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Grothendieck rings of basic classical Lie superalgebras

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Abstract

The Grothendieck rings of finite dimensional representations of the basic classical Lie superalgebras are explicitly described in terms of the corresponding generalized root systems. We show that they can be interpreted as the subrings in the weight group rings invariant under the action of certain groupoids called super Weyl groupoids.

Contents

1. **Introduction**

The classification of finite-dimensional representations of the semisimple complex Lie algebras and related Lie groups is one of the most beautiful pieces of mathematics. In his essay [1] Michael Atiyah mentioned representation theory of Lie groups as an example of a theory, which "can be admired because of its importance, the breadth of its applications, as well as its rational coherence and naturality." This classical theory goes back to fundamental work by Èlie Cartan and Hermann Weyl and is very well presented in various books, of which we would like to mention the famous Serre's lectures [27] and a nicely written Fulton-Harris course [9]. One of its main results can be formulated as follows (see e.g. [9, Th. 23.24]):

The representation ring $R(\mathfrak{g})$ *of a complex semisimple Lie algebra* $\mathfrak{g}$ *is isomorphic to the ring* $\mathbb{Z}[P]^W$ *of* W*-invariants in the integral group ring* $\mathbb{Z}[P]$, *where* P *is the corresponding weight lattice and* W *is the Weyl group. The isomorphism is given by the character map* $\mathrm{Ch} : R(\mathfrak{g}) \to \mathbb{Z}[P]^W$.

The main purpose of the present work is to generalize this result to the case of basic classical complex Lie superalgebras. The class of basic classical Lie superalgebras was introduced by Victor Kac in [12], [13], where the basics of the representation theory of these Lie superalgebras had been also developed. The problem of finding the characters of the finite-dimensional irreducible representations turned out to be very difficult and was not completely resolved (see the important papers by Serganova [18], [20] and Brundan [6] and references therein). Our results may shed some light on these issues.

Recall that a complex simple Lie superalgebra $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ is called the *basic classical* if it admits a nondegenerate invariant (even) bilinear form and the representation of the Lie algebra $\mathfrak{g}_0$ on the odd part $\mathfrak{g}_1$ is completely reducible. The class of these Lie superalgebras can be considered as a natural analogue of the ordinary simple Lie algebras. In particular, they can be described (with the exception of $A(1,1) = \mathfrak{psl}(2,2)$) in terms of Cartan matrix and generalized root systems (see [13], [19]).

Let $\mathfrak{g}$ be such Lie superalgebra different from $A(1,1)$ and $\mathfrak{h}$ be its Cartan subalgebra (which in this case is also Cartan subalgebra of the Lie algebra $\mathfrak{g}_0$). Let $P_0 \subset \mathfrak{h}^*$ be the abelian group of weights of $\mathfrak{g}_0$, W_0 be the Weyl group of $\mathfrak{g}_0$ and $\mathbb{Z}[P_0]^{W_0}$ be the ring of W_0-invariants in the integral group ring $\mathbb{Z}[P_0]$. The decomposition of $\mathfrak{g}$ with respect to the adjoint action of $\mathfrak{h}$ gives the (generalized) root system R of Lie superalgebra $\mathfrak{g}$. By definition $\mathfrak{g}$ has a natural nondegenerate bilinear form on $\mathfrak{h}$ and hence on $\mathfrak{h}^*$, which will be denoted as $(\,,)$. In contrast to the theory of semisimple Lie algebras some of the roots $\alpha \in R$ are isotropic: $(\alpha, \alpha) = 0$. For isotropic roots one cannot define the usual reflection, which explains the difficulty with the notion of Weyl group for Lie superalgebras. A geometric description of the corresponding generalized root systems were found in this case by Serganova [19].

Define the following *ring of exponential super-invariants* $J(\mathfrak{g})$, replacing the algebra of Weyl group invariants in the classical case of Lie algebras:

$$(1) \qquad J(\mathfrak{g}) = \{f \in \mathbb{Z}[P_0]^{W_0} : D_\alpha f \in (e^\alpha - 1) \quad \text{for any isotropic root } \alpha\},$$

where $(e^\alpha - 1)$ denotes the principal ideal in $\mathbb{Z}[P_0]^{W_0}$ generated by $e^\alpha - 1$ and the derivative D_α is defined by the property $D_\alpha(e^\beta) = (\alpha, \beta)e^\beta$. This ring is a variation of the algebra of invariant polynomials investigated for Lie superalgebras in [2], [14], [23], and [24]. For the special case of the Lie superalgebra $A(1,1)$ one should slightly modify the definition because of the multiplicity 2 of the isotropic roots (see §8 below).

Our main result is the following

THEOREM. *The Grothendieck ring $K(\mathfrak{g})$ of finite-dimensional representations of a basic classical Lie superalgebra $\mathfrak{g}$ is isomorphic to the ring $J(\mathfrak{g})$. The isomorphism is given by the supercharacter map* $\mathrm{Sch} : K(\mathfrak{g}) \to J(\mathfrak{g})$.

The fact that the supercharacters belong to the ring $J(\mathfrak{g})$ is relatively simple, but the proof of surjectivity of the supercharacter map is much more involved and based on classical Kac's results [12], [13].

The elements of $J(\mathfrak{g})$ can be described as the invariants in the weight group rings under the action of the following groupoid $\mathfrak{W}$, which we call *super Weyl groupoid*. It is defined as a disjoint union

$$\mathfrak{W}(R) = W_0 \coprod W_0 \ltimes \mathfrak{T}_{\mathrm{iso}},$$

where $\mathfrak{T}_{\mathrm{iso}}$ is the groupoid, whose base is the set R_{iso} of all isotropic roots of $\mathfrak{g}$ and the set of morphisms from $\alpha \to \beta$ with $\beta \neq \alpha$ is nonempty if and only if $\beta = -\alpha$ in which case it consists of just one element τ_α. This notion was motivated by our work on deformed Calogero-Moser systems [25].

The group W_0 is acting on $\mathfrak{T}_{\mathrm{iso}}$ in a natural way and thus defines a semi-direct product groupoid $W_0 \ltimes \mathfrak{T}_{\mathrm{iso}}$ (see details in §9). One can define a natural action of $\mathfrak{W}$ on $\mathfrak{h}$ with τ_α acting as a shift by α in the hyperplane $(\alpha, x) = 0$. If we exclude the special case of $A(1,1)$ our theorem can now be reformulated as the following version of the classical case:

The Grothendieck ring $K(\mathfrak{g})$ of finite-dimensional representations of a basic classical Lie superalgebra $\mathfrak{g}$ is isomorphic to the ring $\mathbb{Z}[P_0]^{\mathfrak{W}}$ of the invariants of the super Weyl groupoid $\mathfrak{W}$.

An explicit description of the corresponding rings $J(\mathfrak{g})$ (and thus the Grothendieck rings) for each type of basic classical Lie superalgebra is given in Sections 7 and 8. For classical series we describe also the subrings, which are the Grothendieck rings of the corresponding natural algebraic supergroups.

2. **Grothendieck rings of Lie superalgebras**

All the algebras and modules in this paper will be considered over the field of complex numbers $\mathbb{C}$.

Recall that a *superalgebra* (or a $\mathbb{Z}_2$*-graded algebra*) is an associative algebra A with a decomposition into direct sum $A = A_0 \oplus A_1$, such that if $a \in A_i$ and $b \in A_j$ then $ab \in A_{i+j}$ for all $i, j \in \mathbb{Z}_2$. We will write $p(a) = i \in \mathbb{Z}_2$ if $a \in A_i$.

A *module over superalgebra* A is a vector space V with a decomposition $V = V_0 \oplus V_1$, such that if $a \in A_i$ and $v \in V_j$ then $av \in V_{i+j}$ for all $i, j \in \mathbb{Z}_2$. A morphism of A-modules $f\colon V \to U$ is module homomorphism preserving their gradings: $f(V_i) \subset U_i$, $i \in \mathbb{Z}_2$.

We have the parity change functor $V \longrightarrow \Pi(V)$, where $\Pi(V)_0 = V_1, \Pi(V)_1 = V_0$, with the A action $a * v = (-1)^{p(a)}av$. If A, B are superalgebras then $A \otimes B$ is a superalgebra with the multiplication

$$(a_1 \otimes b_1)(a_2 \otimes b_2) = (-1)^{p(b_1)p(a_2)} a_1 a_2 \otimes b_1 b_2.$$

The tensor product of A-module V and B-module U is $A \otimes B$-module $V \otimes U$ and

$$(V \otimes U)_0 = (V_0 \otimes U_0) \oplus (V_1 \otimes U_1), \quad (V \otimes U)_1 = (V_1 \otimes U_0) \oplus (V_0 \otimes U_1)$$

with the action $a \otimes b(v \otimes u) = (-1)^{bv} av \otimes bu$.

The *Grothendieck group* $K(A)$ is defined (cf. Serre [28]) as the quotient of the free abelian group with generators given by all isomorphism classes of finite-dimensional $\mathbb{Z}_2$-graded A-modules by the subgroup generated by $[V_1]-[V]+[V_2]$ for all exact sequences

$$0 \longrightarrow V_1 \longrightarrow V \longrightarrow V_2 \longrightarrow 0$$

and by $[V] + [\Pi(V)]$ for all A-modules V.

It is easy to see that the Grothendieck group $K(A)$ is a free $\mathbb{Z}$-module with the basis corresponding to the classes of the irreducible modules.

Let now $A = U(\mathfrak{g})$ be the universal enveloping algebra of a Lie superalgebra $\mathfrak{g}$ (see e.g. [12]) and $K(A)$ be the corresponding Grothendieck group. Consider the map

$$\mathfrak{g} \longrightarrow U(\mathfrak{g}) \otimes U(\mathfrak{g}), \quad x \to x \otimes 1 + 1 \otimes x.$$

One can check that this map is a homomorphism of Lie superalgebras, where on the right-hand side we consider the standard Lie superalgebra structure defined for any associative algebra A by the formula

$$[a, b] = ab - (-1)^{p(a)p(b)} ba.$$

Therefore, for any two $\mathfrak{g}$-modules V and U, one can define the $\mathfrak{g}$-module structure on $V \otimes U$. Using this we define the product on $K(A)$ by the formula

$$[U][V] = [U \otimes V].$$

Since all modules are finite-dimensional this multiplication is well-defined on the Grothendieck group $K(A)$ and introduces the ring structure on it. The corresponding ring is called the *Grothendieck ring of Lie superalgebra* $\mathfrak{g}$ and will be denoted $K(\mathfrak{g})$.

3. Basic classical Lie superalgebras and generalized root systems

Following Kac [12], [13] we call Lie superalgebra $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ *basic classical* if

a) $\mathfrak{g}$ is simple,

b) Lie algebra $\mathfrak{g}_0$ is a reductive subalgebra of $\mathfrak{g}$,

c) there exists a nondegenerate invariant even bilinear form on $\mathfrak{g}$.

Kac proved that the complete list of basic classical Lie superalgebras, which are not Lie algebras, consists of Lie superalgebras of the type

$$A(m,n),\ B(m,n),\ C(n),\ D(m,n),\ F(4),\ G(3),\ D(2,1,\alpha).$$

In full analogy with the case of simple Lie algebras one can consider the decomposition of $\mathfrak{g}$ with respect to the adjoint action of Cartan subalgebra $\mathfrak{h}$:

$$\mathfrak{g} = \mathfrak{h} \oplus (\oplus \mathfrak{g}_\alpha)\,,$$

where the sum is taken over the set R of nonzero linear forms on $\mathfrak{h}$, which are called *roots* of $\mathfrak{g}$. With the exception of the Lie superalgebra of type $A(1,1)$ the corresponding root subspaces $\mathfrak{g}_\alpha$ have dimension 1 (for $A(1,1)$ type the root subspaces corresponding to the isotropic roots have dimension 2).

It turned out that the corresponding root systems admit the following simple geometric description found by Serganova [19].

Let V be a finite-dimensional complex vector space with a nondegenerate bilinear form $(\,,)$.

Definition ([19]). A finite set $R \subset V \setminus \{0\}$ is called a *generalized root system* if the following conditions are fulfilled:

1) R spans V and $R = -R$;
2) If $\alpha, \beta \in R$ and $(\alpha,\alpha) \neq 0$, then $\frac{2(\alpha,\beta)}{(\alpha,\alpha)} \in \mathbb{Z}$ and $s_\alpha(\beta) = \beta - \frac{2(\alpha,\beta)}{(\alpha,\alpha)}\alpha \in R$;
3) If $\alpha \in R$ and $(\alpha,\alpha) = 0$, then there exists an invertible mapping $r_\alpha \colon R \to R$ such that $r_\alpha(\beta) = \beta$ if $(\beta,\alpha) = 0$ and $r_\alpha(\beta) \in \{\beta+\alpha,\ \beta-\alpha\}$ otherwise.

The roots α such that $(\alpha,\alpha) = 0$ are called *isotropic*. A generalized root system R is called *reducible* if it can be represented as a direct orthogonal sum of two nonempty generalized root systems R_1 and R_2: $V = V_1 \oplus V_2$, $R_1 \subset V_1$, $R_2 \subset V_2$, $R = R_1 \cup R_2$. Otherwise the system is called *irreducible*.

Any generalized root system has a partial symmetry described by the finite group W_0 generated by reflections with respect to the nonisotropic roots.

A remarkable fact proved by Serganova [19] is that classification list for the irreducible generalized root systems with isotropic roots coincides with the root systems of the basic classical Lie superalgebras from the Kac list (with the exception of $A(1,1)$) and $B(0,n)$). Note that the superalgebra $B(0,n)$ has no isotropic roots: its root system is the usual nonreduced system of $BC(n)$ type.

Remark. Serganova considered also a slightly wider notion[1] of generalized root systems, when the property 3) is replaced by

3′) If $\alpha \in R$ and $(\alpha, \alpha) = 0$, then for any $\beta \in R$ such that $(\alpha, \beta) \neq 0$ at least one of the vectors $\beta + \alpha$ or $\beta - \alpha$ belongs to R.

This axiomatics includes the root systems of type $A(1, 1)$ as well as the root systems of type $C(m, n)$ and $BC(m, n)$. We have used it in [25] to introduce a class of the deformed Calogero-Moser operators.

4. Ring $J(\mathfrak{g})$ and supercharacters of $\mathfrak{g}$

Let V be a finite-dimensional module over a basic classical Lie superalgebra $\mathfrak{g}$ with Cartan subalgebra $\mathfrak{h}$. Let us assume for the moment that V is a *semisimple* $\mathfrak{h}$-module, which means that V can be decomposed as a sum of the one-dimensional $\mathfrak{h}$-modules:

$$V = \bigoplus_{\lambda \in P(V)} V_\lambda,$$

where $P(V)$ is the set of the corresponding weights $\lambda \in \mathfrak{h}^*$. The *supercharacter* of V is defined as

$$\operatorname{sch} V = \sum_{\lambda \in P(V)} (\operatorname{sdim} V_\lambda) e^\lambda,$$

where sdim is the *superdimension* defined for any $\mathbb{Z}_2$-graded vector space $W = W_0 \oplus W_1$ as the difference of usual dimensions of graded components:

$$\operatorname{sdim} W = \dim W_0 - \dim W_1.$$

By definition the supercharacter $\operatorname{sch} V \in \mathbb{Z}[\mathfrak{h}^*]$ is an element of the integral group ring of $\mathfrak{h}^*$ (considered as an abelian group).

The following proposition shows that in the context of a Grothendieck ring we can restrict ourselves by the semisimple $\mathfrak{h}$-modules. First of all, note that the Grothendieck group has a natural basis consisting of irreducible modules. Indeed any finite-dimensional module has Jordan-Hölder series, so in the Grothendieck group it is equivalent to the sum of irreducible modules.

PROPOSITION 4.1. *Let V be a finite-dimensional irreducible $\mathfrak{g}$-module. Then V is semisimple as $\mathfrak{h}$-module.*

Proof. Let $W \subset V$ be the maximal semisimple $\mathfrak{h}$-submodule. Since V is finite-dimensional W is nontrivial. Let us prove that W is $\mathfrak{g}$-module. We have

$$\mathfrak{g} = \mathfrak{h} \oplus \Big(\bigoplus_{\alpha \in R} \mathfrak{g}_\alpha \Big).$$

[1]Johan van de Leur communicated to us that a similar notion was considered earlier by T. Springer, but his classification results were not complete [29].

Since W is semisimple it is a direct sum of one-dimensional $\mathfrak{h}$-modules. Let $w \in W$ be a generator of one of them, so that $hw = l(h)w$ for any $h \in \mathfrak{h}$. Note that xw is an eigenvector for $\mathfrak{h}$ for any $x \in \mathfrak{g}_\alpha$ since for any $h \in \mathfrak{h}$

$$hxw = [h, x]w + xhw = (\alpha(h) + l(h))xw.$$

Now the fact that xw belongs to W follows from the maximality of W. Since V is irreducible W must coincide with V. $\square$

The following general result (essentially contained in Kac [12], [13]) shows that for basic classical Lie superalgebras an irreducible module is uniquely determined by its supercharacter.

