Asymptotics of characters of symmetric groups related to Stanley character formula

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Abstract

We prove an upper bound for characters of the symmetric groups. In particular, we show that there exists a constant $a > 0$ with a property that for every Young diagram λ with n boxes, $r(\lambda)$ rows and $c(\lambda)$ columns

$$\left| \frac{\operatorname{Tr} \rho^\lambda(\pi)}{\operatorname{Tr} \rho^\lambda(e)} \right| \leq \left[a \max \left(\frac{r(\lambda)}{n}, \frac{c(\lambda)}{n}, \frac{|\pi|}{n} \right) \right]^{|\pi|},$$

where $|\pi|$ is the minimal number of factors needed to write $\pi \in S_n$ as a product of transpositions. We also give uniform estimates for the error term in the Vershik-Kerov's and Biane's character formulas and give a new formula for free cumulants of the transition measure.

1. Introduction

1.1. *Normalized characters.* For a Young diagram λ having n boxes and a permutation $\pi \in S_l$ (where $l \leq n$) we define the *normalized character*

$$(1) \qquad \Sigma^\lambda(\pi) = (n)_l \, \chi^\lambda(\pi),$$

where $(n)_l = n(n-1) \cdots (n-l+1)$ denotes the falling power and where

$$\chi^\lambda(\pi) = \frac{\operatorname{Tr} \rho^\lambda(\pi)}{\operatorname{Tr} \rho^\lambda(e)}$$

is the character rescaled in such a way that $\chi^\lambda(e) = 1$.

1.2. *Short history of the problem.* Unfortunately, the canonical tool for calculating characters, the Murnaghan-Nakayama rule, quickly becomes cumbersome and hence intractable for computing characters corresponding to large Young diagrams. Nevertheless Roichman [Roi96] showed that it is possible to use it to find an upper bound for characters, namely he proved that there exist constants $0 < q < 1$ and $b > 0$ such that

$$(2) \qquad |\chi^\lambda(\pi)| \leq \left[\max \left(\frac{r(\lambda)}{n}, \frac{c(\lambda)}{n}, q \right) \right]^{b \, |\operatorname{supp} \pi|},$$

where $r(\lambda)$, $c(\lambda)$ denote the numbers of rows and columns of λ and $\operatorname{supp} \pi$ denotes the support of a permutation π (the set of its nonfixed points). Inequality (2) is not satisfactory for many practical purposes (such as [MRŚ07]) since it provides rather weak estimates in the case when the Young diagram λ is *balanced*, i.e. $r(\lambda), c(\lambda) = O(\sqrt{n})$.

Another approach to this problem was initiated by Biane [Bia98], [Bia03] who showed that the value of the normalized character $\Sigma^\lambda(\pi)$ can be expressed as a polynomial (called Kerov polynomial) in *free cumulants* of the *transition measure* of a Young diagram λ. The work of Biane was based on previous contributions of Kerov [Ker93], [Ker99] and Vershik. Free cumulants of the transition measure have a nice geometric interpretation therefore Kerov polynomials are a perfect tool for study of the character $\chi^\lambda(\pi)$ in the limit when the permutation π is fixed and the Young diagram λ tends in some sense to infinity.

Unfortunately, despite much progress in this field ([DFŚ10] and references therein) our understanding of Kerov polynomials is still not satisfactory; in particular it is not clear how to use Kerov polynomials in order to obtain nontrivial estimates on the characters $\chi^\lambda(\pi)$ when the length $|\pi|$ of the permutation $\pi \in S_n$ is comparable with n.

In a recent work of one of us with Rattan [RŚ08] we took yet another approach: thanks to the generalized Frobenius formula we showed that the value of a normalized character of a given Young diagram λ can be bounded from above by the value of the normalized character of a rectangular Young diagram $p \times q$ for suitably chosen p, q. For such a rectangular Young diagram the value of the normalized character can be explicitly calculated thanks to the formula of Stanley [Sta04]. In this way we proved that for each C there exists a constant D such that if $r(\lambda), c(\lambda) < C\sqrt{n}$ then

$$(3) \qquad |\chi^\lambda(\pi)| < \left(\frac{D \max(1, \frac{|\pi|^2}{n})}{\sqrt{n}} \right)^{|\pi|},$$

where $|\pi|$ denotes the minimal number of factors necessary to write π as a product of transpositions. Inequality (3) gives a much better estimate than (2) for balanced Young diagrams and a quite short permutation ($|\pi| = o(\sqrt{n})$) but a careful analysis of its proof shows that it gives nontrivial estimates only if $\max\big(r(\lambda), c(\lambda)\big) < O(n^{3/4})$.

1.3. *The main result.* Our main result is the following inequality.

THEOREM 1. *There exists a constant $a > 0$ such that for every Young diagram λ*

$$|\chi^\lambda(\pi)| \leq \left[a \max \left(\frac{r(\lambda)}{n}, \frac{c(\lambda)}{n}, \frac{|\pi|}{n} \right) \right]^{|\pi|},$$

where n denotes the number of boxes of λ.

It is easy to check that (3) is a consequence of this theorem and that it gives better estimates than (2) if $\frac{r(\lambda)}{n}, \frac{c(\lambda)}{n}, \frac{|\pi|}{n}$ are smaller than some positive constant. It is natural to ask what is the optimal value of the constant a. Asymptotics of characters of symmetric groups related to Thoma characters shows that $a \geq 1$.

1.4. *Young diagrams.* In the following we shall identify a Young diagram λ with the set of its boxes which we regard as a subset of $\mathbb{N}^2$ given by a graphical representation of λ according to the French notation; namely, for a partition $\lambda = (\lambda_1, \ldots, \lambda_k)$ it is the set

$$(4) \qquad \lambda = \bigcup_{1 \leq i \leq k} \{1, 2, \ldots, \lambda_i\} \times \{i\} = \{(p, q) \in \mathbb{N}^2 : 1 \leq p \leq \lambda_q\}.$$

1.5. *The main tool*: *reformulation of Stanley character formula.* Our main tool in our investigations will be the following reformulation of Stanley character formula.

The set of cycles of a permutation π is denoted by $C(\pi)$. For given permutations $\sigma_1, \sigma_2 \in S_l$ we shall consider colorings h of the cycles of σ_1 (where each cycle is colored by the number of some column of λ) and of the cycles of σ_2 (where each cycle is colored by the number of some row of λ). Formally, each such coloring can be viewed as a function $h : C(\sigma_1) \sqcup C(\sigma_2) \to \mathbb{N}$. We say that a coloring h is compatible with a Young diagram λ if for all $c_1 \in C(\sigma_1)$ and $c_2 \in C(\sigma_2)$ if $c_1 \cap c_2 \neq \emptyset$, then $(h(c_1), h(c_2)) \in \lambda$; in other words

$$(5) \qquad\qquad\qquad 0 < h(c_1) \leq \lambda_{h(c_2)}$$

holds true for all $c_1 \in C(\sigma_1)$, $c_2 \in C(\sigma_2)$ such that $c_1 \cap c_2 \neq \emptyset$.

