# Cycle integrals of the $j$-function and mock modular forms

By W. Duke, Ö. Imamoḡlu, and Á. Tóth

## Abstract

In this paper we construct certain mock modular forms of weight $1/2$ whose Fourier coefficients are given in terms of cycle integrals of the modular $j$-function. Their shadows are weakly holomorphic forms of weight $3/2$. These new mock modular forms occur as holomorphic parts of weakly harmonic Maass forms. We also construct a generalized mock modular form of weight $1/2$ having a real quadratic class number times a regulator as a Fourier coefficient. As an application of these forms we study holomorphic modular integrals of weight $2$ whose rational period functions have poles at certain real quadratic integers. The Fourier coefficients of these modular integrals are given in terms of cycle integrals of modular functions. Such a modular integral can be interpreted in terms of a Shimura-type lift of a mock modular form of weight $1/2$ and yields a real quadratic analogue of a Borcherds product.

## 1. Introduction

Mock modular forms, especially those of weight $1/2$, have attracted much attention recently. This is mostly due to the discovery of Zwegers [47], [46] that Ramanujan's mock theta functions can be completed to become modular by the addition of a certain nonholomorphic function on the upper half plane $\mathcal{H}$. This complement is associated to a modular form of weight $3/2$, the *shadow* of the mock theta function. Consider, for example, the $q$-series

$$f(\tau) = q^{-1/24} \sum_{n \geq 0} \frac{q^{n^2}}{(1+q)^2 \cdots (1+q^n)^2} \qquad \left(q = e(\tau) = e^{2\pi i \tau}, \tau \in \mathcal{H}\right).$$

Up to the factor $q^{-1/24}$ this is one of Ramanujan's original mock theta functions. The shadow of $f$ is the weight $3/2$ cusp form (a unary theta series)

$$g(\tau) = \sum_{n \in 1 + 6\mathbb{Z}} n \, q^{n^2/24};$$

it is proved in [46] that the completion

$$\hat{f}(\tau) = f(\tau) + g^*(\tau)$$

transforms like a modular form of weight $1/2$ for $\Gamma(2)$, the well-known congruence subgroup of $\Gamma = \mathrm{PSL}(2,\mathbb{Z})$, when

$$g^*(\tau) = \sum_{n \in 1+6\mathbb{Z}} \mathrm{sgn}(n)\, \beta(\tfrac{n^2 y}{6})\, q^{-n^2/24} \qquad \big(y = \mathrm{Im}\,\tau\big).$$

Here $\beta(x)$ is defined for $x > 0$ in terms of the complementary error function and the standard incomplete gamma function by

$$(1.1) \qquad \beta(x) = \mathrm{erfc}(\sqrt{\pi x}) = \frac{1}{\sqrt{\pi}} \Gamma(\tfrac{1}{2}, \pi x), \quad \text{where} \quad \Gamma(s,x) = \int_x^\infty t^s e^{-t}\, \tfrac{dt}{t}.$$

Observe that the Fourier expansion of the nonholomorphic "Eichler integral" $g^*(\tau)$ mirrors that of $g(\tau)$. In addition to leading to a number of new results about mock theta functions, the work of Zwegers has stimulated the study of other kinds of mock modular forms as well (see [33] and [45] for surveys on some of these developments).

In this paper we will consider mock modular forms of weight $1/2$ for $\Gamma_0(4)$. In some sense this is the simplest case, but has not been treated before because the associated shadows, if not zero, cannot be cusp forms. We will show that they are nevertheless quite interesting, and have remarkable connections with cycle integrals of the modular $j$-function and modular integrals having rational period functions. First let us define mock modular forms precisely in this context. Let

$$\theta(\tau) = 1 + 2\,q + 2\,q^4 + 2\,q^9 + 2\,q^{16} + \cdots$$

be the Jacobi theta series, which is a modular form of weight $1/2$ for $\Gamma_0(4)$. Set

$$(1.2) \qquad\qquad j(\gamma, \tau) = \theta(\gamma\tau)/\theta(\tau) \quad \text{for } \gamma \in \Gamma_0(4).$$

For $k \in 1/2 + \mathbb{Z}$ say that $f$ defined on $\mathcal{H}$ has weight $k$ for $\Gamma_0(4)$ (or simply has weight $k$) if

$$f(\gamma\tau) = j(\gamma, \tau)^{2k} f(\tau) \qquad \text{for all} \qquad \gamma \in \Gamma_0(4).$$

Let $M_k^!$ be the space comprising functions holomorphic on $\mathcal{H}$ of weight $k$ for $\Gamma_0(4)$ whose Fourier coefficients $a(n)$ in the expansion $f(\tau) = \sum_n a(n)q^n$ vanish unless $(-1)^{k-1/2}n \equiv 0, 1 \pmod 4$ and $n > N$ for some $N$.

Specializing now to the case of weight $1/2$, let $\mathcal{E}(z)$ be the entire function given by any of the following formulas

$$(1.3) \qquad\qquad \mathcal{E}(z) = \int_0^1 e^{-\pi z u^2}\, du = \frac{\mathrm{erf}(\sqrt{\pi z})}{2\sqrt{z}} = \sum_{n=0}^\infty \frac{(-\pi z)^n}{(2n+1)n!}.$$

For any $g(\tau) = \sum_n b_n q^n \in M_{3/2}^!$ we define the nonholomorphic Eichler integral of $g$ by

$$(1.4) \qquad g^*(\tau) = -4\sqrt{y} \sum_{n \leq 0} b_n \, \mathcal{E}(4ny) \, q^{-n} + \sum_{n>0} \frac{b_n}{\sqrt{n}} \beta(4ny) q^{-n}.$$

Let $f(\tau) = \sum_n a_n q^n$ be holomorphic on $\mathcal{H}$ and such that its coefficients $a_n$ vanish unless $(-1)^{k-1/2} n \equiv 0, 1 (\mathrm{mod}\, 4)$ and $n > N$ for some $N$. We will say that $f(\tau)$ is a mock modular form of weight $1/2$ for $\Gamma_0(4)$ if there exists a $g \in M_{3/2}^!$, its shadow, so that

$$\hat{f}(\tau) = f(\tau) + g^*(\tau)$$

has weight $1/2$ for $\Gamma_0(4)$. Denote by $\mathbb{M}_{1/2}$ the space of all mock modular forms of weight $1/2$ for $\Gamma_0(4)$. Obviously $M_{1/2}^! \subset \mathbb{M}_{1/2}$ but it is not at all clear that there are any nonmodular mock modular forms.

We will show that they do exist and that they are related to the work of Borcherds and Zagier on traces of singular moduli of the classical $j$-function

$$j(\tau) = q^{-1} + 744 + 196884 \, q + \cdots.$$

It is well-known and easily shown that $\mathbb{C}[j]$ has a unique basis $\{j_m\}_{m \geq 0}$ whose members are of the form $j_m(\tau) = q^{-m} + \mathrm{O}(q)$. For example,

$$(1.5) \qquad j_0 = 1, \quad j_1 = j - 744, \quad j_2 = j^2 - 1488j + 159768, \ldots.$$

Here $j_1(\tau)$ is the normalized Hauptmodule for $\Gamma$. In this paper, unless otherwise specified, $d$ is assumed to be an integer $d \equiv 0, 1 (\mathrm{mod}\, 4)$ and is called a discriminant if $d \neq 0$. For each discriminant $d$ let $\mathcal{Q}_d$ be the set of integral binary quadratic forms of discriminant $d$ that are positive definite if $d < 0$. The forms are acted on as usual by $\Gamma$, resulting in finitely many classes $\Gamma \backslash \mathcal{Q}_d$. Let $\Gamma_Q$ be the group of automorphs of $Q$ (see §3 for more details).

Suppose that $d < 0$. For $Q \in \mathcal{Q}_d$ and $\tau_Q$ a root of $Q$ in $\mathcal{H}$, the numbers $j_1(\tau_Q)$ are known by the classical theory of complex multiplication to form a $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-invariant set of algebraic integers, so that their weighted sum

$$(1.6) \qquad \mathrm{Tr}_d(j_1) = \sum_{Q \in \Gamma \backslash \mathcal{Q}_d} |\Gamma_Q|^{-1} j_1(\tau_Q)$$

is an integer. A beautiful theorem of Zagier [44] asserts that these integers give the Fourier coefficients of a weight $3/2$ weakly holomorphic form $\mathbf{T}_-(\tau) \in M_{3/2}^!$:

$$(1.7) \qquad \mathbf{T}_-(\tau) = -q^{-1} + 2 + \sum_{d \leq 0} \mathrm{Tr}_d(j_1)(d) q^{|d|}$$

$$= -q^{-1} + 2 - 248 \, q^3 + 492 \, q^4 - 4119 \, q^7 + 7256 \, q^8 + \cdots.$$

A natural question is whether one can give a similar statement for the numbers $\mathrm{Tr}_d(j_1)$ defined for nonsquare $d > 0$ by

$$(1.8) \qquad \mathrm{Tr}_d(j_1) = \tfrac{1}{2\pi} \sum_{Q \in \Gamma \backslash \mathcal{Q}_d} \int_{C_Q} j_1(\tau) \tfrac{d\tau}{Q(\tau,1)}.$$

Here $C_Q$ is any smooth curve from any $z \in \mathcal{H}$ to $g_Q z$, where $g_Q$ is a certain distinguished generator (see (3.1)) of the infinite cyclic group $\Gamma_Q$ of automorphs of $Q$. Note: $\mathrm{Tr}_d(j_1)$ is well-defined. We will see that the generating function

$$(1.9) \qquad \mathbf{T}_+(\tau) = \sum_{d>0} \mathrm{Tr}_d(j_1) q^d$$

(with a suitable definition of $\mathrm{Tr}_d(j_1)$ when $d$ is a perfect square) defines a mock modular form of weight $1/2$ for $\Gamma_0(4)$ with shadow $\mathbf{T}_-(\tau)$ from (1.7).

THEOREM 1. *The function* $\widehat{\mathbf{T}}_+(\tau)$ *on* $\mathcal{H}$ *defined by*

$$\widehat{\mathbf{T}}_+(\tau) = \mathbf{T}_+(\tau) + \mathbf{T}_-^*(\tau)$$

$$= \sum_{d>0} \mathrm{Tr}_d(j_1)\, q^d + 4\sqrt{y}\, \mathcal{E}(-4y)\, q - 8\sqrt{y} + \sum_{d<0} \frac{\mathrm{Tr}_d(j_1)}{\sqrt{|d|}}\, \beta(4|d|\, y)\, q^d$$

*has weight* $1/2$ *for* $\Gamma_0(4)$.

Zagier [44] showed that $g_1(\tau) = \mathbf{T}_-(\tau)$ from (1.7) is the first member of a basis $\{g_d\}_{0<d\equiv 0,1(4)}$ for $M_{3/2}^!$, where for each $d > 0$ the function $g_d(\tau)$ is uniquely determined by having a $q$-expansion of the form[1]

$$(1.10) \qquad g_d(\tau) = -q^{-d} + \sum_{\substack{n\le 0 \\ n\equiv 0,1(\mathrm{mod}\,4)}} a(d,n)q^{|n|}.$$

We define $a(d,n) = 0$ unless $d, n \equiv 0, 1 \pmod 4$. For $d \le 0$ consider the "dual" form

$$(1.11) \qquad f_d(\tau) = q^d + \sum_{n>0} a(n,d)q^n.$$

As shown in [44], the set $\{f_d\}_{d\le 0}$ coincides with the basis given by Borcherds [2] for $M_{1/2}^!$. Thus $f_0(\tau) = \theta(\tau)$ and the first few terms of the next function are

$$f_{-3}(\tau) = q^{-3} - 248q + 26752q^4 - 85995q^5 + 1707264q^8 + \cdots.$$

We will show that Borcherds' basis extends naturally to a basis for $\mathbb{M}_{1/2}$. The construction of this extension relies heavily on the spectral theory of Maass forms.

---

[1]This is the negative of the $g_d(\tau)$ defined in [44].

THEOREM 2. *For each positive discriminant $d$ there is a unique mock modular form $f_d(\tau) \in \mathbb{M}_{1/2}$ with shadow $g_d(\tau)$ having a Fourier expansion of the form*

$$(1.12) \qquad f_d(\tau) = \sum_{n > 0} a(n, d) q^n.$$

*These Fourier coefficients $a(n, d)$ satisfy $a(n, d) = a(d, n)$. The set $\{f_d\}$ where $d$ runs over all integers $\equiv 0, 1 \pmod 4$ gives a basis for $\mathbb{M}_{1/2}$.*

We have thus defined $a(n, d)$ for all $d, n$ with $n > 0$. We use them to evaluate certain twisted traces, which we now define. Suppose that $D > 0$ is a fundamental discriminant. There is a function $\chi_D : \mathcal{Q}_{dD} \to \{-1, 1\}$ defined below in (3.2) that restricts to a real character (a genus character) on the group of primitive classes and can be used to define a general twisted trace for $dD$ not a square by

$$(1.13) \qquad \mathrm{Tr}_{d,D}(j_m) = \begin{cases} \frac{1}{\sqrt{D}} \sum \chi(Q) |\Gamma_Q|^{-1} j_m(\tau_Q) & \text{if } dD < 0, \\ \frac{1}{2\pi} \sum \chi(Q) \int_{C_Q} j_m(\tau) \frac{d\tau}{Q(\tau,1)} & \text{if } dD > 0, \end{cases}$$

each sum being over $Q \in \Gamma \backslash \mathcal{Q}_{dD}$. We have the following evaluation, which generalizes a well-known result of Zagier [44, (25)] to include positive $d$.

