Serre’s uniformity problem in the split Cartan case

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Abstract

We prove that there exists an integer \( p_0 \) such that \( X_{\text{split}}(p)(\mathbb{Q}) \) is made of cusps and CM-points for any prime \( p > p_0 \). Equivalently, for any non-CM elliptic curve \( E \) over \( \mathbb{Q} \) and any prime \( p > p_0 \) the image of \( \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \) by the representation induced by the Galois action on the \( p \)-division points of \( E \) is not contained in the normalizer of a split Cartan subgroup. This gives a partial answer to an old question of Serre.

1. Introduction

Let \( N \) be a positive integer and \( G \) a subgroup of \( \text{GL}_2(\mathbb{Z}/N\mathbb{Z}) \) such that \( \det G = (\mathbb{Z}/N\mathbb{Z})^\times \). Then the corresponding modular curve \( X_G \), defined as a complex curve as \( \tilde{\mathcal{H}}/\Gamma \), where \( \tilde{\mathcal{H}} \) is the extended Poincaré upper half-plane and \( \Gamma \) is the pullback of \( G \cap \text{SL}_2(\mathbb{Z}/N\mathbb{Z}) \) to \( \text{SL}_2(\mathbb{Z}) \), is actually defined over \( \mathbb{Q} \), that is, it has a geometrically integral \( \mathbb{Q} \)-model. As usual, we denote by \( Y_G \) the finite part of \( X_G \) (that is, \( X_G \) deprived of the cusps). The curve \( X_G \) has a natural (modular) model over \( \mathbb{Z} \) that we still denote by \( X_G \). The cusps define a closed subscheme of \( X_G \) over \( \mathbb{Z} \), and we define the relative curve \( Y_G \) over \( \mathbb{Z} \) as \( X_G \) deprived of the cusps. The set of integral points \( Y_G(\mathbb{Z}) \) consists of those \( P \in Y_G(\mathbb{Q}) \) for which \( j(P) \in \mathbb{Z} \), where \( j \) is, as usual, the modular invariant.

In the special case when \( G \) is the normalizer of a split (or nonsplit) Cartan subgroup of \( \text{GL}_2(\mathbb{Z}/N\mathbb{Z}) \), the curve \( X_G \) is denoted by \( X_{\text{split}}(N) \) (or \( X_{\text{nonsplit}}(N) \), respectively). In this article we focus more precisely on the case when \( G \) is the normalizer of a split Cartan subgroup of \( \text{GL}_2(\mathbb{Z}/p\mathbb{Z}) \) for \( p \) a prime number, that is, \( G \) is conjugate to the set of diagonal and anti-diagonal matrices mod \( p \), and we prove the following theorem.

**Theorem 1.1.** There exists an absolute effective constant \( C \) such that for any prime number \( p \) and any \( P \in Y_{\text{split}}(p)(\mathbb{Z}) \), \( \log |j(P)| \leq 2\pi p^{1/2} + 6 \log p + C \).

This is proved in Section 4, by a variation of the method of Runge after some preparation in Sections 2 and 3. The terms \( 2\pi p^{1/2} \) and \( 6 \log p \) seem to
be optimal for the method. The constant $C$ may probably be replaced by $o(1)$ when $p$ tends to infinity.

We apply Theorem 1.1 to the arithmetic of elliptic curves. Serre proved [23] that for any elliptic curve $E$ without complex multiplication (CM in the sequel), there exists $p_0(E) > 0$ such that for every prime $p > p_0(E)$ the natural Galois representation

$$\rho_{E,p} : \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \text{GL}(E[p]) \cong \text{GL}_2(\mathbb{Z}/p\mathbb{Z})$$

is surjective. Masser and Wüstholz [14], Kraus [10], and Pellarin [21] gave effective versions of Serre’s result; for more recent work, see, for instance, Cojocaru and Hall [6], [7].

Serre asked whether $p_0$ can be made independent of $E$:

**Does there exist an absolute constant $p_0$ such that for any non-CM elliptic curve $E$ over $\mathbb{Q}$ and any prime $p > p_0$ the Galois representation $\rho_{E,p}$ is surjective?**

We refer to this as “Serre’s uniformity problem”. The general guess is that $p_0 = 37$ would probably do.

The group $\text{GL}_2(\mathbb{Z}/p\mathbb{Z})$ has the following types of maximal proper subgroups: normalizers of (split and nonsplit) Cartan subgroups, Borel subgroups, and “exceptional” subgroups (those whose projective image is isomorphic to one of the groups $A_4$, $S_4$ or $A_5$). To solve Serre’s uniformity problem, one has to show that for sufficiently large $p$, the image of the Galois representation is not contained in any of the above listed maximal subgroups. (See [16, §2] for an excellent introduction into this topic.) Serre himself settled the case of exceptional subgroups (see the introduction of [15]), and the work of Mazur [17] on rational isogenies implies Serre uniformity for the Borel subgroups; so to solve Serre’s problem we are left with the Cartan cases. Equivalently, one would like to prove that, for large $p$, the only rational points of the modular curves $X_{\text{split}}(p)$ and $X_{\text{nonsplit}}(p)$ are the cusps and CM points, in which case we will say that the rational points are *trivial*.

In the present article we solve the split Cartan case of Serre’s problem.

**Theorem 1.2.** There exists an absolute constant $p_0$ such that for $p > p_0$ every point in $X_{\text{split}}(p)(\mathbb{Q})$ is either a CM point or a cusp.

In other words, for any non-CM elliptic curve $E$ over $\mathbb{Q}$ and any prime $p > p_0$ the image of the Galois representation $\rho_{E,p}$ is not contained in the normalizer of a split Cartan subgroup.

Several partial results in this direction were available before. In [20], [22] it was proved, by very different techniques, that $X_{\text{split}}(p)(\mathbb{Q})$ is trivial for a (large) positive density of primes; but the methods of loc. cit. have failed to prevent a complementary set of primes from escaping them. In [2] we allowed
ourselves to consider Cartan structures modulo higher powers of primes, and showed that, assuming the Generalized Riemann Hypothesis, $X_{\text{split}}(p^5)(\mathbb{Q})$ is trivial for large enough $p$.