THEOREM 2 (The new formulation of Stanley character formula). *For any Young diagram λ and a permutation $\pi \in S_l$ (where $l \leq n$) the value of the normalized character (1) is given by*

$$(6) \qquad\qquad \Sigma^\lambda(\pi) = \sum_{\substack{\sigma_1, \sigma_2 \in S_l, \\ \sigma_1 \sigma_2 = \pi}} (-1)^{|\sigma_1|} N^\lambda(\sigma_1, \sigma_2),$$

where

$$(7) \quad N^\lambda(\sigma_1, \sigma_2) = \#\{h : h \text{ is a coloring of the cycles of } \sigma_1 \text{ and } \sigma_2$$
$$\text{which is compatible with } \lambda\}.$$

Example 3. For a given factorization $\pi = \sigma_1 \sigma_2$ it is convenient to consider a bipartite graph with the set of vertices $C(\sigma_1) \sqcup C(\sigma_2)$ and with an edge between vertices $c_1 \in C(\sigma_1)$ and $c_2 \in C(\sigma_2)$ if and only if $c_1 \cap c_2 \neq \emptyset$. Notice that the value of $N^\lambda(\sigma_1, \sigma_2)$ does not depend on the exact form of σ_1 and σ_2 but only on the corresponding bipartite graph.
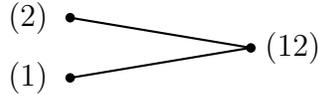
Figure 1. Bipartite graph associated to the factorization $(12) = (1)(2) \cdot (12)$.

Figure 1 presents such a bipartite graph for $\pi = (12)$, $\sigma_1 = (1)(2)$ and $\sigma_2 = (12)$. Now it becomes clear that

$$N^\lambda\big((1)(2),(12)\big) = \sum_i (\lambda_i)^2.$$

Similarly,

$$N^\lambda\big((12),(1)(2)\big) = \sum_i (\lambda_i')^2,$$

where λ' denotes the Young diagram conjugate to λ. In this way, Theorem 2 shows that

$$\Sigma^\lambda(12) = n(n-1)\frac{\operatorname{Tr}\rho^\lambda(12)}{\operatorname{Tr}\rho^\lambda(e)}$$
$$= N^\lambda\big((1)(2),(12)\big) - N^\lambda\big((12),(1)(2)\big) = \sum_i (\lambda_i)^2 - \sum_i (\lambda_i')^2.$$

1.6. *Overview of the paper.* In Section 2 we will prove the new formulation of Stanley character formula, Theorem 2. In Section 3 we present a relation between the characters of symmetric groups and characters of some Gaussian random matrices. We also give a new formula for calculating free cumulants of (the transition measure of) a Young diagram. Section 4 is devoted to the proofs of some techical inequalities. In Section 5 we prove estimates for the characters of the symmetric groups based on Stanley character formula.

2. **Stanley character formula**

In this section, we prove Theorem 2. It is quite easy to show that it is equivalent to a recent formula, conjectured by Stanley [Sta06] and proved by the first author [Fér10]. But the formulation given here is more useful for the purposes of character estimates and its proof is more elementary than the one given in [Fér10].

2.1. *Young symmetrizer.* Let λ be a Young diagram consisting of n boxes. In the following we will distinguish the symmetric group S_n which permutes the elements $\{1, \ldots, n\}$ and the symmetric group $\tilde{S}_n$ which permutes the boxes of λ.

For a box $\square \in \lambda$ we denote by $r(\square) \in \mathbb{N}$ (respectively, $c(\square) \in \mathbb{N}$) the row (respectively, the column) of $\square$; in this way $\square = (c(\square), r(\square))$.

If $\sigma \in \tilde{S}_n$ has a property that if two different boxes $\square_1, \square_2$ are in the same row, then their images $\sigma(\square_1), \sigma(\square_2)$ are not in the same column then we define its number of column inversions $\mathrm{cinv}(\sigma)$ as the number of pairs $\square_1, \square_2$ such that $\sigma(\square_1), \sigma(\square_2)$ are in the same column, $r(\square_1) < r(\square_2)$ and $r(\sigma(\square_1)) > r(\sigma(\square_2))$. If $\sigma \in \tilde{S}_n$ does not have this property, then we define $(-1)^{\mathrm{cinv}(\sigma)} = 0$.

The following theorem gives a very esthetically appealing formula for the characters of the symmetric groups.

THEOREM 4. *Let a Young diagram λ having n boxes and $\pi \in S_n$ be given. Let $\hat{\pi} \in \tilde{S}_n$ be a random permutation distributed with the uniform distribution on the conjugacy class defined by π. Then*

$$\chi^\lambda(\pi) = \mathbb{E}[(-1)^{\mathrm{cinv}(\hat{\pi})}].$$

Proof. We denote

$$P_\lambda = \{\sigma \in \tilde{S}_n \colon \sigma \text{ preserves each row of } \lambda\},$$
$$Q_\lambda = \{\sigma \in \tilde{S}_n \colon \sigma \text{ preserves each column of } \lambda\}$$

and define

$$a_\lambda = \sum_{\sigma \in P_\lambda} \sigma \in \mathbb{C}[\tilde{S}_n],$$
$$b_\lambda = \sum_{\sigma \in Q_\lambda} (-1)^{|\sigma|}\sigma \in \mathbb{C}[\tilde{S}_n],$$
$$c_\lambda = b_\lambda a_\lambda.$$

It is well-known that $p_\lambda = \alpha_\lambda c_\lambda$ is an idempotent for a constant α_λ which will be specified later. Its image $V_\lambda = \mathbb{C}[\tilde{S}_n]p_\lambda$ under multiplication from the right on the regular representation gives a representation ρ^λ (where the symmetric group acts by left multiplication) associated to a Young diagram λ. It turns out that $\alpha_\lambda = \frac{\dim V_\lambda}{n!}$. It follows that for $\tilde{\pi} \in \tilde{S}_n$

$$(8) \qquad n! \frac{\mathrm{Tr}\, \rho^\lambda(\tilde{\pi})}{\dim V_\lambda} = \frac{\mathrm{Tr}\, \rho^\lambda(\tilde{\pi}^{-1})}{\alpha_\lambda} = \frac{1}{\alpha_\lambda} \sum_{\mu \in \tilde{S}_n} \langle \delta_\mu, \tilde{\pi}^{-1}\delta_\mu p_\lambda \rangle$$
$$= \sum_{\mu \in \tilde{S}_n} \sum_{\tilde{\sigma}_1 \in Q_\lambda} \sum_{\tilde{\sigma}_2 \in P_\lambda} (-1)^{|\tilde{\sigma}_1|}[\mu = \tilde{\pi}^{-1}\mu\tilde{\sigma}_1\tilde{\sigma}_2].$$

We define $\hat{\pi} = \mu^{-1}\tilde{\pi}\mu$. For such a permutation $\hat{\pi}$ there exists at most one factorization $\hat{\pi} = \tilde{\sigma}_1\tilde{\sigma}_2$; by Young lemma this factorization exists if and only if $(-1)^{\mathrm{cinv}(\hat{\pi})} \neq 0$. Furthermore, if such a factorization exists then $(-1)^{|\sigma_1|} =$

$(-1)^{\mathrm{cinv}(\hat{\pi})}$. It follows that

$$n! \frac{\mathrm{Tr}\, \rho^\lambda(\tilde{\pi})}{\dim V_\lambda} = \sum_{\mu \in \tilde{S}_n} (-1)^{\mathrm{cinv}(\mu^{-1}\tilde{\pi}\mu)}.$$

As μ runs over all permutations, $\hat{\pi} = \mu^{-1}\tilde{\pi}\mu$ runs over all elements of the conjugacy class defined by π which finishes the proof.   □

For permutations $\sigma_1, \sigma_2 \in S_l$ we define $\tilde{N}^\lambda(\sigma_1, \sigma_2)$ as the number of one-to-one functions f from $\{1, \ldots, l\}$ to the set of boxes of λ such that $r \circ f$ is constant on each cycle of σ_2 and $c \circ f$ is constant on each cycle of σ_1.

