The weak type $(1,1)$ bounds for the maximal function associated to cubes grow to infinity with the dimension

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Abstract

Let M_d be the centered Hardy-Littlewood maximal function associated to cubes in $\mathbb{R}^d$ with Lebesgue measure, and let c_d denote the lowest constant appearing in the weak type (1,1) inequality satisfied by M_d. We show that $c_d \to \infty$ as $d \to \infty$, thus answering, for the case of cubes, a longstanding open question of E. M. Stein and J. O. Strömberg.

1. Introduction and result

By a cube $Q(x,r)$ we mean a closed ℓ_∞ ball of radius r and center x in $\mathbb{R}^d$, that is, a closed cube centered at x, with sides parallel to the coordinate axes, and sidelength $2r$. Let M_d be the centered maximal function

$$(1) \qquad M_d f(x) := \sup_{r>0} \frac{1}{|Q(x,r)|} \int_{Q(x,r)} |f(y)| dy$$

associated to cubes and Lebesgue measure in $\mathbb{R}^d$. A fundamental feature of the Hardy-Littlewood maximal function M is that it satisfies the weak-type $(1,1)$ inequality: There exists a constant $c > 0$ such that for all $\alpha > 0$ and all $f \in L^1$,

$$(2) \qquad \alpha |\{Mf \geq \alpha\}| \leq c\|f\|_1.$$

Denote by c_d the best (i.e. lowest) constant satisfying (2) in $\mathbb{R}^d$.

THEOREM. *Fix $T > 0$. Then there exists a $D = D(T)$ such that for every dimension $d \geq D$, $c_d \geq T$.*

Thus, $c_d \to \infty$ as $d \to \infty$. In fact, these constants approach ∞ in a monotone manner, since $c_{d+1} \geq c_d$ by [AV07, Th. 2].

It is well-known that given $1 < p \leq \infty$, there exists a constant c_p such that for all $f \in L^p(\mathbb{R}^d)$, $\|Mf\|_p \leq c_p\|f\|_p$. When $p = \infty$, trivially $c_p = 1$ in every dimension, for averages never exceed a supremum. Dimension-independent estimates are useful whenever one is interested in extending results from the finite dimensional to the infinite dimensional setting. For the maximal function associated to euclidean balls, E. M. Stein showed that one can take c_p to be independent of d ([Ste82], [Ste85], [SS83]; see also [Ste93]). Stein's result was generalized to the maximal function defined using balls given by arbitrary norms by J. Bourgain ([Bou86a], [Bou86b], [Bou87]) and A. Carbery ([Car86]) when $p > 3/2$. Given $1 \leq q < \infty$, the ℓ_q balls are defined using the norm $\|x\|_q := (|x_1|^q + |x_2|^q + \cdots + |x_d|^q)^{1/q}$, and the ℓ_∞ balls, using $\|x\|_\infty := \max_{1 \leq i \leq d}\{|x_1|, |x_2|, \ldots, |x_d|\}$. For ℓ_q balls, $1 \leq q < \infty$, D. Müller [Mül90] showed that uniform bounds again hold for every $p > 1$. With respect to weak type bounds, in [SS83], E. M. Stein and J. O. Strömberg proved that the best constants in the weak type (1,1) inequality satisfied by the maximal function associated to arbitrary balls grow at most like $O(d \log d)$, while if the balls are euclidean, then the best constants grow at most like $O(d)$. They also asked if uniform bounds could be found. The theorem above shows that in the case of cubes the answer is negative. If the d-dimensional Lebesgue measures are replaced by a sequence of finite, absolutely continuous radial measures with decreasing densities (such as, for instance, the standard Gaussian measures) then best constants grow exponentially with d; cf. [Ald07].