THEOREM 3. *Let $a(n, d)$ be the mock modular coefficients defined in (1.11) and (1.12). Suppose that $m \geq 1$. For $d \equiv 0, 1 \pmod 4$ and fundamental $D > 0$ with $dD$ not a square, we have*

$$(1.14) \qquad \mathrm{Tr}_{d,D}(j_m) = \sum_{n | m} \left( \frac{D}{m/n} \right) n\, a(n^2 D, d).$$

Together with Theorem 2, Theorem 3 implies Theorem 1 after we define $\mathrm{Tr}_d(j_1)$ to be equal to $a(d, 1)$ when $d$ is a perfect square. The proof we give uses Poincaré series and a Kloosterman sum identity that generalizes a well-known result of Salié. In particular, for nonsquare $dD$ with $D > 0$, Theorem 3 gives

$$(1.15) \qquad a(D, d) = \begin{cases} \frac{1}{\sqrt{D}} \sum \chi(Q) |\Gamma_Q|^{-1} j_1(\tau_Q) & \text{if } dD < 0, \\ \frac{1}{2\pi} \sum \chi(Q) \int_{C_Q} j_1(\tau) \frac{d\tau}{Q(\tau,1)} & \text{if } dD > 0, \end{cases}$$

where each sum is over $Q \in \Gamma \backslash \mathcal{Q}_{dD}$.

Concerning the case $m = 0$, there exists an interesting "second order" mock modular form $\mathbf{Z}_+(\tau)$ of weight $1/2$ that is almost, but not quite, in $\mathbb{M}_{1/2}$ with Fourier expansion

$$(1.16) \qquad \mathbf{Z}_+(\tau) = \sum_{d > 0} \mathrm{Tr}_d(1) q^d.$$

Here $\mathrm{Tr}_d(1)$ must be defined suitably for square $d$ while for $d > 1$ a fundamental discriminant we have

$$\mathrm{Tr}_d(1) = \pi^{-1} d^{-1/2} h(d) \log \epsilon_d,$$

where $h(d)$ is the narrow class number of $\mathbb{Q}(\sqrt{d})$ and $\epsilon_d$ is its smallest unit $> 1$ of norm 1. A (generalized) shadow of $\mathbf{Z}_+(\tau)$ is the completion of the mock modular form $\mathbf{Z}_-(\tau)$ of weight $3/2$ with shadow $\theta(\tau)$ discovered by Zagier in 1975 [42] (see also [16]) whose Fourier expansion is

$$(1.17) \qquad \mathbf{Z}_-(\tau) = \sum_{d \leq 0} \mathrm{Tr}_d(1) q^{|d|}.$$

Here for any $d \leq 0$ we have that $\mathrm{Tr}_d(1) = H(|d|)$, the usual Hurwitz class number, whose first few values are given by

$$H(0) = -\tfrac{1}{12}, \quad H(3) = \tfrac{1}{3}, \quad H(4) = \tfrac{1}{2}, \quad H(7) = 1, \dots.$$

The completion of $\mathbf{Z}_-(\tau)$, which has weight $3/2$ for $\Gamma_0(4)$, is given by

$$(1.18) \qquad \widehat{\mathbf{Z}}_-(\tau) = \mathbf{Z}_-(\tau) + \frac{1}{16\pi} \sum_{n \in \mathbb{Z}} \Gamma(-\tfrac{1}{2}, 4\pi n^2 y) q^{-n^2}.$$

Define for $y > 0$ the special function

$$\alpha(y) = \frac{\sqrt{y}}{4\pi} \int_0^\infty t^{-1/2} \log(1+t) e^{-\pi y t} dt.$$

The next result shows that $\mathbf{Z}_+(\tau)$ from (1.16) has $\widehat{\mathbf{Z}}_-(\tau)$ as a generalized shadow (to be made precise later).

THEOREM 4. *The function* $\widehat{\mathbf{Z}}_+(\tau)$ *whose Fourier expansion is given by*

$$(1.19)$$
$$\widehat{\mathbf{Z}}_+(\tau) = \sum_{d>0} \mathrm{Tr}_d(1)\, q^d + \frac{\sqrt{y}}{3} + \sum_{d<0} \frac{\mathrm{Tr}_d(1)}{\sqrt{|d|}} \beta(4|d|y) q^d + \sum_{n \neq 0} \alpha(4n^2 y) q^{n^2} - \tfrac{1}{4\pi} \log y$$

*has weight* $1/2$ *for* $\Gamma_0(4)$.

The automorphic nature of $\widehat{\mathbf{Z}}_+(\tau)$ gives some reason to hope that there might be a connection between the cycle integrals of $j$ and abelian extensions of real quadratic fields. So far this hope has not been realized.

Finally, there is an unexpected connection between mock modular forms of weight $1/2$ and modular integrals having rational period functions. Define for each $d \equiv 0, 1 \pmod 4$

$$(1.20) \qquad F_d(\tau) = -\mathrm{Tr}_d(1) - \sum_{m \geq 1} \left( \sum_{n|m} n\, a(n^2, d) \right) q^m.$$

Note that $F_d(\tau)$ is the derivative of the formal Shimura lift of $f_d$. When $d < 0$ Borcherds showed that $F_d$ is a meromorphic modular form of weight 2 for $\Gamma$ having a simple pole with residue $|\Gamma_Q|^{-1}$ at each point $\tau_Q \in \mathcal{H}$ of discriminant $d$. Thus one has corresponding properties of the infinite product

$$q^{-\mathrm{Tr}_d(1)} \prod_{m \geq 1} (1 - q^m)^{a(m^2, d)}.$$

In case $d = 0$ one finds that this product is $\Delta(\tau)^{1/12}$, and we have that

$$F_0(\tau) = \tfrac{1}{12} - 2 \sum_{n \geq 1} \sigma(m) q^m = \tfrac{1}{12} E_2(\tau).$$

This is a holomorphic modular integral of weight 2 with a rational period function

$$F_0(\tau) - \tau^{-2} F_0(-\tfrac{1}{\tau}) = -\tfrac{1}{2\pi i}\, \tau^{-1}.$$

THEOREM 5. *For each $d > 0$ not a square the function $F_d$ defined in (1.20) is a holomorphic modular integral of weight 2 with a rational period function*

$$(1.21) \qquad F_d(\tau) - \tau^{-2} F_d(-\tfrac{1}{\tau}) = \frac{1}{\pi} \sum_{\substack{c < 0 < a \\ b^2 - 4ac = d}} (a\tau^2 + b\tau + c)^{-1}.$$

*The Fourier expansion of $F_d(\tau)$ can be expressed in the form*

$$F_d(\tau) = - \sum_{m \geq 0} \mathrm{Tr}_d(j_m)\, q^m.$$

Note that the period function has simple poles at certain real quadratic integers of discriminant $d$, in analogy to the behavior of $F_d(\tau)$ when $d < 0$. The existence of a holomorphic $F$ satisfying (1.21) with growth conditions was proved by Knopp [24], [25]. He used a certain Poincaré series built out of cocycles, which was used earlier by Eichler [11], to construct it. However, it appears to be very difficult to compute $F$ explicitly from this construction. At the end of their paper [7], Choie and Zagier raised the problem of explicit construction of a modular integral with a given rational period function. Parson [34] gave a more direct construction in weights $k > 2$ using series of the form

$$\sum_{a > 0} (a\tau^2 + b\tau + c)^{-k/2},$$

which are partial versions of certain hyperbolic Poincaré series studied by Zagier, but they do not converge when $k = 2$. In any case, the expression of the Fourier coefficients as sums of cycle integrals is not immediate from this construction, although it is possible to deduce such expressions this way, at least in higher weights, using methods from this paper. For the rational period functions that occur in (1.21) the modular integral given by $F_d(\tau)$ also gives a real quadratic analogue of (the logarithmic derivative of) the Borcherds product.

It is interesting to examine numerical values of the traces $\mathrm{Tr}_d(j_m)$. We remark that for $d > 1$ fundamental we have the identity

$$(1.22) \qquad\qquad \mathrm{Tr}_d(j_m) = \sum_{n \mid m} \left(\tfrac{d}{m/n}\right) n^{-1}\, \mathrm{Tr}_{n^2 d}(j_1).$$

| m | $\mathrm{Tr}_{-3}(j_m)$ | $\mathrm{Tr}_{-4}(j_m)$ | $\mathrm{Tr}_{-7}(j_m)$ | $\mathrm{Tr}_{-8}(j_m)$ |
|---|---|---|---|---|
| 0 | 1/3 | 1/2 | 1 | 1 |
| 1 | −248 | 492 | −4119 | 7256 |
| 2 | 53256 | 287244 | 16572393 | 52255768 |
| 3 | −12288992 | 153540528 | −67515202851 | 377674781024 |

Table 1. Traces for $d < 0$

| m | $\mathrm{Tr}_5(j_m)$ | $\mathrm{Tr}_8(j_m)$ | $\mathrm{Tr}_{12}(j_m)$ | $\mathrm{Tr}_{13}(j_m)$ | $\mathrm{Tr}_{17}(j_m)$ | $\mathrm{Tr}_{20}(j_m)$ |
|---|---|---|---|---|---|---|
| 0 | 0.13700 | 0.19837 | 0.24202 | 0.21095 | 0.32343 | 0.33097 |
| 1 | −5.16163 | −6.76613 | −8.27912 | −6.49263 | −10.65828 | −8.36253 |
| 2 | −11.56343 | −17.92434 | −21.49601 | −19.16428 | −28.46829 | −23.93151 |
| 3 | −14.31225 | −19.50182 | −24.84575 | −21.99742 | −34.17310 | −29.69296 |

Table 2. Traces for $d > 0$

This result is a consequence of a general identity on Hecke operators proved in [43]. As a numerical check, the reader can verify that the identity (1.22) holds for $d = 5$, $D = 1$ and $m = 2$. For $d = 20$ there are two classes, represented by $[1, 4, -1]$ and the nonprimitive form $[2, 2, -2]$, and (1.22) amounts to the curious identity

$$(1.23) \qquad \int_{C_{[1,4,-1]}} j_1(\tau) \frac{2d\tau}{\tau^2 + 4\tau - 1} = \int_{C_{[1,1,-1]}} j_2(\tau) \frac{d\tau}{\tau^2 + \tau - 1}.$$

In general, a comparison shows that the traces for positive discriminants are generally much smaller than those for negative discriminants and appear to exhibit some regular growth behavior in both $d$ and $m$. For nonsquare $d > 1$ and $m \in \mathbb{Z}^+$, we will see that we have the conditionally (and slowly) convergent expansion

$$(1.24) \qquad \mathrm{Tr}_d(j_m) = -24\sigma_1(m)\mathrm{Tr}_d(1) + d^{-1/2} \sum_{0 < c \equiv 0(4)} S_m(d; c) \sin\left(\frac{4\pi m\sqrt{d}}{c}\right),$$

where

$$S_m(d; c) = \sum_{b^2 \equiv d \pmod c} e\left(\frac{2mb}{c}\right).$$

It is natural to expect that the first term dominates as either $m$ or $d$ gets large, which would in particular imply that as $d \to \infty$ through nonsquares we have the asymptotic formula

$$(1.25) \qquad \mathrm{Tr}_d(j_m) \sim -24\sigma_1(m)\,\mathrm{Tr}_d(1).$$

Such a result is known for negative discriminants (see [8]) except that then there is a large main term that grows like $e^{\pi m\sqrt{|d|}}$. If true, this asymptotic would indicate that there is tremendous cancellation in the integrals of $j$ over

the cycles when $d$ is large, since $\mathrm{Tr}_d(1) \ll_\epsilon d^\epsilon$ and $j_m$ has exponential growth in the cusp.

After seeing our paper, Y. Manin and D. Zagier told us that M. Kaneko also performed calculations of cycle integrals of the $j$ function. Kaneko kindly sent us his paper [22], in which he calculated the individual cycle integrals numerically and observed some interesting behaviour in the distribution of their values. This behavior appears to us to be consistent with the conjectured asymptotic given above.

Finally, we remark that there is an obvious similarity between Theorem 3 and the formula of Katok and Sarnak [23] for the traces of a Maass cusp form. The main difference is that there are no Hecke eigenforms in our setting. Nevertheless, the method we use to prove Theorem 3, which involves Poincaré series and identities between Kloosterman sums, can be applied to give another proof of the Katok-Sarnak formula. It would be interesting to approach our results using a regularized theta lift. (See [14], [6] for the use of regularized theta lifts in the negative discriminants case.) Another interesting problem is to understand the nature of the traces on square discriminants.

## 2. **Weakly harmonic modular forms**

We begin by proving Theorem 2 using the theory of weakly harmonic forms. Set $k \in 1/2 + \mathbb{Z}$. If $f$ of weight $k$ for $\Gamma_0(4)$ is smooth, for example, it will have a Fourier expansion in each cusp. For the cusp at $i\infty$ we have *the* Fourier expansion

$$(2.1) \qquad f(\tau) = \sum_n a(n; y) e(nx)$$

which, if $f$ is holomorphic, has $a(n; y) = a(n)e(niy)$. Set

$$(2.2) \quad f^{\mathrm{e}}(\tau) = \sum_{n \equiv 0(2)} a(n; \tfrac{y}{4}) e(\tfrac{nx}{4}) \quad \text{and} \quad f^{\mathrm{o}}(\tau) = \sum_{n \equiv 1(2)} a(n; \tfrac{y}{4}) e(\tfrac{n}{8}) e(\tfrac{nx}{4}).$$

Suppose that the Fourier coefficients $a(n; y)$ satisfy the *plus space condition*, meaning that they vanish unless $(-1)^{k-1/2}n \equiv 0, 1 \pmod 4$. An easy extension of arguments given in [28, p. 190] shows that such an $f$ satisfies

$$(2.3) \qquad (\tfrac{2\tau}{i})^{-k} f(-\tfrac{1}{4\tau}) = \alpha f^{\mathrm{e}}(\tau) \qquad \text{and} \qquad (\tfrac{2\tau+1}{i})^{-k} f(\tfrac{\tau}{2\tau+1}) = \alpha f^{\mathrm{o}}(\tau),$$

where

$$\alpha = (-1)^{\lfloor \frac{2k+1}{4} \rfloor} 2^{-k+\frac{1}{2}}.$$

In particular, the behavior of such an $f$ at the cusps $0$ and $1/2$ is determined by that at $i\infty$. Thus to check that such a form is weakly holomorphic, meaning it is holomorphic on $\mathcal{H}$ and meromorphic in the cusps, one only needs to look at the Fourier expansion at $i\infty$, as we have done in the introduction. As there, let $M_k^!$ denote the space of all such forms. Let $M_k^+ \subset M_k^!$ denote the subspace of holomorphic forms (having no pole in the cusps) and $S_k^+ \subset M_k^+$ the subspace of cusp forms (having zeros there).

