Annals of Mathematics **173** (2011), 211–249
doi: 10.4007/annals.2011.173.1.6

Dynamical compactifications of $\mathbf{C}^2$

By Charles Favre and Mattias Jonsson

Abstract

We find good dynamical compactifications for arbitrary polynomial mappings of $\mathbf{C}^2$ and use them to show that the degree growth sequence satisfies a linear integral recursion formula. For maps of low topological degree we prove that the Green function is well behaved. For maps of maximum topological degree, we give normal forms.

Introduction

The theory of iteration of rational maps on complex projective varieties has recently seen the introduction of new analytic techniques for constructing invariant currents and measures of dynamical interest, through the work of Bedford-Diller [1], de Thélin-Vigny [13], Diller-Dujardin-Guedj [14], [15], [16], Dinh-Sibony [19], [22], [23], Dujardin [25], [26], Guedj [36], [38], and others. These constructions, however, often require a good birational model in which the dynamical indeterminacy set has a relatively small size, so that the action on cohomology of the rational map is compatible with iteration.

Diller-Favre [17] proved the existence of such models for birational surface maps using the decomposition into blow-ups and blow-downs. There are no other general results, for two reasons. First, it is a delicate task to control a rational map near its indeterminacy set. Second, the indeterminacy set of a noninvertible map tends to grow very rapidly under iteration.

In this paper we prove the existence of good birational models for an important class of rational surface maps, namely polynomial maps.

Theorem A. *Let $F : \mathbf{C}^2 \to \mathbf{C}^2$ be any polynomial mapping. Then there exists a projective compactification $X \supset \mathbf{C}^2$ with at worst quotient singularities and an integer $n \geq 1$ such that the lift $\widetilde{F} : X \dashrightarrow X$ satisfies $\widetilde{F}^{(n+j)*} = \widetilde{F}^{n*}\widetilde{F}^{j*} = \widetilde{F}^{n*}(\widetilde{F}^{*})^{j}$ on the Picard group $\mathrm{Pic}(X)$ for all $j \geq 1$.*

Fix an algebraic embedding $\mathbf{C}^2 \subset \mathbf{P}^2$. Define $\deg(F)$ by the relation $F^*\mathcal{L} = \deg(F)\mathcal{L}$ for $\mathcal{L}$ a generator of $\mathrm{Pic}(\mathbf{P}^2)$. Using Theorem A we prove

Second author supported by the NSF and the Swedish Research Council.

THEOREM B. *For any polynomial mapping $F : \mathbf{C}^2 \to \mathbf{C}^2$, the sequence $(\deg(F^j))_{j\geq 0}$ satisfies an integral linear recursion formula.*

This gives a positive answer to the main conjecture of Bellon-Viallet [6, §5] in our setting. Example 4.5 shows that there may not be a recursion of order one or two, even though the asymptotic degree λ_1 (see below) is always a quadratic integer [30]. In general, the degree growth of rational maps of projective space remains mysterious, despite recent works [2], [3], [4], [7], [39], [45]. Hasselblatt and Propp [39] give examples of rational maps of $\mathbf{C}^2$ for which the degree growth does not satisfy any linear recursion formula. From this perspective, Theorem B is quite remarkable.

In many cases — and always after replacing F by F^2 — X is smooth and can be obtained from $\mathbf{P}^2$ after finitely many blow-ups at infinity. In these cases, Theorem B follows immediately from Theorem A.

Note that the compactifications that we consider here are different in nature from the ones studied by Hubbard, Papadopol, and Veselov [40].

The conclusion in Theorem A is slightly weaker than the condition $(\widetilde{F}^j)^* = (\widetilde{F}^*)^j$ on $\mathrm{Pic}(X)$ for all j; the latter is often referred to as *algebraic stability* [47]. Its importance was first recognized by Fornæss and Sibony [32]. Guedj has conjectured [36, Remark 3.1] that any polynomial mapping of $\mathbf{C}^2$ is algebraically stable on some (smooth) compactification X of $\mathbf{C}^2$. While we suspect this may be too much to ask for (see Remark 6.1), Theorem A shows that the conjecture holds after replacing F by an iterate. In any case, Theorem A is sufficient for all known applications.

