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Global regularity for some classes of large solutions to the Navier-Stokes equations

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Abstract

In previous works by the first two authors, classes of initial data to the three-dimensional, incompressible Navier-Stokes equations were presented, generating a global smooth solution although the norm of the initial data may be chosen arbitrarily large. The main feature of the initial data considered in one of those studies is that it varies slowly in one direction, though in some sense it is "well-prepared" (its norm is large but does not depend on the slow parameter). The aim of this article is to generalize that setting to an "ill prepared" situation (the norm blows up as the small parameter goes to zero). As in those works, the proof uses the special structure of the nonlinear term of the equation.

1. Introduction

We study in this paper the Navier-Stokes equation with initial data which are slowly varying in the vertical variable. More precisely we consider the system

$$
(\mathrm{NS}) \quad \begin{cases} \partial_t u + u\cdot\nabla u - \Delta u = -\nabla p \quad \text{in} \quad \mathbb{R}^+ \times \Omega \\ \operatorname{div} u = 0 \\ u|_{t=0} = u_{0,\varepsilon}, \end{cases}
$$

where $\Omega = \mathbb{T}^2 \times \mathbb{R}$ (the choice of this particular domain will be explained later on), and $u_{0,\varepsilon}$ is a divergence free vector field, whose dependence on the vertical variable x_3 will be chosen to be "slow", meaning that it depends on εx_3 where ε is a small parameter. Our goal is to prove a global existence in time result for the solution generated by this type of initial data, with no smallness assumption on its norm.

1.1. *Recollection of some known results on the Navier-Stokes equations.* The mathematical study of the Navier-Stokes equations has a long history, which we shall describe briefly in this paragraph. We shall first recall the main global wellposedness results and some blow-up criteria. Then we shall concentrate on the case when the special algebraic structure of the system is used, in order to improve those previous results.

To simplify, we shall place ourselves in the whole euclidian space $\mathbb{R}^d$ or in the torus $\mathbb{T}^d$ (or in variants of those spaces, such as $\mathbb{T}^2 \times \mathbb{R}$ in three space dimensions); of course results exist in the case when the equations are posed in domains of the euclidian space, with Dirichlet boundary conditions, but we choose to simplify the presentation by not mentioning explicitly those studies (although some of the theorems recalled below also hold in the case of domains up to obvious modifications of the statements and sometimes much more difficult proofs).

1.1.1. *Global wellposedness and blow-up results.* The first important result on the Navier-Stokes system was obtained by J. Leray in his seminal paper [24] in 1933. He proved that any finite energy initial data (meaning square-integrable data) generates a (possibly nonunique) global in time weak solution which satisfies the energy estimate

$$\frac{1}{2}\|u(t)\|_{L^2}^2 + \int_0^t \|\nabla(t')\|_{L^2}^2 dt' \leq \frac{1}{2}\|u_0\|_{L^2}^2. \tag{1.1}$$

Moreover, he proved in [23] the uniqueness of the solution in two space dimensions. Those results use the structure of the nonlinear terms, in order to obtain the energy inequality. He also proved the uniqueness of weak solutions in three space dimensions, under the additional condition that one of the weak solutions has more regularity properties (say belongs to $L^2(\mathbb{R}^+; L^\infty)$; this would now be qualified as a "weak-strong uniqueness result"). Moreover, he proved under a smallness hypothesis on $\|u_0\|_{L^2}\|\nabla u_0\|_{L^2}$ that global regular solutions exists and thus are unique. The question of the global wellposedness of the three-dimensional Navier-Stokes equations for large initial data was then raised, and has been open ever since. The main difficulty can be explained using the scaling property of the incompressible Navier-Stokes system. If u is a solution of (NS) on $[0, T] \times \mathbb{R}^d$, then $u_\lambda(t, x) = \lambda u(\lambda^2 t, \lambda x)$ is a solution on $[0, T] \times \mathbb{R}^d$. It can be easily checked that in the case when the dimension d is 2, then the energy is scaling invariant and thus can be used for contraction argument. In such a case, the problem is called "critical". It is of course not the case when the dimension d is equal to 3 which is "super-critical", which means that the conserved quantities are less regular than the scaling invariant quantities. We shall now present a few of the historical landmarks in that study.

The Fujita-Kato theorem [11] gives a partial answer to the construction of a global unique solution. Indeed, that theorem provides a unique, local in time solution in the homogeneous Sobolev space $\dot{H}^{\frac{d}{2}-1}$ in d space dimensions, and that solution is proved to be global if the initial data is small in $\dot{H}^{\frac{d}{2}-1}$ (compared to the viscosity, which is chosen equal to one here to simplify).

The result was improved to the Lebesgue space L^d by F. Weissler in [36]; see also [15] and [19]. The method consists in applying a Banach fixed point theorem to the integral formulation of the equation, and was generalized by M. Cannone, Y. Meyer, and F. Planchon in [2] to Besov spaces of negative index of regularity. More precisely they proved that if the initial data is small in the Besov space $\dot{B}_{p,\infty}^{-1+\frac{d}{p}}$ (for $p < \infty$), then there is a unique, global in time solution. Let us emphasize that this result allows to construct global solutions for strongly oscillating initial data which may have a large norm in $\dot{H}^{\frac{d}{2}-1}$ or in L^d. A typical example in three space dimensions is

$$u_0^\varepsilon(x) \stackrel{\text{def}}{=} \varepsilon^{-\alpha} \sin\Big(\frac{x_3}{\varepsilon}\Big)\big(-\partial_2\varphi(x), \partial_1\varphi(x), 0\big), \tag{1.2}$$

where $0 < \alpha < 1$ and $\varphi \in \mathcal{S}(\mathbb{R}^3; \mathbb{R})$. This can be checked by using the definition of Besov norms:

$$\forall s > 0,\ \forall (p,q) \in [1,\infty], \quad \|f\|_{\dot{B}^{-s}_{p,q}} \stackrel{\text{def}}{=} \Big\| t^{\frac{s}{2}} \|e^{t\Delta} f\|_{L^p} \Big\|_{L^q(\mathbb{R}^+;\frac{dt}{t})}.$$

More recently in [20], H. Koch and D. Tataru obtained a unique global in time solution for data small enough in a more general space, consisting of vector fields whose components are derivatives of BMO functions. The norm in that space is given by

$$\|u_0\|^2_{\mathrm{BMO}^{-1}} \stackrel{\text{def}}{=} \sup_{t>0} t\|e^{t\Delta}u_0\|^2_{L^\infty} + \sup_{\substack{x\in\mathbb{R}^d \\ R>0}} \frac{1}{R^d}\int_{P(x,R)} |(e^{t\Delta}u_0)(t,y)|^2 dy, \tag{1.3}$$

where $P(x,R)$ stands for the parabolic set $[0,R^2]\times B(x,R)$ while $B(x,R)$ is the ball centered at x, of radius R. This implies in particular global existence and uniqueness for initial data given by (1.2) in the case when $\alpha = 1$ (provided that φ is small). Those theorems are general results of global existence for small initial data in the sense that they are valid for a very large class of quadratic nonlinearities and do not take into account any particular algebraic properties of the nonlinear terms in the Navier-Stokes equation.

One should notice that spaces where global, unique solutions are constructed for small initial data, are necessarily scaling-invariant spaces. Moreover it can be proved (as observed for instance in [6]) that if B is a Banach space continuously included in the space $\mathcal{S}'$ of tempered distributions such that

$$\text{for any } (\lambda, a) \in \mathbb{R}^+_\star \times \mathbb{R}^d, \quad \|f(\lambda(\cdot - a))\|_B = \lambda^{-1}\|f\|_B,$$

then $\|\cdot\|_B \leq C \sup_{t>0} t^{\frac{1}{2}} \|e^{t\Delta}u_0\|_{L^\infty}$. One recognizes on the right-hand side of the inequality the $\dot{B}^{-1}_{\infty,\infty}$ norm, which is slightly smaller than the BMO^{-1} norm recalled above in (1.3); indeed the BMO^{-1} norm takes into account not only the $\dot{B}^{-1}_{\infty,\infty}$ information, but also the fact that first Picard iterate of the Navier-Stokes equations should be locally square integrable in space and time. It thus seems that the Koch-Tataru theorem is optimal for the wellposedness of the Navier-Stokes equations — note that it was very recently proved (see [14] and [1]) that the equations are in fact ill-posed in $\dot{B}^{-1}_{\infty,\infty}$. This observation also shows that if one wants to go beyond a smallness assumption on the initial data to prove the global existence of unique solutions, one should check that the $\dot{B}^{-1}_{\infty,\infty}$ norm of the initial data may be chosen large.

To conclude this paragraph, let us remark that the fixed-point methods used to prove local in time wellposedness for arbitrarily large data (such results are available in Banach spaces in which the Schwartz class is dense, typically $\dot{B}^{-1+\frac{d}{p}}_{p,q}$ for finite p and q) naturally provide blow-up criteria. For instance, one can prove that if the life span of the solution is finite, then the $L^q\Big([0,T];\dot{B}^{-1+\frac{d}{p}+\frac{2}{q}}_{p,q}\Big)$ norm blows up as T approaches the blow up time. A natural question is to ask if the $\dot{B}^{-1+\frac{d}{p}}_{p,q}$ norm itself blows up. Progress has been made very recently on this question, and uses the specific structure of the equation, which was not the case for the results presented in this paragraph. We therefore postpone the exposition of those results to the next paragraph.

We will not describe more results on the Cauchy problem for the Navier-Stokes equations. We refer the interested reader to the monographs [22] and [28] for more details.

1.1.2. *Results using the specific algebraic structure of the equation.* If one wishes to improve the theory on the Cauchy problem for the Navier-Stokes equations, it seems crucial to use the specific structure of the nonlinear term in the equations, as well as the divergence free assumption. Indeed it was proved by S. Montgomery-Smith in [29] (in a one-dimensional setting, which was later generalized to a 2D and 3D situation by two of the authors in [13]) that some models exist for which finite time blow up can be proved for some classes of large data, despite the fact that the same small-data global wellposedness results hold as for the Navier-Stokes system. Furthermore, the generalization to the 3D case in [13] shows that some large initial data which generate a global solution for the Navier-Stokes equations (namely the data of [5] which will be presented below) actually generate a blowing up solution for the toy model.

In this paragraph, we shall present a number of wellposedness theorems (or blow up criteria) which have been obtained in the past and which specifically concern the Navier-Stokes equations. In order to make the presentation shorter, we choose not to present a number of results which have been proved by various authors under some additional geometrical assumptions on the flow, which imply the conservation of quantities beyond the scaling (namely spherical, helicoidal or axisymmetric conditions). We refer the reader to [21], [26], [32], or [35] for such studies.

To start with, let us recall the question asked in the previous paragraph, concerning the blow up of the $\dot{B}^{-1+\frac{d}{p}}_{p,q}$ norm at blow-up time. A typical example of a solution with a finite $\dot{B}^{-1+\frac{d}{p}}_{p,q}$ norm at blow-up time is a self-similar solution, and the question of the existence of such solutions was actually addressed by J. Leray in [24]. The answer was given 60 years later by J. Neças, M. Ruzįçka, and V. Şverák in [30]. By analyzing the profile equation, they proved that there is no self-similar solution in L^3 in three space dimensions. Later L. Escauriaza, G. Seregin, and V. Şverák were able to prove in [10] that, more generally, if the solution remains bounded in L^3, then it remains regular: in particular any solution blowing up in finite time must blow up in L^3.

Now let us turn to the existence of large, global unique solutions to the Navier-Stokes system in three space dimensions.

An important example where a unique global in time solution exists for large initial data is the case where the domain is thin in the vertical direction (in three space dimensions): that was proved by G. Raugel, and G. Sell in [33]; see also the paper [18] by D. Iftimie, G. Raugel, and G. Sell. The authors obtained the global existence of a strong solution for initial data which are allowed to have a large two-dimensional part (the vertical mean of the initial data) and a small three-dimensional part. Another example of large initial data generating a global solution was obtained by A. Mahalov and B. Nicolaenko in [27]: in that case, the initial data is chosen so as to transform the equation into a rotating fluid equation (for which it is known that global solutions exist for a sufficiently strong rotation).

In both those examples, the global wellposedness of the two-dimensional equation is an important ingredient in the proof. Two of the authors also used such a property to construct in [5] an example of periodic initial data which is strongly oscillating and large in $\dot{B}^{-1}_{\infty,\infty}$ but yet generates a global solution. Such an initial data is given by

$$u_0^N(x) \stackrel{\text{def}}{=} (Nu_h(x_h)\cos(Nx_3), -\operatorname{div}_h u_h(x_h)\sin(Nx_3)),$$

where $\|u_h\|_{L^2(\mathbb{T}^2)} \leq C(\ln N)^{\frac{1}{4}}$, and its $\dot{B}^{-1}_{\infty,\infty}$ norm is typically of the same size. This was generalized to the case of the space $\mathbb{R}^3$ in [6]. The main idea was to obtain the global existence of the solution under a nonlinear smallness assumption on the first iterate $e^{t\Delta}u_0 \cdot \nabla e^{t\Delta}u_0$ instead of a smallness condition directly on $e^{t\Delta}u_0$.

