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Weyl group multiple Dirichlet series, Eisenstein series and crystal bases

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Abstract

We show that the Whittaker coefficients of Borel Eisenstein series on the metaplectic covers of GL_{r+1} can be described as multiple Dirichlet series in r complex variables, whose coefficients are computed by attaching a number-theoretic quantity (a product of Gauss sums) to each vertex in a crystal graph. These Gauss sums depend on "string data" previously introduced in work of Lusztig, Berenstein and Zelevinsky, and Littelmann. These data are the lengths of segments in a path from the given vertex to the vertex of lowest weight, depending on a factorization of the long Weyl group element into simple reflections. The coefficients may also be described as sums over strict Gelfand-Tsetlin patterns. The description is uniform in the degree of the metaplectic cover.

1. Introduction

When F is a local field containing the group μ_n of n^{th} roots of unity, and when G is a split semisimple simply connected algebraic group, Matsumoto [26] defined an n-fold covering group of $G(F)$, that is, a central extension of $G(F)$ by μ_n. Similarly if F is a global field with adele ring $\mathbb{A}_F$ containing μ_n there is a cover $\widetilde{G}(\mathbb{A}_F)$ of $G(\mathbb{A}_F)$ that splits over $G(F)$. The construction is built on ideas of Kubota [22] and makes use of the reciprocity laws of class field theory. It can be extended to reductive and nonsimply connected groups, sometimes at the expense of requiring more roots of unity in F. We will refer to such an extension as a *metaplectic group*. The special case $n = 1$ is contained in this situation, but is simpler and we will refer to this as the *nonmetaplectic* case.

Fourier-Whittaker coefficients of Eisenstein series play a central role in the theory of automorphic forms. In the nonmetaplectic case one has uniqueness of Whittaker models ([31], [33], [17]). Over a global field, this implies that the Whittaker functional is Eulerian, i.e. factors as a product over primes. And at almost all places, the local contribution to the Whittaker coefficient may be computed using the Casselman-Shalika formula, which expresses a value of the spherical Whittaker function as a character of a finite-dimensional representation of the Langlands dual group ${}^L G^\circ$.

In the metaplectic case, one may again define Whittaker functionals, but with the fundamental difference that these are now usually *not* unique. As a consequence, the Whittaker coefficients of metaplectic automorphic forms are not in general Eulerian. The lack of uniqueness of Whittaker models may also be the reason that the Whittaker coefficients of metaplectic Eisenstein series and the extension of the Casselman-Shalika formula to metaplectic groups have not been investigated extensively.

This paper contains a treatment of these topics for metaplectic covers of GL_{r+1}. (For technical reasons we will actually work over SL_{r+1} but the result is best understood as a statement about GL_{r+1}.) We will compute the global Whittaker coefficients of the Borel Eisenstein series. We will prove a "twisted multiplicativity" statement that substitutes for the Eulerian property in showing that one may reconstruct these coefficients from prime-power pieces. We will also determine these local contributions. The local determination may be regarded as a generalization of the Casselman-Shalika formula. For general n the prime-power-supported contribution, or "p-part" for short, is not a character value as it is in the nonmetaplectic case, but it resembles a character value in which the weight monomials are multiplied by products of Gauss sums, computed using crystal bases.

We work over F_S, the product of completions of F at a sufficiently large finite set of places S, and in this setting we will exhibit a representation of these global Whittaker coefficients as Dirichlet series in several complex variables of the form

$$Z(s_1, \dots, s_r; \mathbf{m}) = \sum_{0 \neq C_1, \dots, C_r \in \mathfrak{o}_S/\mathfrak{o}_S^\times} \frac{H(C_1, \dots, C_r; \mathbf{m})\Psi(C_1, \dots, C_r)}{|C_1|^{2s_1} \cdots |C_r|^{2s_r}}. \tag{1}$$

This notation will be treated systematically in the text, but for now we summarize briefly. The C_i are in the ring $\mathfrak{o}_S$ of S-integers, where S is a finite set of places large enough that $\mathfrak{o}_S$ is a principal ideal domain, so that the sum is essentially over nonzero ideals. The norm $|C_i|$ is the cardinality of $\mathfrak{o}_S/C_i\mathfrak{o}_S$. Here $\mathbf{m} = (m_1, \dots, m_r)$ is a vector of nonzero S-integers that parametrizes a nondegenerate character giving a Whittaker functional on $\widetilde{\mathrm{GL}}_{r+1}$ and Ψ is a complex-valued function that depends on the choice of inducing data and varies over a finite-dimensional vector space $\mathcal{M}(\Omega^r)$ defined in Section 6, and H carries the main number theoretic content. It is the product ΨH that is well-defined modulo units, but H is the more interesting of these two functions.

We show that the coefficients H are not in general multiplicative, but possess a generalization of this property which we refer to as *twisted multiplicativity*. More precisely, if $\mathbf{C} = (C_1, \dots, C_r)$ and $\mathbf{C}' = (C_1', \dots, C_r')$ with $\gcd(C_1 \cdots C_r, C_1' \cdots C_r') = 1$, then

$$H(C_1C_1', \dots, C_rC_r'; \mathbf{m}) = \varepsilon_{\boldsymbol{C},\boldsymbol{C}'} H(\mathbf{C}; \mathbf{m})H(\mathbf{C}'; \mathbf{m})$$

where n is the degree of the metaplectic cover and $\varepsilon_{\boldsymbol{C},\boldsymbol{C}'}$ is an n^{th} root of unity given in terms of n^{th} power residue symbols; see Theorem 3. When $n = 1$ (the nonmetaplectic case) we recover the usual multiplicativity which follows from the uniqueness of the Whittaker functional. In addition, if $\mathbf{m} = (m_1, \dots, m_r)$ and $\mathbf{m'} = (m'_1, \dots, m'_r)$ with $\gcd(C_1 \cdots C_r, m'_1 \cdots m'_r) = 1$, then

$$H(\mathbf{C}; \mathbf{mm'}) = \left(\frac{m'_1}{C_1}\right)^{-1} \cdots \left(\frac{m'_r}{C_r}\right)^{-1} H(\mathbf{C}; \mathbf{m})$$

where $\left(\frac{\cdot}{\cdot}\right)$ is n^{th} power residue symbol; see Theorem 2.

Twisted multiplicativity reduces the determination of the general coefficients $H(\mathbf{C}; \mathbf{m})$ to coefficients of the form

$$H(p^{\boldsymbol{k}}; p^{\boldsymbol{l}}) := H(p^{k_1}, \dots, p^{k_r}; p^{l_1}, \dots, p^{l_r})$$

for primes p of $\mathfrak{o}_S$ and nonnegative r-tuples $\mathbf{k} = (k_1, \dots, k_r)$ and $\mathbf{l} = (l_1, \dots, l_r)$. We show that these coefficients may be obtained by attaching number-theoretic quantities to the vertices of a crystal graph and computing the sum over these vertices.

To explain the determination of these coefficients, recall that Kashiwara [19] associated with each dominant weight λ a crystal graph $\mathcal{B}_\lambda$, whose vertices are in bijection with a basis of the irreducible representation of the quantized universal enveloping algebra of $\mathrm{GL}_{r+1}(\mathbb{C})$, the L-group of GL_{r+1}, having λ as its highest weight. The recipe for $H(p^{\boldsymbol{k}}; p^{\boldsymbol{l}})$ interprets $\boldsymbol{l}$ as parametrizing a highest weight λ and $\boldsymbol{k}$ as parametrizing a weight μ, and sums a term $G(v)$ over all elements v of the crystal graph $\mathcal{B}_{\lambda+\rho}$ having weight μ. Here ρ is half the sum of the positive roots. The individual term $G(v)$ is a product of Gauss sums built from data describing a path of shortest length from v to the lowest weight vector of the crystal.

The crystal graph description of this paper was derived from an earlier description of Weyl group multiple Dirichlet series in terms of Gelfand-Tsetlin patterns conjectured by Brubaker, Bump, Friedberg and Hoffstein in [12]. We will prove the equivalence of the Gelfand-Tsetlin and crystal basis descriptions in this paper. We find the crystal graph description preferable to the Gelfand-Tsetlin description since it describes the contributions $G(v)$ in terms of representation-theoretic criteria rather than purely combinatorially. In doing so, it better suggests generalizations to other root systems, potentially including infinite Kac-Moody root systems. The term "Weyl group multiple Dirichlet series," introduced in [6], refers to multiple Dirichlet series with continuation and groups of functional equations, that are ultimately to be shown to agree with metaplectic Whittaker coefficients (as we do here), but whose properties may be developed without making use of automorphic forms on higher rank groups. Functional equations for multiple Dirichlet series whose coefficients were described as sums over Gelfand-Tsetlin patterns were established by the authors in [10], [11] by using combinatorial arguments to reduce them to the rank one case.

Even in the nonmetaplectic case the crystal graph description is a nontrivial reformulation of the Casselman-Shalika formula. The equivalence of the two statements is by means of a combinatorial formula of Tokuyama [35], which is a deformation of the Weyl character formula. These matters will be explained in further detail in the next section.

It is remarkable that there are not one but two distinct generalizations of the Casselman-Shalika formula to the metaplectic case. In the nonmetaplectic case, the Casselman-Shalika formula expresses the Whittaker function as an alternating sum over the Weyl group. In this vein, Chinta and Gunnells [14], [15] gave a formula for p-parts of Weyl group multiple Dirichlet series for arbitrary root systems with global analytic continuation and functional equations.

Both the Chinta-Gunnells description and the crystal graph description are generalizations of the Casselman-Shalika formula, and also (on the L-group side) of the Weyl character formula. But the two generalizations are so different that proving that the Chinta-Gunnells description for type A_r agrees with the definition given in [12], [10], [11] has been an open problem.

The next section precisely defines the way that one attaches a quantity $G(v)$ to a vertex v of the crystal graph. Sections 3, 4, and 5 define Eisenstein series on the metaplectic group induced from data on a maximal parabolic subgroup and compute their Whittaker coefficients. In Section 6, we show how these computations give an expression for the Whittaker coefficients of a minimal parabolic Eisenstein series on a metaplectic cover of SL_{r+1} in terms of Whittaker coefficients on SL_r, and this leads to a recursion relation for the Whittaker coefficients relating rank r to rank $r-1$. In Section 7, we use this relation and induction to prove that the resulting Dirichlet series satisfies the twisted multiplicativity properties. Then in Section 8 we prove (Theorem 4) that the p-part agrees with the conjectured recipe given in [12] in terms of Gelfand-Tsetlin patterns. This is accomplished by showing that the conjectured formula in terms of Gelfand-Tsetlin patterns satisfies the same recursion relation relating r to $r-1$ as the Whittaker coefficients. (For SL_2, it is immediate that the Gelfand-Tsetlin description gives the coefficients of the Eisenstein series.) In Section 9, we explain how to move between the Gelfand-Tsetlin definition and the crystal definition. In the final section, we collect these results and state the main theorem, Theorem 5.

We would like to thank Gautam Chinta and Paul Gunnells for many helpful discussions of these matters, and in particular for calling our attention to Littelmann [24]. The figure was made using SAGE. Both SAGE and Mathematica were used to refine our understanding of the crystal graph and

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Note: (*added September* 2009): Since the writing of this paper, the relationship between the (type A) crystal graph description of the p-part given here and the description via averaging due to Chinta and Gunnells has been established. This follows by combining the work of Chinta and Offen [16] demonstrating that the p-parts in [15] are p-adic metaplectic Whittaker functions in type A with the work of McNamara [27] demonstrating that the p-part definition presented in the next section is indeed a p-adic metaplectic Whittaker function on $\widetilde{\mathrm{SL}}_{r+1}$.

2. Crystal graph description of the p-part

In this section, we define the p-part of a multiple Dirichlet series as in (1). In Theorem 5, we will demonstrate that the resulting multiple Dirichlet series matches a Whittaker coefficient of a metaplectic Eisenstein series. For additional information related to this definition; see [11].

Let

$$\boldsymbol{t} = \begin{pmatrix} t_1 & & \\ & \ddots & \\ & & t_{r+1} \end{pmatrix} \in \mathrm{GL}_{r+1}(\mathbb{C}). \tag{2}$$

(For relations with multiple Dirichlet series, we will choose t_i so that $t_i t_{i+1}^{-1} = |p|^{1-2s_{r+1-i}}$.) We identify the weight lattice Λ of $\mathrm{GL}_{r+1}(\mathbb{C})$ with $\mathbb{Z}^{r+1}$. Thus if $\mu = (\mu_1, \ldots, \mu_{r+1}) \in \Lambda$, then $\boldsymbol{t}^\mu = \prod t_i^{\mu_i}$ is a rational character of the diagonal torus T of GL_{r+1}. The weight is dominant if $\mu_1 \geqslant \mu_2 \geqslant \cdots \geqslant \mu_{r+1}$.

Let $\lambda = (\lambda_1, \ldots, \lambda_r)$ be a dominant weight for the root system Φ, which in this paper will be A_r. Kashiwara associated with λ a *crystal graph* which is a directed graph whose edges are labeled by the simple roots. The crystal graph $\mathcal{B}_\lambda$ comes endowed with a weight map wt to the weight lattice such that $\sum_{v \in \mathcal{B}_\lambda} \boldsymbol{t}^{\mathrm{wt}(v)}$ is the character of the irreducible representation of $\mathrm{GL}_{r+1}(\mathbb{C})$ with highest weight λ. The weight function on $\mathcal{B}_{\lambda+\rho}$ also plays a role in the definition of the Weyl group multiple Dirichlet series. Let $l_i = \lambda_i - \lambda_{i+1}$ when $i < r$ and $l_r = \lambda_r$. Then we will show that the coefficient of $|p|^{-2\sum k_i s_i}$ in the multiple Dirichlet series (1) is

$$H(p^{k_1}, \ldots, p^{k_r}; p^{l_1}, \ldots, p^{l_r}) = \sum_{\substack{v \in \mathcal{B}_{\lambda+\rho} \\ \mathrm{wt}(v) = \mu}} G(v), \tag{3}$$

where μ is the weight related to $(k_1, \ldots, k_r)$ by the condition that

$$\sum_{i=1}^{r} k_i \alpha_i = \lambda + \rho - w_0(\mu), \tag{4}$$

where w_0 is the long Weyl group element, $\alpha_1, \ldots, \alpha_r$ are the simple roots (in the usual order) and the function $G(v)$ will be described presently.

The definition of $G(v)$ depends on the choice of a reduced word representing the long element of the Weyl group. Thus we choose a sequence $\Sigma = (i_1, i_2, \ldots, i_N)$ where N is the number of positive roots, $1 \leqslant i_j \leqslant r$, and

$$w_0 = s_{i_1} \cdots s_{i_N}$$

in terms of the simple reflections s_i. If $1 \leqslant i \leqslant r$, let f_i be the Kashiwara weight lowering operator. Thus f_i maps the crystal graph $\mathcal{B}_\lambda$ to $\mathcal{B}_\lambda \cup \{0\}$ where 0 is an auxiliary element, and $\mathrm{wt}(f_i(v)) = \mathrm{wt}(v) - \alpha_i$ if $f_i(v) \neq 0$. Given a fixed Σ and $v \in \mathcal{B}_\lambda$ we associate a sequence of integers to each v, following Berenstein and Zelevinsky [3], [4] and Littelmann [24] as follows.

- Let b_1 be the largest integer such that $f_{i_1}^{b_1}(v) \neq 0$.
- Let b_2 be the largest integer such that $f_{i_2}^{b_2} f_{i_1}^{b_1} v \neq 0$, etc.