The proof of Theorem A is based on the valuative techniques developed in [30]. These give a framework for studying the dynamics induced by F on the set of divisors at infinity in all compactifications of $\mathbf{C}^2$. This set of divisors can be identified with a dense subset of a metrized $\mathbf{R}$-tree $\mathcal{V}_0$ consisting of all valuations on $\mathbf{C}[x, y]$, centered at infinity and suitably normalized. There are however two difficulties in working directly with $\mathcal{V}_0$. First, a valuation in $\mathcal{V}_0$ is a local object, whereas we are interested in global properties of F. Second, F might be not proper, and in this case it does not preserve $\mathcal{V}_0$. To remedy these problems, we introduced in [30] a subtree $\mathcal{V}_1$ of $\mathcal{V}_0$ consisting of valuations close enough to $-\deg$; see Section 1.5 for a formal definition. This subtree is a fundamental technical tool in our analysis. In op. cit., we showed that valuations in $\mathcal{V}_1$ still capture global information, that F induces a continuous map on $\mathcal{V}_1$, and we proved the existence of a locally attracting valuation $\nu_* \in \mathcal{V}_1$ that we called an *eigenvaluation*. Theorem A is a consequence of a detailed study of the global contracting properties of F on $\mathcal{V}_1$.

Denote by $\lambda_1 := \lim_{n\to\infty} \deg(F^n)^{1/n}$ the *asymptotic degree* of F [46] and by λ_2 the *topological degree* of F. These degrees are invariant under conjugacy

by polynomial automorphisms and satisfy $\lambda_2 \leq \lambda_1^2$. In the case $\lambda_2 < \lambda_1^2$, the Hilbert space methods of Boucksom and the authors [8] apply (see also the work of Hubbard-Papadopol [42], Cantat [11], and Manin [43]). We showed in [8] that $\deg(F^n) \sim \lambda_1^n$. Here we use these techniques to prove that ν_* attracts all valuations in $\mathcal{V}_1$ (with at most one exception).

When $\lambda_2 = \lambda_1^2$, the Hilbert space technique loses some strength. However, this loss is compensated by the built-in rigidity of these maps. A more detailed study of the valuative dynamics allows us to show

THEOREM C. *Let $F : \mathbf{C}^2 \to \mathbf{C}^2$ be a dominant polynomial mapping with $\lambda_2 = \lambda_1^2$. Then we are in one of the following mutually exclusive cases*:

(1) $\deg(F^n) \sim n\lambda_1^n$; *then $\lambda_1 \in \mathbf{N}$ and in suitable affine coordinates, F is a skew product of the form $F(x, y) = (P(x), Q(x, y))$, where $\deg(P) = \lambda_1$ and $Q(x, y) = A(x)y^{\lambda_1} + O_x(y^{\lambda_1 - 1})$ with $\deg(A) \geq 1$.*
(2) $\deg(F^n) \sim \lambda_1^n$; *then there exists a projective compactification $X \supset \mathbf{C}^2$ with at most quotient singularities such that F extends to a holomorphic self-map of X.*

Here, and throughout the paper, the expression "in suitable affine coordinates" means that the statement holds after conjugation by a polynomial automorphism of $\mathbf{C}^2$.

In suitable affine coordinates, X can be chosen as a toric surface and we can give normal forms for all maps occurring in (2); see Section 5.3. When λ_1 is an integer, X can be chosen as a weighted projective plane.

Theorem C extends the Friedland-Milnor classification [33] of polynomial automorphisms with $\lambda_1 = 1$. For automorphisms, only case (2) appears, and X can be chosen as $\mathbf{P}^2$ or a Hirzebruch surface. There is an analogous statement in the general birational surface case (see [17], [35]): $\deg(F^n)$ is then either bounded or grows linearly or quadratically.

Our last result is another application of the dynamical compactifications. One of the basic problems in the iteration of rational maps is the construction and study of an ergodic measure of maximal entropy. When $\lambda_2 > \lambda_1$, such a measure can be defined as a limit of preimages of a generic point [32], [46], [47], and its basic ergodic properties are completely understood; see [9], [19], [21], [38]. In the case of maps with small topological degree $\lambda_2 < \lambda_1$, this construction fails. A different strategy has been proposed for constructing a dynamically interesting invariant measure; see [36]. One first constructs two positive closed $(1, 1)$ currents, invariant by pullback and by pushforward, respectively. The measure is then obtained by taking their intersection. In our setting, the existence of these two currents follows from Theorem A; see [14]. The existence of their intersection, however, is not guaranteed in general except

if one has a good control of the singularities of their potentials. We prove such a control for the pullback invariant current.

Fix affine coordinates (x, y) on $\mathbf{C}^2$ and set $\|(x, y)\| = \max\{1, |x|, |y|\}$.