In recent years evidence has been mounting to the effect that not only weak type $(1, 1)$ inequalities are formally stronger than strong (p, p) inequalities for $1 < p < \infty$ (since the latter are implied by the former via interpolation) but they are also stronger in a substantial way, meaning that the strong type may hold for all $p > 1$ while the weak type $(1, 1)$ may fail. This is the case, for instance, with the uncentered maximal function associated to the standard gaussian measure and euclidean balls. It is shown in [Sjö83] that this maximal function is not of weak type $(1, 1)$, while it is strong (p, p) for all $p > 1$; cf. [FSSU02] (for cubes the strong (p, p) type follows from a more general result in [CF84, cf. Th. 1]). The theorem above may represent another instance of this phenomenon, with respect to uniform bounds in d. However, it is not known for cubes whether uniform bounds hold when $1 < p \leq 3/2$ (it is suggested in [Mül90] that the answer may be negative, and conjectured in [ACPL] that the answer is positive).

Before presenting the proof, we make some comments on the method of discretization for weak type $(1, 1)$ inequalities. It consists in replacing L^1 functions by finite sums of Dirac deltas. This leads to elementary arguments of a combinatorial nature. The fact that one can get lower bounds for c_d using

Dirac deltas instead of functions is obvious, by mollification, and this is all we need here.

We mention for completeness that considering Dirac deltas also suffices to give upper bounds, as shown by M. de Guzmán, see [dG75, Th. 4.1.1]. Furthermore, M. Trinidad Menárguez and F. Soria proved that discretizing does not alter constants; cf. [TMS92, Th. 1], so it can be used to study the precise values of c_d. This method was utilized, for instance, in [Ald98], were it was shown that $37/24 \leq c_1 \leq \frac{9+\sqrt{41}}{8}$, thereby refuting the conjecture that $c_1 = 3/2$ (cf. [BH89, Prob. 7.74c]) and showing that $c_1 < 2$, which is the best constant in the uncentered case. Discretization was also used in [Mel02] and [Mel03], where the exact value of $c_1 = \frac{11+\sqrt{61}}{12}$ was found. No best constants are known for dimensions larger than one.

Let us point out that the configuration of Dirac deltas we will utilize had previously been considered in [TMS92, Th. 6], for the same purpose of bounding c_d from below. It is shown there that $c_d \geq \left(\frac{1+2^{1/d}}{2}\right)^d$. But this yields no information as to whether there is a uniform upper bound for c_d, since $\left(\frac{1+2^{1/d}}{2}\right)^d < 2$.

The first version of this article contained a simple counting error at the beginning of the proof, which rendered large parts of it useless. I am most indebted to professor Keith Rogers for pointing out this mistake to me; substantial modifications of the argument were required in order to fix it. I am also indebted to professors Javier Pérez Lázaro and Peter Sjögren, and to an anonymous referee, for carefully reading this paper and for making several useful suggestions, which led to thorough rewriting and simplification.

2. **Proof**

Given a locally finite measure ν in $\mathbb{R}^d$, the maximal function $M_d\nu$ is defined by

$$(3) \qquad M_d\nu(x) := \sup_{r>0} \frac{\nu Q(x,r)}{|Q(x,r)|}.$$

For notational simplicity we start considering the infinite measure μ^d in $\mathbb{R}^d$ obtained by placing one Dirac delta at each point of the integer lattice $\mathbb{Z}^d$. The finite measure exhibiting a lower bound for c_d will then be obtained by restricting μ^d to a sufficiently large cube. Note that $\mu^d = \mu^1 \times \mu^1 \times \cdots \times \mu^1$. At first, we will work within the unit cube $[0,1]^d$ only.

Given $u \in (0,1)$ and an interval $I \subset \mathbb{R}$, call $y \in I$ centered at level u (more briefly, centered, or u centered) if it belongs to the closed subinterval with the same center and length $(1-u)|I|$, and off center (at level u) otherwise. In particular, for $I = [0,1]$ the centered points are those in $[2^{-1}u, 1 - 2^{-1}u]$. The role of u in the proof is to serve as a discrete parameter, used to describe

which cubes should be considered when estimating the value of the maximal function at a given point.