PROPOSITION 5. *Let λ be a Young diagram having n boxes. For any permutation $\pi \in S_l$ (where $l \leq n$)*

$$\Sigma^\lambda(\pi) = \sum_{\substack{\sigma_1, \sigma_2 \in S_l \\ \sigma_1\sigma_2 = \pi}} (-1)^{|\sigma_1|} \tilde{N}^\lambda(\sigma_1, \sigma_2).$$

Proof. Let us consider the case $l = n$. Let $\tilde{\pi} \in \tilde{S}_n$ be any permutation with the same cycle structure as $\pi \in S_n$. Our starting point is the analysis of (8). Notice that the multiset of the values of $\mu^{-1}\tilde{\pi}\mu$ (over $\mu \in \tilde{S}_n$) coincides with the multiset of the values of $f \circ \pi \circ f^{-1}$ (over bijections f). We define $\sigma_i = f^{-1} \circ \tilde{\sigma}_i \circ f$; then condition $\mu = \tilde{\pi}^{-1}\mu\tilde{\sigma}_1\tilde{\sigma}_2$ is equivalent to $\pi = \sigma_1\sigma_2$. It is easy to check that $\tilde{\sigma}_1 \in Q_\lambda$ if and only if $c \circ f$ is constant on each cycle of σ_1 and $\tilde{\sigma}_2 \in P_\lambda$ if and only if $r \circ f$ is constant on each cycle of σ_2. Thus

(9) $$n! \frac{\mathrm{Tr}\, \rho^\lambda(\tilde{\pi})}{\dim V_\lambda} = \sum_{\substack{\sigma_1, \sigma_2 \in S_n, \\ \sigma_1\sigma_2 = \pi}} (-1)^{|\sigma_1|} \tilde{N}^\lambda(\sigma_1, \sigma_2)$$

and the proof in the case when $l = n$ is finished.

For a permutation $\sigma \in S_n$ we denote by $\mathrm{supp}\,\sigma \subseteq \{1, \ldots, n\}$ the support of a permutation (the set of nonfixed points). We claim that a factorization $\pi = \sigma_1\sigma_2$ has a nonzero contribution to (9) only if $\mathrm{supp}\,\sigma_1, \mathrm{supp}\,\sigma_2 \subseteq \mathrm{supp}\,\pi$. Indeed, if $m \in \mathrm{supp}\,\sigma_1 \setminus \mathrm{supp}\,\pi = \mathrm{supp}\,\sigma_2 \setminus \mathrm{supp}\,\pi$, then for any bijection f at least one of the following conditions hold true: $r(f(m)) \neq r(f(\sigma_2(m)))$ (in this case $r \circ f$ is not constant on the cycles of σ_2) or $c(f(m)) \neq c(f(\sigma_2(m)))$ (in this case $c(f(\sigma_1(\sigma_2(m)))) \neq c(f(\sigma_2(m)))$ hence $c \circ f$ is not constant on the cycles of σ_1).

It follows that if $\pi \in S_l$ then we may restrict the sum in (9) to factorizations $\pi = \sigma_1\sigma_2$, where $\sigma_1, \sigma_2 \in S_l$. It remains to notice that

$$\tilde{N}^\lambda_{S_n}(\sigma_1, \sigma_2) = (n - l)!\, \tilde{N}^\lambda_{S_l}(\sigma_1, \sigma_2),$$

where the quantity on the left-hand side regards σ_1, σ_2 as elements of S_n, while on the quantity on the right-hand side regards them as elements of S_l, and that the analogous relation holds between $\Sigma^\lambda(\pi)$ for $\pi \in S_n$ and $\pi \in S_l$.   □

2.2. *Forgetting injectivity.* For a pair of permutations $\sigma_1, \sigma_2 \in S_l$ we define $\hat{N}^\lambda(\sigma_1, \sigma_2)$ as the number of all functions $f : \{1, \ldots, l\} \to \lambda$ (with values in the set of boxes of λ) such that $r \circ f$ is constant on each cycle of σ_2 and $c \circ f$ is constant on each cycle of σ_1.

LEMMA 6. *For any Young diagram λ and a permutation $\pi \in S_l$*

$$(10) \qquad \sum_{\substack{\sigma_1, \sigma_2 \in S_l \\ \sigma_1 \sigma_2 = \pi,}} (-1)^{|\sigma_1|} \tilde{N}^\lambda(\sigma_1, \sigma_2) = \sum_{\substack{\sigma_1, \sigma_2 \in S_l \\ \sigma_1 \sigma_2 = \pi,}} (-1)^{|\sigma_1|} \hat{N}^\lambda(\sigma_1, \sigma_2).$$

Proof. For a given function $f : \{1, \ldots, l\} \to \lambda$ we will show that it has the same contribution to the left-hand side and to the right-hand side. Clearly, it is enough to consider the case when f is not a one-to-one function. It follows that there exists a transposition $\mu \in S_l$ such that f is constant on the orbits of μ. Function f contributes to the right-hand side with multiplicity

$$(11) \qquad \sum_{\substack{\sigma_1, \sigma_2 \in S_l \\ \sigma_1 \sigma_2 = \pi}} (-1)^{|\sigma_1|},$$

where the sum runs over pairs (σ_1, σ_2) such that $\sigma_1 \sigma_2 = \pi$ and $c \circ f$ is constant on each cycle of σ_1 and $r \circ f$ is constant on each cycle of σ_2. Map $(\sigma_1, \sigma_2) \mapsto (\sigma_1', \sigma_2')$ with $\sigma_1' = \sigma_1 \mu$, $\sigma_2' = \mu \sigma_2$ is an involution of the pairs (σ_1, σ_2) which contribute to (11); the only less trivial condition which should be verified is that $c \circ f$ is constant on each cycle of σ_1' but this is equivalent to $c \circ f$ being constant on each cycle of $\sigma_1'^{-1} = \mu \sigma_1^{-1}$.