Consider the Maass-type differential operator $\xi_k$ defined for any $k \in \mathbb{R}$ through its action on a differentiable function $f$ on $\mathcal{H}$ by

$$\xi_k(f) = 2iy^k \overline{\frac{\partial f}{\partial \bar{\tau}}}.$$

This operator is studied in some detail in [5]. It is easily checked that

$$\xi_k\big((\gamma\tau + \delta)^{-k} f(g\tau)\big) = (\gamma\tau + \delta)^{k-2} (\xi_k f)(g\tau)$$

for any $g \in \mathrm{PSL}(2, \mathbb{R})$. Thus if $f(\tau)$ has weight $k$ for $\Gamma_0(4)$, then $\xi_k f$ has weight $2 - k$ and $\xi_k f = 0$ if and only if $f$ is holomorphic. Also $\xi_k$ preserves the plus space condition. The weight $k$ Laplacian can be conveniently defined by

$$(2.4) \qquad\qquad \Delta_k = -\xi_{2-k} \circ \xi_k.$$

Specializing now to $k = 1/2$, suppose that $h$ is a real analytic function on $\mathcal{H}$ of weight $1/2$ for $\Gamma_0(4)$ that is harmonic on $\mathcal{H}$ in the sense that

$$(2.5) \qquad\qquad \Delta_{1/2} h = 0.$$

By separation of variables every such $h$ has a (unique) Fourier expansion in the cusp at $\infty$ of the form

$$(2.6) \qquad h(\tau) = \sum_n b(n) \mathcal{M}_n(y) e(nx) + \sum_n a(n) \mathcal{W}_n(y) e(nx).$$

The functions $\mathcal{W}_n(y)$ and $\mathcal{M}_n(y)$ in the Fourier expansion (2.6) are defined in terms of the functions $\beta(x)$ and $\mathcal{E}(z)$ from (1.1) and (1.3) by

$$(2.7) \qquad \mathcal{W}_n(y) = e^{-2\pi ny} \begin{cases} |n|^{-\frac{1}{2}} \beta(4|n|y) & \text{if } n < 0, \\ -4y^{\frac{1}{2}} & \text{if } n = 0, \\ n^{-\frac{1}{2}} & \text{if } n > 0, \end{cases}$$

$$(2.8) \qquad \mathcal{M}_n(y) = e^{-2\pi ny} \begin{cases} 1 - \beta(4|n|\, y) & \text{if } n < 0, \\ 1 & \text{if } n = 0, \\ 4(ny)^{\frac{1}{2}} \mathcal{E}(-4ny) & \text{if } n > 0. \end{cases}$$

We remark that $\mathcal{W}_n(y)$ and $\mathcal{M}_n(y)$ are special cases of Whittaker functions (see (2.16)) and we use the notation $\mathcal{W}$ and $\mathcal{M}$ to suggest this relation. More

importantly, definitions (2.7) and (2.8) make possible the complete symmetry of the Fourier coefficients of the basis to be given in the next proposition. It becomes clear after working with them that one can define the normalization for the Fourier coefficients in different reasonable ways, each with advantages and disadvantages. Note that the function $\mathcal{W}_n(y)$ is exponentially decaying while $\mathcal{M}_n(y)$ is exponentially growing in $y$ (see (A.4)).

Let $H_{1/2}^!$ denote the space of all real analytic functions on $\mathcal{H}$ of weight $1/2$ for $\Gamma_0(4)$ that satisfy (2.5), whose Fourier coefficients at $\infty$ are supported on integers $n$ with $n \equiv 0, 1 \pmod 4$ and that have only finitely many nonzero coefficients $b(n)$. As before this is enough to control bad behavior in the other cusps. We will call such an $h \in H_{1/2}^!$ weakly harmonic.[2] This space was identified by Bruinier and Funke [6] as being interesting arithmetically. It follows easily from its general properties that $\xi_{1/2}$ maps $H_{1/2}^!$ to $M_{3/2}^!$ with kernel $M_{1/2}^!$. This is also directly visible after a calculation from (2.8) and (2.7) yields the formulas
(2.9)

$$\xi_{1/2}\big(\mathcal{M}_n(y)e(nx)\big) = 2|n|^{\frac{1}{2}}q^{-n}, \qquad \xi_{1/2}\big(\mathcal{W}_n(y)e(nx)\big) = \begin{cases} 0 & \text{if } n > 0, \\ -2q^{|n|} & \text{if } n \leq 0. \end{cases}$$

Given $h$ in (2.6) with $b(n) = 0$ for all $n$, we infer that $\xi_{1/2}h \in M_{3/2}^+ = \{0\}$. This implies that $h \in S_{1/2}^+ = \{0\}$, and proves the following uniqueness result.

LEMMA 1. *If $h \in H_{1/2}^!$ has Fourier expansion*

$$(2.10) \qquad h(\tau) = \sum_n b(n)\mathcal{M}_n(y)e(nx) + \sum_n a(n)\,\mathcal{W}_n(y)e(nx),$$

*then $h = 0$ if and only if $b(n) = 0$ for all $n \equiv 0, 1 \pmod 4$.*

It is now easy to explain the relation between mock modular forms and weakly harmonic ones (cf. [45]). It follows easily from (2.7), (2.8) and (2.9), or directly, that for $g(\tau) \in M_{3/2}^!$,

$$\xi_{1/2}\,g^*(\tau) = -2\,g(\tau),$$

where $g^*(\tau)$ was defined in (1.4). As a consequence we see that if $f \in \mathbb{M}_{1/2}$ and if $\hat{f} = f + g^*$ is its completion, then $\hat{f} \in H_{1/2}^!$ since $\xi_{1/2}\,\hat{f}(\tau) = -2g(\tau)$; thus $\Delta_{1/2}\hat{f} = 0$. Also $\hat{f}(\tau)$ satisfies the plus space condition. In fact it is easy to see that $f \mapsto \hat{f}$ defines an isomorphism from $\mathbb{M}_{1/2}$ to $H_{1/2}^!$. Given $h \in H_{1/2}^!$ let

---

[2]The definition of harmonic weak Maass forms, for example as given in [3] and elsewhere, is more restrictive and does not apply to the nonholomorphic $h \in H_{1/2}^!$, so we use the terminology *weakly harmonic* to avoid confusion.

$g = -\frac{1}{2}\xi_{1/2}(h) \in M^!_{3/2}$ and take $h^+ = h - g^*$. It is easily checked that $h \mapsto h^+$ gives the inverse of $f \mapsto \hat{f}$. Call $h^+$ the *holomorphic part* of $h$. In terms of the Fourier expansion (2.6),

$$(2.11) \qquad h^+(\tau) = \sum_{n \leq 0} b(n)q^n + \sum_{n > 0} a(n)n^{-1/2}q^n.$$

The next result gives one natural basis for $H^!_{1/2}$.

PROPOSITION 1. *For each $d \equiv 0,1 \pmod 4$ there is a unique $h_d \in H^!_{1/2}$ with Fourier expansion of the form*

$$(2.12) \qquad h_d(\tau) = \mathcal{M}_d(y)e(dx) + \sum_{n \equiv 0,1(4)} a_d(n)\mathcal{W}_n(y)e(nx).$$

*The set $\{h_d\}_{d \equiv 0,1\,(4)}$ forms a basis for $H^!_{1/2}$. The coefficients $a_d(n)$ satisfy the symmetry relation*

$$(2.13) \qquad a_d(n) = a_n(d)$$

*for all integers $n, d \equiv 0,1 \pmod 4$. When $d > 0$ we have*

$$(2.14) \qquad \xi_{1/2}\, h_d(\tau) = -2d^{\frac{1}{2}}\, g_d(\tau),$$

*where $g_d \in M^!_{3/2}$ has Fourier expansion given in (1.10).*

Theorem 2 is an immediate consequence of this proposition. We see that for $d \leq 0$ we have that $h_d = f_d$ from (1.11) and $a(n,d) = n^{-1/2}a_d(n)$ unless $n = d < 0$, in which case $a_d(d) = |d|^{1/2}$. If $d > 0$, let $f_d(\tau) = \sum_{n>0} a(n,d)q^n$ be the holomorphic part of $d^{-1/2}\, h_d$. This gives the $f_d(\tau)$ from Theorem 2 and we find that for $n > 0$ we have

$$(2.15) \qquad a(n,d) = (dn)^{-1/2}a_d(n).$$

We remark that the fact we quoted from [44] — that $\{f_d\}_{d \leq 0}$ from (1.11) gives the Borcherds basis for $M^!_{1/2}$ — also follows from the symmetry relation (2.13) and (2.14) of Proposition 1.

We now turn to the construction of $h_d$. We will give a uniform construction using Poincaré series. Due to some delicate convergence issues that arise from this approach, we will define them through analytic continuation. For fixed $s$ with $\mathrm{Re}(s) > 1/2$ and $n \in \mathbb{Z}$ let

$$\mathcal{M}_n(y,s) = \begin{cases} \Gamma(2s)^{-1}(4\pi|n|y)^{-\frac{1}{4}}M_{\frac{1}{4}\,\mathrm{sgn}\,n,s-\frac{1}{2}}(4\pi|n|y) & \text{if } n \neq 0, \\ y^{s-\frac{1}{4}} & \text{if } n = 0, \end{cases}$$

$$\mathcal{W}_n(y,s) = \begin{cases} |n|^{-\frac{3}{4}}\Gamma(s + \frac{\mathrm{sgn}\,n}{4})^{-1}(4\pi y)^{-\frac{1}{4}}W_{\frac{1}{4}\,\mathrm{sgn}\,n,s-\frac{1}{2}}(4\pi|n|y) & \text{if } n \neq 0, \\ \frac{2^{2s-\frac{1}{2}}}{(2s-1)\Gamma(2s-1/2)}\, y^{\frac{3}{4}-s} & \text{if } n = 0, \end{cases}$$

where $M$ and $W$ are the usual Whittaker functions (see Appendix A). By (A.6) and (A.7), for $n \neq 0$ we have that

$$(2.16) \qquad \mathcal{M}_n(y) = \mathcal{M}_n(y, 3/4) \text{ and } \mathcal{W}_n(y) = \mathcal{W}_n(y, 3/4),$$

where $\mathcal{M}_n(y)$ and $\mathcal{W}_n(y)$ were given in (2.8) and (2.7). However, $\mathcal{M}_0(y) = \mathcal{W}_0(y, 3/4)$ and $\mathcal{W}_0(y) = \mathcal{M}_0(y, 3/4)$.[3] We also need the usual $I$ and $J$-Bessel functions, defined for fixed $\nu$ and $y > 0$ by (see e.g. [29])

$$(2.17) \quad I_\nu(y) = \sum_{k=0}^{\infty} \frac{(y/2)^{\nu+2k}}{k!\,\Gamma(\nu+k+1)} \qquad \text{and} \qquad J_\nu(y) = \sum_{k=0}^{\infty} \frac{(-1)^k (y/2)^{\nu+2k}}{k!\,\Gamma(\nu+k+1)}.$$

For $m \in \mathbb{Z}$ let

$$(2.18) \qquad \psi_m(\tau, s) = \mathcal{M}_m(y, s) e(mx).$$

It follows from (A.3) and (2.4) that

$$\Delta_{1/2} \psi_m(\tau, s) = (s - \tfrac{1}{4})(\tfrac{3}{4} - s) \psi_m(\tau, s).$$

Define the Poincaré series

$$P_m(\tau, s) = \sum_{g \in \Gamma_\infty \backslash \Gamma_0(4)} j(g, \tau)^{-1} \psi_m(g\tau, s),$$

where $\Gamma_\infty$ is the subgroup fixing the cusp $\infty$. By (A.5) this series converges absolutely and uniformly on compacta for $\operatorname{Re} s > 1$. The function $P_0(\tau, s)$ is the usual weight $1/2$ Eisenstein series. It is clear that for $\operatorname{Re}(s) > 1$ and any $m$, the function $P_m(\tau, s)$ has weight $1/2$ and that $P_m$ satisfies

$$\Delta_{1/2} P_m(\tau, s) = (s - \tfrac{1}{4})(\tfrac{3}{4} - s) P_m(\tau, s).$$

As in [27], in order to get forms whose Fourier expansions are supported on $n \equiv 0, 1 \pmod 4$, we will employ the projection operator $\operatorname{pr}^+ = \frac{2}{3}(U_4 \circ W_4) + \frac{1}{3}$, where

$$(U_4 f)(\tau) = \tfrac{1}{4} \sum_{\nu \bmod 4} f\left(\tfrac{\tau+\nu}{4}\right) \qquad \text{and} \qquad (W_4 f)(\tau) = \left(\tfrac{2\tau}{i}\right)^{-1/2} f(-1/4\tau).$$

For each $d \equiv 0, 1 \pmod 4$ and $\operatorname{Re}(s) > 1$ define

$$(2.19) \qquad P_d^+(\tau, s) = \operatorname{pr}^+(P_d(\tau, s)).$$

PROPOSITION 2. *For any $d \equiv 0, 1 \pmod 4$ and* $\operatorname{Re}(s) > 1$ *the function $P_d^+(\tau, s)$ has weight $1/2$ and satisfies*

$$\Delta_{1/2} P_d^+(\tau, s) = (s - \tfrac{1}{4})(\tfrac{3}{4} - s) P_d^+(\tau, s).$$

---

[3]This notational switching is inessential but gives a cleaner statement of Proposition 1 and some other results.

*Its Fourier expansion is given by*

$$(2.20) \qquad P_d^+(\tau, s) = \mathcal{M}_d(y, s)e(dx) + \sum_{n \equiv 0, 1(4)} b_d(n, s)\mathcal{W}_n(y, s)e(nx),$$

*where*

$$(2.21) \qquad b_d(n, s) = \sum_{0 < c \equiv 0(4)} K^+(d, n; c)$$

$$\times \begin{cases} 2^{\frac{1}{2}}\pi|dn|^{\frac{1}{4}}c^{-1}I_{2s-1}\left(\frac{4\pi\sqrt{|dn|}}{c}\right) & \text{if } dn < 0, \\ 2^{\frac{1}{2}}\pi|dn|^{\frac{1}{4}}c^{-1}J_{2s-1}\left(\frac{4\pi\sqrt{|dn|}}{c}\right) & \text{if } dn > 0, \\ \pi^{s+\frac{1}{4}}|d+n|^{s-\frac{1}{4}}c^{-2s} & \text{if } dn = 0, \ d + n \neq 0, \\ 2^{\frac{1}{2}-2s}\pi^{\frac{1}{2}}\Gamma(2s)c^{-2s} & \text{if } d = n = 0, \end{cases}$$

*where $K^+(d, n; c)$ is the modified Kloosterman sum defined in (3.4) below. The sum defining each $b_d(n, s)$ is absolutely convergent.*

*Proof.* The first statement is clear. So is the last statement using the trivial bound for $K^+(d, n; c)$ and the definitions (2.17).