THEOREM D. *Let $F : \mathbf{C}^2 \to \mathbf{C}^2$ be a dominant polynomial mapping with $\lambda_2 < \lambda_1$. Then the limit*

$$G^+(p) = \lim_{n\to\infty} \lambda_1^{-n} \log^+ \|F^n p\|$$

exists locally uniformly on $\mathbf{C}^2$, and defines a continuous, nonzero, nonnegative plurisubharmonic function of logarithmic growth satisfying $G^+ \circ F = \lambda_1 G^+$. The support of the positive closed current $dd^c G^+$ is equal to ∂K^+ where $K^+ = \{G^+ = 0\}$. Further, for each $\varepsilon > 0$, there exists a constant $C > 0$, such that

$$(*) \qquad \log^+ \|F^n p\| \le (\lambda_2 + \varepsilon)^n (\log^+ \|p\| + C)$$

for all $n \ge 0$ and all $p \in K^+$.

Theorem D generalizes classical properties of the Green function both of polynomials in one variable and of Hénon maps [5], [31], [41]. On the locus $\{G^+ = 0\}$ the dynamics can exhibit various speeds of convergence towards infinity; see [18]. Note that $(*)$ is in sharp contrast with the phenomenon described in [48].

The paper is organized as follows. In Sections 1 and 2 we discuss the relationship between compactifications and valuations, and study the induced dynamics on the space of valuations at infinity. We then turn to the proof of refined versions of Theorems A and B in the case $\lambda_2 < \lambda_1^2$: in Section 3 when the eigenvaluation is nondivisorial and in Section 4 when it is divisorial. Polynomial maps with $\lambda_2 = \lambda_1^2$ are handled in Section 5, where we prove Theorem C. Theorems A and B are proved in Section 6, Theorem D in Section 7, where we also provide a list of examples of maps with $\lambda_2 = \lambda_1$. The paper ends with a short appendix outlining an adaptation of the necessary material from [8] to our setting.

Acknowledgments. We thank Sébastien Boucksom, Serge Cantat, and Vincent Guedj for their comments on a preliminary version of this paper. We thank the referee for useful suggestions.

1. Geometry at infinity

We start by discussing compactifications of $\mathbf{C}^2$ together with valuations centered at infinity.

1.1. *Admissible compactifications.* We consider $\mathbf{C}^2$ equipped with a fixed embedding into $\mathbf{P}^2$.

Definition 1.1. An *admissible compactification* of $\mathbf{C}^2$ is a smooth projective surface X admitting a birational morphism $\pi : X \to \mathbf{P}^2$ that is an isomorphism above $\mathbf{C}^2$.

It follows that π is a composition of point blow-ups and that $X \setminus \mathbf{C}^2$ is a connected curve with simple normal crossings. The *primes* of X are the irreducible components of $X \setminus \mathbf{C}^2$. Every X contains a special prime L_∞, the strict transform of the line at infinity in $\mathbf{P}^2$.

1.2. *Valuations* [30, App. A]. Let R be the coordinate ring of $\mathbf{C}^2$. We define $\widehat{\mathcal{V}}_0$ as the set of valuations $\nu : R \to (-\infty, +\infty]$ *centered at infinity*, i.e., $\nu(L) < 0$ for a generic affine function L on $\mathbf{C}^2$. If $\nu \in \widehat{\mathcal{V}}_0$ and X is an admissible compactification of $\mathbf{C}^2$, then the *center* of ν on X is the unique scheme-theoretic point on X such that ν is strictly positive on the maximal ideal of its local ring. Thus the center is either a prime of X or a point on $X \setminus \mathbf{C}^2$. We let $\mathcal{V}_0$ be the subset of $\nu \in \widehat{\mathcal{V}}_0$ that are *normalized* by $\nu(L) = -1$.

There are four kinds of valuations in $\widehat{\mathcal{V}}_0$ that we now describe.

First, if X is an admissible compactification of $\mathbf{C}^2$, then each prime E of X defines a *divisorial* valuation $\operatorname{ord}_E \in \widehat{\mathcal{V}}_0$, the order of vanishing along E. In particular, $\operatorname{ord}_{L_\infty} = -\deg$. Any valuation proportional to some ord_E will also be called divisorial. Thus a valuation is nondivisorial if and only if its center on every admissible compactification is a point. If we set $b_E = -\operatorname{ord}_E(L)$ for a generic affine function L, then $\nu_E := b_E^{-1} \operatorname{ord}_E$ is normalized. We denote by $\widehat{\mathcal{V}}_{\mathrm{div}}$ and $\mathcal{V}_{\mathrm{div}}$ the set of divisorial valuations in $\widehat{\mathcal{V}}_0$ and $\mathcal{V}_0$, respectively.