Since $(-1)^{|\sigma_1|} = (-1) \cdot (-1)^{|\sigma_1'|}$ therefore the contributions of the pairs (σ_1, σ_2) and (σ_1', σ_2') to (11) cancel. In this way we proved that (11) is equal to zero which finishes the proof. $\square$

Proof of Theorem 2. Proposition 5 and Lemma 6 show that

$$\Sigma^\lambda(\pi) = \sum_{\substack{\sigma_1, \sigma_2 \in S_l \\ \sigma_1 \sigma_2 = \pi}} (-1)^{|\sigma_1|} \hat{N}^\lambda(\sigma_1, \sigma_2).$$

Now it is enough to notice that $N^\lambda(\sigma_1, \sigma_2) = \hat{N}^\lambda(\sigma_1, \sigma_2)$; the desired bijection is defined as follows: if $m \in \{1, \ldots, l\}$ fulfills $m \in c_1 \cap c_2$ for $c_i \in C(\sigma_i)$, then we set $f(m) = (h(c_1), h(c_2))$. $\square$

2.3. *Generalization to Young diagrams on $\mathbb{R}_+^2$.* We may identify a Young diagram with a subset of $\mathbb{R}^2$ given by a graphical representation of λ (according to the French notation). For example, for a partition $\lambda = (\lambda_1, \ldots, \lambda_k)$ it is the set

$$(12) \qquad \bigcup_{1 \leq i \leq k} [0, \lambda_i] \times [i - 1, i].$$

In this way we may consider colorings h of the cycles of permutations σ_1 and σ_2 which take real values instead of natural numbers. If we fix some numbering of the cycles in $C = C(\sigma_1) \sqcup C(\sigma_2)$, then any such coloring $h\colon C \to \mathbb{R}_+$ can be identified with an element of $\mathbb{R}_+^{|C|}$.

We define

$$(13) \qquad N^\lambda(\sigma_1, \sigma_2) = \mathrm{vol}\{h \in \mathbb{R}_+^{|C|} : h \text{ compatible with } \lambda\}.$$

Notice that in the case when $\lambda \subset \mathbb{R}^2$ is as prescribed by (12), the set of functions $h \in \mathbb{R}_+^{|C|}$ compatible with λ is a polyhedron hence there is no difficulty in defining its volume; furthermore definitions (7) and (13) give the same value.

The advantage of the definition (13) is that it allows to define characters for any bounded set $\lambda \in \mathbb{R}_+^2$, in particular for *generalized Young diagrams* (see [Ker93]).

It is very natural therefore to ask if Theorem 2 holds true also for skew Young diagrams. Unfortunately, this is not the case as it can be seen for the skew Young diagram $\lambda = (3, 2) \setminus (1)$.

3. Characters of symmetric groups, random matrices and free probability

3.1. *Stanley character formula and random matrices.* Let λ be a Young diagram and $N \geq r(\lambda), c(\lambda)$. We consider a random matrix $T_\lambda = (t_{ij})_{1 \leq i,j \leq N}$ such that

- its entries (t_{ij}) are independent random variables;
- if $(i, j) \in \lambda$ then t_{ij} is a complex centered Gaussian variable, that is to say that

$$\mathbb{E}(t_{ij}) = 0, \quad \mathbb{E}(t_{ij}\overline{t_{ij}}) = 1, \quad \mathbb{E}(t_{ij}^2) = 0;$$

- otherwise, if $(i, j) \notin \lambda$ then $t_{ij} = 0$.

The moments of $T_\lambda T_\lambda^\star$ are given by a formula which is very similar to the Stanley formula for characters (Theorem 2).

THEOREM 7. *With the definitions above and $\pi \in S_l$ with a cycle decomposition $k_1, \ldots, k_r$*

$$(14) \qquad \mathbb{E}\big(\mathrm{Tr}(T_\lambda T_\lambda^\star)^{k_1} \cdots \mathrm{Tr}(T_\lambda T_\lambda^\star)^{k_r} \big) = \sum_{\substack{\sigma_1, \sigma_2 \in S_l \\ \sigma_1 \sigma_2 = \pi}} N^\lambda(\sigma_1, \sigma_2).$$

Proof. The first step is to expand the product and the trace on the left-hand side:

$$\mathbb{E}\big(\operatorname{Tr}(T_\lambda T_\lambda^\star)^{k_1}\ldots\operatorname{Tr}(T_\lambda T_\lambda^\star)^{k_r}\big)$$

$$=\mathbb{E}\Bigg[\Bigg(\sum_{i_1^1,j_1^1,\ldots,i_{k_1}^1,j_{k_1}^1} t_{i_1^1 j_1^1}\overline{t_{i_2^1 j_1^1}}\ldots t_{i_{k_1}^1 j_{k_1}^1}\overline{t_{i_1^1 j_{k_1}^1}}\Bigg)$$

$$\ldots\Bigg(\sum_{i_1^r,j_1^r,\ldots,i_{k_r}^r,j_{k_r}^r} t_{i_1^r j_1^r}\overline{t_{i_2^r j_1^r}}\ldots t_{i_{k_r}^r j_{k_r}^r}\overline{t_{i_1^r j_{k_r}^r}}\Bigg)\Bigg]$$

$$=\sum_{i_1,j_1,\ldots,i_l,j_l}\mathbb{E}\left[\prod_{m=1}^l t_{i_m j_m}\overline{t_{i_{\pi(m)}j_m}}\right].$$

Since random variables (t_{ij}) are Gaussian, we can apply Wick formula (see [Zvo97]) to each summand; in order to do this we need to consider all ways of pairing factors (t_{i_m,j_m}) with factors $(\overline{t_{i_{\pi(m)}j_m}})$. Each such a pairing can be identified with a permutation $\sigma\in S_l$; therefore

$$\mathbb{E}\left[\prod_{m=1}^l t_{i_m j_m}\overline{t_{i_{\pi(m)}j_m}}\right]=\sum_{\sigma\in S_l}\left[\prod_{m=1}^l\mathbb{E}\left(t_{i_{\sigma(m)}j_{\sigma(m)}}\overline{t_{i_{\pi(m)}j_m}}\right)\right].$$

Using the definition of T, this is equal to

$$\sum_{\sigma\in S_l}\prod_{m=1}^l [i_{\sigma(m)}=i_{\pi(m)}]\,[j_{\sigma(m)}=j_m]\,[(i_{\sigma(m)},j_{\sigma(m)})\in\lambda].$$

If we plug this in our calculation, the left-hand side of (14) is equal to

$$\sum_{\sigma\in S_l}\sum_{i_1,j_1,\ldots,i_l,j_l}\left(\prod_{m=1}^l [i_{\sigma(m)}=i_{\pi(m)}]\,[j_{\sigma(m)}=j_m]\,[(i_{\sigma(m)},j_{\sigma(m)})\in\lambda]\right).$$

If we denote $\sigma_1=\pi\sigma_2^{-1}$, $\sigma_2=\sigma$, then the sum over σ can be seen as a sum over all $\sigma_1,\sigma_2\in S_l$ such that $\sigma_1\sigma_2=\pi$. If a sequence $i_1,\ldots,i_l$ (respectively, sequence $j_1,\ldots,j_l$) contributes to the above sum then it must be constant on each cycle of σ_1 (respecively, each cycle of σ_2). It follows that there is a bijective correspondence between sequences $i_1,j_1,\ldots,i_l,j_l$ which contribute to the above sum and colorings of the cycles of σ_1 and σ_2 which are compatible with λ. $\qquad\square$

By comparing the above result with Theorem 2, we obtain the following corollary.