It can be shown that the maximal function is large on the set $E_u \subset [0,1]^d$ of points $x = (x_1, \ldots, x_d)$ with many centered coordinates, where "large" is determined by a fixed $t \gg 1$, and "many" means more than $(1-u)d + t\sqrt{du(1-u)}$. Since $t\sqrt{du(1-u)}$ amounts to t standard deviations of a binomially distributed random variable with parameters d and u, the Central Limit Theorem allows us to bound $|E_u|$ from below, provided d is large enough (we mention that a similar argument can be carried out on the set of points with many uncentered coordinates). For a fixed u, the measure of E_u as $d \to \infty$ turns out to be too small, since we are $t \gg 1$ standard deviations away from the mean. On the other hand, estimates for the size of $M_d\mu^d$ worsen when we have roughly $(1-u)d$ centered coordinates. Changing the value of u by discrete steps and taking the union of many E_u's, we obtain a sufficiently large set over which $M_d\mu$ can be shown to take high values (unlike u, the value of t is fixed throughout the argument, so dependency on t is not indicated in the notation). In order to control the intersections of different E_u's, it is useful to bound also from below the number of uncentered coordinates; so actually we shall slightly modify these sets and call them E^u instead of E_u.

Fix $t \gg 1$. The assumption that t is very large will be used without further mention (save for some occasional reminder). But we emphasize that the value of t remains unchanged throughout the proof; in particular, it does not approach ∞ as $d \to \infty$. So we will assume, again without further mention, that expressions such as $t/\sqrt{d}$ are as small as needed each time they appear.

Recall that the standard deviation of a Bernoulli trial with parameter u is $\sigma_u = \sqrt{u(1-u)}$. Define, for each $u \in [1/8, 1/4]$,

(4) $E^u := \{x \in [0,1]^d : \text{ the number } k \text{ of coordinates } j_1, \ldots, j_k \text{ for which}$

$x_{j_i} \in [0, 2^{-1}u) \cup (1 - 2^{-1}u, 1] \text{ satisfies } ud - (t+t^{-1})\sigma_u\sqrt{d} < k \leq ud - t\sigma_u\sqrt{d}\}.$

The values $1/8$ and $1/4$ are of no special significance; we could have fixed any $0 < a < b < 1$ and chosen $u \in [a, b]$ instead. In order to prove the theorem, we first estimate the size of E^u for each $u = 1/8, 1/8 + t^{-4/3}, 1/8 + 2t^{-4/3}, 1/8 + 3t^{-4/3} \cdots \leq 1/4$, so we consider $\Theta(t^{4/3})$ different values of u, where, as usual, Θ stands for exact order. Second, using the fact that distinct values of u differ by at least $t^{-4/3}$, we prove that different sets E^u have very small intersection. Third, we take the union of the $\Theta(t^{4/3})$ sets E^u and bound the measure of this union from below. And fourth, we show that $M\mu^d$ is large on each E^u, and hence on their union.

Up to here, the argument is carried inside $[0,1]^d$. To complete the proof, we replace μ^d by a finite measure, and apply the estimates obtained within $[0,1]^d$ to several translates of it.

Let $Z \sim N(0,1)$ denote a standard normally distributed random variable.

CLAIM 1. *For all $u \in [1/8, 1/4]$ there exists a $D = D(u)$ such that if $d \geq D$, then*

$$(5) \qquad \frac{e^{-t^2/2}}{2e^2 t\sqrt{2\pi}} < |E^u| < \frac{e^{-t^2/2}}{t\sqrt{\pi}}.$$

Proof. Define a collection of independent Bernoulli random variables $X_{u,i}$ by setting $X_{u,i}(x) = 1$ if the i-th coordinate of $x \in [0,1]^d$ satisfies $x_i \in [0, 2^{-1}u) \cup (1 - 2^{-1}u, 1]$, and $X_{u,i}(x) = 0$ otherwise. Then the probability of having exactly k off center and $d - k$ centered coordinates is

$$\binom{d}{k} u^k (1-u)^{d-k}.$$

Set $S_{u,d} := \sum_{i=1}^{d} X_{u,i}$, so that $S_{u,d}$ counts the number of uncentered coordinates. Then $S_{u,d} \sim B(u,d)$ is binomially distributed with mean $E(S_{u,d}) = ud$ and standard deviation $\sigma(S_{u,d}) = \sqrt{du(1-u)} = \sigma_u \sqrt{d}$, where σ_u is the standard deviation of $X_{u,i}$. Thus, the Lebesgue measure of E^u is given by

$$|E^u| = P\left(ud - (t + t^{-1})\sigma_u\sqrt{d} < S_{u,d} \leq ud - t\sigma_u\sqrt{d}\right).$$