Proposition 4.2. *Let V, U be finite-dimensional irreducible $\mathfrak{g}$-modules. If $\operatorname{sch} V = \operatorname{sch} U$, then V and U are isomorphic as $\mathfrak{g}$-modules.*

Proof. By the previous proposition the modules are semisimple. According to Kac [13] (see Proposition 2.2) every irreducible finite-dimensional module is uniquely determined by its highest weight. Since $\operatorname{sch} V = \operatorname{sch} U$ the modules V and U have the same highest weights and thus are isomorphic as $\mathfrak{g}$-modules. $\square$

Now we are going to explain why the definition (1) of the ring $J(\mathfrak{g})$ is natural in this context.

Recall that in the classical case of semisimple Lie algebras the representation theory of $\mathfrak{sl}(2)$ plays the key role (see e.g. [9]). In the super case it is natural to consider the Lie superalgebra $\mathfrak{sl}(1,1)$, which has three generators H, X, Y (H generates the even part, X, Y are odd) with the following relations:

$$[H, X] = [H, Y] = [Y, Y] = [X, X] = 0,\ [X, Y] = H. \tag{2}$$

However, because of the absence of complete reducibility in the super case, this Lie superalgebra alone is not enough to get the full information. We need to consider the following extension of $\mathfrak{sl}(1,1)$. As before we will use the notation (a) for the principal ideal of the integral group ring $\mathbb{Z}[\mathfrak{h}^*]$ generated by an element $a \in \mathbb{Z}[\mathfrak{h}^*]$.

Proposition 4.3. *Let $\mathfrak{g}(\mathfrak{h}, \alpha)$ be the solvable Lie superalgebra such that $\mathfrak{g}_0 = \mathfrak{h}$ is a commutative finite-dimensional Lie algebra, $\mathfrak{g}_1 = \operatorname{Span}(X, Y)$ and the following relations hold:*

$$[h, X] = \alpha(h)X, \quad [h, Y] = -\alpha(h)Y, \quad [Y, Y] = [X, X] = 0,\ [X, Y] = H, \tag{3}$$

where $H \in \mathfrak{h}$ and $\alpha \neq 0$ is a linear form on $\mathfrak{h}$ such that $\alpha(H) = 0$. Then the Grothendieck ring of $\mathfrak{g}(\mathfrak{h}, \alpha)$ is isomorphic to

$$J(\mathfrak{g}(\mathfrak{h}, \alpha)) = \{f = \sum c_\lambda e^\lambda \mid \lambda \in \mathfrak{h}^*, \quad D_H f \in (e^\alpha - 1)\}, \tag{4}$$

where by definition $D_H e^\lambda = \lambda(H) e^\lambda$. The isomorphism is given by the supercharacter map $\operatorname{Sch} : [V] \longrightarrow \operatorname{sch} V$.

Proof. Every irreducible $\mathfrak{g}(\mathfrak{h},\alpha)$-module V has a unique (up to a multiple) vector v such that $Xv=0$, $hv=\lambda(h)v$ for some linear form λ on $\mathfrak{h}$. This establishes a bijection between the irreducible $\mathfrak{g}(\mathfrak{h},\alpha)$-modules and the elements of $\mathfrak{h}^*$.

There are two types of such modules, depending on whether $\lambda(H)=0$ or not. In the first case the module $V=V(\lambda)$ is one-dimensional and its supercharacter is e^{λ}. If $\lambda(H)\neq 0$, then the corresponding module $V(\lambda)$ is two-dimensional with the supercharacter $\mathrm{sch}(V)=e^{\lambda}-e^{\lambda-\alpha}$. Thus we have proved that the image of the supercharacter map $\mathrm{Sch}(K(\mathfrak{g}(\mathfrak{h},\alpha)))$ is contained in $J(\mathfrak{g}(\mathfrak{h},\alpha))$.

Conversely, let $f=\sum c_\lambda e^\lambda$ belong to $J(\mathfrak{g}(\mathfrak{h},\alpha))$. By subtracting a suitable linear combination of supercharacters of the one-dimensional modules $V(\lambda)$ we can assume that $\lambda(H)\neq 0$ for all λ from f. Then the condition $D_H f\in(e^\alpha-1)$ means that

$$\sum\lambda(H)c_\lambda e^\lambda=\sum d_\mu(e^\mu-e^{\mu-\alpha}). \tag{5}$$

For any $\lambda\in\mathfrak{h}^*$ define the linear functional F_λ on $\mathbb{Z}[\mathfrak{h}^*]$ by

$$F_\lambda(f)=\sum_{k\in\mathbb{Z}}c_{\lambda+k\alpha}.$$

It is easy to see that the conditions $F_\mu(f)=0$ for all $\mu\in\mathfrak{h}^*$ characterise the ideal $(e^\alpha-1)$. Applying F_μ to both sides of relation (5) and using the fact that $\alpha(H)=0$, $\lambda(H)\neq 0$ we deduce that f itself belongs to the ideal. This means that $f=\sum p_\nu(e^\nu-e^{\nu-\alpha})$ for some integers p_ν, which is a linear combination of the supercharacters of the irreducible modules $V(\nu)$. □

Any basic classical Lie superalgebra has a subalgebra isomorphic to (3) corresponding to any isotropic root α. By restricting the modules to this subalgebra we have the following

PROPOSITION 4.4. *For any basic classical Lie superalgebra* $\mathfrak{g}$ *the supercharacter map* Sch *is injective and its image* $\mathrm{Sch}(K(\mathfrak{g}))$ *is contained in* $J(\mathfrak{g})$.

The first claim is immediate consequence of Proposition 4.2. The invariance with respect to the Weyl group W_0 follows from the fact that any finite-dimensional $\mathfrak{g}$-module is also a finite-dimensional $\mathfrak{g}_0$-module.

This gives the proof of an easy part of the theorem. The rest of the proof (surjectivity of the supercharacter map) is much more involved.

5. Geometry of the highest weight set

In this section, which is quite technical, we give the description of the set of highest weights of finite-dimensional $\mathfrak{g}$-modules in terms of the corresponding generalized root systems. Essentially one can think of this as a geometric interpretation of the Kac conditions [12], [13].

Following Kac [13] we split all basic classical Lie superalgebras $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ into two *types*, depending on whether $\mathfrak{g}_0$-module $\mathfrak{g}_1$ is reducible (type I) or not (type II). The Lie superalgebras $A(m,n)$, $C(n)$ have type I, type II list consists of

$$B(m,n),\ D(m,n),\ F(4), G(3), D(2,1,\alpha).$$

In terms of the corresponding root systems type II is characterised by the property that the even roots generate the whole dual space to Cartan subalgebra $\mathfrak{h}$. In many respects Lie superalgebras of type II have more in common with the usual case of simple Lie algebras than Lie superalgebras of type I. In particular, we will see that the corresponding Grothendieck rings in type II can be naturally realised as subalgebras of the polynomial algebras, while in type I it is not the case.

Let us choose a *distinguished system* B of simple roots in R, which contains only one isotropic root γ; this is possible for any basic classical Lie superalgebra except $B(0,n)$, which has no isotropic roots (see [12]). If we take away γ from B the remaining set will give the system of simple roots of the even part $\mathfrak{g}_0$ if and only if $\mathfrak{g}$ has type I. For type II one can replace γ in B by a unique positive even root β (called *special*) to get a basis of simple roots of $\mathfrak{g}_0$.

In the rest of this section we restrict ourselves with the Lie superalgebras of type II. The following fact, which will play an important role in our proof, can be checked case-by-case (see explicit formulas in the last section).

Proposition 5.1. *For any basic classical Lie superagebra of type* II *except* $B(0,n)$ *there exists a unique decomposition of Lie algebra* $\mathfrak{g}_0 = \mathfrak{g}_0^{(1)} \oplus \mathfrak{g}_0^{(2)}$ *such that the isotropic simple root* γ *from distinguished system* B *is the difference*

$$\gamma = \delta - \omega \tag{6}$$

of two weights δ *and* ω *of* $\mathfrak{g}_0^{(1)}$ *and* $\mathfrak{g}_0^{(2)}$ *respectively with the following properties*:

1) $\mathfrak{g}_0^{(2)}$ *is a semisimple Lie algebra and* ω *is its fundamental weight*;

2) *the special root* β *is a root of* $\mathfrak{g}_0^{(1)}$ *and* $\delta = \frac{1}{2}\beta$.

In the exceptional case $B(0,n)$ *we define* $\mathfrak{g}_0^{(1)} = \mathfrak{g}_0$ *and* $\omega = 0$.

Remark. The fundamental weight ω has the following property, which will be very important for us: it has a small orbit in the sense of Serganova (see below).

Let $\mathfrak{a}$ be a semisimple Lie algebra, W be its Weyl group, which acts on the corresponding root system R and weight lattice P (see e.g. [3]). Following Serganova [19] we call the orbit $W\omega$ of weight ω *small* if for any $x, y \in W\omega$

such that $x \neq \pm y$ the difference $x - y$ belongs to the root system R of $\mathfrak{a}$. Such orbits play a special role in the classification of the generalized root systems.

Let $\mathfrak{g}$ be a basic classical Lie superalgebra of type II, $\mathfrak{a} = \mathfrak{g}_0^{(2)}$ and ω as in Proposition 5.1. Define a positive integer $k = k(\mathfrak{g})$ as

$$k = \frac{1}{2} \mid W\omega \mid, \tag{7}$$

where W is the Weyl group of $\mathfrak{a}$ and $\mid W\omega \mid$ is the number of elements in the orbit of the weight ω.

For any positive integer $j \leq k$ consider a subset $L_j \subset P$ of the weight lattice of $\mathfrak{a}$ defined by the relations

$$F(\nu) \neq 0, F(\nu-\omega) = 0, \dots, F(\nu-(j-1)\omega) = 0, \, (\nu, \omega) = (\rho+(j-k)\omega, \omega), \tag{8}$$

where

$$F(\nu) = \prod_{\alpha \in R^+} (\nu, \alpha)$$

and ρ is the half of the sum of positive roots $\alpha \in R^+$ of $\mathfrak{a}$. In particular,

$$L_1 = \{\nu \in P \mid F(\nu) \neq 0, (\nu, \omega) = (\rho + (1-k)\omega, \omega)\}.$$

Let Λ be a highest weight of Lie algebra $\mathfrak{g}_0$ and λ be its projection to the weight lattice of $\mathfrak{a} = \mathfrak{g}_0^{(2)}$ with respect to the decomposition $\mathfrak{g}_0 = \mathfrak{g}_0^{(1)} \oplus \mathfrak{g}_0^{(2)}$. Define an integer $j(\Lambda)$ by the formula

$$j(\Lambda) = k - \frac{(\Lambda, \delta)}{(\delta, \delta)}, \tag{9}$$

where δ is the same as in Proposition 5.1. This number was implicitly used by Kac in [13].

Define the following set $X(\mathfrak{g})$ consisting of the highest weights Λ of $\mathfrak{g}_0$ such that either $j(\Lambda) \leq 0$ or the W-orbit of $\lambda + \rho$ intersects the set L_j for some $j = 1, \dots, k$.

The main result of this section is the following:

Theorem 5.2. *For any basic classical Lie superalgebra $\mathfrak{g}$ of type* II *the set $X(\mathfrak{g})$ coincides with the set of the highest weights of the finite-dimensional representations of $\mathfrak{g}$.*

The rest of the section is the proof of this theorem. Let us define the *support* $\mathrm{Supp}(\varphi)$ of an element $\varphi \in \mathbb{Z}[P]$ as the set of weights $\nu \in P$ in the representation $\varphi = \sum \varphi(\nu) e^{\nu}$, for which $\varphi(\nu)$ is not zero. Define also the *alternation* operation on $\mathbb{Z}[P]$ as

$$\mathrm{Alt}(\varphi) = \sum_{w \in W} \varepsilon(w) w(\varphi), \tag{10}$$

where by definition $w(e^{\nu}) = e^{w\nu}$ and $\varepsilon : W \to \pm 1$ is the sign homomorphism.

Lemma 5.3. *Let ω be a weight such that the orbit $W\omega$ is small. Consider $\varphi \in \mathbb{Z}[P]$ such that* $\mathrm{Alt}(\varphi) = 0$, $\mathrm{Supp}(\varphi)$ *is contained in the hyperplane* $(\nu, \omega) = c$ *for some* c, *and for every* $\nu \in \mathrm{Supp}(\varphi)$

$$F(\nu) = \prod_{\alpha \in R^+} (\nu, \alpha) \neq 0.$$

Then

1) *if* $c \neq 0$ *then for any* $t \in \mathbb{Z}$, $\mathrm{Alt}(\varphi e^{t\omega}) = 0$;
2) *if* $c = 0$ *then the same is true if there exists* $\sigma_0 \in W$ *such that* $\sigma_0\omega = -\omega$ *and* $\sigma_0\varphi = \varphi$, $\varepsilon(\sigma_0) = 1$.

Proof. We have

$$\mathrm{Alt}(\varphi) = \sum_{(\nu,\omega)=c} \varphi(\nu)\mathrm{Alt}(e^{\nu}) = 0.$$

Since $F(\nu) \neq 0$ the elements $\mathrm{Alt}(e^{\nu})$ are nonzero and linearly independent for ν from different orbits of W. Thus the last equality is equivalent to

$$\sum_{\sigma \in W} \varepsilon(\sigma)\varphi(\sigma\nu) = 0 \tag{11}$$

for any ν from the support of φ.

Let $c \neq 0$. Fix $\nu \in \mathrm{Supp}(\varphi)$ and consider $\sigma \in W$ such that $\varphi(\sigma\nu) \neq 0$, in particular $(\sigma\nu, \omega) = c$. We have that $(\nu, \omega - \sigma^{-1}\omega) = 0$, $(\nu, \omega + \sigma^{-1}\omega) = 2c \neq 0$. Since the orbit of ω is small and $F(\nu) \neq 0$ this implies that $\omega = \sigma^{-1}\omega$ and therefore σ belongs to the stabiliser $W_\omega \subset W$ of ω. Thus relation (11) is equivalent to

$$\sum_{\sigma \in W_\omega} \varepsilon(\sigma)\sigma(\varphi) = 0.$$

Since ω is invariant under W_ω this implies

$$\sum_{\sigma \in W_\omega} \varepsilon(\sigma)\sigma(\varphi e^{t\omega}) = 0$$

and thus

$$\sum_{\sigma \in W} \varepsilon(\sigma)\sigma(\varphi e^{t\omega}) = 0.$$

This proves the first part.

When $c = 0$ similar arguments lead to the relation

$$\sum_{\sigma \in W_{\pm\omega}} \varepsilon(\sigma)\sigma(\varphi) = 0,$$

where $W_{\pm\omega}$ is the stabiliser of the set $\pm\omega$. From the conditions of the lemma it follows that $W_{\pm\omega}$ is generated by W_ω and σ_0. Since $\varepsilon(\sigma_0) = 1$ and $\sigma_0\varphi = \varphi$ we can replace in this last formula $W_{\pm\omega}$ by W_ω and repeat the previous arguments to complete the proof. $\square$

Recall that for any $\omega \in P$ the derivative D_ω is determined by the relation $D_\omega e^\lambda = (\omega, \lambda) e^\lambda$. The condition that $D_\omega \varphi = 0$ is equivalent to the support of φ to be contained in the hyperplane $(\omega, \lambda) = 0$.

LEMMA 5.4. *Let $\mathfrak{g}$ be a basic classical Lie superalgebra, $\mathfrak{a} = \mathfrak{g}_0^{(2)}$ and ω as in Proposition* 5.1, *k defined by* (7), *W be the Weyl group of $\mathfrak{a}$ acting on the corresponding weight lattice P. Consider a function of the form*

$$\varphi = \sum_{i=1}^{k} (e^{(k-i)\omega} + e^{-(k-i)\omega}) f_i, \tag{12}$$

where $f_i \in \mathbb{Z}[P]^W$ are some exponential W-invariants. Suppose that $D_\omega \varphi = 0$ and consider the first nonzero coefficient f_j in φ (so that $f_1 = f_2 = \cdots = f_{j-1} = 0$ for some $j \leq k$).

Then f_j is a linear combination of the characters of irreducible representations of $\mathfrak{a}$ with the highest weights λ such that the orbit $W(\lambda + \rho)$ intersects the set L_j defined above.