COROLLARY 8. *Let λ be a Young diagram. Then for any permutation $\pi\in S_l$ with a cycle decomposition $k_1,\ldots,k_r$*

$$|\Sigma^\lambda(\pi)|\le\mathbb{E}\big(\operatorname{Tr}(T_\lambda T_\lambda^\star)^{k_1}\cdots\operatorname{Tr}(T_\lambda T_\lambda^\star)^{k_r}\big).$$

We shall not follow this idea in this article and we will prove all estimates from scratch, but it is worth noticing that the above corollary shows that the asymptotics of characters of symmetric groups can be deduced from the corresponding asymptotics of random matrices. In particular it follows that when the lengths $k_1, \ldots, k_r$ of the cycles of π are big enough then the asymptotics of the corresponding character is related to the limit distribution of the largest eigenvalue of $T_\lambda T_\lambda^*$ [SS98]. Notice, however, that due to the minus sign in Theorem 2 and the resulting cancelations the character $|\Sigma^\lambda(\pi)|$ could, at least in priciple, be much smaller than the appropriate moment of the random matrix $T_\lambda T_\lambda^*$.

3.2. *Free cumulants of the transition measure.* For a (continuous) Young diagram λ we denote by μ^λ its transition measure (which is a probability measure on the real line) [Ker93], [Bia98] and by $R_m^\lambda := R_m(\mu^\lambda)$ we denote the m-th free cumulant of μ^λ. The importance of free cumulants R_m^λ in the study of the asymptotics of symmetric groups was pointed out by Biane [Bia98]. The following theorem gives a new formula for the free cumulants R_m^λ. It has a big advantage that it does not involve the notion of the transition measure and it is related directly with the shape of a Young diagram.

THEOREM 9. *For any Young diagram* λ

(15) $$R_{l+1}^\lambda = \sum_{\substack{\sigma_1, \sigma_2 \in S_l, \\ \sigma_1 \sigma_2 = (1,2,\ldots,l), \\ |\sigma_1| + |\sigma_2| = |(1,2,\ldots,l)|}} (-1)^{|\sigma_1|} \, N^\lambda(\sigma_1, \sigma_2),$$

where the sum runs over minimal factorizations of a cycle of length l.

Proof. For a Young diagram λ and $c > 0$ we denote by $c\lambda$ the (generalized) Young diagram obtained from λ by similarity with scale c. A function f on the set of (generalized) Young diagrams is said to be homogeneous of degree m if

$$f(c\lambda) = c^m f(\lambda)$$

holds true for all choices of λ and c. Each free cumulant R_m is homogenous of degree m.

The value of the normalized character $\Sigma^\lambda(1, 2, \ldots, l)$ on a cycle can be expressed as a polynomial (known as Kerov polynomial) in free cumulants $(R_m^\lambda : m \in \{2, 3, \ldots\})$:

$$\Sigma^\lambda(1, 2, \ldots, l) = R_{l+1}^\lambda + \text{(terms of lower degree)};$$

therefore R_{l+1}^λ is the homogeneous part of $\Sigma^\lambda(1, 2, \ldots, l)$ with degree $l + 1$. We apply (6) for $\pi = (1, 2, \ldots, l)$; it is easy to see that each summand on the right-hand side is homogeneous of degree $|C(\sigma_1)| + |C(\sigma_2)|$ which finishes the proof. □

3.3. *Generalized circular operators.* Let $\mathcal{D}$ be the algebra of continuous functions on $\mathbb{R}_+$. We equip it with an expected value $\phi : \mathcal{D} \to \mathbb{C}$ given by $\phi(f) = \int_0^\infty f(t)dt$.

We consider an operator-valued probability space, which by definition is some $*$-algebra $\mathcal{A}$ such that $\mathcal{D} \subseteq \mathcal{A}$ and equipped with a conditional expectation $\mathbb{E} : \mathcal{A} \to \mathcal{D}$. For a given (generalized) Young diagram λ let $T \in \mathcal{A}$ be a generalized circular operator [VDN92], [Spe98] with a covariance

$$(16) \qquad \left[k(T, fT^*) \right](s) = \int_{t:(t,s)\in\lambda} f(t) \ dt,$$

$$\left[k(T^*, fT) \right](s) = - \int_{t:(s,t)\in\lambda} f(t) \ dt,$$

$$\left[k(T, fT) \right](s) = 0,$$

$$\left[k(T^*, fT^*) \right](s) = 0.$$

THEOREM 10. *For any (generalized) Young diagram λ*

$$R_{l+1}^\lambda = \phi\left[(T^*T)^l \right].$$

Proof. It is easy to check that for permutations σ_1, σ_2 which contribute to (15) the corresponding bipartite graph is a tree therefore the calculation of $N^\lambda(\sigma_1, \sigma_2)$ is particularly simple, namely it is a certain iterated integral. The same iterated integral appears in the nested evaluation of amalgamated free cumulants therefore $N^\lambda(\sigma_1, \sigma_2) = \pm\phi\left[k_{\sigma_2}(T^*, T, \cdots, T^*, T) \right]$. The plus/minus sign is due to the minus sign in the covariance (16). It is easy to check that in fact

$$(-1)^{|\sigma_1|} N^\lambda(\sigma_1, \sigma_2) = \phi\left[k_{\sigma_2}(T^*, T, \cdots, T^*, T) \right].$$

The moment-cumulant formula

$$\mathbb{E}\left[(T^*T)^l \right] = \sum_{\sigma \in \mathrm{NC}_2} k_\sigma(T^*, T, \cdots, T^*, T)$$

finishes the proof since there is a bijective correspondence between noncrossing pair partitions and the minimal factorizations. $\qquad \square$

Due to the analytic machinery of free probability the calculation of the moments of T in a closed form is possible in many cases therefore Theorem 10 gives a practical method of calculating the free cumulants of the Young diagrams.

4. Technical estimates

4.1. *Estimates for the number of colorings $N^\lambda(\sigma_1, \sigma_2)$.* As we already mentioned in Example 3 to permutations σ_1, σ_2, we can associate a bipartite graph $C(\sigma_1) \sqcup C(\sigma_2)$ with an edge between $c_1 \in C(\sigma_1)$ and $c_2 \in C(\sigma_2)$ if $c_1 \cap c_2 \neq \emptyset$.

For a bipartite graph $G = C_1 \sqcup C_2$ (not necessarily arising from the above construction) and a Young diagram λ we define $N^\lambda(G)$ as the number of colorings h of the vertices of $C_1 \sqcup C_2$ which are compatible with the Young diagram λ. (The definition of compatibility in this context is a natural extension of the old one; i.e., we require that if $c_1 \in C_1$ and $c_2 \in C_2$ are connected by an edge then $\big(h(c_1), h(c_2)\big) \in \lambda$.)

We denote by $G_{p,q}$ a full bipartite graph for which $|C_1| = p$ and $|C_2| = q$.