We will call the path

$$v, f_{i_1} v, f_{i_1}^2 v, \ldots, f_{i_1}^{b_1} v, f_{i_2} f_{i_1}^{b_1} v, f_{i_2}^2 f_{i_1}^{b_1} v, \ldots \\ \ldots, f_{i_2}^{b_2} f_{i_1}^{b_1} v, f_{i_3} f_{i_2}^{b_2} f_{i_1}^{b_1} v, \ldots, f_{i_N}^{b_N} \cdots f_{i_2}^{b_2} f_{i_1}^{b_1} v] \tag{5}$$

through the crystal the *canonical path* from v with respect to Σ. Thus $b_1, b_2, \cdots$ are the lengths of the straight segments in the canonical path. And we will call the sequence $\mathrm{BZL}(v) = (b_1, b_2, \ldots, b_N)$ the BZL *string* associated with v with respect to the word Σ. There is a unique element v_- of $\mathcal{B}_\lambda$ such that $\mathrm{wt}(v_-) = w_0(\lambda)$.

PROPOSITION 1. (i) *The right endpoint in the canonical path is v_-. That is,*

$$f_{i_N}^{b_N} \cdots f_{i_2}^{b_2} f_{i_1}^{b_1} v = v_-.$$

(ii) *The string $(b_1, b_2, \ldots, b_N)$ determines v uniquely.*

Proof. See Littelmann [24]. Littelmann makes use of the operators e_i where we use f_i. However the crystal graph admits an involution $\mathrm{Sch} : \mathcal{B}_\lambda \longrightarrow \mathcal{B}_\lambda$ such that $\mathrm{Sch} \circ e_i = f_{r+1-i} \circ \mathrm{Sch}$. (See Schützenberger [32] in the language of tableaux, Berenstein and Kirillov [21] in the language of Gelfand-Tsetlin patterns, and Lusztig [25] and Lenart [23] in the language of crystal bases.) Applying Sch, Littelmann's results are translated from the e_i to the f_i. □

To determine the vertices $v \in \mathcal{B}_{\lambda+\rho}$ in (3) with a given weight $\mathrm{wt}(v) = \mu$, we note that (4) implies

$$k_j = \sum_{i_m = j} b_m, \quad b_m : m^{\mathrm{th}} \text{ element in the BZL string,} \tag{6}$$

and the sum ranges over all indices i_m, $1 \leqslant m \leqslant N$, such that $i_m = j$.

By a *decoration* of $\mathrm{BZL}(v)$ we mean a pair of subsets C, B of $\{1, \cdots, N\}$. If $i \in C$, we say that b_i is *circled* and if $i \in B$ we say it is *boxed*. We will represent these statements graphically by circling or boxing b_i. We will describe below for type A_r and certain particular words Σ particular decorations. For these, we can define

$$G(v) = G_\Sigma(v) = \prod_{b_i \in \mathrm{BZL}(v)} \begin{cases} q^{b_i} & \text{if } b_i \text{ is circled (but not boxed),} \\ g(b_i) & \text{if } b_i \text{ is boxed (but not circled),} \\ h(b_i) & \text{if neither,} \\ 0 & \text{if both.} \end{cases} \tag{7}$$

Here $g(a) = g(p^{a-1}, p^a)$ and $h(a) = g(p^a, p^a)$ are Gauss sums defined below in (32). These are only defined if $a > 0$ but for the decorations that we will use, if $a = 0$ it is always circled, and so $g(0)$ and $h(0)$ will never occur.

We will give an explicit description of the decorations used below, but first let us mention a geometric interpretation. Using the map BZL, we may regard the vertices of $\mathcal{B}_\lambda$ as a set of integral points in $\mathbb{R}^N$ whose convex hull is a polytope cut out by a set of inequalities. The circling or boxing of the components b_i $(i = 1, \cdots, N)$ depends on whether these inequalities are sharp. More precisely, Berenstein and Zelevinsky and Littelmann show that the union over all dominant weights λ and over all $\mathrm{BZL}(v)$ with $v \in \mathcal{B}_\lambda$ are the integer lattice points in a cone $\mathcal{C}$ in $\mathbb{R}^N$, which is cut out by N inequalities $\phi_i(v) \geqslant 0$ where ϕ_i are linear functionals on $\mathbb{R}^N$. The choice of λ determines a further set of N inequalities $\psi_i(v) \geqslant 0$ which (together with those defining the cone) cut out a polytope whose lattice points comprise $\{\mathrm{BZL}(v) | v \in \mathcal{B}_\lambda\}$. Each element b_i of $\mathrm{BZL}(v)$ is circled if $\phi_i(v) = 0$, and it is boxed if $\psi_i(v) = 0$. The element is both boxed and circled if $\phi_i(v) = \psi_i(v) = 0$, in which case $G(v) = 0$. If this is the case, then v is "pinned" somewhere on the boundary of the polytope by two opposing inequalities, a condition analogous to nonstrictness of Gelfand-Tsetlin patterns.

To make this explicit, we will say what particular reduced words we employ, and describe the decoration rules in more concrete terms. One may use either

$$\Sigma = \Sigma_1 := (r, r-1, r, r-2, r-1, r, \ldots, 1, 2, 3, \ldots, r)$$

or

$$\Sigma = \Sigma_2 := (1, 2, 1, 3, 2, 1, \ldots, r, r-1, \ldots 3, 2, 1).$$

Proposition 2. *Given a crystal $\mathcal{B}_\lambda$ and $\Sigma = \Sigma_1$ or Σ_2 as above, for each $v \in \mathcal{B}_\lambda$, arrange the* BZL *string $(b_1, b_2, \ldots, b_N)$ with $N = \frac{1}{2}r(r+1)$ into a triangular array (filling right to left in rows, from the bottom row to the top* *row) as follows*:

$$\mathrm{BZL}(v) = \mathrm{BZL}_\Sigma(v) = \left\{ \begin{array}{ccc} \cdots & \cdots & \cdots \\ b_3 & b_2 & \\ b_1 & & \end{array} \right\}.$$

Then the rows are weakly increasing, i.e.,

$$\begin{array}{ccccc} & & \cdots & & \\ 0 & \leqslant & b_3 & \leqslant & b_2, \\ 0 & \leqslant & b_1. & & \end{array}$$

These are the arrays denoted $\Gamma(\mathfrak{T})$, where $\mathfrak{T}$ is a Gelfand-Tsetlin pattern, in [10] and [11]. Note that the k_i in (4) are now just column sums in the BZL pattern.

Proof. See Littelmann [24], particularly Section 5. □

We decorate the pattern as follows. If b_t is a first entry in its row (so t is a triangular number), then we circle it if $b_t = 0$; otherwise, we circle it if $b_t = b_{t+1}$. On the other hand, we box b_t if $e_{i_t} f_{i_{t-1}}^{b_{t-1}} \cdots f_{i_1}^{b_1} v = 0$. The boxing rule may be made more concrete as follows. If v is any vertex and $1 \leqslant i \leqslant r$, then the *$i$-string through v* is the set of vertices that can be obtained from v by repeatedly applying either e_i or f_i. The boxing condition for b_t is equivalent to the condition that the canonical path contains the entire i_t string through $f_{i_{t-1}}^{b_{t-1}} \cdots f_{i_1}^{b_1} v$. The equivalence of this version of the decoration rule with the geometric version presented earlier requires an explicit description of the polytope attached to a reduced decomposition Σ, which is addressed in Section 9.

We have illustrated this decoration rule in Figure 1, which depicts the crystal with highest weight $(5, 3, 0)$. In the figure, we have labeled each vertex with its BZL pattern. The operator f_1 shifts left along horizontal edges, and the operator f_2 shifts downward along the slanted vertical edges. Consider the case where the vertex v is the one labeled $\binom{2\,2}{1}$. We choose the word $\Sigma_2 = \{1, 2, 1\}$ so that $i_1 = i_3 = 1$ and $i_2 = 2$. By our definitions, the decorations of the BZL pattern are as follows:

$$\mathrm{BZL}(v) = \left\{ \begin{array}{cc} 2 & 2 \\ \boxed{1} & \end{array} \right\}.$$

The decoration rule, together with (7), defines the function $G(v)$, and completes the definition of the function H in the numerator of the multiple Dirichlet series (1) when the parameters there are powers of a single prime p (though we have not yet shown that H arises from a Whittaker coefficient; see Theorem 5 below). In fact, we have given two definitions, since our definition

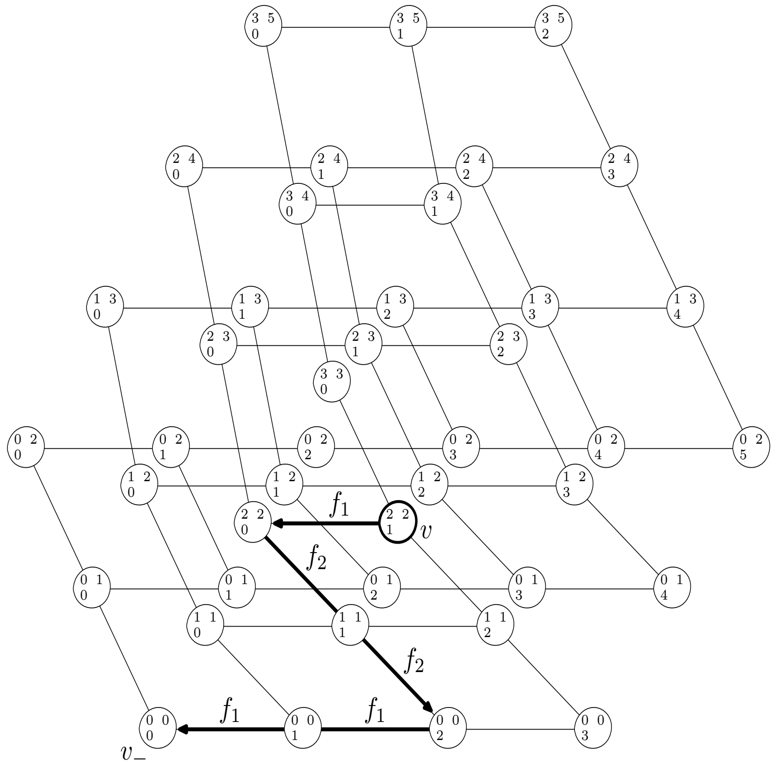

Figure 1. Starting with the vertex v, showing the canonical path to v_-, with respect to the word $\Sigma_2 = (1, 2, 1)$. The lengths of the three straight segments, $1, 2, 2$ comprise the values in $\mathrm{BZL}(v) = \left(\begin{smallmatrix}2 & 2\\ \multicolumn{2}{c}{1}\end{smallmatrix}\right)$. In this example, $\mathrm{wt}(v_-) = (0, 3, 5)$ and $\mathrm{wt}(v) = (3, 2, 3)$.

of $G(v)$ applies for either reduced decomposition of the long word Σ_1 or Σ_2. The equivalence of these definitions is not obvious; to the contrary, it is highly nontrivial. The proof requires both intricate combinatorial arguments and number theoretic input (identities for Gauss sums) and is the subject of [11] and [10].

Remark. As noted in the introduction, the coefficients of the multiple Dirichlet series in [12] were defined in terms of Gelfand-Tsetlin patterns. The analogue of $G(v)$ in that context was derived from a string of integers produced by linear functions on Gelfand-Tsetlin patterns whose definition lacked a representation theoretic interpretation. In the crystal definition presented here, the BZL string used to define $G(v)$ is given in terms of intrinsic representation theoretic data for the associated quantum group, namely paths along Kashiwara operators f_i. In Section 9 of this paper, we will show that the crystal definition presented here is equivalent to the Gelfand-Tsetlin definition of [12] (also given in [10] and [11]).

Using the formulation in terms of crystals, one might try to apply this definition to other root systems. Partial progress has been made for types B_r (n even), as discussed in Brubaker, Bump, Chinta and Gunnells [7] and for type C_r (n odd), as discussed in Beineke, Brubaker and Frechette [2]. More precisely, the definition as given above for a particular decomposition Σ does conjecturally satisfy functional equations and matches an appropriate Whittaker coefficient, though the cited papers prove only special cases within the respective types. There is only one nuance when the root system is not simply laced: the Gauss sums corresponding to root operators for long roots are slightly modified. For the remaining types and cover degrees n, the definition as stated above fails, presumably because the decoration rule becomes more subtle.

Returning to type A_r, the special case $n = 1$ is instructive. In this case, the n^{th} root of unity appearing in the twisted multiplicativity relations is 1, and the series (1) factors as an Euler product. We now show that the definition of the p-part given in this section, specialized to the case $n = 1$, indeed matches the p-part of a Whittaker coefficient of Eisenstein series. This follows from the Shintani-Casselman-Shalika formula, together with a combinatorial identity of Tokuyama.

We recall the Shintani-Casselman-Shalika formula. The Langlands parameters determine a semisimple conjugacy class in the L-group $\mathrm{GL}_{r+1}(\mathbb{C})$, with a representative $\boldsymbol{t}$ as in (2) before. If k is a local field and ψ is an additive character of k with conductor the ring $\mathfrak{o}_k$ of integers, then the unnormalized Whittaker function is

$$W(g) = \int \phi^\circ \left(\begin{pmatrix} 1 & x_{12} & \cdots & x_{1,r+1} \\ & 1 & & \vdots \\ & & \ddots & x_{r,r+1} \\ & & & 1 \end{pmatrix} g \right) \psi\left(\sum x_{i,i+1}\right) \, dx_{i,j}$$

where ϕ° is the spherical vector in the induced model of the principal series representation with Langlands parameters $\boldsymbol{t}$, normalized so $\phi^\circ(1) = 1$. (The integral is either convergent or may be renormalized by analytic continuation in the Langlands parameters $\boldsymbol{t}$ from a region where it is convergent.)

If $a = \mathrm{diag}(a_1, \ldots, a_{r+1}) \in \mathrm{GL}_{r+1}(k)$, let λ_a be $(\mathrm{ord}(a_1), \ldots, \mathrm{ord}(a_{r+1}))$. Let δ be the modular quasicharacter on the standard Borel subgroup. Then $\delta^{-1/2} W(a) = 0$ unless $\lambda = \lambda_a$ is dominant. Assume that λ is dominant, and let χ_λ be the irreducible character of $\mathrm{GL}_{r+1}(\mathbb{C})$ with highest weight λ. Then the formula of Shintani [34] and Casselman and Shalika [13] is

$$\delta^{-1/2}W(a)=\prod_{\alpha\in\Phi^+}(1-q^{-1}\boldsymbol{t}^\alpha)\chi_\lambda(\boldsymbol{t}),\qquad \lambda=\lambda_a,$$

where q is the cardinality of the residue field and Φ^+ denotes the positive roots. The Weyl character formula expresses $\chi_\lambda(\boldsymbol{t})$ as a ratio of a numerator (a sum over the Weyl group) with a denominator. The denominator is (in one normalization) $\prod_{\alpha\in\Phi^+}(1-\boldsymbol{t}^\alpha)$. On the other hand the normalizing factor that appears in the Shintani-Casselman-Shalika formula is $\prod_{\alpha\in\Phi^+}(1-q^{-1}\boldsymbol{t}^\alpha)$.

To connect to the definitions of the p-part of this section, we invoke an identity of Tokuyama [35], who found a deformation of the Weyl character formula that expresses $\chi_\lambda(\boldsymbol{t})$ as a ratio of two quantities. The deformed denominator is $\prod_{\alpha\in\Phi^+}(1-q^{-1}\boldsymbol{t}^\alpha)$. Tokuyama gave his formula in terms of Gelfand-Tsetlin patterns, but we will translate it into the crystal language as

$$\prod_{\alpha\in\Phi^+}(1-q^{-1}\boldsymbol{t}^\alpha)\chi_\lambda(\boldsymbol{t})=\sum_{v\in\mathcal{B}_{\rho+\lambda}}G(v)q^{-\langle\mathrm{wt}(v)-w_0(\lambda+\rho),\rho\rangle}\boldsymbol{t}^{\mathrm{wt}(v)-w_0\rho},$$

where G is given by (7), but the Gauss sums have become Ramanujan sums (since $n=1$) that may be evaluated explicitly: $g(a)=-q^{a-1}$ and $h(a)=(q-1)q^{a-1}$. Thus

$$\boldsymbol{t}^{-w_0(\lambda)}\prod_{\alpha\in\Phi^+}(1-q^{-1}\boldsymbol{t}^\alpha)\chi_\lambda(\boldsymbol{t})$$

is exactly the p-part of H when $n=1$, in agreement with the Shintani-Casselman-Shalika formula.