Similarly in [7], the global wellposedness of the two-dimensional equation was used to prove a global existence result for large data which are slowly varying in one direction. More precisely, if $(v_0^h, 0)$ and w_0 are two smooth divergence free "profile" vector fields, then they proved that the initial data

$$u_{0,\varepsilon}(x_h, x_3) \stackrel{\text{def}}{=} (v_0^h(x_h, \varepsilon x_3), 0) + (\varepsilon w_0^h(x_h, \varepsilon x_3), w_0^3(x_h, \varepsilon x_3)) \tag{1.4}$$

generates, for ε small enough, a global smooth solution. Here, we have denoted $x_h = (x_1, x_2)$. Using the language of the weak compressible limit or fast rotating fluids, this case may be qualified as a "well-prepared" case. Indeed the initial data converges uniformely for x_3 in any compact subset of $\mathbb{R}$ to a two-dimensional vector field which generates global smooth solutions. We shall be coming back to that example in the next paragraph.

As a conclusion of this short (and of course incomplete) survey, let us present some results for the Navier-Stokes system with viscosity vanishing in the vertical direction. Analogous results to the classical Navier-Stokes system in the framework of small data are proved in [4], [17], [31], and [9]. To circumvent the difficulty linked with the absence of vertical viscosity, the key idea, which will be also crucial here (see for instance the proof of the second estimate of Proposition 2.1) is the following: the vertical derivative ∂_3 appears in the nonlinear term of the equation with the prefactor u_3, which has some additional smoothness thanks to the divergence free condition which states that $\partial_3 u_3 = -\partial_1 u_1 - \partial_2 u_2$.

1.2. *Statement of the main result.* In this work, we are interested in generalizing the situation (1.4) to the ill-prepared case. We shall investigate the case of initial data of the form

$$\left(v_0^h(x_h, \varepsilon x_3), \frac{1}{\varepsilon} v_0^3(x_h, \varepsilon x_3)\right), \tag{1.5}$$

where x_h belongs to the torus $\mathbb{T}^2$ and x_3 belongs to $\mathbb{R}$. Following the terminology for weakly compressible fluids or rotating fluids, the above intial data can be qualified as ill-prepared because they do not converge in any sense to a two-dimensional divergence free vector field. The case of ill-prepared initial data studied here is much more difficult than the case of well-prepared data dealt with in [7]. Roughly speaking, this initial data indeed may give rise to instability in the inviscid case or for small viscosity, as for instance in the case of the Prandtl equation arising in the problem of vanishing viscosity in domains with boundaries (see [16]) and the hypothesis of analyticity can make the system solvable (see [34]). This prevents us from working in the framework of Sobolev spaces. However, for analytical initial data we can hope to avoid the possible instability and consequently a good framework in this situation is the analytical class. More precisely, we shall see in the next section that after a change of scale in the vertical variable, the system becomes a Navier-Stokes system with an anisotropic Laplacian of the form $-(\partial_1^2 + \partial_2^2) - \varepsilon^2 \partial_3^2$ and an anisotropic pressure term of the form $(-\nabla_h p, \varepsilon^2 \partial_3 p)$. The equation on the pressure is

$$-(\partial_1^2 + \partial_2^2 + \varepsilon^2 \partial_3^2) p = \sum_{j,k} \partial_j \partial_k (u^j u^k), \tag{1.6}$$

and this equation is not uniformly elliptic and we lose control of one derivative in the vertical variable. This prevents us from working with Sobolev spaces as in the well-prepared case. To overcome this difficulty, we work in the class of analytical functions. The main difficulty is then to control the loss in the radius of analyticity, using a variation of the method introduced in [3]. Our result may in this sense be seen as a Cauchy-Kovalevskaya result, which is global in time. The main theorem of this article is the following.

THEOREM 1. *Let a be a positive number. There are two positive numbers ε_0 and η such that for any divergence free vector field v_0 satisfying*

$$\|e^{a|D_3|}v_0\|_{H^4} \leq \eta,$$

then, for any positive ε smaller than ε_0, the initial data

$$u_{0,\varepsilon}(x) \stackrel{\mathrm{def}}{=} \Big(v_0^h(x_h, \varepsilon x_3), \frac{1}{\varepsilon}v_0^3(x_h, \varepsilon x_3)\Big)$$

generates a global smooth solution u_ε of (NS) *on $\mathbb{T}^2 \times \mathbb{R}$.*

Remarks.

- Such an initial data may be arbitrarily large in $\dot{B}^{-1}_{\infty,\infty}$, more precisely of size ε^{-1}. Indeed it is proved in [7, Prop. 1.1], that if f and g are two functions in $\mathcal{S}(\mathbb{T}^2)$ and $\mathcal{S}(\mathbb{R})$ respectively, then if ε is small enough, $h^\varepsilon(x_h, x_3) \stackrel{\mathrm{def}}{=} f(x_h)g(\varepsilon x_3)$ satisfies
$$\|h^\varepsilon\|_{\dot{B}^{-1}_{\infty,\infty}} \geq \frac{1}{4}\|f\|_{\dot{B}^{-1}_{\infty,\infty}}\|g\|_{L^\infty}.$$
- As in the well-prepared case studied in [7] and recalled in the previous paragraph, the structure of the nonlinear term will have a crucial role to play in the proof of the theorem.
- The reason why we put periodic boundary condition on horizontal variables is technical: in (1.6), in the case of the whole space $\mathbb{R}^3$, it seems difficult to control the very low horizontal frequencies. As we shall see in the proof, in the case of periodic boundary condition in the horizontal variable, we treat separately functions which do not depend on the horizontal variable from the others (for which the horizontal frequencies are greater or equal to 1). In that situation, we are able to solve globally in time the equation (conveniently rescaled in ε) for small analytic-type initial data. We recall that in that spirit, some local in time results for Euler and Prandtl equation with analytic initial data can be found in [34].
- Finally, let us observe that, as pointed out by N. Lerner, Y. Morimoto, and C.-J. Xu in [25], solutions of nonlinear equations obtained by Cauchy-Kovalevskaya type method can be extremely unstable. Here,

each solution u_ε is stable in the following sense: for any fixed ε, we can perturb the initial data $u_{0,\varepsilon}$ by any element δ of the Sobolev space $\dot{H}^{\frac{1}{2}}$ provided $\|\delta\|_{\dot{H}^{\frac{1}{2}}}$ is less or equal to some positive r_ε; then the corresponding initial data still generates a global smooth solution. This comes from the stability results proved in [12].

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2. **Structure of the proof**

2.1. *Reduction to a rescaled problem.* We look for the solution under the form

$$u_\varepsilon(t,x) \stackrel{\text{def}}{=} \Big(v^h(t,x_h,\varepsilon x_3), \frac{1}{\varepsilon} v^3(t,x_h,\varepsilon x_3)\Big).$$

This leads to the following rescaled Navier-Stokes system:

$$(\mathrm{RNS}_\varepsilon) \begin{cases} \partial_t v^h - \Delta_\varepsilon v^h + v\cdot\nabla v^h = -\nabla^h q \\ \partial_t v^3 - \Delta_\varepsilon v^3 + v\cdot\nabla v^3 = -\varepsilon^2\partial_3 q \\ \operatorname{div} v = 0 \\ v_{|t=0} = v_0 \end{cases}$$

with $\Delta_\varepsilon \stackrel{\text{def}}{=} \partial_1^2 + \partial_2^2 + \varepsilon^2\partial_3^2$. As there is no boundary, the rescaled pressure q can be computed with the formula

$$\Delta_\varepsilon q = \sum_{j,k} \partial_j v^k \partial_k v^j = \sum_{j,k} \partial_j\partial_k (v^j v^k). \tag{2.1}$$

It turns out that when ε goes to 0, Δ_ε^{-1} looks like Δ_h^{-1}. In the case of $\mathbb{R}^3$, for low horizontal frequencies, an expression of the type $\Delta_h^{-1}(ab)$ cannot be estimated in L^2 in general. This is the reason why we work in $\mathbb{T}^2\times\mathbb{R}$. In this domain, the problem of low horizontal frequencies reduces to the problem of the horizontal average that we denote by

$$(Mf)(x_3) \stackrel{\text{def}}{=} \bar{f}(x_3) \stackrel{\text{def}}{=} \int_{\mathbb{T}^2} f(x_h,x_3)\,dx_h.$$

Let us also define $M^\perp f \stackrel{\text{def}}{=} (\mathrm{Id} - M)f$. Notice that, because the vector field v is divergence free, we have $\bar{v}^3 \equiv 0$. The system $(\mathrm{RNS}_\varepsilon)$ can be rewritten in the following form:

$$(\mathrm{RNS}_\varepsilon) \begin{cases} \partial_t w^h - \Delta_\varepsilon w^h + M^\perp\big(v\cdot\nabla w^h + w^3\partial_3\bar{v}^h\big) = -\nabla^h q \\ \partial_t w^3 - \Delta_\varepsilon w^3 + M^\perp(v\cdot\nabla w^3) = -\varepsilon^2\partial_3 M^\perp q \\ \partial_t \bar{v}^h - \varepsilon^2\partial_3^2\bar{v}^h = -\partial_3 M(w^3 w^h) \\ \operatorname{div}(\bar{v}+w) = 0 \\ (\bar{v},w)_{|t=0} = (\bar{v}_0, w_0). \end{cases}$$

The problem with solving this sytem is that there is no obvious way to compensate the loss of one vertical derivative which appears in the equation on w_h and $\bar{v}$ and also, but more hidden, in the pressure term. The method we use is inspired by the one introduced in [3] and can be understood as a global Cauchy-Kovalevskaya result. This is the reason why the hypothesis of analyticity in the vertical variable is required in our theorem.

Let us denote by $\mathcal{B}$ the unit ball of $\mathbb{R}^3$ and by $\mathcal{C}$ the annulus of small radius 1 and large radius 2. For nonnegative j, let us denote by L^2_j the space $\mathcal{F}L^2((\mathbb{Z}^2\times\mathbb{R})\cap 2^j\mathcal{C})$ and by L^2_{-1} the space $\mathcal{F}L^2((\mathbb{Z}^2\times\mathbb{R})\cap\mathcal{B})$ respectively equipped with the (semi)norms

$$\|u\|^2_{L^2_j}\stackrel{\mathrm{def}}{=}(2\pi)^{-d}\int_{2^j\mathcal{C}}|\widehat{u}(\xi)|^2d\xi \quad\text{and}\quad \|u\|^2_{L^2_{-1}}\stackrel{\mathrm{def}}{=}(2\pi)^{-d}\int_{\mathcal{B}}|\widehat{u}(\xi)|^2d\xi.$$

Let us now recall the definition of inhomogeneous Besov spaces modeled on L^2.

Definition 2.1. Let s be a nonnegative real number. The space B^s is the subspace of L^2 such that

$$\|u\|_{B^s}\stackrel{\mathrm{def}}{=}\Big\|\Big(2^{js}\|u\|_{L^2_j}\Big)_j\Big\|_{\ell^1}<\infty.$$

We note that $u\in B^s$ is equivalent to writing $\|u\|_{L^2_j}\leq Cc_j 2^{-js}\|u\|_{B^s}$ where (c_j) is a nonnegative series which belongs to the sphere of ℓ^1. Let us notice that $B^{\frac{3}{2}}$ is included in $\mathcal{F}(L^1)$ and thus in the space of continuous bounded functions. Moreover, if we substitute ℓ^2 to ℓ^1 in the above definition, we recover the classical Sobolev space H^s.

The theorem we actually prove is the following.

Theorem 2. *Let a be a positive number. There are two positive numbers ε_0 and η such that for any divergence free vector field v_0 satisfying*

$$\|e^{a|D_3|}v_0\|_{B^{\frac{7}{2}}}\leq\eta,$$

then, for any positive ε smaller than ε_0, the initial data

$$u_{0,\varepsilon}(x)\stackrel{\mathrm{def}}{=}\Big(v_0^h(x_h,\varepsilon x_3),\frac{1}{\varepsilon}v_0^3(x_h,\varepsilon x_3)\Big)$$

generates a global smooth solution of (NS) *on* $\mathbb{T}^2\times\mathbb{R}$.

2.2. *Study of a model problem.* In order to motivate the functional setting and to give a flavour of the method used to prove the theorem, let us study for a moment the following simplified model problem for (RNS_ε), in which we shall see in a rather easy way how the same type of method as that of [3] can be used (as a global Cauchy-Kovalevskaya technique): the idea is to control a nonlinear quantity, which depends on the solution itself. So let us consider the equation

$$\partial_t u + \gamma u + a(D)(u^2) = 0,$$

where u is a scalar, real-valued function, γ is a positive parameter, and $a(D)$ is a Fourier multiplier of order one. We shall sketch the proof of the fact that if the initial data satisfies

$$\|e^{\delta|D|}u_0\|_{B^{\frac{3}{2}}} \leq c\gamma$$

for some positive δ and some small enough constant c, then one has a global smooth solution, say in the space $B^{\frac{3}{2}}$ as well as all its derivatives. The idea of the proof is the following: we want to control the same kind of quantity on the solution, but one expects the radius of analyticity of the solution to decay in time. Let us introduce the following notation, which will be used throughout this article. For any locally bounded function Ψ on $\mathbb{R}^+ \times \mathbb{Z}^2 \times \mathbb{R}$ and for any function f, we define

$$f_\Psi(t) \stackrel{\text{def}}{=} \mathcal{F}^{-1}\Big(e^{\Psi(t,\cdot)}\widehat{f}(t,\cdot)\Big).$$

Let us notice that this notation does not make sense for any f; the following can be made rigourous by a cut-off in Fourier space. This will be done in the next section, in the proof of Theorem 2.