LEMMA 11. *Let G be a finite bipartite graph such that the degree of any vertex is nonzero. It is possible (not necessarily in a unique way) to remove some of the edges of G in such a way that the resulting graph $\widetilde{G}$ is a disjoint union of the graphs of the form $G_{1,1}$, $G_{k,1}$, $G_{1,k}$.*

Assume that a Young diagram λ consists of n boxes. Then, for any $A \geq r(\lambda), c(\lambda)$

$$(17) \quad N^\lambda(G) \leq A^{(\text{number of vertices of } G)} \left(\frac{n}{A^2} \right)^{(\text{number of connected components of } \widetilde{G})}.$$

Proof. If the graph G contains an edge which connects two vertices of degree bigger than one we remove it and iterate this procedure; if no such edge exists then the resulting graph $\widetilde{G}$ has the desired property.

Clearly, $N^\lambda(G) \leq N^\lambda(\widetilde{G})$ therefore it is enough to find a suitable upper bound for $N^\lambda(\widetilde{G})$. Since both sides of (17) are multiplicative with respect to the disjoint sum of graphs it is enough to prove (17) for $\widetilde{G} \in \{G_{1,1}, G_{k,1}, G_{1,k}\}$. Now, the lemma follows from:

$$N^\lambda(G_{k,1}) = \sum_i \lambda_i^k \leq \sum_i \lambda_i A^{k-1} = nA^{k-1}$$

and the analogous inequality for $N^\lambda(G_{1,k})$. $\square$

PROPOSITION 12. *Suppose that $r(\lambda), c(\lambda) \leq A \leq n$, $\sigma_1, \sigma_2 \in S_l$ and $\pi = \sigma_1 \sigma_2$. Then*

$$(18) \qquad N^\lambda(\sigma_1, \sigma_2) \leq A^{|C(\sigma_1)| + |C(\sigma_2)|} \left(\frac{n}{A^2} \right)^{\text{orbits}(\sigma_1, \sigma_2)}$$

$$\leq A^{l - |C(\pi)|} n^{|C(\pi)|} \left(\frac{1}{A} \right)^{o(\sigma_1, \sigma_2)},$$

where $\text{orbits}(\sigma_1, \sigma_2)$ denotes the number of orbits in the action of $\langle \sigma_1, \sigma_2 \rangle$ on the set $\{1, \ldots, n\}$ and

$$o(\sigma_1, \sigma_2) = l - |C(\sigma_1)| - |C(\sigma_2)| + \text{orbits}(\sigma_1, \sigma_2).$$

Proof. The first inequality is a simple corollary from Lemma 11 since $A^2 \geq r(\lambda)c(\lambda) \geq n$ and the number of connected components of $\widetilde{G}$ is bounded

from below by orbits(σ_1, σ_2). The second inequality follows by multiplying by

$$\left(\frac{n}{A}\right)^{|C(\pi)|-\text{orbits}(\sigma_1,\sigma_2)} \geq 1. \qquad \square$$

4.2. *Estimates for the number of factorizations.* Now, we have to find a bound of the number of factorizations of π with a given value of the statistic $o(\sigma_1, \sigma_2)$.

LEMMA 13. *Let* $\pi, \sigma_1, \sigma_2 \in S_l$ *be such that* $\pi = \sigma_1\sigma_2$. *There exist permutations* $\sigma_1', \sigma_2' \in S_l$ *such that* $\pi = \sigma_1'\sigma_2'$, $|\sigma_1'| + |\sigma_2'| = |\sigma_1| + |\sigma_2|$, $|\sigma_2'| = |\sigma_2'\sigma_2^{-1}| + |\sigma_2|$ *and every cycle of* σ_1' *is contained in some cycle of* σ_2'. *Furthermore,* $|\sigma_1'| = o(\sigma_1, \sigma_2)$.

Proof. If every cycle of σ_1 is contained in some cycle of σ_2 then $\sigma_1' = \sigma_1$ and $\sigma_2' = \sigma_2$ have the required property.

Otherwise, there exist $a, b \in \{1, \ldots, n\}$ such that a and b belong to the same cycle of σ_1 but not to the same cycle of σ_2. We define $\sigma_1' = \sigma_1(a, b)$, $\sigma_2' = (a, b)\sigma_2$. Notice that $|\sigma_1'| = |\sigma_1| - 1$, $|\sigma_2'| = |\sigma_2| + 1$, and the orbits under the action of the subgroups $\langle \sigma_1, \sigma_2 \rangle$ and $\langle \sigma_1', \sigma_2' \rangle$ are the same.

We iterate this procedure if necessary (it will finish after a finite number of steps because the length of σ_1 decreases in each step). It remains to prove that $|\sigma_2'| \geq |\sigma_2'\sigma_2^{-1}| + |\sigma_2|$ (the opposite inequality follows from the triangle inequality): notice that $|\sigma_2'| - |\sigma_2|$ is equal to k (where k is the number of steps after which the procedure has terminated) and $\sigma_2'\sigma_2^{-1}$ is a product of k transpositions, hence $|\sigma_2'\sigma_2^{-1}| \leq k$.

Furthermore, as every cycle of σ_1' is contained in some cycle of σ_2', the set of orbits of the group $\langle \sigma_1', \sigma_2' \rangle$ is equal to $C(\sigma_2')$, so

$$o(\sigma_1, \sigma_2) = o(\sigma_1', \sigma_2') = l - |C(\sigma_1')| = |\sigma_1'|. \qquad \square$$

LEMMA 14. *For any integers* $l \geq 1$ *and* $i \geq 0$ *and for any* $\pi \in S_l$

$$(19) \qquad \#\{\sigma \in S_l : |\sigma| = i\} \leq \frac{l^{2i}}{i!}.$$

Proof. Since every permutation in S_l appears exactly once in the product

$$[1 + (12)][1 + (13) + (23)] \cdots [1 + (1l) + \cdots + (l-1, l)],$$

we have

$$\sum_i x^i \#\{\sigma \in S_l : |\sigma| = i\} = (1 + x)(1 + 2x) \cdots \left(1 + (l-1)x\right).$$

Each of the coefficients of x^k on the right-hand side is bounded from above by the corresponding coefficient of $e^x e^{2x} \cdots e^{(l-1)x} = e^{\frac{l(l-1)x}{2}}$, finishing the proof.
$$\square$$

LEMMA 15. *There exists a constant C_0 with a property that for any k the number of minimal factorisations $\sigma_1 \sigma_2 = \pi$, $|\sigma_1| + |\sigma_2| = |\pi|$ of a cycle $\pi = (1, \ldots, k)$ and such that the associated graph $\widetilde{G}$ consists of $s \geq 2$ components is bounded from above by $\frac{(C_0 k)^{2s-2}}{(2s-2)!}$.*

Proof. Since the factorization is minimal therefore the graph G associated to σ_1, σ_2 is a tree. We will give to G a structure of planted planar tree: the root is the cycle of σ_1 containing 1 and his left-most edge links it to the cycle of σ_2 containing 1.

In each connected component of $\widetilde{G}$ there is at most one vertex of degree higher than one and we shall decorate this vertex. If in some connected component of $\widetilde{G}$ there are no such vertices we decorate any of them which is not a leaf in G. In this way the decorated vertices can be identified with connected components of $\widetilde{G}$.