Proof. Since $D_\omega \varphi = 0$ the support of φ is contained in the hyperplane $(\omega, \mu) = 0$. We can write the function φ as the sum $\varphi = \varphi_1 + \cdots + \varphi_j + \psi_j$, where the support of $\varphi_j e^{\rho + (j-k)\omega}$ is contained in L_j and the support of $\psi_j e^{\rho + (i-k)\omega}$ is not contained in L_i for all $i = 1, \dots, j$. Let us multiply (12) consequently by $e^{\rho + (1-k)\omega}$, $e^{\rho + (2-k)\omega}$, $\dots$, $e^{\rho + (j-k)\omega}$ and then apply the alternation operation (10). Then from the definition of the sets L_j we have

$$\begin{gathered}
\mathrm{Alt}(e^\rho) f_1 = \mathrm{Alt}(\varphi_1 e^{\rho + (1-k)\omega}), \\
\mathrm{Alt}(e^{\rho+\omega}) f_1 + \mathrm{Alt}(e^\rho) f_2 = \mathrm{Alt}(\varphi_1 e^{\rho + (2-k)\omega}) + \mathrm{Alt}(\varphi_2 e^{\rho + (2-k)\omega}), \\
\cdots\cdots\cdots \\
\mathrm{Alt}(e^{\rho + (j-1)\omega}) f_1 + \cdots + \mathrm{Alt}(e^\rho) f_j = \mathrm{Alt}(\varphi_1 e^{\rho + (j-k)\omega}) + \cdots + \mathrm{Alt}(\varphi_j e^{\rho + (j-k)\omega}).
\end{gathered}$$

Suppose that $f_1 = f_2 = \cdots = f_{j-1} = 0$. Then from the first equation we see that $\mathrm{Alt}(\varphi_1 e^{\rho + (1-k)\omega}) = 0$. One can verify that $(\rho + (j-k)\omega, \omega) = 0$ if and only if $j = 1$ and $(\mathfrak{a}, \omega)$ must be either $(D(m), \varepsilon_1)$ or $(B_3, 1/2(\varepsilon_1 + \varepsilon_2 + \varepsilon_3)$. In both of these cases we can find σ_0 such that $\varepsilon(\sigma_0) = 1$, $\sigma_0(\omega) = -\omega$, $\sigma_0 \varphi = \varphi$, so we can apply Lemma 5.3 to show that $\mathrm{Alt}(\varphi_1 e^{\rho - (k-i)\omega}) = 0$ for $i = 1, \dots, j$. Similarly, from the second equation $\mathrm{Alt}(\varphi_2 e^{\rho + (2-k)\omega}) = 0$ and by applying Lemma 5.3 again, we have $\mathrm{Alt}(\varphi_2 e^{\rho + (i-k)\omega}) = 0$ for $i = 2, \dots, j$ and eventually

$$\mathrm{Alt}(e^\rho) f_j = \mathrm{Alt}(\varphi_j e^{\rho + (j-k)\omega}).$$

Now the claim follows from the classical *Weyl character formula* (see e.g. [27]) for the representation with highest weight λ:

$$\mathrm{ch}\, V^\lambda = \frac{\mathrm{Alt}(e^{\lambda+\rho})}{\mathrm{Alt}(e^\rho)}. \tag{13}$$

□

Now we need the conditions on the highest weights of the finite-dimensional representations, which were found by Kac [12]. In the following lemma, which is a reformulation of Proposition 2.3 from [12], we use the basis of the weight lattice of $\mathfrak{g}_0$ described in Section 7.

Lemma 5.5 (Kac [12]). *For the basic classical Lie superalgebras $\mathfrak{g}$ of type* II, *a highest weight ν of $\mathfrak{g}_0$ is a highest weight of finite-dimensional irreducible $\mathfrak{g}$-module if and only if one of the corresponding conditions is satisfied*:

1) $\mathfrak{g} = B(m,n), \quad \Lambda = (\mu_1, \dots, \mu_n, \lambda_1, \dots, \lambda_m)$

- $\mu_n \geq m$
- $\mu_n = m - j,\ 0 < j \leq m$ *and* $\lambda_m = \lambda_{m-1} = \cdots = \lambda_{m-j+1} = 0.$

2) $\mathfrak{g} = D(m,n), \quad \Lambda = (\mu_1, \dots, \mu_n, \lambda_1, \dots, \lambda_m)$

- $\mu_n \geq m$
- $\mu_n = m - j,\ 0 < j \leq m$ *and* $\lambda_m = \lambda_{m-1} = \cdots = \lambda_{m-j+1} = 0.$

3) $\mathfrak{g} = G(3),\ \Lambda = (\mu, \lambda_1, \lambda_2)$

- $\mu \geq 3$
- $\mu = 2,\ \lambda_2 = 2\lambda_1$
- $\mu = 0,\ \lambda_1 = \lambda_2 = 0.$

4) $\mathfrak{g} = F(4), \quad \Lambda = (\mu, \lambda_1, \lambda_2, \lambda_3)$

- $\mu \geq 4$
- $\mu = 3,\ \lambda_1 = \lambda_2 + \lambda_3 - 1/2$
- $\mu = 2,\ \lambda_1 = \lambda_2,\ \lambda_3 = 0$
- $\mu = 0,\ \lambda_1 = \lambda_2 = \lambda_3 = 0.$

5) $\mathfrak{g} = D(2,1,\alpha), \quad \Lambda = (\lambda_1, \lambda_2, \lambda_3)$

- $\lambda_1 \geq 2$
- $\lambda_1 = 1$, α *is rational and* $\lambda_2 - 1 = |\alpha|(\lambda_3 - 1)$
- $\lambda_1 = 0, \lambda_2 = \lambda_3 = 0.$

Now we are ready to prove Theorem 5.2. Let $\mathfrak{g}$ be a basic classical Lie superalgebra of type II, $\mathfrak{a}$, ω, k, and W be the same as in Lemma 5.4, Λ be a highest weight of Lie algebra $\mathfrak{g}_0$, λ be its projection to the weight lattice of $\mathfrak{a}$, and $j = j(\Lambda)$ is defined by the formula (9).

We are going to show that the conditions defining the set $X(\mathfrak{g})$ are equivalent to Kac's conditions from the previous lemma. First of all an easy check shows that in each case the condition $j(\Lambda) \leq 0$ is equivalent to the first of Kac's conditions.

Let us consider now the condition that $W(\lambda+\rho)$ intersects the set L_j. We will see that in that case $j = j(\Lambda)$.

By definition L_j is described by the following system for the weights ν of $\mathfrak{a}$

$$\begin{cases} F(\nu) \neq 0 \\ F(\nu-\omega) = 0 \\ F(\nu - 2\omega) = 0 \\ \dots \\ F(\nu - (j-1)\omega) = 0 \\ (\rho - (k-j)\omega, \omega) = (\nu, \omega). \end{cases}$$

Consider this system in each case separately.

1) Let $\mathfrak{g} = B(m,n)$ with $m > 0$, $\mathfrak{a} = B(m)$, then $k = m$, $\omega = \varepsilon_1$, $\rho = \sum_{i=1}^m (m-i+1/2)\varepsilon_i$ and

$$F(\nu) = \prod_{p=1}^m \nu_p \prod_{p<q} (\nu_p^2 - \nu_q^2).$$

Since $F(\nu) \neq 0$ all ν_i are nonzero and pairwise different. The condition that $(\rho - (k-j)\omega, \omega) = (\nu, \omega)$ means that $\nu_1 = j - 1/2$. Then we have the following system:

$$\begin{cases} (\nu_2^2 - (j-3/2)^2)(\nu_3^2 - (j-3/2)^2)\cdots(\nu_m^2 - (j-3/2)^2) = 0 \\ (\nu_2^2 - (j-5/2)^2)(\nu_3^2 - (j-5/2)^2)\cdots(\nu_m^2 - (j-5/2)^2) = 0 \\ \dots \\ (\nu_2^2 - (1/2)^2)(\nu_3^2 - (1/2)^2)\cdots(\nu_m^2 - (1/2)^2) = 0. \end{cases}$$

The first equation implies that one of ν_i equals to $j - 3/2$, the second one implies that one of them is $j - 5/2$ and so on. So if $W(\lambda+\rho) \cap L_j \neq \emptyset$ then $\lambda_m = \lambda_{m-1} = \dots = \lambda_{m-j+1} = 0$, which is one of the corresponding conditions in Lemma 5.5. In the case $B(0,n)$ we have the only condition $j(\Lambda) \leq 0$, which is equivalent to $\mu_n \geq 0$.

2) If $\mathfrak{g} = D(m,n)$, $\mathfrak{a} = D(m)$, then $k = m$, $\omega = \varepsilon_1$, $\rho = \sum_{i=1}^m (m-i)\varepsilon_i$, and

$$F(\nu) = \prod_{p<q} (\nu_p^2 - \nu_q^2).$$

In that case the condition $(\rho - (k-j+1)\omega, \omega) = (\nu, \omega)$ implies that $\nu_1 = j-1$ and we have the following system

$$\begin{cases} (\nu_2^2 - (j-2)^2)(\nu_3^2 - (j-2)^2)\cdots(\nu_m^2 - (j-2)^2) = 0 \\ (\nu_2^2 - (j-3)^2)(\nu_3^2 - (j-3)^2)\cdots(\nu_m^2 - (j-3)^2) = 0 \\ \dots \\ \nu_2^2 \nu_3^2 \cdots \nu_m^2 = 0. \end{cases}$$

If $W(\lambda+\rho) \cap L_j \neq \emptyset$, then we have similarly again $\lambda_m = \lambda_{m-1} = \dots = \lambda_{m-j+1} = 0$.

3) Let $\mathfrak{g} = G(3)$, $\mathfrak{a} = G(2)$, then $k = 3$, $\omega = \varepsilon_1 + \varepsilon_2$, $\rho = 2\varepsilon_1 + 3\varepsilon_2$ and

$$F(\nu) = \nu_1\nu_2(\nu_2 - \nu_1)(\nu_1 + \nu_2)(2\nu_1 - \nu_2)(2\nu_2 - \nu_1).$$

The condition $(\rho-(k-j)\omega,\omega)=(\nu,\omega)$ means that $\nu_1+\nu_2=2j-1$. If $j=1$, then we have $\nu_1+\nu_2=1$, $F(\nu)\neq 0$. One check that in that case $\nu\in W(\lambda+\rho)$ only if $\lambda_2=2\lambda_1$.

If $j=2$, then we have the conditions $\nu_1+\nu_2=3$, $F(\nu_1-1,\nu_2-1)=0$, $F(\nu)\neq 0$, which cannot be satisfied for $\nu\in W(\lambda+\rho)$.

If $j=3$ we have $\nu_1+\nu_2=3$, $F(\nu_1-1,\nu_2-1)=0$, $F(\nu_1-2,\nu_2-2)=0$, $F(\nu)\neq 0$, which imply that if $\nu\in W(\lambda+\rho)$, then $\lambda_2=\lambda_1=0$ in agreement with Lemma 5.5.

4) If $\mathfrak{g}=F(4)$, $\mathfrak{a}=B(3)$, then $k=4$, $\omega=\frac{1}{2}(\varepsilon_1+\varepsilon_2+\varepsilon_3)$, $\rho=\frac{5}{2}\varepsilon_1+\frac{3}{2}\varepsilon_2+\frac{1}{2}\varepsilon_3$,

$$F(\nu)=\nu_1\nu_2\nu_3(\nu_1^2-\nu_2^2)(\nu_1^2-\nu_3^2)(\nu_2^2-\nu_3^2).$$

The condition $(\rho-(3-j)\omega,\omega)=(\nu,\omega)$ means that $\nu_1+\nu_2+\nu_3=\frac{3}{2}(j-1)$. If $j=1$ we have the conditions $\nu_1+\nu_2+\nu_3=0$, $F(\nu)\neq 0$, which imply that if $\nu\in W(\lambda+\rho)$, then $\lambda_1=\lambda_2+\lambda_3-1/2$.

If $j=2$ we have $\nu_1+\nu_2+\nu_3=3/2$, $F(\nu_1-1/2,\nu_2-1/2,\nu_3-1/2)=0$, $F(\nu)\neq 0$. One can check that if $\nu\in W(\lambda+\rho)$, then $\lambda_1=\lambda_2$, $\lambda_3=0$.

If $j=3$, then we have the conditions $\nu_1+\nu_2+\nu_3=3$, $F(\nu_1-1/2,\nu_2-1/2,\nu_3-1/2)=0$, $F(\nu_1-1,\nu_2-1,\nu_3-1)=0$, $F(\nu)\neq 0$, which cannot be satisfied for $\nu\in W(\lambda+\rho)$.

If $j=4$ we have $\nu_1+\nu_2+\nu_3=9/2$, $F(\nu_1-1/2,\nu_2-1/2,\nu_3-1/2)=0$, $F(\nu_1-1,\nu_2-1,\nu_3-1)=0$, $F(\nu_1-3/2,\nu_2-3/2,\nu_3-3/2)=0$, $F(\nu)\neq 0$. In that case $\nu\in W(\lambda+\rho)$ only if $\lambda_1=\lambda_2=\lambda_3=0$.

5) Let $\mathfrak{g}=D(2,1,\alpha)$, $\mathfrak{a}=A_1\oplus A_1$; then $k=2$, $\omega=\varepsilon_2+\varepsilon_3$, $\rho=-\varepsilon_2-\varepsilon_3$ and $F(\nu)=4\alpha\nu_1\nu_2$. The condition $(\rho-(2-j)\omega,\omega)=(\nu,\omega)$ means that $\nu_1+\alpha\nu_2=j-1$. If $j=1$ we have $\nu_1+\alpha\nu_2=0$, $\nu_1\nu_2\neq 0$. If α is irrational, then the system has no integer solution. If α is rational and $\nu\in W(\lambda+\rho)$, then $(\lambda_1+1)=|\alpha|(\lambda_2+1)$.

If $j=2$ the conditions $\nu_1+\nu_2=1$, $(\nu_1-1)(\nu_2-1)=0$, $\nu_1\nu_2\neq 0$ imply that if $\nu\in W(\lambda+\rho)$, then $\lambda_1=\lambda_2=0$ in agreement with Lemma 5.5.

This completes the proof of Theorem 5.2.[2]

6. **Proof of the main theorem**

Let $\mathfrak{g}$ be a basic classical Lie superalgebra of type II, $\mathfrak{g}_0=\mathfrak{g}_0^{(1)}\oplus\mathfrak{g}_0^{(2)}$ be the decomposition of the corresponding Lie algebra $\mathfrak{g}_0$ from Proposition 5.1, and $\gamma=\delta-\omega$ be the same as in (6). The root system R_0 of $\mathfrak{g}_0$ is a disjoint union $R_0^{(1)}\cup R_0^{(2)}$ of root systems of $\mathfrak{g}_0^{(1)}$ and $\mathfrak{g}_0^{(2)}$.

[2]As we have recently learnt from Serganova a different description of the set of highest weights can be found in [22].

Let us introduce the following *partial order* $\succ$ on the weight lattice $P(R_0^{(1)})$: we say that $\mu \succeq 0$ if and only if μ is a sum of simple roots from $R_0^{(1)}$ and the weight δ with nonnegative integer coefficients.

LEMMA 6.1. *Let V^Λ be an irreducible finite-dimensional $\mathfrak{g}$-module with highest weight Λ and μ, λ be the projections of Λ on $P(R_0^{(1)})$ and $P(R_0^{(2)})$ respectively. Then the supercharacter of V^Λ can be represented as*

$$\mathrm{sch}(V^\Lambda) = e^\mu \mathrm{ch}(V^\lambda) + \sum_{\tilde{\mu} \prec \mu} e^{\tilde{\mu}} F_{\tilde{\mu}}, \quad F_{\tilde{\mu}} \in \mathbb{Z}[P(R_0^{(2)})], \tag{14}$$

where $\prec$ means partial order introduced above and V^λ is the irreducible $\mathfrak{g}_0^{(2)}$-module with highest weight λ.

Proof. Consider V^Λ as $\mathfrak{g}_0^{(1)}$-module and introduce the subspace $W \subset V^\Lambda$ consisting of all vectors of weight μ. Let us prove that W as a module over Lie algebra $\mathfrak{g}_0^{(2)}$ is irreducible. It is enough to prove that it is a highest weight module over $\mathfrak{g}_0^{(2)}$. Let $v \in W$ be a vector of weight $\tilde{\Lambda}$ with respect to the Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$. Then $v = uv_\Lambda$, where v_Λ is the highest weight vector of V^Λ and u is a linear combination of the elements of the form

$$\prod_{\alpha \in (R_0^1)^+} X_{-\alpha}^{n_\alpha} \prod_{\gamma \in R_1^+} X_{-\gamma}^{n_\gamma} \prod_{\beta \in (R_0^2)^+} X_{-\beta}^{n_\beta},$$

where R_1 is the set of roots of $\mathfrak{g}_1$ and X_α is an element from the corresponding root subspace of $\mathfrak{g}$. We have that

$$\Lambda - \tilde{\Lambda} = \sum_{\alpha \in (R_0^1)^+} n_\alpha \alpha + \sum_{\gamma \in R_1^+} n_\gamma \gamma + \sum_{\beta \in (R_0^2)^+} n_\beta \beta.$$

Let $\tilde{\mu}$ be the projection of $\tilde{\Lambda}$ to $P(R_0^{(1)})$. It is easy to check case by case that the condition $\tilde{\mu} = \mu$ implies $n_\alpha = n_\gamma = 0$ for any $\alpha \in (R_0^1)^+$, $\gamma \in R_1^+$. This proves the irreducibility of W and justifies the first term in the right-hand side of (14).

To prove the form of the remainder in (14) we note that if $\tilde{\mu} \neq \mu$ then $\tilde{\Lambda} < \Lambda$ with respect to the partial order defined by R^+ and hence $\tilde{\mu} \prec \mu$ with respect to the partial order defined above. The lemma is proved. □

The following key lemma establishes the link between the ring $J(\mathfrak{g})$ and the supercharacters of $\mathfrak{g}$.

LEMMA 6.2. *Consider any $f = \sum_\mu e^\mu F_\mu \in J(\mathfrak{g})$, $\mu \in P(R_0^{(1)})$, $F_\mu \in \mathbb{Z}[P(R_0^{(2)})]$. Let μ_* be a maximal with respect to the partial order $\succ$ among all μ such that $F_\mu \neq 0$ and $j = j(\mu_*)$ be defined by* (9).

If $j > 0$, then F_{μ_*} is a linear combination of the characters of irreducible representations of $\mathfrak{a} = \mathfrak{g}_0^{(2)}$ with the highest weights λ such that the orbit $W(\lambda+\rho)$ intersects the set L_j defined by (8).