By the Central Limit Theorem, for all d large enough we have

$$(6) \qquad 2^{-1}P(-t - t^{-1} < Z \leq -t) < |E^u| < \sqrt{2}P(-t - t^{-1} < Z \leq -t).$$

Since $P(-t - t^{-1} < Z \leq -t) = \frac{1}{\sqrt{2\pi}} \int_t^{t+t^{-1}} e^{-y^2/2} dy$ and $e^{-2}e^{-t^2/2} \leq e^{-y^2/2} \leq e^{-t^2/2}$ for every $y \in [t, t + t^{-1}]$, we obtain (5). $\qquad\square$

We show next that if u and v are "far apart", then $|E^v \cap E^u|$ is small relative to $|E^u|$.

CLAIM 2. *Fix $u, v \in [1/8, 1/4]$ with $u - v \geq t^{-4/3}$. Then there exists a $D = D(u)$ such that for all $d \geq D$,*

$$(7) \qquad |E^v \cap E^u| \leq t^{-1/3} e^{-2t^{2/3}/9} |E^u|.$$

Proof. We partition the sets E^u into subsets $A_{u,K}$, consisting of all points with coordinates x_j off center if and only if $j \in K$. More precisely, let us fix a subset $K \subset \{1, \dots, d\}$ of cardinality k, with k satisfying

$$(8) \qquad ud - (t + t^{-1})\sigma_u\sqrt{d} < k \leq ud - t\sigma_u\sqrt{d}.$$

We define
$$(9)$$
$$A_{u,K} := \left\{ x \in [0,1]^d : x_j \in \left[0, 2^{-1}u\right) \cup \left(1 - 2^{-1}u, 1\right] \text{ if and only if } j \in K \right\}.$$

Since $A_{u,K} \cap A_{u,M} = \emptyset$ unless $K = M$, and $E^u = \cup_K A_{u,K}$, we do have a partition of E^u.

Let $v \leq u - t^{-4/3}$. In order to estimate $|A_{u,K} \cap E^v|$, consider an arbitrary set $A_{v,M}$ in the partition of E^v. We may suppose that $M \subset K$, for otherwise $A_{u,K} \cap A_{v,M} = \emptyset$. Let M have cardinality m, and let $x \in A_{u,K} \cap A_{v,M}$. Observe that $x_i \in [0, 2^{-1}v) \cup (1 - 2^{-1}v, 1]$ for every $i \in M$, $x_j \in [2^{-1}v, 2^{-1}u) \cup (1 - 2^{-1}u, 1 - 2^{-1}v]$ for every $j \in K \setminus M$, and $x_r \in [2^{-1}u, 1 - 2^{-1}u]$ for the remaining $d - k$ coordinates. Thus

$$|A_{u,K} \cap A_{v,M}| = v^m \times (u-v)^{k-m} \times (1-u)^{d-k} = u^k(1-u)^{d-k}\left(\frac{v}{u}\right)^m\left(1 - \frac{v}{u}\right)^{k-m}.$$

The lower bound on k given by (8) allows us to conclude that $k > m$ for sufficiently large d, since in that case $ud - (t+t^{-1})\sigma_u\sqrt{d} > vd - t\sigma_v\sqrt{d}$. Only the upper bound $m \leq [vd - t\sigma_v\sqrt{d}]$ (where $[w]$ denotes the integer part of w) is needed in the next estimate. Summing first over all sets $M \subset K$ of fixed cardinality m, and then over all $m \leq [vd - t\sigma_v\sqrt{d}]$, we get

$$(10) \qquad |A_{u,K} \cap E^v| \leq u^k(1-u)^{d-k}\sum_{m=0}^{[vd-t\sigma_v\sqrt{d}]}\binom{k}{m}\left(\frac{v}{u}\right)^m\left(1 - \frac{v}{u}\right)^{k-m}.$$