3. **The Metaplectic group and Whittaker functionals**

Our foundations will be similar to those in Brubaker and Bump [5] and Brubaker, Bump and Friedberg [8], [9]. We refer to those papers as well as Brubaker, Bump, Chinta, Friedberg and Hoffstein [6] for amplification.

Let F be a totally complex number field containing the group μ_{2n} of $2n^{\text{th}}$ roots of unity. Let S be a finite set of places containing the archimedean ones. Let $F_S=\prod_{v\in S}F_v$. The ring $\mathfrak{o}_S$ of S-integers consists of $x\in F$ such that $|x|_v=1$ for $v\notin S$. We assume that S contains those places ramified over $\mathbb{Q}$ (in particular those dividing n) and enough others such that the ring $\mathfrak{o}_S$ of S-integers is a principal ideal domain and the residue field has at least four elements for all $v\notin S$.

Let S_∞ (resp. S_{fin}) be the set of archimedean (resp. nonarchimedean) places in S. We may factor $F_S=F_\infty\times F_{\text{fin}}$ where $F_\infty=\prod_{v\in S_\infty}F_v$ and $F_{\text{fin}}=\prod_{v\in S_{\text{fin}}}F_v$. We embed F and $\mathfrak{o}_S$ diagonally in F_S.

Let $(x,y)_S=\prod_{v\in S}(x,y)_v$ be the S-Hilbert symbol. As in [5] we will take our Hilbert symbol to be the inverse of the symbol used in Neukirch [30].

Based on earlier work of Kubota [22] and Matsumoto [26], Kazhdan and Patterson described an explicit "metaplectic" cocycle in $H^2(\mathrm{GL}_{r+1}(F_S), \mu_n)$; see [20]. We note that a correction to this cocycle was made by Banks, Levy and Sepanski [1], also based directly on Matsumoto [26]. However the Kazhdan-Patterson cocycle is correct under our assumption that $\mu_{2n} \subset F$. We will not work with the cocycle described in [20] but on a modification, which is obtained by composing that cocycle with the outer automorphism of GL_{r+1}:

$$g \longmapsto J_{r+1}\, {}^t g^{-1} J_{r+1}, \qquad J_{r+1} = \begin{pmatrix} & & & (-1)^r \\ & & \cdot^{\cdot^{\cdot}} & \\ & -1 & & \\ 1 & & & \end{pmatrix}.$$

This will result in nicer formulas. Let $\sigma = \sigma_{r+1} : \mathrm{GL}_{r+1}(F_S)\times\mathrm{GL}_{r+1}(F_S) \to \mu_n$ denote this cocycle, which is described as follows.

We will identify the Weyl group W with the subgroup of GL_{r+1} consisting of permutation matrices. Let N be the group of upper triangular unipotent elements of $\mathrm{GL}_{r+1}(F_S)$, T be the diagonal subgroup, and let Φ^+ (resp. Φ^-) denote the set of positive roots (resp. negative roots) with respect to the standard Borel subgroup of upper triangular matrices. We have

$$\sigma\left(\begin{pmatrix} x_1 & & \\ & \ddots & \\ & & x_{r+1} \end{pmatrix}, \begin{pmatrix} y_1 & & \\ & \ddots & \\ & & y_{r+1} \end{pmatrix}\right) = \prod_{i>j} (x_i, y_j)_S,$$

$$\sigma\left(\begin{pmatrix} x_1 & & \\ & \ddots & \\ & & x_{r+1} \end{pmatrix}, w\right) = 1,$$

$$\sigma\left(w, \begin{pmatrix} x_1 & & \\ & \ddots & \\ & & x_{r+1} \end{pmatrix}\right) = \prod_{\substack{\alpha = \alpha_{i,j} \in \Phi^+ \\ w(\alpha) \in \Phi^-}} (x_i, x_j)_S,$$

if $w \in W$, where $\alpha = \alpha_{i,j}$ is the root $\boldsymbol{t}^\alpha = t_i t_j^{-1}$, and

$$\sigma(w, w') = 1, \qquad w, w' \in W.$$

(Without our assumption that $-1 \in (F_S^\times)^n$, this last equality would be limited to the case $l(ww') = l(w) + l(w')$ as in Banks, Levy and Sepanski [1].)

With these definitions, the cocycle is extended to monomial matrices by

$$\sigma\left(h_1 w_1, h_2 w_2\right) = \sigma(h_1, w_1 h_2 w_1^{-1})\sigma(w_1, h_2), \qquad \text{if } h_i \in T,\ w_i \in W. \tag{8}$$

To extend it to the whole group, let R be the map from G to the subgroup generated by the monomial matrices such that $R(ngn') = R(g)$ for $n, n' \in N$. Then

$$\sigma(ng, g'n') = \sigma(g, g') \text{ if } n, n' \in N, \tag{9}$$

and

$$\sigma(h,g)=\sigma(h,R(g)),\qquad h\in T,$$

$$\sigma(s,g)=\sigma(R(sg)R(g)^{-1},R(g)) \text{ if } s \text{ is a simple reflection in } W.$$

The following lemma shows how to compute $\sigma(g,g')$ algorithmically for any g,g'. We will use it without comment later wherever we assert that a cocycle has a certain value.

LEMMA 1. *Write $g=g_1\cdots g_m$ where g_1 and g_m are in N, g_2 is in H and $g_3,\dots,g_{m-1}$ are simple reflections. Then*

$$\sigma(g,g')=\prod_{i=2}^{m-1}\sigma(g_i,g_{i+1}\cdots g_m g'), \tag{10}$$

Proof. The cocycle property $\sigma(xy,z)\sigma(x,y)=\sigma(x,yz)\sigma(y,z)$ implies

$$\prod_{i=1}^{m}\sigma(g_i,g_{i+1}\cdots g_m g')=\left[\prod_{i=1}^{m-1}\sigma(g_1\cdots g_i,g_{i+1})\right]\sigma(g,g').$$

In the product on the left, the first and last terms are 1 since $g_1,g_m\in N$. On the other hand by (9) and the special case $\sigma(h_1w_1,w_2)=1$ of (8) we have

$$\prod_{i=1}^{m-1}\sigma(g_1\cdots g_i,g_{i+1})=\prod_{i=1}^{m-1}\sigma(g_2\cdots g_i,g_{i+1})=1.$$

The statement follows. □

The cocycle satisfies a *block compatibility property* emphasized by Banks, Levy and Sepanski [1]. If $g,g'\in\mathrm{GL}_k(F_S)$ and $h,h'\in\mathrm{GL}_l(F_S)$ where $k+l=r+1$, then

$$\sigma_{r+1}\left(\begin{pmatrix} g & \\ & h\end{pmatrix},\begin{pmatrix} g' & \\ & h'\end{pmatrix}\right)=\sigma_k(g,g')\sigma_l(h,h')(\det(g),\det(h'))_S. \tag{11}$$

Now if $G\subseteq\mathrm{GL}_{r+1}(F_S)$ let $\widetilde{G}$ be the central extension of G by μ_n determined by σ. Thus μ_n is embedded in $\widetilde{G}$ as a subgroup, and $G\cong\widetilde{G}/\mu_n$ with $p:\widetilde{G}\longrightarrow G$ the projection and a section $\boldsymbol{s}:G\longrightarrow\widetilde{G}$ satisfying $\boldsymbol{s}(g)\boldsymbol{s}(h)=\sigma(g,h)\boldsymbol{s}(gh)$. We will call a function $f:\widetilde{G}\longrightarrow\mathbb{C}$ *genuine* if $f(\varepsilon g)=\varepsilon f(g)$ for $\varepsilon\in\mu_n$. Thus if f is genuine we have

$$f(\boldsymbol{s}(g)\boldsymbol{s}(g')\tilde{g})=\sigma(g,g')f(\boldsymbol{s}(gg')\tilde{g}),\qquad g,g'\in G,\tilde{g}\in\widetilde{G}. \tag{12}$$

If $x\in F_S$, we will sometimes factor $|x|=|x|_\infty\cdot|x|_{\mathrm{fin}}$, where $|x|_\infty=\prod_{v\in S_\infty}|x_v|_v$ and $|x|_{\mathrm{fin}}=\prod_{v\in S_{\mathrm{fin}}}|x_v|_v$. Let $s_1,\dots,s_r$ be complex numbers. Let $t_1,\dots,t_{r+1}$ satisfy $\prod t_i=1$ and define

$$\mathfrak{I}(t_1,\dots,t_{r+1})=\prod_{i+j\leqslant r+1}|t_i|^{2s_j}.$$

We also define $\mathfrak{I}_{\mathrm{fin}}$ and $\mathfrak{I}_\infty$ to be the functions in which $|\,|$ is replaced by $|\,|_{\mathrm{fin}}$ and $|\,|_\infty$, respectively, so that $\mathfrak{I} = \mathfrak{I}_{\mathrm{fin}}\mathfrak{I}_\infty$.

We say that a subgroup of $F_S^\times$ is *isotropic* if $(x, y)_S = 1$ for elements of that subgroup. The subgroup $\Omega = \mathfrak{o}_S^\times (F_S^\times)^n$ is then maximal isotropic ([5, Lemma 2]). This implies the irreducibility (for s_i in general position) of the representation $\pi(s_1, \ldots, s_r)$ of $\widetilde{\mathrm{SL}}_{r+1}(F_S)$, acting by right translation on the space of all smooth genuine functions f on $\widetilde{\mathrm{SL}}_{r+1}(F_S)$ such that

$$f\left(\boldsymbol{s}\begin{pmatrix} t_1 & * & \cdots & * \\ & t_2 & \cdots & * \\ & & \ddots & \vdots \\ & & & t_{r+1}\end{pmatrix} g\right) = \mathfrak{I}(t_1, \ldots, t_{r+1}) f(g), \qquad t_i \in \Omega. \tag{13}$$

If $\mathrm{re}(s_i)$ are sufficiently large and $m_i \in \mathfrak{o}_S$ are nonzero, define the Whittaker functionals

$$\begin{aligned}\Lambda_{m_1,\ldots,m_r}^{d_1,\ldots,d_r}(f) &= \mathfrak{I}(d_r^{-1}, d_{r-1}^{-1}, \ldots, d_1^{-1}, d_1\cdots d_r)^{-1} \\ &\times \int_{F_S^{r(r+1)/2}} f\left(\boldsymbol{s}\begin{pmatrix} d_r^{-1} & & & \\ & \ddots & & \\ & & d_1^{-1} & \\ & & & d_1\cdots d_r\end{pmatrix}\boldsymbol{s}(J_{r+1})\boldsymbol{s}\begin{pmatrix} 1 & x_{12} & \cdots & x_{1,r+1} \\ & 1 & \cdots & x_{2,r+1} \\ & & \ddots & \vdots \\ & & & 1\end{pmatrix}\right) \\ &\times \psi\left(\sum_{i=1}^r m_i x_{i,i+1}\right) dx_{i,j}.\end{aligned} \tag{14}$$

Here $\psi : F_S \to \mathbb{C}$ is a fixed additive character trivial on $\mathfrak{o}_S$ but no larger fractional ideal. As usual, this integral is convergent for $\mathrm{re}(s_i)$ sufficiently large but has analytic continuation to all s_i in a suitable sense. See Jacquet [18] and Kazhdan and Patterson [20].

4. **The Kubota symbol**

If $\alpha \in \Phi^+$ is a positive root of GL_{r+1}, let $i_\alpha : \mathrm{GL}_2 \longrightarrow \mathrm{GL}_{r+1}$ be the canonical embedding. We will parametrize the elements of Φ^+ by pairs (i, j) with $1 \leqslant i < j \leqslant r+1$, so that if $\alpha = \alpha_{i,j}$, then

$$i_\alpha\begin{pmatrix} a & b \\ c & d\end{pmatrix} = \begin{pmatrix} I_{i-1} & & & & \\ & a & & b & \\ & & I_{j-i-1} & & \\ & c & & d & \\ & & & & I_{r+1-j}\end{pmatrix}.$$

We will denote $s_\alpha = i_\alpha\begin{pmatrix} 0 & -1 \\ 1 & 0\end{pmatrix}$.

If c, d are coprime elements of $\mathfrak{o}_S$, let $\left(\frac{c}{d}\right) := \left(\frac{c}{d}\right)_n$ denote the n^{th} power residue symbol satisfying the reciprocity law

$$\left(\frac{c}{d}\right) = (d, c)_S\left(\frac{d}{c}\right) \tag{15}$$

and other familiar properties that are summarized in [5]. The reciprocity law is Theorem 8.3 on page 415 of Neukirch. We bear in mind that our n^{th} power Hilbert symbol is the inverse of his.

LEMMA 2. *There exists a map* $\kappa : \mathrm{SL}_{r+1}(\mathfrak{o}_S) \longrightarrow \mu_n$ *such that*

$$\kappa(\gamma\gamma') = \sigma(\gamma, \gamma')\kappa(\gamma)\kappa(\gamma'). \tag{16}$$

If α *is any positive root, then*

$$\kappa\left(i_\alpha\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = \begin{cases} \left(\frac{d}{c}\right) & \text{if } c \neq 0, \\ 1 & \text{if } c = 0. \end{cases} \tag{17}$$

This is the *Kubota symbol*. We can show, using Kazhdan and Patterson [20, Prop. 0.1.2], that the symbol can be extended to $\mathrm{GL}_{r+1}(\mathfrak{o})$.

Proof. For each place v of F, let σ_v be the local cocycle on $\mathrm{SL}_{r+1}(F_v)$ defined by the formulas in Section 3. Thus if $g, g' \in F_S$, then $\sigma(g, g') = \prod_{v\in S}\sigma_v(g_v, g'_v)$, but we will make use of σ_v also for $v \notin S$.

We will use the fact that the metaplectic cover splits over $\mathrm{SL}_{r+1}(\mathfrak{o}_v)$ when $v \notin S$. This is a consequence of Lemma 11.3 of Moore [29] which is applicable since our assumptions on S imply that the residue field at v has cardinality $\geqslant 4$ for all $v \notin S$. Let $\kappa_v : \mathrm{SL}_{r+1}(\mathfrak{o}_v) \longrightarrow \mu_n$ be a splitting, so that

$$\sigma_v(g_v, g'_v) = \frac{\kappa_v(g_v)\kappa_v(g'_v)}{\kappa_v(g_v g'_v)}.$$

We say that $g \in \mathrm{SL}_{r+1}(\mathfrak{o}_S)$ is *locally finite* if $\kappa_v(g_v) = 1$ for almost all v. At the end we will show that all $g \in \mathrm{SL}_{r+1}(\mathfrak{o}_S)$ are locally finite. If g is locally finite, let

$$\kappa(g) = \prod_{v\notin S}\kappa_v(g_v).$$

If $g, g' \in \mathrm{SL}_{r+1}(F)$, then $\sigma_v(g_v, g'_v) = 1$ for almost all v and $\prod_v \sigma_v(g_v, g'_v) = 1$. This is because we can reduce σ_v to a product of Hilbert symbols using (10), and then use the Hilbert reciprocity law $\prod_v (a, b)_v = 1$ for $a, b \in F$. Thus if $g, g' \in \mathrm{SL}_{r+1}(\mathfrak{o}_S)$ are locally finite, then so is gg' and

$$\sigma(g, g') = \prod_{v\in S}\sigma_v(g_v, g'_v) = \prod_{v\notin S}\sigma_v(g_v, g'_v)^{-1} = \frac{\kappa(gg')}{\kappa(g)\kappa(g')}.$$

Thus (16) will be proved when we show all g are locally finite.

We will make use of the fact that the pullback of the cocycle under i_α to $\mathrm{SL}_2(F_S)$ is Kubota's cocycle. This is clear if α is a simple root, and in general one may conjugate i_α to a simple root by a series of simple reflections. Showing that these do not change, we see that the cocycle requires a computation, which we omit. (It might fail without our assumption that -1 is an n^{th} power.)

