Annals of Mathematics **173** (2011), 907–945
doi: 10.4007/annals.2011.173.2.7

Hermitian integral geometry

By Andreas Bernig and Joseph H. G. Fu

Abstract

We give in explicit form the principal kinematic formula for the action of the affine unitary group on $\mathbb{C}^n$, together with a straightforward algebraic method for computing the full array of unitary kinematic formulas, expressed in terms of certain convex valuations introduced, essentially, by H. Tasaki. We introduce also several other canonical bases for the algebra of unitary-invariant valuations, explore their interrelations, and characterize in these terms the cones of positive and monotone elements.

1. Introduction

1.1. *General background.* In [17], it was shown that if $\overline{G}$ is a Lie group acting transitively on the sphere bundle of a Riemannian manifold M, then there exist kinematic formulas (cf. (1) below) for certain geometric quantities associated to subspaces $A, B \subset M$; the case $\overline{G} = \mathrm{SO}(n) \ltimes \mathbb{R}^n, M = \mathbb{R}^n$ being the classical kinematic formulas of Blaschke-Santaló-Federer-Chern [26], [30]. The proof was a distillation of the geometric method used in [14] and [16] to establish the classical case.

A different and in some ways more incisive proof of the classical case was provided by [22]. Restricting formally to the case where the subspaces are convex sets, Hadwiger displayed a concrete finite basis for the vector space of continuous convex valuations invariant under the euclidean group. The existence of the kinematic formulas is then a simple consequence, and the precise numerical values of the coefficients involved may be calculated using the "template method", i.e., by evaluating the relevant integrals for enough conveniently chosen A, B. This approach leaves the impression that the values of the coefficients are in some way accidental. However, A. Nijenhuis [27] showed by direct calculation that under a suitable renormalization of the Hadwiger basis all of the coefficients are equal to unity.

More recently, S. Alesker [1] gave another proof of the theorem of [17] as part of a far-reaching reconceptualization of the theory of convex valuations. He showed that if G is a compact Lie group acting transitively on the sphere of a euclidean space V, then the space $\mathrm{Val}^G(V)$ of continuous convex valuations

invariant under the group $\overline{G} := G \ltimes V$, generated by translations and the action of G, is finite dimensional. Just as in the case of the full euclidean group, the theorem of [17] follows directly (at least in the euclidean case). In these terms, the result may be stated as follows.

THEOREM 1.1. *Let $\phi_1, \dots, \phi_N$ be a basis for $\mathrm{Val}^G(V)$. Given $\mu \in \mathrm{Val}^G(V)$, there are constants $c_{ij}^{\mu}, 1 \le i, j \le N$, such that for any two compact convex bodies $A, B \subset V$*

$$\int_{\overline{G}} \mu(A \cap \bar{g}B)\, d\bar{g} = \sum_{ij} c_{ij}^{\mu} \phi_i(A) \phi_j(B). \tag{1}$$

Moreover, in [1] Alesker gave an explicit basis (in fact two of them) for the space $\mathrm{Val}^{U(n)}(\mathbb{C}^n)$ of unitary-invariant valuations on $\mathbb{C}^n$. Although this in itself gives a lot of information about the kinematic formulas, a complete determination of the formulas using the template method appears intractable (although H. Park [28] used it successfully in the cases $n = 2, 3$).

Meanwhile, H. Tasaki [31], [32], building on previous work of R. Howard [23], established a more detailed description of the unitary kinematic formula, which he stated in the restricted case where A, B are compact submanifolds of complementary dimension. He showed first of all that if $k \le n$, then the unitary orbits of the (real) dimension k (resp. codimension k) Grassmannian $\mathrm{Gr}_k(\mathbb{C}^n)$ (resp. $\mathrm{Gr}_{2n-k}(\mathbb{C}^n)$) are naturally parametrized by the $p := \lfloor \frac{k}{2} \rfloor$-simplex $\{(\theta_1, \dots, \theta_p) : 0 \le \theta_1 \le \cdots \le \theta_p \le \frac{\pi}{2}\}$. Put $\Theta(E)$ for this "multiple Kähler angle" of $E \in \mathrm{Gr}_{k \text{ or } 2n-k}(\mathbb{C}^n)$, and $\cos^2 \Theta(E)$ for the vector with components $\cos^2 \theta_i$. Tasaki's theorem may then be restated as follows.

THEOREM 1.2 (Tasaki [32]). *Given $k \le n$, there is a symmetric $(p+1) \times (p+1)$ matrix $T = T_k^n$ such that whenever $A^k, B^{2n-k} \subset \mathbb{C}^n$ are C^1 submanifolds of dimension and codimension k respectively,*

$$\int_{\overline{U(n)}} \#(A \cap \bar{g}B)\, d\bar{g} = \sum_{ij} T_{ij} \int_A \sigma_i(\cos^2 \Theta(T_x A))\, dx \int_B \sigma_j(\cos^2 \Theta(T_y B))\, dy, \tag{2}$$

where σ_i is the i^{th} elementary symmetric function and $\overline{U(n)} = U(n) \ltimes \mathbb{C}^n$ is the affine unitary group.

As Tasaki noted, this formula also holds verbatim if $\mathbb{C}^n$ is replaced by either of the complex space forms $\mathbb{C}P^n, \mathbb{C}H^n$, with their full groups of isometries. This is an instance of the *transfer principle*, which we discuss in Section 2.5 below.

1.2. *Results of the present paper.* In the pages to follow we bring more of the algebraic machinery introduced by Alesker to bear on the problem of the integral geometry of the unitary group. The key underlying observation

(Theorem 2.1 below) is that the graded multiplication introduced in [3] on the space of convex valuations is intimately related to the various G-kinematic formulas. This illuminates even the classical $\mathrm{SO}(n)$ theory, explaining the result of Nijenhuis cited above (cf. [20, §2.3], and also [19]). Our point of entry is the determination in [20] of the multiplicative structure of $\mathrm{Val}^{U(n)}(\mathbb{C}^n)$. Here we give a more or less complete set of answers to the questions posed in Section 4 of [20]. We now describe our present results as they relate to those questions, in the order given there.

(1) *Explicit kinematic formulas for $U(n)$.* The paper [20] posed the problem of computing the kinematic formulas explicitly in terms of the *monomial basis* (cf. §3.1). This boils down to computing the inverses Q_k^n of certain symmetric matrices P_k^n. It turns out that the P_k^n are Hankel matrices with ascending entries of the form $\binom{2i}{i}$. Thus the expansion of the inverse as a polynomial in the matrix entries gives some kind of answer to this question, but it seems unreasonably complicated. It would be interesting to have a closed form.

In the present paper we take a different approach, showing how to determine completely the unitary kinematic formulas (cf. Theorem 5.12, Corollary 5.14 and §5.4) in terms of the *Tasaki basis* (cf. Proposition 3.7 below) for $\mathrm{Val}^{U(n)}$, obtaining in this way the *Tasaki matrices* T_k^n, which may be obtained in principle by a change of basis from the Q_k^n. Although the formulas remain complicated, they are an order of magnitude less so than the naïve formulas for Q_k^n described above. Using this approach we can show, for instance, that the Q_k^n are positive definite (Corollary 5.13), answering another question of [20]. Furthermore, the Tasaki valuations are more amenable to calculation in concrete geometric situations. Strictly speaking we carry this out in full detail only for the principal kinematic formula $k_{U(n)}(\chi)$ (cf. (8) below), then show how the general formulas may be computed in an essentially straightforward way.

Among the many special bases for $\mathrm{Val}^{U(n)}$ (cf. the next item) the Tasaki valuations seem to enjoy a privileged status. For example, if $k = 2p$ is even then, in addition to the usual diagonal symmetry $(T_k^n)_{ij} = (T_k^n)_{ji}$, the Tasaki matrices T_{2p}^n display the unexpected antidiagonal symmetry $\left(T_{2p}^n\right)_{ij} = \left(T_{2p}^n\right)_{p-i,p-j}$ (Theorem 3.10).

(2) *Canonical bases and their interrelations.* We explore with varying degrees of depth several canonical bases for $\mathrm{Val}^{U(n)}(\mathbb{C}^n)$: the monomial basis and its Fourier transform, the hermitian intrinsic volumes $\mu_{k,q}$ (which correspond in a natural way with certain differential forms on the tangent bundle of $\mathbb{C}^n$), their "Crofton duals" $\nu_{k,q}$, and the Tasaki valuations $\tau_{k,q}$ and their Fourier transforms $\widehat{\tau_{k,q}}$. Although we explicitly study their interrelations only to the extent necessary to answer our other concerns, there is enough information here to give a complete (though again complicated) dictionary among them.

(2) *Special cones.* We show that the cone P of nonnegative elements of $\mathrm{Val}^{U(n)}$ is generated by the hermitian intrinsic volumes. Stimulated by the fact (due to Kazarnovskii) that the "Kazarnovskii pseudo-volume" $\mu_{n,0}$ is at once nonnegative and nonmonotone, we give a complete characterization of the cone M of monotone elements of $\mathrm{Val}^{U(n)}$.

As a concluding general remark, we have taken care to give precise and complete values whenever possible. Beyond the obvious motive of providing solid information for possible applications, we mean to make the point (in the only way possible) that this algebraic approach is sufficient to formulate these results in complete detail, in an area historically plagued by statements of the form "There exists a formula such that…."

In the latter respect, however, things are not yet in a satisfactory state. Some results are given in terms of sums for which we have not found closed forms. Whether or not such closed forms exist, their nature suggests that there might exist some combinatorial model that generates them, perhaps something like the devices that occur in Schubert calculus. Indeed much of the approach in the following pages is inspired by the principle that the algebras $\mathrm{Val}^G(V)$ are similar to the cohomology algebras of Kähler manifolds — it is even the case that the main subject of this paper, $\mathrm{Val}^{U(n)}(\mathbb{C}^n)$, has the same Betti numbers as the even-degree cohomology of the Grassmann manifold of complex 2-planes in $\mathbb{C}^{n+2}$, although the algebras themselves are not isomorphic.

Acknowledgements. We wish to thank Semyon Alesker, Ludwig Bröcker, Dan Nakano, Jason Parsley, Ted Shifrin and Robert Varley for helpful discussions, and the Universities of Georgia (USA) and Fribourg (Switzerland) for hosting our mutual visits as we worked out this material. We thank also the anonymous referee, whose many useful remarks greatly improved the text at key points, and who in particular suggested the proof of Theorem 2.12 in the nonsmooth case.

2. **Valuations and curvature measures**

Throughout most of this section we let V be an oriented euclidean vector space of dimension $n < \infty$. We note, however, that for much of the discussion the euclidean assumption is not strictly necessary if we substitute the dual space V^* for V in appropriate spots.

We put

$$\omega_k := \frac{\pi^{\frac{k}{2}}}{\Gamma(\frac{k}{2}+1)}$$

for the volume of the k-dimensional euclidean ball. In particular

$$\omega_{2l} = \frac{\pi^l}{l!},$$

which also happens to be the volume of the complex projective space $\mathbb{C}P^l$ under the Fubini-Study metric.

2.1. *Basics.* For definiteness we will work formally in the context of convex valuations on V. However, many statements apply also to other geometrically valid subsets (e.g. C^2 submanifolds, or in the case of the Crofton formulas even C^1 submanifolds) of smooth manifolds, in terms of the formalism of valuations on manifolds [4], [5], [10], [7], [6]. Since these notions intervene only at the stage of interpretation of our main results, and never in an essential technical way, we will say no more about them.

We put $\mathcal{K} = \mathcal{K}(V)$ for the space of all compact convex subsets of V, endowed with the Hausdorff metric, and $\mathcal{K}^{\mathrm{sm}}(V) \subset \mathcal{K}(V)$ for the subspace consisting of subsets with nonempty interior and smooth boundary, and for which all principal curvatures are nonzero. We refer to [13] and the sources cited there for the definition and basic properties of the vector space $\mathrm{Val} = \mathrm{Val}(V)$ of continuous translation-invariant valuations on V, and of the dense subspace $\mathrm{Val}^{\mathrm{sm}}(V)$ of smooth valuations. Basic examples of these objects include the Euler characteristic χ and the volume measure vol_n.

Recall that a valuation ϕ is of *degree* k if $\phi(tK) = t^k\phi(K)$ for all $t \geq 0$ and *even* if $\phi(-K) = \phi(K)$ for all $K \in \mathcal{K}$. The corresponding subspace of Val is denoted by Val_k^+. It is known [25] that the restriction of an even valuation μ of degree k to a k-dimensional subspace $E \subset V$ is a multiple of the restriction of the usual Hausdorff measure vol_k to E. Putting $\mathrm{Kl}_\mu(E)$ for the proportionality factor, we obtain the *Klain function* $\mathrm{Kl}_\mu \in C(\mathrm{Gr}_k(V))$ of μ. In other words, Kl_μ is uniquely characterized by the relation

$$\mu(K) = \mathrm{Kl}_\mu(E)\,\mathrm{vol}_k(K) \text{ for } E \in \mathrm{Gr}_k, \quad K \in \mathcal{K}(E). \tag{3}$$

A theorem of Klain [25] states that the resulting map Kl from the space of even valuations of degree k to $C(\mathrm{Gr}_k(V))$ is injective.

Every even $\mu \in \mathrm{Val}_k^{\mathrm{sm}}(V)$ admits a *Crofton measure*, i.e. a signed measure m on $\mathrm{Gr}_k(V)$ such that

$$\mu(A) = \int_{\mathrm{Gr}_k(V)} \mathrm{vol}_k(\pi_E(A))\, dm(E),$$

where π_E is the orthogonal projection to E. This follows from the Alesker-Bernstein theorem [9] (compare also §1 in [1]).

We recall also Alesker's *Fourier transform* $\mathbb{F} : \mathrm{Val}_k^{\mathrm{sm}}(V) \to \mathrm{Val}_{n-k}^{\mathrm{sm}}(V)$ (cf. [8]). In the present paper we will denote the Fourier transform of a valuation ϕ by

$$\widehat{\phi} := \mathbb{F}\phi. \tag{4}$$

We will only make use of it for even valuations, in which case it is uniquely characterized in terms of the Klain embedding by

$$\mathrm{Kl}_{\widehat{\phi}}(E) = \mathrm{Kl}_{\phi}(E^{\perp}), \quad E \in \mathrm{Gr}_{n-k}(V) \tag{5}$$

for even $\phi \in \mathrm{Val}_k^{\mathrm{sm}}$. In this form, the Alesker-Fourier transform was denoted by $\mathbb{D}$ in [1], [13] and in several other papers.

Alesker has defined in [3] a commutative graded product on $\mathrm{Val}^{\mathrm{sm}}(V)$, with the property that the symmetric pairing

$$(\phi, \psi) := \text{ degree } n \text{ part of } \phi \cdot \psi \tag{6}$$

is perfect. We recall [13] that the related pairing

$$\langle \phi, \psi \rangle := (\phi, \widehat{\psi}) \tag{7}$$

is symmetric. We will see later that the restriction of the pairing $\langle \cdot, \cdot \rangle$ to $\mathrm{Val}^{U(n)}$ is positive definite. However this is not true of the unrestricted pairing— it is shown in [11] that if n is odd then the index of the restriction of the pairing to $\mathrm{Val}^{SU(n)}(\mathbb{C}^n)$ is 1.