So let us introduce the function $\theta(t)$ which describes the "loss of analyticity" of the solution. We define

$$\dot\theta(t) \stackrel{\text{def}}{=} \|u_\Phi(t)\|_{B^{\frac{3}{2}}} \quad \text{with} \quad \theta(0) = 0 \quad \text{and} \quad \Phi(t,\xi) = (\delta - \lambda\theta(t))|\xi|. \tag{2.2}$$

The parameter λ will be chosen large enough at the end, and we shall prove that $\delta - \lambda\theta(t)$ remains positive for all times. The computations that follow hold as long as that assumption is true (and a bootstrap will prove that in fact it does remain true for all times). Taking the Fourier transform of the equation gives

$$|\widehat{u}(t,\xi)| \leq e^{-\gamma t}|\widehat{u}_0(\xi)| + C\int_0^t e^{-\gamma(t-t')}\,|\xi|\,|\mathcal{F}(u^2)(t',\xi)|\,dt'.$$

Using the fact that

$$\begin{aligned}\gamma t + (\delta - \lambda\theta(t))\,|\xi| \leq \gamma(t-t') - \lambda|\xi| &\int_{t'}^t \dot\theta(t'')\,dt'' \\ &+ \gamma t' + (\delta - \lambda\theta(t'))\,|\xi-\eta| + (\delta - \lambda\theta(t'))\,|\eta|\,,\end{aligned}$$

we infer that

$$e^{\gamma t}|\widehat{u}_\Phi(t,\xi)| \leq e^{\delta|\xi|}|\widehat{u}_0(\xi)| + C\int_0^t e^{-\lambda|\xi|\int_{t'}^t \dot\theta(t'')\,dt''}|\xi|\,e^{\gamma t'}|\mathcal{F}(u_\Phi^2)|(t',\xi)\,dt'.$$

Thus, for any ξ in $2^j\mathcal{C}$,

$$e^{\gamma t}|\widehat{u}_\Phi(t,\xi)| \leq e^{\delta|\xi|}|\widehat{u}_0(\xi)| + C\int_0^t e^{-\lambda 2^j\int_{t'}^t \dot\theta(t'')\,dt''}2^j\,e^{\gamma t'}|\mathcal{F}(u_\Phi^2)|(t',\xi)\,dt'.$$

Taking the $L^2(2^j\mathcal{C}, d\xi)$ norm gives

(2.3)
$$e^{\gamma t}\|u_\Phi(t,\cdot)\|_{L^2_j} \leq \|e^{\delta|D|}u_0\|_{L^2_j} + C\int_0^t e^{-\lambda 2^j \int_{t'}^t \dot{\theta}(t'')\,dt''} 2^j \,\|e^{\gamma t'} u^2_\Phi(t',\cdot)\|_{L^2_j}\,dt'.$$

Now, we need a lemma of paradifferential calculus type. The statement requires the following spaces, introduced in [8].

Definition 2.2. Let s be a real number. We define the space $\widetilde{L}^\infty_T(B^s)$ as the subspace of functions f of $L^\infty_T(B^s)$ such that the following quantity is finite:

$$\|f\|_{\widetilde{L}^\infty_T(B^s)} \stackrel{\text{def}}{=} \sum_j 2^{js}\|f\|_{L^\infty_T(L^2_j)}.$$

Let us notice that $\widetilde{L}^\infty_T(B^s)$ is obviously included in $L^\infty_T(B^s)$.

We shall also use a very basic version of Bony's decomposition: let us define

$$T_a b \stackrel{\text{def}}{=} \mathcal{F}^{-1}\sum_j \int_{2^j\mathcal{C}\cap\mathcal{B}(\xi,2^j)} \widehat{a}(\xi-\eta)\widehat{b}(\eta)d\eta$$

and

$$R_a b \stackrel{\text{def}}{=} \mathcal{F}^{-1}\sum_j \int_{2^j\mathcal{C}\cap\mathcal{B}(\xi,2^{j+1})} \widehat{a}(\xi-\eta)\widehat{b}(\eta)d\eta.$$

We obviously have $ab = T_a b + R_b a$.

Lemma 2.1. *For any positive s, a constant C exists which satisfies the following properties. For any function Ψ satisfying*

$$\Psi(t,\xi) \leq \Psi(t,\xi-\eta) + \Psi(t,\eta) \tag{2.4}$$

for any function b and any positive T, a positive sequence $(c_j)_{j\in\mathbb{Z}}$ exists in the sphere of $\ell^1(\mathbb{Z})$ (and depending only on T and b) such that, for any a and any $t \in [0,T]$,

$$\|(T_a b)_\Psi(t)\|_{L^2_j} + \|(R_a b)_\Psi(t)\|_{L^2_j} \leq C c_j 2^{-js}\|a_\Psi(t)\|_{B^{\frac{3}{2}}}\|b_\Psi\|_{\widetilde{L}^\infty_T(B^s)}.$$

Proof. We prove only the lemma for R, the proof for T being strictly identical. Let us first investigate the case when the function Ψ is identically 0. We first observe that for any ξ in the annulus $2^j\mathcal{C}$, we have

$$\mathcal{F}(R_a b(t))(\xi) = \sum_{j'\geq j-2}\int_{2^{j'}\mathcal{C}\cap B(\xi,2^{j'+1})} \widehat{a}(t,\xi-\eta)\widehat{b}(t,\eta)d\eta.$$

By definition of $\|\cdot\|_{\widetilde{L}^\infty_T(B^s)}$, we infer that

$$\|R_a b(t)\|_{L^2_j} \leq C\|a(t)\|_{\mathcal{F}(L^1)}\sum_{j'\geq j-2} c_{j'}2^{-j's}\|b\|_{\widetilde{L}^\infty_T(B^s)}.$$

Defining $\tilde{c}_j = \sum_{j' \geq j-2} 2^{(j-j')s} c_{j'}$ which satisfies $\sum_j \tilde{c}_j \leq C_s$, we obtain

$$\|R_a b(t)\|_{L^2_j} \leq C \tilde{c}_j 2^{-js} \|a(t)\|_{\mathcal{F}(L^1)} \|b\|_{\widetilde{L}^\infty_T(B^s)}. \tag{2.5}$$

Since $B^{\frac{3}{2}}$ is included in $\mathcal{F}(L^1)$, the lemma is then proved in the case when the function Ψ is identically 0. In order to treat the general case, let us write that

$$\begin{aligned} |e^{\Psi(t,\xi)} \mathcal{F}(R_a b)(t,\xi)| &= e^{\Psi(t,\xi)} \sum_j \int_{2^j \mathcal{C} \cap B(\xi, 2^j)} |\widehat{a}(t, \xi - \eta)| \, |\widehat{b}(t,\eta)| d\eta \\ &\leq \sum_j \int_{2^j \mathcal{C} \cap B(\xi, 2^j)} e^{\Psi(t,\xi-\eta)} |\widehat{a}(t,\xi-\eta)| e^{\Psi(t,\eta)} |\widehat{b}(t,\eta)| d\eta. \end{aligned}$$

Estimate (2.5) implies the lemma. □

Now let us return to (2.3). We write

$$e^{\gamma t'} u^2_\Phi(t') = T_{u_\Phi(t')} e^{\gamma t'} u_\Phi(t') + R_{u_\Phi(t')} e^{\gamma t'} u_\Phi(t').$$

Lemma 2.1 gives

$$\|e^{\gamma t'} u^2_\Phi(t', \cdot)\|_{L^2_j} \leq C c_j(T) 2^{-j\frac{3}{2}} \|u_\Phi(t')\|_{B^{\frac{3}{2}}} \|e^{\gamma t'} u_\Phi(t')\|_{\widetilde{L}^\infty_T(B^{\frac{3}{2}})}$$

for all $t' \leq T$, as long as the function Φ is positive. By definition of the function θ, this gives

$$\|e^{\gamma t'} u^2_\Phi(t', \cdot)\|_{L^2_j} \leq C c_j(T) 2^{-j\frac{3}{2}} \dot{\theta}(t') \|e^{\gamma t'} u_\Phi(t')\|_{\widetilde{L}^\infty_T(B^{\frac{3}{2}})}.$$

Plugging this inequality in (2.3) (after multiplication by $2^{j\frac{3}{2}}$) gives

$$\begin{aligned} & 2^{j\frac{3}{2}} e^{\gamma t} \|u_\Phi(t,\cdot)\|_{L^2_j} \\ & \leq 2^{j\frac{3}{2}} \|e^{\delta |D|} u_0\|_{L^2_j} + C c_j(T) \|e^{\gamma t} u_\Phi(t)\|_{\widetilde{L}^\infty_T(B^{\frac{3}{2}})} \int_0^t e^{-\lambda 2^j \int_{t'}^t \dot{\theta}(t'')\, dt''} 2^j \, \dot{\theta}(t') \, dt', \end{aligned}$$

for all $t \leq T$, as long as the function Φ is positive. Since

$$\int_0^t e^{-\lambda 2^j \int_{t'}^t \dot{\theta}(t)\, dt''} 2^j \, \dot{\theta}(t') \, dt' \leq \frac{1}{\lambda},$$

we get

$$2^{j\frac{3}{2}} e^{\gamma t} \|u_\Phi(t,\cdot)\|_{L^2_j} \leq 2^{j\frac{3}{2}} \|e^{\delta |D|} u_0\|_{L^2_j} + \frac{C}{\lambda} c_j(T) \|e^{\gamma t} u_\Phi(t)\|_{\widetilde{L}^\infty_T(B^{\frac{3}{2}})}.$$

Taking the supremum for $t \leq T$ and summing over j, we get

$$\|e^{\gamma t} u_\Phi(t,\cdot)\|_{\widetilde{L}^\infty_T(B^{\frac{3}{2}})} \leq \|e^{\delta |D|} u_0\|_{B^{\frac{3}{2}}} + \frac{C}{\lambda} \|e^{\gamma t} u_\Phi(t)\|_{\widetilde{L}^\infty_T(B^{\frac{3}{2}})}$$

as long as the function Φ is positive. Thus, choosing $\lambda = 2C$ we infer that

$$\|e^{\gamma t} u_\Phi(t,\cdot)\|_{\widetilde{L}^\infty_T(B^{\frac{3}{2}})} \leq 2 \|e^{\delta |D|} u_0\|_{B^{\frac{3}{2}}}.$$

Since $\|a\|_{L^\infty_T(B^{\frac32})} \leq \|a\|_{\widetilde{L}^\infty_T(B^{\frac32})}$, by definition of θ, we get

$$\dot\theta(t) \leq 2e^{-\gamma t}\|e^{\delta|D|}u_0\|_{B^{\frac32}},$$

as long as the function Φ is positive. This gives $\gamma\theta(t) \leq 2\|e^{\delta|D|}u_0\|_{B^{\frac32}}$. If

$$\|e^{\delta|D|}u_0\|_{B^{\frac32}} \leq \frac{\delta\gamma}{8C},$$

then we get that the function Φ remains positive for all time and the global regularity is proved.

2.3. *Functional setting for the study of* (RNS$_\varepsilon$). In the light of the computations of the previous section, let us introduce the functional setting we are going to work with to prove the theorem. The proof relies on exponential decay estimates for the Fourier transform of the solution. Let us define the key quantity we wish to control in order to prove the theorem. To do so, let us consider the Friedrichs approximation of the original (NS) system

$$\begin{cases} \partial_t u - \Delta u + \mathbb{P}_n(u\cdot\nabla u + \nabla p) = 0 \\ \operatorname{div} u = 0 \\ u|_{t=0} = \mathbb{P}_n\, u_{0,\varepsilon}, \end{cases}$$

where $\mathbb{P}_n$ denotes the orthogonal projection of L^2 on functions the Fourier transform of which is supported in the ball B_n centered at the origin and of radius n. Thanks to the L^2 energy estimate, this approximated system has a global solution the Fourier transform of which is supported in B_n. Of course, this provides an approximation of the rescaled system namely

$$(\mathrm{RNS}_{\varepsilon,n}) \quad \begin{cases} \partial_t w^h - \Delta_\varepsilon w^h + \mathbb{P}_{n,\varepsilon}\, M^\perp\big(v\cdot\nabla w^h + w^3\partial_3\bar v + \nabla^h q\big) = 0 \\ \partial_t w^3 - \Delta_\varepsilon w^3 + \mathbb{P}_{n,\varepsilon}\, M^\perp\big(v\cdot\nabla w^3 + \varepsilon^2\partial_3 q\big) = 0 \\ \partial_t \bar v^h - \varepsilon^2\partial_3^2\bar v^h + \mathbb{P}_{n,\varepsilon}\,\partial_3 M(w^3 w^h) = 0 \\ \operatorname{div}(\bar v + w) = 0 \\ (\bar v, w)_{|t=0} = (\bar v_0, w_0), \end{cases}$$

where $\mathbb{P}_{n,\varepsilon}$ denotes the orthogonal projection of L^2 on functions the Fourier transform of which is supported in $B_{n,\varepsilon} \stackrel{\text{def}}{=} \{\xi \,/\, |\xi_\varepsilon|^2 \stackrel{\text{def}}{=} |\xi_h|^2 + \varepsilon^2\xi_3^2 \leq n^2\}$. We shall prove analytic type estimates here, meaning exponential decay estimates for the solution of the above approximated system. In order to make notation not too heavy we shall drop the fact that the solutions we deal with are in fact approximate solutions and not solutions of the original system. *A priori* bounds on the approximate sequence will be derived, which will clearly yield the same bounds on the solution. In the spirit of [3] (see also (2.2) in the previous section), we define the function θ (we drop also the fact that θ depends on ε

in all that follows) by

$$\dot\theta(t) = \|w^3_\Phi(t)\|_{B^{\frac{7}{2}}} + \varepsilon\|w^h_\Phi(t)\|_{B^{\frac{7}{2}}} \quad \text{and} \quad \theta(0)=0, \tag{2.6}$$

where

$$\Phi(t,\xi) = t^{\frac{1}{2}}|\xi_h| + a|\xi_3| - \lambda\theta(t)|\xi_3| \tag{2.7}$$

for some λ that will be chosen later on (see §2.5). Notice that the definition of θ takes into account the particular algebraic structure of $(\mathrm{RNS}_{\varepsilon,n})$. Since the Fourier transform of w is compactly supported, the above differential equation has a unique global solution on $\mathbb{R}^+$. If we prove that

$$\forall t\in\mathbb{R}^+\,,\ \theta(t)\leq \frac{a}{\lambda}\,, \tag{2.8}$$

this will imply that the sequence of approximated solutions of the rescaled system is a bounded sequence of $L^1(\mathbb{R}^+;\mathrm{Lip})$. So is, for a fixed ε, the family of approximation of the original Navier-Stokes equations. This is (more than) enough to imply that a global smooth solution exists.