Second, we have *irrational* valuations. To define them, consider any two primes E, E' in X intersecting at a point p and local coordinates (z, w) at p such that $E = \{z = 0\}$ and $E' = \{w = 0\}$. To any pair $(s, t) \in \mathbf{R}_+^2$ we attach the valuation ν defined on the ring $\mathcal{O}_p$ of holomorphic germs at p by

$$\nu\left(\sum a_{ij} z^i w^j\right) = \min\{si + tj \mid a_{ij} \neq 0\};$$

it does not depend on the choice of coordinates (z, w). By first extending ν to the common fraction field $\mathbf{C}(X)$ of $\mathcal{O}_p$ and R, then restricting to R, we obtain a valuation in $\widehat{\mathcal{V}}_0$, called *quasimonomial*. (It is monomial in the local coordinates (z, w) at p.) The valuation ν is normalized if and only if $sb_E + tb_{E'} = 1$. It is divisorial if and only if either $t = 0$ or the ratio s/t is a rational number. Any divisorial valuation is quasimonomial. An irrational valuation is by definition a nondivisorial quasimonomial valuation.

Third, pick a point p on the line at infinity $L_\infty \subset \mathbf{P}^2$, a formal irreducible curve C at p, not contained in L_∞, and a constant $\gamma > 0$. Then $P \mapsto \gamma^{-1} \operatorname{ord}_p(P|_C)$, for polynomials $P \in R$, defines a valuation in $\widehat{\mathcal{V}}_0$ called a *curve valuation*. It is normalized if and only if γ is chosen as the intersection

number $(C \cdot L_\infty)$. Note that a curve valuation can take the value $+\infty$ when the curve is algebraic.

In suitable affine coordinates (x, y), a curve valuation can be computed using a Puiseux parametrization $y = h(x^{-1})$ of the curve: the value on a polynomial P is proportional to $\mathrm{ord}_{x=\infty} P(x, h(x^{-1}))$. Now replace the Puiseux series h by a formal series of the form $h(\zeta) = \sum a_k \zeta^{\beta_k}$ with $a_k \in \mathbf{C}^*$, and β_k an increasing sequence of rational numbers with unbounded denominators. Then $P \mapsto \mathrm{ord}_{x=\infty} P(x, h(x^{-1}))$ defines a valuation of the fourth and last kind, namely an *infinitely singular valuation*.

1.3. *Tree structure* [30, App. A]. The space $\mathcal{V}_0$ of normalized valuations is equipped with a partial ordering: $\nu \leq \mu$ if and only if $\nu(P) \leq \mu(P)$ for all $P \in R$, naturally turning it into a rooted *tree*. In particular, every two elements $\mu, \nu \in \mathcal{V}_0$ admit a minimum $\mu \wedge \nu \in \mathcal{V}_0$. The valuation $-\deg$ is the minimal element of $\mathcal{V}_0$. For any admissible compactification X, one can map a prime E to the normalized valuation $\nu_E := b_E^{-1} \mathrm{ord}_E$. In this way, we get an embedding of the set of primes of X into $\mathcal{V}_0$. The partial ordering coming from $\mathcal{V}_0$ on the set of primes coincides with the one coming from the tree structure of the dual graph of $X \setminus \mathbf{C}^2$.

The *ends* of $\mathcal{V}_0$, i.e. the maximal elements in the partial ordering, are the curve and infinitely singular valuations, i.e. the valuations that are not quasimonomial.

We topologize $\mathcal{V}_0$ by declaring $\nu_n \to \nu$ if and only if $\nu_n(P) \to \nu(P)$ for all P. This topology is compact and admits two important characterizations. First, it is the weakest topology such that the natural retraction map $r_I : \mathcal{V}_0 \to I$ is continuous for any given closed segment $I \subset \mathcal{V}_0$. Second, given any admissible compactification X and any point $p \in X \setminus \mathbf{C}^2$, let $U(p) \subset \mathcal{V}_0$ be the set of valuations whose center on X is p. Then the topology on $\mathcal{V}_0$ is the weakest one in which $U(p) \subset \mathcal{V}_0$ is always open. Each of the four subsets of divisorial, irrational, curve, and infinitely singular valuations is dense in $\mathcal{V}_0$.

There is a unique, decreasing, upper semicontinuous *skewness* function $\alpha : \mathcal{V}_0 \to [-\infty, 1]$ satisfying $\alpha(-\deg) = 1$ and $|\alpha(\nu_E) - \alpha(\nu_{E'})| = (b_E b_{E'})^{-1}$ whenever E and E' are intersecting primes in some admissible compactification X; see [30, §A.1] and [28, §6.6.3, §6.8]. One has $b_E^2 \alpha(\nu_E) \in \mathbf{Z}$; see Lemma A.2.