2.2. *Grassmannians.* We denote the Grassmannian of k-dimensional subspaces of the real vector space V by $\mathrm{Gr}_k(V)$. If $V = \mathbb{C}^n$ (considered as a *real* vector space) we consider the (k,p)-*Grassmannian* $\mathrm{Gr}_{k,p}(\mathbb{C}^n) \subset \mathrm{Gr}_k(\mathbb{C}^n)$ to be the submanifold of all k-dimensional real subspaces that may be expressed as the orthogonal direct sum of a p-dimensional complex subspace and a $(k-2p)$-dimensional real subspace that is isotropic with respect to the standard symplectic (Kähler) structure on $\mathbb{C}^n$. A general element of $\mathrm{Gr}_{k,p}$ will be denoted $E^{k,p}$. It is easy to see that $\mathrm{Gr}_{k,p}$ is the orbit of $\mathbb{C}^p \oplus \mathbb{R}^{k-2p}$ under the standard action of $U(n)$. In particular, $\mathrm{Gr}_{2p,p}$ is the Grassmannian of p-dimensional complex subspaces and $\mathrm{Gr}_{n,0}(\mathbb{C}^n)$ is the Lagrangian Grassmannian.

2.3. *Global and local.* We recall that the family of algebras $\mathrm{Val}^{U(n)} := \mathrm{Val}^{U(n)}(\mathbb{C}^n)$ is related by the sequence of surjective restriction homomorphisms $\mathrm{Val}^{U(n)} \to \mathrm{Val}^{U(n-1)}, n \geq 1$. The algebra $\mathrm{Val}^{U(\infty)}$ of *global valuations* is the inverse limit of this system; abusing terminology we will identify a global valuation with its images in the various $\mathrm{Val}^{U(n)}$. An expression for an element of $\mathrm{Val}^{U(n)}$ that does not hold in $\mathrm{Val}^{U(\infty)}$ will be called *local*, or *local at* n. Likewise we will refer to global and local relations among valuations.

2.4. *Poincaré duality and kinematic formulas.* We recall from [1] that if $G \subset O(V)$ is a compact group acting transitively on the sphere of V, then the space $\mathrm{Val}^G(V)$ of G-invariant and translation invariant valuations on V is finite dimensional. It follows (cf. [13]) that there is a linear injection k_G :

$\mathrm{Val}^G \to \mathrm{Val}^G \otimes \mathrm{Val}^G$ such that whenever $A, B \in \mathcal{K}$ and $\phi \in \mathrm{Val}^G$,

$$k_G(\phi)(A,B) = \int_{\overline{G}} \phi(A \cap \bar{g}B)\, d\bar{g}. \tag{8}$$

Here $\overline{G} := G \ltimes V$ is the group generated by G and the translation group of V and $d\bar{g}$ is the Haar measure, normalized so that

$$d\bar{g}\,(\{\bar{g} : \bar{g}o \in S\}) = \mathrm{vol}_n(S), \quad S \subset V \text{ measurable}, \tag{9}$$

where $o \in V$ is an arbitrarily chosen point. If $\phi \in \mathrm{Val}_k^G$, then $k_G(\phi) \in \bigoplus_{i+j=n+k} \mathrm{Val}_i^G \otimes \mathrm{Val}_j^G$. Taking A to be a point, it is clear that the term of $k_G(\chi)$ of bidegree $(0,n)$ is $\chi \otimes \mathrm{vol}_n$.

The algebraic approach to the kinematic formula is based on the following statement from [13]. Let $p : \mathrm{Val}^G \to \mathrm{Val}^{G^*}$ denote the linear isomorphism induced by the Poincaré duality pairing (6), $m_G : \mathrm{Val}^G \otimes \mathrm{Val}^G \to \mathrm{Val}^G$ the restriction of the multiplication map to Val^G, and $m_G^* : \mathrm{Val}^{G^*} \to \mathrm{Val}^{G^*} \otimes \mathrm{Val}^{G^*}$ its adjoint.

THEOREM 2.1.

$$(p \otimes p) \circ k_G = m_G^* \circ p. \tag{10}$$

To state this in more sensible terms:

THEOREM 2.2. *Let $\phi_1, \dots, \phi_N$ and $\psi_1, \dots, \psi_N$ be bases of Val^G, and let M be the $N \times N$ matrix*

$$M_{ij} := (\phi_i, \psi_j),$$

where the right-hand side is given by the Poincaré duality pairing (6). *Let $K := M^{-1}$. Then*

$$k_G(\chi) = \sum_{i,j} K_{ij}\, \phi_i \otimes \psi_j. \tag{11}$$

If the $\psi_i = \widehat{\phi_i}$, then M and K are symmetric. More generally, for any $\mu \in \mathrm{Val}^G$,

$$k_G(\mu) = \sum_{i,j} K_{ij}\, (\mu \cdot \phi_i) \otimes \psi_j = \sum_{i,j} K_{ij}\, \phi_i \otimes (\mu \cdot \psi_j). \tag{12}$$

The symmetry assertion is of course the same as the symmetry of the pairing (7).

These formulas also apply to other types of geometric subsets of V, as described in [17], [23], and [18]. The simplest case occurs when A, B are smooth compact submanifolds of complementary dimensions $k, n-k$. It is advantageous to use bases for Val^G comprised of bases for the components Val_k^G of the grading by degree. Given an even valuation $\phi \in \mathrm{Val}_k^G$, and a

compact C^1 k-dimensional submanifold $A \subset V$, it is natural to put

$$\phi(A) := \int_A \mathrm{Kl}_\phi(T_x A)\, dx$$

and the kinematic formula yields the Crofton formula

$$\int_{\overline{G}} \#(A \cap \bar{g}B)\, d\bar{g} = \sum_{\deg \phi_i = k, \deg \psi_j = n-k} K_{ij}\, \phi_i(A)\psi_j(B), \tag{13}$$

where $\#$ denotes the cardinality.

2.5. *The transfer principle for Crofton formulas.* R. Howard has established a general Crofton formula for Riemannian homogeneous spaces $M := G/K$. Put $\overline{\mathrm{Gr}_m}(M)$ for the dimension m Grassmann bundle over M.

THEOREM 2.3 ([23]). *Let G be a unimodular Lie group and $M := G/K$ a Riemannian homogeneous space of G, and let $m + n \geq \dim M$. Let the Haar measure on M be given by* (9). *Then there exists a nonnegative function $f_{M,G,K} \in C^\infty(\overline{\mathrm{Gr}_m}(M) \times \overline{\mathrm{Gr}_n}(M))$, invariant under the action of $G \times G$ on $\overline{\mathrm{Gr}_m}(M) \times \overline{\mathrm{Gr}_n}(M)$, such that if $A^m, B^n \subset M$ are C^1 submanifolds, then*

$$\int_G \mathrm{vol}_{\dim M - m - n}(A \cap gB)\, dg = \int\int_{A \times B} f_{M,G,K}(T_x A, T_y B)\, dx\, dy. \tag{14}$$

Note that the function $f_{M,G,K}$ is completely determined by its restriction $\bar{f}_{M,G,K}$to $\mathrm{Gr}_m(T_oM) \times \mathrm{Gr}_n(T_oM)$, where $o = [K] \in M$ is a representative point, and that this restriction is $K \times K$ invariant. Under this correspondence, the function $f_{M,G,K}$ is in a certain sense universal:

THEOREM 2.4 (Transfer principle [23]). *Suppose G' is another unimodular Lie group containing K, and $M' = G'/K$ an associated Riemannian homogeneous space, such that for representative points $o \in M, o' \in M'$ there exists an isometric K-map $T_oM \to T_{o'}M'$. If we identify these two spaces via this map, then $\bar{f}_{M,G,K} = \bar{f}_{M',G',K}$.*

Heuristic proof. Given $A^m, B^n \subset M$, we may think of A, B as being made up of infinitesimal pieces of linear elements $E \in \mathrm{Gr}_m(V), F \in \mathrm{Gr}_n(V)$, where $V := T_oM \simeq T_xM$ for any $x \in M$. Taking Riemann sums, it follows that $\bar{f}_{M,G,K} = \bar{f}_{V,K \ltimes V,K}$. □

2.6. *The normal cycle, curvature measures and the first variation of a valuation.* Let $S(V)$ denote the unit sphere of V and set $SV := V \times S(V)$, the sphere bundle over V. Given a smooth translation-invariant form $\beta \in \Omega^{n-1}(SV)^V$ we define $\Psi_\beta \in \mathrm{Val}^{\mathrm{sm}}(V)$ by

$$\Psi_\beta(A) := \int_{N(A)} \beta \tag{15}$$

for $A \in \mathcal{K}(V)$, where $N(A)$ is the normal cycle of A. Conversely, any element of $\mathrm{Val}^{\mathrm{sm}}(V)$ may be expressed as $c\,\mathrm{vol}_n + \Psi_\beta$ for some constant c and some β as above. This was proved by Alesker [4, Thm. 5.2.1].

The map $\Psi : \Omega^{n-1}(SV)^V \to \mathrm{Val}^{\mathrm{sm}}(V)$ from forms to valuations may be factored through the *curvature measure map* Φ as follows. The *curvature measure* Φ_β is defined to be the assignment to any $A \in \mathcal{K}(V)$, of a signed measure supported on ∂A given by

$$\Phi^A_\beta(S) := \int_{\pi^{-1}(S)\cap N(A)} \beta$$

for measurable subsets $S \subset V$, where $\pi : SV \to V$ is the projection. Thus $\Psi_\beta(A) = \Phi^A_\beta(A)$. We say that the curvature measure Φ_β is *nonnegative* if the measure $\Phi^A_\beta \geq 0$ for all $A \in \mathcal{K}(V)$. We observe that if the boundary of A is a smooth hypersurface, then the last integral may be expressed as the integral over S of a function, determined by β, which at each point $x \in \partial A$ is polynomial in the second fundamental form of ∂A at x (cf. Lemma 2.8 below).

Recall that SV is a contact manifold with the global contact form α defined by $\alpha|_{(x,v)}(w) = \langle v, d\pi(w)\rangle$. The unique vector field T on SV with $i_T\alpha = 1, \mathcal{L}_T\alpha = 0$ is called the *Reeb vector field* (here $\mathcal{L}$ denotes the Lie derivative). Given a form $\beta \in \Omega^{n-1}(SV)$ there exists a unique vertical form $\alpha \wedge \xi$ such that $d(\beta + \alpha \wedge \xi)$ is vertical, i.e., is a multiple of α. The *Rumin operator* D, introduced in [29], is the second order differential operator $D\beta := d(\beta + \alpha \wedge \xi)$.

Consider now the first variation of a valuation $\mu \in \mathrm{Val}^{\mathrm{sm}}(V)$: given $A \in \mathcal{K}^{\mathrm{sm}}$ and a smooth vector field ξ on V, we put

$$\delta_\xi \mu(A) := \left.\frac{d}{dt}\right|_{t=0} \mu(F_t(A))$$

where F_t is the flow of ξ. The following implies that $\delta_\xi\mu$ extends by continuity to a smooth (but not translation-invariant) valuation in the sense of [4], [5], [7], [6], [10] (although we will not make use of this fact).

Lemma 2.5. *If $\mu = \Psi_\beta$, then*

$$\delta_\xi \mu(A) = \int_{N(A)} \langle \xi, \pi_* T\rangle \, i_T(D\beta). \tag{16}$$

Since ∂A is smooth this may be rephrased as

Corollary 2.6. *Suppose $A \in \mathcal{K}^{\mathrm{sm}}(V)$, and let n be the outward pointing normal field to ∂A. Then*

$$\delta_\xi \Psi_\beta(A) = \int_{\partial A} \langle \xi, n\rangle \, d\Phi^A_{i_T(D\beta)}\,.$$

Proof of Lemma 2.5. Let $\tilde{\xi}$ denote the complete lift of ξ to SV, i.e., the vector field whose flow $\tilde{F}_t$ consists of contact transformations and which covers F_t ([33]). Put $A_t := F_t(A)$. Then $N(A_t) = \tilde{F}_{t*}(N(A))$, whence

$$\begin{aligned}\delta_\xi \Psi_\beta(A) &= \frac{d}{dt}\Big|_{t=0} \left(\int_{N(A_t)} \beta \right) \\ &= \int_{N(A)} \mathcal{L}_{\tilde{\xi}} \beta \\ &= \int_{N(A)} \alpha(\tilde{\xi})\, i_T D\beta \\ &= \int_{N(A)} \langle \xi, \pi_* T \rangle\, i_T \left(D\beta \right),\end{aligned}$$

as claimed. □

The kernel of the map Ψ of (15) has been characterized in [12]. This result may be restated in the vector space setting as follows. Define the map δ from $\mathrm{Val}^{\mathrm{sm}}$ to the space of curvature measures by

$$\delta(\Psi_\beta) := \Phi_{i_T(D\beta)}, \quad \delta(\mathrm{vol}_n) := P, \tag{17}$$

where $P^K(S) = \mathrm{vol}_{n-1}(S \cap \partial K)$ for $K \in \mathcal{K}^{\mathrm{sm}}$ and $S \subset V$ measurable. We recall that $\mu \in \mathrm{Val}(V)$ is said to be *monotone* if $\mu(K) \leq \mu(L)$ whenever $K \subset L, K, L \in \mathcal{K}(V)$.

THEOREM 2.7. *The mapping δ is well-defined, with kernel equal to the one-dimensional subspace spanned by the Euler characteristic χ. A valuation $\mu \in \mathrm{Val}^{\mathrm{sm}}(V)$ is monotone if and only if $\delta\mu \geq 0$ and $\mu(\{\mathrm{point}\}) \geq 0$.*

Proof. That δ is well-defined follows from Lemma 2.5. Corollary 2.6 implies that if $\mu \in \ker \delta$ then $\delta_\xi \mu \equiv 0$ for all smooth vector fields ξ. Taking $\xi := -\sum x_i \frac{\partial}{\partial x_i}$ to be the Euler vector field generating the homothetic flow towards the origin, continuity implies that $\mu(K) = \mu(\{0\}) =: c$ for all $K \in \mathcal{K}$. It follows that $\mu = c\chi$.

To prove the last assertion, by continuity of μ it is enough to show that μ is monotone if and only if $\mu(\{\mathrm{point}\}) \geq 0$ and $\delta\mu^K \geq 0$ for all $K \in \mathcal{K}^{\mathrm{sm}}$.

Suppose μ is monotone and $K \in K^{\mathrm{sm}}$. Then $\mu(\{\mathrm{point}\}) \geq 0$ since $\mu(\emptyset) = 0$. Furthermore, if $f : \partial K \to \mathbb{R}$ is smooth and ≥ 0 then by Corollary 2.6 $0 \leq \delta_{fn}\mu(K) = \int_{\partial K} f\, d(\delta\mu)^K$. This implies that $(\delta\mu)^K \geq 0$, as claimed.

To prove the converse, it is enough to show that if $K, L \in \mathcal{K}^{\mathrm{sm}}$ and $K \supset L$ then $\mu(K) \geq \mu(L)$. Under these conditions there is a smooth deformation $F_t : V \to V$ such that $F_0 = Id$, $F_1(L) = K$ and $\langle \frac{\partial F_t}{\partial t}(t), n \rangle \geq 0$ for all outward normals n to $F_t(L)$ (for example, the deformation arising from the linear interpolation between the support functions of the two bodies). Integrating the result of Corollary 2.6 completes the proof. □

2.7. *Constant coefficient valuations.* If β extends to a translation-invariant form $\overline{\beta} \in \Omega^{n-1}(TV)^V \simeq \Omega^{n-1}(V \times V)^V$, then Stokes' theorem gives

$$\int_{N(A)} \beta = \int_{N_1(A)} d\overline{\beta},$$

where $N_1(A)$ is the "disk bundle" defined in (41) of [13]. We consider here the case where $\phi := d\overline{\beta}$ has constant coefficients, i.e. $\phi \in \Lambda^n(V \oplus V)$.