Regarding possible generalizations, note that Runge’s method applies to the study of integral points on an affine curve $Y$, defined over $\mathbb{Q}$, if the following Runge condition is satisfied:

\[(R) \quad \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \text{ acts nontransitively on the set } X \setminus Y,\]

where $X$ is the projectivization of $Y$. The Runge condition is satisfied for the curve $X_{\text{split}}(p)$ because it has two Galois orbits of cusps over $\mathbb{Q}$. Runge’s method also applies to other modular curves such as $X_{\text{nonsplit}}(p)$, because all cusps of this curve are conjugate over $\mathbb{Q}$ and the Runge condition fails. Moreover, we need a weak version of Mazur’s method to obtain integrality of rational points, and this is believed not to apply to $X_{\text{nonsplit}}(p)$, because (the parity part of) the Birch and Swinnerton-Dyer conjecture predicts that the Jacobian of the latter curve has no nontrivial quotient of rank 0 over $\mathbb{Q}$; see [5] for more details. Actually, it is of interest that the Euler system constructed by Kato [9] to prove the triviality of the rank of Jacobian quotients in the modular cases relies on the same Siegel functions as those we use in Runge’s method; so it seems that both obstructions in applying our method to the nonsplit case come from the lack of sufficiently many Galois orbits of cusps over $\mathbb{Q}$. Several other applications of our techniques are however possible, and at present we work on applying Runge’s method to general modular curves over general number fields; see [2], [3]. For more on Runge’s method the reader may consult [4], [12].

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Convention. Everywhere in this article the $O(\cdot)$-notation, as well as the Vinogradov notation “$\ll$” implies absolute effective constants.

2. Siegel functions

As above, we denote by $\mathcal{H}$ the Poincaré upper half-plane and put $\mathcal{H} = \mathbb{H} \cup \mathbb{Q} \cup \{i\infty\}$. For $\tau \in \mathcal{H}$, as usual we put $q = q(\tau) = e^{2\pi i \tau}$. For a rational number $a$ we define $q^a = e^{2\pi i a \tau}$. Let $a = (a_1, a_2) \in \mathbb{Q}^2$ be such that $a \notin \mathbb{Z}^2$,
and let \( g_\mathbf{a} : \mathcal{H} \to \mathbb{C} \) be the corresponding Siegel function \([11, \S 2.1]\). Then we have the following infinite product presentation for \( g_\mathbf{a} \) \([11, \text{p. 29}]\):

\[
(1) \quad g_\mathbf{a}(\tau) = -q^{B_2(a_1)/2}e^{\pi i a_2(a_1-1)} \prod_{n=0}^{\infty} \left( 1 - q^{n+a_1}e^{2\pi i a_2} \right) \left( 1 - q^{n+1-a_1}e^{-2\pi i a_2} \right),
\]

where \( B_2(T) = T^2 - T + 1/6 \) is the second Bernoulli polynomial. We also have \([11, \text{pp. 27–30}]\) the relations

\[
(2) \quad g_\mathbf{a} \circ \gamma = g_{\mathbf{a}\gamma} \cdot (\text{a root of unity}) \quad \text{for} \quad \gamma \in \text{SL}_2(\mathbb{Z}),
\]

\[
(3) \quad g_\mathbf{a} = g_{\mathbf{a}'} \cdot (\text{a root of unity}) \quad \text{when} \quad \mathbf{a} \equiv \mathbf{a}' \mod \mathbb{Z}^2.
\]

Note that the root of unity in (2) is of order dividing 12, and in (3) of order dividing \( 2N \), where \( N \) is the denominator of \( \mathbf{a} \) (the common denominator of \( a_1 \) and \( a_2 \)). (For (2) use properties \( K_0 \) and \( K_1 \) of loc. cit., and for (3) use \( K_3 \) and the fact that \( \Delta \) is modular of weight 12.) Moreover,

\[
(4) \quad g_\mathbf{a} \circ \gamma = g_{\mathbf{a}} \cdot (\text{a root of unity}) \quad \text{for} \quad \gamma \in \Gamma(N),
\]

the root of unity being of order dividing \( 12N \), because \( g_{12N}\mathbf{a} \) is \( \Gamma(N) \)-automorphic, where, as above, \( N \) is the denominator of \( \mathbf{a} \). Since \( g_{\mathbf{a}} \) is holomorphic and does not vanish on the upper half-plane \( \mathcal{H} \) (again by Theorem 1.2 of loc. cit.), both \( g_{\mathbf{a}} \) and \( g_{\mathbf{a}}^{-1} \) must be integral over the ring \( \mathbb{C}[j] \). Actually, a stronger assertion holds.

**Proposition 2.1.** Assume that \( 0 \leq a_1 < 1 \). Then for \( \tau \in \mathcal{H} \) satisfying \( |q(\tau)| \leq 0.1 \),

\[
\log |g_\mathbf{a}(\tau)| = \frac{1}{2} B_2(a_1) \log |q| + \log \left| 1 - q^{a_1}e^{2\pi i a_2} \right| + \log \left| 1 - q^{1-a_1}e^{-2\pi i a_2} \right| + O(|q|)
\]

(where we recall that, throughout this article, the notation \( O(\cdot) \) as well as \( \ll \) imply absolute effective constants).

For \( \mathbf{a} \in \mathbb{Q}^2 \setminus \mathbb{Z}^2 \) the Siegel function \( g_{\mathbf{a}} \) is algebraic over the field \( \mathbb{C}(j) \). This again follows from the fact that \( g_{12N}\mathbf{a} \) is \( \Gamma(N) \)-automorphic, where, as above, \( N \) is the denominator of \( \mathbf{a} \). Since \( g_{\mathbf{a}} \) is holomorphic and does not vanish on the upper half-plane \( \mathcal{H} \) (again by Theorem 1.2 of loc. cit.), both \( g_{\mathbf{a}} \) and \( g_{\mathbf{a}}^{-1} \) must be integral over the ring \( \mathbb{C}[j] \). Actually, a stronger assertion holds.

**Proposition 2.2.** Both \( g_{\mathbf{a}} \) and \( (1-\zeta_N)g_{\mathbf{a}}^{-1} \) are integral over \( \mathbb{Z}[j] \). Here \( N \) is the denominator of \( \mathbf{a} \) and \( \zeta_N \) is a primitive \( N \)-th root of unity.