Proof. Since μ_* is maximal with respect to partial order $\succ$ it is also maximal with respect to the partial order defined by $(R_0^{(1)})^+$. Because of the symmetry of f with respect to the Weyl group of the root system $R_0^{(1)}$ the weight μ_* is dominant. From the definition of the ring $J(\mathfrak{g})$ we have that

$$D_\gamma\Bigg(\sum_{\mu^\perp=\mu_*^\perp} e^\mu F_\mu\Bigg) \in (e^\gamma - 1),$$

where γ is the same as in Proposition 5.1 and $\mu^\perp$ is the component of μ perpendicular to δ. This can be rewritten as

$$D_\gamma \phi \in (e^\gamma - 1), \tag{15}$$

where

$$\phi = \sum_{\mu^\perp=\mu_*^\perp}\left(e^{\frac{(\mu,\delta)}{(\delta,\delta)}\delta} + e^{-\frac{(\mu,\delta)}{(\delta,\delta)}\delta}\right)F_\mu$$

(we have used the symmetry with respect to the root 2δ).

Let φ be the restriction of ϕ on the hyperplane $\gamma = 0$, where we consider weights as linear functions on Cartan subalgebra $\mathfrak{h}$. Using the relation $\gamma = \delta - \omega$ we can rewrite (15) as $D_\omega\varphi = 0$. The conditions $\mu \prec \mu_*$, $\mu^\perp = \mu_*^\perp$ imply that $\frac{(\mu_*,\delta)}{(\delta,\delta)} > \frac{(\mu,\delta)}{(\delta,\delta)}$. We have that

$$\varphi = (e^{(k-j)\omega} + e^{-(k-j)\omega})F_{\mu_*} + \sum_{0\le l<k-j}(e^{l\omega} + e^{-l\omega})F_l,\ j = k - \frac{(\mu_*,\delta)}{(\delta,\delta)}.$$

Since F_{μ_*}, F_l are invariant with respect to the Weyl group of $R_0^{(2)}$ for $0 \le l < k-j$ we can apply now Lemma 5.4 to conclude the proof. □

Now we are ready to prove our main theorem from the introduction for the basic simple Lie superalgebras of type II.

Consider any element $f \in J(\mathfrak{g})$ and write it as in Lemma 6.2 in the form $f = \sum_\mu e^\mu F_\mu$, where $\mu \in P(R_0^{(1)})$, $F_\mu \in \mathbb{Z}[P(R_0^{(2)})]$. Let $H(f) = \{\mu_1, \dots, \mu_N\}$ be the set consisting of all the maximal elements among all μ such that $F_\mu \neq 0$ with respect to the partial order introduced above. Let $S(f)$ be the finite set of highest weights of the Lie algebra $\mathfrak{g}_0^{(1)}$, which are less or equal to some of μ_i from $H(f)$, and $M = M(f)$ be the number of elements in the set $S(f)$.

According to Theorem 5.2 and Lemmas 6.1, 6.2 there are irreducible finite-dimensional $\mathfrak{g}$-modules $V^{\Lambda_1}, \dots, V^{\Lambda_K}$ and integers $n_1, \dots, n_K$ such that

$$\tilde f = f - \sum_{l=1}^K n_l \mathrm{sch}(V^{\Lambda_l}) = \sum e^{\tilde\mu}\tilde F_{\tilde\mu},$$

where in the last sum all $\tilde{\mu}$ are strictly less than some of μ_i. In particular this implies that none of μ_i belongs to $S(\tilde{f}) \subset S(f)$ and therefore $M(\tilde{f}) < M(f)$. Induction in M completes the proof of the theorem for type II.

Example. Let us illustrate the proof in the case of $G(3)$. In this case $\mathfrak{g}_0 = \mathfrak{g}_0^{(1)} \oplus \mathfrak{g}_0^{(2)}$, where $\mathfrak{g}_0^{(1)} = \mathfrak{sl}(2)$, $\mathfrak{g}_0^{(2)} = G(2)$. Therefore $P(R_0^{(1)}) = \mathbb{Z}$ and the partial order introduced above coincides with the natural order on $\mathbb{Z}$. We have $\gamma = \delta - \omega$, where δ is the only fundamental weight of $\mathfrak{sl}(2)$ and ω is the second fundamental weight of $G(2)$. Thus for any $f \in J(\mathfrak{g})$ the set $H(f)$ contains only one element $l\delta$ with some integer $l \geq 0$ and the corresponding $M(f) = l + 1$.

To prove the theorem for type I we use the explicit description of the ring $J(\mathfrak{g})$ given in the next section and the following notion of a Kac module.

If $\mathfrak{g}$ is a basic classical Lie superalgebra of type I then $\mathfrak{g}_0$-module $\mathfrak{g}_1$ is a direct sum of two irreducible modules $\mathfrak{g}_1 = \mathfrak{g}_1^+ \oplus \mathfrak{g}_1^-$, where $\mathfrak{g}_1^-$ is linearly generated by negative odd root vectors and $\mathfrak{g}_1^+$ is linearly generated by positive odd root vectors. One can check that $\mathfrak{g}_0 \oplus \mathfrak{g}_1^+$ is a subalgebra of $\mathfrak{g}$, so for every irreducible finite-dimensional $\mathfrak{g}_0$-module V_0 we can define the *Kac module*

$$K(V_0) = U(\mathfrak{g}) \otimes_{U(\mathfrak{g}_0 \oplus \mathfrak{g}_1^+)} V_0$$

where $\mathfrak{g}_1^+$ acts trivially on V_0 (see [12]). A Kac module is a finite-dimensional analogue of a Verma module. Namely, if λ is the highest weight of V_0, then every finite-dimensional $\mathfrak{g}$-module with the same highest weight λ is the quotient of $K(V_0)$. It is easy to see that the character of a Kac module can be given by the following formula

$$\operatorname{sch} KV_0) = \prod_{\alpha \in R_1^+} (1 - e^{-\alpha}) \operatorname{ch} V_0. \tag{16}$$

Let us proceed with the proof now.

Consider first the case $A(n,m)$ with $m \neq n$. The corresponding ring $J(\mathfrak{g})$ is described by Proposition 7.3 and can be represented as a sum $J(\mathfrak{g}) = \bigoplus_{a \in \mathbb{C}/\mathbb{Z}} J(\mathfrak{g})_a$. Comparing formulae (16) and (19) we see that the components $J(\mathfrak{g})_a$ with $a \notin \mathbb{Z}$ are spanned over $\mathbb{Z}$ by the supercharacters of Kac modules. According to the last statement of Proposition 7.3 the component $J(\mathfrak{g})_0$ is generated over $\mathbb{Z}$ by h_k and h_k^*, which are the supercharacters of k-th symmetric power of the standard representation and its dual. This proves the theorem in this case.

In the $A(n,n)$ case with $n \neq 1$ according to Proposition 7.4 the ring $J(\mathfrak{g})_0$ is spanned over $\mathbb{Z}$ by the products $h_1^{m_1} h_2^{m_2} \cdots h_1^{*n_1} h_2^{*n_2} \cdots$ with the condition that the total degree $m_1 + 2m_2 + \cdots - n_1 - 2n_2 - \cdots$ is equal to 0. It is easy to see that if V is the standard representation of $\mathfrak{gl}(n+1, n+1)$, then such a product is the supercharacter of the tensor product

$$S^1(V)^{\otimes m_1} \otimes S^2(V)^{\otimes m_2} \otimes \cdots \otimes S^1(V^*)^{\otimes n_1} \otimes S^2(V^*)^{\otimes n_2} \ldots,$$

considered as a module over $A(n,n)$. When $i \neq 0$ the component $J(\mathfrak{g})_i$ is linearly generated by supercharacters of Kac modules. The special case of $A(1,1)$ is considered separately in Section 8.

In the $C(n)$ case due to Proposition 7.5 $J(\mathfrak{g}) = \bigoplus_{a \in \mathbb{C}/\mathbb{Z}} J(\mathfrak{g})_a$, where again the components $J(\mathfrak{g})_a$ with $a \notin \mathbb{Z}$ are spanned over $\mathbb{Z}$ by the supercharacters of Kac modules $K(\chi)$ with

$$\chi = a\varepsilon + \sum_{j=1}^{n} \mu_j \delta_j, \mu_1 \geq \mu_2 \geq \cdots \geq \mu_n,\ \mu_j \in \mathbb{Z}_{\geq 0}.$$

The zero component is the direct sum $J(\mathfrak{g})_0 = J(\mathfrak{g})_0^+ \oplus J(\mathfrak{g})_0^-$, where $J(\mathfrak{g})_0^-$ is spanned over $\mathbb{Z}$ by the supercharacters of Kac modules $K(\chi)$ with

$$\chi = \lambda\varepsilon + \sum_{j=1}^{n} \mu_j \delta_j, \mu_1 \geq \mu_2 \geq \cdots \geq \mu_n,\ \lambda \in \mathbb{Z}, \mu_j \in \mathbb{Z}_{\geq 0}$$

and $J(\mathfrak{g})_0^+$ is generated over $\mathbb{Z}$ by h_k, which are the supercharacters of symmetric powers of the standard representation.

The proof of our main theorem is now complete.

7. Explicit description of the rings $J(\mathfrak{g})$

In this section we describe explicitly the rings $J(\mathfrak{g})$ for all basic classical superalgebras except $A(1,1)$ case, which is to be considered separately in the next section. We start with the case of Lie superalgebra $\mathfrak{gl}(n,m)$, which will be used for the investigation of the $A(n,m)$ case.

$\mathfrak{gl}(n,m)$. In this case $\mathfrak{g}_0 = \mathfrak{gl}(n) \oplus \mathfrak{gl}(m)$ and $\mathfrak{g}_1 = V_1 \otimes V_2^* \oplus V_1^* \otimes V_2$, where V_1 and V_2 are the identical representations of $\mathfrak{gl}(n)$ and $\mathfrak{gl}(m)$) respectively. Let $\varepsilon_1, \ldots, \varepsilon_{n+m}$ be the weights of the identical representation of $\mathfrak{gl}(n,m)$. Then the root system of $\mathfrak{g}$ is expressed in terms of linear functions $\varepsilon_i,\ 1 \leq i \leq n$, and $\delta_p = \varepsilon_{p+n},\ 1 \leq p \leq m$i, as follows:

$$R_0 = \{\varepsilon_i - \varepsilon_j, \delta_p - \delta_q : \ i \neq j \ : \ 1 \leq i,j \leq n \ , p \neq q,\ 1 \leq p,q \leq m\},$$

$$R_1 = \{\pm(\varepsilon_i - \delta_p), \quad 1 \leq i \leq n,\ 1 \leq p \leq m\} = R_{\text{iso}}.$$

The invariant bilinear form is determined by the relations

$$(\varepsilon_i, \varepsilon_i) = 1,\ (\varepsilon_i, \varepsilon_j) = 0,\ i \neq j,\ (\delta_p, \delta_q) = -1,\ (\delta_p, \delta_q) = 0,\ p \neq q,\ (\varepsilon_i, \delta_p) = 0.$$

The Weyl group $W_0 = S_n \times S_m$ acts on the weights by separately permuting $\varepsilon_i,\ i = 1, \ldots, n$, and $\delta_p,\ p = 1, \ldots, m$. Recall that the weight group of Lie algebra $\mathfrak{g}_0$ is defined as

$$P_0 = \{\lambda \in \mathfrak{h}^* \mid \frac{2(\lambda, \alpha)}{(\alpha, \alpha)} \in \mathbb{Z} \text{ for any } \alpha \in R_0\}.$$

In this case we have

$$P_0 = \{\lambda \in \mathfrak{h}^* \mid \lambda = \sum_{i=1}^{m} \lambda_i \varepsilon_i + \sum_{p=1}^{n} \mu_p \delta_p,\ \lambda_i - \lambda_j \in \mathbb{Z} \text{ and } \mu_p - \mu_q \in \mathbb{Z}\}. \tag{17}$$

Choose the following distinguished (in the sense of §5) system of simple roots

$$B = \{\varepsilon_1 - \varepsilon_2, \dots, \varepsilon_{n-1} - \varepsilon_n, \varepsilon_n - \delta_1, \delta_1 - \delta_2, \dots, \delta_{m-1} - \delta_m\}.$$

Note that the only isotropic root is $\varepsilon_n - \delta_1$. The weight λ is a highest weight for $\mathfrak{g}_0$ if $\frac{2(\lambda,\alpha)}{(\alpha,\alpha)} \geq 0$ for every nonisotropic root α from B.

Let $x_i = e^{\varepsilon_i}, y_p = e^{\delta_p}$ be the elements of the group ring of $\mathbb{Z}[P_0]$, which can be described as the direct sum $\mathbb{Z}[P_0] = \bigoplus_{a,b \in \mathbb{C}/\mathbb{Z}} \mathbb{Z}[P_0]_{a,b}$, where

$$\mathbb{Z}[P_0]_{a,b} = (x_1 \cdots x_n)^a (y_1 \cdots y_m)^b \mathbb{Z}[x_1^{\pm 1}, \dots, x_n^{\pm 1}, y_1^{\pm 1}, \dots, y_m^{\pm 1}]^{W_0}.$$

By definition the ring $J(\mathfrak{g})$ is the subring

$$J(\mathfrak{g}) = \{f \in \mathbb{Z}[P_0] \mid y_p \frac{\partial f}{\partial y_p} + x_i \frac{\partial f}{\partial x_i} \in (y_p - x_i), \quad p = 1, \dots, m, \quad i = 1, \dots, n\}.$$

Consider the rational function

$$\chi(t) = \frac{\prod_{p=1}^{m}(1 - y_p t)}{\prod_{i=1}^{n}(1 - x_i t)}$$

and expand it into Laurent series at zero and at infinity[3]

$$\chi(t) = \sum_{k=0}^{\infty} h_k t^k = \sum_{k=n-m}^{\infty} h_k^{\infty} t^{-k}.$$

Let us introduce

$$\Delta = \frac{y_1 \cdots y_m}{x_1 \cdots x_n}, \ \ \Delta^* = \frac{x_1 \cdots x_n}{y_1 \cdots y_m} = \Delta^{-1}, \ \ h_k^* = h_k(x_1^{-1}, \dots, x_n^{-1}, y_1^{-1}, \dots, y_m^{-1}).$$

It is easy to see that $h_k^{\infty} = \Delta h_{k+m-n}^*$. We define also h_k (and thus h_k^{∞}) for all $k \in \mathbb{Z}$ by assuming that $h_k \equiv 0$ for negative k.

Proposition 7.1. *The ring $J(\mathfrak{g})$ for the Lie superalgebra $\mathfrak{gl}(n, m)$ is a direct sum*

$$J(\mathfrak{g}) = \bigoplus_{a,b \in \mathbb{C}/\mathbb{Z}} J(\mathfrak{g})_{a,b},$$

[3]The importance of considering the Laurent series both at zero and infinity in this context was first understood by Khudaverdian and Voronov [15]. They used this to write down some interesting relations in the Grothendieck ring of finite-dimensional representations of $\mathrm{GL}(m, n)$.

where

$$J(\mathfrak{g})_{a,b} = (x_1 \cdots x_n)^a (y_1 \cdots y_m)^b \times \prod_{i,p} (1 - x_i/y_p)\, \mathbb{Z}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}, y_1^{\pm 1}, \ldots, y_m^{\pm 1}]^{S_n \times S_m}$$

if $a + b \notin \mathbb{Z}$;

$$J(\mathfrak{g})_{a,b} = (x_1 \cdots x_n)^a (y_1 \cdots y_m)^{-a} J(\mathfrak{g})_{0,0}$$

if $a + b \in \mathbb{Z}$, $\quad a \notin \mathbb{Z}$ *and*

$$J(\mathfrak{g})_{0,0} = \{f \in \mathbb{Z}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}, y_1^{\pm 1}, \ldots, y_m^{\pm 1}]^{S_n \times S_m} \mid y_p \frac{\partial f}{\partial y_p} + x_i \frac{\partial f}{\partial x_i} \in (y_p - x_i)\}.$$

The proof easily follows from the definition of $J(\mathfrak{g})$.

Proposition 7.2. *The subring $J(\mathfrak{g})_{0,0}$ is generated over $\mathbb{Z}$ by Δ, Δ^*, h_k, h_k^*, $k \in \mathbb{N}$ and can be interpreted as the Grothendieck ring of finite-dimensional representations of algebraic supergroup* $\mathrm{GL}(n,m)$.

Proof. We use the induction in $n + m$. When $n + m = 1$ it is obvious. Assume that $n + m > 1$. If $m = 0$ or $n = 0$ the statement follows from the theory of symmetric functions [16]. So we can assume that $n > 0$ and $m > 0$. Consider a homomorphism

$$\tau \colon J((\mathfrak{gl}(n,m))_{0,0} \longrightarrow J(\mathfrak{gl}(n-1, m-1))_{0,0}$$

such that $\tau(x_n) = \tau(y_m) = t$ and identical on others x_i and y_p. From the definition of the ring $J((\mathfrak{gl}(n,m))$ it follows that the image indeed belongs to $J(\mathfrak{gl}(n-1, m-1))$. By induction we may assume that $J(\mathfrak{gl}(n-1, m-1))_{0,0}$ is generated by Δ, Δ^* and h_k, h_k^* for $k = 1, 2, \ldots$. We have that

$$\tau(\Delta)(x_1, \ldots, x_{n-1}, t, y_1, \ldots, y_{m-1}, t) = \Delta(x_1, \ldots, x_{n-1}, y_1, \ldots, y_{m-1})$$

$$\tau(h_k)(x_1, \ldots, x_{n-1}, t, y_1, \ldots, y_{m-1}, t) = h_k(x_1, \ldots, x_{n-1}, y_1, \ldots, y_{m-1})$$

and the same for Δ^* and h_k^*, $k = 1, 2, \ldots$. Therefore homomorphism τ is surjective. So now we need only to prove that the kernel of τ is generated by $\Delta, \Delta^*, h_k, h_k^*$ for $k = 1, 2, \ldots$.