This subject is relevant here for two related reasons. First, it turns out (cf. Theorem 3.2 below) that all unitary-invariant valuations belong to this class. Second, constant coefficient valuations are important even in the general theory of valuations: from (17), we know that the first variation $\delta\mu$ of any valuation μ on V^n corresponds to a translation-invariant differential form γ of degree $n-1$ on the sphere bundle SV, which is a contact manifold. At each point $(x, v) \in SV$, the contact hyperplane $Q_{x,v}$ may be naturally identified with $P_v \oplus P_v$, where $P_v := v^\perp$. Thus if we fix (x, v) and restrict $\gamma_{x,v}$ to $Q_{x,v}$, we obtain an element of $\Lambda^{n-1}(P_v \oplus P_v)$. We may now regard $\gamma_{x,v}$ as giving a constant coefficient valuation on the vector space P_v. It turns out that the positivity of this family of "infinitesimal" constant coefficient valuations (parametrized by $(x, v) \in SV$) is equivalent to the monotonicity (in the sense defined in the remarks preceding Theorem 2.7 above) of μ. This has the following consequence: in view of the fact (Corollary 2.10) that a constant coefficient valuation is positive if and only if its homogeneous components are positive, a general translation-invariant valuation is monotone if and only if its homogeneous components are monotone (Theorem 2.12).

Strictly speaking, the positivity of the constant coefficient valuation determined by $\gamma_{x,v}$ is not the relevant concern for the monotonicity question—instead, the matter turns on the positivity of the functional on symmetric bilinear forms defined in equation (18). However, Lemma 2.9 and and Proposition 2.11 show that these two conditions are equivalent. This is a help when we want to determine the monotone cone in the space of $U(n)$-invariant valuations: the family of infinitesimal constant coefficient valuations that arise in calculating their first variations may be expressed in terms of the invariant valuations in dimension $n-1$. Thus the determination of the (invariant) positive cone translates at once into a criterion (Proposition 4.5) for the monotone cone.

Put Σ for the vector space of self-adjoint linear maps $V \to V$. We identify Σ in the usual way with the space of symmetric bilinear forms on V. Given $\phi \in \Lambda^n(V \oplus V)$, consider the map $\lambda_\phi : \Sigma \to \mathbb{R}$ given by

$$\lambda_\phi(\sigma) := \overline{\sigma}^* \phi, \tag{18}$$

where $\overline{\sigma}(v) := (v, \sigma v)$ is the graphing map, and we identify $\Lambda^n V$ with $\mathbb{R}$ by $t \cdot \mathrm{vol} \simeq t$. Given a euclidean space W of dimension $n+1$, together with $A \in \mathcal{K}^{\mathrm{sm}}(W)$ and $\beta \in \Omega(SW)^W$, it is convenient to express the curvature measure

Φ^A_β in these terms by taking $V := T_x\partial A$, where $x \in \partial A$. Let $n : \partial A \to S(W)$ denote the Gauss map and $\sigma_x : V \to V$ the Weingarten map. As above, the contact hyperplane $Q_{x,n(x)}$ is naturally identified with $V \oplus V$, and (after restriction) $\beta_{x,n(x)} \in \Lambda^n(V \oplus V)$. The following is immediate.

LEMMA 2.8. *Let $\beta \in \Omega(SV)^V$. For $A \in \mathcal{K}^{\mathrm{sm}}$ the curvature measure determined by β may be expressed as the curvature integral*

$$\Phi^A_\beta(S) = \int_{S\cap\partial A} \lambda_{\beta_{x,n(x)}}(\sigma_x)\, dx. \tag{19}$$

We say that $\lambda_\phi \geq 0$ if $\lambda_\phi(\sigma) \geq 0$ whenever σ is nonnegative semidefinite. Put ν_ϕ for the valuation

$$\nu_\phi(K) := \int_{N_1(K)} \phi. \tag{20}$$

Put $\Lambda^n_k(V \oplus V)$ for the space of forms of bidegree $(k, n-k)$ and $\Sigma_k \subset \Sigma$ for the cone of maps of corank k.

Observe that if $\phi \in \Lambda^n_k(V \oplus V)$ then the Klain function Kl_{ν_ϕ} is given as follows. Given $E \in \mathrm{Gr}_k(V)$, let $\bar e_1, \dots, \bar e_n$ be a basis *adapted to* E, i.e., an orthonormal basis for V such that $\bar e_1, \dots, \bar e_k$ span E. Put $e_i := (\bar e_i, 0), \varepsilon_i := (0, \bar e_i)$. Then

$$\mathrm{Kl}_{\nu_\phi}(E) = \pm\omega_{n-k}\, \phi(e_1, \dots, e_k, \varepsilon_{k+1}, \dots, \varepsilon_n), \tag{21}$$

where the sign is positive or negative accordingly as the ordered basis $\bar e_i$ determines the correct orientation of V or not.

LEMMA 2.9. *Suppose $\phi \in \Lambda^n_k(V \oplus V)$. The following are equivalent*:

(1) $\nu_\phi \geq 0$.
(2) $\mathrm{Kl}_{\nu_\phi} \geq 0$.
(3) $\lambda_\phi \geq 0$.
(4) $\lambda_\phi(\sigma) \geq 0$ *for all* $\sigma \in \Sigma_k$ *with* $\sigma \geq 0$.

Proof. (1) $\iff$ (2): That (1) $\implies$ (2) is obvious. To prove the converse it is enough to observe that if $P \subset V$ is a compact convex polytope then

$$\nu_\phi(P) = \sum_{F\in P^k} \mathrm{Kl}_{\nu_\phi}(\langle F\rangle)\, \mathrm{vol}_k(F)\angle(P, F),$$

where P^k is the k-skeleton of P, $\langle F\rangle$ is the k-plane spanned by F, and $\angle(P, F)$ is the normalized exterior angle of P along F. (4) $\iff$ (3): That (3) $\implies$ (4) is obvious. To prove the converse, let $\tau \in \Sigma, \tau \geq 0$. We may assume that τ is diagonal. The restriction of λ_ϕ to the subspace of diagonal maps τ, with entries $t_1, \dots, t_n \geq 0$, may be expressed as $\lambda_\phi(\tau) = \sum_{i_1<\dots<i_{n-k}} a_{i_1\dots i_{n-k}} t_{i_1} \cdots t_{i_{n-k}}$ for some coefficients $a_{i_1\dots i_{n-k}}$. Setting suitable subsets of the t_i to be zero, the

hypothesis implies that all $a_{i_1\cdots i_{n-k}} \geq 0$. (4) $\Longleftrightarrow$ (2): Given $\sigma \in \Sigma_k$, $\sigma \geq 0$, let $\bar{e}_i$ be a positively oriented basis of V adapted to $E := \ker \sigma$. Then

$$\begin{aligned}\lambda_\phi(\sigma) &= \phi(e_1, \ldots, e_k, e_{k+1} + a_{k+1}\varepsilon_{k+1}, \ldots, e_n + a_n\varepsilon_n)\\ &= \phi(e_1, \ldots, e_k, (\bar{e}_{k+1}, \sigma\bar{e}_{k+1}), \ldots, (\bar{e}_n, \sigma\bar{e}_n))\\ &= \det(\sigma|_{E^\perp})\, \phi(e_1, \ldots, e_k, \varepsilon_{k+1}, \ldots, \varepsilon_n).\end{aligned}$$

Since $\sigma \geq 0$, the determinant is nonnegative. Thus both conditions are equivalent to the assertion that the right-hand side of (21) is nonnegative on such a basis. □

COROLLARY 2.10. *A constant coefficient valuation is positive if and only if its homogeneous components are positive.*

Proof. Let $\mu = \nu_\phi, \phi \in \Lambda^n(V \oplus V)$ be a constant coefficient valuation. Let $\phi = \sum_k \phi_k,\ \phi_k \in \Lambda^n_k(V \oplus V)$. Suppose some $\nu_{\phi_k} \not\geq 0$. By Lemma 2.9, there is $E \in \mathrm{Gr}_k$ such that $\mathrm{Kl}_{\nu_{\phi_k}}(E) < 0$. Since the restrictions to E of the $\nu_{\phi_j}, j > k$, all vanish, it follows that $\nu_\phi(E \cap B_R) < 0$ for balls of sufficiently large radius R. □

PROPOSITION 2.11. *Let $\phi \in \Lambda^n(V \oplus V)$. Then $\nu_\phi \geq 0$ if and only if $\lambda_\phi \geq 0$.*

Proof. This follows from Lemma 2.9, Corollary 2.10 and the fact that

$$\lambda_\phi \geq 0 \text{ if and only if each } \lambda_{\phi_k} \geq 0,$$

whose proof is similar to that of Corollary 2.10, substituting an appropriate nonnegative symmetric bilinear form of rank k in place of E. □

THEOREM 2.12. *A valuation $\mu \in \mathrm{Val}(V)$ is monotone if and only if all of its homogeneous components are monotone.*

Proof. First we prove the statement in the smooth case.

By (17), given $\mu \in \mathrm{Val}^{\mathrm{sm}}$, the first variation measure of μ may be expressed as $\delta\mu = \Phi_\gamma$ for some $\gamma \in \Omega(SV)^V$ (if μ is a multiple of vol then γ is the corresponding multiple of the form κ_{n-1} of [17]). Since the second fundamental form of a smooth convex hypersurface is nonnegative semidefinite, and conversely every nonnegative semidefinite bilinear form may be realized as such at some point of the boundary of such a hypersurface, from Lemma 2.8 it follows that the curvature measure Φ_γ is nonnegative if and only if $\lambda_{\gamma_{(x,v)}} \geq 0$ as an element of $\Lambda^{n-1}_k(Q_{x,v})$ for every $(x, v) \in SV$. By Lemma 2.9, this is the case if and only if $\lambda_{\gamma^k_{(x,v)}} \geq 0$, $k = 0, \ldots, n-1$, where $\gamma^k_{(x,v)} \in \Lambda^{n-1}_k(Q_{x,v})$ are the homogeneous components of $\gamma_{(x,v)}$. Thus by assertion (2) from the proof of Proposition 2.11, the present proof will be completed by showing that these correspond to the homogeneous components of μ.

This amounts to showing: if $\deg\mu = k$ then $\gamma_{x,v} \in \Lambda^{n-1}_{k-1}(Q_{x,v})$ for all $(x,v) \in SV$. Since this is clearly true when μ is a multiple of vol, we may assume that $\deg\mu < n$, and hence $\mu = \Psi_\beta$ for some translation-invariant form β of bidegree $(k, n-k-1)$. Note that $d\beta$ then has bidegree $(k, n-k)$. By the construction of [29], $D\beta = d(\beta + \alpha\wedge\xi)$, where ξ is the unique form such that $i_T\xi = 0$ and $(d\beta + d\alpha\wedge\xi)|_{Q_{x,v}} = 0$. In particular ξ is translation-invariant and of bidegree $(k-1, n-k-1)$, so $\gamma := i_T D\beta$ has bidegree $(k-1, n-k)$, as claimed.

Next, let μ be any continuous translation invariant valuation. Let m_1, $m_2, \ldots$ be a sequence of smooth compactly supported probability measures on $GL(V)$ whose supports converge to the identity. The valuations

$$\mu * m_i := \int_{Gl(V)} g\mu \, dm_i(g),$$

where $g\mu(A) := \mu(g^{-1}A)$, are then smooth and monotone, with $\mu * m_i \to \mu$. Thus the homogeneous components of each μ_i are monotone by what we have shown above, and the resulting sequences converge, respectively, to the homogeneous components of μ. Since monotonicity is clearly a closed condition, the result follows. □

PROPOSITION 2.13. *Suppose W is the orthogonal direct sum $\mathbb{R}\oplus V$, with orientation induced by those of $\mathbb{R}, V$, and let $t, \tau : W\times W \to \mathbb{R}$ be the projections to the two $\mathbb{R}$ factors respectively. Let $\phi \in \Lambda^n(V\times V)$. Then the three conditions*

$$\nu_\phi \geq 0, \quad \nu_{dt\wedge\phi} \geq 0, \quad \nu_{d\tau\wedge\phi} \geq 0$$

are equivalent. If $\phi \in \Lambda^n_k(V\times V)$ and $\psi \in \Lambda^n_{k-1}(V\times V)$, then $\nu_{d\tau\wedge\phi + dt\wedge\psi} \geq 0$ if and only if both $\nu_\phi, \nu_\psi \geq 0$.

Proof. We may assume that $\phi \in \Lambda^n_k(V\times V)$, so that

$$dt\wedge\phi \in \Lambda^{n+1}_{k+1}(W\times W), \; d\tau\wedge\phi \in \Lambda^{n+1}_k(W\times W).$$

Given $E \in \mathrm{Gr}_{k+1}(W)$, there is a positively oriented basis of W adapted to E of the form

$$c\frac{\partial}{\partial t} + s\bar{e}_{k+1}, \bar{e}_1, \ldots, \bar{e}_k, -s\frac{\partial}{\partial t} + c\bar{e}_{k+1}, \bar{e}_{k+2}, \ldots, \bar{e}_n,$$

where $\bar{e}_1, \ldots, \bar{e}_n$ is a positively oriented orthonormal basis for V and $c^2 + s^2 = 1$. Similarly, given any $F \in \mathrm{Gr}_k(W)$ there is a positively oriented basis of W adapted to F of the form

$$c\frac{\partial}{\partial t} + s\bar{e}_1, \ldots, \bar{e}_k, -s\frac{\partial}{\partial t} + c\bar{e}_1, \bar{e}_{k+1}, \ldots, (-1)^{k-1}\bar{e}_n.$$

By Lemma 2.9, we may check the nonnegativity of $\nu_{dt\wedge\phi}$ by evaluating

$$\begin{aligned}(22)\qquad dt\wedge\phi\Big(c\frac{\partial}{\partial t}+se_{k+1},e_1,\dots,e_k,-s\frac{\partial}{\partial\tau}+c\varepsilon_{k+1},\varepsilon_{k+2},\dots,\varepsilon_n\Big)\\ =dt\wedge\phi\Big(c\frac{\partial}{\partial t},e_1,\dots,e_k,c\varepsilon_{k+1},\varepsilon_{k+2},\dots,\varepsilon_n\Big)\\ =c^2\phi(e_1,\dots,e_k,\varepsilon_{k+1},\dots,\varepsilon_n),\end{aligned}$$

and of $\nu_{d\tau\wedge\phi}$ by evaluating

$$\begin{aligned}(23)\qquad d\tau\wedge\phi\Big(c\frac{\partial}{\partial t}+se_1,\dots,e_k,-s\frac{\partial}{\partial\tau}+c\varepsilon_1,\varepsilon_{k+1},\dots,(-1)^{k-1}\varepsilon_n\Big)\\ =d\tau\wedge\phi\Big(se_1,\dots,e_k,-s\frac{\partial}{\partial\tau},\varepsilon_{k+1},\dots,(-1)^{k-1}\varepsilon_n\Big)\\ =s^2\phi(e_1,\dots,e_k,\varepsilon_{k+1},\varepsilon_{k+2},\dots,\varepsilon_n).\end{aligned}$$

By (21), each of these expressions is nonnegative precisely when $\nu_\phi\geq 0$, which proves the first assertion.

To prove the second assertion, it is enough to show that the first condition implies the second. But (22) and (23) imply that for $E\in\mathrm{Gr}_k(V)$, $F\in\mathrm{Gr}_{k-1}(V)$

$$\omega_{n+1-k}^{-1}\,\mathrm{Kl}_{\nu_{d\tau\wedge\phi+dt\wedge\psi}}(\{0\}\times E)=\omega_{n-k}^{-1}\,\mathrm{Kl}_{\nu_\phi}(E),\tag{24}$$

$$\mathrm{Kl}_{\nu_{d\tau\wedge\phi+dt\wedge\psi}}(\mathbb{R}\times F)=\mathrm{Kl}_{\nu_\psi}(F),\tag{25}$$

from which this follows at once. □

3. **Special bases for** $\mathrm{Val}^{U(n)}$

Every valuation in $\mathrm{Val}^{U(n)}(\mathbb{C}^n)$ is even and smooth.