This is, essentially, established in \([11]\), but is not stated explicitly therein. Therefore we briefly indicate the proof here. A \( \Gamma(N) \)-automorphic function \( f : \mathcal{H} \to \mathbb{C} \) admits the infinite \( q \)-expansion

\[
(5) \quad f(\tau) = \sum_{k \in \mathbb{Z}} a_k q^{k/N}.
\]

We call the \( q \)-series (5) \emph{algebraic integral} if the following two conditions are satisfied: the negative part of (5) has only finitely many terms (that is, \( a_k = 0 \)
for large negative $k$), and the coefficients $a_k$ are algebraic integers. Algebraic integral $q$-series form a ring. The invertible elements of this ring are $q$-series with invertible leading coefficient. By the leading coefficient of an algebraic integral $q$-series we mean $a_m$, where $m \in \mathbb{Z}$ is defined by $a_m \neq 0$, but $a_k = 0$ for $k < m$.

**Lemma 2.3.** Let $f$ be a $\Gamma(N)$-automorphic function regular on $\mathcal{H}$ such that for every $\gamma \in \Gamma(1)$ the $q$-expansion of $f \circ \gamma$ is algebraic integral. Then $f$ is integral over $\mathbb{Z}[j]$.

**Proof.** This is, essentially, Lemma 2.1 from [11, §2.2]. Since $f$ is $\Gamma(N)$-automorphic, the set $\{f \circ \gamma : \gamma \in \Gamma(1)\}$ is finite. The coefficients of the polynomial $F(T) = \prod (T - f \circ \gamma)$ (where the product is taken over the finite set above) are $\Gamma(1)$-automorphic functions with algebraic integral $q$-expansions. Since they have no pole on $\mathcal{H}$, they belong to $\mathbb{C}[j]$ and even to $\mathbb{Z}[j]$, where $\mathbb{Z}$ is the ring of all algebraic integers, because the coefficients of their $q$-expansions are algebraic integers. It follows that $f$ is integral over $\mathbb{Z}[j]$, hence over $\mathbb{Z}[j]$. \[\square\]

**Proof of Proposition 2.2.** The function $g_{12N}^a$ is automorphic of level $N$ and its $q$-expansion is algebraic integral (as one can easily see by transforming the infinite product (1) into an infinite series). By (2), the same is true for every $(g_{a \circ \gamma})^{12N}$. Lemma 2.3 now implies that $g_{12N}^a$ is integral over $\mathbb{Z}[j]$, and so is $g_a$.

Further, the $q$-expansion of $g_a$ is invertible if $a_1 \notin \mathbb{Z}$ and is $1 - e^{\pm 2\pi ia_2}$ times an invertible $q$-series if $a_1 \in \mathbb{Z}$. Hence the $q$-expansion of $g_a^{-1}$ is algebraic integral when $a_1 \notin \mathbb{Z}$, and if $a_1 \in \mathbb{Z}$ the same is true for $(1 - e^{\pm 2\pi ia_2}) g_a^{-1}$.

In the latter case $N$ is the exact denominator of $a_2$, which implies that $(1 - \zeta_N)/(1 - e^{\pm 2\pi ia_2})$ is an algebraic unit. Hence, in any case, $(1 - \zeta_N) g_a^{-1}$ has algebraic integral $q$-expansion, and the same is true with $g_a$ replaced by $g_{a \circ \gamma}$ for any $\gamma \in \Gamma(1)$. (We again use (2) and notice that $a$ and $a\gamma$ have the same order in $\mathbb{Q}/\mathbb{Z}$.) Applying Lemma 2.3 to the function $((1 - \zeta_N) g_a^{-1})^{12N}$, we complete the proof. \[\square\]

### 3. A modular unit

In this section we define a special “modular unit” (in the spirit of [11]) and study its asymptotic behavior at infinity. With the common abuse of speech, the modular invariant $j$, as well as the other modular functions used below, may be viewed, depending on the context, as either automorphic functions on the Poincaré upper half-plane, or rational functions on the corresponding modular curves.

Since the root of unity in (3) is of order dividing $2N$, where $N$ is a denominator of $a$, the function $g_{12N}^a$ will be well-defined if we select $a$ in the...
set \((N^{-1}\mathbb{Z}/\mathbb{Z})^2\). Thus, fix a positive integer \(N\) and for a nonzero element \(a\) of \((N^{-1}\mathbb{Z}/\mathbb{Z})^2\) put \(u_a = g_a^{12N}\). After fixing a choice for \(\zeta_N\) in \(\mathbb{C}\) (for instance \(\zeta_N = e^{2i\pi/N}\)), we see that the analytic modular curve \(X(N)(\mathbb{C}) := \overline{\mathcal{H}}/\Gamma(N)\) has a modular model over \(\mathbb{Q}(\zeta_N)\), parametrizing isomorphism classes of generalized elliptic curves endowed with a basis \((S,T)\) such that the Weil pairing of \(S\) with \(T\) is \(\zeta_N\). As already noticed, the function \(u_a\) is \(\Gamma(N)\)-automorphic and hence defines a rational function on the modular curve \(X(N)(\mathbb{C})\); in fact, it belongs to the field \(\mathbb{Q}(\zeta_N)(X(N))\). The Galois group of the latter field over \(\mathbb{Q}(j)\) is isomorphic to \(\text{GL}_2(\mathbb{Z}/N\mathbb{Z})/\{\pm 1\}\), and we may identify the two groups to make the Galois action compatible with the natural action of \(\text{GL}_2(\mathbb{Z}/N\mathbb{Z})\) on \((N^{-1}\mathbb{Z}/\mathbb{Z})^2\) in the following sense: for any \(\sigma \in \text{Gal}(\mathbb{Q}(X(N))/\mathbb{Q}(j)) = \text{GL}_2(\mathbb{Z}/N\mathbb{Z})/\{\pm 1\}\) and any nonzero \(a \in (N^{-1}\mathbb{Z}/\mathbb{Z})^2\) we have \(u_{\sigma a} = u_{a}\), where \(\sigma \in \text{GL}_2(\mathbb{Z}/N\mathbb{Z})\) is a pullback of \(\sigma\). Notice that \(u_a = u_{-a}\), which follows from (2). For the proof of the statements above the reader may consult [11, pp. 31–36], and especially Theorem 1.2, Proposition 1.3 and the beginning of Section 2.2 therein.