Let $a_0 = 1$, $a_i = (-1)^i \sigma_i(x)$, $i = 1, \ldots, n$, where σ_i are the elementary symmetric polynomials in $x_1, \ldots, x_n$. We have that

$$\prod_{j=1}^{m} (1 - y_j t) = \chi(t) \sum_{i=0}^{n} a_i t^i = \sum_{i=0}^{n} a_i t^i \sum_{k \in \mathbb{Z}} h_k t^k = \sum_{i=0}^{n} a_i t^i \sum_{k \in \mathbb{Z}} h_{-k}^{\infty} t^k.$$

We see that

$$\sum_{k \in \mathbb{Z}} \left(\sum_{i=0}^{n} h_{k-i} a_i \right) t^k = \sum_{k \in \mathbb{Z}} \left(\sum_{i=0}^{n} h_{-k+i}^{\infty} a_i \right) t^k,$$

so we have the following infinite system of linear equations (see Khudaverdian and Voronov [15]):

$$\sum_{i=0}^{n}(h_{k-i} - h^{\infty}_{i-k})a_i = 0, \quad k \in \mathbb{Z}.$$

Introducing the elements $\tilde{h}_k = h_k - h^{\infty}_{-k}$, we have

$$\sum_{i=0}^{n} \tilde{h}_{k+n-i}a_i = 0, \quad k = 0, \pm 1, \pm 2, \dots .$$

Considering this as a linear system for the unknown $a_1, \dots, a_n$ with given $a_0 = 1$, we have, by Cramer's rule for any pairwise different $k_1, \dots, k_n$, that

$$\begin{vmatrix} \tilde{h}_{k_1} & \tilde{h}_{k_1+1} & \dots & \tilde{h}_{k_1+n-1} \\ \tilde{h}_{k_2} & \tilde{h}_{k_2+1} & \dots & \tilde{h}_{k_2+n-1} \\ \vdots & \vdots & \ddots & \vdots \\ \tilde{h}_{k_n} & \tilde{h}_{k_n+1} & \dots & \tilde{h}_{k_n+n-1} \end{vmatrix} a_n = (-1)^n \begin{vmatrix} \tilde{h}_{k_1+1} & \tilde{h}_{k_1+2} & \dots & \tilde{h}_{k_1+n} \\ \tilde{h}_{k_2+1} & \tilde{h}_{k_2+2} & \dots & \tilde{h}_{k_2+n} \\ \vdots & \vdots & \ddots & \vdots \\ \tilde{h}_{k_n+1} & \tilde{h}_{k_n+2} & \dots & \tilde{h}_{k_n+n} \end{vmatrix}$$

and, more generally for any integer l, that

(18)
$$\begin{vmatrix} \tilde{h}_{k_1} & \tilde{h}_{k_1+1} & \dots & \tilde{h}_{k_1+n-1} \\ \tilde{h}_{k_2} & \tilde{h}_{k_2+1} & \dots & \tilde{h}_{k_2+n-1} \\ \vdots & \vdots & \ddots & \vdots \\ \tilde{h}_{k_n} & \tilde{h}_{k_n+1} & \dots & \tilde{h}_{k_n+n-1} \end{vmatrix} a_n^l = (-1)^{nl} \begin{vmatrix} \tilde{h}_{k_1+l} & \tilde{h}_{k_1+l+1} & \dots & \tilde{h}_{k_1+n+l-1} \\ \tilde{h}_{k_2+l} & \tilde{h}_{k_2+l+1} & \dots & \tilde{h}_{k_2+n+l-1} \\ \vdots & \vdots & \ddots & \vdots \\ \tilde{h}_{k_n+l} & \tilde{h}_{k_n+l+1} & \dots & \tilde{h}_{k_n+n+l-1} \end{vmatrix}.$$

Any element from kernel of τ has a form

$$f = R(x,y)g(x,y), \quad g \in \mathbb{Z}[x_1^{\pm 1}, \dots, x_n^{\pm 1}, y_1^{\pm 1}, \dots, y_m^{\pm 1}]^{S_n \times S_m},$$

where

$$R(x,y) = \prod_{i=1}^{n}\prod_{p=1}^{m}\left(1 - \frac{y_p}{x_i}\right).$$

Let $s_\lambda(x), s_\mu(y)$ be the Schur functions corresponding to the sequences of non-increasing integers $\lambda = (\lambda_1 \geq \dots \geq \lambda_n)$, $\mu = (\mu_1 \geq \dots \geq \mu_m)$ (see [16]). It is easy to see that the products $s_\lambda(x)s_\mu(y)$ give a basis in

$$\mathbb{Z}[x_1^{\pm 1}, \dots, x_n^{\pm 1}, y_1^{\pm 1}, \dots, y_m^{\pm 1}]^{S_n \times S_m}.$$

Thus we need to show that $f_{\lambda,\mu} = s_\lambda(x)s_\mu(y)R(x,y)$ can be expressed in terms of h_k, h_k^*, Δ, Δ^*. Multiplying $f_{\lambda,\mu}$ by an appropriate power of Δ we can assume that $f_{\lambda,\mu} = a_n^l s_\lambda(x)s_\mu(y)R(x,y)$, where l is an integer and λ, μ are partitions (i.e. λ_n and μ_m are nonnegative) such that $\lambda_n \geq m$. But in this case we can use the well-known formula (see e.g. [16, I.3, Ex. 23])

$$s_\lambda(x)s_\mu(y)R(x,y) = \begin{vmatrix} h_{\lambda_1} & h_{\lambda_1+1} & \dots & h_{\lambda_1+p+n-1} \\ h_{\lambda_2-1} & h_{\lambda_2} & \dots & h_{\lambda_2+p+n-2} \\ \vdots & \vdots & \ddots & \vdots \\ h_{\lambda_n-n+1} & h_{\lambda_n-n+2} & \dots & h_{\lambda_n+p} \\ h_{\mu'_1-n} & h_{\mu'_1-n+1} & \dots & h_{\mu'_1+p-1} \\ \vdots & \vdots & \ddots & \vdots \\ h_{\mu'_p-p-n+1} & h_{\mu'_p-p-n+2} & \dots & h_{\mu'_p} \end{vmatrix},$$

where $\mu'_1,\dots,\mu'_p$ be the partition conjugated to $\mu_1,\dots,\mu_m$. Since $\lambda_n \geq m$ for any h_k from the first n rows we have $h_k = \tilde{h}_k$. Let us multiply this equality by a_n^l and then expand the determinant with respect to the first n rows by Laplace's rule. Using (18) we get

$$f_{\lambda,\mu} = \begin{vmatrix} \tilde{h}_{\lambda_1+l} & \tilde{h}_{\lambda_1+l+1} & \dots & \tilde{h}_{\lambda_1+l+p+n-1} \\ \tilde{h}_{\lambda_2+l-1} & \tilde{h}_{\lambda_2+l} & \dots & \tilde{h}_{\lambda_2+l+p+n-2} \\ \vdots & \vdots & \ddots & \vdots \\ h_{\mu'_p-p-n+1} & h_{\mu'_p-p-n+2} & \dots & h_{\mu'_p} \end{vmatrix}.$$

Thus we have shown that $J(\mathfrak{g})_{0,0}$ is generated by Δ, Δ^*, h_k, h_k^*. Since all these elements are the supercharacters of some representations of the algebraic supergroup $\mathrm{GL}(n,m)$ (see e.g. [7]) we see that $J(\mathfrak{g})_{0,0}$ is a subring of the Grothendieck ring of this supergroup. Other elements of $J(\mathfrak{g})$ cannot be extended already to the algebraic subgroup $\mathrm{GL}(n)\times\mathrm{GL}(m)$, so $J(\mathfrak{g})_{0,0}$ coincides with the Grothendieck ring of $\mathrm{GL}(n,m)$. $\square$

Now we are going through the list of basic classical Lie superalgebras.

$A(n-1,m-1)$.

PROPOSITION 7.3. *The ring $J(\mathfrak{g})$ for the Lie superalgebra $\mathfrak{sl}(n,m)$ with $(n,m)\neq(2,2)$ is a direct sum $J(\mathfrak{g}) = \bigoplus_{a\in\mathbb{C}/\mathbb{Z}} J(\mathfrak{g})_a$,*

$$J(\mathfrak{g})_a = \{f \in (x_1\cdots x_n)^a \prod_{i,p}(1-x_i/y_p)\mathbb{Z}[x^{\pm1},y^{\pm1}]_0^{S_n\times S_m} \tag{19}$$

if $a\notin\mathbb{Z}$ and

$$J(\mathfrak{g})_0 = \{f\in\mathbb{Z}[x^{\pm1},y^{\pm1}]_0^{S_n\times S_m} \mid y_j\frac{\partial f}{\partial y_j} + x_i\frac{\partial f}{\partial x_i} \in (y_j - x_i)\}, \tag{20}$$

where $\mathbb{Z}[x^{\pm1},y^{\pm1}]_0^{S_n\times S_m}$ is the quotient of the ring

$$\mathbb{Z}[x_1^{\pm1},\dots,x_n^{\pm1},y_1^{\pm1},\dots,y_m^{\pm1}]^{S_n\times S_m}$$

by the ideal generated by $x_1\cdots x_n - y_1\cdots y_m$.

The subring $J(\mathfrak{g})_0$ is generated over $\mathbb{Z}$ by h_k, h_k^*, $k \in \mathbb{N}$ and can be interpreted as the Grothendieck ring of finite-dimensional representations of algebraic supergroup $\mathrm{SL}(n,m)$.

The first part easily follows from Proposition 7.1, the description of $J(\mathfrak{g})_0$ is based on Proposition 7.2. The case $m = n$ is special.

$A(n-1, n-1) = \mathfrak{psl}(n,n)$, $n > 2$. The root system of $A(n-1, n-1)$ is

$$R_0 = \{\tilde{\varepsilon}_i - \tilde{\varepsilon}_j, \tilde{\delta}_p - \tilde{\delta}_q : \ i \neq j \ 1 \leq i,j \leq n \ , p \neq q, \, 1 \leq p,q \leq n\}$$
$$R_1 = \{\pm(\tilde{\varepsilon}_i - \tilde{\delta}_p), \quad 1 \leq i \leq n, \, 1 \leq p \leq n\} = R_{\mathrm{iso}},$$

where

$$\tilde{\varepsilon}_1 + \cdots + \tilde{\varepsilon}_n = 0, \, \tilde{\delta}_1 + \cdots + \tilde{\delta}_n = 0.$$

These weights are related to the weights of $\mathfrak{sl}(n,n)$ by the formulas

$$\tilde{\varepsilon}_i = \varepsilon_i - \frac{1}{n}\sum_{j=1}^{n} \varepsilon_j, \quad \tilde{\delta}_i = \delta_i - \frac{1}{n}\sum_{j=1}^{n} \delta_j, \, i = 1, \dots, n.$$

The bilinear form is defined by the relations

$$(\tilde{\varepsilon}_i, \tilde{\varepsilon}_i) = 1 - 1/n, \, (\tilde{\varepsilon}_i, \tilde{\varepsilon}_j) = -1/n, \qquad i \neq j,$$
$$(\tilde{\delta}_p, \tilde{\delta}_p) = -1 + 1/n, \, (\tilde{\delta}_p, \tilde{\delta}_q) = 1/n, \qquad p \neq q, \, (\tilde{\varepsilon}_i, \tilde{\delta}_p) = 0.$$

The Weyl group $W_0 = S_n \times S_n$ acts on the weights by permuting separately $\tilde{\varepsilon}_i, \ i = 1, \dots, n$ and $\tilde{\delta}_p, \ p = 1, \dots, n$. A distinguished system of simple roots can be chosen as

$$B = \{\tilde{\varepsilon}_1 - \tilde{\varepsilon}_2, \dots, \tilde{\varepsilon}_{n-1} - \tilde{\varepsilon}_n, \tilde{\varepsilon}_n - \tilde{\delta}_1, \tilde{\delta}_1 - \tilde{\delta}_2, \dots, \tilde{\delta}_{n-1} - \tilde{\delta}_n\}.$$

The weight lattice of the Lie algebra $\mathfrak{g}_0$ is

$$P_0 = \left\{ \sum_{i=1}^{n-1} \lambda_i \tilde{\varepsilon}_i + \sum_{p=1}^{n-1} \mu_p \tilde{\delta}_p \mid \lambda_i, \mu_p \in \mathbb{Z} \right\}.$$

Proposition 7.4. *The ring $J(\mathfrak{g})$ for Lie superalgebra $\mathfrak{g} = \mathfrak{psl}(n,n)$ with $n > 2$ is a direct sum*

$$J(\mathfrak{g}) = \bigoplus_{i=0}^{n-1} J(\mathfrak{g})_i,$$

where for $i \neq 0$

$$J(\mathfrak{g})_i = \left\{ f = (x_1 \cdots x_n)^{\frac{i}{n}} \prod_{j,p}^{n} (1 - x_j/y_p) g, \, g \in \mathbb{Z}[x^{\pm 1}, y^{\pm 1}]_0^{S_n \times S_n}, \, \deg g = -i \right\},$$

and $J(\mathfrak{g})_0$ is the subring of (20) *with $m = n$, consisting of elements of degree* 0.

The ring $J(\mathfrak{g})_0$ is linearly generated by the products

$$h_1^{m_1} h_2^{m_2} \cdots (h_1^*)^{n_1} (h_2^*)^{n_2} \cdots$$

such that $m_1 + 2m_2 + \cdots = n_1 + 2n_2 + \cdots$ *and can be interpreted as the Grothendieck ring of finite-dimensional representations of the algebraic supergroup* $\mathrm{PSL}(n, n)$.

Proof. From the definition of the ring $J(A(n-1, n-1))$ it follows that this ring can be identified with the subring in $J(\mathfrak{sl}(n, n))$ consisting of the linear combinations of

$$e^{\lambda_1\varepsilon_1+\cdots+\lambda_n\varepsilon_n+\mu_1\delta_1+\cdots+\mu_n\delta_n}$$

such that $\lambda_1+\cdots+\lambda_n+\mu_1+\cdots+\mu_n = 0$. This subring can be also characterised as the ring of invariants with respect to the automorphism

$$\theta_t(x_i) = tx_i,\ \theta_t(y_i) = ty_i.$$

Now the proposition easy follows from these formulas and Proposition 7.3. $\square$

$C(n) = \mathfrak{osp}(2, 2n)$. In this case $\mathfrak{g}_0 = \mathfrak{so}(2)\oplus\mathrm{sp}(2n)$ and $\mathfrak{g}_1 = V_1\otimes V_2$, where V_1 and V_2 are the identical representations of $\mathrm{so}(2)$ and $\mathrm{sp}(2n)$ respectively.

Let $\varepsilon_1, \dots, \varepsilon_{n+1}$ be the weights of the identical representation of $C(n)$ and define $\varepsilon = \varepsilon_1$, $\delta_j = \varepsilon_{j+1}$, $1 \le j \le n$. The root system is

$$R_0 = \{\pm\delta_i \pm \delta_j,\ \pm 2\delta_i,\ i \ne j,\ 1 \le i, j \le n\}$$

$$R_1 = \{\pm\varepsilon \pm \delta_j,\ \pm\delta_j\},\quad R_{\mathrm{iso}} = \{\pm\varepsilon \pm \delta_j\},$$

with the bilinear form

$$(\varepsilon, \varepsilon) = 1,\ (\delta_i, \delta_i) = -1,\ (\delta_i, \delta_j) = 0,\ i \ne j,\ (\varepsilon, \delta_k) = 0.$$

The Weyl group W_0 is the semi-direct product of S_n and Z_2^n. It acts on the weights by permuting and changing the signs of $\delta_j, j = 1, \dots, n$. As a distinguished system of simple roots we select

$$B = \{\varepsilon - \delta_1, \delta_1 - \delta_2, \dots, \delta_{n-1} - \delta_n, 2\delta_n\}.$$

The weight group has the form

$$P_0 = \{\nu = \lambda\varepsilon + \sum_{j=1}^{n} \mu_j\delta_j,\ \lambda \in \mathbb{C},\ \mu_j \in \mathbb{Z}\}.$$

Let $e^{\varepsilon} = x$, $e^{\delta_j} = y_j$, $u = x + x^{-1}$, $v_j = y_j + y_j^{-1}$, $j = 1, \dots, n$. Consider the Taylor expansion at zero of the following rational function

$$\chi(t) = \frac{\prod_{j=1}^{m}(1 - y_j t)(1 - y_j^{-1} t)}{(1 - xt)(1 - x^{-1}t)} = \sum_{k=0}^{\infty} h_k t^k.$$

Proposition 7.5. *The ring* $J(\mathfrak{g})$ *for the Lie superalgebra* $C(n)$ *is a direct sum*

$$J(\mathfrak{g}) = \bigoplus_{a\in\mathbb{C}/\mathbb{Z}} J(\mathfrak{g})_a,$$

where

$$J(\mathfrak{g})_a = x^a \prod_{j=1}^{n}(1-x/y_j)(1-xy_j)\,\mathbb{Z}[x^{\pm 1}, y_1^{\pm 1}, \ldots, y_n^{\pm 1}]^{W_0}$$

if $a \notin \mathbb{Z}$ *and*

$$J(\mathfrak{g})_0 = \left\{ f \in \mathbb{Z}[x^{\pm 1}, y_1^{\pm 1}, \ldots, y_n^{\pm 1}]^{W_0} \mid y_j \frac{\partial f}{\partial y_j} + x\frac{\partial f}{\partial x} \in (y_j - x),\ j = 1, \ldots, n \right\}.$$

More explicitly, $J(\mathfrak{g})_0 = J(\mathfrak{g})_0^+ \oplus J(\mathfrak{g})_0^-$, *where*

$$J(\mathfrak{g})_0^- = \left\{ f = x\prod_{j=1}^{n}(u - v_j)g \mid\ g \in \mathbb{Z}[u, v_1, \ldots, v_n]^{S_n} \right\},$$

$$J(\mathfrak{g})_0^+ = \left\{ f \in \mathbb{Z}[u, v_1, \ldots, v_n]^{S_n} \mid u\frac{\partial f}{\partial u} + v_j \frac{\partial f}{\partial v_j} \in (u - v_j),\ j = 1, \ldots n \right\}.$$

The subring $J(\mathfrak{g})_0^+$ *is generated over* $\mathbb{Z}$ *by* h_k, $k \in \mathbb{N}$ *and can be interpreted as the Grothendieck ring of finite-dimensional representations of the algebraic supergroup* $\mathrm{OSP}(2, 2n)$.