Now let us argue that $\kappa_v \circ i_\alpha$ is given by Kubota's formula

$$\kappa_v \circ i_\alpha \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{cases} (c,d)_v^{-1} & \text{if } c \neq 0 \text{ and } n \nmid \operatorname{ord}_v(c), \\ 1 & \text{otherwise.} \end{cases} \tag{18}$$

Indeed, since Kubota [22] shows that the right-hand side is a function splitting the cocycle on $\mathrm{SL}_2(\mathfrak{o}_v)$, its ratio to $\kappa_v \circ i_\alpha$ is a character. By Lemma 11.1 of Moore [29], the abelianization of $\mathrm{SL}_2(\mathfrak{o}_v)$ is trivial, so this identity is proved.

Now we see that $i_\alpha(g)$ when $g \in \mathrm{SL}_2(\mathfrak{o}_S)$ is locally finite and since these elements generate $\mathrm{SL}_{r+1}(\mathfrak{o}_S)$, using Proposition V.3.4 on page 335 of Neukirch [30] (remembering that his Hilbert symbol is the inverse of ours) we see that

$$\kappa \circ i_\alpha \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \prod_{\substack{v \notin S \\ \pi_v | c}} (\pi_v, d)^{-\operatorname{ord}_v(c)} = \prod_{\substack{v \notin S \\ \pi_v | c}} \left(\frac{d}{\pi_v}\right)^{\operatorname{ord}_v(c)} = \left(\frac{d}{c}\right).$$

Here π_v is a prime element at the place v. We have used the fact that c and d are coprime so that d is a unit in $\mathfrak{o}_v$ when $\pi_v | c$. $\square$

We observe that if f is genuine, then combining (16) with (12) we have

$$\kappa(\gamma)\kappa(\gamma')f(\boldsymbol{s}(\gamma)\boldsymbol{s}(\gamma')\tilde{g}) = \kappa(\gamma\gamma')f(\boldsymbol{s}(\gamma\gamma')\tilde{g}). \tag{19}$$

5. Whittaker coefficients of Eisenstein series

Let $f \in \pi(s_1, \ldots, s_r)$ be as in Section 3. Define the Eisenstein series

$$E_f^{r+1}(g) = \sum_{\gamma \in B(\mathfrak{o}_S) \backslash \mathrm{SL}_{r+1}(\mathfrak{o}_S)} \kappa(\gamma) f(\boldsymbol{s}(\gamma) g), \qquad g \in \widetilde{\mathrm{SL}}_{r+1}(F_S), \tag{20}$$

where $B := B_{\mathrm{SL}_{r+1}}$ is the Borel subgroup of upper triangular matrices in SL_{r+1}.

Theorem 1. *When $\mathbf{m} = (m_1, \ldots, m_r)$, an r-tuple of nonzero S-integers, there exist constants $H(C_1, \ldots, C_r; \mathbf{m})$ independent of f such that*

(21)

$$\int_{(\mathfrak{o}_S \backslash F_S)^{r(r+1)/2}} E_f^{r+1}\left(\boldsymbol{s}(J_{r+1})\boldsymbol{s}\begin{pmatrix} 1 & x_{12} & \cdots & x_{1,r+1} \\ & 1 & \cdots & x_{2,r+1} \\ & & \ddots & \vdots \\ & & & 1 \end{pmatrix}\right) \times \psi(m_1 x_{12} + \cdots + m_r x_{r,r+1}) dx_{ij}$$
$$= \sum_{0 \neq C_i \in \mathfrak{o}_S / \mathfrak{o}_S^\times} H(C_1, \ldots, C_r; m_1, \ldots, m_r) \prod_i |C_i|^{-2s_i} \Lambda_{m_1,\ldots,m_r}^{C_1/C_2,\ldots,C_{r-1}/C_r,C_r}(f).$$

We may express $H =: H_{r+1}$ for $\widetilde{\mathrm{SL}}_{r+1}$ recursively in terms of the H_r for $\widetilde{\mathrm{SL}}_r$ by

(22)

$$\begin{aligned}&H_{r+1}(C_1,\dots,C_r;m_1,\dots,m_r)\\&=\sum_{\substack{0\neq D_i\in\mathfrak{o}_S/\mathfrak{o}_S^\times\\0\neq d_i\in\mathfrak{o}_S/\mathfrak{o}_S^\times\\C_i=D_i\prod_{i=j}^r d_j\\d_{i+1}|m_{i+1}d_i}}\ \sum_{c_i\ mod\ d_i}\left(\frac{c_1}{d_1}\right)\cdots\left(\frac{c_r}{d_r}\right)\psi\left(\frac{m_1c_1}{d_1}+\frac{m_2u_1c_2}{d_2}+\cdots+\frac{m_ru_{r-1}c_r}{d_r}\right)\\&\times\prod_{i<j}(d_i,d_j)_S\prod_{i=2}^r(d_i,D_i)_S\,|d_2d_3^2\cdots d_r^{r-1}|\,H_r\left(D_2,\dots,D_r;\frac{m_2d_1}{d_2},\dots,\frac{m_rd_{r-1}}{d_r}\right).\end{aligned}$$

Here the sum is over $d_1,\dots,d_r$ and $D_2,\dots,D_r$, the integers $u_1,\dots,u_{r-1}$ are determined by $c_iu_i\equiv 1$ mod d_i, and we set $D_1=1$ for a uniform expression of C_i.

Since $\mathfrak{o}_S$ is a principal ideal domain the sum over d_i and D_i is essentially a sum over ideals. The notation H was used earlier in this paper (3). The two definitions for H will be shown to be the same in Section 9.

Proof. By induction we may assume that the statement is true for $\widetilde{\mathrm{SL}}_r$. We may write

$$E_f^{r+1}(g)=\sum_{\gamma\in P(\mathfrak{o}_S)\backslash\mathrm{SL}_{r+1}(\mathfrak{o}_S)}\kappa(\gamma)\Theta(\boldsymbol{s}(\gamma)g),\tag{23}$$

where P is the standard maximal parabolic subgroup of SL_{r+1} with Levi factor $\mathrm{SL}_r\times\{1\}$ and

$$\Theta(g)=\sum_{\gamma\in B_{\mathrm{SL}_r}(\mathfrak{o}_S)\backslash\mathrm{SL}_r(\mathfrak{o}_S)}\kappa(\gamma)f\left(\boldsymbol{s}\begin{pmatrix}\gamma&\\&1\end{pmatrix}g\right).$$

We will parametrize the coset of γ in $P(\mathfrak{o}_S)\backslash\mathrm{SL}_{r+1}(\mathfrak{o}_S)$ by the bottom row of each matrix, which is a vector of coprime integers that is determined modulo multiplication by a unit. Let the bottom row of γ be $(B_{r+1},B_r,\dots,B_1)$. Writing the left-hand side of (21) as

$$\int_{(\mathfrak{o}_S\backslash F_S)^{r(r+1)/2}}\sum_\gamma\kappa(\gamma)f\left(\boldsymbol{s}(\gamma)\boldsymbol{s}(J_{r+1})\boldsymbol{s}\begin{pmatrix}1&x_{12}&\cdots&x_{1,r+1}\\&1&\cdots&x_{2,r+1}\\&&\ddots&\vdots\\&&&1\end{pmatrix}\right)$$
$$\times\psi(-m_1x_{12}-\cdots-m_rx_{r,r+1})\prod dx_{ij},$$

we see that only γ with γJ_{r+1} in the big Bruhat cell give a nonzero contribution. Indeed if γJ_{r+1} is in another cell BwB, we can find a simple root $\alpha_{i,i+1}$ such that $w^{-1}(\alpha_{i,i+1})$ is a positive root, and then the integration with respect to

$x_{i,i+1}$ kills the term. Thus we may assume that $B_1 \neq 0$. Let $d_1, \ldots, d_r$ be determined by the conditions

$$\begin{gathered} d_r = \gcd(B_1, B_2, \ldots, B_r), \\ d_{r-1} d_r = \gcd(B_1, B_2, \ldots, B_{r-1}),\ \ldots,\ d_1 d_2 \cdots d_r = B_1. \end{gathered}$$

Let $c_r = B_{r+1}$, $c_{r-1} d_r = B_r$, $c_{r-2} d_{r-1} d_r = B_{r-1}$, ... In this way we parametrize the bottom row of γ:

$$(B_{r+1}, B_r, \ldots, B_1) = (c_r, c_{r-1} d_r, c_{r-2} d_{r-1} d_r, \ldots, c_1 d_2 \cdots d_r, d_1 d_2 \cdots d_r).$$

Because we are assuming that $\mathfrak{o}_S$ is a principal ideal domain, we may find a_k, u_k such that $a_k d_k + u_k c_k = 1$ for $k = 1, \ldots, r$. Now the matrix

$$\gamma = \gamma_r \gamma_{r-1} \cdots \gamma_1, \quad \text{with} \quad \gamma_i = \begin{pmatrix} I_{r-i} & & & \\ & a_i & & -u_i \\ & & I_{i-1} & \\ & c_i & & d_i \end{pmatrix}$$

has the prescribed bottom row and we may choose this to be the coset representative γ. Thus using (17) and (19) we have, for genuine f,

$$\kappa(\gamma) f(\boldsymbol{s}(\gamma)\tilde{g}) = \prod_{k=1}^{r} \left(\frac{d_k}{c_k}\right) f(\boldsymbol{s}(\gamma_r) \cdots \boldsymbol{s}(\gamma_1)\tilde{g}).$$

Let α be a positive root. By abuse of notation, if $g \in \mathrm{SL}_2(F_S)$ let us temporarily write $\boldsymbol{s}(g)$ for $\boldsymbol{s}(i_\alpha(g))$. We have

$$\boldsymbol{s}\begin{pmatrix} a & -u \\ c & d \end{pmatrix} = (d, c)_S \boldsymbol{s}\begin{pmatrix} 1 & -u/d \\ & 1 \end{pmatrix} \boldsymbol{s}\begin{pmatrix} d^{-1} & \\ & d \end{pmatrix} \boldsymbol{s}\begin{pmatrix} 1 & \\ c/d & 1 \end{pmatrix}. \tag{24}$$

Substitute this into the definition of γ_k. We rearrange $\boldsymbol{s}(\gamma_r) \cdots \boldsymbol{s}(\gamma_1)\boldsymbol{s}(J_{r+1})$ by pulling the upper triangular matrices involving $-u_k/d_k$ to the left. Conjugating them by the diagonal matrices changes their entries but leaves them in the last column. Conjugating them by the lower triangular matrices involving c_j/d_j produces some commutators that are lower triangular; we only need to keep track of the subdiagonal entries and some cocycles. We obtain

$$\begin{aligned} &\boldsymbol{s}(\gamma_r) \cdots \boldsymbol{s}(\gamma_1) \boldsymbol{s}(J_{r+1}) \\ &\quad = \prod_{k=1}^{r} (d_k, c_k)_S \prod_{i<j} (d_i, d_j)_S\, \boldsymbol{s}\begin{pmatrix} 1 & 0 & \cdots & 0 & * \\ & 1 & & & * \\ & & \ddots & & \vdots \\ & & & 1 & * \\ & & & & 1 \end{pmatrix} \mathfrak{D}_{d_r, \cdots, d_1} \boldsymbol{s}(J_{r+1}) \boldsymbol{s}(n_+), \end{aligned} \tag{25}$$

where

$$\mathfrak{D}_{d_r, \ldots, d_1} = \boldsymbol{s}\begin{pmatrix} d_r^{-1} & & & & \\ & \ddots & & & \\ & & d_2^{-1} & & \\ & & & d_1^{-1} & \\ & & & & \prod d_i \end{pmatrix}$$

and

$$n_+ = \begin{pmatrix} 1 & -c_1/d_1 & * & \cdots & & * & * \\ & 1 & -u_1c_2/d_2 & & & & * \\ & & 1 & & & & \\ & & & \ddots & \ddots & \vdots & \vdots \\ & & & & 1 & -u_{r-2}c_{r-1}/d_{r-1} & * \\ & & & & & 1 & -u_{r-1}c_r/d_r \\ & & & & & & 1 \end{pmatrix}.$$

The Hilbert symbols $\prod_{k=1}^r (d_k, c_k)_S$ come from (24). The symbols $(d_i, d_j)_S$ arise from combining the diagonal matrices. It may be checked that there are no other nontrivial contributions from the various cocycles.

Now substitute (23) into the left-hand side of (21). We may collapse the integration over $x_{i,j}$ with $i = 1$ with the summation over the bottom row entries B_i, $i > 1$. In other words with $B_1 \neq 0$ fixed

$$\sum_{B_2,\ldots,B_{r+1} \bmod B_1} \int_{\substack{(F_S/\mathfrak{o}_S)^r \\ (i=1)}} = \int_{\substack{(F_S)^r \\ (i=1)}} .$$

Moreover $\kappa(\gamma)$ depends only on c_k mod d_k, and for $k = 1, \ldots, r$ if we sum over c_k mod d_k and then multiply the result by $|d_1 \cdots d_{k-1}|$, this has the same result as summing over B_{k+1} mod B_1. Since $\prod_{k=2}^r |d_1 \cdots d_{k-1}| = |d_1^{r-1} d_2^{r-2} \cdots d_{r-1}|$, we obtain this factor. The n_+ we eliminate by a change of variables in the x_{ij} producing a factor $\psi\left(\frac{m_1c_1}{d_1} + \frac{m_2u_1c_2}{d_2} + \cdots\right)$. We also make use of the reciprocity law (15) and obtain

$$\begin{aligned} (26) \qquad & \sum_{d_k} |d_1^{r-1} d_2^{r-2} \cdots d_{r-1}| \prod_{i<j} (d_i, d_j)_S \sum_{c_k \bmod d_k} \left(\frac{c_k}{d_k}\right) \\ & \times \psi\Big(\frac{m_1c_1}{d_1} + \frac{m_2u_1c_2}{d_2} + \cdots + \frac{m_r u_{r-1} c_r}{d_r}\Big) \\ & \times \int_{\substack{(F_S)^r \\ (i=1)}} \int_{\substack{(F_S/\mathfrak{o}_S)^{r(r-1)/2} \\ (i>1)}} \Theta\left(\mathfrak{D}_{d_r,\ldots,d_1} \boldsymbol{s}(J_{r+1}) \begin{pmatrix} 1 & x_{12} & \cdots & x_{1,r+1} \\ & 1 & & \vdots \\ & & \ddots & x_{r,r+1} \\ & & & 1 \end{pmatrix}\right) \\ & \times \psi(m_1x_{12} + \cdots + m_r x_{r,r+1}) \prod_{i,j} dx_{ij}. \end{aligned}$$

Since Θ is invariant under lower triangular matrices in $P(\mathfrak{o}_S)$, the integral in (26) is 0 unless $d_{i+1} | m_{i+1} d_i$ for all $i = 1, \ldots, r-1$. Indeed, this is seen by moving a general lower triangular matrix in $P(\mathfrak{o}_S)$ to the right and changing variables. To proceed further let us define, for $g' \in \widetilde{\mathrm{SL}}_r(F_S)$

$$\begin{aligned} & f'_{d_1,\ldots,d_r}(g') \\ & = \int_{(F_S)^r} f\left(i(g') \mathfrak{D}_{d_1,\ldots,d_r} \boldsymbol{s} \begin{pmatrix} & -I_r \\ 1 & \end{pmatrix} \boldsymbol{s} \begin{pmatrix} 1 & x_{12} & x_{13} & \cdots & x_{1,r+1} \\ & 1 & 0 & \cdots & 0 \\ & & \ddots & & \vdots \\ & & & 1 & 0 \\ & & & & 1 \end{pmatrix} \right) \psi(m_1x_{12}) \prod_j dx_{1j}, \end{aligned}$$

where $i : \widetilde{\mathrm{SL}}_r(F_S) \longrightarrow \widetilde{\mathrm{SL}}_{r+1}(F_S)$ is the embedding in the upper left corner. This function is in $\pi(s_2, \ldots, s_r)$ and

$$\prod_{i<j}(d_i, d_j)_S \int_{\substack{(F_S)^r \\ (i=1)}} \Theta\left(\mathfrak{D}_{d_r,\ldots,d_1}\boldsymbol{s}(J_{r+1})\boldsymbol{s}\begin{pmatrix} 1 & x_{12} & \cdots & x_{1,r+1} \\ & 1 & & \vdots \\ & & \ddots & x_{r,r+1} \\ & & & 1 \end{pmatrix}\right)\psi(m_1 x_{12})\prod_j dx_{1j}$$
$$= E^r_{f'_{d_1,\cdots,d_r}}\left(\boldsymbol{s}(J_r)\boldsymbol{s}\begin{pmatrix} 1 & x'_{23} & \cdots & x'_{2,r} \\ & 1 & & \vdots \\ & & \ddots & x'_{r-1,r} \\ & & & 1 \end{pmatrix}\right),$$

where $x'_{ij} = d_{i-1}^{-1} d_{j-1} x_{ij}$. Here we have used the identity

$$\boldsymbol{s}(J_{r+1}) = i(\boldsymbol{s}(J_r))\boldsymbol{s}\begin{pmatrix} & -I_r \\ 1 & \end{pmatrix}.$$

The order of the d_i in $\mathfrak{D}$ was switched when the matrix moved past $\boldsymbol{s}(J_r)$, which also accounts for the symbols $(d_i, d_j)_S$.