For the calculation of the Fourier expansion we employ the following lemma, whose proof is standard and follows from an application of Poisson summation using an integral formula found in [13, p. 176]. See [26, Lemma 2, p. 20] or [27] for the prototype result.

LEMMA 2. *Let $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}(2, \mathbb{R})$ have $c > 0$ and suppose that $\mathrm{Re}(s) > 1/2$. Then for $\psi_m$ defined in (2.18) with any $m \in \mathbb{Z}$, we have*

$$\sum_{r \in Z}(c(\tau + r) + d)^{-1/2}\psi_m\left(\frac{a(\tau + r) + b}{c(\tau + r) + d}, s\right)$$

$$= 2\pi i^{-1/2}\sum_{n \in \mathbb{Z}} e\left(\frac{am+nd}{c}\right)\mathcal{W}_n(y, s)e(nx)$$

$$\times \begin{cases} c^{-1}|mn|^{\frac{1}{4}}J_{2s-1}(4\pi\sqrt{|mn|}c^{-1}) & \text{if } mn > 0, \\ c^{-1}|mn|^{\frac{1}{4}}I_{2s-1}(4\pi\sqrt{|mn|}c^{-1}) & \text{if } mn < 0, \\ 2^{-\frac{1}{2}}\pi^{s-\frac{3}{4}}c^{-2s}|m+n|^{s-\frac{1}{4}} & \text{if } mn = 0, \ m + n \neq 0, \\ \pi^{-\frac{1}{2}}(2c)^{-2s}\Gamma(2s) & \text{if } m = n = 0, \end{cases}$$

*where both sides of the identity converge uniformly on compact subsets of $\mathcal{H}$.*

With this lemma, since the computation of the Fourier coefficients parallels so closely that given in [26, pp. 18–27] in the holomorphic case, we will omit the details. $\qquad\square$

It is a well-known consequence of the theory of the resolvent kernel that $P_d(\tau, s)$ has an analytic continuation in $s$ to $\mathrm{Re}(s) > 1/2$ except for possibly

finitely many simple poles in $(1/2, 1)$. These poles may only occur at points of the discrete spectrum of $\Delta_{1/2}$ on the Hilbert space consisting of weight $1/2$ functions $f$ on $\mathcal{H}$ that satisfy

$$\int_{\Gamma \backslash \mathcal{H}} |f(\tau)|^2 y \, d\mu < \infty \qquad (d\mu = y^{-2} dx \, dy),$$

and this space contains the residues.[4] It is easily seen from (2.19) that $P_d^+(\tau, s)$ also has an analytic continuation to $\mathrm{Re}(s) > 1/2$ with at most finitely many simple poles in $(1/2, 1)$. Actually, such poles can only occur in $(\frac{1}{2}, \frac{3}{4}]$, since by (3.8) and (2.17) the series in (2.21) giving the Fourier coefficient $b_d(n, s)$ converges absolutely for $\mathrm{Re}(s) > 3/4$. Thus for $\mathrm{Re}(s) > 1/2$ away from these poles, the function $P_d^+(\tau, s)$ has weight $1/2$ and satisfies

$$\Delta_{1/2} P_d^+(\tau, s) = (s - \tfrac{1}{4})(\tfrac{3}{4} - s) P_d^+(\tau, s).$$

Furthermore, a residue at $s = 3/4$ is a weight $1/2$ weakly harmonic form $f \in H_{1/2}^!$. In fact, the Fourier expansion of $f$ can be obtained from that of $P_d^+$ in (2.20) by taking residues term by term, a process that is easily justified using the integral representations for the Fourier coefficients since the convergence is uniform on compacta. This shows that the Fourier expansion of $f$ is supported on $n$ with $n \equiv 0, 1 \pmod 4$ and that it can have no exponentially growing terms. Another way to see these facts is to observe that $f$ is the projection of the residue of $P_d$, which comes from the discrete spectrum. Thus by Lemma 1 applied to $f - b(0)\theta$, we obtain the following result.

LEMMA 3. *For each $d$ and each $\tau \in \mathcal{H}$ the function $P_d^+(\tau, s)$ has an analytic continuation around $s = 3/4$ with at most a simple pole there with residue*

$$(2.22) \qquad \mathrm{res}_{s=3/4} P_d^+(\tau, s) = \rho_d \, \theta(\tau),$$

*where $\rho_d \in \mathbb{C}$.*

When $d = 0$, this result is well-known. In fact, $b_0(n, s)$ can be computed in terms of Dirichlet $L$-functions. We have the following formulas (see e.g. [19]).

LEMMA 4. *For $m \in \mathbb{Z}^+$ and $D$ a fundamental discriminant we have that*

$$(2.23) \quad \sum_{n \mid m} \left(\tfrac{D}{n}\right) b_0(\tfrac{Dm^2}{n^2}, s) = 2^{2-4s} \pi^{s+\frac{1}{4}} m^{\frac{3}{2}-2s} |D|^{s-\frac{1}{4}} \sigma_{4s-2}(m) \frac{L_D(2s - \frac{1}{2})}{\zeta(4s - 1)}$$

---

[4]See [13, p. 179 ] and its references, especially [35] and [12]. A very clear treatment when the weight is 0 and the multiplier is trivial is given in [31]. In particular, see Satz 6.8 p. 60 in [31]; the case of weight $1/2$ is similar.

*and*

$$(2.24) \qquad b_0(0, s) = \pi^{\frac{1}{2}} 2^{\frac{5}{2} - 6s} \Gamma(2s) \frac{\zeta(4s - 2)}{\zeta(4s - 1)},$$

*where $L_D$ is the Dirichlet L-function.*

By Möbius inversion, (2.23) gives for $m \neq 0$ the identity
(2.25)
$$b_0(Dm^2, s) = 2^{2-4s} \pi^{s + \frac{1}{4}} |D|^{s - \frac{1}{4}} \frac{L_D(2s - 1/2)}{\zeta(4s - 1)} \sum_{n|m} \mu(m/n) \left(\frac{D}{m/n}\right) n^{\frac{3}{2} - 2s} \sigma_{4s-2}(n).$$

This yields a direct proof of Lemma 3 in case $d = 0$. Since $b_0(d, s) = b_d(0, s)$, which is clear from (3.5) and (2.21), a calculation using (2.25) and the (2.24) also gives the constant $\rho_d$ in (2.22):

$$(2.26) \qquad \rho_d = \begin{cases} \frac{3}{4\pi} & \text{if } d = 0, \\ \frac{6}{\pi} \sqrt{d} & \text{if } d \text{ is a nonzero square,} \\ 0 & \text{otherwise.} \end{cases}$$

We are finally ready to define the basis functions $h_d$. For $d \neq 0$ let

$$(2.27) \qquad h_d(\tau, s) = P_d^+(\tau, s) - \frac{b_d(0, s)}{b_0(0, s)} P_0^+(\tau, s).$$

It has the Fourier expansion
(2.28)
$$h_d(\tau, s) = \mathcal{M}_d(y, s) e(dx) - \frac{b_d(0, s)}{b_0(0, s)} y^{s - \frac{1}{4}} + \sum_{0 \neq n \equiv 0, 1(4)} a_d(n, s) \mathcal{W}_n(y, s) e(nx),$$

where

$$(2.29) \qquad a_d(n, s) = b_d(n, s) - \frac{b_d(0, s) b_0(n, s)}{b_0(0, s)}.$$

LEMMA 5. *For each nonzero $d \equiv 0, 1 \pmod 4$ the function $h_d(\tau, s)$ defined in (2.27) has an analytic continuation to $s = 3/4$ and*

$$h_d(\tau, 3/4) = h_d(\tau) \in H_{1/2}^!.$$

*The Fourier expansion of each such $h_d$ at $\infty$ has the form (2.12), where for each nonzero $n \equiv 0, 1 \pmod 4$ we have*

$$a_d(n) = \lim_{s \to 3/4^+} a_d(n, s).$$

*Furthermore, $a_d(0) = 2\sqrt{d}$ if $d$ is a square and $a_d(0) = 0$ otherwise.*

*Proof.* Observe that $h_d(\tau, s)$ defined in (2.27) is holomorphic at $s = 3/4$, since otherwise, by Proposition 3 it would have as residue there a nonzero

multiple of $\theta(\tau)$, which cannot happen since (2.28) does not yield the constant term in $\theta$. From (2.28) its Fourier expansion is given by

$$h_d(\tau) = \mathcal{M}_d(y)e(dx) + \sum_{n \equiv 0,1(4)} a_d(n)\mathcal{W}_n(y)e(nx),$$

where $a_d(n) = \lim_{s \to 3/4^+} a_d(n,s)$ for $n \neq 0$ and, after recalling the definition of $\mathcal{W}_0(y)$ from (2.7), we have that

$$(2.30) \qquad\qquad a_d(0) = \lim_{s \to 3/4^+} \frac{b_d(0,s)}{4b_0(0,s)}.$$

Here again we use the integral representations for the Fourier coefficients and the fact that $h_d(\tau,s) \to h_d(\tau)$ uniformly on compacta as $s \to 3/4^+$. Thus $h_d \in H_{1/2}^!$ for all $d \neq 0$. The last statement of Lemma 5 can easily be obtained from (2.30), (2.25) and (2.24). □

Continuing with the proof of Proposition 1, we next show that the symmetry relation (2.13) holds. By (3.5) and (2.21) we have that $b_d(n,s) = b_n(d,s)$; hence by (2.29)

$$(2.31) \qquad\qquad a_d(n,s) = a_n(d,s).$$

Now (2.13) follows from Lemma 5 and (2.31), where we use that $h_0 = \theta$ in case $nd = 0$. Note that $a_0(0) = 0$. A direct calculation using (2.9) together with (2.13) yields (2.14). This completes the proof of Proposition 1 and hence of Theorem 2.

## 3. Binary quadratic forms and Kloosterman sums

Before turning to the proof of Theorem 3, we need to give some basic results about binary quadratic forms and Kloosterman sums. Recall that $\mathcal{Q}_d$ is the set of integral binary quadratic forms $Q(x,y) = ax^2 + bxy + cy^2 = [a,b,c]$ of discriminant $d = b^2 - 4ac$ that are positive definite if $d < 0$. Let $\mathcal{Q}_d^+ \subset \mathcal{Q}_d$ be those forms with $a > 0$, so that $\mathcal{Q}_d = \mathcal{Q}_d^+$ when $d < 0$. Let $Q \mapsto gQ$ be the usual action of $\Gamma$ that is compatible with linear fractional action on the roots of $Q(\tau,1) = 0$. Explicitly,

$$(gQ)(x,y) = Q(\delta x - \beta y, -\gamma x + \alpha y), \qquad \text{where} \quad g = \pm \left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right) \in \Gamma.$$

As is well-known, the resulting set of classes $\Gamma \backslash \mathcal{Q}_d$ is finite and those classes consisting of primitive forms make up an abelian group under composition. Let $\Gamma_Q = \{g \in \Gamma; gQ = Q\}$ be the group of automorphs of $Q$. If $d < 0$ then $|\Gamma_Q| = 1$ unless $Q \sim a(x^2 + y^2)$ or $Q \sim a(x^2 + xy + y^2)$, in which case $|\Gamma_Q| = 2$ or 3, respectively. If $d > 0$ is not a square, then $\Gamma_Q$ is infinite cyclic with a

distinguished generator denoted by $g_Q$, which for primitive $Q$ is given by

$$(3.1) \qquad g_Q = \pm \begin{pmatrix} \frac{t+bu}{2} & cu \\ -au & \frac{t-bu}{2} \end{pmatrix},$$

where $t, u$ are the smallest positive integral solutions of $t^2 - du^2 = 4$. If $\delta = \gcd(a, b, c)$, then $g_Q = g_{Q/\delta}$.

Suppose that $D$ is a fundamental discriminant, i.e. the discriminant of $\mathbb{Q}(\sqrt{D})$, and that $d$ is a discriminant. For $Q = [a, b, c]$ with discriminant $dD$ let

$$(3.2) \ \ \chi(Q) = \chi_D(Q)$$

$$= \begin{cases} \left(\frac{D}{r}\right) & \text{if } (a, b, c, D) = 1, \text{ where } Q \text{ represents } r \text{ and } (r, D) = 1, \\ 0, & \text{if } (a, b, c, D) > 1. \end{cases}$$

Here $\left(\frac{\cdot}{\cdot}\right)$ is the Kronecker symbol. It is well-known that $\chi$ is well-defined on classes $\Gamma \backslash \mathcal{Q}_{dD}$, that $\chi$ restricts to a real character (a genus character) on the group of primitive classes, and that all such characters arise this way. We have the identity

$$(3.3) \qquad \chi_D(-Q) = (\text{sgn } D)\chi_D(Q).$$

If $d$ is also fundamental we have that $\chi_D = \chi_d$ on $\Gamma \backslash \mathcal{Q}_{dD}$. A good reference for the basic theory of these characters is [15, p. 508].