From now on we assume that \(N = p \geq 3\) is an odd prime number, and that \(G\) is the normalizer of the diagonal subgroup of \(\text{GL}_2(\mathbb{F}_p)\). In this case the curve \(X_G = X_{\text{split}}(p)\) has two Galois orbits of cusps over \(\mathbb{Q}\), the first being the cusp at infinity, which is \(\mathbb{Q}\)-rational (we denote it by \(\infty\)), and the second consisting of the \((p-1)/2\) other cusps (denoted by \(P_1, \ldots, P_{(p-1)/2}\)), which are defined over the real cyclotomic field \(\mathbb{Q}(\zeta_p)^+\). According to the theorem of Manin-Drinfeld, there exists \(U \in \mathbb{Q}(X_G)\) such that the principal divisor \((U)\) is of the form

\[
m((p-1)/2 \cdot \infty - (P_1 + \cdots + P_{(p-1)/2}))
\]

with some positive integer \(m\). Below we use Siegel functions to find such \(U\) explicitly with \(m = 2p(p-1)\). See Remark 3.4 for a more precise statement.

**Remark 3.1.** (a) The general form of units we build is more ripe for generalization, but in the present case, using the \(\mathbb{Q}\)-isomorphism between \(X_{\text{split}}(p)\) and \(X_0(p^2)/w_p\), our unit could probably be expressed in terms of \((\text{products of})\) modular forms of shape \(\Delta(nz)\).

(b) The assumption that \(p \geq 3\) is purely technical: the content of this section extends, with insignificant changes, to \(p = 2\).

Denote by \(p^{-1}\mathbb{F}_p^\times\) the set of nonzero elements of \(p^{-1}\mathbb{Z}/\mathbb{Z}\). Then the set

\[
A = \{(a, 0) : a \in p^{-1}\mathbb{F}_p^\times\} \cup \{(0, a) : a \in p^{-1}\mathbb{F}_p^\times\}
\]

is \(G\)-invariant. Hence the function

\[
U = \prod_{a \in A} u_a
\]
belongs to the field \( \mathbb{Q}(X_G) \). In particular, viewed as a function on \( \mathcal{H} \), it is \( \Gamma \)-automorphic, where \( \Gamma \) is the pullback to \( \Gamma(1) \) of \( G \cap \text{SL}_2(\mathbb{F}_p) \).

More generally, for \( c \in \mathbb{Z} \) put
\[
\beta_c = \begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix}, \quad U_c = U \circ \beta_c = \prod_{a \in A\beta_c} u_a
\]
(so that \( U = U_0 \)).

Let \( D \) be the familiar fundamental domain of \( \text{SL}_2(\mathbb{Z}) \); that is, the hyperbolic triangle with vertices \( e^{\pi i/3}, e^{2\pi i/3} \) and \( i\infty \), together with the geodesic segments \([i, e^{2\pi i/3}]\) and \([e^{2\pi i/3}, i\infty]\). Let \( D + \mathbb{Z} \) be the union of all translates of \( D \) by the rational integers. Recall also that \( j \) denotes the modular invariant.

**Lemma 3.2.** For any \( P \in Y_1(\mathbb{C}) \) there exists \( c \in \mathbb{Z} \) (even \( c \in \{0, \ldots, (p-1)/2\} \)) and \( \tau \in D + \mathbb{Z} \) such that \( j(\tau) = j(P) \) and \( U_c(\tau) = U(\tau) \).

**Proof.** Let \( \tau' \in \mathcal{H} \) be such that \( j(\tau') = j(P) \) and \( U(\tau') = U(P) \). There exists \( \beta \in \Gamma(1) \) such that \( \beta^{-1}(\tau') \in D \). Now observe that the set \( \{\beta_0, \ldots, \beta_{(p-1)/2}\} \) is a full system of representatives of the double cosets \( \Gamma \setminus \Gamma(1)/\Gamma_{\infty} \), where \( \Gamma_{\infty} \) is the subgroup of \( \Gamma(1) \) stabilizing \( \infty \). Thus we may write \( \beta = \gamma_0 \beta_1 \kappa \) with \( \gamma \in \Gamma \), \( c \in \{0, \ldots, [p/2]\} \) and \( \kappa \in \Gamma_{\infty} \). Then \( \tau = \kappa \beta^{-1}(\tau') \) is as desired. \( \square \)

**Proposition 3.3.** For \( \tau \in \mathcal{H} \) such that \( |q(\tau)| \leq 1/p \),

\[(6) \quad \bigg| \log |U_c(\tau)| - (p-1)^2 \log |q(\tau)| \bigg| \leq 4\pi^2 \frac{p^2}{\log |q(\tau)| - 1} + O(p \log p) \]

if \( p \nmid c \), and

\[(7) \quad \bigg| \log |U_c(\tau)| + 2(p-1) \log |q(\tau)| \bigg| \leq 8\pi^2 \frac{p^2}{\log |q(\tau)| - 1} + O(p) \]

if \( p \nmid c \).

**Remark 3.4.** As suggested by the referee, it might perhaps be illuminating to re-state this proposition not in terms of \( U_c \) and \( q \), but in terms of the original function \( U \) and the “\( q \)-parameter” \( q_c = q \circ \beta_c^{-1} \) at the cusp \( \beta_c(\infty) \). From this point of view (which is systematically taken in [3]) the proposition means that \( U \) behaves like \( q_c^{(p-1)/2} \) near the cusp at infinity and like \( q_c^{-2(p-1)} \) near the other cusps. Since \( q_c^{1/p} \) is a uniformizer at the cusp \( \beta_c(\infty) \), this implies, in particular, that the principal divisor \( (U) \) is \( m((p-1)/2 \cdot \infty - (P_1 + \cdots + P_{(p-1)/2})) \) with \( m = 2p(p-1) \), as indicated above.