We make the variable change $x_{ij} \longmapsto d_{i-1} d_{j-1}^{-1} x_{ij}$ and interpret

$$\int_{F_S/\mathfrak{o}_S} = \lim_{\mathfrak{a}} \frac{1}{|\mathfrak{a}|}\int_{F_S/\mathfrak{a}}$$

for sufficiently large fractional ideals $\mathfrak{a}$. The change in measure is compensated by a change in the norm of $\mathfrak{a}$, so this change of variables has no effect on the measure. With the above, (26) may be rewritten

(27)

$$\sum_{\substack{0 \neq d_i \in \mathfrak{o}_S/\mathfrak{o}_S^\times \\ d_{i+1}|m_{i+1}d_i}} |d_1^{r-1} d_2^{r-2} \cdots d_{r-1}| \sum_{c_k \bmod d_k} \left(\frac{c_k}{d_k}\right)\psi\left(\tfrac{m_1 c_1}{d_1} + \tfrac{m_2 u_1 c_2}{d_2} + \cdots + \tfrac{m_r u_{r-1} c_r}{d_r}\right)$$
$$\times \int_{\substack{(F_S/\mathfrak{o}_S)^{r(r-1)/2} \\ (i>1)}} E^r_{f'}\left(\boldsymbol{s}(J_r)\boldsymbol{s}\begin{pmatrix} 1 & x_{23} & \cdots & x_{2,r} \\ & 1 & & \vdots \\ & & \ddots & x_{r-1,r} \\ & & & 1 \end{pmatrix}\right)\psi\left(\tfrac{m_2 d_1}{d_2}x_{23} + \cdots + \tfrac{m_r d_{r-1}}{d_r}x_{r-1,r}\right)\prod dx_{ij},$$

$f' = f'_{(d_1,\ldots,d_r)}$. Now we use the induction hypothesis and write the integral here

$$\sum_{0 \neq D_i \in \mathfrak{o}_S/\mathfrak{o}_S^\times} H\left(D_2, \ldots, D_r; \frac{m_2 d_1}{d_2}, \ldots, \frac{m_r d_{r-1}}{d_r}\right)$$
$$\times \prod_{i=2}^r |D_i|^{-2s_i} \Lambda^{D_2/D_3,\ldots,D_{r-1}/D_r,D_r}_{m_2 d_1/d_2,\ldots,m_r d_{r-1}/d_r}(f'_{d_1,\ldots,d_r}),$$

where we recall that Λ is given by (14). The result will follow from substituting the above evaluation into (27) and using the identity

(28)

$$\begin{aligned}&\mathfrak{I}(D_r^{-1}, D_rD_{r-1}^{-1}, \dots, D_3D_2^{-1}, D_2)\Lambda^{D_2/D_3,\dots,D_{r-1}/D_r,D_r}_{m_2d_1/d_2,\dots,m_rd_{r-1}/d_r}(f'_{d_1,\dots,d_r})\\&= \mathfrak{I}(C_r^{-1}, C_rC_{r-1}^{-1}, \dots, C_2C_1^{-1}, C_1)\\&\quad\times \prod_{i<j}(d_i,d_j)_S\prod_{i=2}^{r}(d_i,D_i)_S|d_1^{r-1}d_2^{r-3}\cdots d_r^{-(r-1)}|^{-1}\Lambda^{C_1/C_2,\dots,C_{r-1}/C_r,C_r}_{m_1,\dots,m_r}(f).\end{aligned}$$

Here $C_i = D_i\prod_{j=i}^r d_j$. Indeed the left-hand side equals

$$\int_{F_S^{r(r+1)/2}} f\left(\boldsymbol{s}\begin{pmatrix} D_r^{-1} &&&&\\ & D_{r-1}^{-1}D_r &&&\\ && \ddots &&\\ &&& D_3D_2^{-1} &&\\ &&&& D_2 &\\ &&&&& 1\end{pmatrix} i(\boldsymbol{s}(J_r))\boldsymbol{s}\begin{pmatrix}1 & x_{23} & \cdots & x_{2,r+1} & 0\\ & 1 & & \vdots & \vdots\\ && \ddots & x_{r,r+1} & 0\\ &&& 1 & 0\\ &&&& 1\end{pmatrix}\right.$$

$$\left.\times\, \mathfrak{D}_{d_1,\dots,d_r}\boldsymbol{s}\begin{pmatrix} & -I_r\\ 1 & \end{pmatrix}\boldsymbol{s}\begin{pmatrix}1 & x_{12} & x_{13} & \cdots & x_{1,r+1}\\ & 1 & 0 & \cdots & 0\\ && \ddots && \vdots\\ &&& 1 & 0\\ &&&& 1\end{pmatrix}\right)$$

$$\times\,\psi\left(m_1x_{12} + \frac{m_2d_1}{d_2}x_{23} + \cdots + \frac{m_rd_{r-1}}{d_r}x_{r,r+1}\right)\prod dx_{ij}.$$

Moving the $\mathfrak{D}_{d_1,\dots,d_r}$ past $\boldsymbol{s}(J_r)$ and then reparametrizing the x_{ij} produces a measure change and a cocycle. Combining the diagonal matrices produces another cocycle, and we obtain (28). $\square$

6. A special vector

Since we want to construct a particular multiple Dirichlet series, we will specialize f. Let us immediately impose one condition. We note that the metaplectic cover splits over $G_\infty = \mathrm{SL}_{r+1}(F_\infty)$ and on G_∞ the section $\boldsymbol{s}$ is a splitting homomorphism. Let $K = \prod_{v\in S_\infty} U(r+1)$ be the standard maximal compact subgroup of G_∞. The condition that we impose immediately is that $f_\infty(g_\infty) = 1$ when $g_\infty \in \boldsymbol{s}(K)$. Since $\Omega \supset F_\infty^\times$ the eigenvalues t_i can be arbitrary elements of $F_\infty^\times$ in (13) and the archimedean component is just $\frac{1}{2}[F:\mathbb{Q}]$ copies of a standard principal series representation of $\mathrm{SL}_{r+1}(\mathbb{C})$, and we have chosen the normalized spherical function at these places. We express this by saying that f is *spherical at the archimedean places*. Then we may write $f(g) = f_\infty^\circ(g_\infty)f_{\mathrm{fin}}(g_{\mathrm{fin}})$ where we factor $g = g_\infty g_{\mathrm{fin}}$ with $g_\infty \in \widetilde{G}_\infty = p^{-1}\,\mathrm{SL}_{r+1}(F_\infty)$ and $g_{\mathrm{fin}} \in \widetilde{G}_{\mathrm{fin}} = p^{-1}\,\mathrm{SL}_{r+1}(F_{\mathrm{fin}})$. Here f_∞° is the standard spherical function on $\widetilde{G}_\infty$ and f_{fin} is as yet unspecified.

Choose a nontrivial character ψ of F_S that is trivial on $\mathfrak{o}_S$ but on no larger fractional ideal. We will consider Whittaker functions associated to f. The relevant archimedean integral is

$$W^\circ(s_1,\dots,s_r) = W^\circ_{m_1,\dots,m_r}(s_1,\dots,s_r)$$
$$= \int_{F_\infty^{r(r+1)/2}} f^\circ_\infty\left(\boldsymbol{s}(J_{r+1})\boldsymbol{s}\begin{pmatrix} 1 & x_{12} & \cdots & x_{1,r+1} \\ & 1 & \cdots & x_{2,r+1} \\ & & \ddots & \vdots \\ & & & 1\end{pmatrix}\right)\psi\left(\sum_{i=1}^r m_i x_{i,i+1}\right)\prod_{i,j} dx_{i,j}.$$

At the finite places, define

(29)

$$\Psi(c_1,c_2,\dots,c_r) := \Psi_{m_1,\dots,m_r;f}(c_1,c_2,\dots,c_r) = \mathfrak{I}_{\mathrm{fin}}(c_r^{-1}, c_r c_{r-1}^{-1},\dots,c_2c_1^{-1},c_1)^{-1}$$
$$\times\int_{F_{\mathrm{fin}}^{r(r+1)/2}} f_{\mathrm{fin}}\left(\mathfrak{T}_{c_1,\dots,c_r}\,\boldsymbol{s}(J_{r+1})\boldsymbol{s}\begin{pmatrix} 1 & x_{12} & \cdots & x_{1,r+1} \\ & 1 & \cdots & x_{2,r+1} \\ & & \ddots & \vdots \\ & & & 1\end{pmatrix}\right)\psi\left(\sum_{i=1}^r m_i x_{i,i+1}\right)\prod_{i,j} dx_{i,j},$$

where we set

$$\mathfrak{T}_{c_1,\dots,c_r} = \boldsymbol{s}\begin{pmatrix} c_r^{-1} & & & & \\ & c_r c_{r-1}^{-1} & & & \\ & & \ddots & & \\ & & & c_2c_1^{-1} & \\ & & & & c_1\end{pmatrix}.$$

It follows from Jacquet [18] that the integrals are convergent if $\mathrm{re}(s_i)$ is sufficiently large, but have meromorphic continuation to all s_i.

The notation makes explicit the dependence of W° on the s_i, and emphasizes the dependence of Ψ on the c_i. This point requires comment. First regarding W°, we could have written W° with an expression identical to (29) replacing fin by ∞. However this expression would be independent of the c_i by (13) since the infinite components of the c_i are in $\Omega \supset (F_S^\times)^n \supset F_\infty^\times$. Thus the notation shows the s_i dependence of W° but not the c_i dependence. The value

$$\left\{\prod_{1\leqslant i<j\leqslant r+1}\Gamma_{\mathbb{C}}(2s_i+2s_{i+1}+\cdots+2s_j-j+i+1)\right\}W^\circ_{m_1,\dots,m_r}(s_1,\dots,s_r)$$

is the normalized Jacquet-Whittaker function at the identity. By Jacquet [18] it is entire and if ψ is chosen suitably at the archimedean places it is invariant, up to an exponential factor that depends on ψ and the m_i, under the Weyl group action described in [8].

Regarding Ψ, it too depends on the s_i. However, the following lemma shows that we may choose f varying analytically so that Ψ is constant, that is, independent of the s_i. In view of this, we will suppress the s_i from the notation.

As in [8], let $\mathcal{M}(\Omega^r)$ be the finite-dimensional vector space of functions $\Psi : F_{\mathrm{fin}}^r \longrightarrow \mathbb{C}$ such that for any $\varepsilon_1, \dots, \varepsilon_r$ in Ω, we have

$$\Psi(\varepsilon_1 c_1, \dots, \varepsilon_r c_r) = \prod_{i=1}^{r} (\varepsilon_i, c_i)_S \left\{ \prod_{i<j} (\varepsilon_i, c_j)_S^{-1} \right\} \Psi(c_1, \dots, c_r). \tag{30}$$

Proposition 3. *If Ψ is defined by* (29), *then $\Psi \in \mathcal{M}(\Omega^r)$. Conversely, if $\Psi \subset \mathcal{M}(\Omega^r)$ is given, then we may choose the function f depending analytically on $s_1, \dots, s_r$ so that the integral* (29) *is independent of $s_1, \dots, s_r$ and equal to the given Ψ.*

Proof. It is easy to check that Ψ given by (29) satisfies (30). On the other hand, suppose that $\Psi \in \mathcal{M}(\Omega^r)$ is given. Let $N(\mathfrak{a})$ be the subgroup of N consisting of elements whose entries above the diagonal lie in an ideal $\mathfrak{a}$. Let f be a function in $\pi(s_1, \cdots, s_r)$ (in particular, genuine) with support in the big Bruhat cell of $\mathrm{SL}_{r+1}(F_{\mathrm{fin}})$ that satisfies

$$f\left(\boldsymbol{s}(n)\mathfrak{T}(c_1, \dots, c_r)\boldsymbol{s}(n')\right) = \begin{cases} \mathrm{vol}(N(\mathfrak{a}))^{-1}\mathfrak{I}_{\mathrm{fin}}(c_r^{-1}, c_r c_{r-1}^{-1}, \dots, c_1)\Psi(c_1, \dots, c_r) & \text{if } n' \in N(\mathfrak{a}), \\ 0 & \text{otherwise}. \end{cases}$$

Then it is easy to see that if $\mathfrak{a}$ is sufficiently small (depending on m_1 and m_2) that (29) is satisfied. $\square$

Combining the archimedean and nonarchimedean integrals, we have

$$W^\circ(s_1, \dots, s_r)\Psi(c_1, \dots, c_r) = (\mathfrak{I}(c_r^{-1}, c_r c_{r-1}^{-1}, \dots, c_1))^{-1}$$
$$\times \int_{F_S^{r(r+1)/2}} f\left(\mathfrak{T}_{c_1, \cdots, c_r}\, \boldsymbol{s}(J_{r+1})\boldsymbol{s}\begin{pmatrix} 1 & x_{12} & \cdots & x_{1,r+1} \\ & 1 & \cdots & x_{2,r+1} \\ & & \ddots & \vdots \\ & & & 1 \end{pmatrix}\right) \psi\left(\sum_{i=1}^{r} m_i x_{i,i+1}\right) \prod_{i,j} dx_{i,j}.$$

If $\prod t_i = 1$, then rewriting the last displayed formula in terms of t_i,

$$\int_{F_S^{r(r+1)/2}} f\left(\mathfrak{V}_{t_1, \dots, t_{r+1}}\boldsymbol{s}(J_{r+1})\boldsymbol{s}\begin{pmatrix} 1 & x_{12} & \cdots & x_{1,r+1} \\ & 1 & \cdots & x_{2,r+1} \\ & & \ddots & \vdots \\ & & & 1 \end{pmatrix}\right) \psi\left(\sum_{i=1}^{r} m_i x_{i,i+1}\right) \prod_{i,j} dx_{i,j}$$
$$= \mathfrak{I}(t_1, \dots, t_{r+1})\Psi(t_{r+1}, t_r t_{r+1}, \dots, t_2 \cdots t_{r+1})W^\circ(s_1, \dots, s_r), \tag{31}$$

where

$$\mathfrak{V}_{t_1, \dots, t_{r+1}} = \boldsymbol{s}\begin{pmatrix} t_1 & & \\ & \ddots & \\ & & t_{r+1} \end{pmatrix}.$$

7. Twisted multiplicativity

In this section, we prove two twisted multiplicativity statements for the coefficients of the multiple Dirichlet series. Taken together, these imply that the value of a general coefficient $H(C_1, \dots, C_r; m_1, \dots, m_r)$ is determined from the values of the coefficients where all parameters C_i and m_i are powers of a single prime p.