As before, we control the sum above by using the Central Limit Theorem, applied to the binomially distributed random variable $S_{v/u,k} \sim B(v/u, k)$. We shall need a lower bound for $E(S_{v/u,k})$ and an upper bound for $\sigma(S_{v/u,k})$. From $ud - (t+t^{-1})\sigma_u\sqrt{d} < k \leq ud$ we obtain

$$(11) \qquad E(S_{v/u,k}) = k\left(\frac{v}{u}\right) > vd - \left(t+t^{-1}\right)v\sqrt{d}\sqrt{u^{-1} - 1}$$

and

$$(12) \qquad \sigma(S_{v/u,k}) = \sqrt{k\left(\frac{v}{u}\right)\left(1 - \frac{v}{u}\right)} < \sqrt{vd\left(1 - \frac{v}{u}\right)}.$$

Since $t \gg 1$, $2t^{-2/3} \leq \sqrt{1 - v/u}$, and $\sqrt{v}\sqrt{u^{-1} - 1} + \sqrt{1-v} < \sqrt{7}$,

$$\frac{2t^{1/3}}{3} < \frac{t\sqrt{1-\frac{v}{u}}}{\sqrt{7}} - \frac{\sqrt{7}}{t^{1/3}} < \frac{t\left(1-\frac{v}{u}\right)}{\left(\sqrt{v}\sqrt{\frac{1}{u} - 1} + \sqrt{1-v}\right)\sqrt{1-\frac{v}{u}}} - \frac{\sqrt{v}\sqrt{\frac{1}{u} - 1}}{t\sqrt{1-\frac{v}{u}}}$$

$$= \frac{t\left(\sqrt{1-v} - \sqrt{v}\sqrt{\frac{1}{u} - 1}\right) - t^{-1}\sqrt{v}\sqrt{\frac{1}{u} - 1}}{\sqrt{1-\frac{v}{u}}}$$

$$= \frac{t\sqrt{vd}\sqrt{1-v} - \left(t+t^{-1}\right)v\sqrt{d}\sqrt{\frac{1}{u} - 1} + vd - vd}{\sqrt{vd}\sqrt{1-\frac{v}{u}}}$$

$$< \frac{E(S_{v/u,k}) - vd + t\sigma_v\sqrt{d}}{\sigma(S_{v/u,k})},$$

where the last inequality follows from (11) and (12). Hence, by the Central Limit Theorem we have, for all sufficiently large d,

$$(13) \qquad P\left(S_{v/u,k} \leq vd - t\sigma_v\sqrt{d}\right) \leq P\left(\frac{S_{v/u,k} - E(S_{v/u,k})}{\sigma(S_{v/u,k})} \leq -\frac{2t^{1/3}}{3}\right)$$

$$(14) \qquad \leq \frac{2\sqrt{2\pi}}{3} P\left(Z \leq -\frac{2t^{1/3}}{3}\right) = \frac{2}{3}\int_{\frac{2t^{1/3}}{3}}^{\infty} e^{-y^2/2} dy$$

$$\leq \frac{1}{t^{1/3}}\int_{\frac{2t^{1/3}}{3}}^{\infty} ye^{-y^2/2} dy = \frac{e^{-2t^{2/3}/9}}{t^{1/3}}.$$

Thus, from (10) and (13), (14) we get

$$(15) \qquad |A_{u,K} \cap E^v| \leq u^k(1-u)^{d-k}t^{-1/3}e^{-2t^{2/3}/9},$$

and now (7) follows by adding up over all the sets $A_{u,K}$ in the partition of E^u. $\qquad\square$

Next, we write $u(j) := 1/8 + jt^{-4/3}$, letting u range over $[1/8, 1/4]$ in discrete steps of size $t^{-4/3}$.