For the proof of Proposition 3.3 we need an elementary, but crucial lemma.
Lemma 3.5. Let $z$ be a complex number, $|z| < 1$, and $N$ a positive integer. Then

\[ \left| \sum_{k=1}^{N} \log |1 - z^k| \right| \leq \frac{\pi^2}{6} \frac{1}{\log |z|} + O(1). \]

Proof. We have $|\log |1 + z|| \leq -\log |1 - |z||$ for $|z| < 1$. Applying this with $-z^k$ instead of $z$, we conclude that it suffices to bound $-\sum_{k=1}^{\infty} \log |1 - q^k|$ with $q = |z|$. Since the left-hand side of (8) is bounded (independently of $N$) for $|z| \leq 1/2$, we may assume that

\[ \frac{1}{2} \leq q < 1. \]

Put $\tau = \log q/(2\pi i)$. Then

\[ -\sum_{k=1}^{\infty} \log |1 - q^k| = \frac{1}{24} \log q - \log |\eta(\tau)|, \]

where $\eta(\tau)$ is the Dedekind $\eta$-function. Since $|\eta(\tau)| = |\tau|^{-1/2}|\eta(-\tau^{-1})|$, we have

\[ -\sum_{k=1}^{\infty} \log |1 - q^k| = -\frac{1}{24} \log |Q| + \frac{1}{24} \log q + \frac{1}{2} \log |\tau| - \sum_{k=1}^{\infty} \log |1 - Q^k| \]

with $Q = e^{-2\pi i \tau^{-1}} = e^{4\pi^2/\log q}$. The first term on the right-hand side of (10) is exactly $(\pi^2/6)/\log |z|$, the second term is negative, the third term is again negative (here we use (9)), and the infinite sum is $O(1)$, again by (9). The lemma is proved. \(\square\)

Proof of Proposition 3.3. Write $q = q(\tau)$. Recall that for a rational number $\alpha$ we define $q^\alpha = e^{2\pi i \alpha \tau}$. For $a \in \mathbb{Q}/\mathbb{Z}$ we denote by $\tilde{a}$ the lifting of $a$ to the interval $[0, 1)$. Then for $\tau \in \mathcal{H}$ satisfying $|q| \leq 0.1$ we deduce from Proposition 2.1 that

\[ \log |U_c(\tau)| = 6p \sum_{a \in A_{\beta_c}} B_2(\tilde{a}_1) \log |q| + 12p \sum_{a \in A_{\beta_c}} \left( \log |1 - q^{\tilde{a}_1} e^{2\pi i a^2}| + \log |1 - q^{1-\tilde{a}_1} e^{-2\pi i a^2}| \right) + O(p^2 |q|). \]

The rest of the proof splits into two cases and relies on the identity

\[ \sum_{k=1}^{N-1} B_2 \left( \frac{k}{N} \right) = -\frac{(N-1)}{6N}. \]
The first case: \( p \mid c \). In this case \( A \beta = A \). Hence

\[
(12) \quad \sum_{a \in A \beta} B_2(\tilde{a}_1) = \sum_{k=1}^{p-1} B_1 \left( \frac{k}{p} \right) + (p-1)B_2(0) = \frac{(p-1)^2}{6p}.
\]

Further,

\[
(13) \quad \sum_{a \in A \beta} \left( \log|1 - q^{\tilde{a}_1}e^{2\pi ia_2}| + \log|1 - q^{1-\tilde{a}_1}e^{-2\pi ia_2}| \right)
\]

\[= 2 \sum_{k=1}^{p-1} \log|1 - q^{k/p}| + \log \left| \frac{1 - q}{1 - q} \right| + \log p.
\]

Lemma 3.5 with \( z = q^{1/p} \) implies that

\[
\sum_{k=1}^{p-1} \log|1 - q^{k/p}| \leq \frac{\pi^2}{6} \log \left| \frac{p}{q - 1} \right| + O(1).
\]

Also, \( \log|1 - q| \ll |q|^p \) and \( \log|1 - q| \ll |q| \). Combining all this with (11), (12) and (13), we obtain (6).

The second case: \( p \nmid c \). In this case

\[
A \beta = \{(a, 0) : a \in p^{-1}F_p^\times \} \cup \{(a, ab) : a \in p^{-1}F_p^\times \},
\]

where \( b \in \mathbb{Z} \) satisfies \( bc \equiv 1 \mod p \). Hence

\[
\sum_{a \in A \beta} B_2(\tilde{a}_1) = 2 \sum_{k=1}^{p-1} B_2 \left( \frac{k}{p} \right) = -\frac{p-1}{3p}.
\]

Further,

\[
\sum_{a \in A \beta} \left( \log|1 - q^{\tilde{a}_1}e^{2\pi ia_2}| + \log|1 - q^{1-\tilde{a}_1}e^{-2\pi ia_2}| \right)
\]

\[= 2 \sum_{k=1}^{p-1} \log|1 - q^{k/p}| + 2 \sum_{k=1}^{p-1} \log \left| 1 - (q^{1/p}e^{2\pi ib/p})^k \right|.
\]

Again using Lemma 3.5, we complete the proof. \( \square \)

4. Proof of Theorem 1.1

In this section \( p \) is a prime number and \( G \) is the normalizer of the diagonal subgroup of \( GL_2(\mathbb{Z}/p\mathbb{Z}) \). Define the "modular units" \( U_c \) as in Section 3. Recall that \( U = U_0 \) belongs to the field \( \mathbb{Q}(X_G) \). Theorem 1.1 is a consequence of the following two statements.
**Proposition 4.1.** Assume that \( p \geq 3 \). For any \( P \in Y_G(\mathbb{C}) \) we have either
\[
\log |j(P)| \leq 2\pi p^{1/2} + 6 \log p + O(1)
\]
or
\[
\log |j(P)| \leq \frac{1}{2(p-1)} \left| \log |U(P)| \right| + 2\pi p^{1/2} - 6 \log p + O(1).
\]

**Proposition 4.2.** For \( P \in Y_G(\mathbb{Z}) \) we have
\[
0 \leq \log |U(P)| \leq 24 \pi \log p.
\]

Combining the two propositions, we find that for \( P \in Y_{\text{split}}(p)(\mathbb{Z}) \) we have
\[
\log |j(P)| \leq 2\pi p^{1/2} + 6 \log p + O(1),
\]
which proves Theorem 1.1 for \( p \geq 3 \).

A similar approach can be used for \( p = 2 \) as well, but in this case it is easier to appeal to the general Runge theorem: If an affine curve \( Y \), defined over \( \mathbb{Q} \), has 2 (or more) rational points at infinity, then integral points on \( Y \) are effectively bounded; see, for instance, [4], [12].