If m, c are nonzero elements of $\mathfrak{o}_S$, define the n^{th} order Gauss sum

$$g(m,c) = \sum_{\substack{d \bmod c \\ \gcd(d,c)=1}} \left(\frac{d}{c}\right)_n \psi\left(\frac{md}{c}\right), \tag{32}$$

formed with n^{th} power residue symbol and additive character ψ trivial on $\mathfrak{o}_S$ as before. Properties of these Gauss sums are summarized in [5]. We suppress the dependence on n in the notation, and understand all power residue symbols and Hilbert symbols to be n^{th} power symbols for a fixed integer n, the degree of our metaplectic cover.

Theorem 2. *If* $\gcd(m_1 \cdots m_r, C_1 \cdots C_r) = 1$, *then*

$$H(C_1, \dots, C_r; m_1 n_1, \dots, m_r n_r) = \left(\frac{m_1}{C_1}\right)^{-1} \cdots \left(\frac{m_r}{C_r}\right)^{-1} H(C_1, \dots, C_r; n_1, \dots, n_r). \tag{33}$$

Proof. We induct on r. For $r = 1$, since $H(C_1, m_1 n_1) = g(m_1 n_1, C_1)$, equation (33) follows by the usual properties of Gauss sums.

For general r, since $d_1 \cdots d_r = C_1$, we have $\gcd(d_1 \cdots d_r, m_1 \cdots m_r) = 1$. Hence in formula (22) for $H(C_1, \dots, C_r; m_1 n_1, \dots, m_r n_r)$, the condition $d_{i+1} \mid m_{i+1} n_{i+1} d_i$ holds if and only if $d_{i+1} \mid n_{i+1} d_i$ for each $1 \leqslant i \leqslant r-1$. In the inner sum in (22), we may make the variable changes $c_i \mapsto (\prod_{\ell=1}^{i} m_\ell)^{-1} c_i$ for $i = 1, \dots, r$, where the inverses are multiplicative inverses modulo C_1 (and hence modulo d_i for each i). Note that this changes u_i to $(\prod_{\ell=1}^{i} m_\ell) u_i$. This variable change removes all m_i's from the exponential sum and contributes the factor

$$\prod_{i=1}^{r} \left(\frac{\prod_{\ell=1}^{i} m_\ell}{d_i}\right)^{-1} = \prod_{\ell=1}^{r} \left(\frac{m_\ell}{\prod_{i=\ell}^{r} d_i}\right)^{-1}. \tag{34}$$

Also, we have

$$\gcd\left(m_2 \cdots m_r, \frac{C_2}{d_2 \cdots d_r} \cdot \frac{C_3}{d_3 \cdots d_r} \cdots \frac{C_r}{d_r}\right) = 1.$$

So we may apply induction to simplify the coefficient H on the right-hand side of (22). Pulling out $m_2, \dots, m_r$ contributes a factor of

$$\prod_{\ell=2}^{r} \left(\frac{m_\ell}{C_\ell / \prod_{i=\ell}^{r} d_i}\right)^{-1}. \tag{35}$$

Multiplying (34) and (35) and simplifying, one obtains (33). □

THEOREM 3. *If* $\gcd(C_1\cdots C_r, C'_1\cdots C'_r)=1$, *then*

$$H(C_1C'_1,\dots,C_rC'_r;n_1,\dots,n_r)$$
$$=\varepsilon_{(C_1,\dots,C_r),(C'_1,\dots,C'_r)}H(C_1,\dots,C_r;n_1,\dots,n_r)H(C'_1,\dots,C'_r;n_1,\dots,n_r),$$

where

$$\varepsilon_{(C_1,\dots,C_r),(C'_1,\dots,C'_r)}=\prod_{i=1}^{r}\left(\frac{C_i}{C'_i}\right)\left(\frac{C'_i}{C_i}\right)\prod_{j=1}^{r-1}\left(\frac{C_j}{C'_{j+1}}\right)^{-1}\left(\frac{C'_j}{C_{j+1}}\right)^{-1}.$$

Here the n^{th} root of unity $\epsilon_{(C_1,\dots,C_r),(C'_1,\dots,C'_r)}$ reflects the root system A_r, whose Dynkin diagram has r nodes with the j^{th} node connected to the $(j+1)^{\text{st}}$ node for $1\leqslant j\leqslant r-1$. Indeed up to a product of Hilbert symbols this is

$$\prod_{i=1}^{r}\left(\frac{C_i}{C'_i}\right)^2\prod_{j=i\pm1}\left(\frac{C_i}{C'_j}\right)^{-1}$$

and the exponents here are the coefficients in the Cartan matrix of type A_r. This phenomenon extends to other root systems as in [8] or [9].

Proof. Suppose $\gcd(C_1\cdots C_r, C'_1\cdots C'_r)=1$. We begin with the sum of the form (22) for the coefficient $H(C_1C'_1,\dots,C_rC'_r;n_1,\dots,n_r)$. We must sum over d_i, $1\leqslant i\leqslant r$, such that $d_1\cdots d_r=C_1C'_1$ and such that the d_i satisfy the divisibility conditions

$$d_{i+1}|n_{i+1}d_i \text{ for } 1\leqslant i\leqslant r-1,\qquad d_j\cdots d_r|C_jC'_j \text{ for } 2\leqslant j\leqslant r. \tag{36}$$

Since $\gcd(C_1,C'_1)=1$, there is a unique way to factor each d_i, $d_i=e_ie'_i$, such that

$$e_1\cdots e_r=C_1,\qquad e'_1\cdots e'_r=C'_r. \tag{37}$$

Doing so, $\gcd(e_i,e'_j)=1$ for all i,j, and the divisibility conditions also break up:

$$\begin{aligned}&e_{i+1}|n_{i+1}e_i, e'_{i+1}|n_{i+1}e'_i \text{ for } 1\leqslant i\leqslant r-1,\\&e_j\cdots e_r|C_j, e'_j\cdots e'_r|C'_j \text{ for } 2\leqslant j\leqslant r.\end{aligned} \tag{38}$$

Conversely, given e_i, e'_i satisfying conditions (37), (38), set $d_i=e_ie'_i$. Then $d_1\cdots d_r=C_1C'_1$ and the divisibility conditions (36) hold. For example, since $e_{i+1}\mid n_{i+1}e_i$ and $e'_{i+1}\mid n_{i+1}e'_i$, and since $\gcd(e_i,e'_i)=1$, we have

$$e_{i+1}\mid n_{i+1}e_i/\gcd(n_{i+1},e'_{i+1}) \text{ and } e'_{i+1}\mid n_{i+1}e'_i/\gcd(n_{i+1},e_{i+1}).$$

From this it easily follows that $e_{i+1}e'_{i+1}\mid n_{i+1}e_ie'_i$, or $d_{i+1}\mid n_{i+1}d_i$. Thus there is a one-to-one correspondence between the d_i satisfying $d_1\cdots d_r=C_1C'_1$ and (36) and the pairs e_i,e'_i satisfying (37), (38), and we may split up the sum over the d_i into sums over e_i and over e'_i.

When we do so, we must split the inner sum in (22), using the Chinese Remainder Theorem. It is convenient to do so as follows. Let $c_i=x'_ie_1\cdots e_i+$

$x_i e'_1 \cdots e'_i$. Then since $e_1 \cdots e_{i-1}$ is a unit modulo e'_i and $e'_1 \cdots e'_{i-1}$ is a unit modulo e_i, as x'_i varies modulo e'_i and x_i varies modulo e_i, c_i varies modulo d_i. With this parametrization, the c_i that are invertible modulo d_i are those with $\gcd(x_i, e_i) = \gcd(x'_i, e'_i) = 1$, and for such c_i, u_i is determined by the equations $u_i x'_i e_1 \cdots e_i \equiv 1 \bmod e'_i$, $u_i x_i e'_1 \cdots e'_i \equiv 1 \bmod e_i$. Let v_i modulo e_i (resp. v'_i modulo e'_i) satisfy $v_i x_i \equiv 1 \bmod e_i$ (resp. $v'_i x'_i \equiv 1 \bmod e'_i$). We have $\psi(n_1 c_1/d_1) = \psi(n_1 x_1/e_1)\psi(n_1 x'_1/e'_1)$ and, for $i \geqslant 2$,

$$\begin{aligned}\psi(n_i u_{i-1} c_i/d_i) &= \psi(n_i u_{i-1}(x'_i e_1 \cdots e_i + x_i e'_1 \cdots e'_i)/e_i e'_i)\\ &= \psi(n_i u_{i-1} x'_i e_1 \cdots e_{i-1}/e'_i)\psi(n_i u_{i-1} x_i e'_1 \cdots e'_{i-1}/e_i)\\ &= \psi(n_i v'_{i-1} x'_i/e'_i)\psi(n_i v_{i-1} x_i/e_i).\end{aligned}$$

Here the last equality follows from the congruences above since $e_i \mid n_i e_{i-1}$, $e'_i \mid n_i e'_{i-1}$.

Thus the exponential sum in (22) factors into two sums with similar divisibility conditions and similar exponentials. We now compute the power residue symbols that arise in doing so. Throughout this computation, we will be working with pairs of numbers of the form A, A' and B, B' such that $\gcd(A, B') = \gcd(A', B) = 1$. For convenience, let us introduce the notation $f(A, B) = \left(\frac{A}{B'}\right)\left(\frac{A'}{B}\right)$. Then we have

$$\begin{aligned}&f(A_1 A_2, B_1 B_2) = f(A_1, B_1) f(A_1, B_2) f(A_2, B_1) f(A_2, B_2),\\ &f(A, B^{-1}) = f(A, B)^{-1}.\end{aligned} \tag{39}$$

With the notation as above, for $1 \leqslant i \leqslant r$ we have

$$\left(\frac{c_i}{d_i}\right) = \left(\frac{x'_i e_1 \cdots e_i + x_i e'_1 \cdots e'_i}{e_i e'_i}\right) = \left(\frac{x'_i}{e'_i}\right)\left(\frac{x_i}{e_i}\right) f(e_1 \cdots e_i, e_i). \tag{40}$$

Power residue symbols also arise when we use induction to decompose the coefficient

$$H\left(\frac{C_2 C'_2}{d_2 \cdots d_r}, \frac{C_3 C'_3}{d_3 \cdots d_r}, \ldots, \frac{C_r C'_r}{d_r}; \frac{n_2 d_1}{d_2}, \ldots, \frac{n_r d_{r-1}}{d_r}\right)$$

on the right-hand side of (22). Let D_i, D'_i be defined by $C_i = D_i \prod_{j=i}^r e_j$, $C'_i = D'_i \prod_{j=i}^r e'_j$. Then $\gcd(D_2 \cdots D_r, D'_2 \cdots D'_r) = 1$, and we obtain by induction

(41)

$$\begin{aligned}&H\left(D_2 D'_2, D_3 D'_3, \ldots, D_r D'_r; \frac{n_2 d_1}{d_2}, \ldots, \frac{n_r d_{r-1}}{d_r}\right)\\ &\quad = \prod_{j=2}^{r} f(D_j, D_j) \prod_{k=2}^{r-1} f(D_k, D_{k+1})^{-1}\\ &\quad \times H\left(D_2, \ldots, D_r; \frac{n_2 d_1}{d_2}, \ldots, \frac{n_r d_{r-1}}{d_r}\right) H\left(D'_2, \ldots, D'_r; \frac{n_2 d_1}{d_2}, \ldots, \frac{n_r d_{r-1}}{d_r}\right).\end{aligned}$$

To split up the d_i's on the right-hand side of (41), write $n_i = m_i m_i'$ such that $m_i e_{i-1}/e_i$ and $m_i' e_{i-1}'/e_i'$ are integral and $\gcd(m_i, C_1') = \gcd(m_i', C_1) = 1$ for all i. Then $\gcd(m_i, D_j') = \gcd(m_i', d_j) = 1$ for all i, j, so that by Theorem 2,

$$H\left(D_2, \ldots, D_r; \frac{n_2 d_1}{d_2}, \ldots, \frac{n_r d_{r-1}}{d_r}\right) = \prod_{j=2}^{r} \left(\frac{m_j' e_{j-1}'/e_j'}{D_j}\right)^{-1} H\left(D_2, \ldots, D_r; \frac{m_2 e_1}{e_2}, \ldots, \frac{m_r e_{r-1}}{e_r}\right).$$

But $\left(\frac{m_j' e_{j-1}'/e_j'}{D_j}\right)^{-1} = \left(\frac{m_j'}{D_j}\right)^{-1} \left(\frac{e_{j-1}'}{D_j}\right)^{-1} \left(\frac{e_j'}{D_j}\right)$. Since $\gcd(m_j', D_j) = 1$ for all j, we may then put the m_j back into the coefficients H by using Theorem 2 "in reverse." Doing so, and making a similar argument with

$$H\left(D_2', \ldots, D_r'; \frac{n_2 d_1}{d_2}, \ldots, \frac{n_r d_{r-1}}{d_r}\right),$$

we obtain

(42)
$$\begin{aligned} &H\left(D_2 D_2', D_3 D_3', \ldots, D_r D_r'; \frac{n_2 d_1}{d_2}, \ldots, \frac{n_r d_{r-1}}{d_r}\right) \\ &\quad = \prod_{j=2}^{r} f(D_j, , D_j) f(e_{j-1}, D_j)^{-1} f(e_j, D_j) \prod_{k=2}^{r-1} f(D_k, D_{k+1})^{-1} \\ &\qquad \times H\left(D_2, \ldots, D_r; \frac{n_2 e_1}{e_2}, \ldots, \frac{n_r e_{r-1}}{e_r}\right) H\left(D_2', \ldots, D_r'; \frac{n_2 e_1'}{e_2'}, \ldots, \frac{n_r e_{r-1}'}{e_r'}\right). \end{aligned}$$

Also Hilbert symbols arise from the factorizations of d_i and D_i in (22). We have

$$(d_i, d_j)_S = (e_i, e_j)_S (e_i', e_j')_S (e_i, e_j')_S (e_i', e_j)_S$$

and similarly for $(e_i e_i', D_i D_i')_S$.

Finally, we collect the residue symbols that arise in (40) and (42) that are independent of x_i, x_i'. Call this quantity ϵ. Then

$$\begin{aligned} \epsilon = &\prod_{k=1}^{r} \prod_{\ell=1}^{k} f(e_\ell, e_k) \prod_{j=2}^{r} f(D_j, D_j) f(e_{j-1}, D_j)^{-1} f(e_j, D_j) \\ &\times \prod_{k=2}^{r-1} f(D_k, D_{k+1})^{-1} \prod_{i<j} (e_i, e_j')_S (e_i', e_j)_S \prod_{j=2}^{r} (e_j, D_j')_S (e_j', D_j)_S. \end{aligned} \tag{43}$$

By contrast,

$$\begin{aligned}\epsilon_{(C_1,\ldots,C_r),(C'_1,\ldots,C'_r)} &= \prod_{i=1}^{r} f(C_i,C_i)\prod_{j=1}^{r-1} f(C_j,C_{j+1})^{-1}\\ &= f(C_r,C_r)\prod_{j=1}^{r-1} f(C_j,C_jC_{j+1}^{-1}).\end{aligned}$$

Since $C_jC_{j+1}^{-1} = D_jD_{j+1}^{-1}e_j$, using the properties (39), rearranging and cancelling terms, and recalling that $D_1 = 1$, so that $f(D_1, a) = f(a, D_1) = 1$ for all a, we see that

$$\begin{aligned}\epsilon_{(C_1,\ldots,C_r),(C'_1,\ldots,C'_r)} &= f(D_re_r,D_re_r)\prod_{j=1}^{r-1}\prod_{k=j}^{r} f(D_je_k,D_jD_{j+1}^{-1}e_j)\\ &= \prod_{j=2}^{r} f(D_j,D_j)\prod_{k=2}^{r-1} f(D_k,D_{k+1})^{-1}\prod_{k=1}^{r}\prod_{\ell=1}^{k} f(e_k,e_\ell)\\ &\quad\times\prod_{j=2}^{r} f(D_j,e_j)f(e_{j-1},D_j)^{-1}.\end{aligned}\tag{44}$$

However, according to the reciprocity law (15), we have $f(A,B)(A,B')_S(A',B)_S = f(B,A)$ for all A, B, A', B' as above. Comparing (43) and (44) we have that $\epsilon = \epsilon_{(C_1,\ldots,C_r),(C'_1,\ldots,C'_r)}$. This completes the proof of Theorem 3. □

8. Evaluation of the p-parts

In Section 2 of [12], we associated the p-part of a multiple Dirichlet series of type A_r indexed by $p^{\mathbf{l}} = (p^{l_1},\ldots,p^{l_r})$ to the set of all Gelfand-Tsetlin patterns with top row:

$$\lambda+\rho = (L_1,\ldots,L_r,0) := (l_1+\cdots+l_r+r,\ldots,l_r+1,0),$$

where $\rho = (r, r-1,\ldots,0)$. Thus, setting $L_{r+1} = 0$ for uniformity of notation, we have $L_i - L_{i+1} - 1 = l_i$ for $i = 1,\ldots,r$. Let $\mathrm{GT}(\lambda+\rho)$ denote the set of all Gelfand-Tsetlin patterns with this fixed top row.