2.4. *Main steps of the proof.* The proof of intequality (2.8) (and of Theorem 2) will be a consequence of the following two propositions which provide estimates on v^h, w^h, and w^3. They will be proved in the coming sections.

The first one uses only the fact that the function Φ is subadditive.

Proposition 2.1. *A constant $C_0^{(1)}$ exists such that, for any positive λ, for any initial data v_0, and for any T satisfying $\theta(T)\leq a/\lambda$, we have*

$$\theta(T)\leq\varepsilon\|e^{a|D_3|}w_0^h\|_{B^{\frac{7}{2}}} + \|e^{a|D_3|}w_0^3\|_{B^{\frac{7}{2}}} + C_0^{(1)}\|v_\Phi\|_{\widetilde L^\infty_T(B^{\frac{7}{2}})}\theta(T).$$

Moreover, we have the following L^∞-type estimate on the vertical component:

$$\|w^3_\Phi\|_{\widetilde L^\infty_T(B^{\frac{7}{2}})} \leq \|e^{a|D_3|}w_0^3\|_{B^{\frac{7}{2}}} + C_0^{(1)}\|v_\Phi\|^2_{\widetilde L^\infty_T(B^{\frac{7}{2}})}.$$

The second proposition is more subtle to prove, and it shows that the use of the analytic-type norm actually allows to recover the missing vertical derivative on v^h, in a L^∞-type space. It should be compared to the methods described in the model case above.

Proposition 2.2. *A constant $C_0^{(2)}$ exists such that, for any positive λ, for any initial data v_0, and for any T satisfying $\theta(T)\leq a/\lambda$, we have*

$$\|v^h_\Phi\|_{\widetilde L^\infty_T(B^{\frac{7}{2}})} \leq \|e^{a|D_3|}v_0^h\|_{B^{\frac{7}{2}}} + C_0^{(2)}\Big(\frac{1}{\lambda}+\|v_\Phi\|_{\widetilde L^\infty_T(B^{\frac{7}{2}})}\Big)\|v^h_\Phi\|_{\widetilde L^\infty_T(B^{\frac{7}{2}})}.$$

2.5. *Proof of the theorem assuming the two propositions.* Let us assume these two propositions are true for the time being and conclude the proof of Theorem 2. It relies on a continuation argument.

For any positive λ and η, let us define

$$\mathcal{T}_\lambda \stackrel{\text{def}}{=} \big\{T \,/\, \max\{\|v_\Phi\|_{\widetilde{L}^\infty_T(B^{\frac{7}{2}})}, \theta(T)\} \leq 4\eta\big\}.$$

Since the two functions involved in the definition of $\mathcal{T}_\lambda$ are nondecreasing, $\mathcal{T}_\lambda$ is an interval. Since θ is an increasing function which vanishes at 0, a positive time T_0 exists such that $\theta(T_0) \leq 4\eta$. Moreover, if $\|e^{a|D|_3|}v_0\|_{B^{\frac{7}{2}}} \leq \eta$ then, since $\partial_t v = \mathbb{P}_n F(v)$ (recall that we are considering Friedrich's approximations), a positive time T_1 (possibly depending on n) exists such that $\|v_\Phi\|_{\widetilde{L}^\infty_{T_1}(B^{\frac{7}{2}})}$ $\leq 4\eta$. Thus $\mathcal{T}_\lambda$ is of the form $[0, T^\star)$ for some positive $T^\star$. Our purpose is to prove that $T^\star = \infty$. As we want to apply Propositions 2.1 and 2.2, we need that $\lambda\theta(T) \leq a$. This leads to the condition

$$4\lambda\eta \leq a. \tag{2.9}$$

From Proposition 2.1, defining $C_0 \stackrel{\text{def}}{=} C_0^{(1)} + C_0^{(2)}$, we have

$$\|v_\Phi\|_{\widetilde{L}^\infty_T(B^{\frac{7}{2}})} \leq \|e^{a|D_3|}v_0\|_{B^{\frac{7}{2}}} + \frac{C_0}{\lambda}\|v_\Phi\|_{\widetilde{L}^\infty_T(B^{\frac{7}{2}})} + C_0\|v_\Phi\|^2_{\widetilde{L}^\infty_T(B^{\frac{7}{2}})}$$

for all $T \in \mathcal{T}_\lambda$. Let us choose $\lambda = \dfrac{1}{2C_0}\cdot$ This gives

$$\|v_\Phi\|_{\widetilde{L}^\infty_T(B^{\frac{7}{2}})} \leq 2\|e^{a|D_3|}v_0\|_{B^{\frac{7}{2}}} + 4C_0\eta\|v_\Phi\|_{\widetilde{L}^\infty_T(B^{\frac{7}{2}})}.$$

Choosing $\eta = \dfrac{1}{12C_0}$, we infer that, for any $T \in \mathcal{T}_\lambda$,

$$\|v_\Phi\|_{\widetilde{L}^\infty_T(B^{\frac{7}{2}})} \leq 3\|e^{a|D_3|}v_0\|_{B^{\frac{7}{2}}}. \tag{2.10}$$

Propositions 2.1 and 2.2 imply that, for all $T \in \mathcal{T}_\lambda$,

$$\theta(T) \leq \varepsilon\|e^{a|D_3|}w_0^h\|_{B^{\frac{7}{2}}} + \|e^{a|D_3|}w_0^3\|_{B^{\frac{7}{2}}} + C_0\eta\theta(T).$$

This implies that

$$\theta(T) \leq 2\varepsilon\|e^{a|D_3|}w_0^h\|_{B^{\frac{7}{2}}} + 2\|e^{a|D_3|}w_0^3\|_{B^{\frac{7}{2}}}.$$

If $2\varepsilon\|e^{a|D_3|}w_0^h\|_{B^{\frac{7}{2}}} + 2\|e^{a|D_3|}w_0^3\|_{B^{\frac{7}{2}}} \leq \eta$ and $\|e^{a|D_3|}v_0^h\|_{B^{\frac{7}{2}}} \leq \eta$, then the above estimate and intequality (2.10) ensure (2.8). This concludes the proof of Theorem 2.

3. The action of the phase Φ on the heat operator

The purpose of this section is the study of the action of the multiplier e^Φ on $\mathbb{E}_\varepsilon f$. Let us recall that the function Φ is defined in (2.7) by $\Phi(t,\xi) = t^{\frac12}|\xi_h| + a|\xi_3| - \lambda\theta(t)|\xi_3|$. This action is described by the following lemma.

LEMMA 3.1. *A constant C_0 exists such that, for any function f with compact spectrum, for any s we have*

$$\|(\mathbb{E}_\varepsilon M^\perp f)_\Phi\|_{\widetilde{L}^\infty_T(B^s)} \leq C_0 \|g_\Phi\|_{\widetilde{L}^\infty_T(B^s)} \quad \text{and}$$
$$\|(\mathbb{E}_\varepsilon M^\perp f)_\Phi\|_{L^1_T(B^s)} \leq C_0 \|g_\Phi\|_{L^1_T(B^s)} \quad \text{where} \quad g \stackrel{\text{def}}{=} \mathcal{F}^{-1}\Big(\frac{1}{|\xi_h|}|\mathcal{F}M^\perp f|\Big).$$

Proof. It will be useful to consider, for any function f, the inverse Fourier transform of $|\widehat{f}|$, defined as

$$f^+ \stackrel{\text{def}}{=} \mathcal{F}^{-1}|\widehat{f}|.$$

Let us notice that the map $f \mapsto f^+$ preserves the norm of all B^s spaces.

Let us write $\mathbb{E}_\varepsilon$ in terms of the Fourier transform. For any $\xi \in (\mathbb{Z}^2 \setminus \{0\}) \times \mathbb{R}$,

$$\mathcal{F}\left(\mathbb{E}_\varepsilon f\right)_\Phi(t,\xi) = e^{\Phi(t,\xi)} \int_0^t e^{-(t-t')|\xi_\varepsilon|^2} f(t',\xi)dt',$$

with $|\xi_\varepsilon|^2 \stackrel{\text{def}}{=} |\xi_h|^2 + \varepsilon^2|\xi_3|^2$, as in all that follows. Thus we infer, for any $\xi \in (\mathbb{Z}^2 \setminus \{0\}) \times \mathbb{R}$,

$$|\mathcal{F}\left((\mathbb{E}_\varepsilon f)_\Phi\right)(t,\xi)| \leq \int_0^t e^{-(t-t')|\xi_\varepsilon|^2 + \Phi(t,\xi) - \Phi(t',\xi)} \mathcal{F}(f^+_\Phi)(t',\xi)dt'.$$

By definition of Φ, we have (see [3, estimate (24)])

$$\Phi(t,\xi) - \Phi(t',\xi) \leq -\lambda|\xi_3| \int_{t'}^t \dot\theta(t'')dt'' + \frac{t-t'}{2}|\xi_h|^2. \tag{3.1}$$

Thus, for any $\xi \in (\mathbb{Z}^2 \setminus \{0\}) \times \mathbb{R}$,

$$|\mathcal{F}\left((\mathbb{E}_\varepsilon f)_\Phi\right)(t,\xi)| \leq \int_0^t e^{-\frac{(t-t')}{2}|\xi_h|^2 - \varepsilon^2(t-t')|\xi_3|^2} \mathcal{F}(f^+_\Phi)(t',\xi)dt'. \tag{3.2}$$

Let us define $\mathcal{C}_h \stackrel{\text{def}}{=} \{1 \leq |\xi_h| \leq 2\} \times \mathbb{R}$. The above inequality means that for any ξ in $2^j\mathcal{C} \cap 2^k\mathcal{C}_h$, we have

$$|\mathcal{F}((\mathbb{E}_\varepsilon f)_\Phi)(t,\xi)| \leq C \int_0^t e^{-c(t-t')2^{2k}} 2^k \widehat{g}_\Phi(t',\xi)dt'.$$

Taking the L^2 norm in ξ in that inequality gives

$$\|(\mathbb{E}_\varepsilon f)_\Phi(t)\|_{\mathcal{F}L^2(2^j\mathcal{C}\cap 2^k\mathcal{C}_h)} \leq \int_0^t e^{-c(t-t')2^{2k}} 2^k \|g_\Phi(t')\|_{L^2_j} dt'. \tag{3.3}$$

By definition of the $\widetilde{L}^\infty_T(B^s)$ norm, this gives, for any $t \leq T$,

$$2^{js}\|(\mathbb{E}_\varepsilon\, f)_\Phi\|_{L^\infty_T(L^2(2^j\mathcal{C}\cap 2^k\mathcal{C}_h))} \leq Cc_j\|g_\Phi\|_{\widetilde{L}^\infty_T(B^s)}\int_0^t e^{-c(t-t')2^{2k}}2^k dt'$$
$$\leq Cc_j 2^{-k}\|g_\Phi\|_{\widetilde{L}^\infty_T(B^s)}.$$

Now, writing that

$$\|(\mathbb{E}_\varepsilon\, M^\perp f)_\Phi\|_{L^\infty_T(L^2_j)} \leq \sum_{k=0}^\infty \|\,\mathbb{E}_\varepsilon(f_\Phi)\|_{L^\infty_T(L^2(2^j\mathcal{C}\cap 2^k\mathcal{C}_h))}$$

gives the first inequality of the lemma.

In order to prove the second one, let us use the definition of the norm of the space B^s and (3.3); this gives

$$\sum_j 2^{js}\|(\mathbb{E}_\varepsilon\, f)_\Phi\|_{L^1_T(L^2_j)} \leq \sum_{j,k} 2^{js}\|\,\mathbb{E}_\varepsilon(f_\Phi)\|_{L^1_T(\mathcal{F}L^2(2^j\mathcal{C}\cap 2^k\mathcal{C}_h))}$$
$$\leq C\sum_{j,k}\int_{[0,T]^2}\mathbf{1}_{t\geq t'}e^{-c2^{2k}(t-t')}2^k c_j(t')\|g_\Phi(t')\|_{B^s}dt'dt.$$

Integrating first in t gives

$$\sum_j 2^{js}\|(\mathbb{E}_\varepsilon\, f_\Phi)\|_{L^1_T(L^2_j)} \leq C\sum_{j,k}\int_{[0,T]} 2^{-k}c_j(t')\|g_\Phi(t')\|_{B^s}dt.$$

As the index k is nonnegative, we get the second estimate of the lemma. □

The following proposition will be of frequent use.