A crucial ingredient in what follows is an identity connecting the weight $1/2$ Kloosterman sum with a certain exponential sum taken over solutions to a quadratic congruence, a quadratic Weyl sum. In a special case this identity is due to Salié and variants have found many applications in the theory of modular forms. We shall use a general version due essentially to Kohnen [27]. To define the weight $1/2$ Kloosterman sum we need an explicit formula for the theta multiplier in $j(\gamma, \tau)$, which was defined in (1.2). This may be found in [36, p. 447]. As usual, for nonzero $z \in \mathbb{C}$ and $v \in \mathbb{R}$ we define $z^v = |z|^v \exp(iv \arg z)$ with $\arg z \in (-\pi, \pi]$. We have

$$j(\gamma, \tau) = (c\tau + a)^{1/2} \varepsilon_a^{-1} \left(\frac{c}{a}\right) \quad \text{for} \quad \gamma = \pm \left(\begin{smallmatrix} * & * \\ c & a \end{smallmatrix}\right) \in \Gamma_0(4),$$

where $\left(\frac{c}{a}\right)$ is the extended Kronecker symbol and

$$\varepsilon_a = \begin{cases} 1 & \text{if } a \equiv 1 \pmod 4, \\ i & \text{if } a \equiv 3 \pmod 4. \end{cases}$$

For $c \in \mathbb{Z}^+$ with $c \equiv 0 \pmod 4$ and $m, n \in \mathbb{Z}$ let

$$K_{1/2}(m, n; c) = \sum_{a(\text{mod } c)} \left(\frac{c}{a}\right) \varepsilon_a e \left(\frac{ma + n\bar{a}}{c}\right)$$

be the weight $1/2$ Kloosterman sum. Here $\bar{a} \in \mathbb{Z}$ satisfies

$$a\bar{a} \equiv 1 \pmod c.$$

It is convenient to define the modified Kloosterman sum

$$(3.4) \qquad K^+(m,n;c) = (1-i)K_{1/2}(m,n;c) \times \begin{cases} 1 & \text{if } c/4 \text{ is even,} \\ 2 & \text{otherwise.} \end{cases}$$

It is easily checked that

$$(3.5) \qquad K^+(m,n;c) = K^+(n,m;c) = \overline{K^+(n,m;c)}.$$

The associated exponential sum is defined for $d \equiv 0, 1 \pmod 4$ and fundamental $D$ by

$$(3.6) \qquad S_m(d,D;c) = \sum_{\substack{b(\mathrm{mod}\,c) \\ b^2 \equiv Dd\,(\mathrm{mod}\,c)}} \chi\left(\left[\tfrac{c}{4}, b, \tfrac{b^2-Dd}{c}\right]\right) e\left(\tfrac{2mb}{c}\right),$$

where $\chi$ was defined in (3.2). Clearly

$$S_{-m}(d,D;c) = \overline{S_m(d,D;c)} = S_m(d,D;c).$$

The following identity is proved by a slight modification of the proof given by Kohnen in [27, Prop. 5, p. 259] (see also [8], [21] and [37]).

PROPOSITION 3. *For positive* $c \equiv 0 \pmod 4$, $d, m \in \mathbb{Z}$ *with* $d \equiv 0, 1 \pmod 4$ *and* $D$ *a fundamental discriminant, we have*

$$S_m(d,D;c) = \sum_{n \mid (m, \frac{c}{4})} \left(\tfrac{D}{n}\right) \sqrt{\tfrac{n}{c}}\, K^+\left(d, \tfrac{m^2D}{n^2}; \tfrac{c}{n}\right).$$

By Möbius inversion in two variables this can be written in the form

$$(3.7) \qquad c^{-1/2} K^+(d, m^2D, c) = \sum_{n \mid (m, \frac{c}{4})} \mu(n) \left(\tfrac{D}{n}\right) S_{m/n}\left(d, D; \tfrac{c}{n}\right).$$

Note that this gives an identity for $K^+(d, d', c)$ for any pair $d, d' \equiv 0, 1 \pmod 4$. An immediate consequence of (3.7) and the obvious upper bound

$$S_m(d,D;c) \ll_\epsilon c^\epsilon$$

is the upper bound

$$(3.8) \qquad K^+(d,d',c) \ll_\epsilon c^{1/2+\epsilon},$$

which holds for any $\epsilon > 0$. Furthermore, since for any $m, n \in \mathbb{Z}$ we have

$$K_{1/2}(m,n;c) = \tfrac{1}{4} K_{1/2}(4m, 4n; 4c),$$

(3.8) implies that for any $m, n \in \mathbb{Z}$

$$K_{1/2}(m,n,c) \ll_\epsilon c^{1/2+\epsilon}.$$

This elementary bound corresponds to Weil's bound for the ordinary (weight 0) Kloosterman sum

$$K_0(m, n; c) = \sum_{\substack{a(\mathrm{mod}\ c) \\ (a,c)=1}} e\left(\tfrac{ma+n\bar{a}}{c}\right),$$

which states that (see [39], [17, Lemma 2])

(3.9) $$K_0(m, n; c) \ll_\epsilon (m, n, c)^{1/2} c^{1/2+\epsilon}.$$

## 4. Cycle integrals of Poincaré series

As further preparation for the proof of Theorem 3, in this section we will compute the cycle integrals of certain general Poincaré series, which we will then specialize in order to treat $j_m$. To begin we need to make some elementary observations about cycle integrals. For $Q \in \mathcal{Q}_d$ with $d > 0$ not a square let $S_Q$ be the oriented semi-circle defined by

(4.1) $$a|\tau|^2 + b\,\mathrm{Re}\,\tau + c = 0,$$

directed counterclockwise if $a > 0$ and clockwise if $a < 0$. Clearly

(4.2) $$S_{gQ} = gS_Q,$$

for any $g \in \Gamma$. Given $z \in S_Q$ let $C_Q$ be the directed arc on $S_Q$ from $z$ to $g_Q z$, where $g_Q$ was defined in (3.1). It can easily be checked that $C_Q$ has the same orientation as $S_Q$. It is convenient to define

(4.3) $$d\tau_Q = \frac{\sqrt{d}\,d\tau}{Q(\tau, 1)}.$$

If $\tau' = g\tau$ for some $g \in \Gamma$ we have

(4.4) $$d\tau'_{gQ} = d\tau_Q.$$

For any $\Gamma$-invariant function $f$ on $\mathcal{H}$ the integral $\int_{C_Q} f(\tau)d\tau_Q$ is both independent of $z \in S_Q$ and is a class invariant. This is an immediate consequence of the following lemma that expresses this cycle integral as a sum of integrals over arcs in a fixed fundamental domain for $\Gamma$. This lemma will also be used in the proof of Theorem 5. Let $\mathcal{F}$ be the standard fundamental domain for $\Gamma$

$$\mathcal{F} = \{\tau \in \mathcal{H}; -\tfrac{1}{2} \le \mathrm{Re}\,\tau \le 0, |\tau| \ge 1\} \cup \{\tau \in \mathcal{H}; 0 < \mathrm{Re}\,\tau < \tfrac{1}{2}, |\tau| > 1\}.$$

LEMMA 6. *Let $Q \in \mathcal{Q}_d$ be a form with $d > 0$ not a square and $\mathcal{F}' = g\mathcal{F}$ be the image of $\mathcal{F}$ under any fixed $g \in \Gamma$. Suppose that $f$ is $\Gamma$-invariant and continuous on $S_Q$. Then for any $z \in S_Q$ we have*

(4.5) $$\int_{C_Q} f(\tau)d\tau_Q = \sum_{q \in (Q)} \int_{S_q \cap \mathcal{F}'} f(\tau)d\tau_q,$$

*where $(Q)$ denotes the class of $Q$.*

*Proof.* Let $\tilde{f}(\tau) = f(\tau)$ if $\tau \in \mathcal{F}'$ and $\tilde{f}(\tau) = 0$ otherwise, so that $f(\tau) = \sum_{g \in \Gamma} \tilde{f}(g\tau)$ with only a discrete set of exceptions. Thus

$$\int_{C_Q} f(\tau)d\tau_Q = \int_{C_Q} \sum_{g \in \Gamma} \tilde{f}(g\tau)d\tau_Q$$

$$= \sum_{g \in \Gamma/\Gamma_Q} \sum_{\sigma \in \Gamma_Q} \int_{C_Q} \tilde{f}(g\sigma\tau)d\tau_Q = \sum_{g \in \Gamma/\Gamma_Q} \int_{S_Q} \tilde{f}(g\tau)d\tau_Q.$$

Take $g\tau$ as a new variable. By (4.2) and (4.4) we get

$$\int_{C_Q} f(\tau)d\tau_Q = \sum_{g \in \Gamma/\Gamma_Q} \int_{S_{gQ}} \tilde{f}(\tau)d\tau_{gQ},$$

which immediately yields (4.5). $\qquad\qquad\square$

The general Poincaré series are built from a test function $\phi : \mathbb{R}^+ \to \mathbb{C}$ assumed to be smooth and to satisfy that for some $a > 1$ we have $\phi(y) = O(y^a)$ as $y \to 0$. For any $m \in \mathbb{Z}$ let

(4.6)
$$G_m(\tau, \phi) = \sum_{g \in \Gamma_\infty \backslash \Gamma} e\left(m \operatorname{Re} g\tau\right) \phi(\operatorname{Im} g\tau).$$

This sum converges uniformly on compacta and defines a smooth $\Gamma$-invariant function on $\mathcal{H}$. We express its cycle integrals in terms of the sum $S_m(d, D; c)$ from (3.6). Define for $t > 0$ the integral transform.

$$\Phi_m(t) = \int_0^\pi \cos(2\pi m t \cos\theta)\phi(t \sin\theta)\frac{d\theta}{\sin\theta}.$$

For $\phi$ as above, we see that this integral converges absolutely and that $\Phi_m(t) = O_\epsilon(t^{1+\epsilon})$. As we have seen, we may assume without loss that $d, D > 0$.

LEMMA 7. *Suppose that $d, D > 0$ with $dD$ not a square. Then for all $m \in \mathbb{Z}$,*

$$\sum_{Q \in \Gamma \backslash \mathcal{Q}_{dD}} \chi(Q) \int_{C_Q} G_m(\tau, \phi)d\tau_Q = \sum_{0 < c \equiv 0(4)} S_m(d, D; c)\Phi_m\left(\tfrac{2\sqrt{dD}}{c}\right).$$

*Proof.* For each $Q$, interchanging the sum defining $G_m$ and the integral yields

(4.7)
$$\int_{C_Q} G_m(\tau, \phi)d\tau_Q = \sum_{g \in \Gamma_\infty \backslash \Gamma} \int_{C_Q} e(m \operatorname{Re} g\tau)\phi(\operatorname{Im} g\tau)d\tau_Q.$$

Now $\Gamma_Q$, the group of automorphs of $Q$, acts freely on $\Gamma_\infty \backslash \Gamma$ so we have that

$$\sum_{g \in \Gamma_\infty \backslash \Gamma} \int_{C_Q} \mathrm{e}(m \operatorname{Re} g\tau) \phi(\operatorname{Im} g\tau) d\tau_Q$$

$$= \sum_{g \in \Gamma_\infty \backslash \Gamma / \Gamma_Q} \sum_{\sigma \in \Gamma_Q} \int_{C_Q} \mathrm{e}(m \operatorname{Re} g\sigma\tau) \phi(\operatorname{Im} g\sigma\tau) d\tau_Q$$

$$= \sum_{g \in \Gamma_\infty \backslash \Gamma / \Gamma_Q} \int_{S_Q} \mathrm{e}(m \operatorname{Re} g\tau) \phi(\operatorname{Im} g\tau) d\tau_Q.$$

Applying (4.4) and (4.2) in the last expression, we get from (4.7) that

$$\int_{C_Q} G_m(\tau, \phi) d\tau_Q = \sum_{g \in \Gamma_\infty \backslash \Gamma / \Gamma_Q} \int_{S_{gQ}} \mathrm{e}(m \operatorname{Re} \tau) \phi(\operatorname{Im} \tau) d\tau_{gQ}$$

and hence that

$$\sum_{Q \in \Gamma \backslash \mathcal{Q}_{dD}} \chi(Q) \int_{C_Q} G_m(\tau, \phi) d\tau_Q = \sum_{Q \in \Gamma_\infty \backslash \mathcal{Q}_{dD}} \chi(Q) \int_{S_Q} \mathrm{e}(m \operatorname{Re} \tau) \phi(\operatorname{Im} \tau) d\tau_Q.$$

We now need to parametrize the cycle explicitly. Let

(4.8)
$$\tau_Q = \frac{-b}{2a} + \frac{i\sqrt{d}}{2|a|},$$

which is easily seen to be the apex of the circle $S_Q$. We can parametrize $S_Q$ by $\theta \in (0, \pi)$ via

$$\tau = \begin{cases} \operatorname{Re} \tau_Q + e^{i\theta} \operatorname{Im} \tau_Q & \text{if } a > 0, \\ \operatorname{Re} \tau_Q - e^{-i\theta} \operatorname{Im} \tau_Q & \text{if } a < 0. \end{cases}$$

With this parametrization we find that

$$Q(\tau, 1) = \frac{d}{4a} \cdot \begin{cases} e^{2i\theta} - 1 & \text{if } a > 0, \\ e^{-2i\theta} - 1 & \text{if } a < 0, \end{cases}$$

and hence that $d\tau_Q = d\theta / \sin \theta$. Since $\chi(Q) = \chi(-Q)$, we arrive at the identity

$$\sum_{Q \in \Gamma \backslash \mathcal{Q}_{dD}} \chi(Q) \int_{C_Q} G_m(\tau, \phi) d\tau_Q = 2 \sum_{Q \in \Gamma_\infty \backslash \mathcal{Q}_{dD}^+} \chi(Q) \, \mathrm{e}(m \operatorname{Re} \tau_Q) \Phi_m(\operatorname{Im} \tau_Q).$$

The proof of Lemma 7 is thus reduced to the following lemma. □

LEMMA 8. *Let $\phi$ be as above and suppose that $dD$ is not a square. Then for all $m \in \mathbb{Z}$ we have the identity*

$$\sum_{\Gamma_\infty \backslash \mathcal{Q}_{dD}^+} \chi(Q) \, \mathrm{e}(m \operatorname{Re} \tau_Q) \phi(\operatorname{Im} \tau_Q) = \tfrac{1}{2} \sum_{0 < c \equiv 0(4)} S_m(d, D; c) \phi\left(\frac{2\sqrt{|dD|}}{c}\right),$$

*where $\tau_Q$ is defined in (4.8).*

*Proof.* Under the growth condition on $\phi$, both series are absolutely convergent and can be rearranged at will. Consider the left-hand side. For $g = \pm(\begin{smallmatrix} 1 & k \\ 0 & 1 \end{smallmatrix}) \in \Gamma_\infty$ and $Q = [a, b, c] \in \mathcal{Q}_{dD}$, $gQ = [a, b - 2ka, *]$ and so the map

$$[a, b, c] \mapsto (a, b \bmod 2a)$$

is $\Gamma_\infty$-invariant. Thus

$$\sum_{\Gamma_\infty \backslash \mathcal{Q}_{dD}^+} \chi(Q) e(m \operatorname{Re} \tau_Q) \phi(\operatorname{Im} \tau_Q) = \sum_{a=1}^{\infty} \phi\left(\frac{\sqrt{|dD|}}{2a}\right) \sum_{b \, (2a)} \chi([a, b, \tfrac{b^2 - dD}{4a}]) \, \mathrm{e}(-\tfrac{mb}{2a}).$$

The sum in $b$ is restricted to those values for which $\frac{b^2 - dD}{4a}$ is an integer. This happens exactly when $b^2 \equiv dD \pmod{4a}$. Thus the inner sum is

$$\sum_{\substack{b \, (2a) \\ b^2 \equiv dD \, (4a)}} \chi([a, b, \tfrac{b^2 - dD}{4a}]) e(-\tfrac{mb}{2a}) = \tfrac{1}{2} \sum_{\substack{b \, (4a) \\ b^2 \equiv dD \, (4a)}} \chi([a, b, \tfrac{b^2 - dD}{4a}]) \, \mathrm{e}(-\tfrac{2mb}{4a})$$

$$= \tfrac{1}{2} S_m(d, D; 4a).$$

Replace $4a$ by $c$ to finish the proof. $\qquad\square$

We remark that the positive definite version of Lemma 7 is following the well-known formula for $dD < 0$:

$$(4.9) \qquad \sum_{Q \in \Gamma \backslash \mathcal{Q}_{dD}} w_Q^{-1} \chi(Q) G_m(\tau, \phi) = \tfrac{1}{2} \sum_{0 < c \equiv 0(4)} S_m(d, D; c) \phi\left(\tfrac{2\sqrt{|dD|}}{c}\right).$$

This formula is an immediate consequence of Lemma 8.