Given a fixed prime p of norm $|p| = q$, then we set

$$\begin{aligned}H_{\mathrm{GT}}(p^{\mathbf{k}},p^{\mathbf{l}}) &= H_{\mathrm{GT}}(p^{k_1},\ldots,p^{k_r};p^{l_1},\ldots,p^{l_r})\\ &= \sum_{\substack{\mathfrak{T}\,\in\,\mathrm{GT}(\lambda+\rho)\\ k(\mathfrak{T})=\mathbf{k}}} G(\mathfrak{T})q^{-2k_1(\mathfrak{T})s_1-\ldots-2k_r(\mathfrak{T})s_r},\end{aligned}\tag{45}$$

where the two functions on Gelfand-Tsetlin patterns,

$$G(\mathfrak{T})\quad\text{and}\quad k(\mathfrak{T}) = (k_1(\mathfrak{T}),\ldots,k_r(\mathfrak{T})),$$

will be defined presently. Let us denote the entries of the Gelfand-Tsetlin pattern as follows:

$$\left\{\begin{array}{ccccccccccc} a_{0,0} & & a_{0,1} & & a_{0,2} & & \cdots & a_{0,r-1} & & a_{0,r} \\ & a_{1,1} & & a_{1,2} & & a_{1,3} & \cdots & & a_{1,r} & \\ & & \ddots & & & & & & & \\ & & & & a_{r,r} & & & & & \end{array}\right\} \tag{46}$$

with $a_{0,j} = L_{j+1}$ for $0 \leqslant j \leqslant r$. Then define

$$e_{i,j} = \sum_{k=j}^{r} a_{i,k} - a_{i-1,k} \quad \text{for all } 1 \leqslant i \leqslant j \leqslant r \tag{47}$$

and

$$G(\mathfrak{T}) = \prod_{1 \leqslant i \leqslant j \leqslant r} \gamma(a_{i,j}), \tag{48}$$

$$\gamma(a_{i,j}) = \begin{cases} q^{e_{i,j}} & \text{if } a_{i-1,j-1} > a_{i,j} = a_{i-1,j}, \\ g(p^{e_{i,j}}, p^{e_{i,j}}) & \text{if } a_{i-1,j-1} > a_{i,j} > a_{i-1,j}, \\ g(p^{e_{i,j}-1}, p^{e_{i,j}}) & \text{if } a_{i-1,j-1} = a_{i,j} > a_{i-1,j}, \\ 0 & \text{if } a_{i-1,j-1} = a_{i,j} = a_{i-1,j}, \end{cases}$$

where the Gauss sums $g(p^a, p^b)$ are as defined in (32). Further define for $i = 1, \ldots, r$,

$$k_i(\mathfrak{T}) = \sum_{j=i}^{r} (a_{i,j} - a_{0,j}). \tag{49}$$

Note that these are identical to the functions presented in [12, eq. (28) and (31)].

We now prove that the Gelfand-Tsetlin pattern description of H_{GT} given in (45) satisfies the same recursion at prime-power supported coefficients as the one asserted in Theorem 1 for the coefficients of Eisenstein series H. To do this, we parametrize the set of Gelfand-Tsetlin patterns with top row $\lambda+\rho$ by pairs $(\tau, GT(\lambda+\rho,\tau))$ with τ of form

(50)

$$\tau = \left\{\begin{array}{ccccccccccc} L_1 & & L_2 & & L_3 & & \cdots & L_r & & L_{r+1} \\ & L_2+t_1 & & L_3+t_2 & & \cdots & L_r+t_{r-1} & & L_{r+1}+t_r & \end{array}\right\}$$

(i.e. given by a choice of vector $(t_1, \ldots, t_r)$ satisfying certain inequalities) and let $GT(\lambda+\rho,\tau)$ denote the set of patterns of rank $r-1$ with fixed top row $(L_2+t_1, \ldots, L_r+t_{r-1}, L_{r+1}+t_r)$. Our proofs will use induction on the rank r. To emphasize the dependence on the rank, let $k^{(r-1)}(\mathfrak{T})$ be the function $k(\mathfrak{T})$ defined above with respect to the rank $r-1$ pattern and similarly for $G^{(r-1)}$. In order to behave well with respect to induction, we number the components for $k^{(r-1)}(\mathfrak{T})$ by $k_2^{(r-1)}(\mathfrak{T}), \ldots, k_r^{(r-1)}(\mathfrak{T})$.

Proposition 4. *Let* $\mathbf{l} = (l_1, \dots, l_r)$ *and* $\mathbf{k} = (k_1, \dots, k_r)$ *be* r*-tuples of nonnegative integers. Then*

(51)

$$H_{\mathrm{GT}}(p^{\mathbf{k}}, p^{\mathbf{l}}) = \sum_{\substack{\mathfrak{T} \in \mathrm{GT}(\lambda+\rho) \\ k(\mathfrak{T}) = \mathbf{k}}} G(\mathfrak{T}) = \sum_{\substack{t_1, \dots, t_r \\ 0 \leqslant t_j \leqslant L_j - L_{j+1} \\ t_1 + \cdots + t_r = k_1}} \prod_{i=1}^{r} q^{(i-1)t_i} \sum_{\substack{c_1, \dots, c_r \\ c_i \ mod\ p^{t_i} \\ u_i : c_i u_i \equiv 1\ mod\ p^{t_i}}} \left(\frac{c_1}{p^{t_1}}\right) \cdots$$
$$\cdots \left(\frac{c_r}{p^{t_r}}\right) \psi\left(\frac{p^{l_1} c_1}{p^{t_1}} + \frac{p^{l_2} u_1 c_2}{p^{t_2}} + \cdots + \frac{p^{l_r} u_{r-1} c_r}{p^{t_r}}\right)$$
$$\times H_{\mathrm{GT}}^{(r-1)}(p^{k_2 - t_2 - \cdots - t_r}, \dots, p^{k_r - t_r}; p^{l_2 + t_1 - t_2}, \dots, p^{l_r + t_{r-1} - t_r}).$$

Here we understand that the lower rank $H_{\mathrm{GT}}^{(r-1)}(\,\cdot\,; \dots, p^{l_i + t_{i-1} - t_i}, \dots) = 0$ *if any of the exponents* $l_i + t_{i-1} - t_i < 0$ *for* $i = 2, \dots, r$.

Proof. We begin by rewriting each of the Gelfand-Tsetlin patterns $\mathfrak{T}$ with top row $\lambda + \rho$ and $k(\mathfrak{T}) = \mathbf{k}$ in terms of pairs $(\tau, \mathfrak{T}')$ where τ consists of two rows as in (50), and $\mathfrak{T}'$ is a pattern of rank one less with top row matching the bottom row of τ. As defined in (49), the condition $k(\mathfrak{T}) = \mathbf{k}$ implies that row sums in $\mathfrak{T}$, and hence row sums in the corresponding τ and $\mathfrak{T}'$, are fixed. More precisely, one immediately checks that in τ, $t_1 + \cdots + t_r = k_1$ and

$$k_i(\mathfrak{T}) = k_i^{(r-1)}(\mathfrak{T}') + t_i + \cdots + t_r \qquad \text{for } i = 2, \dots, r.$$

Hence we may write

(52)
$$\sum_{\substack{\mathfrak{T} \in \mathrm{GT}(\lambda+\rho) \\ k(\mathfrak{T}) = \mathbf{k}}} G(\mathfrak{T}) = \sum_{\substack{t_1, \dots, t_r \\ 0 \leqslant t_j \leqslant L_j - L_{j-1} \\ t_1 + \cdots + t_r = k_1}} \left[\prod_{1 \leqslant j \leqslant r} \gamma(a_{1,j})\right] \sum_{\substack{\mathfrak{T}' \in \mathrm{GT}(\lambda+\rho, \tau) \\ k^{(r-1)}(\mathfrak{T}') = \mathbf{k}^{(r-1)}}} G^{(r-1)}(\mathfrak{T}'),$$

where $a_{1,j} = L_{j+1} + t_j$ and $\mathbf{k}^{(r-1)} = (k_2 - t_2 - \cdots - t_r, \dots, k_r - t_r)$. The summation conditions on the t_i guarantee the second row entries of $\mathfrak{T}$ interleave, but it may still happen that two adjacent second row entries $a_{1,j-1}$ and $a_{1,j}$ are equal. Then according to our definition of $\gamma(a_{i,j})$ in (48), $\gamma(a_{2,j}) = 0$ and hence $G(\mathfrak{T}) = 0$. To show the right-hand side of (51) is also 0 in this case, note that the definition of τ implies that if $a_{1,j-1} = a_{1,j}$, then $t_{j-1} = 0$, $t_j = L_j - L_{j+1} = l_j + 1$ and hence $l_j + t_{j-1} - t_j = -1$. So we understand $H_{\mathrm{GT}}^{(r-1)}$ to be 0 as noted in the statement of the proposition. Henceforth, we may assume the entries $a_{1,j}$ are strictly decreasing for $j = 1, \dots, r$.

Returning to (52), we must show that

$$\prod_{1\leqslant j\leqslant r}\gamma(a_{1,j})=\prod_{i=1}^{r}q^{(i-1)t_i}\sum_{\substack{c_1 \bmod p^{t_1}\\ c_1u_1\equiv 1 \bmod p^{t_1}}}\cdots\sum_{\substack{c_{r-1} \bmod p^{t_{r-1}}\\ c_{r-1}u_{r-1}\equiv 1 \bmod p^{t_{r-1}}}}\sum_{c_r \bmod p^{t_r}}$$
$$\left(\frac{c_1}{p^{t_1}}\right)\cdots\left(\frac{c_r}{p^{t_r}}\right)\psi\left(\frac{p^{l_1}c_1}{p^{t_1}}+\frac{p^{l_2}u_1c_2}{p^{t_2}}+\cdots+\frac{p^{l_r}u_{r-1}c_r}{p^{t_r}}\right),$$

which is an exercise in elementary number theory. We give an outline leaving the details to the reader. If $t_r=0$, then both the additive and multiplicative characters in terms of t_r are trivial as is the sum over p^{t_r}, corresponding to the fact that $\gamma(a_{1,r})=1$ in this case. If $t_r>0$, then we rewrite the inner sum via the automorphism $c_r\mapsto c_{r-1}c_r$. Note that the multiplicative character modulo $p^{t_{r-1}}$ guarantees that the sum is only nonzero when $\gcd(c_{r-1},p)=1$. In this case, the inner sum contributes $g(p^{l_r},p^{t_r})$ again matching the contribution $\gamma(a_{1,r})$. One may then repeat this case analysis for each successive sum with the substitution $c_i\mapsto c_{i-1}c_i$ to obtain the above identity. Note that in the i^{th} sum, the modulus of the multiplicative character associated to c_i will be $p^{t_i+\cdots+t_r}$ after successive changes of variable. Though the i^{th} sum remains over c_i modulo p^{t_i}, we may express the contribution of this sum as a Gauss sum with modulus $p^{t_i+\cdots+t_r}$ by borrowing $q^{t_{i+1}+\cdots+t_r}$ from $\prod_{i=1}^{r}q^{(i-1)t_i}$ and rewriting the additive character accordingly.

Hence we may rewrite the right-hand side of (52) as

$$\sum_{\substack{t_1,\dots,t_r\\ 0\leqslant t_j\leqslant L_j-L_{j-1}\\ t_1+\cdots+t_r=k_1}}\prod_{i=1}^{r}q^{(i-1)t_i}\sum_{\substack{c_1,\dots,c_r\\ c_i \bmod p^{t_i}\\ c_iu_i\equiv 1 \bmod p^{t_i}}}$$
$$\times\left(\frac{c_1}{p^{t_1}}\right)\cdots\left(\frac{c_r}{p^{t_r}}\right)\psi\left(\frac{p^{l_1}c_1}{p^{t_1}}+\frac{p^{l_2}u_1c_2}{p^{t_2}}+\cdots+\frac{p^{l_r}u_{r-1}c_r}{p^{t_r}}\right)$$
$$\times\sum_{\substack{\mathfrak{T}\in\mathrm{GT}(\lambda+\rho,\tau)\\ k^{(r-1)}(\mathfrak{T})=(k_2-t_2-\cdots-t_r,\dots,k_r-t_r)}}G^{(r-1)}(\mathfrak{T}).$$

To finish the proposition, note that $(L_j+t_{j-1})-(L_{j+1}+t_j)-1=l_j+t_{j-1}-t_j$, so that we may rewrite the inner sum above as

$$H_{\mathrm{GT}}^{(r-1)}(p^{k_2-t_2-\cdots-t_r},\dots,p^{k_r-t_r};p^{l_2+t_1-t_2},\dots,p^{l_r+t_{r-1}-t_r})$$

and substitute the result into the right-hand side of (52). $\square$

As a consequence, we establish the following determination of the p-part of the Whittaker coefficients in terms of Gelfand-Tsetlin patterns.

THEOREM 4. *Given any r-tuples of nonnegative integers $\mathbf{l} = (l_1, \ldots, l_r)$ and $\mathbf{k} = (k_1, \ldots, k_r)$ and any fixed prime p, we have*

$$H_{\mathrm{GT}}(p^{\mathbf{k}}, p^{\mathbf{l}}) = H(p^{\mathbf{k}}, p^{\mathbf{l}}),$$

where $H(p^{\mathbf{k}}, p^{\mathbf{l}})$ is as defined in Theorem 1 (§5).

Proof. The proof follows from the above proposition, since the reader can immediately check that this recursion for H_{GT} is identical to that of H given in Theorem 1, with $d_i = p^{t_i}$ and $n_i = p^{l_i}$. Moreover, the two descriptions agree at prime powers in rank 1 as both produce a single Gauss sum, and this uniquely determines a solution to the recursion. □

Combining this result with Theorems 2 and 3 we conclude that the $\mathbf{m}^{\mathrm{th}}$ Whittaker coefficient of the metaplectic Eisenstein series is a multiple Dirichlet series of the form (1) whose coefficients H may be computed using Gelfand-Tsetlin patterns as descibed in [12]. This establishes Conjecture 2 of [12].

9. Gelfand-Tsetlin patterns and crystal bases

In what follows, we demonstrate that the two definitions of our p-parts of H, presented in terms of Gelfand-Tsetlin patterns (in the preceding section) and crystal graphs (in §2), agree. A more extensive discussion of this matching and the combinatorial connections to crystals, Gelfand-Tsetlin patterns, and tableaux can be found in [11].

Given a semisimple algebraic group G, Littelmann [24] associates to any irreducible G-module V_λ of highest weight λ a combinatorial model for the crystal graph $\mathcal{B}_\lambda$ of V_λ (or more properly the corresponding simple module for the quantum group $U_q(\mathrm{Lie}(G))$) described in the introduction and Section 2. In this combinatorial model, the basis vectors of $\mathcal{B}_\lambda$ are parametrized by BZL patterns associated to a reduced decomposition Σ of the long element w_0 of the Weyl group. However, Littelmann's model differs from the one presented in Section 2 in one way. He uses Kashiwara raising operators e_i to the highest weight vector (applying them as before in order of simple reflections appearing in Σ), whereas we use lowering operators f_i to the lowest weight vector, which we find more compatible with the description of our resulting Dirichlet series.