**Proof of Proposition 4.1.** According to Lemma 3.2, there exist \( \tau \in D + \mathbb{Z} \) and \( c \in \mathbb{Z} \) with \( U_c(\tau) = U(P) \) and \( j(\tau) = j(P) \). (As in Remark 3.4, one may say here that \( P \) is “close” to the cusp \( \beta_c(\infty) \) with respect to the archimedean metric on our curve.) We write \( q = q(\tau) \). Since \( \tau \in D + \mathbb{Z} \), we have
\[
j(\tau) = q^{-1} + O(1),
\]
which implies that either \( \log |j(P)| \leq 2\pi p^{1/2} + 6 \log p + O(1) \) or \( \log |q^{-1}| \geq 2\pi p^{1/2} + 6 \log p \). In the latter case we apply Proposition 3.3. When \( p \nmid c \) it yields
\[
\left| \log |q| + \frac{1}{2(p-1)} \log |U_c(\tau)| \right| \leq \frac{8\pi^2 p^2}{2(p-1)(2\pi p^{1/2} + 6 \log p)} + O(1)
\]
\[
= 2\pi p^{1/2} - 6 \log p + O(1),
\]
which, together with (15), implies the result. In the case \( p \mid c \) Proposition 3.3 gives
\[
\left| \log |q| - \frac{1}{(p-1)^2} \log |U_c(\tau)| \right| \leq \frac{4\pi^2 p^2}{(p-1)^2(2\pi p^{1/2} + 6 \log p)} + O(1) = O(1),
\]
which implies an even better bound than needed.

**Proof of Proposition 4.2.** Since \( U \) belongs to \( \mathbb{Q}(X_G) \) and has no pole or zero outside the cusps, \( U(P) \) is a nonzero rational number. Let \( \zeta = \zeta_p \) be a primitive \( p \)-th root of unity. Since \( U \) is a product of \( 24p(p-1) \) Siegel functions, Proposition 2.2 implies that both \( U \) and \( (1 - \zeta)^{24p(p-1)}U^{-1} \) are integral over \( \mathbb{Z}[j] \). Hence for \( P \in Y_G(\mathbb{Z}) \) both the numbers \( U(P) \) and \( (1 - \zeta)^{24p(p-1)}U(P)^{-1} \) are algebraic integers. Since \( U(P) \in \mathbb{Q}^* \), it is a nonzero rational integer; in
particular, \( \log |U(P)| \geq 0 \). Further, \( U(P) \) divides \( (1 - \zeta)^{24p(p-1)} \). Taking the \( \mathbb{Q}(\zeta)/\mathbb{Q} \)-norm, we see that \( U(P)^{p-1} \) divides \( p^{24p(p-1)} \). This proves the proposition. \( \Box \)

\section{5. Proof of Theorem 1.2}

First of all, recall the following \textit{integrality property} of the \( j \)-invariant.

\textbf{Theorem 5.1} (Mazur, Momose, Merel). \textit{For a prime \( p = 11 \) or \( p \geq 17 \), the \( j \)-invariant \( j(P) \) of any noncuspidal point of \( X_{\text{split}}(p)(\mathbb{Q}) \) belongs to \( \mathbb{Z} \).

This is a combination of results of Mazur [17], Momose [19], and Merel [18]. For more details see the Appendix (§6), where we give a short unified proof.

Denote by \( h(\alpha) \) the absolute logarithmic height of an algebraic number \( \alpha \). If \( \alpha \) is a nonzero rational integer, then \( h(\alpha) = \log |\alpha| \). It follows from \textbf{Theorem 5.1} that if \( E \) is an elliptic curve over \( \mathbb{Q} \) endowed with a normalizer of split Cartan mod \( p \) structure\(^1\) with \( p \geq 17 \), then \( h(j_E) = \log |j_E| \).

In view of \textbf{Theorem 5.1}, \textbf{Theorem 1.2} is a straightforward consequence of \textbf{Theorem 1.1} and the following proposition, whose proof will be the goal of this section.

\textbf{Proposition 5.2}. \textit{There exists an absolute effective constant \( \kappa \) such that the following holds. Let \( p \) be a prime number, and \( E \) a non-CM elliptic curve over \( \mathbb{Q} \), endowed with a structure of normalizer of split Cartan subgroup in level \( p \). Then}

\begin{equation}
(16) \quad h(j_E) = \log |j_E| \geq \kappa p.
\end{equation}


\textbf{Theorem 5.3} (Masser-Wüstholz, Pellarin). \textit{Let \( E \) be an elliptic curve defined over a number field \( K \) of degree \( d \). Let \( E' \) be another elliptic curve, defined over \( K \) and isogenous to \( E \). Then there exists an isogeny \( \psi: E \to E' \) of degree at most \( \kappa(d) \left( 1 + h(j_E) \right)^2 \), where the constant \( \kappa(d) \) depends only on \( d \) and is effective.}

Masser and Wüstholz had exponent 4 (they actually proved similar statements for general abelian varieties) and Pellarin reduced it to 2, which is crucial for us; in fact, any exponent below 4 would do. Pellarin gave an explicit expression for \( \kappa(d) \) of the shape \( \lambda d^4 (1 + \log d)^2 \) with an absolute constant \( \lambda \). See

\(^1\)That is, whose mod \( p \) Galois representation has image contained in a normalizer of a split Cartan.
also the work [25] of E. Viada, who obtains exponent 3, but smaller \( \kappa(d) \). In [1, App. B] Bertrand remarks (refering to the exponent as \( C \)):

\[ \text{En fait, tout porte à croire [...] que du point de vue transcendant, la valeur optimale de } C \text{ est 2. La tradition folklorique veut sans doute que } C \text{ vaille 0 [...] mais cela paraît sans espoir du côté transcendant.} \]

**Corollary 5.4.** Let \( E \) be a non-CM elliptic curve defined over a number field \( K \) of degree \( d \), and admitting a cyclic isogeny over \( K \) of degree \( \delta \). Then
\[
\delta \leq \kappa(d) \left( 1 + h(j_E) \right)^2.
\]

**Proof.** Let \( \phi \) be a cyclic isogeny from \( E \) to \( E' \), and let \( \phi^D: E' \to E \) be the dual isogeny. Let \( \psi: E \to E' \) be an isogeny of degree bounded by \( \kappa(d) \left( 1 + h(j_E) \right)^2 \); without loss of generality, \( \psi \) may be assumed cyclic. As \( E \) has no CM, the composed map \( \phi^D \circ \psi \) must be multiplication by some integer, so that \( \phi = \pm \psi \).