PROPOSITION 3.1. *If s is positive, then for any function Ψ satisfying* (2.4),

$$\|(T_a b)_\Psi\|_{\widetilde{L}^\infty_T(B^s)} + \|(R_a b)_\Psi\|_{\widetilde{L}^\infty_T(B^s)} \leq C\|a_\Psi\|_{L^\infty_T(B^{\frac{3}{2}})}\|b_\Psi\|_{\widetilde{L}^\infty_T(B^s)} \quad \textit{and}$$
$$\|(T_a b)_\Psi\|_{L^1_T(B^s)} + \|(R_a b)_\Psi\|_{L^1_T(B^s)} \leq C\min\Big\{\|a_\Psi\|_{L^1_T(B^{\frac{3}{2}})}\|b_\Psi\|_{\widetilde{L}^\infty_T(B^s)},$$
$$\|a_\Psi\|_{L^\infty_T(B^{\frac{3}{2}})}\|b_\Psi\|_{L^1_T(B^s)}\Big\}.$$

Proof. Taking the L^∞ norm in time in the inequality of Lemma 2.1 gives that

$$\|(T_a b)_\Psi\|_{L^\infty_T(L^2_j)} + \|(R_a b)_\Psi\|_{L^\infty_T(L^2_j)} \leq Cc_j 2^{-js}\|a\|_{L^\infty_T(B^{\frac{3}{2}})}\|b\|_{\widetilde{L}^\infty_T(B^s)},$$

which is the first inequality of the corollary. Taking the L^1 norm in time on the inequality of Lemma 2.1 gives

$$\|(T_a b)_\Psi\|_{L^1_T(L^2_j)} + \|(R_a b)_\Psi\|_{L^1_T(L^2_j)} \leq Cc_j 2^{-js}\|a\|_{L^1_T(B^{\frac{3}{2}})}\|b\|_{\widetilde{L}^\infty_T(B^s)},$$

while the other estimate is a consequence of product rules in Besov spaces. □

The following lemma is a key one. It is here that the function θ allows the gain of the vertical derivative, in the spirit of the method followed in the model example presented above.

LEMMA 3.2. *Let $a(D)$ and $b(D)$ be two Fourier multipliers such that $|a(\xi)| \leq C|\xi_3|$ and $|b(\xi)| \leq C|\xi|^2$. We have*

$$\|(\mathbb{E}_\varepsilon\, a(D) R_{b(D)w^3} f)_\Phi\|_{\widetilde{L}^\infty_T(B^{\frac72})} + \|(\mathbb{E}_\varepsilon\, a(D) T_{b(D)w^3} f)_\Phi\|_{\widetilde{L}^\infty_T(B^{\frac72})} \leq C\Big(\frac{1}{\lambda} + \|w^3_\Phi\|_{\widetilde{L}^\infty_T(B^{\frac72})}\Big)\|f_\Phi\|_{\widetilde{L}^\infty_T(B^{\frac72})}.$$

Proof. We give only the proof for the first term. The second term is estimated exactly along the same lines. Let us write $\mathbb{E}_\varepsilon$ in Fourier variables. We have

$$\mathcal{F}(\mathbb{E}_\varepsilon\, a(D) R_{b(D)w^3} f)_\Phi(t,\xi) = e^{\Phi(t,\xi)} \int_0^t e^{-(t-t')|\xi_\varepsilon|^2} a(\xi) \mathcal{F}(R_{b(D)w^3} f)(t',\xi)dt'.$$

Thus, using the fact that $|a(\xi)| \leq C|\xi_3|$, we obtain that

$$|\mathcal{F}(\mathbb{E}_\varepsilon\, a(D) R_{w^3} f)_\Phi(t,\xi)| \leq C\int_0^t e^{-(t-t')|\xi_\varepsilon|^2+\Phi(t,\xi)-\Phi(t',\xi)} |\xi_3|\, |\mathcal{F}((R_{b(D)w^3} f)_\Phi)(t',\xi)|dt'.$$

Taking into account Inequality (3.1),

$$|\mathcal{F}(\mathbb{E}_\varepsilon\, a(D) R_{w^3} f)_\Phi(t,\xi)| \leq C\int_0^t e^{-\frac{t-t'}{2}|\xi_\varepsilon|^2-\lambda|\xi_3|\int_{t'}^t \dot\theta(t'')dt''} |\xi_3|\, |\mathcal{F}((R_{b(D)w^3} f)_\Phi)(t',\xi)|dt'.$$

Let us denote by Ψ the Fourier multiplier $\Psi a \stackrel{\text{def}}{=} \mathcal{F}^{-1}(\mathbf{1}_{|\xi_h|\leq 2|\xi_3|}\widehat{a})$. If $|\xi_h| \leq 2|\xi_3|$ and ξ is in $2^j\mathcal{C}$, then we have that $|\xi_3| \sim 2^j$. Thus, for any ξ in $2^j\mathcal{C}$, we infer that

$$|\mathcal{F}\Psi(\mathbb{E}_\varepsilon\, a(D) R_{b(D)w^3} f)_\Phi(t,\xi)| \leq \int_0^t e^{-c\lambda 2^j \int_{t'}^t \dot\theta(t'')dt''} 2^j |\mathcal{F}((R_{b(D)w^3} f)_\Phi)(t',\xi)|dt'.$$

Taking the L^2 norm gives

$$\|\Psi(\mathbb{E}_\varepsilon\, a(D) R_{b(D)w^3} f)_\Phi(t,\cdot)\|_{L^2_j} \leq \int_0^t e^{-c\lambda 2^j \int_{t'}^t \dot\theta(t'')dt''} 2^j \|(R_{b(D)w^3} f)_\Phi)(t')\|_{L^2_j}dt'.$$

Using Lemma 2.1, we get

$$
\begin{aligned}
2^{j\frac{7}{2}}\|\Psi(\mathbb{E}_\varepsilon\, a(D)R_{b(D)w^3}f)_\Phi(t,\cdot)\|_{L^2_j} &\leq Cc_j\|f_\Phi(t)\|_{\widetilde{L}^\infty_T(B^{\frac{7}{2}})}\\
&\qquad\times\int_0^t e^{-c\lambda 2^j\int_{t'}^t\dot\theta(t'')dt''}2^j\|b(D)w^3_\Phi(t',\cdot)\|_{B^{\frac{3}{2}}}dt'\\
&\leq Cc_j\|f_\Phi(t)\|_{\widetilde{L}^\infty_T(B^{\frac{7}{2}})}\\
&\qquad\times\int_0^t e^{-c\lambda 2^j\int_{t'}^t\dot\theta(t'')dt''}2^j\|w^3_\Phi(t',\cdot)\|_{B^{\frac{7}{2}}}dt'\\
&\leq Cc_j\|f_\Phi(t)\|_{\widetilde{L}^\infty_T(B^{\frac{7}{2}})}\int_0^t e^{-c\lambda 2^j\int_{t'}^t\dot\theta(t'')dt''}2^j\dot\theta(t')dt'\\
&\leq \frac{C}{\lambda}c_j\|f_\Phi(t)\|_{\widetilde{L}^\infty_T(B^{\frac{7}{2}})}.
\end{aligned}
$$

By summation in j, we deduce that

$$
\|\Psi(\mathbb{E}_\varepsilon\, a(D)R_{w^3}f)_\Phi\|_{\widetilde{L}^\infty_T(B^{\frac{7}{2}})}\leq\frac{C}{\lambda}\|f_\Phi(t)\|_{\widetilde{L}^\infty_T(B^{\frac{7}{2}})}. \tag{3.4}
$$

If $2|\xi_3|\leq|\xi_h|$, then, for any ξ in $2^j\mathcal{C}$, $|\xi_h|$ is equivalent to 2^j and $|\xi_3|$ is less than 2^j. So we infer that for any ξ in $2^j\mathcal{C}$,

$$
\begin{aligned}
&|\mathcal{F}(\mathrm{Id}-\Psi)(\mathbb{E}_\varepsilon\, a(D)R_{b(D)w^3}f)_\Phi(t,\xi)|\\
&\qquad\qquad\leq\int_0^t e^{-c(t-t')2^{2j}}2^j|\mathcal{F}((R_{b(D)w^3}f)_\Phi)(t',\xi)|dt'.
\end{aligned}
$$

By definition of $\|\cdot\|_{\widetilde{L}^\infty_T(B^{\frac{7}{2}})}$, taking the L^2 norm of the above inequality gives

$$
\begin{aligned}
&2^{j\frac{7}{2}}\|(\mathrm{Id}-\Psi)(\mathbb{E}_\varepsilon\, a(D)R_{b(D)w^3}f)_\Phi(t,\cdot)\|_{L^2_j}\\
&\qquad\leq\int_0^t e^{-c2^{2j}(t-t')}2^j2^{j\frac{7}{2}}\|(R_{b(D)w^3}f)_\Phi)(t')\|_{L^2_j}dt'\\
&\qquad\leq Cc_j\|(R_{b(D)w^3}f)_\Phi\|_{\widetilde{L}^\infty_T(B^{\frac{7}{2}})}.
\end{aligned}
$$

After a summation in j, Proposition 3.1 implies that

$$
\begin{aligned}
\|(\mathrm{Id}-\Psi)(\mathbb{E}_\varepsilon\, a(D)R_{b(D)w^3}f)_\Phi\|_{\widetilde{L}^\infty_T(B^{\frac{7}{2}})}&\leq C\|b(D)w^3_\Phi\|_{\widetilde{L}^\infty_T(B^{\frac{3}{2}})}\|f_\Phi\|_{\widetilde{L}^\infty_T(B^{\frac{7}{2}})}\\
&\leq C\|w^3_\Phi\|_{\widetilde{L}^\infty_T(B^{\frac{7}{2}})}\|f_\Phi\|_{\widetilde{L}^\infty_T(B^{\frac{7}{2}})}.
\end{aligned}
$$

Together with (3.4), this concludes the proof of the lemma. □

LEMMA 3.3. *A constant C_0 exists such that, for any function f with compact spectrum, we have for α in $\{1,2\}$,*

$$
\begin{aligned}
&\left\|\left(\mathbb{E}_\varepsilon(\varepsilon\partial_3)^\alpha M^\perp f\right)_\Phi\right\|_{\widetilde{L}^\infty_T(B^s)}\leq C_0\|f_\Phi\|_{\widetilde{L}^\infty_T(B^s)}\quad\textit{and}\\
&\left\|\left(\mathbb{E}_\varepsilon(\varepsilon\partial_3)^\alpha M^\perp f\right)_\Phi\right\|_{L^1_T(B^s)}\leq C_0\|f_\Phi\|_{L^1_T(B^s)}.
\end{aligned}
$$

Proof. Let us start with the case when $\alpha = 2$. Recalling (3.2), we have that (for $0 < \varepsilon < 1$)

$$\varepsilon^2 |\mathcal{F}(\mathbb{E}_\varepsilon \, \partial_3^2 f)_\Phi(t,\xi)| \leq \int_0^t e^{-\varepsilon^2 \frac{(t-t')}{2}|\xi|^2} \varepsilon^2 \xi_3^2 \mathcal{F}(f_\Phi^+)(t',\xi) dt'.$$

Writing that $|\xi_3| \leq |\xi|$, we infer that

$$\varepsilon^2 \Big\| (\mathbb{E}_\varepsilon \, \partial_3^2 f)_\Phi(t) \Big\|_{L^2_j} \leq \int_0^t e^{-c\varepsilon^2 (t-t') 2^{2j}} \varepsilon^2 2^{2j} \|f(t')\|_{L^2_j} dt'.$$

The estimates follow directly by applying Young's inequality in t.

In the case when $\alpha = 1$, we decompose f into two parts,

$$f = f^{(1)} + f^{(2)}, \quad \text{with} \quad f^{(1)} = \mathcal{F}^{-1}(\mathbf{1}_{\varepsilon|\xi_3| \leq |\xi_h|} \widehat{f}).$$

Let us start by studying the first contribution. We simply write that

$$\begin{aligned}
\varepsilon \Big| \mathcal{F}\Big(\mathbb{E}_\varepsilon \, \partial_3 f^{(1)}\Big)_\Phi(t,\xi) \Big| &\leq \int_0^t e^{-\frac{(t-t')}{2}|\xi_\varepsilon|^2} \varepsilon |\xi_3| \mathcal{F}(f_\Phi^{(1)})^+(t',\xi) dt' \\
&\leq \int_0^t e^{-\frac{(t-t')}{2}|\xi_\varepsilon|^2} |\xi_h| \mathcal{F}(f_\Phi^{(1)})^+(t',\xi) dt'
\end{aligned}$$

which amounts exactly to the computation (3.3), with g replaced by $f^{(1)}$. On the other hand, for $f^{(2)}$ we can write

$$\widehat{g}^{(2)}(\xi) \stackrel{\text{def}}{=} \frac{1}{|\xi_h|} \mathbf{1}_{\varepsilon|\xi_3| \geq |\xi_h|} |\mathcal{F} M^\perp f^{(2)}(\xi)|$$

so that

$$\varepsilon \Big| \mathcal{F}\Big(\mathbb{E}_\varepsilon \, \partial_3 M^\perp f^{(2)}\Big)_\Phi(t,\xi) \Big| \leq \int_0^t e^{-\frac{(t-t')}{2}|\xi_h|^2 - \varepsilon^2 (t-t')|\xi_3|^2} \varepsilon |\xi_3| |\xi_h| \widehat{g}^{(2)}(t',\xi) dt'.$$

Since $|\xi_h| \leq \varepsilon|\xi_3|$, we are reduced to the case when $\alpha = 2$ and the conclusion comes from the fact that $\|g_\Phi^{(2)}\|_{B^s} \leq \|M^\perp f_\Phi^{(2)}\|_{B^s} \leq \|f_\Phi\|_{B^s}$. That proves the lemma. $\square$

4. Classical analytic-type parabolic estimates

The purpose of this section is to prove Proposition 2.1. We shall use the algebraic structure of the Navier-Stokes system and the fact that the function Φ is subadditive.