In particular, Littelmann shows that the integer sequences comprising the BZL patterns for all elements of the crystal base of V_λ, regarded as integer lattice points in $\mathbb{R}^\nu$ where ν is the number of positive roots, are integral points of a polytope P_λ. For particular "good enumerations" Σ of the long element w_0, the inequalities describing this polytope (in terms of the group G, enumeration Σ, and the highest weight λ) are given explicitly. Good enumerations of w_0 are associated to a sequence of Levi subgroups $G \supset L_1 \supset \cdots \supset L_n = T$, where the Levi subgroups correspond to so-called braidless fundamental weights; see Section 4 of [24] for this definition.

In Section 5 of [24], Littelmann gives an explicit description of the highest weight polytope P_λ for irreducible $\mathrm{SL}(r+1)$-modules using the enumeration $w_0 = s_1(s_2s_1)\cdots(s_rs_{r-1}\cdots s_1)$. We require an explicit description for a different good enumeration, so we outline the proof briefly in the following result. For $\underline{c} \in \mathbb{R}^{\frac{1}{2}r(r+1)}$, let $\Delta(\underline{c})$ denote the filling of a triangular array with r rows and r columns from bottom to top and in each row from right to left. For example,

$$\underline{c} = (1,3,2,5,\ldots) \mapsto \Delta(\underline{c}) = \begin{array}{ccc} & \cdots & \\ \hline \multicolumn{1}{|c|}{} & \multicolumn{1}{c|}{} & \multicolumn{1}{c|}{5} \\ \hline \multicolumn{1}{|c|}{2} & \multicolumn{1}{c|}{3} & \\ \cline{1-2} \multicolumn{1}{|c|}{1} & & \\ \cline{1-1} \end{array} .$$

We further identify $\Delta(\underline{c})$ with $(c_{i,j})$, $1 \leqslant i \leqslant r$ and $1 \leqslant j \leqslant r+1-i$ where $c_{i,j}$ denotes the j^{th} element in the i^{th} row down, as usual. Thus, columns in $\Delta(\underline{c})$ correspond to the same simple reflection.

Lemma 3. *Given the good enumeration of the long element*

$$w_0 = s_r(s_{r-1}s_r)\cdots(s_1s_2\cdots s_r), \tag{53}$$

for the Weyl group of $\mathrm{SL}(r+1)$, *and a dominant weight*

$$\lambda = \lambda_1\epsilon_1 + \cdots + \lambda_r\epsilon_r \quad \epsilon_i\text{: \textit{fundamental weights.}}$$

The integral points of the polytope P_λ (parametrizing the crystal base of the highest weight module V_λ) consist of all sequences formed from triangular arrays $\Delta = (c_{i,j})$ which are nonnegative and weakly increasing in rows and bounded above by the inequalities (for all $1 \leqslant i \leqslant r$ and $1 \leqslant j \leqslant r+1-i$)

$$c_{i,j} \leqslant \lambda_{r+1-j} - \lambda_{r+2-j} + s(c_{i,j-1}) - 2s(c_{i-1,j}) + s(c_{i-1,j+1}), \tag{54}$$

where $s(c_{i,j}) = \sum_{k=1}^{i} c_{k,j}$. We understand $c_{i,j} = 0$ if $i+j > r+1$, $i = 0$, or $j = 0$.

This follows easily from results in Littelmann's paper, using Theorem 4.2 in [24] to show the $c_{i,j}$ are weakly increasing in rows (i.e. are members of a cone in $\mathbb{R}^{r(r+1)/2}$) and Proposition 1.5(b) to demonstrate the upper bound inequalities in (54) coming from the highest weight vector λ.

Now we define the following bijection between the Gelfand-Tsetlin basis and the BZL patterns (using raising operators e_i) associated to the enumeration in (53), both associated to a highest weight representation V_λ of highest weight λ.

LEMMA 4. *The map β from Gelfand-Tsetlin patterns $\{(a_{i,j})\}$ with top row $\lambda = (\lambda_1, \dots, \lambda_{r+1}) = (l_1 + \cdots + l_r, \dots, l_r, 0)$ to Littelmann patterns $(c_{i,j})$ with highest weight λ defined by*

$$c_{i,j} = \sum_{k=r+1-j}^{r} (a_{i,k} - a_{i-1,k}) \qquad \text{for all } 1 \leqslant i \leqslant r, 1 \leqslant j \leqslant r+1-i \tag{55}$$

is a bijection. Here the labeling for Gelfand-Tsetlin patterns is as in (46).

Note that the right-hand side of (55) is $e_{i,r+1-j}$ with $e_{i,j}$ as defined in (47). In short, the *entries* of the Littelmann pattern are precisely the data used in the definition of $G(\mathfrak{T})$ in (48). It is easy to check that the inverse map is then

$$a_{i,j} = \lambda_{j+1} + \sum_{k=1}^{i} (c_{k,r+1-j} - c_{k,r-j}) \, .$$

As an example for SL(5), observe that

$$\mathfrak{T} = \left\{ \begin{array}{ccccccccc} 11 & & 8 & & 7 & & 2 & & 0 \\ & 9 & & 7 & & 4 & & 1 & \\ & & 9 & & 5 & & 2 & & \\ & & & 8 & & 2 & & & \\ & & & & 3 & & & & \end{array} \right\} \leftrightarrow \Delta(\underline{c}) = \begin{array}{|c|c|c|c|} \hline 1 & 3 & 3 & 4 \\ \hline 1 & 2 & 4 & \multicolumn{1}{c}{} \\ \cline{1-3} 0 & 3 & \multicolumn{2}{c}{} \\ \cline{1-2} 1 & \multicolumn{3}{c}{} \\ \cline{1-1} \end{array} \, .$$

Proof. To verify that β gives a bijection, we must show that the resulting $c_{i,j}$ satisfy the polytope inequalities listed in Lemma 3. First note that $c_{i,j} \leqslant c_{i,j+1}$, since

$$c_{i,j+1} - c_{i,j} = \sum_{k=r-j}^{r} (a_{i,k} - a_{i-1,k}) - \sum_{k=r+1-j}^{r} (a_{i,k} - a_{i-1,k}) = a_{i,r-j} - a_{i-1,r-j} \geqslant 0$$

according to the interleaving rules for Gelfand-Tsetlin patterns. Further, note

$$c_{i,j} \leqslant \lambda_{r+1-j} - \lambda_{r+2-j} + s(c_{i,j-1}) - 2s(c_{i-1,j}) + s(c_{i-1,j+1})$$

since the interleaving rules of the Gelfand-Tsetlin pattern imply $a_{i,r+1-j} \leqslant a_{i-1,r-j}$ so that

$$c_{i,j} \leqslant a_{i-1,r-j} - a_{i-1,r+1-j} + c_{i,j-1}.$$

Using the inverse map to the bijection, the right-hand side can be rewritten as

$$\lambda_{r+1-j} - \lambda_{r+2-j} + \sum_{k=1}^{i-1} (c_{k,j+1} - c_{k,j}) - \sum_{k=1}^{i-1} (c_{k,j} - c_{k,j-1}) + c_{i,j-1},$$

which, upon substitution, gives the desired upper bound on $c_{i,j}$. $\square$

Proposition 5. *For $v \in \mathcal{B}_{\lambda+\rho}$, let $\mathfrak{T}$ be the Gelfand-Tsetlin pattern such that* $\mathrm{BZL}(v) = \beta(\mathfrak{T})$. *Then*

$$G(\mathfrak{T}) = G(v),$$

where $G(\mathfrak{T})$ is as defined in (48) *and $G(v)$ is as defined in Section* 2.

Proof. We first note that the set of all BZL patterns for a crystal graph $\mathcal{B}_\lambda$ made with the decomposition of the long word (53) and by raising operators to the highest weight vector are identical to those BZL patterns for $\mathcal{B}_\lambda$ made with decomposition

$$w_0 = s_1(s_2 s_1) \cdots (s_r s_{r-1} \cdots s_1),$$

and Kashiwara lowering operators to the lowest weight vector. This latter recipe is used in Section 2, where the decomposition is labeled Σ_2.

Indeed, the two descriptions can be related as described by Lenart in Proposition 2.3 of [23] (which is a recasting of Proposition 7.1 in [25]). There exists an involution η_λ on the crystal graph $\mathcal{B}_\lambda$ such that η_λ maps the highest weight vector to the lowest weight vector, and where

$$\eta_\lambda(e_i(v)) = f_{i^*}(\eta_\lambda(v)),$$

and the i^* indicates the root operator corresponding to the root $-w_0(\alpha_i) = \alpha_{r+1-i}$. (This map η_λ was shown by Berenstein and Zelevinsky [3] to coincide with the Schützenberger involution on tableaux in type A.) Hence, we obtain the same polytope $P_{\lambda+\rho}$ and corresponding patterns $(c_{i,j})$ using either recipe for constructing BZL patterns, and so we may take the results of Lemmas 3 and 4 above, initially applied to BZL patterns obtained from raising operators to the highest weight vector, to hold for the BZL patterns used in Section 2.

The definition of the bijection β in Lemma 4 guarantees that the entries of $\mathrm{BZL}(v)$ will match the $e_{i,j}$ as defined in (47). To see that the decoration rule corresponds to the cases in (48), note that under the bijection, $a_{i,j} = a_{i-1,j}$ if and only if $c_{i,r+1-j} = c_{i,r-j}$. The latter condition implies that $c_{i,r+1-j}$ is circled, and so the component $\gamma(a_{i,j})$ of $G(\mathfrak{T})$ in (48) matches the component of $G(v)$ in (7). Similarly, $a_{i,j} = a_{i-1,j-1}$ if and only if the inequality (54) for $c_{i,r+1-j}$ is sharp (which implies that $c_{i,r+1-j}$ is boxed) and the cases in (48) and (7) again match. $\square$

Proposition 6. *Let $v \in \mathcal{B}_{\lambda+\rho}$ correspond to the Gelfand-Tsetlin pattern $\mathfrak{T}$ under* $\mathrm{BZL}(v) = \beta(\mathfrak{T})$, *and let* $\mathrm{wt}(v) = \mu$. *Then, with $k(\mathfrak{T})$ as in* (49),

(56)
$$k(\mathfrak{T}) = (k_1, \dots, k_r), \quad \textit{where the } k_i \textit{ are defined by} \quad \sum_{i=1}^{r} k_i \alpha_i = \lambda + \rho - w_0(\mu),$$

with α_i the simple roots. That is, the bijection β takes Gelfand-Tsetlin patterns $\mathfrak{T}$ contributing to $H_{\mathrm{GT}}(p^{\mathbf{k}}, p^{\mathbf{l}})$ to BZL*patterns* $\mathrm{BZL}(v)$ *contributing to $H(p^{\mathbf{k}}, p^{\mathbf{l}})$*

as in (3). *Hence,*

$$H_{\mathrm{GT}}(p^{\mathbf{k}}, p^{\mathbf{l}}) = \sum_{\substack{v \in \mathcal{B}_{\lambda+\rho} \\ \mathrm{wt}(v) = \mu}} G(v),$$

with $G(v)$ *as defined in Section* 2.

Proof. Recall that the k_j appearing on the right-hand side of (56) are equal to the column sums $\sum_{i=1}^{j} c_{i,r+1-j}$ in $\mathrm{BZL}(v)$, according to (6). By the definition of β in Lemma 4,

$$\sum_{i=1}^{j} c_{i,r+1-j} = \sum_{i=1}^{j} \sum_{k=j}^{r} (a_{i,k} - a_{i-1,k}) = \sum_{i=j}^{r} (a_{i,j} - a_{0,j}),$$

so that column sums in $\mathrm{BZL}(v)$ indeed match the definition in (49). The resulting identity for H_{GT} in terms of the crystal graph and the function $G(v)$ follows from this identification of $k(\mathfrak{T})$ together with Proposition 5. $\square$

Because H_{GT} was shown to match the function H given in the Whittaker coefficient of Theorem 1 in the previous section, this at last confirms the description of the Whittaker coefficients of the metaplectic Eisenstein series as multiple Dirichlet series whose coefficients are computed using crystal graphs as presented in Section 2.

10. **The main theorem**

We end by collecting the pieces. Let N denote the standard upper triangular unipotent subgroup of SL_{r+1}, $\mathbf{m} = (m_1, \ldots, m_r)$ be a vector of nonzero S-integers, ψ be an additive character of F_S with conductor $\mathfrak{o}_S$, and $\psi_{\mathbf{m}}$ be the character of $N(F_S)$

$$\psi_{\mathbf{m}}(x) = \psi\left(\sum_{j=1}^{r} m_j x_{j,j+1}\right).$$

Let J_{r+1} represent the long element w_0 of the Weyl group and $\boldsymbol{s} : G \longrightarrow \widetilde{G}$ a section satisfying $\boldsymbol{s}(g)\boldsymbol{s}(h) = \sigma(g,h)\boldsymbol{s}(gh)$, as in Section 3. Then we have proved:

THEOREM 5. *Let* $f \in \pi(s_1, \ldots, s_r)$ *be spherical at the archimedean places, and let* $E_f^{r+1}(g)$ *be the corresponding Borel Eisenstein series on the* n*-fold metaplectic cover of* SL_{r+1} *as in* (20). *Then the* $\mathbf{m}^{\mathrm{th}}$ *Whittaker coefficient of* E_f^{r+1},

$$\int_{N(\mathfrak{o}_S)\backslash N(F_S)} E_f^{r+1}(\boldsymbol{s}(J_{r+1})\boldsymbol{s}(n))\, \psi_{\mathbf{m}}(n)\, dn,$$

is equal to

$$W^\circ(s_1,\dots,s_r)\sum_{0\neq C_1,\dots,C_r\in\mathfrak{o}_S/\mathfrak{o}_S^\times}\frac{H(C_1,\dots,C_r;\mathbf{m})\Psi_{\mathbf{m};f}(C_1,\dots,C_r)}{|C_1|^{2s_1}\cdots|C_r|^{2s_r}},$$

where W° is an archimedean Whittaker function (§6) *and $\Psi_{\mathbf{m};f}\in\mathcal{M}(\Omega^r)$ is given by* (29). *Any particular $\Psi\in\mathcal{M}(\Omega^r)$ occurs as $\Psi_{\mathbf{m};f}$ for a suitable choice of f. The coefficients H are characterized by the following two properties. First, they are twisted multiplicative in both the C_i and the m_i* (*Theorems* 2 *and* 3). *Second, if the C_i and m_i are powers of a given prime p of $\mathfrak{o}_S$, then the coefficient H is given by*

$$H(p^{\mathbf{k}};p^{\mathbf{l}})=\sum_{\substack{v\in\mathcal{B}_{\lambda+\rho}\\ \mathrm{wt}(v)=\mu}}G(v),$$

where the sum is taken over crystal graph vertices $v\in\mathcal{B}_{\lambda+\rho}$ with weight μ such that $\sum k_i\alpha_i=\lambda+\rho-w_0(\mu)$ and $G(v)$ is defined as in Section 2.

Proof. This follows by combining Theorems 1, 2, 3 and 4 with Propositions 3, 6 and equation (31). □

As a consequence of the Main Theorem, we obtain the analytic continuation and functional equation of the series (1) for any $\Psi\in\mathcal{M}(\Omega^r)$, which is part of Conjecture 1 of [12]. This follows from the corresponding properties of the Eisenstein series themselves, which were established by Mœglin and Waldspurger [28] in generality that includes metaplectic groups. Since the Eisenstein series have functional equations, the Main Theorem could also be used to establish functional equations for the series (1). Some additional work with the inducing data would be needed to establish functional equations as precise as those in [9], along the lines of the rank 1 case which is treated in Section 4 of [5]. We do not carry this out here, but instead note that an alternative proof of the analytic continuation and functional equations has been given by the authors in [11].

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