We consider the graph G' obtained from G by removing the leaves (except the root) and the graph G'' which consists of the decorated vertices of $\widetilde{G}$; we connect two vertices $A, B \in G''$ by an edge if vertices A, B are connected in G (or, equivalently, G') by a *direct path*, i.e. a path which does not pass through any connected component of $\widetilde{G}$ other than the ones specified by A and B. It is easy to see that G'' inherits the structure of a plane rooted tree from G (we define the root of G'' to be the connected component of $\widetilde{G}$ of the root of G) and it has s vertices. It follows that the number of such trees G'' is bounded from above by the Catalan number $\frac{1}{s+1}\binom{2s}{s} < 4^s$.

In order to reconstruct the tree G' from G'' we have to specify for each edge of G'' if it comes from a single edge of G' or from a pair or a triple of consecutive edges of G'; it follows that we have (at most) 3^{s-1} choices. It might happen also for two adjacent (with respect to the planar structure) edges e_1, e_2 of G'' that each of these edges e_i corresponds to a pair or a triple of consecutive edges $f_i = (f_{i1}, f_{i2}[, f_{i3}])$ of G' and these tuples f_1 and f_2 have one edge in common. There are at most $2s - 3$ such pairs of adjacent edges which accounts for at most 2^{2s-3} choices. If the root of G is not a decorated vertex, it might happen that it is a leaf or that it belongs to the left-most and/or to the right-most edge of the root of G': there are four choices for that.

In order to reconstruct tree G from G' we have to specify whether the root of G is a decorated vertex or not. Furthermore we have to specify places in which we will add missing l leaves to the tree G' (note that $l \leq k + 1 - s$); it is easy to see that this is equivalent to specifying a partition $l = a_1 + \cdots + a_{2s-1}$, where $a_1, \ldots, a_{2s-1} \geq 0$ are integers. It follows that the number of choices is bounded from above by

$$2\binom{l + 2s - 2}{2s - 2} \leq 2\binom{k + s - 1}{2s - 2} \leq 2\frac{\left(k + \frac{1}{2}\right)^{2s-2}}{(2s - 2)!}.$$

A minimal factorisation is determined by its bicolored map with a marked edge, for example the one linking the two cycles containing 1 [GJ92]. With our construction, the coloring is determined by the root which always belongs to $C(\sigma_1)$ and the marked edge is its left-most edge.

It follows that the total number of choices is bounded from above by

$$2 \cdot 3^{s-1} \cdot 2^{2s-1} \cdot 4^s \frac{\left(k + \frac{1}{2}\right)^{2s-2}}{(2s-2)!}. \qquad \square$$

5. Asymptotics of characters

5.1. *Upper bound for characters*: *Proof of Theorem* 1.

Proof of Theorem 1. Let $k_1, \ldots, k_r \geq 2$ be the lengths of the nontrivial cycles in the cycle decomposition of $\pi \in S_n$. It follows that $l := k_1 + \cdots + k_r = |\operatorname{supp} \pi|$ and in the following we will regard π as an element of S_l. We denote $A = \max(l, r(\lambda), c(\lambda))$.

We consider a map which to a pair (σ_1, σ_2) associates any pair (σ_1', σ_2') as prescribed by Lemma 13. For any fixed σ_2' the permutations σ_2 such that $|\sigma_2'| = |\sigma_2' \sigma_2^{-1}| + |\sigma_2|$ can be identified with noncrossing partitions of the cycles of σ_2' (see [Bia97, §1.3]). It follows that the number of such permutations σ_2 is equal to the product of appropriate Catalan numbers and, hence, this product is bounded from above by 4^l. Therefore Theorem 2 and Proposition 12 show that

$$|\Sigma^\lambda(\pi)| \leq \sum_{\substack{\sigma_1,\sigma_2 \in S_l, \\ \sigma_1\sigma_2=\pi}} A^{l-r} n^r \left(\frac{1}{A}\right)^{o(\sigma_1,\sigma_2)} \leq 4^l \sum_{\substack{\sigma_1',\sigma_2' \in S_l, \\ \sigma_1'\sigma_2'=\pi}} A^{l-r} n^r \left(\frac{1}{A}\right)^{|\sigma_1'|}$$

$$\leq 4^l A^{l-r} n^r \sum_{\sigma_1' \in S_l} \left(\frac{1}{A}\right)^{|\sigma_1'|} \leq 4^l A^{l-r} n^r \sum_{i \geq 0} \frac{l^{2i}}{A^i i!},$$

where the last inequality follows from Lemma 14. It follows that

$$(20) \qquad |\Sigma^\lambda(\pi)| \leq 4^l A^{l-r} n^r e^{\frac{l^2}{A}} \leq (4e)^l A^{l-r} n^r.$$

After dividing by $(n)_l \geq \left(\frac{n}{e}\right)^l$, we obtain $|\chi^\lambda(\pi)| \leq (4e^2)^l (A/n)^{l-r}$, which finishes the proof because $l = |\operatorname{supp} \pi| \leq 2|\pi|$ and $l - r = |\pi|$. $\qquad \square$

5.2. *Error term for balanced Young diagrams.* Biane [Bia98] proved that if the permutation π is fixed, its nontrivial cycle lengths are equal to $k_1, \ldots, k_r \geq 2$, and the Young diagram λ is balanced (i.e. $r(\lambda), c(\lambda) = O(\sqrt{n})$), then

$$\chi^\lambda(\pi) = \frac{1}{(n)_{|\operatorname{supp}(\pi)|}} \prod_{i=1}^r R_{k_i+1}(\lambda) + O\left(n^{-\frac{|\pi|+2}{2}}\right).$$

The following theorem gives a uniform estimate for the error term in Biane's formula. (Note that if π is fixed and the Young diagram balanced, with $\varepsilon = \frac{C}{\sqrt{n}}$ and $A = D\sqrt{n}$ we recover the result of Biane.)

THEOREM 16. *There exists a constant a such that, for any $0 < \varepsilon < 1$, any Young diagram λ of size n and any permutation $\pi \in S_n$ such that $|\operatorname{supp}(\pi)|^2 \leq \varepsilon A$ and $r(\lambda), c(\lambda) \leq A \leq n$ we have*

$$\left| \chi^\lambda(\pi) - \frac{1}{(n)_{|\operatorname{supp}(\pi)|}} \prod_{i=1}^r R_{k_i+1}(\lambda) \right| \leq \left(\varepsilon^2 + \frac{A}{n}\varepsilon \right) \left(\frac{aA}{n} \right)^{|\pi|},$$

where the k_i are the lengths of the nontrivial cycles of π.

Proof. We can assume that $\pi \in S_l$ has no fixpoints. Using Theorems 2 and 9 together with the fact that any minimal factorisation of π is a product of minimal factorisations of its cycles, we can write

$$\Sigma^\lambda(\pi) - \prod_{i=1}^r R_{k_i+1}^\lambda = \sum_{\substack{\sigma_1, \sigma_2 \in S_l \\ \sigma_1\sigma_2 = \pi \\ |\sigma_1| + |\sigma_2| > |\pi|}} (-1)^{|\sigma_1|} N^\lambda(\sigma_1, \sigma_2).$$

To such a pair (σ_1, σ_2) of permutations we can associate one of the pairs of permutations (σ_1', σ_2') given by Lemma 13 with $|\sigma_1'| \geq 1$.

