, but the conjecture is likely to be true for all $n \geq 2$, as the second order derivative estimates for (4.1) may hold for any $n \geq 2$.

We conclude this paper with some interesting questions related to our Theorems 1.1–1.3. For Theorem 1.1 a question is whether an entire solution of (1.1) in $\mathbf{R}^2$ is convex. For Theorem 1.2 a question is whether a convex solution u of (1.1) is rotationally symmetric if $|Du(x)| \to \infty$ as $|x| \to \infty$. We expect affirmative answers to both questions.

For Theorem 1.3, an interesting question is whether a nonrotationally symmetric ancient convex solution can occur as a limit flow in $\mathbf{R}^n$. We believe that any limit flow to a mean convex flow at isolated singularities in space-time is rotationally symmetric, otherwise nonrotationally symmetric convex limit flow may occur if the following situations arise: (a) if there exists a

mean convex flow in $\mathbf{R}^n$ ($n \geq 4$) which develops first-time type II singularities simultaneously on a nonsmooth curve (say a polygon) in a 2-plane; (b) if a mean curvature flow in $\mathbf{R}^n$ ($n \geq 4$) develops a first-time singularity on a smooth curve, and the singularity is type I, except one type II singular point. In case (a) we expect a nonrotationally symmetric convex translating solution at the vertices of the polygon. In case (b) a blow-up solution near the type II singular point may not be rotationally symmetric.

References

[1] S. BERNSTEIN, Sur un théorème de géométrie et son application aux equations aux dérivées partielles du type elliptique, *Comm. Soc. Math. Kharkvo* **15** (1915), 38–45.

[2] _____, Über ein geometrisches Theorem und seine Anwendung auf die partiellen Differentialgleichungen vom elliptischen Typus, *Math. Z.* **26** (1927), 551–558. MR 1544873. doi: `10.1007/BF01475472`.

[3] Y. G. CHEN, Y. GIGA, and S. GOTO, Uniqueness and existence of viscosity solutions of generalized mean curvature flow equations, *J. Differential Geom.* **33** (1991), 749–786. MR 1100211. Zbl 0696.35087. Available at http://projecteuclid.org/getRecord?id=euclid.jdg/1214446564.

[4] K.-S. CHOU and X.-J. WANG, Entire solutions of the Monge-Ampère equation, *Comm. Pure Appl. Math.* **49** (1996), 529–539. MR 1377561. Zbl 0851.35035. doi: `10.1002/(SICI)1097-0312(199605)49:5<529::AID-CPA2>3.3.CO;2-G`.

[5] L. C. EVANS, Classical solutions of fully nonlinear, convex, second-order elliptic equations, *Comm. Pure Appl. Math.* **35** (1982), 333–363. MR 0649348. Zbl `0469.35022`. doi: `10.1002/cpa.3160350303`.

[6] L. C. EVANS and J. SPRUCK, Motion of level sets by mean curvature. I, *J. Differential Geom.* **33** (1991), 635–681. MR 1100206. Zbl 0726.53029. Available at http://projecteuclid.org/getRecord?id=euclid.jdg/1214446559.

[7] M. GAGE and R. S. HAMILTON, The heat equation shrinking convex plane curves, *J. Differential Geom.* **23** (1986), 69–96. MR 0840401. Zbl 0621.53001. Available at http://projecteuclid.org/getRecord?id=euclid.jdg/1214439902.

[8] D. GILBARG and N. S. TRUDINGER, *Elliptic Partial Differential Equations of Second Order*, second ed., *Grundl. Math. Wissen.* **224**, Springer-Verlag, New York, 1983. MR 0737190. Zbl 0562.35001.

[9] G. HUISKEN, Flow by mean curvature of convex surfaces into spheres, *J. Differential Geom.* **20** (1984), 237–266. MR 0772132. Zbl 0556.53001. Available at http://projecteuclid.org/getRecord?id=euclid.jdg/1214438998.