Now we will specialize the Poincaré series $G_m$ from (4.6) and construct the modular functions $j_m$. Let $G_m(\tau, s) = G_m(\tau, \phi_{m,s})$, where

$$\phi_{m,s}(y) = \begin{cases} y^s & \text{if } m = 0, \\ 2\pi |m|^{\frac{1}{2}} y^{\frac{1}{2}} I_{s-\frac{1}{2}}(2\pi |m| y) & \text{if } m \neq 0, \end{cases}$$

with $I_{s-1/2}$ the Bessel function as before. The resulting $\Gamma$-invariant function satisfies

$$\triangle_0 G_m(\tau, s) = s(1 - s) G_m(\tau, s).$$

The function $G_0$ is the usual Eisenstein series while $G_m$ for $m \neq 0$ was studied by Neunhöffer [31] and Niebur [32], among others. The required analytic properties of $G_m(\tau, s)$ in $s$ are most easily obtained from their Fourier expansions. For the Eisenstein series we have the well-known formulas (see e.g. [20])

$$G_0(\tau, s) = y^s + c_0(0, s) y^{1-s} + \sum_{n \neq 0} c_0(n, s) K_{s-\frac{1}{2}}(2\pi |n| y) e(nx),$$

where $K_{s-\frac{1}{2}}$ is the $K$-Bessel function (see e.g. [29]),

$$c_0(0,s) = \frac{\xi(2s-1)}{\xi(2s)} \qquad \text{and for } n \neq 0 \qquad c_0(n,s) = \frac{2y^{1/2}}{\xi(2s)}|n|^{s-1/2}\sigma_{1-2s}(|n|),$$

with $\xi(s) = \pi^{-\frac{s}{2}}\Gamma(s/2)\zeta(s)$. For $m \neq 0$ the Fourier expansion of $G_m$ can be found in [13], and is given by

$$G_m(\tau,s) = 2\pi|m|^{\frac{1}{2}}y^{\frac{1}{2}}I_{s-\frac{1}{2}}(2\pi|m|y)e(mx) + c_m(0,s)y^{1-s}$$
$$+ 4\pi|m|^{1/2}y^{1/2}\sum_{n\neq 0}|n|^{1/2}c_m(n,s)K_{s-\frac{1}{2}}(2\pi|n|y)e(nx),$$

where

$$c_m(0,s) = \frac{4\pi|m|^{1-s}\sigma_{2s-1}(|m|)}{(2s-1)\xi(2s)}$$

and

$$c_m(n;s) = \sum_{c>0}c^{-1}K_0(m,n;c) \cdot \begin{cases} I_{2s-1}(4\pi\sqrt{|mn|}\,c^{-1}) & \text{if } mn < 0, \\ J_{2s-1}(4\pi\sqrt{|mn|}\,c^{-1}) & \text{if } mn > 0. \end{cases}$$

Define for $m \in \mathbb{Z}^+$ and $\mathrm{Re}(s) > 1$

$$(4.10) \qquad j_m(\tau,s) = G_{-m}(\tau,s) - \frac{2m^{1-s}\sigma_{2s-1}(m)}{\pi^{-(s+\frac{1}{2})}\Gamma(s+\frac{1}{2})\zeta(2s-1)}G_0(\tau,s).$$

It follows from its Fourier expansion, Weil's bound (3.9) and (2.17) that $j_m(\tau,s)$ has an analytic continuation to $\mathrm{Re}(s) > 3/4$. Furthermore, since a bounded harmonic function is constant, for $m \in \mathbb{Z}^+$ we have

$$(4.11) \qquad\qquad\qquad j_m(\tau,1) = j_m(\tau),$$

where $j_m$ was defined above (1.5) (cf. [32]). Alternatively, we could apply the theory of the resolvent kernel in weight 0 to get the analytic continuation of $j_m(\tau,s)$ up to $\mathrm{Re}\,s > 1/2$.

In view of (4.11), in order to compute the traces of $j_m(\tau,s)$ it is enough to compute them for $G_m(\tau,s)$. We have the following identities, which are known when $m = 0$ (Dirichlet/Hecke) and when $dD < 0$ (see e.g. [9], [8], [3]). For the convenience of the reader we will give a uniform proof.

PROPOSITION 4. *Let* $\mathrm{Re}(s) > 1$ *and* $m \in \mathbb{Z}$. *Suppose that* $d$ *and* $D$ *are not both negative and that* $dD$ *is not a square. Then, when* $dD < 0$ *we have*

$$\sum_{Q\in\Gamma\backslash Q_{dD}} \frac{\chi(Q)}{w_Q}G_m(\tau_Q,s)$$

$$= \begin{cases} \sqrt{2}\pi|m|^{\frac{1}{2}}|dD|^{\frac{1}{4}}\sum_{c\equiv 0(4)}\frac{S_m(d,D;c)}{c^{1/2}}I_{s-\frac{1}{2}}\left(\frac{4\pi\sqrt{m^2|dD|}}{c}\right) & \text{if } m \neq 0, \\ 2^{s-1}|dD|^{\frac{s}{2}}\sum_{c\equiv 0(4)}\frac{S_0(d,D;c)}{c^s} & \text{if } m = 0, \end{cases}$$

*while when $dD > 0$ we have*

$$\sum_{Q \in \Gamma \backslash Q_{dD}} \frac{\chi(Q)}{B(s)} \int_{C_Q} G_m(\tau, s) d\tau_Q$$

$$= \begin{cases} \sqrt{2}\pi |m|^{\frac{1}{2}} |dD|^{\frac{1}{4}} \sum_{c \equiv 0(4)} \frac{S_m(d,D;c)}{c^{1/2}} J_{s-\frac{1}{2}}\left(\frac{4\pi\sqrt{m^2|dD|}}{c}\right) & \text{if } m \neq 0, \\ 2^{s-1} |dD|^{\frac{s}{2}} \sum_{c \equiv 0(4)} \frac{S_0(d,D;c)}{c^s} & \text{if } m = 0, \end{cases}$$

*where* $B(s) = 2^s \Gamma(\frac{s}{2})^2/\Gamma(s)$.

*Proof.* By (4.9) the proof of Proposition 4 reduces to the case $dD > 0$. Applying Lemma 7 when $m = 0$ we use the well-known evaluation

$$\int_0^\pi (\sin\theta)^{s-1} d\theta = 2^{s-1} \frac{\Gamma(\frac{s}{2})^2}{\Gamma(s)}.$$

When $m \neq 0$ we need the following not-so-well-known evaluation to finish the proof. $\qquad\square$

LEMMA 9. *For* $\text{Re}(s) > 0$ *we have*

$$\int_0^\pi \cos(t\cos\theta) I_{s-\frac{1}{2}}(t\sin\theta) \frac{d\theta}{(\sin\theta)^{1/2}} = 2^{s-1} \frac{\Gamma(\frac{s}{2})^2}{\Gamma(s)} J_{s-1/2}(t).$$

*Proof.* Denote the left-hand side by $L_s(t)$. We use the definition of $I_{s-\frac{1}{2}}$ in (2.17) to get

$$L_s(t) = \sum_{k=0}^\infty \frac{(t/2)^{s+2k-1/2}}{k!\Gamma(s+k+\frac{1}{2})} \int_0^\pi \cos(t\cos\theta)(\sin\theta)^{s+2k-1} d\theta.$$

Lommel's integral representation [38, p. 47] gives for $\text{Re}\, v > -1/2$ that

$$J_\nu(y) = \frac{(y/2)^\nu}{\Gamma(\nu + \frac{1}{2})\Gamma(\frac{1}{2})} \int_0^\pi \cos(y\cos\theta)(\sin\theta)^{2\nu} d\theta.$$

Thus for $\text{Re}(s) > 0$ we have that

$$L_s(t) = \Gamma(\tfrac{1}{2}) \sum_{k=0}^\infty \frac{\Gamma(\frac{s}{2}+k)}{k!\Gamma(s+k+\frac{1}{2})} (t/2)^{s/2+k} J_{(s-1)/2+k}(t).$$

This Neumann series can be evaluated (see [38, p. 143, eq. 1]) giving for $\text{Re}(s) > 0$

$$L_s(t) = \frac{\Gamma(\frac{1}{2})\Gamma(\frac{s}{2})}{\Gamma(\frac{s}{2}+\frac{1}{2})} J_{s-1/2}(t).$$

The result follows by the duplication formula for $\Gamma(s)$. $\qquad\square$

## 5. **The traces in terms of Fourier coefficients**

In this section we complete the proofs of Theorems 3 and 4. We need to express the traces of $j_m$ in terms of the Fourier coefficients of our basis $h_d$. This is first done for $j_m(\tau, s)$ with $\mathrm{Re}(s) > 1$ by applying Proposition 3 to transform the sum of exponential sums in Proposition 4 into a sum of Kloosterman sums, which is then related to the coefficients of $h_d(\tau, s)$. The method of using Kloosterman sums in this way was first applied by Zagier [41] to base change, then by Kohnen [27] to the Shimura lift and more recently to weakly holomorphic forms in [4], [8], [21] and [3].

Theorem 3 follows from Lemma 5, (4.11) and the next result by taking the limit as $s \to 1^+$ of both sides of (5.1). Also we use the relationship between $a(n, d)$ and $a_d(n)$ given in and above equation (2.15). We remark that we actually get a slightly more general result than Theorem 3 in that we may allow $D < 0$, but the general result is best left in terms of the coefficients $a_d(n)$.

PROPOSITION 5. *Let* $m \in \mathbb{Z}^+$ *and* $\mathrm{Re}(s) > 1$. *Suppose that* $d$ *and* $D$ *are not both negative and that* $dD$ *is not a square. Then*
(5.1)
$$
\sum_{n|m} \left(\tfrac{D}{n}\right) a_d\left(\tfrac{m^2 D}{n^2}, \tfrac{s}{2} + \tfrac{1}{4}\right) = 
\begin{cases}
\sum_Q \chi(Q) w_Q^{-1} j_m(\tau_Q, s) & \text{if } dD < 0, \\
\\
B(s)^{-1} \sum_Q \chi(Q) \int_{C_Q} j_m(\tau, s) d\tau_Q & \text{if } dD > 0,
\end{cases}
$$
*where each sum on the right-hand side is over* $Q \in \Gamma \backslash \mathcal{Q}_{dD}$.