3.1. *The monomial basis and its Fourier transform.* We recall the global valuations $s\in\mathrm{Val}_2^{U(n)},t\in\mathrm{Val}_1^{U(n)}$ from [20]. The monomials

$$s^pt^{k-2p},\quad 0\leq p\leq\min\left\{\left\lfloor\frac{k}{2}\right\rfloor,\left\lfloor\frac{2n-k}{2}\right\rfloor\right\}$$

constitute a basis of $\mathrm{Val}^{U(n)}$. In Alesker's [1] notation,

$$s^pt^{k-2p}=\frac{n!(k-2p)!\omega_{k-2p}}{(n-p)!\pi^{k-2p}}U_{k,p},$$

where

$$U_{k,p}(K):=\int_{\overline{\mathrm{Gr}}_{2n-2p,n-p}}\mu_{k-2p}(K\cap\bar E)\,d\bar E$$

and the integral is over the corresponding affine Grassmannian with Haar measure $d\bar E$ normalized as in (19) of [20]. By [13] and [1], their Fourier transforms are given by

$$\widehat{s^pt^{k-2p}}=\widehat{(s^p)}*\widehat{(t^{k-2p})}=\mathrm{const.}s^{n-p}*t^{2n-k+2p}=\mathrm{const.}C_{2n-k,n-p},$$

where $*$ is the convolution product of [13] and

$$C_{k,q}(K) := \int_{\mathrm{Gr}_{2q,q}} \mu_k(\pi_E(K))\, dE.$$

We recall from [20]:

THEOREM 3.1. *The ideal of polynomials p such that $p(s,t) = 0$ locally at n is the ideal (f_{n+1}, f_{n+2}), where $\deg f_k(s,t) = k$ and $\log(1+s+t) = \sum_k f_k(s,t)$. The f_k satisfy the relations*

$$\begin{aligned} f_1 &= t, \\ f_2 &= s - \frac{t^2}{2}, \\ ksf_k + (k+1)tf_{k+1} + (k+2)f_{k+2} &= 0, \quad k \geq 1. \end{aligned} \tag{26}$$

3.2. *The hermitian intrinsic volumes.*

THEOREM 3.2. *There exist global valuations $\mu_{k,q} \in \mathrm{Val}_k^{U(\infty)}$ uniquely determined by the relations*

$$\mathrm{Kl}_{\mu_{k,q}}(E^{k',q'}) = \delta_{k,q}^{k',q'}. \tag{27}$$

The valuations $\mu_{k,q}$, $\max(0, k-n) \leq q \leq \lfloor \frac{k}{2} \rfloor$, comprise a basis for the vector space $\mathrm{Val}_k^{U(n)}$.

The $\mu_{k,q}$ are all constant coefficient valuations in the sense of Section 2.7, and satisfy the local relations

$$\widehat{\mu_{k,q}} = \mu_{2n-k,n-k+q}. \tag{28}$$

Proof. Let $(z_1, \dots, z_n, \zeta_1, \dots, \zeta_n)$ be canonical coordinates on $T\mathbb{C}^n \simeq \mathbb{C}^n \times \mathbb{C}^n$, where $z_i = x_i + \sqrt{-1}y_i$ and $\zeta_i = \xi_i + \sqrt{-1}\eta_i$. The natural action of $U(n)$ on $T\mathbb{C}^n$ corresponds to the diagonal action on $\mathbb{C}^n \times \mathbb{C}^n$.

Following Park [28], we consider the elements

$$\begin{aligned} \theta_0 &:= \sum_{i=1}^n d\xi_i \wedge d\eta_i, \\ \theta_1 &:= \sum_{i=1}^n \left(dx_i \wedge d\eta_i - dy_i \wedge d\xi_i \right), \\ \theta_2 &:= \sum_{i=1}^n dx_i \wedge dy_i \end{aligned}$$

in $\Lambda^2(\mathbb{C}^n \oplus \mathbb{C}^n)^*$. Thus θ_2 is the pullback via the projection map $T\mathbb{C}^n \to \mathbb{C}^n$ of the Kähler form of $\mathbb{C}^n$, and $\theta_0 + \theta_1 + \theta_2$ is the pullback of the Kähler form under the exponential map $\exp(z, \zeta) := z + \zeta$. Together with the symplectic form $\theta_s = \sum_{i=1}^n (dx_i \wedge d\xi_i + dy_i \wedge d\eta_i)$, the θ_i generate the algebra of all $U(n)$-invariant elements in $\Lambda^*(\mathbb{C}^n \times \mathbb{C}^n)$ (cf. [28]).

For positive integers k, q with $\max\{0, k-n\} \le q \le \frac{k}{2} \le n$, we now set

$$\theta_{k,q} := c_{n,k,q}\theta_0^{n+q-k} \wedge \theta_1^{k-2q} \wedge \theta_2^q,$$

where

$$c_{n,k,q} := \frac{1}{q!(n-k+q)!(k-2q)!\omega_{2n-k}}.$$

Note that $\theta_{k,q} \in \Lambda_k^{2n}(\mathbb{C}^n \times \mathbb{C}^n)$. We put

$$\mu_{k,q}(K) := \int_{N_1(K)} \theta_{k,q}. \tag{29}$$

Since this valuation has constant coefficients in the sense of Section 2.7, we may evaluate its Klain function using (21). We write $E^{k,p}$ for a generic element of $\mathrm{Gr}_{k,p}(\mathbb{C}^n)$. By invariance we may assume that $E^{k,p} = \mathbb{C}^p \oplus \mathbb{R}^{k-2p}$, with adapted basis

$$\frac{\partial}{\partial z_1}, \dots, \frac{\partial}{\partial z_p}, \frac{\partial}{\partial x_{p+1}}, \dots, \frac{\partial}{\partial x_{k-p}}, \frac{\partial}{\partial y_{p+1}}, \dots, \frac{\partial}{\partial y_{k-p}}, \frac{\partial}{\partial z_{k-p+1}}, \dots, \frac{\partial}{\partial z_n}, \tag{30}$$

where $\frac{\partial}{\partial z_i}$ stands for the pair $\frac{\partial}{\partial x_i}, \frac{\partial}{\partial y_i}$. We evaluate

(31)

$$\begin{aligned}
\theta_{k,q}\Big(\frac{\partial}{\partial z_1}, \dots, \frac{\partial}{\partial z_p}, \frac{\partial}{\partial x_{p+1}}, \dots, \frac{\partial}{\partial x_{k-p}}, \frac{\partial}{\partial \eta_{p+1}}, \dots, \frac{\partial}{\partial \eta_{k-p}}, \frac{\partial}{\partial \zeta_{k-p+1}}, \dots, \frac{\partial}{\partial \zeta_n}\Big) \\
&= \delta_q^p p!(n-k+p)!\theta_1^{k-2p}\left(\frac{\partial}{\partial x_{p+1}}, \dots, \frac{\partial}{\partial x_{k-p}}, \frac{\partial}{\partial \eta_{p+1}}, \dots, \frac{\partial}{\partial \eta_{k-p}}\right) \\
&= \frac{\pm\delta_q^p}{\omega_{2n-k}},
\end{aligned}$$

where the sign is that of the basis $\frac{\partial}{\partial x_{p+1}}, \dots, \frac{\partial}{\partial x_{k-p}}, \frac{\partial}{\partial y_{p+1}}, \dots, \frac{\partial}{\partial y_{k-p}}$ relative to the standard orientation of $\mathbb{C}^{k-2p}$, i.e., the same as that of the basis (30).

This proves (27). In particular, for fixed n the $\mu_{k,p}$ in the given range are linearly independent, since their Klain functions are. Since their number equals the dimension of $\mathrm{Val}^{U(n)}$ they form a basis. Finally, since $(E^{k,q})^\perp = E^{2n-k,n-k+q}$, relation (28) is immediate, concluding the proof of Theorem 3.2. □

As a final remark about the hermitian intrinsic volumes, we recall from Theorem 3.1 that the kernel of the map $\mathrm{Val}_{n+1}^{U(\infty)} \to \mathrm{Val}_{n+1}^{U(n)}$ is spanned by the polynomial f_{n+1}. At the same time it is clear that $\mu_{n+1,0} = 0$ locally at n. This implies the following global relation.

LEMMA 3.3. *There are constants $\gamma_k \neq 0$ such that*

$$\mu_{k,0} = \gamma_k f_k.$$

The valuation $\mu_{n,0} \in \mathrm{Val}^{U(n)}$ was originally discovered by Kazarnovskii, and is called the *Kazarnovskii pseudo-volume*.

3.3. *Hermitian curvature measures.* Next we consider the $\overline{U(n)}$-invariant curvature measures, which correspond to invariant $(2n-1)$-forms on the sphere bundle $S\mathbb{C}^n \simeq \mathbb{C}^n \times S^{2n-1} \subset \mathbb{C}^n \times \mathbb{C}^n \simeq T\mathbb{C}^n$. Consider first the following three invariant 1-forms on $T\mathbb{C}^n$ and their exterior derivatives:

$$\alpha = \sum_{i=1}^{n} \xi_i dx_i + \eta_i dy_i, \quad d\alpha = -\theta_s,$$
$$\beta = \sum_{i=1}^{n} \xi_i dy_i - \eta_i dx_i, \quad d\beta = \theta_1,$$
$$\gamma = \sum_{i=1}^{n} \xi_i d\eta_i - \eta_i d\xi_i, \quad d\gamma = 2\theta_0,$$

where θ_s is the symplectic form of $\mathbb{C}^n \times \mathbb{C}^n \simeq T^*\mathbb{C}^n$. The restrictions of these forms to the sphere bundle $\mathbb{C}^n \times S^{2n-1}$, together with that of θ_2, generate the algebra of invariant forms on that space (we will not distinguish notationally the forms from their restrictions) [28]. Thus each form of degree $2n-1$ that is a product of these forms gives rise to a $\overline{U(n)}$-invariant curvature measure. Since the contact form α and its exterior derivative θ_s vanish identically on any normal cycle, it is enough to consider the products of $\beta, \gamma, \theta_0, \theta_1, \theta_2$. Since $\partial N_1(K) = N(K)$, from Stokes' theorem one easily computes that

PROPOSITION 3.4. *Set* $B_{k,q} := \Phi_{\beta_{k,q}}, \Gamma_{k,q} := \Phi_{\gamma_{k,q}}$ *to be the curvature measures corresponding to the invariant forms*

$$\beta_{k,q} := c_{n,k,q}\, \beta \wedge \theta_0^{n-k+q} \wedge \theta_1^{k-2q-1} \wedge \theta_2^q, \quad k > 2q, \tag{32}$$
$$\gamma_{k,q} := \frac{c_{n,k,q}}{2}\, \gamma \wedge \theta_0^{n-k+q-1} \wedge \theta_1^{k-2q} \wedge \theta_2^q, \quad n > k-q \tag{33}$$

on the sphere bundle $\mathbb{C}^n \times S^{2n-1}$. *Then both of these curvature measures give rise to the hermitian intrinsic volume* $\mu_{k,q}$, *i.e., for* $K \in \mathcal{K}$

$$\mu_{k,q}(K) = B^K_{k,q}(K) = \Gamma^K_{k,q}(K).$$

3.4. *Tasaki valuations.* Tasaki ([31], [32]) was the first to give explicit Poincaré-Crofton formulas for submanifolds in complex space forms. As a preparatory step, Tasaki showed that if $k \leq n$ then the family of $U(n)$ orbits of $\mathrm{Gr}_k(\mathbb{C}^n)$ is in natural one-to-one correspondence with the p-dimensional simplex

$$0 \leq \theta_1 \leq \cdots \leq \theta_p \leq \frac{\pi}{2}, \quad p := \left\lfloor \frac{k}{2} \right\rfloor$$

The invariant $\Theta(E) := (\theta_1(E), \dots, \theta_p(E))$ is called the *multiple Kähler angle* of $E \in \mathrm{Gr}_k(\mathbb{C}^n)$, and is characterized by the condition that there is an orthonormal basis $\alpha_1, \dots, \alpha_k$ of the dual space E^* such that the restriction of the Kähler form of $\mathbb{C}^n$ to E is

$$\sum_{i=1}^{\lfloor \frac{k}{2} \rfloor} \cos\theta_i \ \alpha_{2i-1} \wedge \alpha_{2i}.$$

Thus a subspace E is of type (k, q) if and only if

$$\Theta(E) = \Big(\underbrace{0, \dots, 0}_{q}, \underbrace{\frac{\pi}{2}, \dots, \frac{\pi}{2}}_{p-q} \Big).$$

With this definition, the multiple Kähler angle is a global invariant in the sense of Section 2.3, in that it remains the same under the natural embedding $\mathrm{Gr}_k(\mathbb{C}^n) \to \mathrm{Gr}_k(\mathbb{C}^{n+1})$. On the other hand it is easy to see that if $k > n$ then

$$\Theta(E) = \Big(\underbrace{0, \dots, 0}_{k-n}, \Theta(E^\perp) \Big).$$

We remark that Tasaki defined the multiple Kähler angle to be $\Theta(E^\perp)$ in this case.

Tasaki ([31, Prop. 3]) observed that if $k = 2p \leq n$ is even then there is an orthonormal basis $e_1, \dots, e_n$ of the hermitian vector space $\mathbb{C}^n$, such that

$$e_1, e_3, \dots, e_{2p-1}, \cos\theta_1 \sqrt{-1} e_1 + \sin\theta_1 \, e_2, \dots, \cos\theta_p \sqrt{-1} e_{2p-1} + \sin\theta_p \, e_{2p} \tag{34}$$

is an orthonormal basis for E as a real euclidean vector space, and

$$\begin{aligned} &\sqrt{-1}\, e_2, \dots, \sqrt{-1}\, e_{2p}, e_{2p+1}, \sqrt{-1} e_{2p+1}, \dots, e_n, \sqrt{-1} e_n, \\ &\qquad -\sin\theta_1 \sqrt{-1} e_1 + \cos\theta_1 \, e_2, \dots, -\sin\theta_p \sqrt{-1} e_{2p-1} + \cos\theta_p \, e_{2p} \end{aligned}$$

is an orthonormal basis for $E^\perp$, and similarly if k is odd and/or larger than n. By (21), it is now easy to see

LEMMA 3.5. *For each k, q as above, the Klain function $\mathrm{Kl}_{\mu_{k,q}}(E)$ is a linear combination of the elementary symmetric functions in*

$$\cos^2\theta_1(E), \dots, \cos^2\theta_p(E).$$

Proof. Referring to the basis (34) and the expression (21) for the Klain function, the latter is symmetric in these quantities, and of degree at most one in each of them. □

We now define the *Tasaki valuations* $\tau_{k,q} \in \mathrm{Val}^{U(\infty)}, 0 \leq q \leq p := \lfloor \frac{k}{2} \rfloor$ by the condition

$$\mathrm{Kl}_{\tau_{k,q}}(E) = \sigma_q(\Theta(E)) := \sigma_q(\cos^2\theta_1(E), \dots, \cos^2\theta_p(E)) \tag{35}$$

where σ_q is the the q^{th} elementary symmetric function.