Let us first bound the horizontal component. We recall that

$$\begin{aligned}
w_\Phi^h(t) = {} & e^{t\Delta_\varepsilon + \Phi(t,D)} w^h(0) \\
& - \Big(\mathbb{E}_\varepsilon \, M^\perp (v \cdot \nabla w^h)\Big)_\Phi(t) - \Big(\mathbb{E}_\varepsilon \, M^\perp (w^3 \partial_3 \bar{v}^h)\Big)_\Phi(t) - \Big(\mathbb{E}_\varepsilon (\nabla_h q)\Big)_\Phi(t).
\end{aligned}$$

We note that $v\cdot\nabla w^h=\mathrm{div}_h(v^h\otimes w^h)+\partial_3(w^3w^h)$, recalling that $v^3=w^3$. On the one hand, using Lemma 3.1 and Corollary 3.1, we can write

$$\begin{aligned}\varepsilon\|\,\mathbb{E}_\varepsilon(\mathrm{div}_h(v^hw^h))_\Phi\|_{L^1_T(B^{\frac72})}&\le C\varepsilon\|(v^hw^h)_\Phi\|_{L^1_T(B^{\frac72})}\\&\le C\|v_\Phi\|_{\widetilde{L}^\infty_T(B^{\frac72})}\varepsilon\|w^h\|_{L^1_T(B^{\frac72})}.\end{aligned}$$

By definition of θ, we infer that

$$\varepsilon\|\,\mathbb{E}_\varepsilon(\mathrm{div}_h(v^hw^h))_\Phi\|_{L^1_T(B^{\frac72})}\le C\theta(T)\|v^h_\Phi\|_{\widetilde{L}^\infty_T(B^{\frac72})}.\tag{4.1}$$

On the other hand, Lemma 3.3 and Proposition 3.1 imply that

$$\begin{aligned}\|\,\mathbb{E}_\varepsilon\left(\varepsilon\partial_3M^\perp(w^3w^h)\right)_\Phi\|_{L^1_T(B^{\frac72})}&\le C\|w^3w^h\|_{L^1_T(B^{\frac72})}\\&\le C\theta(T)\|v^h_\Phi\|_{\widetilde{L}^\infty_T(B^{\frac72})}.\end{aligned}\tag{4.2}$$

For the second term, we use paradifferential calculus which gives

$$\begin{aligned}w^3\partial_3\bar v^h&=T_{w^3}\partial_3\bar v^h+R_{\partial_3\bar v^h}w^3\\&=\partial_3T_{w^3}\bar v^h-T_{\partial_3w^3}\bar v^h+R_{\partial_3\bar v^h}w^3.\end{aligned}$$

Using again Lemma 3.3 and Proposition 3.1, we get

$$\begin{aligned}\|\,\mathbb{E}_\varepsilon\left(\varepsilon\partial_3M^\perp T_{w^3}\bar v^h\right)_\Phi\|_{L^1_T(B^{\frac72})}&\le C\|(T_{w^3}\bar v^h)_\Phi\|_{L^1_T(B^{\frac72})}\\&\le C\|w^3_\Phi\|_{L^1_T(B^{\frac72})}\|\bar v^h\|_{\widetilde{L}^\infty_T(B^{\frac72})}.\end{aligned}$$

By definition of θ, we infer that

$$\|\,\mathbb{E}_\varepsilon\left(\varepsilon\partial_3M^\perp T_{w^3}\bar v^h\right)_\Phi\|_{L^1_T(B^{\frac72})}\le C\theta(T)\|v^h_\Phi\|_{\widetilde{L}^\infty_T(B^{\frac72})}.\tag{4.3}$$

By Lemma 3.1 and Proposition 3.1, we can write that

$$\begin{aligned}\|\,\mathbb{E}_\varepsilon\left(\varepsilon M^\perp T_{\partial_3w^3}\bar v^h\right)_\Phi\|_{L^1_T(B^{\frac72})}&\le C\varepsilon\|(T_{\partial_3w^3}\bar v^h)_\Phi\|_{L^1_T(B^{\frac72})}\\&\le C\varepsilon\|w^3_\Phi\|_{L^1_T(B^{\frac72})}\|\bar v^h\|_{\widetilde{L}^\infty_T(B^{\frac72})}.\end{aligned}$$

Thus,

$$\|\,\mathbb{E}_\varepsilon\left(\varepsilon M^\perp T_{\partial_3w^3}\bar v^h\right)_\Phi\|_{L^1_T(B^{\frac72})}\le C\theta(T)\|v^h_\Phi\|_{\widetilde{L}^\infty_T(B^{\frac72})},\tag{4.4}$$

and finally, along the same lines,

$$\|\,\mathbb{E}_\varepsilon\left(\varepsilon M^\perp R_{\partial_3\bar v^h}w^3\right)_\Phi\|_{L^1_T(B^{\frac72})}\le C\theta(T)\|v^h_\Phi\|_{\widetilde{L}^\infty_T(B^{\frac72})}.\tag{4.5}$$

Now we are left with the study of the pressure. Some of its properties are described in the following lemma.

LEMMA 4.1. *Let us define* $\nabla_\varepsilon \stackrel{\mathrm{def}}{=} (\nabla_h, \varepsilon\partial_3)$. *The following two inequalities on the rescaled pressure hold*:

$$\varepsilon\|(\mathbb{E}_\varepsilon\, \nabla_\varepsilon M^\perp q)_\Phi\|_{\widetilde{L}^\infty_T(B^{\frac72})} \leq C\|v_\Phi\|^2_{\widetilde{L}^\infty_T(B^{\frac72})}$$

and

$$\varepsilon\|(\mathbb{E}_\varepsilon\, \nabla_\varepsilon M^\perp q)_\Phi\|_{L^1_T(B^{\frac72})} \leq C\|v_\Phi\|_{\widetilde{L}^\infty_T(B^{\frac72})}\theta(T).$$

Proof. Using the formula (2.1) on the rescaled pressure and the divergence free condition on v, let us decompose it as $\varepsilon q = q_{1,\varepsilon} - q_{2,\varepsilon}$ with

$$q_{1,\varepsilon} \stackrel{\mathrm{def}}{=} \sum_{k=1}^{2} \partial_k\partial_\ell\Delta_\varepsilon^{-1}(\varepsilon w^k v^\ell) + \sum_{1\leq k\leq 2} \partial_k(\varepsilon\partial_3)\Delta_\varepsilon^{-1}(w^3v^k)$$

and

$$q_{2,\varepsilon} \stackrel{\mathrm{def}}{=} 2\varepsilon\partial_3\Delta_\varepsilon^{-1}(w^3 \operatorname{div}_h w^h).$$

Let us start with $q_{1,\varepsilon}$. We have

$$\nabla_\varepsilon q_{1,\varepsilon} = \sum_{k=1}^{2} \partial_k\biggl(\sum_{\ell=1}^{2} \nabla_\varepsilon\partial_\ell\Delta_\varepsilon^{-1}(\varepsilon w^k v^\ell) + \nabla_\varepsilon(\varepsilon\partial_3)\Delta_\varepsilon^{-1}(w^3v^k)\biggr).$$

Since $\nabla_\varepsilon^2\Delta_\varepsilon^{-1}$ is a family of bounded Fourier multipliers (uniformly with respect to ε), from Lemma 3.1 and Proposition 3.1, we infer that

$$\|(\mathbb{E}_\varepsilon(\nabla_\varepsilon M^\perp q_{1,\varepsilon}))_\Phi\|_{\widetilde{L}^\infty_T(B^{\frac72})} \leq C\|v_\Phi\|^2_{\widetilde{L}^\infty_T(B^{\frac72})} \tag{4.6}$$

and

$$\|(\mathbb{E}_\varepsilon(\nabla_\varepsilon M^\perp q_{1,\varepsilon}))_\Phi\|_{L^1_T(B^{\frac72})} \leq C\|v_\Phi\|_{\widetilde{L}^\infty_T(B^{\frac72})}\theta(T). \tag{4.7}$$

In order to study $q_{2,\varepsilon}$, let us observe that

$$\begin{aligned} w^3 \operatorname{div}_h w^h &= R_{\operatorname{div}_h w^h} w^3 + T_{w^3} \operatorname{div}_h w^h \\ &= R_{\operatorname{div}_h w^h} w^3 + \sum_{k=1}^{2}\bigl(\partial_k T_{w^3} w^k - T_{\partial_k w^3} w^k\bigr). \end{aligned} \tag{4.8}$$

As above, using Lemma 3.1 and Proposition 3.1 we get

$$\|(\mathbb{E}_\varepsilon(\nabla_\varepsilon M^\perp q_{2,\varepsilon}))_\Phi\|_{\widetilde{L}^\infty_T(B^{\frac72})} \leq C\|v_\Phi\|^2_{\widetilde{L}^\infty_T(B^{\frac72})}$$

and

$$\|(\mathbb{E}_\varepsilon(\nabla_\varepsilon M^\perp q_{2,\varepsilon}))_\Phi\|_{L^1_T(B^{\frac72})} \leq C\|v_\Phi\|_{\widetilde{L}^\infty_T(B^{\frac72})}\theta(T).$$

Together with estimates (4.6) and (4.7), this concludes the proof of the lemma. □

The above Lemma 4.1, together with estimates (4.1) to (4.4), implies that

$$\varepsilon\|w^h\|_{L^1_T(B^{\frac72})} \leq \varepsilon\|e^{a|D_3|}w_0^h\|_{B^{\frac72}} + C_0^{(1)}\|v_\Phi\|_{\widetilde{L}^\infty_T(B^{\frac72})}\theta(T). \tag{4.9}$$

Let us prove the estimates on the vertical component. It turns out that it is better behaved because of the special structure of the system. Indeed, thanks to the divergence free condition, almost no vertical derivatives appear in the equation of w^3: we have (since $w^3 = v^3$)

$$\partial_t w^3 - \Delta_\varepsilon w^3 = -v^h \cdot \nabla_h w^3 + w^3 \operatorname{div}_h w^h - \varepsilon^2 \partial_3 q. \tag{4.10}$$

The Duhamel formula reads

$$w^3(t) = e^{t\Delta_\varepsilon} w^3(0) + \mathbb{E}_\varepsilon\, M^\perp (w^3 \operatorname{div}_h w^h - v^h \cdot \nabla_h w^3)(t) - \mathbb{E}_\varepsilon\, M^\perp (\varepsilon^2 \partial_3 q)(t).$$

Applying the Fourier multiplier $e^{\Phi(t,D)}$ to the above relation gives

$$\begin{aligned} w^3_\Phi(t) = e^{t\Delta_\varepsilon + \Phi(t,D)} w^3(0) &+ \Big(\mathbb{E}_\varepsilon\, M^\perp (w^3 \operatorname{div}_h w^h - v^h \cdot \nabla_h w^3)\Big)_\Phi(t) \\ &- \Big(\mathbb{E}_\varepsilon\, M^\perp \varepsilon^2 \partial_3 q\Big)_\Phi(t). \end{aligned} \tag{4.11}$$

Using (4.8) and then Lemma 3.1 and Proposition 3.1, we get

$$\Big\|\Big(\mathbb{E}_\varepsilon\, M^\perp (w^3 \operatorname{div}_h w^h)\Big)_\Phi\Big\|_{\widetilde{L}^\infty_T(B^{\frac72})} \leq C \|w^3_\Phi\|_{\widetilde{L}^\infty_T(B^{\frac72})} \|w^h_\Phi\|_{\widetilde{L}^\infty_T(B^{\frac72})} \tag{4.12}$$

and

$$\Big\|\Big(\mathbb{E}_\varepsilon\, M^\perp (w^3 \operatorname{div}_h w^h)\Big)_\Phi\Big\|_{L^1_T(B^{\frac72})} \leq C \|w^3_\Phi\|_{L^1_T(B^{\frac72})} \|w^h_\Phi\|_{\widetilde{L}^\infty_T(B^{\frac72})}. \tag{4.13}$$

Writing

$$\begin{aligned} v^h \cdot \nabla_h a &= \sum_{k=1}^2 \Big(T_{v^k} \partial_k a + R_{\partial_k a} v^k\Big) \\ &= -T_{\operatorname{div}_h w^h} a + \sum_{k=1}^2 \Big(\partial_k T_{v^k} a + R_{\partial_k a} v^k\Big) \end{aligned}$$

and using Lemma 3.1 and Proposition 3.1, we get

$$\Big\|\Big(\mathbb{E}_\varepsilon\, M^\perp (v^h \cdot \nabla_h w^3)\Big)_\Phi\Big\|_{\widetilde{L}^\infty_T(B^{\frac72})} \leq C \|w^3_\Phi\|_{\widetilde{L}^\infty_T(B^{\frac72})} \|w^h_\Phi\|_{\widetilde{L}^\infty_T(B^{\frac72})}$$

and

$$\Big\|\Big(\mathbb{E}_\varepsilon\, M^\perp (v^h \cdot \nabla_h w^3)\Big)_\Phi\Big\|_{L^1_T(B^{\frac72})} \leq C \|w^3_\Phi\|_{L^1_T(B^{\frac72})} \|w^h_\Phi\|_{\widetilde{L}^\infty_T(B^{\frac72})}.$$

Together with estimates (4.12) and (4.13), and Lemma 4.1, this gives

$$\|w^3_\Phi\|_{L^1_T(B^{\frac72})} \leq \|e^{a|D_3|} w^3_0\|_{B^{\frac72}} + C_0^{(1)} \|v_\Phi\|_{\widetilde{L}^\infty_T(B^{\frac72})} \theta(T)$$

and

$$\|w^3_\Phi\|_{\widetilde{L}^\infty_T(B^{\frac72})} \leq \|e^{a|D_3|} w^3_0\|_{B^{\frac72}} + C_0^{(1)} \|v_\Phi\|^2_{\widetilde{L}^\infty_T(B^{\frac72})}.$$

Together with (4.9), this concludes the proof of Proposition 2.1.