*Proof.* It is convenient to set for any $m \in \mathbb{Z}$
$$
T_m(s) = 
\begin{cases}
\sum_Q \chi(Q) w_Q^{-1} G_m(\tau_Q, s) & \text{if } dD < 0, \\
B(s)^{-1} \sum_Q \chi(Q) \int_{C_Q} G_m(\tau, s) d\tau_Q & \text{if } dD > 0,
\end{cases}
$$
where each sum is over $Q \in \Gamma \backslash \mathcal{Q}_{dD}$. By Propositions 4 and 3 we have for $m \neq 0$ and $\mathrm{Re}(s) > 1$ that
$$
T_m(s) = \pi |2m|^{\frac{1}{2}} |dD|^{\frac{1}{4}} \sum_{n|m} \left(\tfrac{D}{n}\right) n^{-\frac{1}{2}} \sum_{c \equiv 0(4)} c^{-1} K^+\left(d, \tfrac{m^2 D}{n^2}; c\right)
$$
$$
\cdot
\begin{cases}
I_{s-\frac{1}{2}}\left(\tfrac{4\pi}{c} \sqrt{\tfrac{m^2}{n^2} |Dd|}\right) & \text{if } dD < 0, \\
J_{s-\frac{1}{2}}\left(\tfrac{4\pi}{c} \sqrt{\tfrac{m^2}{n^2} |Dd|}\right) & \text{if } dD > 0,
\end{cases}
$$
while when $m = 0$ we have
$$
T_0(s) = 2^{s-1} |dD|^{\frac{s}{2}} L_D(s) \sum_{c \equiv 0(4)} c^{-s-1/2} K^+\left(d, 0; c\right).
$$

Thus by (2.21) of Proposition 2 we derive that

$$(5.2) \qquad T_m(s) = \begin{cases} \sum_{n|m} \left(\frac{D}{n}\right) b_d\left(\frac{m^2 D}{n^2}, \frac{s}{2} + \frac{1}{4}\right) & \text{if } m \neq 0, \\ 2^{s-1} \pi^{-\frac{s+1}{2}} |D|^{\frac{s}{2}} L_D(s) b_d(0, \frac{s}{2} + \frac{1}{4}) & \text{if } m = 0. \end{cases}$$

In view of (4.10), in order to prove Proposition 5 it is enough to show that

$$(5.3) \qquad \sum_{n|m} \left(\frac{D}{n}\right) a_d\left(\frac{m^2 D}{n^2}, \frac{s}{2} + \frac{1}{4}\right) = T_m(s) - \frac{2m^{1-s}\sigma_{2s-1}(m)}{\pi^{-(s+\frac{1}{2})}\Gamma(s+\frac{1}{2})\zeta(2s-1)} T_0(s).$$

By (2.29) and the first formula of (5.2) the left-hand side of (5.3) is

$$T_m(s) - \frac{b_d(0, \frac{s}{2} + \frac{1}{4})}{b_0(0, \frac{s}{2} + \frac{1}{4})} \sum_{n|m} \left(\frac{D}{n}\right) b_0\left(\frac{m^2 D}{n^2}, \frac{s}{2} + \frac{1}{4}\right).$$

Hence by the second formula of (5.2) we are reduced to showing that

$$b_0(0, \tfrac{s}{2} + \tfrac{1}{4})^{-1} \sum_{n|m} \left(\frac{D}{n}\right) b_0\left(\frac{m^2 D}{n^2}, \frac{s}{2} + \frac{1}{4}\right) = \frac{2^s \pi^{s/2} |D|^{s/2} m^{1-s} \sigma_{2s-1}(m) L_D(s)}{\Gamma(s+\frac{1}{2})\zeta(2s-1)},$$

which follows by Lemma 4. This finishes the proof of Proposition 5, hence of Theorem 3.  $\qquad\square$

We now give a quick proof of Theorem 4. By (2.26) we have

$$\operatorname{Res}_{s=\frac{3}{4}} P_0^+(\tau, s) = \frac{3}{4\pi}\theta(\tau).$$

The function $\widehat{\mathbf{Z}}_+(\tau)$ can now be defined through the limit formula[5]

$$(5.4) \qquad \widehat{\mathbf{Z}}_+(\tau) = \frac{1}{3} \lim_{s \to \frac{3}{4}} \left( P_0^+(\tau, s) - \frac{\frac{3}{4\pi}\theta(\tau)}{s - 3/4} \right).$$

It follows from (5.4) that $\widehat{\mathbf{Z}}_+(\tau)$ has weight $1/2$ and satisfies

$$(5.5) \qquad \Delta_{1/2}(\widehat{\mathbf{Z}}_+) = -\frac{1}{8\pi}\theta.$$

Finally, using (5.2) when $m = 0$ and the fact that $G_0(\tau, s)$ has a simple pole at $s = 1$ with residue $3/\pi$, one shows that $\widehat{\mathbf{Z}}_+(\tau)$ has a Fourier expansion of the form (1.19).

The statement that $\widehat{\mathbf{Z}}_+(\tau)$ has generalized shadow $\widehat{\mathbf{Z}}_-(\tau)$ from (1.18) can now be made precise since it follows from (5.5) and the easily established identity

$$\xi_{3/2}\,\widehat{\mathbf{Z}}_= - \tfrac{1}{4\pi}\theta,$$

that

$$\xi_{1/2}\,\widehat{\mathbf{Z}}_+ = -2\widehat{\mathbf{Z}}_-.$$

---

[5]We remark that a similar limit formula was considered in [10].

## 6. **Rational period functions**

We now prove Theorem 5. First we give a rough bound for the traces in terms of $m$ when $d > 0$ is not a square that is sufficient to show that $F_d$ is holomorphic in $\mathcal{H}$.

PROPOSITION 6. *For $d > 0$ not a square and $m \in \mathbb{Z}^+$ we have for all $\epsilon > 0$ that*

$$\mathrm{Tr}_d(j_m) \ll_{d,\epsilon} m^{5/4+\epsilon}.$$

*Proof.* It follows from [18, Thm. 1, p. 110] that for fixed $d$ not a square and $x > 0$, we have for all $\epsilon > 0$ that

$$(6.1) \qquad \sum_{0<n<x} S_m(d,1;4n) \ll_{d,\epsilon} (mx)^\epsilon (m^{5/4} + x^{3/4}),$$

after replacing $d$ by $4d$ if necessary. For $1 < s < 2$ we have by the trivial bound for $S_m(d,1;4n)$ and the well-known bound (see e.g. [29, p. 122])

$$J_v(y) \ll_\nu y^{-1/2}$$

that

$$\sum_{0<n\leq m} S_m(d,1;4n)\sqrt{\tfrac{m}{n}} J_{s-\frac{1}{2}}\left(\pi\sqrt{|d|}\tfrac{m}{n}\right) \ll_{d,\epsilon} m^{1+\epsilon}.$$

By (2.17) we have for $x > m$

$$\sum_{m<n<x} S_m(d,1;4n)\sqrt{\tfrac{m}{n}} J_{s-\frac{1}{2}}\left(\pi\sqrt{|d|}\tfrac{m}{n}\right) \ll_{d,\epsilon} m^s \left| \sum_{m<n<x} S_m(d,1;4n)n^{-s} \right| + m^{1+\epsilon}.$$

Summation by parts and (6.1) give

$$m^s \sum_{m<n<x} S_m(d,1;4n)n^{-s} \ll_{d,\epsilon} m^{5/4+\epsilon}.$$

Now Proposition 6 follows by Proposition 4 and (4.10) by taking $s \to 1^+$ in the resulting uniform inequality

$$\sum_{Q\in\Gamma\backslash\mathcal{Q}_d} \int_{C_Q} j_m(\tau,s)d\tau_Q \ll_{d,\epsilon} m^{5/4+\epsilon}$$

and using (4.11). $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

It follows from Theorem 3 and Proposition 6 that the function $F_d$ defined in (1.20) for $d > 0$ not a square can be represented by the series

$$(6.2) \qquad\qquad F_d(\tau) = -\sum_{m\geq 0} \mathrm{Tr}_d(j_m)\, q^m,$$

which gives a holomorphic function on $\mathcal{H}$. The basis $\{j_m\}_{m \geq 0}$ has a generating function that goes back to Faber (see e.g. [1]):

$$(6.3) \qquad \sum_{m \geq 0} j_m(z) q^m = \frac{j'(\tau)}{j(z) - j(\tau)}, \quad \text{where} \quad j'(\tau) = \frac{1}{2\pi i} \frac{d\,j}{d\,\tau}.$$

Note that this formal series converges when $\text{Im}(\tau) > \text{Im}(z)$ and that for fixed $\tau$ not a zero of $j'$ it has a simple pole at $z = \tau$ with residue $(2\pi i)^{-1}$. It follows from (6.3) and (6.2) that for $\text{Im}(\tau)$ sufficiently large we have

$$(6.4) \qquad F_d(\tau) = \frac{1}{2\pi} \sum_{Q \in \Gamma \backslash \mathcal{Q}_d} \int_{C_Q} \frac{j'(\tau)}{j(\tau) - j(z)} \frac{dz}{Q(z,1)},$$

where we take for $C_Q$ an arc on $S_Q$, the semi-circle defined in (4.1). Let $\mathcal{F}' = -\mathcal{F}^{-1}$ be the image of the standard fundamental domain under inversion $z \mapsto -1/z$. By (6.4) and Lemma 6 applied to each class of $\mathcal{Q}_d$ and to each fundamental domain $\mathcal{F}$ and $\mathcal{F}'$, we can write

$$F_d(\tau) = \frac{1}{4\pi} \sum_{Q \in \mathcal{Q}_d} \left( \int_{S_Q \cap \mathcal{F}} \frac{j'(\tau)}{j(\tau) - j(z)} \frac{dz}{Q(z,1)} + \int_{S_Q \cap \mathcal{F}'} \frac{j'(\tau)}{j(\tau) - j(z)} \frac{dz}{Q(z,1)} \right).$$

Now it is easily seen that each of these integrals is invariant under $Q \mapsto -Q$, so we may restrict the sum to $\mathcal{Q}_d^+$, giving
$$(6.5)$$
$$F_d(\tau) = \frac{1}{2\pi} \sum_{Q \in \mathcal{Q}_d^+} \left( \int_{S_Q \cap \mathcal{F}} \frac{j'(\tau)}{j(\tau) - j(z)} \frac{dz}{Q(z,1)} + \int_{S_Q \cap \mathcal{F}'} \frac{j'(\tau)}{j(\tau) - j(z)} \frac{dz}{Q(z,1)} \right).$$

Recall from [7] that an indefinite quadratic form $Q = [a,b,c]$ is called *simple* if $c < 0 < a$. It is easily seen that $Q \in \mathcal{Q}_d$ is simple if and only if $Q \in \mathcal{Q}_d^+$ and $S_Q$ intersects $\mathcal{F}'' = \mathcal{F} \cup \mathcal{F}'$. For simple $Q$ let $A_Q = S_Q \cap \mathcal{F}''$ be the arc in $\mathcal{F}''$ oriented from right to left. Clearly $A_Q$ must connect the two "vertical" sides of $\mathcal{F}''$.[6] Thus from (6.5) we obtain the identity

$$F_d(\tau) = \frac{1}{2\pi} \sum_{\substack{Q \text{ simple} \\ b^2 - 4ac = d}} \int_{A_Q} \frac{j'(\tau)}{j(\tau) - j(z)} \frac{dz}{Q(z,1)}.$$

Now we deform each arc $A_Q$ in the sum of integrals to $B_Q$, which is within $\mathcal{F}''$ and has the same endpoints as $A_Q$, but travels above $\tau$. By evaluating each

---

[6]For example, when $d = 12$ the simple forms are $[1, 0, -3]$, $[1, -2, -2]$, $[1, 2, -2]$, $[3, 0, -1]$, $[2, 2, -1]$, $[2, -2, -1]$. A diagram showing the corresponding arcs $A_Q$ in this case is given in Figure 1.
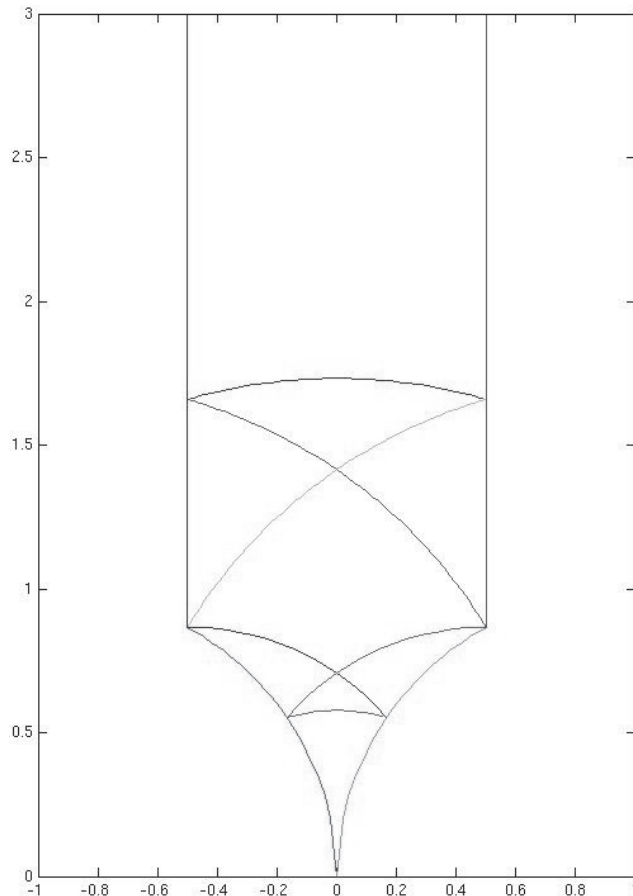
Figure 1.   Arcs $A_Q$ when $d = 12$.

resulting residue at $\tau$, we get the formula

$$F_d(\tau) = \frac{1}{2\pi} \sum_{\substack{Q \text{ simple} \\ b^2 - 4ac = d}} \int_{B_Q} \frac{j'(\tau)}{j(\tau) - j(z)} \frac{dz}{Q(z,1)} + \frac{1}{2\pi} \sum_{\substack{Q \text{ simple} \\ b^2 - 4ac = d}} Q(\tau,1)^{-1},$$

which is also valid at $-1/\tau$. A simple calculation now shows that (1.21) holds in a neighborhood of $\tau$, hence for all $\tau \in \mathcal{H}$. Thus Theorem 5 follows.

Finally, for fixed $m \in \mathbb{Z}^+$ the inequality (6.1) can be used to show that the series in Proposition 4 converges when $s = 1$. They yield the formula (1.24) upon using the elementary evaluation

$$J_{1/2}(y) = \sqrt{\tfrac{2}{\pi y}} \, \sin y.$$

## Appendix A. **Whittaker functions**

A standard reference for the theory of Whittaker functions is [40, Chap. 16]. Another good reference is [30]. For the convenience of the reader we will record here some of the properties of these special functions that we need.