DEFINITION 3.6.

$$u := 4s - t^2.$$

PROPOSITION 3.7. *The Tasaki valuations are well-defined, and are given by*

$$\tau_{k,q} = \sum_{i=q}^{\lfloor k/2 \rfloor} \binom{i}{q} \mu_{k,i} \tag{36}$$

$$= \frac{\pi^k}{\omega_k (k-2q)!(2q)!} t^{k-2q} u^q. \tag{37}$$

Furthermore the polynomials from Theorem 3.1 *may be expressed*

$$f_k = \frac{1}{k(-2)^{k-1}} \sum_{q=0}^{\lfloor k/2 \rfloor} (-1)^q \binom{k}{2q} t^{k-2q} u^q. \tag{38}$$

Proof. Since the elementary symmetric functions corresponding to the $p+1$ hermitian intrinsic volumes are linearily independent, relation (36) is a straightforward calculation, using the defining relations (27).

To prove (38), we introduce the formal complex variable $z := t + \sqrt{-1}v$, where v is formally real and $v^2 = u$. Then

$$\begin{aligned} \sum_k f_k = \log(1+s+t) &= \log\left(1 + t + \frac{t^2}{4} + \frac{v^2}{4}\right) \\ &= \log\left(\left|1 + \frac{z}{2}\right|^2\right) \\ &= 2\,\mathrm{Re}\left(\log\left(1 + \frac{z}{2}\right)\right) \\ &= \mathrm{Re} \sum_k \frac{1}{k(-2)^{k-1}} \left(t + \sqrt{-u}\right)^k. \end{aligned}$$

We postpone the proof of (37) to Section 5.2. □

COROLLARY 3.8. *The global Tasaki valuations* $\tau_{k,q}, 0 \le q \le \frac{k}{2}$, *constitute a basis for* $\mathrm{Val}_k^{U(\infty)}$, *and in fact*

$$\mu_{k,q} = \sum_{i=q}^{\lfloor k/2 \rfloor} (-1)^{i+q} \binom{i}{q} \tau_{k,i}. \tag{39}$$

If $k \le n$ *then the local Tasaki valuations* $\widehat{\tau_{k,q}}, 0 \le q \le \frac{k}{2}$ *constitute a basis for* $\mathrm{Val}_{2n-k}^{U(n)}$.

If we now write out the $U(n)$ kinematic formula in terms of the basis $\{\tau_{k,q}, \widehat{\tau_{k,q}} : k \le n, 0 \le q \le \frac{k}{2}\}$ of $\mathrm{Val}^{U(n)}$, then the general Crofton formula (13) now yields the main theorem of [32].

THEOREM 3.9 (Tasaki). *For $0 \le k \le n$, there is a $\left(\lfloor \frac{k}{2} \rfloor + 1\right) \times \left(\lfloor \frac{k}{2} \rfloor + 1\right)$ symmetric matrix T_k^n, such that whenever $M, N \subset \mathbb{C}^n$ are C^1 compact submanifolds of dimensions $k, 2n-k$ respectively,*

(40)
$$\int_{\overline{U(n)}} \#(M \cap \bar{g}N)\, d\bar{g} = \sum_{i,j} (T_k^n)_{i,j} \int_M \sigma_i(\Theta(T_x M))\, dx \int_N \sigma_j(\Theta(T_y N^\perp))\, dy.$$

The symmetry of T_k^n follows from that of the pairing (7). In fact these formulas exhibit a further remarkable symmetry:

THEOREM 3.10. *If $k = 2l$ is even then*

$$(T_k^n)_{i,j} = (T_k^n)_{l-i,l-j}, \quad 0 \le i, j \le l. \tag{41}$$

To prove Theorem 3.10 we introduce the linear involution $\iota : \mathrm{Val}_{2*}^{U(\infty)} \to \mathrm{Val}_{2*}^{U(\infty)}$ on the subspace of valuations of even degree, determined by its action on Tasaki valuations:

$$\iota(\tau_{2l,q}) := \tau_{2l,l-q}.$$

LEMMA 3.11. (1) *ι is an algebra automorphism.*
(2) *ι covers an algebra automorphism of every $\mathrm{Val}_{2*}^{U(n)}$.*
(3) *The action of ι on the top degree component $\mathrm{Val}_{2n}^{U(n)}$ is trivial.*
(4) *ι commutes with the Fourier transform.*

Proof of Lemma 3.11. (1) Any element of $\mathrm{Val}_{2*}^{U(\infty)}$ may be expressed as polynomial in t and v, involving only even powers of each variable. We may regard this as a (real) polynomial function $p(z)$ in the complex variable $z = t + \sqrt{-1}v$. From the expression (37), in these terms $\iota(p(z)) = p(\sqrt{-1}\, z)$, which is of course an algebra isomorphism.

(2) To prove that ι descends to an automorphism of $\mathrm{Val}^{U(n)}$ it is enough to show that ι stabilizes the kernel of the map $\mathrm{Val}_{2*}^{U(\infty)} \to \mathrm{Val}_{2*}^{U(n)}$. This kernel consists of the even degree elements of the ideal (f_{n+1}, f_{n+2}). By (1), it is enough to show that $\iota(f_{2k}) \in (f_{2k})$ and that $\iota(t f_{2k-1}) \in (t f_{2k-1}, f_{2k})$. But by the proof of (38),

$$\begin{aligned}
\iota(f_{2k}) &= -\frac{1}{k\, 2^{2k}}\, \iota(\mathrm{Re}\, z^{2k}) \\
&= -\frac{1}{k\, 2^{2k}}\, \mathrm{Re}(\sqrt{-1} z)^{2k} \\
&= \frac{(-1)^{k+1}}{k\, 2^{2k}}\, \mathrm{Re}\, z^{2k} \\
&= (-1)^k f_{2k}
\end{aligned}$$

and

$$\begin{aligned}\iota(tf_{2k-1}) &= \frac{1}{(2k-1)\,2^{2k-2}}\,\iota(\mathrm{Re}[tz^{2k-1}])\\ &= \frac{1}{(2k-1)\,2^{2k-2}}\,\mathrm{Re}[-v(\sqrt{-1}\,z)^{2k-1}]\\ &= \frac{1}{(2k-1)\,2^{2k-2}}\,\mathrm{Re}\left[(\sqrt{-1}\,z)^{2k}-\sqrt{-1}\,t(\sqrt{-1}\,z)^{2k-1}\right]\\ &= (-1)^{k+1}\frac{4k}{2k-1}f_{2k}+\frac{(-1)^{k+1}}{(2k-1)2^{2k-2}}\,\mathrm{Re}(tz^{2k-1})\\ &= (-1)^{k+1}\left(\frac{4k}{2k-1}f_{2k}+tf_{2k-1}\right).\end{aligned}$$

(3) Since locally $\mu_{2n,k}=0$ for $k<n$, (36) shows that $\tau_{2n,k}=\tau_{2n,n-k}=\binom{n}{k}\mu_{2n,n}$ locally.

(4) Put Σ_p for the vector space spanned by the elementary symmetric polynomials $\sigma_{p,0}:=1,\sigma_{p,1}:=x_1+\cdots+x_p,\dots,\sigma_{p,p}:=x_1x_2\dots x_p$ in the p variables $x_1,\dots,x_p$. As noted above, Σ_p is canonically isomorphic to $\mathrm{Val}_{2p}^{U(\infty)}$ via $\sigma_{p,q}\mapsto\tau_{2p,q}$, where the map ι corresponds to $\sigma_{p,q}\mapsto\sigma_{p,p-q}$, which we again denote by ι.

Fixing $n\geq 2p$, the Fourier transform $\widehat{\ }\colon \mathrm{Val}_{2n-2p}^{U(n)}\to\mathrm{Val}_{2p}^{U(n)}$ corresponds to the linear surjection $r:\Sigma_{n-p}\to\Sigma_p$ given by

$$r(f)=f(x_1,\dots,x_p,1,\dots,1).$$

The assertion thus reduces to the claim that for $m=n-p\geq p$ the diagram

$$\begin{array}{ccc}\Sigma_m & \xrightarrow{\ \iota\ } & \Sigma_m\\ \big\downarrow r & & \big\downarrow r\\ \Sigma_p & \xrightarrow{\ \iota\ } & \Sigma_p\end{array}$$

commutes. It is enough to prove this for $m=p+1$, in which case $r(\sigma_{p+1,i})=\sigma_{p,i}+\sigma_{p,i-1}$. Hence for $i=0,\dots,p+1$,

$\iota\circ r(\sigma_{p+1,i})=\iota(\sigma_{p,i}+\sigma_{p,i-1})=\sigma_{p,p-i}+\sigma_{p,p-i+1}=r(\sigma_{p+1,p-i+1})=r\circ\iota(\sigma_{p+1,i})$. □

Proof of Theorem 3.10. By Lemma 3.11,

$$\begin{aligned}\tau_{2p,i}\cdot\widehat{\tau_{2p,j}} &= \tau_{2p,i}\cdot\widehat{(\iota\tau_{2p,p-j})}\\ &= \tau_{2p,i}\cdot\iota(\widehat{\tau_{2p,p-j}})\\ &= \iota\left(\tau_{2p,i}\cdot\iota(\widehat{\tau_{2p,p-j}})\right)\\ &= \iota(\tau_{2p,i})\cdot\widehat{\tau_{2p,p-j}}\\ &= \tau_{2p,p-i}\cdot\widehat{\tau_{2p,p-j}}.\end{aligned}$$

With Theorem 2.2, this implies the result. □

4. **The positive, monotone and Crofton-positive cones**

We wish to determine the cones $CP \subset M \subset P \subset \mathrm{Val}^{U(n)}$ given by

$$P := \{\phi : \phi(K) \geq 0 \text{ for all } K \in \mathcal{K}\}, \tag{42}$$

$$M := \{\phi : \phi(K) \geq \phi(L) \text{ whenever } K, L \in \mathcal{K} \text{ and } K \supset L\}, \tag{43}$$

$$CP := \{\phi : \text{ the homogeneous components of } \phi \text{ each admit a nonnegative Crofton measure}\}. \tag{44}$$

We recall from [13] that if $\phi, \psi \in \mathrm{Val}_k^{sm,+}$ are even, and m_ψ is a Crofton measure for ψ, then the pairing (7) is given by

$$\langle \phi, \psi \rangle = \int_{\mathrm{Gr}_k} \mathrm{Kl}_\phi \, dm_\psi. \tag{45}$$

Proposition 4.1. *The cone P is generated by the hermitian intrinsic volumes $\mu_{k,q}$. The cone CP is the cone $P^* := \{\phi : \langle \phi, \mu \rangle \geq 0$ for all $\mu \in P\}$ dual to P with respect to the pairing $\langle \cdot, \cdot \rangle$ of* (7).

Proof. By the equivalence (1) $\iff$ (3) $\iff$ (4) in Lemma 2.9, a constant coefficient valuation belongs to P if and only if its homogeneous components do; and by the equivalence (1) $\iff$ (2), a homogeneous constant coefficient valuation belongs to P if and only if its Klain function is nonnegative. By Lemma 3.5, the first assertion of Proposition 4.1 is equivalent to the following statement. Consider the vector space Σ spanned by the elementary symmetric functions in the variables $x_1, \ldots, x_p$, and let C denote the cube $0 \leq x_1, \ldots, x_p \leq 1$ (we think of $x_i = \cos^2 \theta_i$). Let $f \in \Sigma$ be given. Then $f|_C \geq 0$ if and only if its value at each vertex of $C \geq 0$. This is easily proved by induction on the dimension of the faces of C, using the observation that f is affine in each variable separately if the others are held fixed.

Moving on to CP, put $\nu_{k,p} \in \mathrm{Val}^{U(n)}$ for the dual basis to $\mu_{k,p}$ with respect to the pairing (7), i.e.,

$$\langle \nu_{k,p}, \mu_{l,q} \rangle := \delta_{l,q}^{k,p}.$$

Thus, by (45), $\nu_{k,p}$ is the valuation determined by the Crofton measure that is $U(n)$-invariant, is supported on $\mathrm{Gr}_{k,p}$, and has total mass 1; furthermore it is clear that the dual cone P^* is generated by the $\nu_{k,p} \in CP$, so $P^* \subset CP$. To prove the opposite inclusion, we note that (45) implies that if $\psi \in CP$ then $\langle \phi, \psi \rangle \geq 0$ for all ϕ with nonnegative Klain function. Taking ψ to have degree k and writing $\psi = \sum_p b_p \nu_{k,p}$, we find that

$$0 \leq \left\langle \sum_p a_p \mu_{k,p}, \psi \right\rangle = \sum_p a_p b_p, \tag{46}$$

whenever all $a_p \geq 0$, which implies that all $b_p \geq 0$, i.e. $\psi \in P^*$. □

This discussion invites the following brief excursion. Define the norms $\|\cdot\|_\infty$ and $\|\cdot\|_1$ on $\mathrm{Val}^{sm,+}$ by

$$\|\phi\|_\infty := \|\mathrm{Kl}_\phi\|_\infty , \tag{47}$$

$$\|\phi\|_1 := \min\{\mathrm{mass}\, m : m \text{ is a Crofton measure for } \phi\}. \tag{48}$$

By (45), the norm dual to $\|\cdot\|_\infty$ with respect to the pairing $\langle\cdot,\cdot\rangle$ satisfies

$$\|\cdot\|_\infty^* \leq \|\cdot\|_1 . \tag{49}$$

PROPOSITION 4.2. *Restricted to* $\mathrm{Val}_k^{U(n)}$ *the norms* $\|\cdot\|_1$ *and* $\|\cdot\|_\infty$ *are dual to one another with respect to the pairing* (7)*, with*

$$\left\|\sum_p a_p\mu_{k,p}\right\|_\infty = \max_p |a_p|, \tag{50}$$

$$\left\|\sum_p b_p\nu_{k,p}\right\|_1 = \sum_p |b_p|. \tag{51}$$

Proof. Relation (50) follows from the argument in the first paragraph of the proof of Proposition 4.1, and by (46),

$$\left\|\sum_p b_p\nu_{k,p}\right\|_\infty^* = \sum_p |b_p| = \mathrm{mass}\left(\sum_p b_p\nu_{k,p}\right) \geq \left\|\sum_p b_p\nu_{k,p}\right\|_1$$

which, with (49), completes the proof. □

4.0.1. *The monotone cone.*

THEOREM 4.3. *A valuation* $\mu \in \mathrm{Val}_k^{U(n)}$ *is monotone if and only if*

$$\mu = \sum_{q=\max\{0,k-n\}}^{\lfloor k/2\rfloor} a_q\mu_{k,q},$$

where

$$(k-2q)a_q \geq (k-2q-1)a_{q+1}, \quad \max\{0,k-n\} \leq q \leq \left\lfloor \frac{k-1}{2}\right\rfloor \tag{52}$$

(53)

$$(n+q-k+1)a_q \leq (n+q-k+3/2)a_{q+1}, \quad \max\{0,k-n-1\} \leq q \leq \left\lfloor \frac{k-2}{2}\right\rfloor .$$

COROLLARY 4.4. *The inclusions* $CP \subset M \subset P$ *are strict.*

By Theorem 2.7, in order to prove Theorem 4.3 we need to characterize the cone of nonnegative hermitian curvature measures.

PROPOSITION 4.5. *Given constants* $a_{k,q}, b_{l,p} \in \mathbb{R}$, $k > 2q, n > l-p$, *the curvature measure* $\sum a_{k,q}B_{k,q} + \sum b_{l,p}\Gamma_{l,p} \geq 0$ *if and only if all* $a_{k,q}, b_{l,p} \geq 0$.

Proof. Each tangent space $T_x\partial A$ is naturally isomorphic to the orthogonal direct sum $\mathbb{R}\oplus\mathbb{C}^{n-1}$, where the first summand corresponds to the distinguished line spanned by $\sqrt{-1}\,n(x)$ and the second summand to the maximal complex subspace of $T_x\partial A$. Thus the 1-forms β,γ correspond respectively to $dt, d\tau$ in Proposition 2.13. In view of the characterization in Proposition 4.1 of the nonnegative elements of $\mathrm{Val}^{U(n-1)}(\mathbb{C}^{n-1})$, the result now follows from Propositions 2.11 and 2.13. □

Recall from Theorem 2.7 the first variation map δ from valuations to curvature measures.