CLAIM 3. *Let $j = 0, 1, 2, \ldots, M$, where M is the largest integer j satisfying $jt^{-4/3} \leq 1/8$. Then*

$$(16) \qquad \left|\bigcup_{j=0}^{M} E^{u(j)}\right| \geq \Theta(t^{1/3}e^{-t^2/2}).$$

Proof. Let $0 \leq k < j \leq M$ be natural numbers. We apply Claim 2 and Claim 1 to all pairs $u(k) < u(j)$, obtaining

$$(17) \qquad \sum_{0 \leq k < j \leq M} \left|E^{u(k)} \cap E^{u(j)}\right| < t^{8/3}\left(\frac{e^{-2t^{2/3}/9}}{t^{1/3}}\right)\left(\frac{e^{-t^2/2}}{t}\right)$$

$$= t^{4/3}e^{-2t^{2/3}/9}e^{-t^2/2} = O\left(t^{-1}e^{-t^2/2}\right).$$

Using the inclusion-exclusion principle, together with (17) and (5), we obtain

$$(18) \qquad \left|\bigcup_{j=0}^{M} E^{u(j)}\right| \geq \sum_{j=0}^{M}\left|E^{u(j)}\right| - \sum_{0 \leq k < j \leq M}\left|E^{u(k)} \cap E^{u(j)}\right|$$

$$(19) \qquad \geq \left(\frac{t^{4/3}}{8}\right)\Theta(t^{-1}e^{-t^2/2}) - O\left(t^{-1}e^{-t^2/2}\right) = \Theta(t^{1/3}e^{-t^2/2}).$$

This proves Claim 3. $\qquad\square$

After having estimated the measure of $\bigcup_{j=1}^{M} E^{u(j)}$, it remains to show that $M\mu^d$ takes high values on this set. We do this next.

CLAIM 4. *Fix $u \in [1/8, 1/4]$. Then $E^u \subset \{M\mu^d > 2^{-1}e^{t^2/2}\}$.*

Proof. Let $x_j \in [2^{-1}u, 1 - 2^{-1}u]$. Given any integer $s > 0$, we have that $\mu^1[x_j - (s - 2^{-1}u), x_j + s - 2^{-1}u] = 2s$. Suppose $y \in [0, 1]$ is off center, say for instance $y > 1 - 2^{-1}u$. Shifting the interval $[x_j - (s - 2^{-1}u), x_j + s - 2^{-1}u]$ to the right by $y - x_j$ (so now it is centered at y) loses at most one Dirac delta on the left. Thus, $\mu^1([y - (s - 2^{-1}u), y + s - 2^{-1}u]) \geq 2s - 1$. Suppose $x \in [0, 1]^d$ has r off center and $d - r$ centered coordinates, where $r \leq r_0 := ud - t\sigma_u\sqrt{d}$. Then for every $s = 1, 2, 3, \ldots$,

$$
(20) \qquad M_d\mu^d(x) \geq \frac{(2s)^{d-r}(2s-1)^r}{\left(2\left(s - 2^{-1}u\right)\right)^d} = \frac{\left(1 - \frac{1}{2s}\right)^r}{\left(1 - \frac{u}{2s}\right)^d} \geq \frac{\left(1 - \frac{1}{2s}\right)^{r_0}}{\left(1 - \frac{u}{2s}\right)^d}.
$$

The next step consists in showing that for some suitably chosen s, the right-hand side of (20) is bounded below by $2^{-1}e^{t^2/2}$. Set $f(s) := (1 - \frac{1}{2s})^{r_0}/(1 - \frac{u}{2s})^d$. An elementary calculus argument shows that $f(s)$ is maximized over $s \geq 1$ when

$$
(21) \qquad s = \frac{u(d - r_0)}{2(ud - r_0)} =: s_0,
$$

and this is the only critical point, so f decreases as we move away from s_0. In particular, $f(s_0) \geq f([s_0]) \geq f(s_0 - 1)$ (where $[s_0]$ denotes the integer part of s_0) so for convenience we shall use $s_0 - 1$ instead of $[s_0]$ in (20). Thus,
(22)
$$
\log M_d\mu^d(x) \geq \left(ud - t\sigma_u\sqrt{d}\right)\log\left(1 - \frac{1}{2(s_0 - 1)}\right) - d\log\left(1 - \frac{u}{2(s_0 - 1)}\right).
$$