For fixed $\mu, \nu$ with $\mathrm{Re}(\nu \pm \mu + 1/2) > 0$, the Whittaker functions may be defined for $y > 0$ by [30, pp. 311, 313]

$$(A.1) \quad M_{\mu,\nu}(y) = y^{\nu+\frac{1}{2}} e^{\frac{y}{2}} \frac{\Gamma(1+2\nu)}{\Gamma(\nu+\mu+\frac{1}{2})\Gamma(\nu-\mu+\frac{1}{2})} \int_0^1 t^{\nu+\mu-\frac{1}{2}}(1-t)^{\nu-\mu-\frac{1}{2}} e^{-yt}\, dt$$

and

$$(A.2) \quad W_{\mu,\nu}(y) = y^{\nu+\frac{1}{2}} e^{\frac{y}{2}} \frac{1}{\Gamma(\nu-\mu+\frac{1}{2})} \int_1^\infty t^{\nu+\mu-\frac{1}{2}}(t-1)^{\nu-\mu-\frac{1}{2}} e^{-yt}\, dt.$$

Both $M_{\mu,\nu}(y)$ and $\mathcal{W}_{\mu,\nu}(y)$ satisfy the second order linear differential equation

$$(A.3) \quad \frac{d^2 w}{dy^2} + \left(-\tfrac{1}{4} + \mu y^{-1} + (\tfrac{1}{4} - \nu^2)y^{-2}\right)w = 0.$$

Their asymptotic behavior as $y \to \infty$ for fixed $\mu, \nu$ is easily found from (A.1) and (A.2) by changing variable $t \mapsto t/y$:

$$(A.4) \quad M_{\mu,\nu}(y) \sim \frac{\Gamma(1+2\nu)}{\Gamma(\nu-\mu+\frac{1}{2})}\, y^{-\mu} e^{y/2} \quad \text{and} \quad W_{\mu,\nu}(y) \sim y^{\mu} e^{-y/2}.$$

In particular, they are linearly independent. For small $y$ we get directly from (A.1) that

$$(A.5) \quad M_{\mu,\nu}(y) = y^{\nu+\frac{1}{2}}\left(1 + \mathrm{O}_{\mu,\nu}(y)\right).$$

It is also apparent from (A.1) and (A.2) that when $\nu - \mu = 1/2$ we have

$$(A.6) \quad M_{\mu,\nu}(y) + (2\mu+1)W_{\mu,\nu}(y) = \Gamma(2\mu+2)y^{-\mu} e^{y/2},$$

while when $\nu + \mu = 1/2$ we have from (A.2) that

$$(A.7) \quad W_{\mu,\nu}(y) = y^{\mu} e^{-y/2}.$$

The $I$-Bessel and $K$-Bessel functions are special Whittaker functions [30]:

$$I_\nu(y) = 2^{-2\nu-\frac{1}{2}}\Gamma(\nu+1)^{-1} y^{-\frac{1}{2}} M_{0,\nu}(2y) \qquad \text{and} \qquad K_\nu(y) = \sqrt{\tfrac{\pi}{2y}}\, W_{0,\nu}(2y).$$

Their asymptotic properties for large $y$ thus follow from (A.4).

## References

[1] T. ASAI, M. KANEKO, and H. NINOMIYA, Zeros of certain modular functions and an application, *Comment. Math. Univ. St. Paul* **46** (1997), 93–101. MR 1448475. Zbl 0907.11013.

[2] R. E. BORCHERDS, Automorphic forms on $\mathrm{O}_{s+2,2}(\mathbf{R})$ and infinite products, *Invent. Math.* **120** (1995), 161–213. MR 1323986. Zbl 0932.11028. doi: 10.1007/BF01241126.

[3]  K. BRINGMANN and K. ONO, Arithmetic properties of coefficients of half-integral
     weight Maass-Poincaré series, *Math. Ann.* **337** (2007), 591–612. MR 2274544.
     Zbl 1154.11015. doi: 10.1007/s00208-006-0048-0.

[4]  J. H. BRUINIER, Borcherds products and Chern classes of Hirzebruch-Zagier
     divisors, *Invent. Math.* **138** (1999), 51–83. MR 1714336. Zbl 1011.11027. doi:
     10.1007/s002220050341.

[5]  J. H. BRUINIER and J. FUNKE, On two geometric theta lifts, *Duke
     Math. J.* **125** (2004), 45–90. MR 2097357. Zbl 1088.11030. doi: 10.1215/
     S0012-7094-04-12513-8.

[6]  _____ , Traces of CM values of modular functions, *J. Reine Angew. Math.* **594**
     (2006), 1–33. MR 2248151. Zbl 1104.11021. doi: 10.1515/CRELLE.2006.034.

[7]  Y. CHOIE and D. ZAGIER, Rational period functions for PSL(2, **Z**), in *A Tribute
     to Emil Grosswald*: *Number Theory and Related Analysis*, *Contemp. Math.* **143**,
     Amer. Math. Soc., Providence, RI, 1993, pp. 89–108. MR 1210514. Zbl 0790.
     11044.

[8]  W. DUKE, Modular functions and the uniform distribution of CM points,
     *Math. Ann.* **334** (2006), 241–252. MR 2207698. Zbl 1091.11012. doi: 10.1007/
     s00208-005-0706-7.

[9]  W. DUKE, J. B. FRIEDLANDER, and H. IWANIEC, Equidistribution of roots of
     a quadratic congruence to prime moduli, *Ann. of Math.* **141** (1995), 423–441.
     MR 1324141. Zbl 0840.11003. doi: 10.2307/2118527.

[10] W. DUKE and Ö. IMAMOḠLU, A converse theorem and the Saito-Kurokawa lift,
     *Internat. Math. Res. Notices* (1996), 347–355. MR 1389957. Zbl 0849.11039.
     doi: 10.1155/S1073792896000220.

[11] M. EICHLER, Grenzkreisgruppen und kettenbruchartige Algorithmen, *Acta Arith.*
     **11** (1965), 169–180. MR 0228436. Zbl 0148.32503.

[12] J. ELSTRODT, Die Resolvente zum Eigenwertproblem der automorphen Formen
     in der hyperbolischen Ebene. I, *Math. Ann.* **203** (1973), 295–300. MR 0360472.
     Zbl 0239.10014. doi: 10.1007/BF01351910.

[13] J. D. FAY, Fourier coefficients of the resolvent for a Fuchsian group, *J. Reine
     Angew. Math.* **293/294** (1977), 143–203. MR 0506038. Zbl 0352.30012. doi:
     10.1515/crll.1977.293-294.143.

[14] J. FUNKE, Heegner divisors and nonholomorphic modular forms, *Compositio
     Math.* **133** (2002), 289–321. MR 1930980. Zbl 1010.11023. doi: 10.1023/A:
     1020002121978.

[15] B. GROSS, W. KOHNEN, and D. ZAGIER, Heegner points and derivatives of *L*-
     series. II, *Math. Ann.* **278** (1987), 497–562. MR 0909238. Zbl 0079.03302. doi:
     10.1007/BF01458081.

[16] F. HIRZEBRUCH and D. ZAGIER, Intersection numbers of curves on Hilbert mod-
     ular surfaces and modular forms of Nebentypus, *Invent. Math.* **36** (1976), 57–113.
     MR 0453649. Zbl 0332.14009. doi: 10.1007/BF01390005.

[17] C. HOOLEY, An asymptotic formula in the theory of numbers, *Proc. London
     Math. Soc.* **7** (1957), 396–413. MR 0090613. Zbl 0079.27301. doi: 10.1112/
     plms/s3-7.1.396.

[18] C. HOOLEY, On the number of divisors of a quadratic polynomial, *Acta Math.* **110** (1963), 97–114. MR 0153648. Zbl 0116.03802. doi: 10.1007/BF02391856.

[19] T. IBUKIYAMA and H. SAITO, On zeta functions associated to symmetric matrices (II): Functional equations and special values, MPI preprint 1997-37; http://www.mpim-bonn.mpg.de/node/263.

[20] H. IWANIEC and E. KOWALSKI, *Analytic Number Theory, Amer. Math. Soc. Colloq. Publ.* **53**, Amer. Math. Soc., Providence, RI, 2004. MR 2061214. Zbl 0641.14013.

[21] P. JENKINS, Kloosterman sums and traces of singular moduli, *J. Number Theory* **117** (2006), 301–314. MR 2213767. Zbl 1177.11038. doi: 10.1016/j.jnt.2005.06.009.

[22] M. KANEKO, Observations on the 'values' of the elliptic modular function $j(\tau)$ at real quadratics, *Kyushu J. Math.* **63** (2009), 353–364. MR 2568779. doi: 10.2206/kyushujm.63.353.

[23] S. KATOK and P. SARNAK, Heegner points, cycles and Maass forms, *Israel J. Math.* **84** (1993), 193–227. MR 1244668. Zbl 0787.11016. doi: 10.1007/BF02761700.

[24] M. I. KNOPP, Some new results on the Eichler cohomology of automorphic forms, *Bull. Amer. Math. Soc.* **80** (1974), 607–632. MR 0344454. Zbl 1059.11001. doi: 10.1090/S0002-9904-1974-13520-2.

[25] _____ , Rational period functions of the modular group, *Duke Math. J.* **45** (1978), 47–62, With an appendix by Georges Grinstein. MR 0485700. Zbl 0374.10014. Available at http://projecteuclid.org/getRecord?id=euclid.dmj/1077312686.

[26] W. KOHNEN, *Beziehungen Zwischen Modulformen Halbganzen Gewichts und Modulformen Ganzen Gewichts, Bonner Mathematische Schriften [Bonn Math. Publ.]* **131**, Universität Bonn Mathematisches Institut, Bonn, 1981, Dissertation, Rheinische Friedrich-Wilhelms-Universität, Bonn, 1980. MR 0633060. Zbl 0451.10016.

[27] _____ , Fourier coefficients of modular forms of half-integral weight, *Math. Ann.* **271** (1985), 237–268. MR 0783554. Zbl 0542.10018. doi: 10.1007/BF01455989.

[28] W. KOHNEN and D. ZAGIER, Values of $L$-series of modular forms at the center of the critical strip, *Invent. Math.* **64** (1981), 175–198. MR 0629468. Zbl 0468.10015. doi: 10.1007/BF01389166.

[29] N. N. LEBEDEV, *Special Functions and Their Applications*, Dover Publications, New York, 1972, revised edition, translated from the Russian and edited by Richard A. Silverman; unabridged and corrected republication. MR 0350075. Zbl 0271.33001.

[30] W. MAGNUS, F. OBERHETTINGER, and R. P. SONI, *Formulas and Theorems for the Special Functions of Mathematical Physics, Third enlarged edition. Grundlehren Math. Wissen.* **52**, Springer-Verlag, New York, 1966. MR 0232968. Zbl 0143.08502.

[31] H. NEUNHÖFFER, Über die analytische Fortsetzung von Poincaréreihen, *S.-B. Heidelberger Akad. Wiss. Math.-Natur. Kl.* (1973), 33–90. MR 0352007. Zbl 0272.10015.

[32] D. Niebur, A class of nonanalytic automorphic functions, *Nagoya Math. J.* **52** (1973), 133–145. MR 0337788. Zbl 0288.10010. Available at http://projecteuclid.org/getRecord?id=euclid.nmj/1118794882.

[33] K. Ono, Unearthing the visions of a master: harmonic Maass forms and number theory, in *Current Developments in Mathematics*, 2008, Internat. Press, Somerville, MA, 2009, pp. 347–454. MR 2555930. Zbl 05635202.

[34] L. A. Parson, Modular integrals and indefinite binary quadratic forms, in *A Tribute to Emil Grosswald*: *Number Theory and Related Analysis*, *Contemp. Math.* **143**, Amer. Math. Soc., Providence, RI, 1993, pp. 513–523. MR 1210538. Zbl 0795.11022.

[35] W. Roelcke, Das Eigenwertproblem der automorphen Formen in der hyperbolischen Ebene. I, II. (German), *Math. Ann.* **167** (1966), 292–337, *ibid.* **168** (1966), 261–324. MR 0243062. Zbl 0152.07705. doi: 10.1007/BF01364540. doi: 10.1007/BF01361556.

[36] G. Shimura, On modular forms of half integral weight, *Ann. of Math.* **97** (1973), 440–481. MR 0332663. Zbl 0266.10022. doi: 10.2307/1970831.

[37] Á. Tóth, On the evaluation of Salié sums, *Proc. Amer. Math. Soc.* **133** (2005), 643–645. MR 2113909. Zbl 1092.11036. doi: 10.1090/S0002-9939-04-07768-8.

[38] G. N. Watson, *A Treatise on the Theory of Bessel Functions*, Cambridge Univ. Press, Cambridge, England, 1944. MR 0010746. Zbl 0063.08184.

[39] A. Weil, On some exponential sums, *Proc. Nat. Acad. Sci. U. S. A.* **34** (1948), 204–207. MR 0027006. Zbl 0032.26102.

[40] E. T. Whittaker and G. N. Watson, *A Course of Modern Analysis. An Introduction to the General Theory of Infinite Processes and of Analytic Functions*: *With an Account of the Principal Transcendental Functions*, Fourth; reprinted ed., Cambridge Univ. Press, New York, 1962. MR 0178117. Zbl 0105.26901.

[41] D. Zagier, Modular forms associated to real quadratic fields, *Invent. Math.* **30** (1975), 1–46. MR 0382174. Zbl 0308.10014. doi: 10.1007/BF01389846.

[42] _____, Nombres de classes et formes modulaires de poids 3/2, *C. R. Acad. Sci. Paris Sér. A* **281** (1975), 883–886. MR 0429750. Zbl 0323.10021.

[43] _____, Eisenstein series and the Riemann zeta function, in *Automorphic Forms, Representation Theory and Arithmetic* (Bombay, 1979), *Tata Inst. Fund. Res. Studies in Math.* **10**, Tata Inst. Fundamental Res., Bombay, 1981, pp. 275–301. MR 0633666. Zbl 0484.10019.

[44] _____, Traces of singular moduli, in *Motives, Polylogarithms and Hodge Theory*, *Part* I, *Internat. Press Lect. Ser.* **3**, Internat. Press, Somerville, MA, 2002, pp. 211–244. MR 1977587. Zbl 1048.11035.

[45] _____, Ramanujan's mock theta functions and their applications [d'après Zwegers and Bringmann-Ono], *Sem. Bourbaki* **326** (2009), 143–164, Exp. No. 986. MR 2605321. Zbl 1198.11046.

[46] S. P. Zwegers, Mock $\theta$-functions and real analytic modular forms, in *q-Series with Applications to Combinatorics, Number Theory, and Physics*, *Contemp.*

*Math.* **291**, Amer. Math. Soc., Providence, RI, 2001, pp. 269–277. MR 1874536. Zbl 1044.11029.

[47] S. P. ZWEGERS, Mock theta functions, 2002, Utrecht Ph.D. thesis. Available at http://igitur-archive.library.uu.nl/dissertations/2003-0127-094324/inhoud.htm.

UNIVERSITY OF CALIFORNIA, LOS ANGELES, CA
*E-mail*: wdduke@ucla.edu

ETH, ZÜRICH, SWITZERLAND
*E-mail*: ozlem@math.ethz.ch

EOTVOS LORAND UNIVERSITY, DEPARTMENT OF ANALYSIS, BUDAPEST, HUNGARY
*E-mail*: toth@cs.elte.hu