PROPOSITION 4.6.

(54)

$$\delta\mu_{k,q} = 2c_{n,k,q}(c^{-1}_{n,k-1,q}(k-2q)^2\Gamma_{k-1,q} - c^{-1}_{n,k-1,q-1}(n+q-k)q\Gamma_{k-1,q-1}$$
$$+c^{-1}_{n,k-1,q-1}(n+q-k+\frac{1}{2})qB_{k-1,q-1} - c^{-1}_{n,k-1,q}(k-2q)(k-2q-1)B_{k-1,q}).$$

Proof. By definition of the $\mu_{k,q}$, this valuation is represented by some $(2n-1)$-form $\omega_{k,q}$ with

$$d\omega_{k,q} = c_{n,k,q}\theta_0^{n+q-k}\wedge\theta_1^{k-2q}\wedge\theta_2^q, \tag{55}$$

i.e., $\mu_{k,q} = \Psi_{\omega_{k,q}}$. To compute $D\omega_{k,q}$, we must solve for ξ in the equation

$$D\omega_{k,q} = d(\omega_{k,q}+\alpha\wedge\xi) \equiv 0 \mod \alpha. \tag{56}$$

Fixing a point $(x,v)\in S\mathbb{C}^n = \mathbb{C}^n\times S^{2n-1}$, let $Q\subset T_{(x,v)}S\mathbb{C}^n$ denote the contact hyperplane $\alpha^\perp_{(x,v)}$. Thus $Q\simeq\mathbb{R}\oplus\mathbb{C}^{n-1}\oplus\mathbb{R}\oplus\mathbb{C}^{n-1}$ in a natural way, and carries a symplectic structure (cf. [29]). Let L denote the Lefschetz operator on Λ^*Q (i.e., multiplication by the symplectic form $\theta_s = -d\alpha$) and Λ the dual Lefschetz operator. By [24], they induce an $\mathfrak{sl}_2$-structure on Λ^*Q, i.e., $[L,\Lambda] = k+1-2n$ on Λ^kQ.

To solve (56) amounts to finding $\xi\in\Lambda^{2n-2}Q$ with

$$L\xi = d\omega_{k,q}|_Q.$$

We write $d\omega_{k,q}|_Q$ in terms of its Lefschetz decomposition

$$d\omega_{k,q}|_Q = \sum_{i=0}^{n-1} L^{n-i}\pi_{2i}. \tag{57}$$

Here π_{2i} is a primitive form of degree $2i$, i.e. $\Lambda\pi_{2i}=0$, where Λ is the dual Lefschetz operator. The sum terminates with $i=n-1$ (and not with $i=n$) since there are no primitive forms of degree $2n$. Clearly

$$\xi = \sum_{i=0}^{n-1} L^{n-i-1}\pi_{2i}$$

solves (56).

We apply Λ to both sides of (57) and use the fact that

$$[L^i, \Lambda] = i(k + i - 2n)L^{i-1} \quad \text{on } \Lambda^k Q$$

to deduce that

$$\Lambda d\omega_{k,q}|_Q = \sum_{i=0}^{n-1} (n-i)^2 L^{n-i-1}\pi_{2i} \equiv \xi \mod d\alpha. \tag{58}$$

From this point on we drop the $\wedge$ notation, with all products of forms understood to be wedge products.

Lemma 4.7.

$$\Lambda\left(\theta_0^a\theta_1^b\theta_2^c\right) \equiv \beta\gamma\theta_0^{a-1}\theta_1^{b-2}\theta_2^{c-1}\left(ac\theta_1^2 - b(b-1)\theta_0\theta_2\right) \mod (\alpha, d\alpha).$$

Proof. Since everything is $U(n)$-invariant, it suffices to do the computation at the point $(0, e_1) \in S\mathbb{C}^n$, i.e., where $\xi_1 = 1, \xi_2 = \cdots = \eta_n = 0$. At this point, $d\xi_1 = 0$, $\frac{\partial}{\partial\xi_1} = 0$ since $\sum(\xi_j^2 + \eta_j^2) = 1$, and $\beta = dy_1, \gamma = d\eta_1$.

Next, using the abbreviation $i_{x_j} := i_{\frac{\partial}{\partial x_j}}$, we compute that

$$\begin{aligned} i_{\xi_j} \circ i_{x_j}\left(\theta_0^a\theta_1^b\theta_2^c\right) &= i_{\xi_j}\left(b\theta_0^a d\eta_j\theta_1^{b-1}\theta_2^c + c\theta_0^a\theta_1^b dy_j\theta_2^{c-1}\right) \\ &= dy_j d\eta_j\left(b(b-1)\theta_0^a\theta_1^{b-2}\theta_2^c - ac\theta_0^{a-1}\theta_1^b\theta_2^{c-1}\right) \end{aligned}$$

and similarly

$$i_{\eta_j} \circ i_{y_j}\left(\theta_0^a\theta_1^b\theta_2^c\right) = dx_j d\xi_j\left(b(b-1)\theta_0^a\theta_1^{b-2}\theta_2^c - ac\theta_1^{a-1}\theta_1^b\theta_2^{c-1}\right).$$

Since $\Lambda = i_{\eta_1} \circ i_{y_1} + \sum_{j=2}^n (i_{\xi_j} \circ i_{x_j} + i_{\eta_j} \circ i_{y_j})$ at the selected point, and $\beta\gamma + \sum_{j=2}(dx_j d\xi_j + dy_j d\eta_j) = -d\alpha$, the result follows. □

With (58) and the defining relation (55), this yields

$$\xi \equiv c_{n,k,q}\beta\gamma\theta_0^{n+q-k-1}\theta_1^{k-2q-2}\theta_2^{q-1}\left((n+q-k)q\theta_1^2 - (k-2q)(k-2q-1)\theta_0\theta_2\right) \mod (\alpha, d\alpha).$$

Replacing this into (56) we find

$$\begin{aligned} i_T D\omega_{k,q} &\equiv i_T d\omega_{k,q} - d\xi \\ &\equiv c_{n,k,q}\theta_0^{n+q-k-1}\theta_1^{k-2q-2}\theta_2^{q-1} \\ &\quad \times \left((k-2q)^2\gamma\theta_0\theta_1\theta_2 - (n+q-k)q\,\gamma\theta_1^3\right. \\ &\quad + 2(n+q-k+1/2)q\,\beta\theta_0\theta_1^2 \\ &\quad \left. -2(k-2q)(k-2q-1)\beta\theta_0^2\theta_2\right) \mod (\alpha, d\alpha). \end{aligned} \tag{59}$$

The proposition now follows from Theorem 2.7 and the definition of B, Γ from Proposition 3.4. □

Proof of Theorem 4.3. Let $\mu = \sum_{k,q} a_q \mu_{k,q}$. The coefficient of $\Gamma_{k-1,q}$ with $\max\{0, k-n\} \leq q \leq \left\lfloor \frac{k-1}{2} \right\rfloor$ in $\delta\mu$ is given by

$$2\frac{c_{n,k,q}(k-2q)^2}{c_{n,k-1,q}} a_q - 2\frac{c_{n,k,q+1}(n+q-k+1)(q+1)}{c_{n,k-1,q}} a_{q+1};$$

it has the same sign as $(k-2q)a_q - (k-2q-1)a_{q+1}$.

Similarly, the coefficient of $B_{k-1,q}$ with $\max\{0, k-n-1\} \leq q \leq \left\lfloor \frac{k-2}{2} \right\rfloor$ in $\delta\mu$ is given by

$$\frac{2c_{n,k,q+1}(n+q-k+3/2)(q+1)}{c_{n,k-1,q}} a_{q+1} - \frac{2c_{n,k,q}(k-2q)(k-2q-1)}{c_{n,k-1,q}} a_q,$$

which has the same sign as $(n+q-k+3/2)a_{q+1} - (n-k+q+1)a_q$. By Theorem 2.7 and Proposition 4.5, the valuation μ is monotone if and only if inequalities (52) and (53) are satisfied. □

5. **Explicit kinematic formulas**

Our goal in this section is to give explicit forms for the kinematic formulas (12) in terms of the basis of Tasaki valuations and their Fourier transforms. Our approach is based on the explicit calculation of the structure of $\mathrm{Val}^{U(n)}$ as an $\mathfrak{sl}(2)$ module. The existence of such a structure follows from general considerations (the Jacobson-Morozov theorem [15]) and the fact, originally established by Alesker [1], [2], that $\mathrm{Val}^{\mathrm{sm}}$ satisfies the hard Lefschetz property with respect to either of two different operators of degrees ± 1 respectively. Using the results of [12], [13] we compute explicitly how these operators act on the Tasaki valuations, and show that together they yield a representation of $\mathfrak{sl}(2)$ on $\mathrm{Val}^{U(n)}$ (although Alesker has pointed out that this is not the case when these operators are regarded as acting on the entire space $\mathrm{Val}^{\mathrm{sm}}$). We then calculate explicitly the primitive elements of $\mathrm{Val}^{U(n)}$ with respect to this representation, giving rise to one more canonical basis $\pi_{k,p}$ for $\mathrm{Val}^{U(n)}$. Since the Poincaré duality multiplication table of $\mathrm{Val}^{U(n)}$ in terms of this basis is antidiagonal (Proposition 5.5 below), we can then easily express the kinematic formulas in these terms.

5.1. *The* $\mathfrak{sl}(2)$ *action.* We recall [1], [12], [13] the two operators $\tilde{L}, \tilde{\Lambda} : \mathrm{Val}^{\mathrm{sm}}(\mathbb{C}^n) \to \mathrm{Val}^{\mathrm{sm}}(\mathbb{C}^n)$, of degrees ± 1 respectively:

$$\tilde{L}\phi := \mu_1 \cdot \phi, \tag{60}$$

$$\tilde{\Lambda}\phi := 2\,\mu_{2n-1} * \phi = \left.\frac{d}{dt}\right|_{t=0} \phi(\cdot + tB), \tag{61}$$

where B is the unit ball of $\mathbb{C}^n$. (Note that $\tilde{\Lambda}\phi$ is the valuation corresponding to the curvature measure $\delta\phi$, i.e. $\tilde{\Lambda}\phi(A) = (\delta\phi)^A(A)$.)

Since $\tilde{L}$ is a multiplication operator in a commutative algebra, the following point is obvious:

LEMMA 5.1. *For* $\phi, \psi \in \mathrm{Val}^{U(n)}$,

$$(\tilde{L}\phi) \cdot \psi = \phi \cdot (\tilde{L}\psi).$$

We renormalize these operators by taking

$$L := \frac{2\omega_k}{\omega_{k+1}} \tilde{L}, \tag{62}$$

$$\Lambda := \frac{\omega_{2n-k}}{\omega_{2n-k+1}} \tilde{\Lambda} \tag{63}$$

on each homogeneous component $\mathrm{Val}_k^{\mathrm{sm}}$.

LEMMA 5.2.

$$L\tau_{k,p} = (k - 2p + 1)\, \tau_{k+1,p}, \tag{64}$$

$$\Lambda\tau_{k,p} = (2n - 2p - k + 1)\, \tau_{k-1,p} + (k - 2p + 1)\, \tau_{k-1,p-1}. \tag{65}$$

Proof. We show first that

$$\Lambda\mu_{k,q} = 2(n - k + q + 1)\mu_{k-1,q} + (k - 2q + 1)\mu_{k-1,q-1}, \tag{66}$$

$$L\mu_{k,q} = 2(q + 1)\mu_{k+1,q+1} + (k - 2q + 1)\mu_{k+1,q}. \tag{67}$$

Recall from [12] that if $\mu(K), \mu \in \mathrm{Val}^{\mathrm{sm}}(V)$, is obtained by integration over $N_1(K)$ of a differential form ψ on TV then $\tilde{\Lambda}\mu(K)$ is obtained by integration of the Lie derivative $\mathcal{L}_T\psi$ with respect to the Reeb vector field T; i.e. in the notation of (20),

$$\tilde{\Lambda}\nu_\psi = \nu_{\mathcal{L}_T\psi}.$$

The Lie derivatives of the θ_i with respect to T are

$$\mathcal{L}_T\theta_0 = 0, \mathcal{L}_T\theta_1 = 2\theta_0, \mathcal{L}_T\theta_2 = \theta_1,$$

from which one computes that

$$\mathcal{L}_T\theta_{k,q} = \frac{\omega_{2n-k+1}}{\omega_{2n-k}} \left(2(n - k + q + 1)\theta_{k-1,q} + (k - 2q + 1)\theta_{k-1,q-1}\right).$$

Relation (66) now follows at once. Relation (67) follows from (66) using equation (28) and the fact (which follows at once from Corollary 1.9 of [13]) that the Fourier transform intertwines the operators L, Λ:

$$\widehat{\ } \circ L = \Lambda \circ \widehat{\ }. \tag{68}$$

The assertions of the lemma now follow from (36) and (39). □

THEOREM 5.3. *Let X, Y, H with $[X,Y] = H, [H,X] = 2X, [H,Y] = -2Y$ be generators of $\mathfrak{sl}(2,\mathbb{R})$. The map*

$$\begin{aligned} H &\mapsto 2k - 2n, \\ X &\mapsto L, \\ Y &\mapsto \Lambda \end{aligned}$$

defines a representation of $\mathfrak{sl}(2,\mathbb{R})$ on $\mathrm{Val}^{U(n)}$.

Proof. This is a direct calculation, using Lemma 5.2. □

The following corollary is a standard fact for $\mathfrak{sl}(2)$ representations, compare [24] or [21].

COROLLARY 5.4.

$$[H, L^i] = 2iL^i, \tag{69}$$

$$[L^i, \Lambda] = iL^{i-1} \circ H + i(i-1)L^{i-1}. \tag{70}$$

We recall that an element π in degree $k \leq n$ of such a representation is called *primitive* if $\Lambda\pi = 0$, or equivalently, if $L^{2n-2k+1}\pi = 0$. By the Hard Lefschetz Theorem of Alesker [1], and comparing dimensions, it follows that there exists a unique (up to a multiplicative constant) primitive valuation in $\mathrm{Val}^{U(n)}$ in each even degree not larger than n.

In the following, we use the standard notation

$$(2k+1)!! = (2k+1) \cdot (2k-1) \cdot (2k-3) \cdots 1$$

and set formally $(-1)!! := 1$. For $0 \leq 2r \leq n$, using Lemma 5.2 we put

$$\pi_{2r,r} := (-1)^r (2n-4r+1)!! \sum_{i=0}^{r} (-1)^i \frac{(2r-2i-1)!!}{(2n-2r-2i+1)!!} \tau_{2r,i} \tag{71}$$

to be the unique primitive valuation of degree $2r$ whose expansion in terms of the Tasaki valuations has leading term $\tau_{2r,r}$, and define for $k \geq 2r$

$$\pi_{k,r} := L^{k-2r} \pi_{2r,r} \tag{72}$$

$$= (-1)^r (2n-4r+1)!! \sum_{i=0}^{r} (-1)^i \frac{(k-2i)!}{(2r-2i)!} \frac{(2r-2i-1)!!}{(2n-2r-2i+1)!!} \tau_{k,i} \tag{73}$$

by (64).