5. The gain of one vertical derivative on the horizontal part

In this section we shall prove Proposition 2.2. The proof will be separated into two parts: first we shall consider the case of the horizontal average $\bar{v}^h_\Phi$, and then the remainder w^h_Φ.

5.1. *The gain of one vertical derivative on the horizontal average.* We shall study in this section the equation on the horizontal average of the solution. We emphasize that in the equation on $\bar{v}$ we cannot recover the vertical derivative appearing in the force term by the regularizing effect. The fundamental idea to gain a vertical derivative is to use the analyticity of the solution and therefore to estimate $\bar{v}_\Phi$. The lemma is the following.

LEMMA 5.1. *A constant C_0 exists such that, for any positive λ, for any initial data v_0, and for any T satisfying $\theta(T) \leq a/\lambda$,*

$$\|\bar{v}^h_\Phi\|_{\widetilde{L}^\infty_T(B^{\frac{7}{2}})} \leq \|e^{a|D_3|}\bar{v}^h_0\|_{B^{\frac{7}{2}}} + C_0\Big(\frac{1}{\lambda} + \|v_\Phi\|_{\widetilde{L}^\infty_T(B^{\frac{7}{2}})}\Big)\|v^h_\Phi\|_{\widetilde{L}^\infty_T(B^{\frac{7}{2}})}.$$

Proof. The horizontal average $\bar{v}$ satisfies

$$\partial_t \bar{v} - \varepsilon^2 \partial_3^2 \bar{v} = -\partial_3 M(w^3 w^h) \quad \text{and} \quad \bar{v}_{|t=0} = \bar{v}_0. \tag{5.1}$$

Let us define $G \stackrel{\text{def}}{=} -\partial_3 M(w^3 w^k)$. Writing the solution of (5.1) in terms of the Fourier transform, using (3.1) with $\xi_h = 0$, we get

$$|\mathcal{F}(\bar{v}_\Phi)(t,\xi)| \leq |\mathcal{F}\bar{v}_0(\xi)|e^{a|\xi_3|} + \int_0^t e^{-\lambda|\xi_3|\int_{t'}^t \dot{\theta}(t'')dt''}|\mathcal{F}(G_\Phi)(t',\xi)|dt'.$$

Then, taking the L^2_j norm, we infer that

$$\|\bar{v}_\Phi(t)\|_{L^2_j} \leq \|e^{a|D_3|}\bar{v}_0\|_{L^2_j} + \int_0^t e^{-c\lambda 2^j\int_{t'}^t \dot{\theta}(t'')dt''}\|G_\Phi(t')\|_{L^2_j}dt'. \tag{5.2}$$

Now, let us estimate $\|G_\Phi(t')\|_{L^2_j}$. For any function a, using the fact that the vector field w is divergence free, let us write that

$$\begin{aligned}\partial_3(w^3 a) &= \partial_3\big(T_{w^3}a + R_a w^3\big)\\ &= \partial_3 T_{w^3}a + R_{\partial_3 a}w^3 - R_a \operatorname{div}_h w^h\\ &= \partial_3 T_{w^3}a + R_{\partial_3 a}w^3 - \sum_{\ell=1}^2 \partial_\ell R_a w^\ell + \sum_{\ell=1}^2 R_{\partial_\ell a}w^\ell.\end{aligned} \tag{5.3}$$

Thus, we infer that

$$\begin{aligned}G &= -\partial_3 M T_{w^3}w^k - M\Big(R_{\partial_3 w^k}w^3 + \sum_{\ell=1}^2 R_{\partial_\ell w^k}w^\ell - \sum_{\ell=1}^2 \partial_\ell R_{w^3}w^\ell\Big)\\ &= -\partial_3 M T_{w^3}w^k - M\Big(R_{\partial_3 w^k}w^3 + \sum_{\ell=1}^2 R_{\partial_\ell w^k}w^\ell\Big).\end{aligned} \tag{5.4}$$

Now, let us study $\mathcal{F}M(T_a b)_\Phi$ and $\mathcal{F}M(R_a b)_\Phi$ for two functions a and b which have 0 horizontal average. As the two terms are identical, let us study the first one. By definition, we have

$$\mathcal{F}\big(T_a b\big)(t,(0,\xi_3)) = \sum_j \int_{2^j\mathcal{C}\cap B((0,\xi_3),2^j)} \widehat{a}((0,\xi_3)-\eta)\widehat{b}(\eta)d\eta.$$

Since $\theta(T)\leq \lambda^{-1}a$, by definition of Φ, for any $\eta\in(\mathbb{Z}^2\setminus\{0\})\times\mathbb{R}$, we gave

$$\begin{aligned}\Phi(t,(0,\xi_3)) &\leq \Phi(t,(0,\xi_3-\eta_3))+\Phi(t,(0,\eta_3))\\ &\leq -2t^{\frac12}+\Phi(t,((0,\xi_3)-\eta)+\Phi(t,-\eta).\end{aligned}$$

Thus we have

$$|(\mathcal{F}M(T_a b)_\Phi)(t,\xi)| \leq e^{-2t^{\frac12}}(\mathcal{F}MT_{a^+_\Phi}b^+_\Phi)(t,\xi).$$

Applied to (5.4), this implies that

$$\begin{aligned}\Big|\mathcal{F}G_\Phi(t,\xi)\Big| &\leq |\xi_3|\mathcal{F}\Big(T_{(w^3_\Phi)^+}(w^k_\Phi)^+\Big)(t,(0,\xi_3))\\ &\quad + e^{-2t^{\frac12}}\mathcal{F}\Big(R_{(\partial_3 w^k_\Phi)^+}(w^3_\Phi)^+ + \sum_{\ell=1}^2 R_{(\partial_\ell w^k_\Phi)^+}(w^\ell_\Phi)^+\Big)(t,(0,\xi_3)).\end{aligned}$$

Inequality (2.1) then implies that, for any $t\in[0,T]$,

$$2^{j\frac72}\|G_\Phi(t)\|_{L^2_j} \leq Cc_j\|v^h_\Phi\|_{\widetilde{L}^\infty_T(B^{\frac72})}\big(2^j\|w^3_\Phi(t)\|_{B^{\frac72}}+e^{-2t^{\frac12}}\|v_\Phi\|_{\widetilde{L}^\infty_T(B^{\frac72})}\big). \tag{5.5}$$

Then, by definition of θ, inequalities (5.2) and (5.5) imply that

$$\begin{aligned}2^{j\frac72}\|(\bar v_\Phi)(t)\|_{L^2_j} &\leq 2^{j\frac72}\|e^{a|D_3|}\bar v_0\|_{L^2_j}\\ &+Cc_j\|v_\Phi\|_{\widetilde{L}^\infty_T(B^{\frac72})}\bigg(\int_0^t e^{-c\lambda 2^j\int_{t'}^t\dot\theta(t'')dt''}2^j\dot\theta(t')dt' + \|v_\Phi\|_{\widetilde{L}^\infty_T(B^{\frac72})}\int_0^t e^{-2t'^{\frac12}}dt'\bigg).\end{aligned}$$

This gives

$$2^{j\frac72}\|\bar v_\Phi\|_{L^\infty_T(L^2_j)} \leq 2^{j\frac72}\|e^{a|D_3|}\bar v_0\|_{L^2_j}+Cc_j\|v^h_\Phi\|_{\widetilde{L}^\infty_T(B^{\frac72})}\Big(\frac1\lambda+\|v_\Phi\|_{\widetilde{L}^\infty_T(B^{\frac72})}\Big).$$

Taking the sum over j concludes the proof of the lemma. □

5.2. *The gain of the vertical derivative on the whole horizontal term.* Now let us estimate the rest of the horizontal term, that is $\|w^h_\Phi\|_{\widetilde{L}^\infty_T(B^{\frac72})}$. As in Section 5.1, the function θ will play a crucial role.

LEMMA 5.2. *A constant C_0 exists such that, for any λ, for any initial data v_0, and for any T satisfying $\theta(T)\leq a/\lambda$, we have*

$$\|w^h_\Phi\|_{\widetilde{L}^\infty_T(B^{\frac72})} \leq \|e^{a|D_3|}w^h_0\|_{B^{\frac72}} + C_0\Big(\frac1\lambda+\|v_\Phi\|_{\widetilde{L}^\infty_T(B^{\frac72})}\Big)\|v^h_\Phi\|_{\widetilde{L}^\infty_T(B^{\frac72})}.$$

Proof. The Duhamel formula writes

$$w^h(t) = e^{t\Delta_\varepsilon} w^h(0) - \mathbb{E}_\varepsilon \operatorname{div}_h(v^h \otimes v^h)(t) - \mathbb{E}_\varepsilon M^\perp \partial_3(w^3 v^h)(t) - \mathbb{E}_\varepsilon(\nabla_h q)(t).$$

Lemma 3.1 and Proposition 3.1 imply that

$$\begin{aligned} \|(\mathbb{E}_\varepsilon \operatorname{div}_h(v^h \otimes v^h))_\Phi\|_{\widetilde{L}^\infty_T(B^{\frac{7}{2}})} &\leq C\|(v^h \otimes v^h)_\Phi\|_{\widetilde{L}^\infty_T(B^{\frac{7}{2}})} \\ &\leq C\|v^h_\Phi\|^2_{\widetilde{L}^\infty_T(B^{\frac{7}{2}})}. \end{aligned} \tag{5.6}$$

Then using (5.4), thanks to Leibnitz formula, we get

$$\begin{aligned} M^\perp \partial_3(w^3 v^k) &= M^\perp \partial_3 T_{w_3} v^k + F^k \quad \text{with} \\ &F^k \stackrel{\text{def}}{=} M^\perp \bigg(R_{\partial_3 v^k} w^3 - \sum_{\ell=1}^{2} \Big(\partial_\ell R_{v^k} w^\ell - R_{\partial_\ell v_k} w^\ell \Big) \bigg). \end{aligned}$$

Thanks to Lemma 3.1 and Proposition 3.1, we get

$$\begin{aligned} \|(\mathbb{E}_\varepsilon M^\perp F^k)_\Phi\|_{\widetilde{L}^\infty_T(B^{\frac{7}{2}})} &\leq \|(F^k)_\Phi\|_{\widetilde{L}^\infty_T(B^{\frac{7}{2}})} \\ &\leq C\|v_\Phi\|_{\widetilde{L}^\infty_T(B^{\frac{7}{2}})} \|v^h_\Phi\|_{\widetilde{L}^\infty_T(B^{\frac{7}{2}})}. \end{aligned}$$

Together with Lemma 3.2, this gives

$$\Big\| \Big(\mathbb{E}_\varepsilon M^\perp \partial_3(w^3 v^h) \Big)_\Phi \Big\|_{\widetilde{L}^\infty_T(B^{\frac{7}{2}})} \leq C_0 \Big(\frac{1}{\lambda} + \|v_\Phi\|_{\widetilde{L}^\infty_T(B^{\frac{7}{2}})} \Big) \|v^h_\Phi\|_{\widetilde{L}^\infty_T(B^{\frac{7}{2}})}. \tag{5.7}$$

Now let us study the pressure term. Formula (2.1) together with the divergence free condition leads to the decomposition $q = q_h(w^h) + q_3(v)$ with

$$q_h(w^h) \stackrel{\text{def}}{=} \Delta_\varepsilon^{-1} \Big((\operatorname{div}_h w^h)^2 + \sum_{1 \leq k, \ell \leq 2} \partial_k w^\ell \partial_\ell w^k \Big) \tag{5.8}$$

and

$$q_3(v) \stackrel{\text{def}}{=} \Delta_\varepsilon^{-1} \Big(\sum_{1 \leq \ell \leq 2} \partial_3 v^\ell \partial_\ell w^3 \Big). \tag{5.9}$$

For the first term we use Bony's decomposition in order to obtain

$$\partial_k w^\ell \partial_\ell w^k = T_{\partial_k w^\ell} \partial_\ell w^k + R_{\partial_\ell w^k} \partial_k w^\ell.$$