[10] _____, Local and global behaviour of hypersurfaces moving by mean curvature, in *Differential Geometry*: *Partial Differential Equations on Manifolds* (Los Angeles, CA, 1990), *Proc. Sympos. Pure Math.* **54**, Amer. Math. Soc., Providence, RI, 1993, pp. 175–191. MR 1216584. Zbl 0791.58090.

[11] G. HUISKEN and C. SINESTRARI, Convexity estimates for mean curvature flow and singularities of mean convex surfaces, *Acta Math.* **183** (1999), 45–70. MR 1719551. Zbl 0992.53051. doi: `10.1007/BF02392946`.

[12] G. HUISKEN and C. SINESTRARI, Mean curvature flow singularities for mean convex surfaces, *Calc. Var. Partial Differential Equations* **8** (1999), 1–14. MR 1666878. Zbl 0992.53052. doi: 10.1007/s005260050113.

[13] _____, Mean curvature flow with surgeries of two-convex hypersurfaces, *Invent. Math.* **175** (2009), 137–221. MR 2461428. Zbl 1170.53042. doi: 10.1007/s00222-008-0148-4.

[14] N. IVOCHKINA, N. TRUDINGER, and X.-J. WANG, The Dirichlet problem for degenerate Hessian equations, *Comm. Partial Differential Equations* **29** (2004), 219–235. MR 2038151. Zbl 1140.35418. doi: 10.1081/PDE-120028851.

[15] B. KAWOHL, *Rearrangements and Convexity of Level Sets in* PDE, *Lecture Notes in Math.* **1150**, Springer-Verlag, New York, 1985. MR 0810619. Zbl 0593.35002.

[16] N. V. KRYLOV, *Nonlinear Elliptic and Parabolic Equations of the Second Order*, *Math. Appl.* (*Soviet Series*) **7**, D. Reidel Publishing Co., Dordrecht, 1987. MR 0901759. Zbl 0619.35004.

[17] _____, Smoothness of the payoff function for a controllable diffusion process in a domain, *Izv. Akad. Nauk SSSR Ser. Mat.* **53** (1989), 66–96. MR 0992979.

[18] _____, Lectures on fully nonlinear elliptic equations, 1994, Lipschitz Lectures, Univ. of Bonn.

[19] D. G. MIGUEL, *Differentiation of Integrals in* $\mathbf{R}^n$, *Lecture Notes in Math.* **481**, Springer-Verlag, New York, 1977.

[20] G. PERELMAN, The entropy formula for the ricci flow and its geometric applications. arXiv math.DG/0211159.

[21] _____, Ricci flow with surgery on three manifolds. arXiv math.DG/0303109.

[22] W. SHENG and X.-J. WANG, Singularity profile in the mean curvature flow, *Methods Appl. Anal.* **16** (2009), 139–155. MR 2563745. Zbl 1184.53071.

[23] L. SIMON, The minimal surface equation, in *Geometry*, V, *Encyclopaedia Math. Sci.* **90**, Springer-Verlag, New York, 1997, pp. 239–272. MR 1490041. Zbl 0905.53003.

[24] N. S. TRUDINGER, On the Dirichlet problem for Hessian equations, *Acta Math.* **175** (1995), 151–164. MR 1368245. Zbl 0887.35061. doi: 10.1007/BF02393303.

[25] N. S. TRUDINGER and X.-J. WANG, The Bernstein problem for affine maximal hypersurfaces, *Invent. Math.* **140** (2000), 399–422. MR 1757001. Zbl 0978.53021. doi: 10.1007/s002220000059.

[26] B. WHITE, The size of the singular set in mean curvature flow of mean-convex sets, *J. Amer. Math. Soc.* **13** (2000), 665–695. MR 1758759. Zbl 0961.53039. doi: 10.1090/S0894-0347-00-00338-6.

[27] _____, The nature of singularities in mean curvature flow of mean-convex sets, *J. Amer. Math. Soc.* **16** (2003), 123–138. MR 1937202. Zbl 1027.53078. doi: 10.1090/S0894-0347-02-00406-X.

AUSTRALIAN NATIONAL UNIVERSITY, CANBERRA, AUSTRALIA
E-mail: wang@maths.anu.edu.au