For further use, we note that by equation (36),

$$\pi_{2r,r} \equiv (-1)^r \frac{(2n-4r+1)!!(2r-1)!!}{(2n-2r+1)!!} \left(\mu_{2r,0} + \frac{2(2r-n-1)}{2r-1} \mu_{2r,1} \right) \mod \langle \mu_{2r,i} : i > 1 \rangle, \quad 2r \leq n. \tag{74}$$

PROPOSITION 5.5. *For each* $0 \leq k \leq 2n$ *the valuations* $\pi_{k,r}$, $0 \leq r \leq \frac{\min(k,2n-k)}{2}$ *constitute a basis of* $\mathrm{Val}_k^{U(n)}$. *Furthermore,*

$$\pi_{k,r} \cdot \pi_{2n-k,s} = 0, \quad r \neq s.$$

Proof. The fact that these elements constitute a base of $\mathrm{Val}_k^{U(n)}$ follows at once from the Lefschetz decomposition of the $\mathfrak{sl}(2)$-representation $\mathrm{Val}^{U(n)}$. If $r \neq s$, say $r > s$, then

$$\pi_{k,r} \cdot \pi_{2n-k,s} = L^{k-2r}\pi_{2r,r} \cdot L^{k-2s}\pi_{2s,s} = C \cdot L^{2n-2r-2s}\pi_{2r,r} \cdot \pi_{2s,s} = 0 \tag{75}$$

since $L^{2n-4r+1}\pi_{2r,r} = 0$. □

LEMMA 5.6. *For* $0 \leq 2r \leq k \leq 2n - 2r$,

$$\widehat{\pi_{k,r}} = \frac{(k-2r)!}{(2n-2r-k)!}\pi_{2n-k,r}. \tag{76}$$

Proof. We assume, as we may, that $k \leq n$. By the Hard Lefschetz Theorem of Alesker [1], $\Lambda^{n-k} : \mathrm{Val}_{2n-k}^{U(n)} \to \mathrm{Val}_n^{U(n)}$ is injective, so it is enough to show that $\Lambda^{n-k}\widehat{\pi_{k,r}} = \frac{(k-2r)!}{(2n-2r-k)!}\Lambda^{n-k}\pi_{2n-k,r}$. By (68) and the fact that the Fourier transform acts trivially on $\mathrm{Val}_n^{U(n)}$, the left-hand side is just $\pi_{n,r}$. On the other hand, relation (70) yields

$$\Lambda\pi_{l,r} = (l-2r)(2n-2r-l+1)\pi_{l-1,r},$$

which after iterating $n-k$ times gives

$$\Lambda^{n-k}\pi_{2n-k,r} = \frac{(2n-2r-k)!}{(k-2r)!}\pi_{n,r},$$

as claimed. □

Remark. Comparing the algebra of $\mathrm{Val}^{U(n)}$ to the cohomology of Kähler manifolds, (76) corresponds to the *magic formula* relating primitive forms, the Lefschetz operator and the Hodge star operator ([24, Prop. 1.2.31]).

5.2. *Two loose ends.* We tie up two loose ends from Sections 3.2 and 3.4.

PROPOSITION 5.7. *The constants* γ_k *from Lemma* 3.3 *are given by*

$$\mu_{k,0} = (-1)^{k+1}\frac{(2\pi)^k}{2\omega_k(k-1)!}f_k.$$

To this end we will make use of two lemmas. We say that a valuation in $\mathrm{Val}_k^{U(n)}$ is *anisotropic* if its Klain function vanishes on the isotropic k-Grassmannian $\mathrm{Gr}_{k,0}$. Thus the space of anisotropic valuations is spanned by the $\mu_{k,p}, p \geq 1$.

LEMMA 5.8. *The space of anisotropic valuations is an ideal in* $\mathrm{Val}^{U(n)}$.

Proof of Lemma 5.8. Let $\phi \in \mathrm{Val}_k^{U(n)}$ be anisotropic, and $\psi \in \mathrm{Val}^{U(n)}$ of degree l. By [13, §1.2.2], we may write

$$\psi(K) = \int_{\overline{\mathrm{Gr}}_{2n-l}(\mathbb{C}^n)} \chi(K \cap \bar{E}) d\mu(\bar{E})$$

with some smooth measure μ on the affine Grassmannian $\overline{\mathrm{Gr}}_{2n-l}(\mathbb{C}^n)$, and the product $\phi \cdot \psi$ may be expressed

$$\phi \cdot \psi(K) = \int_{\overline{\mathrm{Gr}}_{2n-l}(\mathbb{C}^n)} \phi(K \cap \bar{E}) d\mu(\bar{E}).$$

If K is contained in an isotropic subspace, then the same trivially holds true for $K \cap \bar{E}$. Since ϕ is anisotropic, the integrand on the right-hand side vanishes. It follows that $\phi \cdot \psi$ is anisotropic. □

Remark. In fact the ideal of anisotropic valuations equals the principal ideal $(u) = (\tau_{2,1}) = (\mu_{2,1})$.

Lemma 5.9.

$$t^i = \frac{i!\omega_i}{\pi^i}\mu_i, \tag{77}$$

$$s = \frac{1}{\pi}\left(\mu_{2,1} + \frac{1}{2}\mu_{2,0}\right), \tag{78}$$

$$u = \frac{2}{\pi}\,\mu_{2,1}. \tag{79}$$

Proof. Clearly $t^0 = \chi = \mu_0$ and $t = \frac{2}{\pi}\mu_1$ by equations (46) and (48) of [20]. Relation (77) now follows by induction using equation (67) (cf. also [20, Cor. 3.4]).

Theorem 3.1 implies that $s = \frac{1}{2}t^2$ locally at $n = 1$. This implies that the value of s on a complex disc is 1. Thus $s = \frac{1}{\pi}(\mu_{2,1} + a\mu_{2,0})$ for some $a \in \mathbb{R}$. Meanwhile, $-st + \frac{1}{3}t^3 = f_3 = 0$ locally at $n = 2$. Therefore $-s + \frac{1}{3}t^2$ is primitive in $\mathrm{Val}^{U(2)}$ with respect to the given $\mathfrak{sl}(2,\mathbb{R})$-representation. Since $\mu_{3,0} = \mu_{3,2} = 0$ locally at $n = 2$, this implies that

$$\begin{aligned} 0 = \pi L\left(-s + \frac{1}{3}t^2\right) &= L\left(-\mu_{2,1} - a\mu_{2,0} + \frac{2}{3}(\mu_{2,0} + \mu_{2,1})\right) \\ &= -\frac{1}{3}\mu_{3,1} + \left(\frac{2}{3} - a\right)2\mu_{3,1} \end{aligned}$$

by (67). Thus $a = \frac{1}{2}$. □

Proof of Proposition 5.7. By the recursion (26) we have

(80)

$$\begin{aligned} kf_k &= -t(k-1)f_{k-1} - s(k-2)f_{k-2} \\ &= -t\gamma_{k-1}^{-1}(k-1)\mu_{k-1,0} - \frac{t^2}{4}\gamma_{k-2}^{-1}(k-2)\mu_{k-2,0} - \frac{\gamma_{k-2}^{-1}(k-2)}{4}u\mu_{k-2,0}. \end{aligned}$$

Since $u = \frac{2}{\pi}\mu_{2,1}$ is anisotropic, the same holds true for $u \cdot \mu_{k-2,0}$ by Lemma 5.8. Comparing the coefficients of $\mu_{k,0}$ in (80), we obtain using (64)

$$k\gamma_k^{-1} = -(k-1)\gamma_{k-1}^{-1}\frac{\omega_k}{\pi\omega_{k-1}}k - \frac{\gamma_{k-2}^{-1}(k-2)}{4}\frac{\omega_k}{\pi^2\omega_{k-2}}k(k-1),$$

from which the proposition follows by induction. □

The next loose end is

Proof of (37) *from Section* 3.4. We proceed by induction on q. Since $\tau_{k,0}$ is the k^{th} intrinsic volume μ_k, the case $q = 0$ is (77). For the inductive step we observe first that since (64) may be reformulated as

$$\tau_{k+1,p} = \frac{\pi\omega_k}{(k-2p+1)\omega_{k+1}}t \cdot \tau_{k,p},$$

it is enough to prove the desired relation for $\tau_{2r,r}$. To accomplish this we compare the expressions (38) for the f_k with

$$\begin{aligned} f_k &= (-1)^{k+1}\frac{2\omega_k(k-1)!}{(2\pi)^k}\,\mu_{k,0} \\ &= (-1)^{k+1}\frac{2\omega_k(k-1)!}{(2\pi)^k}\sum_{i=0}^{\lfloor k/2\rfloor}(-1)^i\tau_{k,i}, \end{aligned}$$

which follows from Corollary 3.8 and Proposition 5.7. Taking $k = 2r$ and equating the two expressions, (37) follows from the inductive hypothesis. □

COROLLARY 5.10. *If* $2p \le k$ *then*

$$\begin{aligned} u \cdot \mu_{k,p} \equiv \frac{4(p+1)}{\pi(k+2)}&((2p+1)\mu_{k+2,p+1} - 2(p+2)\mu_{k+2,p+2}) \\ &\mod \langle \mu_{k+2,i} : i > p+2\rangle. \end{aligned} \tag{81}$$

Proof. Since $u \cdot \tau_{k,p} = \frac{2(2p+1)(2p+2)}{\pi(k+2)}\tau_{k+2,p+1}$ by (37), the desired relation (81) may be computed from relations (36), (39) between the τ and the μ. □

Remark. The two sides of (81) are in reality precisely equal, although we will not use this fact.

5.3. *The main computation.*

PROPOSITION 5.11. *For all* $k \geq 2r$

$$(\pi_{k,r}, \widehat{\pi_{k,r}}) = \frac{8^r \pi^n}{\omega_k \omega_{2n-k}} \binom{n}{2r} \frac{(k-2r)!(2n-4r)!}{(n-r)!(2n-2r-k)!} \frac{(2n-4r+1)!!}{(2n-2r+1)!!}. \tag{82}$$

Proof. We show first that for $2r \leq n$, the value of the Poincaré pairing (6) of u^r and $\widehat{\pi_{2r,r}}$ is

$$(u^r, \widehat{\pi_{2r,r}}) = \left(\frac{8}{\pi}\right)^r \frac{n!}{(n-2r)!} \frac{(2n-4r+1)!!}{(2n-2r+1)!!}. \tag{83}$$

This follows in turn from the relation

$$u \cdot \widehat{\pi_{2r,r}} = \frac{8(2n-2r+3)(n-2r+1)(n-2r+2)}{\pi(2n-4r+3)(2n-4r+5)} \widehat{\pi_{2r-2,r-1}}, \tag{84}$$

after r iterations, since $\widehat{\pi_{0,0}} = \widehat{\mu_0} = \mu_{2n}$. Both sides of (84) lie in the kernel of the map $L : \mathrm{Val}^{U(n)}_{2n-2r+2} \to \mathrm{Val}^{U(n)}_{2n-2r+3}$, which is one-dimensional. In order to fix the proportionality factor, it suffices to compare the coefficients of $\mu_{2n-2r+2,n-2r+2}$ on the two sides (note that locally $\mu_{2n-2r+2,n-2r+1} = 0$). It is straightforward to carry this out using (81) and (74).

To prove (82) observe first that by (37),

$$\pi_{2r,r} \equiv \tau_{2r,r} = \frac{\pi^{2r}}{\omega_{2r}(2r)!} u^r = \frac{\pi^r r!}{(2r)!} u^r \mod t.$$

Since $t \cdot \widehat{\pi_{2r,r}} = \text{const.}\, t \cdot \pi_{2n-2r,r} = 0$, the case $k = 2r$ follows from (83), the definition (71) of $\pi_{2r,r}$, and (37). If $k > 2r$ we use Lemmas 5.1 and 5.6 to compute

$$\begin{aligned}
\pi_{k,r} \cdot \widehat{\pi_{k,r}} &= \frac{(k-2r)!}{(2n-2r-k)!} \pi_{k,r} \cdot \pi_{2n-k,r} \\
&= \frac{(k-2r)!}{(2n-2r-k)!} (L^{k-2r} \pi_{2r,r}) \cdot (L^{2n-k-2r} \pi_{2r,r}) \\
&= \frac{(k-2r)!}{(2n-2r-k)!} \frac{\omega_{2r}\omega_{2n-2r}}{\omega_k \omega_{2n-k}} \pi_{2r,r} \cdot L^{2n-4r} \pi_{2r,r} \\
&= \frac{(k-2r)!(2n-4r)!}{(2n-2r-k)!} \frac{\omega_{2r}\omega_{2n-2r}}{\omega_k \omega_{2n-k}} \pi_{2r,r} \cdot \widehat{\pi_{2r,r}},
\end{aligned}$$

which with the previous case yields (82). □

Using Theorem 2.2, relation (82), Proposition 5.5 and Lemma 5.6 now yield at once

THEOREM 5.12. *Set* $p := \min\left\{\lfloor \frac{k}{2} \rfloor, \lfloor \frac{2n-k}{2} \rfloor\right\}$.

(85)

$$\begin{aligned} k_{U(n)}(\chi) = \frac{1}{\pi^n} \sum_{k=0}^{2n} \omega_k \omega_{2n-k} \\ \sum_{r=0}^{p} \frac{(2n-2r-k)!(n-r)!}{8^r (k-2r)!(2n-4r)!} \frac{(2n-2r+1)!!}{(2n-4r+1)!!} \binom{n}{2r}^{-1} \pi_{k,r} \otimes \widehat{\pi_{k,r}} \\ = \frac{1}{\pi^n} \sum_{k=0}^{2n} \omega_k \omega_{2n-k} \\ \sum_{r=0}^{p} \frac{(n-r)!}{8^r (2n-4r)!} \frac{(2n-2r+1)!!}{(2n-4r+1)!!} \binom{n}{2r}^{-1} \pi_{k,r} \otimes \pi_{2n-k,r}. \end{aligned}$$

COROLLARY 5.13. *The Tasaki matrices* T_k^n, *and the matrices* Q_k^n *of* [20], *are positive definite.*

Proof. These matrices are the inverses of those arising respectively by expressing the bilinear forms

$$(\phi, \psi) \mapsto \phi \cdot \widehat{\psi}, \quad (\phi, \psi) \mapsto \phi \cdot t^{2n-k} \psi$$

on $\mathrm{Val}_k^{U(n)}$ in terms of specific bases (the Tasaki valuations in the first case and the monomials in s and t in the second). Both of these diagonalize upon change of basis to the $\pi_{k,r}$, and the diagonal entries are the inverses of the (positive) coefficients of (85) in the first case, and positive multiples of these in the second (by Lemma 5.6 and the definition (72) of the $\pi_{k,r}$). □

Expanding via (73) we obtain

COROLLARY 5.14. *The* (i,j) *entry of the Tasaki matrix* T_k^n *is*

$$\begin{aligned} (T_k^n)_{ij} = (-1)^{i+j} \frac{\omega_k \omega_{2n-k}}{\pi^n} \\ \sum_{r=\max(i,j)}^{\lfloor \frac{k}{2} \rfloor} \left[\binom{n}{2r}^{-1} \frac{(2n-2r-k)!(n-r)!(k-2i)!(k-2j)!}{8^r (k-2r)!(2n-4r)!(2r-2i)!(2r-2j)!} \right. \\ \left. \times \frac{(2n-2r+1)!!(2n-4r+1)!!(2r-2i-1)!!(2r-2j-1)!!}{(2n-2r-2i+1)!!(2n-2r-2j+1)!!} \right]. \end{aligned}$$

We have not been able to simplify this expression further. However, for fixed k, the above sum is finite and can be computed in a closed form. Thus it is straightforward (albeit messy) to calculate

$$T_2^n = \frac{1}{4n(n-1)} \begin{pmatrix} 2n-1 & -1 \\ -1 & 2n-1 \end{pmatrix}, \tag{86}$$

$$T_3^n = \frac{2^{n-2}(n-3)!}{n\pi(2n-3)!!}\begin{pmatrix} 2n-3 & -1 \\ -1 & \frac{2n-1}{3} \end{pmatrix}, \tag{87}$$

(88)
$$T_4^n = \frac{(n-4)!}{16\,n!}\begin{pmatrix} 3(2n-5)(2n-3) & -3(2n-3) & 9 \\ -3(2n-3) & 2n^2-4n+3 & -3(2n-3) \\ 9 & -3(2n-3) & 3(2n-5)(2n-3) \end{pmatrix},$$

etc. The matrices T_2^n, T_3^3 had previously been computed in [32] using the template method.