Consider separately the case $|\sigma_1'| = 1$. Then $\operatorname{orbits}(\sigma_1, \sigma_2) = \operatorname{orbits}(\sigma_1', \sigma_2') \geq |C(\pi)| - 1$ and the first inequality in (18) shows that

$$N^\lambda(\sigma_1, \sigma_2) \leq A^{|C(\sigma_1)| + |C(\sigma_2)|} \left(\frac{n}{A^2} \right)^{|C(\pi)|-1}.$$

Therefore the estimate given by Proposition 12 can be improved to the following one:

$$N^\lambda(\sigma_1, \sigma_2) \leq A^{l-|C(\pi)|} n^{|C(\pi)|} \frac{1}{n}.$$

Clearly $A \geq l$ therefore by the same argument as in the proof of Theorem 1, we obtain the inequality

$$\left| \Sigma^\lambda(\pi) - \prod_{i=1}^r R_{k_i+1}(\lambda) \right| \leq 4^l A^{l-r} n^r \left(\frac{l^2}{n} + \sum_{i\geq 2} \frac{l^{2i}}{A^i i!} \right).$$

The proof is now finished thanks to the remarks of the previous subsection and the inequality $\exp(z) - 1 - z \leq z^2$ for $0 < z < 1$. $\qquad \square$

5.3. *Characters of symmetric groups related to Thoma characters.* Vershik and Kerov [VK81] proved that if π is a fixed permutation with the lengths of nontrivial cycles $k_1, \ldots, k_r$ then for any Young diagram λ with n boxes

$$\chi^\lambda(\pi) = \prod_{i=1}^r \left[\sum_j \alpha_j^{k_i} - \sum_j (-\beta_j)^{k_i} \right] + O\left(\frac{1}{n} \right),$$

where $\alpha_j = \frac{\lambda_j}{n}$, $\beta_j = \frac{\lambda'_j}{n}$. We prefer to write this formula in the following equivalent form:

$$(21) \qquad \chi^\lambda(\pi) = \frac{n^l}{(n)_l} \prod_{i=1}^{r} \left[\sum_j \alpha_j^{k_i} - \sum_j (-\beta_j)^{k_i} \right] + O\left(\frac{1}{n}\right).$$

In this section we will prove Theorem 17 which together with Theorem 16 gives a uniform estimate for the error term in the formula (21). In particular, for $A = n$ and $\varepsilon = \frac{D}{n}$ we recover the result of Vershik and Kerov.

THEOREM 17. *There exist constants $a, C > 0$ with the following property. Let $k_1, \ldots, k_r$ be positive integers; we denote $k_1 + \ldots + k_r = l$. If λ is a Young diagram having n boxes with less than A rows and columns and such that $\varepsilon = \frac{(k_1^2 + \cdots + k_r^2)n}{A^2} < C$, then*

$$(22) \qquad \left| \frac{\prod_{i=1}^{r} R_{k_i+1}(\lambda)}{n^l} - \prod_{i=1}^{r} \left[\sum_j \alpha_j^{k_i} - \sum_j (-\beta_j)^{k_i} \right] \right| \leq \varepsilon \left(\frac{A}{n}\right)^{l-r} a^r,$$

where $\alpha_j = \frac{\lambda_j}{n}$, $\beta_i = \frac{\lambda'_j}{n}$.

Proof. Firstly, let us consider the case $r = 1$. Note that

$$N^\lambda\big(e, (1, \ldots, k)\big) = \sum_j (n\alpha_j)^k, \quad N^\lambda\big((1, \ldots, k), e\big) = \sum_j (n\beta_j)^k.$$

Therefore Theorem 9 implies that the left-hand side of (22) is equal to

$$\left| \frac{1}{n^k} \sum_{\substack{\sigma_1, \sigma_2 \in S_k \setminus \{e\} \\ \sigma_1 \sigma_2 = \pi \\ |\sigma_1| + |\sigma_2| = |\pi|}} (-1)^{|\sigma_1|} N^\lambda(\sigma_1, \sigma_2) \right|.$$

For a pair of permutations σ_1, σ_2 which contributes to the above sum we consider the bipartite graph G and the graph $\widetilde{G}$ given by Lemma 11. Clearly, in this case graph $\widetilde{G}$ has more than one component. With Lemmas 15 and 11

$$(23) \qquad \left(\frac{n}{A}\right)^{k-1} \left| \frac{R_{k+1}^\lambda}{n^k} - \sum_j \left(\alpha_j^k - (-\beta_j)^k \right) \right| \leq \sum_{s \geq 2} \frac{(C_0 k)^{2s-2}}{(2s-2)!} \left(\frac{n}{A^2}\right)^{s-1}$$

$$\leq \frac{2C_0^2 k^2 n}{A^2} = 2C_0^2 \varepsilon,$$

where the last inequality holds true if $\frac{2C_0^2 k^2 n}{A^2} = 2C_0^2 \varepsilon$ is smaller than some positive constant and the proof is finished in the case $r = 1$.

For the general case, we put $\epsilon_i = \frac{k_i^2 n}{A^2}$. We denote

$$X_i = \frac{1}{a}\left(\frac{n}{A}\right)^{k_i-1}\frac{R_{k_i+1}^\lambda}{n^{k_i}},$$

$$Y_i = \frac{1}{a}\left(\frac{n}{A}\right)^{k_i-1}\sum_j\left(\alpha_j^{k_i}-(-\beta_j)^{k_i}\right).$$

Let us fix $a > 2$. Clearly, $|Y_i| < \frac{2}{a} < 1$ hence (23) shows that $|X_i| < 1$ if ε is smaller than some positive constant. Telescopic summation

$$X_1\cdots X_r - Y_1\cdots Y_r = X_1\cdots X_{r-1}(X_r-Y_r)+$$
$$X_1\cdots X_{r-2}(X_{r-1}-Y_{r-1})Y_r+\cdots+(X_1-Y_1)Y_2\cdots Y_r$$

shows that

$$\frac{1}{a^r}\left(\frac{n}{A}\right)^{l-r}\left|\frac{\prod_{i=1}^r R_{k_i+1}^\lambda}{n^l}-\prod_{i=1}^r\left[\sum_j\alpha_j^{k_i}-\sum_j(-\beta_j)^{k_i}\right]\right|$$
$$\leq\frac{2C_0(\epsilon_1+\cdots+\epsilon_r)}{a}.\quad\square$$

5.4. *Concluding remarks.* In the case where π is a fixed permutation, we only need Lemma 11 and we can avoid most of the technicalities. Therefore, our method gives a unified, simple way to reprove three important results on asymptotics of character values on fixed permutations as well as new results: the intermediate case between balanced diagrams ($A = \Theta(\sqrt{n})$) and diagrams with long rows and/or columns ($A = \Theta(n)$) has, to our knowledge, not been studied until now. Moreover, it is interesting to note that our method can be extended to quite long permutations.

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