Then the Leibnitz formula implies that

$$\partial_k w^\ell \partial_\ell w^k = \partial_\ell T_{\partial_k w^\ell} w^k + \partial_k R_{\partial_\ell w^k} w^\ell - T_{\partial_\ell \partial_k w^\ell} w^k - R_{\partial_k \partial_\ell w^k} w^\ell. \tag{5.10}$$

On the other hand, again by paradifferential calculus, we can write that

$$\begin{aligned} r(\operatorname{div}_h w^h)^2 &= T_{\operatorname{div}_h w^h} \operatorname{div}_h w^h + R_{\operatorname{div}_h w^h} \operatorname{div}_h w^h \\ &= \operatorname{div}_h \Big(T_{\operatorname{div}_h w^h} w^h + R_{\operatorname{div}_h w^h} w^h \Big) \\ &\quad - \sum_{k=1}^{2} \Big(T_{\partial_k \operatorname{div}_h w^h} w^k + R_{\partial_k \operatorname{div}_h w^h} w^k \Big). \end{aligned} \tag{5.11}$$

Then Lemma 3.1 implies that

$$\|(\mathbb{E}_\varepsilon \nabla_h q_h(w^h))_\Phi\|_{\widetilde{L}^\infty_T(B^{\frac{7}{2}})} \leq C_0 \|(M^\perp q_h(w^h))_\Phi\|_{\widetilde{L}^\infty(B^{\frac{7}{2}})}.$$

Using Proposition 3.1 and the fact that the operators $\nabla_h \Delta_\varepsilon^{-1} M^\perp$ and $\Delta_\varepsilon^{-1} M^\perp$ are bounded (uniformly in ε) Fourier multipliers, we obtain

$$\left\|\left(\mathbb{E}_\varepsilon(\nabla_h q_h(w^h))\right)_\Phi\right\|_{\widetilde{L}^\infty_T(B^{\frac{7}{2}})} \leq C_0 \|v^h_\Phi\|^2_{\widetilde{L}^\infty_T(B^{\frac{7}{2}})}. \tag{5.12}$$

For the second term, let us decompose $q_3(v)$ in the following way:

$$\begin{aligned}\partial_3 v^\ell \partial_\ell w^3 &= T_{\partial_3 v^\ell} \partial_\ell w^3 + R_{\partial_\ell w^3} \partial_3 v^\ell \\ &= \partial_\ell T_{\partial_3 v^\ell} w^3 + \partial_3 R_{\partial_\ell w^3} v^\ell - T_{\partial_3 \partial_\ell v^\ell} w^3 - R_{\partial_3 \partial_\ell w^3} v^\ell.\end{aligned}$$

Using now Lemma 3.1 together with Proposition 3.1 and Lemma 3.2, we obtain

$$\left\|\left(\mathbb{E}_\varepsilon(\nabla_h q_3(v))\right)_\Phi\right\|_{\widetilde{L}^\infty_T(B^{\frac{7}{2}})} \leq C_0\Big(\frac{1}{\lambda} + \|v_\Phi\|_{\widetilde{L}^\infty_T(B^{\frac{7}{2}})}\Big)\|v^h\|_{\widetilde{L}^\infty_T(B^{\frac{7}{2}})}. \tag{5.13}$$

The expected result is obtained putting together estimates (5.12) and (5.13) on the pressure with estimates (5.6) and (5.7) on the nonlinear terms. □

References

[1] J. Bourgain and N. Pavlović, Ill-posedness of the Navier-Stokes equations in a critical space in 3D, *J. Funct. Anal.* **255** (2008), 2233–2247. MR 2473255. Zbl 1161.35037. doi: 10.1016/j.jfa.2008.07.008.

[2] M. Cannone, Y. Meyer, and F. Planchon, Solutions auto-similaires des équations de Navier-Stokes, in *Séminaire sur les Équations aux Dérivées Partielles*, 1993–1994, École Polytech., Palaiseau, 1994, pp. Exp. No. VIII, 12. MR 1300903. Zbl 0882.35090.

[3] J.-Y. Chemin, Le système de Navier-Stokes incompressible soixante dix ans après Jean Leray, in *Actes des Journées Mathématiques à la Mémoire de Jean Leray*, *Sémin. Congr.* **9**, Soc. Math. France, Paris, 2004, pp. 99–123. MR 2145938. Zbl 1075.35035.

[4] J.-Y. Chemin, B. Desjardins, I. Gallagher, and E. Grenier, Fluids with anisotropic viscosity, *M2AN Math. Model. Numer. Anal.* **34** (2000), 315–335. MR 1765662. Zbl 0954.76012. doi: 10.1051/m2an:2000143.

[5] J.-Y. Chemin and I. Gallagher, On the global wellposedness of the 3-D Navier-Stokes equations with large initial data, *Ann. Sci. École Norm. Sup.* **39** (2006), 679–698. MR 2290141. Zbl 1124.35052. doi: 10.1016/j.ansens.2006.07.002.

[6] ———, Wellposedness and stability results for the Navier-Stokes equations in $\mathbf{R}^3$, *Ann. Inst. H. Poincaré Anal. Non Linéaire* **26** (2009), 599–624. MR 2504045. Zbl 1165.35038. doi: 10.1016/j.anihpc.2007.05.008.

[7] ———, Large, global solutions to the Navier-Stokes equations, slowly varying in one direction, *Trans. Amer. Math. Soc.* **362** (2010), 2859–2873. MR 2592939. Zbl 1189.35220. doi: 10.1090/S0002-9947-10-04744-6.

[8] J.-Y. Chemin and N. Lerner, Flot de champs de vecteurs non lipschitziens et équations de Navier-Stokes, *J. Differential Equations* **121** (1995), 314–328. MR 1354312. Zbl 0878.35089. doi: 10.1006/jdeq.1995.1131.

[9] J.-Y. Chemin and P. Zhang, On the global wellposedness to the 3-D incompressible anisotropic Navier-Stokes equations, *Comm. Math. Phys.* **272** (2007), 529–566. MR 2300252. Zbl 1132.35068. doi: 10.1007/s00220-007-0236-0.

[10] L. Escauriaza, G. A. Serëgin, and V. Şverák, $L_{3,\infty}$-solutions to the Navier-Stokes equations and backward uniqueness, *Russ. Math. Surv.* **58** (2003), 211–250. MR 1992563. Zbl 1064-35134. doi: 10.1070/RM2003v058n02ABEH000609.

[11] H. Fujita and T. Kato, On the Navier-Stokes initial value problem. I, *Arch. Rational Mech. Anal.* **16** (1964), 269–315. MR 0166499. Zbl 0126.42301. doi: 10.1007/BF00276188.

[12] I. Gallagher, D. Iftimie, and F. Planchon, Asymptotics and stability for global solutions to the Navier-Stokes equations, *Ann. Inst. Fourier* (*Grenoble*) **53** (2003), 1387–1424. MR 2032938. Zbl 1038.35054. Available at http://aif.cedram.org/item?id=AIF_2003__53_5_1387_0.

[13] I. Gallagher and M. Paicu, Remarks on the blow-up of solutions to a toy model for the Navier-Stokes equations, *Proc. Amer. Math. Soc.* **137** (2009), 2075–2083. MR 2480289. Zbl 1162.76016. doi: 10.1090/S0002-9939-09-09765-2.

[14] P. Germain, The second iterate for the Navier-Stokes equation, *J. Funct. Anal.* **255** (2008), 2248–2264. MR 2473256. Zbl 1173.35097. doi: 10.1016/j.jfa.2008.07.014.

[15] Y. Giga and T. Miyakawa, Solutions in L_r of the Navier-Stokes initial value problem, *Arch. Rational Mech. Anal.* **89** (1985), 267–281. MR 786550. Zbl 0587.35078. doi: 10.1007/BF00276875.

[16] E. Grenier, On the nonlinear instability of Euler and Prandtl equations, *Comm. Pure Appl. Math.* **53** (2000), 1067–1091. MR 1761409. Zbl 1048.35081. doi: 10.1002/1097-0312(200009)53:9<1067::AID-CPA1>3.3.CO;2-H.

[17] D. Iftimie, A uniqueness result for the Navier-Stokes equations with vanishing vertical viscosity, SIAM *J. Math. Anal.* **33** (2002), 1483–1493. MR 1920641. Zbl 1011.35105. doi: 10.1137/S0036141000382126.

[18] D. Iftimie, G. Raugel, and G. R. Sell, Navier-Stokes equations in thin 3D domains with Navier boundary conditions, *Indiana Univ. Math. J.* **56** (2007), 1083–1156. MR 2333468. Zbl 1129.35056. doi: 10.1512/iumj.2007.56.2834.

[19] T. Kato, Strong L^p-solutions of the Navier-Stokes equation in $\mathbf{R}^m$, with applications to weak solutions, *Math. Z.* **187** (1984), 471–480. MR 0760047. Zbl 0545.35073. doi: 10.1007/BF01174182.

[20] H. Koch and D. Tataru, Well-posedness for the Navier-Stokes equations, *Adv. Math.* **157** (2001), 22–35. MR 1808843. Zbl 0972.35084. doi: 10.1006/aima.2000.1937.

[21] O. A. Ladyzhenskaya, *The Mathematical Theory of Viscous Incompressible Flow*, *Math. Appl.* **2**, Gordon and Breach Science Publishers, New York, 1969, Second English edition, revised and enlarged. Translated from the Russian by Richard A. Silverman and John Chu. MR 0254401. Zbl 0184.52603.

[22] P. G. Lemarié-Rieusset, *Recent Developments in the Navier-Stokes Problem*, *Chapman & Hall/CRC Research Notes in Math.* **431**, Chapman & Hall/CRC, Boca Raton, FL, 2002. MR 1938147. Zbl 1034.35093.

[23] J. Leray, Étude de diverses équations intégrales non linéaires et de quelques problèmes que pose l'hydrodynamique, *Journal Math. Pures et Appliquées* **12** (1933), 1–82. Zbl 0006.16702.

[24] ———, Sur le mouvement d'un liquide visqueux emplissant l'espace, *Acta Matematica* **63** (1933), 193–248. MR 1555394. JFM 59.0763.02. doi: 10.1007/BF02547354.

[25] N. Lerner, Y. Morimoto, and C.-J. Xu, Instability of the Cauchy-Kovalevskaya solution for a class of non-linear systems, *Amer. J. Math.* **132** (2010), 99–123. MR 2597507. Zbl 1194.35104. doi: 10.1353/ajm.0.0096.

[26] A. Mahalov, E. S. Titi, and S. Leibovich, Invariant helical subspaces for the Navier-Stokes equations, *Arch. Rational Mech. Anal.* **112** (1990), 193–222. MR 1076072. Zbl 0708.76044. doi: 10.1007/BF00381234.

[27] A. S. Makhalov and V. P. Nikolaenko, Global solvability of three-dimensional Navier-Stokes equations with uniformly high initial vorticity, *Uspekhi Mat. Nauk* **58** (2003), 79–110. MR 1992565. Zbl 1059.35099. doi: 10.1070/RM2003v058n02ABEH000611.

[28] Y. Meyer, Wavelets, paraproducts, and Navier-Stokes equations, in *Current Developments in Mathematics*, International Press, Somerville, MA, 1996, pp. 104–212. MR 1724946. Zbl 0926.35115.

[29] S. Montgomery-Smith, Finite time blow up for a Navier-Stokes like equation, *Proc. Amer. Math. Soc.* **129** (2001), 3025–3029. MR 1840108. Zbl 0970.35100. doi: 10.1090/S0002-9939-01-06062-2.

[30] J. Nečas, M. Ruzįçka, and V. Šverák, On Leray's self-similar solutions of the Navier-Stokes equations, *Acta Math.* **176** (1996), 283–294. MR 1397564. Zbl 0884.35115. doi: 10.1007/BF02551584.

[31] M. Paicu, Équation anisotrope de Navier-Stokes dans des espaces critiques, *Rev. Mat. Iberoamericana* **21** (2005), 179–235. MR 2155019. Zbl 1110.35060. Available at http://projecteuclid.org/getRecord?id=euclid.rmi/1114176232.

[32] G. Ponce, R. Racke, T. C. Sideris, and E. S. Titi, Global stability of large solutions to the 3D Navier-Stokes equations, *Comm. Math. Phys.* **159** (1994), 329–341. MR 1256992. Zbl 0795.35082. Available at http://projecteuclid.org/getRecord?id=euclid.cmp/1104254602.

[33] G. Raugel and G. R. Sell, Navier-Stokes equations on thin 3D domains. I. Global attractors and global regularity of solutions, *J. Amer. Math. Soc.* **6** (1993), 503–568. MR 1179539. Zbl 0787.34039. doi: 10.2307/2152776.

[34] M. Sammartino and R. E. Caflisch, Zero viscosity limit for analytic solutions, of the Navier-Stokes equation on a half-space. I. Existence for Euler and Prandtl equations, *Comm. Math. Phys.* **192** (1998), 433–461. MR 1617542. Zbl 0913.35103. doi: 10.1007/s002200050304.

[35] M. R. Ukhovskii and V. I. Iudovich, Axially symmetric flows of ideal and viscous fluids filling the whole space, *J. Appl. Math. Mech.* **32** (1968), 52–61. MR 0239293. Zbl 0172.53405. doi: 10.1016/0021-8928(68)90147-0.

[36] F. B. Weissler, The Navier-Stokes initial value problem in L^p, *Arch. Rational Mech. Anal.* **74** (1980), 219–230. MR 591222. Zbl 0454.35072. doi: 10.1007/BF00280539.

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