Note that since $k = 2, 4$ are even, the matrices T_2^n, T_4^n display both the expected diagonal symmetry and the antidiagonal symmetry predicted by Theorem 3.10. In fact that theorem gives a family of identities among the values given in Corollary 5.14 whenever k is even. From a practical perspective this is an aid in computing closed forms for these expressions, since for larger values of i, j the sum in Corollary 5.14 is shorter.

5.4. *Other kinematic formulas.* Of course the whole point of the computations above is to give explicit forms for the kinematic formulas

$$k_{U(n)}(\chi)(A, B) = \int_{\overline{U(n)}} \chi(A \cap \bar{g}B)\, d\bar{g},$$

which in turn specialize to Crofton formulas when $A, B \subset \mathbb{C}^n$ are compact C^1 submanifolds (or even rectifiable sets) of complementary dimension. By the transfer principle (Theorem 2.4), the latter formulas hold verbatim if $\mathbb{C}^n$ and $\overline{U(n)}$ are replaced by the spaces $\mathbb{C}P^n$ or $\mathbb{C}H^n$ of constant holomorphic sectional curvature together with their groups of isometries, with measures $d\bar{g}$ given by the standard convention (9).

In the case of $\mathbb{C}P^n$, however, another natural convention is to take $d\bar{g}$ to be a probability measure. The resulting Crofton formulas may then be viewed as a generalization of Bézout's theorem. Normalizing the metric to be the standard Fubini-Study metric (i.e. with holomorphic sectional curvature 4), they are obtained by dividing the constants above by $\mathrm{vol}_{2n}(\mathbb{C}P^n) = \frac{\pi^n}{n!}$. It is reassuring to recover Bézout's theorem for pairs (algebraic curve, algebraic hypersurface) and (algebraic surface, algebraic variety of codimension 2) from the matrices (86), (88), using the fact that for varieties $V^k, W^{n-k} \subset \mathbb{C}P^n$

$$\tau_{2k,q}(V) = \binom{k}{q}\mu_{2k,k}(V) = \binom{k}{q}\mu_{2k}(V) = \binom{k}{q}\frac{\pi^k}{k!}\deg(V),$$
$$\widehat{\tau_{2k,q}}(W) = \binom{k}{q}\mu_{2n-2k,n-k}(W) = \binom{k}{q}\mu_{2n-2k}(W) = \binom{k}{q}\frac{\pi^{n-k}}{(n-k)!}\deg(W),$$

as may be computed via (36).

The calculations above also permit us to compute in explicit form the kinematic formulas $k_{U(n)}(\tau_{k,p})$, using the fundamental relation (12) and the product formula

$$\tau_{k,p}\cdot\tau_{l,q}=\frac{\omega_{k+l}}{\omega_k\omega_l}\binom{k+l-2p-2q}{k-2p}\binom{2p+2q}{2p}\tau_{k+l,p+q}, \tag{89}$$

which is a simple consequence of (37). Rather than write down further messy general formulas, we illustrate by computing the expected value of the length of the curve given by the intersection of a real 4-fold and a real 5-fold in $\mathbb{C}P^4$.

THEOREM 5.15. *Let $M^4, N^5 \subset \mathbb{C}P^4$ be real C^1 submanifolds of dimension $4,5$ respectively. Let θ_1,θ_2 be the Kähler angles of the tangent plane to M at a general point x and ψ the Kähler angle of the orthogonal complement to the tangent plane to N at y. Let dg denote the invariant probability measure on $U(5)$. Then*

$$\begin{aligned}\int_{U(5)}&\operatorname{length}(M\cap gN)\,dg\\ &=\frac{1}{5\pi^4}\times\Big[30\operatorname{vol}_4(M)\operatorname{vol}_5(N)-6\operatorname{vol}_4(M)\int_N\cos^2\psi\,dy\\ &\qquad-3\int_M(\cos^2\theta_1+\cos^2\theta_2)\,dx\cdot\operatorname{vol}_5(N)\\ &\qquad+7\int_M(\cos^2\theta_1+\cos^2\theta_2)\,dx\cdot\int_N\cos^2\psi\,dy\Big].\end{aligned}$$

Proof. If $l\subset\mathbb{C}P^n$ is a real curve, then $\tau_{1,0}(l)=\operatorname{length}(l)$. Thus by the transfer principle we wish to compute the terms of bidegree $(4,5)$ in $\frac{4!}{\pi^4}k_{U(4)}(\tau_{1,0})$. Since

$$\tau_{4,0}\cdot\tau_{1,0}=\frac{8}{3}\tau_{5,0},\quad \tau_{4,1}\cdot\tau_{1,0}=\frac{8}{5}\tau_{5,1},\quad \tau_{4,2}\cdot\tau_{1,0}=\frac{8}{15}\tau_{5,2}$$

the matrix giving the relevant terms is

$$\frac{1}{10\pi^4}\begin{pmatrix}75&-15&3\\-25&19&-5\\15&-15&15\end{pmatrix},$$

where the columns are indexed by the $\tau_{5,i}$ and the rows by the $\widehat{\tau_{4,j}}=\tau_{4,j}$. Locally at $n=4$,

$$\tau_{5,0}=\widehat{\tau_{3,0}},\quad \tau_{5,1}=\widehat{\tau_{3,0}}+\widehat{\tau_{3,1}},\quad \tau_{5,2}=\widehat{\tau_{3,1}}$$

so with respect to the bases $\widehat{\tau_{3,i}},\tau_{4,j}$ one computes the pairing matrix to be

$$\frac{1}{5\,\pi^4}\begin{pmatrix}30&-6\\-3&7\\0&0\end{pmatrix}. \tag{90}$$

□

By [13], we can also translate this result to give an additive kinematic formula for the average 7-dimensional volume of the Minkowski sum of two convex subsets in $\mathbb{C}^4$ of dimensions 3 and 4 respectively.

THEOREM 5.16. *Let $E \in \mathrm{Gr}_4(\mathbb{C}^4), F \in \mathrm{Gr}_3(\mathbb{C}^4)$; let θ_1, θ_2 be the Kähler angles of E and ψ the Kähler angle of F. Let dg be the invariant probability measure on $U(4)$. If $A \in \mathcal{K}(E), B \in \mathcal{K}(F)$ then*

$$\int_{U(4)} \mathrm{vol}_7(A + gB)\, dg = \frac{1}{120}\,\mathrm{vol}_4(A)\,\mathrm{vol}_3(B) \times \left[30 - 6\cos^2\psi - 3(\cos^2\theta_1 + \cos^2\theta_2) + 7\cos^2\psi(\cos^2\theta_1 + \cos^2\theta_2)\right].$$

Proof. Recall that the additive kinematic operator $a_{U(4)} : \mathrm{Val}^{U(4)}(\mathbb{C}^4) \to \mathrm{Val}^{U(4)}(\mathbb{C}^4) \otimes \mathrm{Val}^{U(4)}(\mathbb{C}^4)$ is given by

$$a_{U(4)}(\phi)(A, B) = \int_{U(4)} \phi(A + gB)\, dg.$$

By Theorem 1.7 of [13],

$$a_{U(4)}(\mu_7) = \widehat{k_{U(4)}(\widehat{\mu_7})} = \widehat{k_{U(4)}(\mu_1)}.$$

Thus the bidegree $(3, 4)$ terms of $a_{U(4)}(\mu_7)$ are given with respect to the bases $\tau_{3,i}, \tau_{4,j}$ by the matrix $\frac{\pi^4}{4!} \times$ (90). □

References

[1] S. ALESKER, Hard Lefschetz theorem for valuations, complex integral geometry, and unitarily invariant valuations, *J. Differential Geom.* **63** (2003), 63–95. MR 2015260. Zbl 1073.52004. Available at http://projecteuclid.org/getRecord?id=euclid.jdg/1080835658.

[2] ———, Hard Lefschetz theorem for valuations and related questions of integral geometry, in *Geometric Aspects of Functional Analysis*, *Lecture Notes in Math.* **1850**, Springer-Verlag, New York, 2004, pp. 9–20. MR 2087146. Zbl 1070.52007.

[3] ———, The multiplicative structure on continuous polynomial valuations, *Geom. Funct. Anal.* **14** (2004), 1–26. MR 2053598. Zbl 1072.52011. doi: 10.1007/s00039-004-0450-2.

[4] ———, Theory of valuations on manifolds. I. Linear spaces, *Israel J. Math.* **156** (2006), 311–339. MR 2282381. Zbl 1132.52017. doi: 10.1007/BF02773837.

[5] ———, Theory of valuations on manifolds. II, *Adv. Math.* **207** (2006), 420–454. MR 2264077. Zbl 1117.52016. doi: 10.1016/j.aim.2005.11.015.

[6] ———, Theory of valuations on manifolds: a survey, *Geom. Funct. Anal.* **17** (2007), 1321–1341. MR 2373020. Zbl 1132.52018. doi: 10.1007/s00039-007-0631-x.

[7] ———, Theory of valuations on manifolds. IV. New properties of the multiplicative structure, in *Geometric Aspects of Functional Analysis*, *Lecture Notes in Math.* **1910**, Springer-Verlag, New York, 2007, pp. 1–44. MR 2347038. Zbl 1127.52010. doi: 10.1007/978-3-540-72053-9_1.

[8] S. Alesker, A Fourier type transform on translation invariant valuations on convex sets, preprint. arXiv math/0702842.

[9] S. Alesker and J. Bernstein, Range characterization of the cosine transform on higher Grassmannians, *Adv. Math.* **184** (2004), 367–379. MR 2054020. Zbl 1059.22013. doi: 10.1016/S0001-8708(03)00149-X.

[10] S. Alesker and J. H. G. Fu, Theory of valuations on manifolds. III. Multiplicative structure in the general case, *Trans. Amer. Math. Soc.* **360** (2008), 1951–1981. MR 2366970. Zbl 1130.52008. doi: 10.1090/S0002-9947-07-04489-3.

[11] A. Bernig, A Hadwiger-type theorem for the special unitary group, *Geom. Funct. Anal.* **19** (2009), 356–372. MR 2545241. Zbl 1180.53076. doi: 10.1007/s00039-009-0008-4.

[12] A. Bernig and L. Bröcker, Valuations on manifolds and Rumin cohomology, *J. Differential Geom.* **75** (2007), 433–457. MR 2301452. Zbl 1117.58005. Available at http://projecteuclid.org/getRecord?id=euclid.jdg/1175266280.

[13] A. Bernig and J. H. G. Fu, Convolution of convex valuations, *Geom. Dedicata* **123** (2006), 153–169. MR 2299731. Zbl 1117.53054. doi: 10.1007/s10711-006-9115-7.

[14] S.-S. Chern, On the kinematic formula in integral geometry, *J. Math. Mech.* **16** (1966), 101–118. MR 0198406. Zbl 0142.20704.

[15] P. Deligne, La conjecture de Weil. II, *Inst. Hautes Études Sci. Publ. Math.* (1980), 137–252. MR 0601520. Zbl 0456.14014. Available at http://www.numdam.org/item?id=PMIHES_1980__52__137_0.

[16] H. Federer, Curvature measures, *Trans. Amer. Math. Soc.* **93** (1959), 418–491. MR 0110078. Zbl 0089.38402. doi: 10.2307/1993504.

[17] J. H. G. Fu, Kinematic formulas in integral geometry, *Indiana Univ. Math. J.* **39** (1990), 1115–1154. MR 1087187. Zbl 0703.53059. doi: 10.1512/iumj.1990.39.39052.

[18] ———, Curvature measures of subanalytic sets, *Amer. J. Math.* **116** (1994), 819–880. MR 1287941. Zbl 0818.53091. doi: 10.2307/2375003.

[19] ———, Integral geometry and Alesker's theory of valuations, in *Integral Geometry and Convexity*, World Sci. Publ., Hackensack, NJ, 2006, pp. 17–27. MR 2240970. Zbl 1121.52029. doi: 10.1142/9789812774644_0002.

[20] ———, Structure of the unitary valuation algebra, *J. Differential Geom.* **72** (2006), 509–533. MR 2219942. Zbl 1096.52003. Available at http://projecteuclid.org/getRecord?id=euclid.jdg/1143593748.

[21] P. Griffiths and J. Harris, *Principles of Algebraic Geometry*, *Pure and Applied Math.*, Wiley-Interscience [John Wiley & Sons], New York, 1978. MR 0507725. Zbl 0408.14001.

[22] H. Hadwiger, *Vorlesungen über Inhalt*, *Oberfläche und Isoperimetrie*, *Die Grundl. Math. Wissen.* **93**, Springer-Verlag, New York, 1957. MR 0102775. Zbl 0078.35703.

[23] R. Howard, The kinematic formula in Riemannian homogeneous spaces, *Mem. Amer. Math. Soc.* **106** (1993), vi+69. MR 1169230. Zbl 0810.53057.

[24] D. Huybrechts, *Complex Geometry*, *Universitext*, Springer-Verlag, New York, 2005. MR 2093043. Zbl 1055.14001. doi: 10.1007/3-540-26687-9_2.

[25] D. A. Klain, Even valuations on convex bodies, *Trans. Amer. Math. Soc.* **352** (2000), 71–93. MR 1487620. Zbl 0940.52002. doi: 10.1090/S0002-9947-99-02240-0.

[26] D. A. Klain and G.-C. Rota, *Introduction to Geometric Probability*, *Lezioni Lincee.* [*Lincei Lectures*], Cambridge Univ. Press, Cambridge, 1997. MR 1608265. Zbl 0896.60004.

[27] A. Nijenhuis, On Chern's kinematic formula in integral geometry, *J. Differential Geometry* **9** (1974), 475–482. MR 0345047. Zbl 0284.53045. Available at http://projecteuclid.org/getRecord?id=euclid.jdg/1214432423.

[28] H. Park, Kinematic formulas for the real subspaces of complex space forms of dimension 2 and 3, Ph.D. thesis, University of Georgia, 2002.

[29] M. Rumin, Formes différentielles sur les variétés de contact, *J. Differential Geom.* **39** (1994), 281–330. MR 1267892. Zbl 0973.53524. Available at http://projecteuclid.org/getRecord?id=euclid.jdg/1214454873.

[30] L. A. Santaló, *Integral Geometry and Geometric Probability*, Second ed., *Cambridge Math. Library*, Cambridge Univ. Press, Cambridge, 2004. MR 2162874. Zbl 1116.53050.

[31] H. Tasaki, Generalization of Kähler angle and integral geometry in complex projective spaces, in *Steps in Differential Geometry*, Inst. Math. Inform., Debrecen, 2001, pp. 349–361. MR 1859314. Zbl 0984.53030.

[32] ———, Generalization of Kähler angle and integral geometry in complex projective spaces. II, *Math. Nachr.* **252** (2003), 106–112. MR 1903043. Zbl 1038.53069. doi: 10.1002/mana.200310040.

[33] K. Yano and S. Ishihara, *Tangent and Cotangent Bundles*: *Differential Geometry*, *Pure Appl. Math.* **16**, Marcel Dekker, New York, 1973. MR 0350650. Zbl 0262.53024.

(Received: January 14, 2008)
(Revised: July 2, 2009)

Goethe-Universität Frankfurt, Frankfurt, Germany
E-mail: bernig@math.uni-frankfurt.de

University of Georgia, Athens, GA
E-mail: fu@math.uga.edu