Proof. The first claim is obvious. To prove the second one note that $x^2 - xu + 1 = 0$. Therefore any element f from $J(\mathfrak{g})_0$ can be uniquely written in the form $f_0 + xf_1$, where $f_0,\ f_1 \in \mathbb{Z}[u, v_1, \ldots, v_n]^{S_n}$. Condition $y_j \frac{\partial f}{\partial y_j} + x\frac{\partial f}{\partial x} \in (y_j - x)$ means that after substitution $y_j = x$ the polynomial $f = f_0 + xf_1$ does not depend on x. Because of the symmetry $y_j \to y_j^{-1}$ the same must be true for $f_0 + x^{-1}f_1$. This means that f_1 is zero after substitution $y_j = x$, which implies the claim.

The fact that $J(\mathfrak{g})_0^+$ is generated by h_k follows from the theory of supersymmetric functions [16].

Since h_k are the supercharacters of the symmetric powers of the standard representation all elements of $J(\mathfrak{g})_0^+$ give rise to representations of the supergroup $\mathrm{OSP}(2, 2n)$. The elements of $J(\mathfrak{g})_0^-$ cannot be extended already to the subgroup $O(2)$. □

$B(m, n) = \mathfrak{osp}(2m+1, 2n)$. Here $\mathfrak{g}_0 = \mathrm{so}(2m+1) \oplus \mathrm{sp}(2n)$ and $\mathfrak{g}_1 = V_1 \otimes V_2$, where V_1 and V_2 are the identical representations of $\mathrm{so}(2m+1)$ and $\mathrm{sp}(2n)$ respectively. Let $\pm\varepsilon_1, \ldots, \pm\varepsilon_m, \pm\delta_1, \ldots, \pm\delta_n$ be the nonzero weights of the identical representation of $B(m, n)$. Then the root system of $B(m, n)$ is

$$R_0 = \{\pm\varepsilon_i \pm \varepsilon_j,\ \pm\varepsilon_i,\ \pm\delta_p \pm \delta_q,\ \pm 2\delta_p,\ i \neq j,\ 1 \le i, j \le m\ , p \neq q,\ 1 \le p, q \le n\}$$

$$R_1 = \{\pm\varepsilon_i \pm \delta_p,\ \pm\delta_p\}, \quad R_{\mathrm{iso}} = \{\pm\varepsilon_i \pm \delta_p\}.$$

The invariant bilinear form is

$$(\varepsilon_i, \varepsilon_i) = 1,\ (\varepsilon_i, \varepsilon_j) = 0,\ i \neq j,\ (\delta_p, \delta_p) = -1,\ (\delta_p, \delta_q) = 0,\ p \neq q,\ (\varepsilon_i, \delta_p) = 0.$$

The Weyl group $W_0 = (S_n \ltimes \mathbb{Z}_2^n) \times (S_m \ltimes \mathbb{Z}_2^m)$ acts on the weights by separately permuting $\varepsilon_i,\ j = 1, \dots, m$ and $\delta_p,\ p = 1, \dots, n$, and changing their signs. The weight lattice of the Lie algebra $\mathfrak{g}_0$ is

$$P_0 = \left\{ \nu = \sum_{i=1}^{m} \lambda_i \varepsilon_i + \sum_{p=1}^{n} \mu_p \delta_p,\ \lambda_i \in \mathbb{Z} \text{ or } \lambda_i \in \mathbb{Z} + \frac{1}{2} \text{ for all } i,\ \ \mu_p \in \mathbb{Z} \right\}.$$

A distinguished system of simple roots can be chosen as

$$B = \{\delta_1 - \delta_2, \dots, \delta_{n-1} - \delta_n, \delta_n - \varepsilon_1, \varepsilon_1 - \varepsilon_2, \dots, \varepsilon_{m-1} - \varepsilon_m, \varepsilon_m\}.$$

The weight λ is a highest weight of $\mathfrak{g}_0$ if $\frac{2(\lambda,\alpha)}{(\alpha,\alpha)} \geq 0$ for any simple root of $\mathfrak{g}_0$

$$\alpha \in \{\delta_1 - \delta_2, \dots, \delta_{n-1} - \delta_n, 2\delta_n, \varepsilon_1 - \varepsilon_2, \dots, \varepsilon_{m-1} - \varepsilon_m, \varepsilon_m\}.$$

Introduce the variables $x_i = e^{\varepsilon_i}$, $x_i^{1/2} = e^{\varepsilon_i/2}$, $u_i = x_i + x_i^{-1}$, $i = 1, \dots, m$ and $y_p = e^{\delta_p}$, $v_p = y_p + y_p^{-1}$, $p = 1, \dots, n$. Consider the Taylor series at zero of the following function

$$\chi(t) = \frac{\prod_{p=1}^{n}(1 - y_p t)(1 - y_p^{-1} t)}{(1-t)\prod_{i=1}^{m}(1 - x_i t)(1 - x_i^{-1} t)} = \sum_{k=0}^{\infty} h_k(x,y) t^k.$$

Proposition 7.6. *The ring $J(\mathfrak{g})$ of Lie superalgebra of type $B(m,n)$ is a direct sum*

$$J(\mathfrak{g}) = J(\mathfrak{g})_0 \oplus J(\mathfrak{g})_{1/2},$$

where

$$J(\mathfrak{g})_{1/2} = \prod_{i=1}^{m}(x_i^{1/2} + x_i^{-1/2}) \prod_{i,p}(u_i - v_p) g \mid g \in \mathbb{Z}[u_1, \dots, u_m, v_1, \dots, v_n]^{S_m \times S_n}$$

and

$$J(\mathfrak{g})_0 = \left\{ f \in \mathbb{Z}[u_1, \dots, u_m, v_1, \dots, v_n]^{S_m \times S_n} \mid u_i \frac{\partial f}{\partial u_i} + v_p \frac{\partial f}{\partial v_p} \in (u_i - v_p) \right\}.$$

The subring $J(\mathfrak{g})_0$ is generated over $\mathbb{Z}$ by $h_k(x,y)$, $k \in \mathbb{Z}$ and can be interpreted as the Grothendieck ring of finite-dimensional representations of the algebraic supergroup $\mathrm{OSP}(2m+1, 2n)$.

Proof. The decomposition $J(\mathfrak{g}) = J(\mathfrak{g})_0 \oplus J(\mathfrak{g})_{1/2}$ reflects the fact that all λ_i in the weight lattice P_0 are either all integer or half-integers. Consider $f \in J(\mathfrak{g})$ and suppose first that all the corresponding λ_i are half integer. Write f as a Laurent polynomial with respect to x_1, y_1

$$f = \sum c_{i,j} x_1^i y_1^j,$$

where the coefficients $c_{i,j}$ depend on the remaining variables. The condition $x_1 \frac{\partial f}{\partial x_1} + y_1 \frac{\partial f}{\partial y_1} \in (x_1 - y_1)$ means that $\sum (i+j) c_{i,j} = 0$. Since i is not an integer but j is an integer, we conclude that $\sum c_{i,j} = 0$. This means that f is divisible by $(x_1 - y_1)$ and hence by the symmetry by $\prod_{i,p}(u_i - v_p)$. The factor

$\prod_{i=1}^m(x_i^{1/2}+x_i^{-1/2})$ is due to the Weyl group symmetry of $B(m)$. The last part is similar to the previous case. $\square$

$D(m,n) = \mathfrak{osp}(2m,2n)$, $m>1$. In this case $\mathfrak{g}_0 = \mathrm{so}(2m)\oplus\mathrm{sp}(2n)$ and $\mathfrak{g}_1 = V_1\otimes V_2$, where V_1 and V_2 are the identical representations of $\mathrm{so}(2m)$ and $\mathrm{sp}(2n)$ respectively. Let $\pm\varepsilon_1,\dots,\pm\varepsilon_m$, $\pm\delta_1,\dots,\pm\delta_n$ be the weights of the identical representation of $D(m,n)$. The root system is

$$R_0 = \{\pm\varepsilon_i\pm\varepsilon_j,\ \pm\delta_p\pm\delta_q,\ \pm2\delta_p,\ \ i\neq j,\ 1\le i,j\le m\ ,p\neq q,\ 1\le p,q\le n\}$$

$$R_1 = \{\pm\varepsilon_i\pm\delta_p\} = R_{\mathrm{iso}}.$$

The bilinear form is defined by the relations

$$(\varepsilon_i,\varepsilon_i)=1,\ (\varepsilon_i,\varepsilon_j)=0,\ i\neq j,\ (\delta_p,\delta_p)=-1,\ (\delta_p,\delta_q)=0,\ p\neq q,\ (\varepsilon_i,\delta_p)=0.$$

The Weyl group $W=\left(S_m\ltimes\mathbb{Z}_2^{m-1}\right)\times(S_n\ltimes\mathbb{Z}_2^n)$ acts on the weights by separately permuting $\varepsilon_i,\ i=1,\dots,m$ and $\delta_p,\ p=1,\dots,n$ and changing their signs such that the total change of signs of ε_i is even.

The weight lattice of Lie algebra $\mathfrak{g}_0$ is the same as in the previous case:

$$P_0=\left\{\nu=\sum_{i=1}^m\lambda_i\varepsilon_i+\sum_{p=1}^n\mu_p\delta_p,\ \lambda_i\in\mathbb{Z}\,\text{or}\,\lambda_i\in\mathbb{Z}+\frac{1}{2}\,\text{for all}\ i,\ \ \mu_p\in\mathbb{Z}\right\}.$$

A distinguished system of simple roots is

$$B=\{\delta_1-\delta_2,\dots,\delta_{n-1}-\delta_n,\delta_n-\varepsilon_1,\varepsilon_1-\varepsilon_2,\dots,\varepsilon_{m-1}-\varepsilon_m,\varepsilon_{m-1}+\varepsilon_m\}.$$

The weight ν is a highest weight for $\mathfrak{g}_0$ if $\frac{2(\nu,\alpha)}{(\alpha,\alpha)}\ge0$ for all simple roots of $\mathfrak{g}_0$

$$\alpha\in\{\delta_1-\delta_2,\dots,\delta_{n-1}-\delta_n,2\delta_n,\varepsilon_1-\varepsilon_2,\dots,\varepsilon_{m-1}-\varepsilon_m,\varepsilon_{m-1}+\varepsilon_m\},$$

which is equivalent to

$$\mu_1\ge\cdots\ge\mu_n\ge0,\ \lambda_1\ge\cdots\ge\lambda_{m-1}\ge|\lambda_m|.$$

Introduce the variables $x_i=e^{\varepsilon_i}$, $x_i^{1/2}=e^{\varepsilon_i/2}$, $u_i=x_i+x_i^{-1}$, $i=1,\dots,m$ and $y_p=e^{\delta_p}$, $v_p=y_p+y_p^{-1}$, $p=1,\dots,n$ and consider the following Taylor series

$$\chi(t)=\frac{\prod_{p=1}^n(1-y_pt)(1-y_j^{-1}t)}{\prod_{i=1}^m(1-x_it)(1-x_i^{-1}t)}=\sum_{k=0}^\infty h_k(x,y)t^k.$$

We will need also the following invariant of the Weyl group $D(m)$:

$$\omega=\sum x_1^{\pm1}\cdots x_m^{\pm1},$$

where the sum is over all possible combinations of ±1 with even sum.

Proposition 7.7. *The ring $J(\mathfrak{g})$ of Lie superalgebra of type $D(m,n)$ is a direct sum*

$$J(\mathfrak{g})=J(\mathfrak{g})_0\oplus J(\mathfrak{g})_{1/2},$$

where

$$J(\mathfrak{g})_{1/2} = \left\{ \prod_{i,p}(u_i - v_p) \left((x_1 \cdots x_m)^{1/2}\mathbb{Z}[x_1^{\pm 1}, \dots, x_m^{\pm 1}, y_1^{\pm 1}, \dots, y_n^{\pm 1}]\right)^{W_0} \right\}$$

and

$$J(\mathfrak{g})_0 = \left\{ f \in \mathbb{Z}[x_1^{\pm 1}, \dots, x_m^{\pm 1}, y_1^{\pm 1}, \dots, y_n^{\pm 1}]^{W_0} \mid y_p \frac{\partial f}{\partial y_p} + x_i \frac{\partial f}{\partial x_i} \in (y_p - x_i) \right\}.$$

More explicitly,

$$J(\mathfrak{g})_0 = J(\mathfrak{g})_0^+ \oplus J(\mathfrak{g})_0^-,$$

where

$$J(\mathfrak{g})_0^- = \left\{ \omega \prod_{i,p}(u_i - v_p)\mathbb{Z}[u_1, \dots, u_m, v_1, \dots, v_n]^{S_m \times S_n} \right\},$$

$$J(\mathfrak{g})_0^+ = \left\{ f \in \mathbb{Z}[u_1, \dots, u_m, v_1, \dots, v_n]^{S_m \times S_n} \mid u_i \frac{\partial f}{\partial u_i} + v_p \frac{\partial f}{\partial v_p} \in (u_i - v_p) \right\}.$$

The subring $J(\mathfrak{g})_0^+$ is generated over $\mathbb{Z}$ by $h_k(x, y)$, $k \in \mathbb{Z}$ and can be interpreted as the Grothendieck ring of finite-dimensional representations of algebraic supergroup $\mathrm{OSP}(2m, 2n)$.