Replacing r_0 by its value in (21) we see that

$$
s_0 = \frac{\sigma_u\sqrt{d}}{2t} + \frac{u}{2} = \Theta(\sqrt{d}) \quad \text{and} \quad 2(s_0 - 1) = \frac{\sigma_u\sqrt{d}}{t}\left(1 - \frac{(2-u)t}{\sigma_u\sqrt{d}}\right).
$$

Thus, using $(1 - w)^{-1} = 1 + w + O(w^2)$ applied to $w_1 := \frac{(2-u)t}{\sigma_u\sqrt{d}} = \Theta(1/\sqrt{d})$ we get

$$
(23) \qquad \frac{1}{2(s_0 - 1)} = \frac{t}{\sigma_u\sqrt{d}} + \frac{(2-u)t^2}{\sigma_u^2 d} + O\left(\frac{1}{d^{3/2}}\right).
$$

Finally, from $\log(1 - w) = -w - w^2/2 + O(w^3)$ applied to $w_2 := \frac{1}{2(s_0 - 1)} = \Theta(1/\sqrt{d})$ and to $w_3 := uw_2$, we obtain, by substituting in (22) and simplifying,

$$
(24) \qquad \log M_d\mu^d(x) \geq \frac{t^2}{2} + O\left(\frac{1}{\sqrt{d}}\right). \qquad \qquad \square
$$

Completing the argument. The last step consists in fixing d (so large that all the preceding estimates hold) and replacing the infinite measure μ^d by a finite measure μ_R^d, such that the ratio of unit volume cubes to Dirac deltas is close to 1. The measure μ_R^d is obtained by keeping only the point masses of

μ^d contained in the cube $\left[-\sqrt{d}, R + \sqrt{d}\right]^d$. This part of the proof (save for a small modification) already appears in [TMS92, Th. 6].

Let f be an integrable function and ν a finite sum of Dirac deltas. By discretization, any lower bound for c in $\alpha\left|\{Mf \geq \alpha\}\right| \leq c\|f\|_1$ is a lower bound for C in $\alpha\left|\{M\nu \geq \alpha\}\right| \leq C\nu(\mathbb{R}^d)$, and vice versa. Here $\nu(\mathbb{R}^d)$ simply counts the number of point masses in ν. Observe that the cubes used in Claim 4 to estimate the size of $M\mu^d(x)$, for $x \in [0,1]^d$, never exceed a sidelength of $2\sqrt{d}$. Let $\mu_R^d := \sum_i \delta_{x_i}$, where $R = R(d) \gg d$ is a natural number and $x_i \in \mathbb{Z}^d$ ranges over all the points with integer coordinates in the cube $[-\sqrt{d}, R + \sqrt{d}]^d$. Using the fact that the estimates in the preceding claims hold for every unit subcube of $[0,R]^d$ with vertices in $\mathbb{Z}^d$ (by the same argument presented for $[0,1]^d$) we have

$$
\begin{aligned}
c_d &\geq \sup_{R>0} \frac{2^{-1}e^{t^2/2}\left|\left\{M_d\mu_R^d > 2^{-1}e^{t^2/2}\right\} \cap [0,R]^d\right|}{\mu_R^d(\mathbb{R}^d)} \\
&\geq \sup_{R>0}\left(\frac{R^d}{(R + 2\sqrt{d} + 1)^d}\right) \Theta\left(t^{1/3}e^{-t^2/2}e^{t^2/2}\right) = \Theta\left(t^{1/3}\right). \qquad \square
\end{aligned}
$$

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