**Proof of Proposition 5.2.** For an elliptic curve \( E \) endowed with a structure of normalizer of split Cartan subgroup in level \( p \) over \( \mathbb{Q} \), write \( C_1 \) and \( C_2 \) for the obvious two independent \( p \)-subgroups in \( E[p] \) which are Galois conjugates over a quadratic extension \( K/\mathbb{Q} \). Set \( \varphi_i: E \to E_i := E/C_i \) and recall that there is a cyclic \( p^2 \)-isogeny over \( K \) from \( E_1 \) to \( E_2 \), factorizing as the product:
\[
\varphi: E_1 \xleftarrow{\varphi_1^i} E \xrightarrow{\varphi_2} E_2.
\]

It follows from Corollary 5.4 that \( h(j_{E_i}) \geq \kappa_1 p \) for \( i = 1, 2 \), where \( \kappa_1 \) is some constant independent of \( p \) and \( E \).

A result of Faltings [8, Lemma 5] asserts that \( h_F(E_1) \leq h_F(E) + \frac{1}{2} \log p \), where \( h_F \) is Faltings’ semistable height. Finally, for any elliptic curve \( E \) over a number field we have
\[
\left| h(j_E) - 12h_F(E) \right| \leq 6 \log \left( 1 + h(j_E) \right) + O(1);
\]
see [24, Prop. 2.1]. (Pellarin shows that \( O(1) \) can be replaced by 47.15; see [21, eq. (51), p. 240].) This completes the proof of Proposition 5.2 and of Theorem 1.2.

6. **Appendix: Integrality of the \( j \)-invariant**

Here we prove that rational points on \( X_{\text{split}}(p) \) are, in fact, integral.

**Theorem 6.1** (Mazur, Momose, Merel). *For a prime \( p = 11 \) or \( p \geq 17 \), the \( j \)-invariant \( j(P) \) of any noncuspidal point of \( X_{\text{split}}(p)(\mathbb{Q}) \) belongs to \( \mathbb{Z} \).*

The proof of this theorem is somehow scattered in the literature. Mazur [17, Cor. 4.8] proved that a prime divisor \( \ell \) of the denominator of \( j(P) \) must
either be 2, or \( p \), or satisfy \( \ell \equiv \pm 1 \mod p \). The cases \( \ell \equiv \pm 1 \mod p \) and 
\( \ell = p \) were settled by Momose [19, Prop. 3.1], together with the case \( \ell = 2 \) when 
\( p \equiv 1 \mod 8 \) [19, Cor. 3.6]. Finally the case \( \ell = 2 \) with \( p \not\equiv 1 \mod 8 \) was treated by Merel [18, Th. 5]. The aim of this appendix is to present a
short unified proof. To avoid some technicalities occurring only for small \( p \), we assume in the sequel that \( p \geq 37 \).

Recall that the curve \( X_{\text{split}}(p) \) parametrizes (isomorphism classes of) elliptic curves endowed with an unordered pair of independent \( p \)-isogenies. Let 
\( P = (E, \{A, B\}) \in X_{\text{split}}(p)(\mathbb{Q}) \), which we may assume to be non-CM. Then the isogenies \( A \) and \( B \) are defined over a number field \( K \) with degree at most 2.

**Proposition 6.2.** Let 
\( P = (E, \{A, B\}) \in X_{\text{split}}(p)(\mathbb{Q}) \) and \( K \) be defined as above. Let \( \mathcal{O}_K \) be its ring of integers. Then we have the following:

(a) The curve \( E \) is not potentially supersingular at \( p \).

(b) The points \( (E, B) \) and \( (E/A, E[p]/A) = (E/A, A^*) \), where \( A^* \) is the isogeny dual to \( A \), coincide in the fibers of characteristic \( p \) of \( X_0(p)/\mathcal{O}_K \).

**Proof.** Part (a) is proved in [19, Lemma 1.3]. Part (b) follows from [20, proof of Prop. 3.1]. For the convenience of the reader we sketch somewhat different (and simpler) arguments.

It follows from Serre’s study of the action of inertia groups \( I_p \) at \( p \) on the formal group of elliptic curves that if \( E \) is potentially supersingular then \( I_p \) (potentially) acts via a “fundamental character of level 2” (at least if \( E \) has \( j \)-invariant different from \( 1728 \mod p \)), so that the image of inertia contains a subgroup of index 4 or 6 in a nonsplit Cartan subgroup of \( \text{GL}(E[p]) \) (see [23, Paragraph 1]). This gives a contradiction to the fact that a subgroup of index 2 in the absolute Galois group of \( \mathbb{Q} \) preserves two lines in \( E[p] \); for the remaining case of \( j = 1728 \mod p \) we refer to the article of Momose, loc. cit., whence part (a).

For (b) we remark that we may assume the schematic closure of \( A \) to be étale over \( \mathcal{O} \) (the ring of integers of a completion \( K_{\mathfrak{P}} \) of \( K \) at a prime \( \mathfrak{P} \) above \( p \), whose residue field we denote by \( k_{\mathfrak{P}} \)); indeed, as \( E \) is not potentially supersingular at \( \mathfrak{P} \), at most one line in \( E[p] \) can be purely radicial over \( k_{\mathfrak{P}} \). Up to replacing \( K_{\mathfrak{P}} \) by a finite ramified extension, we shall also assume \( E \) is semistable over \( K_{\mathfrak{P}} \). Now \( E/A \) is isomorphic over \( k_{\mathfrak{P}} \) to \( E^{(p)} \) via the Verschiebung isogeny, and the latter is in turn isomorphic to \( E/k_{\mathfrak{P}} \) as \( E \) has a model over \( \mathbb{Z} \). Moreover the isomorphism between \( B \) and \( E[p]/A \) as \( K \)-group schemes induced by the projection \( E \to E/A \) extends to an isomorphism over \( \mathcal{O} \) by Raynaud’s theorem on group schemes of type \( (p, \ldots, p) \), as recalled in [19, Proof of Lemma 1.3]. It follows that \( (E, B)_{k_{\mathfrak{P}}} \) is isomorphic to 
\( (E/A, E[p]/A)_{k_{\mathfrak{P}}} = (w_{p}(E, A))_{k_{\mathfrak{P}}} \), whence (b). This completes the proof. \( \square \)
The curve $X_{\text{split}}(p)$ admits an obvious double covering by the curve $X_{\text{sp,Car}}(p)$, parametrizing elliptic curves endowed with an ordered pair of $p$-isogenies. We denote by $w$ the generator of the Galois group of this covering; that is, $w$ modularly exchanges the two $p$-isogenies. If $(E, (A, B))$ is a point on $X_{\text{sp,Car}}(p)$, then $w(E, (A, B)) = (E, (B, A))$. We recall certain properties of the modular Jacobian $J_0(p)$ and its Eisenstein quotient $\tilde{J}(p)$ (see [15]).