Proof. The proof of the first claim is similar to the previous case. Let us explain the decomposition of $J(\mathfrak{g})_0$. It is well known (see e.g. [9]) that any element from $\mathbb{Z}[x_1^{\pm 1}, \dots, x_m^{\pm 1}, y_1^{\pm 1}, \dots, y_n^{\pm 1}]^{W_0}$ can be written uniquely in the form $f_0 + \omega f_1$, where $f_0, f_1 \in \mathbb{Z}[u_1, \dots, u_m, v_1, \dots, v_n]^{S_m \times S_n}$. The condition $y_1 \frac{\partial f}{\partial y_1} + x_1 \frac{\partial f}{\partial x_1} \in (x_1 - y_1)$ means that after the substitution $y_1 = x_1 = t$ the polynomial f does not depend on t. Because of the symmetry $y_1 \to y_1^{-1}$ the same must be true for $f_0 + \tau(\omega) f_1$, where the transformation τ maps x_1 to x_1^{-1} and leaves the remaining variables invariant. Therefore $(\omega - \tau(\omega)) f_1$ does not depend on t after the substitution $y_1 = x_1 = t$. This implies that f_1 is zero after this substitution, which explains the form of $J(\mathfrak{g})_0^-$. The last part is standard by now. □

$G(3)$. In this case $\mathfrak{g}_0 = G(2) \oplus \mathfrak{sl}(2)$ and $\mathfrak{g}_1 = U \otimes V$, where U is the first fundamental representation of $G(2)$ (see [17] or [4]), and V is the identity representation of $\mathfrak{sl}(2)$. Let $\pm\varepsilon_i, i = 1, 2, 3,\ \varepsilon_1 + \varepsilon_2 + \varepsilon_3 = 0$ be the nonzero weights of U and $\pm\delta$ be the weights of identity representation of $\mathfrak{sl}(2)$. Then the root system of $G(3)$ is

$$R_0 = \{\varepsilon_i - \varepsilon_j,\ \pm\varepsilon_i,\ \pm 2\delta\}, \quad R_1 = \{\pm\varepsilon_i \pm \delta,\ \pm\delta\}, \quad R_{\mathrm{iso}} = \{\pm\varepsilon_i \pm \delta\}$$

with the bilinear form defined by

$$(\varepsilon_i, \varepsilon_i) = 2 \quad (\varepsilon_i, \varepsilon_j) = -1,\ i \neq j, \quad (\delta, \delta) = -2.$$

The Weyl group $W_0 = D_6 \times \mathbb{Z}_2$, where D_6 is the dihedral group of order 12 acting on ε_i by permutations and simultaneously changing their signs, while

$\mathbb{Z}_2$ is acting by changing the sign of δ. The weight lattice of the Lie algebra $\mathfrak{g}_0$ can be written as

$$P_0 = \{\nu = \lambda_1\varepsilon_1 + \lambda_2\varepsilon_2 + \mu\delta,\ \lambda_1, \lambda_2, \mu \in \mathbb{Z}\}.$$

A distinguished system of simple roots is

$$B = \{\varepsilon_3 + \delta,\ \varepsilon_1,\ \varepsilon_2 - \varepsilon_1\}.$$

The weight λ is a highest weight for $\mathfrak{g}_0$ if $(\lambda, \alpha) \geq 0$ for $\alpha \in \{\varepsilon_1,\ \varepsilon_2 - \varepsilon_1, \delta\}$, which is equivalent to the following conditions $\lambda_1 \geq \lambda_2 - \lambda_1 \geq 0$. Let $x_1 = e^{\varepsilon_1}$, $x_2 = e^{\varepsilon_2}$, $y = e^{\delta}$. By definition the ring $J(\mathfrak{g})$ is

$$J(\mathfrak{g}) = \left\{f \in \mathbb{Z}[x_1^{\pm 1}, x_2^{\pm 1}, y^{\pm 1}]^{W_0} \mid x_1\frac{\partial f}{\partial x_1} + x_2\frac{\partial f}{\partial x_2} + 2y\frac{\partial f}{\partial y} \in (y - x_1x_2)\right\},$$

where the action of W_0 is generated by the permutation of x_1 and x_2 and by the transformations $x_1 \to (x_1x_2)^{-1}$, $x_2 \to x_2$ and $x_1 \to x_1^{-1}$, $x_2 \to x_2^{-1}$. Let $u_1 = x_1 + x_1^{-1}$, $u_2 = x_2 + x_2^{-1}$, $u_3 = x_1x_2 + x_1^{-1}x_2^{-1}$, $v = y + y^{-1}$ and introduce

$$w = v^2 - v(u_1 + u_2 + u_3 + 1) + u_1u_2 + u_1u_3 + u_2u_3,$$

which (up to additional constant 1) is the supercharacter of the adjoint representation, and hence belongs to $J(\mathfrak{g})$.

Proposition 7.8. *The ring $J(\mathfrak{g})$ of Lie superalgebra of type $G(3)$ can be described as*

$$J(\mathfrak{g}) = \{f = g(w) + (v - u_1)(v - u_2)(v - u_3)h \mid h \in \mathbb{Z}[u_1, u_2, u_3, v]^{S_3},\ g \in \mathbb{Z}[w]\}.$$

Proof. It is not difficult to verify that

$$\mathbb{Z}[x_1^{\pm 1}, x_2^{\pm 1}, y^{\pm 1}]^{\mathbb{Z}_2 \times \mathbb{Z}_2} = \mathbb{Z}[u_1, u_2, u_3, v],$$

where the generators of $\mathbb{Z}_2 \times \mathbb{Z}_2$ are acting by changing $y \to y^{-1}$ and $x_1 \to x_1^{-1}$, $x_2 \to x_2^{-1}$. For any $f \in J(\mathfrak{g})$ consider $q = f(x_1, x_2, x_1x_2)$, then we have

$$x_1\frac{\partial f}{\partial x_1} + x_2\frac{\partial f}{\partial x_2} + 2y\frac{\partial f}{\partial y} = x_1\frac{\partial q}{\partial x_1} + x_2\frac{\partial q}{\partial x_2} = 0$$

when $y = x_1x_2$. This means that q has degree 0. Since q is also invariant under the transformation $x_1 \to x_1^{-1}$, $x_2 \to x_2^{-1}$ there exists a polynomial of one variable g such that $q = g(\frac{x_1}{x_2} + \frac{x_2}{x_1})$. But it is easy to check that when $y = x_2x_3$ then $w = \frac{x_1}{x_2} + \frac{x_2}{x_1}$. Therefore the difference $f - g(w)$ is divisible by $(y - x_1x_2)$ and by the symmetry it is also divisible by

$$(y - x_1x_2)(y - x_1^{-1}x_2^{-1})(y - x_1)(y - x_1^{-1})(y - x_2)(y - x_2^{-1}) \\ = y^3(v - u_1)(v - u_2)(v - u_3).$$

A simple check shows that any polynomial of the form

$$(v - u_1)(v - u_2)(v - u_3)h,\ h \in \mathbb{Z}[u_1, u_2, u_3, v]^{S_3}$$

belongs to $J(\mathfrak{g})$. ∎

$F(4)$. In this case $\mathfrak{g}_0 = B_3 \oplus \mathfrak{sl}(2)$ and $\mathfrak{g}_1 = U \otimes V$, where U is the spin representation of B_3 (see [17] or [4]) and V is the identity representation of $\mathfrak{sl}(2)$. Let $\pm\varepsilon_1, \pm\varepsilon_2, \pm\varepsilon_3$ are the nonzero weights of the identity representation of B_3 and $\pm\frac{1}{2}\delta$ be the weights of identity representation of $\mathfrak{sl}(2)$. The root system of $\mathfrak{g}$ is

$$R_0 = \{\pm\varepsilon_i \pm \varepsilon_j,\ \pm\varepsilon_i\ \pm \delta\}, \quad R_1 = \left\{\frac{1}{2}(\pm\varepsilon_1 \pm \varepsilon_2 \pm \varepsilon_3 \pm \delta)\right\} = R_{\text{iso}}$$

with the bilinear form defined by

$$(\varepsilon_i, \varepsilon_i) = 1 \quad (\varepsilon_i, \varepsilon_j) = 0,\ i \neq j, \quad (\delta, \delta) = -3.$$

The Weyl group W_0 is $(S_3 \ltimes \mathbb{Z}_2^3) \times \mathbb{Z}_2$, where $S_3 \ltimes \mathbb{Z}_2^3$ acts on ε_i's by permutations and changing their signs while the second factor $\mathbb{Z}_2$ changes the sign of δ. The weight lattice of the Lie algebra $\mathfrak{g}_0$ is

$$P_0 = \{\nu = \lambda_1\varepsilon_1 + \lambda_2\varepsilon_2 + \lambda_3\varepsilon_3 + \mu\delta,\ \lambda_i \in \mathbb{Z} \text{ or } \lambda_i \in \mathbb{Z} + 1/2,\ 2\mu \in \mathbb{Z}\}.$$

As a distinguished system of simple root we choose

$$B = \left\{\frac{1}{2}(\delta - \varepsilon_1 - \varepsilon_2 - \varepsilon_3), \varepsilon_1 - \varepsilon_2, \varepsilon_2 - \varepsilon_3, \varepsilon_3\right\}.$$

The weight λ is a highest weight for $\mathfrak{g}_0$ if $\frac{(\nu,\alpha)}{(\alpha,\alpha)} \geq 0$ for $\alpha \in \{\varepsilon_1 - \varepsilon_2,\ \varepsilon_2 - \varepsilon_3,\ \varepsilon_3,\ \delta\}$, which is equivalent to the following conditions: $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq 0$, $\mu \geq 0$. Let $x_1 = e^{\frac{1}{2}\varepsilon_1}$, $x_2 = e^{\frac{1}{2}\varepsilon_2}$, $x_3 = e^{\frac{1}{2}\varepsilon_3}$, $y = e^{\frac{1}{2}\delta}$, $u_i = x_i + x_i^{-1} (i = 1, 2, 3)$, $v = y + y^{-1}$. By definition, the ring $J(\mathfrak{g})$ consists of polynomials $f \in \mathbb{Z}[x_1^{\pm 2}, x_2^{\pm 2}, x_3^{\pm 2}, (x_1x_2x_3)^{\pm 1}, y^{\pm 1}]^{W_0}$ such that

$$3y\frac{\partial f}{\partial y} + x_1\frac{\partial f}{\partial x_1} + x_2\frac{\partial f}{\partial x_2} + x_3\frac{\partial f}{\partial x_3} \in (y - x_1x_2x_3).$$

Introduce

$$Q = (v - x_1x_2x_3 - x_1^{-1}x_2^{-1}x_3^{-1})\prod_{i=1}^{3}\left(v - \frac{x_1x_2x_3}{x_i^2} - \frac{x_i^2}{x_1x_2x_3}\right)$$

and

$$w_k = \sum_{i\neq j}\frac{x_i^{2k}}{x_j^{2k}} + \sum_{i=1}^{3}(x_i^{2k} + x_i^{-2k}) + y^{2k} + y^{-2k} - (y^k + y^{-k})\prod_{i=1}^{3}(x_i^k + x_i^{-k}),\ k = 1, 2.$$

It is easy to check that Qh belongs to the ring $J(\mathfrak{g})$ for any polynomial h from $\mathbb{Z}[x_1^{\pm 2}, x_2^{\pm 2}, x_3^{\pm 2}, (x_1x_2x_3)^{\pm 1}, y^{\pm 1}]^{W_0}$. The element w_1 up to a constant is the supercharacter of the adjoint representation, w_2 can be expressed as a linear combination of the supercharacters of the tensor square of the adjoint representation and its second symmetric power, so both of them also belong to the ring.

Proposition 7.9. *The ring $J(\mathfrak{g})$ of Lie superalgebra of type $F(4)$ can be described as*

$$J(\mathfrak{g}) = \{f = g(w_1, w_2) + Qh \mid h \in \mathbb{Z}[x_1^{\pm 2}, x_2^{\pm 2}, x_3^{\pm 2}, (x_1x_2x_3)^{\pm 1}, y^{\pm 1}]^{W_0}, \\ g \in \mathbb{Z}[w_1, w_2]\}.$$

Proof. Let $f \in J(\mathfrak{g})$ and consider $q = f(x_1, x_2, x_3, x_1x_2x_3)$. We have

$$3y\frac{\partial f}{\partial y} + x_1\frac{\partial f}{\partial x_1} + x_2\frac{\partial f}{\partial x_2} + x_3\frac{\partial f}{\partial x_3} = x_1\frac{\partial g}{\partial x_1} + x_2\frac{\partial g}{\partial x_2} + x_3\frac{\partial g}{\partial x_3} = 0$$

when $y = x_1x_2x_3$. This means as before that q has degree 0 and therefore it is a Laurent polynomial in x_1^2, x_2^2, x_3^2. Since q is invariant under the transformations $x_i \to x_i^{-1}$ and the permutation group S_3 there exists a polynomial g of two variables such that $q = g(u_1, u_2)$, where

$$u_1 = \sum_{i \neq j} \frac{x_i^2}{x_j^2},\ u_2 = \sum_{i \neq j} \frac{x_i^4}{x_j^4}.$$

But it is easy to check that when $y = x_1x_2x_3$ we have $w_1 = u_1$, $w_2 = u_2$. Therefore the difference $f - g(w_1, w_2)$ is divisible by $(y - x_1x_2x_3)$ and by symmetry is divisible by Q. □

$D(2,1,\alpha)$. In this case $\mathfrak{g}_0 = \mathfrak{sl}(2) \oplus \mathfrak{sl}(2) \oplus \mathfrak{sl}(2)$, $\mathfrak{g}_1 = V_1 \otimes V_2 \otimes V_3$, where V_i are the identity representations of the corresponding $\mathfrak{sl}(2)$. Let $\pm\varepsilon_1, \pm\varepsilon_2, \pm\varepsilon_3$ be their weights. The root system of $\mathfrak{g}$ is

$$R_0 = \{\pm 2\varepsilon_1, \pm 2\varepsilon_2, \pm 2\varepsilon_3\} \quad R_1 = \{\pm\varepsilon_1 \pm \varepsilon_2 \pm \varepsilon_3\}, \quad R_{\text{iso}} = R_1.$$

The bilinear form depends on the parameter α:

$$(\varepsilon_1, \varepsilon_1) = -1 - \alpha,\ (\varepsilon_2, \varepsilon_2) = 1,\ (\varepsilon_3, \varepsilon_3) = \alpha.$$

The Weyl group $W_0 = \mathbb{Z}_2^3$ acts on the ε_i's by changing their signs. The weight lattice of the Lie algebra $\mathfrak{g}_0$ is

$$P_0 = \{\lambda = \lambda_1\varepsilon_1 + \lambda_2\varepsilon_2 + \lambda_3\varepsilon_3,\ \lambda_1, \lambda_2, \lambda_3 \in \mathbb{Z}\}.$$

Choose the following distinguished system of simple roots

$$B = \{\varepsilon_1 + \varepsilon_2 + \varepsilon_3,\ -2\varepsilon_2,\ -2\varepsilon_3\};$$

then the highest weights λ satisfy following conditions: $\lambda_1 \geq 0$, $\lambda_2 \leq 0$, $\lambda_3 \leq 0$.

Let $x_i = e^{\varepsilon_i}$, $u_i = x_i + x_i^{-1}$, $i = 1, 2, 3$. By definition we have

$$J(\mathfrak{g}) = \left\{ f \in \mathbb{Z}[x_1^{\pm 1}, x_2^{\pm 1}, x_3^{\pm 1}]^{\mathbb{Z}_2^3} \mid (1+\alpha)x_1\frac{\partial f}{\partial x_1} + x_2\frac{\partial f}{\partial x_2} + \alpha x_3\frac{\partial f}{\partial x_3} \in (x_1 - x_2x_3) \right\}.$$

Introduce

$$\begin{aligned} Q &= (x_1 - x_2x_3)(x_2 - x_1x_3)(x_3 - x_1x_2)(1 - x_1x_2x_3)x_1^{-2}x_2^{-2}x_3^{-2} \\ &= u_1^2 + u_2^2 + u_3^2 - u_1u_2u_3 - 4, \end{aligned}$$

which up to a constant is the supercharacter of the adjoint representation. For the rational nonzero values of the parameter $\alpha = p/q$, $p \in \mathbb{Z}$, $q \in \mathbb{N}$ we will need the additional element

$$w_\alpha = (x_1 + x_1^{-1} - x_2x_3 - x_2^{-1}x_3^{-1})\frac{(x_2^p - x_2^{-p})(x_3^q - x_3^{-q})}{(x_2 - x_2^{-1})(x_3 - x_3^{-1})} + x_2^p x_3^{-q} + x_2^{-p} x_3^q,$$

which also belongs to $J(\mathfrak{g})$ as one can check directly.

Proposition 7.10. *If α is not rational, then the ring $J(\mathfrak{g})$ of the Lie superalgebra $D(2,1,\alpha)$ can be described as follows*:

$$J(\mathfrak{g}) = \{f = c + Qh \mid c \in \mathbb{Z},\ h \in \mathbb{Z}[u_1, u_2, u_3]\}.$$

If $\alpha = p/q$ is rational, then

$$J(\mathfrak{g}) = \{f = g(w_\alpha) + Qh \mid h \in \mathbb{Z}[u_1, u_2, u_3],\ g \in \mathbb{Z}[w]\}.$$

Proof. First note that $\mathbb{Z}[x_1^{\pm1}, x_2^{\pm1}, x_3^{\pm1}]^{\mathbb{Z}_2^3} = \mathbb{Z}[u_1, u_2, u_3]$. Take $f \in J(\mathfrak{g})$ and consider the function $\phi(x_2, x_3) = f(x_2x_3, x_2, x_3)$; then

$$(1+\alpha)x_1\frac{\partial f}{\partial x_1} + x_2\frac{\partial f}{\partial x_2} + \alpha x_3\frac{\partial f}{\partial x_3} = x_2\frac{\partial \phi}{\partial x_2} + \alpha x_3\frac{\partial \phi}{\partial x_3} = 0$$

when $x_1 = x_2x_3$. If α is irrational, then ϕ must be a constant. If $\alpha = p/q$ is rational, then $\phi = g(x_2^p x_3^{-q} + x_2^{-p} x_3^q)$ for some polynomial $g \in \mathbb{Z}[w]$ since it is invariant under the transformation $x_2 \to x_2^{-1}, x_3 \to x_3^{-1}$. But, when $x_1 = x_2x_3$, the element $w_\alpha = x_2^p x_3^{-q} + x_2^{-p} x_3^q$. Therefore the difference $f - g(w_\alpha)$ is divisible by $(x_1 - x_2x_3)$ and by symmetry by Q. $\square$

8. **Special case** $A(1,1)$

This case is special because the isotropic roots have multiplicity 2. The definition of the ring $J(\mathfrak{g})$ should be modified in this case as follows:

$$J(\mathfrak{g}) = \{f \in \mathbb{Z}[P]^{W_0} : D_\alpha f \in ((e^\alpha - 1)^2) \quad \text{for any isotropic root } \alpha\}, \tag{21}$$

where $((e^\alpha - 1)^2)$ denotes the principal ideal in $\mathbb{Z}[P]$ generated by $(e^\alpha - 1)^2$. We would like to note that the property (21) can be rewritten as

$$D_\alpha \frac{1}{(e^\alpha - 1)} D_\alpha f \in (e^\alpha - 1),$$

which is similar to the condition proposed for the quantum Calogero-Moser systems by Chalykh and one of the authors in [8].

Theorem 8.1. *The Grothendieck ring $K(\mathfrak{g})$ of finite-dimensional representations of Lie superalgebra $\mathfrak{g} = A(1,1) = \mathfrak{psl}(2,2)$ is isomorphic to the ring $J(\mathfrak{g})$. The isomorphism is given by the supercharacter map* $\mathrm{Sch} : K(\mathfrak{g}) \to J(\mathfrak{g})$.

Now we are going to prove this result. We have in this case $\mathfrak{g}_0 = \mathfrak{sl}(2) \oplus \mathfrak{sl}(2)$, $\mathfrak{g}_1 = V_1 \otimes V_2 \oplus V_1 \otimes V_2$, where V_1, V_2 are the identity representations of the corresponding $\mathfrak{sl}(2)$. Let $\{\varepsilon, -\varepsilon, \delta, -\delta\}$ be the corresponding weights. The roots of $A(1,1)$ are

$$R_0 = \{2\varepsilon,\ -2\varepsilon,\ 2\delta,\ -2\delta\}$$
$$R_{\text{iso}} = R_1 = \{\varepsilon+\delta,\ \varepsilon-\delta,\ -\varepsilon+\delta,\ -\varepsilon-\delta\}$$

and the invariant bilinear form is

$$(\varepsilon,\varepsilon) = 1,\ (\delta,\delta) = -1,\ (\varepsilon,\delta) = 0.$$

The important fact is that the multiplicity of any isotropic root equals to two. Note that in this case R is not a generalized root system in the sense of the definition given in Section 3: one can check that the third property is not satisfied. However it is the case if we use a more general definition proposed by Serganova [19] and used in our previous work [25].

The weight lattice of the Lie algebra $\mathfrak{g}_0$ is

$$P_0 = \{\nu = \lambda\varepsilon + \mu\delta,\ \lambda, \mu \in \mathbb{Z}\}.$$

The Weyl group $W_0 = \mathbb{Z}_2 \times \mathbb{Z}_2$ is acting on the weights by changing the signs of ε and δ. A distinguished system of simple roots is

$$B = \{\varepsilon - \delta, 2\delta\}.$$

The weight $\nu = \lambda\varepsilon + \mu\delta$ is a highest weight for $\mathfrak{g}_0$ if $\lambda, \mu \geq 0$.

The following result generalizes the Proposition 4.3 to the case when multiplicities of the isotropic roots are equal to 2.

Proposition 8.2. *Let $\mathfrak{g}$ be the solvable Lie superalgebra such that $\mathfrak{g}_0 = \mathfrak{h}$ is a commutative finite-dimensional Lie algebra, $\mathfrak{g}_1 = \mathrm{Span}(X_1, X_2, Y_1, Y_2)$, and the following relations hold*:

$$[h, X_i] = \alpha(h)X_i, \quad [h, Y_i] = -\alpha(h)Y_i, \quad [Y_i, Y_j] = [X_i, X_j] = 0,$$
$$[X_i, Y_j] = \delta_{i,j}H, \quad i, j = 1, 2,$$

where $H \in \mathfrak{h}$ and $\alpha \neq 0$ is a linear form on $\mathfrak{h}$ such that $\alpha(H) = 0$. Then the Grothendieck ring of $\mathfrak{g}$ is isomorphic to

$$J(\mathfrak{g}) = \{f = \sum c_\lambda e^\lambda \mid \lambda \in \mathfrak{h}^*, \quad D_H f \in ((e^\alpha - 1)^2))\}. \tag{22}$$

The isomorphism is given by the supercharacter map $\mathrm{Sch} : [V] \longrightarrow \mathrm{sch}\, V$.

Proof. Every irreducible finite-dimensional $\mathfrak{g}$-module V has unique (up to a multiple) vector v such that $X_1 v = X_2 v = 0$, $hv = \lambda(h)v$ for some linear form λ on $\mathfrak{h}$. This establishes a bijection between the irreducible $\mathfrak{g}$-modules and the elements of $\mathfrak{h}^*$.

There are two types of such modules, depending on whether $\lambda(H) = 0$ or not. In the first case the module $V = V(\lambda)$ is one-dimensional and its supercharacter is e^{λ}. If $\lambda(H) \neq 0$, then the corresponding module $V(\lambda)$ is four-dimensional with the supercharacter $\mathrm{sch}(V) = e^{\lambda}(1-e^{-\alpha})^2$. In both cases the supercharacters belong to the ring $J(\mathfrak{g})$. Thus we have proved that the image of $\mathrm{Sch}(K(\mathfrak{g}))$ is contained in $J(\mathfrak{g})$.

Conversely, let $f = \sum c_\lambda e^\lambda$ belong to $J(\mathfrak{g})$. By subtracting a suitable linear combination of supercharacters of the one-dimensional modules $V(\lambda)$ we can assume that $\lambda(H) \neq 0$ for all λ from f. The condition $D_H f \in ((e^\alpha - 1)^2))$ implies that $D_H f \in (e^\alpha - 1)$. Using the same arguments as in the proof of Proposition 4.3, we deduce that f itself belongs to the ideal generated by $(e^\alpha - 1)$. This means that $f = (e^\alpha - 1)h$ for some $h \in \mathbb{Z}[\mathfrak{h}^*]$. It is easy to see that $D_H f \in ((e^\alpha - 1)^2)$ is equivalent to the condition $D_H h \in (e^\alpha - 1)$. Therefore as before $h \in (e^\alpha - 1)$, so $f \in ((e^\alpha - 1)^2)$. From the proof of the first part we conclude that f is a linear combination of the supercharacters of the irreducible $\mathfrak{g}$-modules. $\square$

Let $x = e^{\varepsilon}$, $y = e^{\delta}$, $u = x + x^{-1}$, $v = y + y^{-1}$. By definition we have

$$J(\mathfrak{g}) = \left\{ f \in \mathbb{Z}[x^{\pm 1}, y^{\pm 1}]^{\mathbb{Z}_2 \times \mathbb{Z}_2} \mid x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} \in ((x-y)^2) \right\}.$$

Proposition 8.3. *The ring $J(\mathfrak{g})$ of Lie superalgebra of type $A(1,1)$ can be described as*

$$J(\mathfrak{g}) = \{ f = c + (u-v)^2 g(u,v) \mid c \in \mathbb{Z},\ g \in \mathbb{Z}[u,v] \}.$$

The subring $J(\mathfrak{g})^+$ of polynomials of even degree in $J(\mathfrak{g})$ can be interpreted as the Grothendieck ring of finite-dimensional representations of algebraic supergroup $\mathrm{PSL}(2,2)$.

Proof. The isomorphism

$$\mathbb{Z}[x^{\pm 1}, y^{\pm 1}]^{\mathbb{Z}_2 \times \mathbb{Z}_2} = \mathbb{Z}[u,v]$$

is standard. Take any $f \in J(\mathfrak{g})$. We can write f in the form

$$f = c + (u-v)q(v) + (u-v)^2 g(u,v)$$

for some $c \in \mathbb{Z}$, $q \in \mathbb{Z}[v]$, $g \in \mathbb{Z}[u,v]$. From the identity $u - v = (x-y)(1-1/xy)$ it follows that $(u-v)^2 g(u,v) \in J(\mathfrak{g})$. Therefore $(u-v)q(v) \in J(\mathfrak{g})$. But it is easy to verify that in this case q must be zero.

Let us prove the statement about $\mathrm{PSL}(2,2)$. From the isomorphism $A(1,1) = \mathfrak{psl}(2,2)$ we have the natural imbedding

$$J(A(1,1)) = K(A(1,1)) \longrightarrow K(\mathfrak{sl}(2,2)) = J(\mathfrak{gl}(2,2))/I,$$

such that

$$u \to \left(\frac{x_1}{x_2}\right)^{\frac{1}{2}} + \left(\frac{x_2}{x_1}\right)^{\frac{1}{2}}, \ \ v \to \left(\frac{y_1}{y_2}\right)^{\frac{1}{2}} + \left(\frac{y_2}{y_1}\right)^{\frac{1}{2}},$$

and I is the ideal generated by $1 - e^{a(\varepsilon_1+\varepsilon_2-\delta_1-\delta_2)}$, $a \in \mathbb{C}$. In the same way as in Proposition 7.4, one can prove that the ring $K(\mathrm{PSL}(2,2))$ can be identified with the subring in $J(\mathfrak{sl}(2,2))$ linearly generated by $h_{i_1}\cdots h_{i_s}h^*_{j_1}\cdots h^*_{j_r}$ such that $i_1+\cdots+i_s = j_1+\cdots+j_r$. But it is not difficult to verify that this subring coincides with the image of $J^+(\mathfrak{g})$. The proposition is proved. □

Now the Theorem 8.1 follows from the fact that any polynomial of the form $(u-v)^2\chi_k(u)\chi_l(v)$, where $\chi_k(u)$, $\chi_l(v)$ are the characters of the irreducible $A(1)$-modules with the highest weights k and l, is the supercharacter of a Kac module over $A(1,1)$.

9. **Super Weyl groupoid**

In this section we associate to any generalized root system (in Serganova's sense) $R \subset V$ a certain groupoid $\mathfrak{W} = \mathfrak{W}(R)$, which we will call super Weyl groupoid.[4] The corresponding Grothendieck ring can be interpreted as the invariant ring of a natural action of this groupoid.

For a nice introduction to the theory of groupoids, including some history, we refer to the surveys by Brown [5] and Weinstein [30]. Recall that a *groupoid* can be defined as a small category with all morphisms being invertible. The set of objects is denoted as $\mathfrak{B}$ and is called the *base* while the set of morphisms is denoted as $\mathfrak{G}$. We will follow the common tradition to use the same notation $\mathfrak{G}$ for the groupoid itself.

If the base $\mathfrak{B}$ consists of one element, then $\mathfrak{G}$ has a group structure. More generally, for any $x \in \mathfrak{B}$ one can associate an *isotropy group* $\mathfrak{G}_x$ consisting of all morphisms $g \in \mathfrak{G}$ from x into itself. For any groupoid we have a natural equivalence relation on the base $\mathfrak{B}$, when $x \sim y$, if there exists a morphism $g \in \mathfrak{G}$ from x to y. One can think, therefore, of groupoids as generalisations of both groups and the equivalence relations. In fact, any finite groupoid is a disjoint union of its subgroupoids called *components*, corresponding to the equivalence classes called *orbits*. Each such component up to an isomorphism is uniquely determined by the orbit and its isotropy group (see [5]).

A standard example of groupoid comes from the action of a group Γ on a set X: the base $\mathfrak{B} = X$ and set $\mathfrak{G}$ of morphisms from x to y consists of the elements $\gamma \in \Gamma$ such that $\gamma(x) = y$.

One can generalize this example in the following way. Let $\mathfrak{G}$ be a groupoid and the group Γ is acting on it by the automorphisms of the corresponding

[4] We should note that the possibility of a groupoid version of the Weyl group for Lie superalgebras was contemplated by Serganova [21], but she had a different picture in mind (see [22]). Recently, Heckenberger and Yamane [11] introduced a groupoid related to basic classical Lie superalgebras motivated by Serganova's work and the notion of the Weyl groupoid for Nichols algebras [10]. Our super Weyl groupoid has no direct relation with these notions.

category. In particular, Γ acts on the base $\mathfrak{B}$ of $\mathfrak{G}$ (for convenience, on the right). Then one can define a *semi-direct product groupoid* $\Gamma \ltimes \mathfrak{G}$ with the same base $\mathfrak{B}$ and the morphisms from x to y being pairs $(\gamma,\, f)$, $\gamma \in \Gamma, f \in \mathfrak{G}$ such that $f\colon \gamma(x) \to y$. The composition is defined in a natural way: $(\gamma_1,\, f_1) \circ (\gamma_2,\, f_2) = (\gamma_1\gamma_2,\, \gamma_2(f_1) \circ f_2)$.

Now we are ready to define the super Weyl groupoid $\mathfrak{W}(R)$ corresponding to generalized root system R. Recall that the reflections with respect to the nonisotropic roots generate a finite group denoted W_0.

Consider first the following groupoid $\mathfrak{T}_{\mathrm{iso}}$ with the base R_{iso}, which is the set of all the isotropic roots in R. The set of morphisms from $\alpha \to \beta$ is nonempty if and only if $\beta = \pm\alpha$ in which case it consists of just one element. We will denote the corresponding morphism $\alpha \to -\alpha$ as $\tau_\alpha, \alpha \in R_{\mathrm{iso}}$. The group W_0 is acting on $\mathfrak{T}_{\mathrm{iso}}$ in a natural way: $\alpha \to w(\alpha)$, $\tau_\alpha \to \tau_{w(\alpha)}$. We define now the *super Weyl groupoid*

$$\mathfrak{W}(R) = W_0 \coprod W_0 \ltimes \mathfrak{T}_{\mathrm{iso}}$$

as a disjoint union of the group W_0 considered as a groupoid with a single point base $[W_0]$ and the semi-direct product groupoid $W_0 \ltimes \mathfrak{T}_{\mathrm{iso}}$ with the base R_{iso}. Note that the disjoint union is a well-defined operation on the groupoids.

There is a natural action of the groupoid $\mathfrak{W}(R)$ on the ambient space V of generalized root system R in the following sense.

For any set X one can define the following groupoid $\mathfrak{S}(X)$, whose base consists of all possible subsets $Y \subset X$ and the morphisms are all possible bijections between them. By the *action of a groupoid* $\mathfrak{G}$ *on a set* X we will mean the homomorphism of $\mathfrak{G}$ into $\mathfrak{S}(X)$ (which is a functor between the corresponding categories). In the case if $X = V$ is a vector space and $Y \subset X$ are the affine subspaces with morphisms being affine bijections, then we will talk about *affine action.*

Let $X = V$ and define the following affine action π of the super Weyl groupoid $\mathfrak{W}(R)$ on it. The base point $[W_0]$ maps to the whole space V, while the base element corresponding to an isotropic root α maps to the hyperplane Π_α defined by the equation $(\alpha, x) = 0$. The elements of the group W_0 are acting in a natural way and the element τ_α acts as a shift

$$\tau_\alpha(x) = x + \alpha,\; x \in \Pi_\alpha.$$

Note that since α is isotropic $x + \alpha$ also belongs to Π_α. One can easily check that this indeed defines an affine action of $\mathfrak{W}(R)$ on V.

A version of this action can be seen in the definition of the algebra $\Lambda_{R,B}$ of quantum integrals of the deformed Calogero-Moser systems introduced in our paper [25]: in that case the element τ_α acts as a shift between two different affine hyperplanes (see formula (7) in [25]). The above defintion of the super Weyl groupoid was mainly motivated by this action.

The following reformulation of our main theorem shows that the super Weyl groupoid may be considered as a substitute of the Weyl group in the theory of Lie superalgebras.

Let $V = \mathfrak{h}^*$ be the dual space to a Cartan subalgebra $\mathfrak{h}$ of a basic classical Lie superalgebra $\mathfrak{g}$ with generalized root system R. Using the invariant bilinear form we can identify V and $V^* = \mathfrak{h}$ and consider the elements of the group ring $\mathbb{Z}[\mathfrak{h}^*]$ as functions on V. A function f on V is *invariant under the action of groupoid* $\mathfrak{W}$ if for any $g \in \mathfrak{W}$ we have $f(g(x)) = f(x)$ for all x from the definition domain of the action map of g.

Let $P_0 \subset \mathfrak{h}^*$ be the abelian group of weights of $\mathfrak{g}_0$ and $\mathbb{Z}[P_0]$ be the corresponding integral group ring. It is easy to see that the ring $J(\mathfrak{g})$ is nothing else but the invariant elements of $\mathbb{Z}[P_0]$ invariant under the action π of the super Weyl groupoid described above. Thus our main result can be reformulated as follows.

THEOREM 9.1. *The Grothendieck ring $K(\mathfrak{g})$ of the finite-dimensional representations of a basic classical Lie superalgebra $\mathfrak{g}$ except $A(1,1)$ is isomorphic to the ring $\mathbb{Z}[P_0]^{\mathfrak{W}}$ of invariants of the super Weyl groupoid $\mathfrak{W}$ under the action defined above.*

10. **Concluding remarks**

Thus we have now a description of the Grothendieck rings of finite-dimensional representations for all basic classical Lie superalgebras. The fact that the corresponding rings can be described by simple algebraic conditions seems to be remarkable. We believe that these rings, as well as the corresponding super Weyl groupoids, will play important roles in representation theory.

An important problem is to describe "good" bases of the rings $K(\mathfrak{g})$ as modules over $\mathbb{Z}$ and transition matrices between them. For example, in the classical case of Lie algebra of type $A(n)$, we have various bases labeled by Young diagrams λ: Schur polynomials s_λ (or characters of the irreducible representations), symmetric functions h_λ and e_λ (see [16] for the details).

We also hope that our result could lead to a better understanding of the algorithms of computing the characters proposed by Serganova and Brundan (see [6], [18]). The investigation of the deformations of the Grothendieck rings, and the spectral decompositions of the corresponding analogues of the deformed Calogero-Moser and Macdonald operators [25], [26], may help to clarify the situation.

One can also define the Grothendieck ring $P(\mathfrak{g})$ of projective finite-dimensional $\mathfrak{g}$-modules (cf. Serre [28]). It can be shown that $P(\mathfrak{g}) \subset K(\mathfrak{g})$ is an ideal in the Grothendieck ring $K(\mathfrak{g})$. An interesting problem is to describe the structure of $P(\mathfrak{g})$ as a $K(\mathfrak{g})$-module.

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