**Proposition 6.3.** Let $p$ be a prime number. Then we have the following.

(a) [15, Th. 1] The group $J_0(p)(\mathbb{Q})_{\text{tors}}$ is cyclic and generated by $\text{cl}(0 - \infty)$, where $0$ and $\infty$ are the cusps of $X_0(p)$. Its order is equal to the numerator of the quotient $(p - 1)/12$.

(b) [15, Th. 4] The group $\tilde{J}(p)(\mathbb{Q})$ is finite. Moreover, the natural projection $J_0(p) \to \tilde{J}(p)$ defines an isomorphism $J_0(p)(\mathbb{Q})_{\text{tors}} \to \tilde{J}(p)(\mathbb{Q})$.

As Mazur remarks, Raynaud’s theorem on group schemes of type $(p, \ldots, p)$ insures that $J_0(p)(\mathbb{Q})_{\text{tors}}$ defines a $\mathbb{Z}$-group scheme which, being constant in the generic fiber, is étale outside 2, and which at 2 has étale quotient of rank at least half that of $J_0(p)(\mathbb{Q})_{\text{tors}}$.

**Proof of Theorem 6.1.** For an element $t$ in the $\mathbb{Z}$-Hecke algebra for $\Gamma_0(p)$, define the morphism $g_t$ from $X_{\text{sp,Car}}(p)_{\mathbb{Z}}$ to $J_0(p)_{\mathbb{Z}}$ which extends the morphism on generic fibers:

$$g_t: \begin{cases} X_{\text{sp,Car}}(p) & \to J_0(p) \\ Q = (E, (A, B)) & \to t \cdot \text{cl}( (E, A) - (E/B, E[p]/B) ) \end{cases}$$

Let $J_0(p) \to \tilde{J}(p)$ be the projection to the Eisenstein quotient, and $\tilde{g}_t := \pi \circ g_t$. One checks that $g_t \circ w = -w_p \circ g_t$ and one knows that $(1 + w_p)$ acts trivially on $\tilde{J}(p)$ from [15, Prop. 17.10]. Therefore $\tilde{g}_t$ actually factorizes through a $\mathbb{Q}$-morphism from $X_{\text{split}}(p)$ to $\tilde{J}(p)$, which we extend by the universal property of Néron models to a map from $X_{\text{split}}(p)_{\mathbb{Z}}$ to $\tilde{J}(p)_{\mathbb{Z}}$. We still denote this morphism by $\tilde{g}_t$ and we put $\tilde{g} = \tilde{g}_1$.

Let $P$ be a rational point on $X_{\text{split}}(p)$, and $\ell$ a prime divisor of the denominator of $j(P)$. Then $P$ specializes to a cusp at $\ell$. Recall that $X_{\text{split}}(p)$ has one cusp defined over $\mathbb{Q}$ (the rational cusp), and $(p - 1)/2$ other cusps, conjugate over $\mathbb{Q}$. We first claim that $P$ specializes to the rational cusp. Indeed, it follows from Proposition 6.2 (a) that $P$ does extend to a section of $X_{\text{split}}(p)_{\mathbb{Z}_p}$, from Proposition 6.2 (b) that $\tilde{g}(P)(\mathbb{F}_p) = 0(\mathbb{F}_p)$, and from the remark after Proposition 6.3 that $\tilde{g}(P)(\mathbb{Q}) = 0(\mathbb{Q})$ (recall $p \neq 2$). The nonrational cusps of $X_{\text{split}}(p)(\mathbb{C})$ map to $\text{cl}(0 - \infty)$ in $J_0(p)(\mathbb{C})$ (this can be seen with the above modular interpretation of $\tilde{g}_t$, by the fact that the nonrational cusps specialize at $p$ to a generalized elliptic curve endowed with a pair of étale isogenies. Or, if $f$ denotes the map $f: X_{\text{sp,C}}(p) \to X_0(p)$, $(E, (A, B)) \mapsto (E, A)$,
one has $g_1 = \text{cl}(f - w_p f w)$, and as $f(c_i) = 0 \in X_0(p)$ for $c_i$ a nonrational cusp and $w$ permutes the $c_i$s, one sees that $\tilde{g}(c_i) = \text{cl}(0 - \infty)$. For more details see, for instance, the proof of Proposition 2.5 in [19]). Therefore, as we assumed $p \geq 37$, Proposition 6.3 implies that if $P$ specializes to a nonrational cusp at $\ell$ then $\tilde{g}(P)$ would not be 0 at $\ell$, a contradiction.

Now we use the winding quotient (see, for instance, [18]). Take an $\ell$-adically maximal element $t$ in the Hecke algebra which kills the winding ideal $I_e$. Again, as $t(1+w_p) = 0$, the above morphism $g_t$ factorizes through a morphism $g_t^+$ from $X_{\text{split}}^\text{smooth}(p)/\mathbb{Z}$ to $t \cdot J_0(p)/\mathbb{Z}$. Moreover $g_t^+(P)$ belongs to $t \cdot J_0(p)(\mathbb{Q})$, hence is a torsion point, as $t \cdot J_0(p)$ is isogenous to a quotient of the winding quotient of $J_0(p)$. As above, by looking at the fiber at $p$, we see that $g_t^+(P) = 0$ at $p$, hence at the generic fiber as well. We then easily check by use of the $q$-expansion principle, as in [18, Th. 5], that $g_t^+$ is a formal immersion at the specialization $\infty(\mathbb{F}_\ell)$ of the rational cusp on $X_{\text{split}}^\text{smooth}(p)$. This allows us to apply the classical argument of Mazur (see e.g. [17, proof of Cor. 4.3]), yielding a contradiction; therefore $P$ is not cuspidal at $\ell$. □

References


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