Similarly, there is a unique, increasing, lower semicontinuous *thinness* function $A : \mathcal{V}_0 \to [-2, +\infty]$ such that $A(\nu_E) = a_E / b_E$, where $a_E = 1 + \mathrm{ord}_E(dx \wedge dy)$ and $b_E = -\mathrm{ord}_E(L)$ as above; see [30, §A.1]. An irrational valuation has irrational skewness and thinness.

1.4. *The subtree* $\mathcal{V}_1$. Define $\mathcal{V}_1$ as the set of valuations $\nu \in \mathcal{V}_0$ with skewness $\alpha(\nu) \geq 0$ and thinness $A(\nu) \leq 0$. Then $\mathcal{V}_1$ is a subtree of $\mathcal{V}_0$ of crucial importance to our study. Its properties are spelled out in detail in [30, Th. A.7].

Here we note that the fundamental fact is that any quasimonomial valuation in $\mathcal{V}_1$ is dominated by a *pencil valuation*.

To define the latter, consider an affine curve $C = \{P = 0\} \subset \mathbf{C}^2$ with *one place at infinity*, that is, the closure of C in $\mathbf{P}^2$ is an irreducible curve intersecting the line at infinity in a single point and is locally irreducible there. Consider the pencil $|C|$ consisting of the affine curves $C_\lambda = \{P = \lambda\} \subset \mathbf{C}^2$ for $\lambda \in \mathbf{C}$. It is a theorem by Moh (see [44], [10]) that C_λ has one place at infinity for every $\lambda \in \mathbf{C}$. The (normalized) pencil valuation $\nu_{|C|} \in \mathcal{V}_0$ associated to $|C|$ is then defined by $\nu_{|C|}(Q) := (\deg C)^{-1} \operatorname{ord}_\infty(Q|_{C_\lambda})$ for λ generic. By Bézout, $-\nu_{|C|}(Q)\deg(C)$ equals the number of intersection points in $\mathbf{C}^2$ of C with the curve $\{Q = \rho\}$ for generic $\rho \in \mathbf{C}$. See [30, §A.2] for more details.

This crucial fact that any quasimonomial valuation in $\mathcal{V}_1$ is dominated by a pencil valuation is the affine analog of the (easier) local result that any quasimonomial valuation in the valuative tree is dominated by a curve valuation; see [28, Prop. 3.20].

A pencil valuation $\nu_{|C|}$ is divisorial but does not itself belong to $\mathcal{V}_1$ in general. Indeed, we have $\alpha(\nu_{|C|}) = 0$ and $A(\nu_{|C|}) = (2g-1)/\deg(C)$, where g is the geometric genus of the closure of a generic curve C_λ in the pencil. Thus $\nu_{|C|}$ belongs to $\mathcal{V}_1$ if and only if $C_\lambda \simeq \mathbf{C}$ for generic λ. We then call $\nu_{|C|}$ *rational pencil valuation*.

It follows that if ν is a quasimonomial valuation in $\mathcal{V}_1$, then either $\alpha(\nu) > 0$ or ν is a rational pencil valuation.

1.5. *Tight compactifications.* Associated to the subtree $\mathcal{V}_1$ is an important class of compactifications of $\mathbf{C}^2$.

Definition 1.2. An admissible compactification X of $\mathbf{C}^2$ is *tight* if the normalized divisorial valuations associated to the primes of X belong to the subtree $\mathcal{V}_1$ of $\mathcal{V}_0$.

Remark 1.3. An admissible compactification X is tight if and only if $\operatorname{ord}_E(dx \wedge dy) < 0$ and $\operatorname{ord}_E(P) \leq 0$ for any prime E of X and any polynomial $P \in \mathbf{C}[x, y]$. The second condition is equivalent to Z_{ord_E} being nef; see Section A.2. This implies that the nef and psef cones of an admissible compactification X of $\mathbf{C}^2$ are simplicial whenever Z_{ord_E} is nef for every prime E of X. A compactification of $\mathbf{C}^2$ associated to a curve with one place at infinity as defined in [10] always has the latter property and is tight if and only if the curve is rational.

Lemma 1.4. *Let X be a tight compactification of $\mathbf{C}^2$. Pick a point $p \in X \setminus \mathbf{C}^2$ and let X' be the admissible compactification of $\mathbf{C}^2$ obtained by blowing up p. Then X' is tight if and only if p does not lie on a unique prime of X,*

whose associated normalized divisorial valuation has skewness zero or thinness zero.

Proof. Let $\nu \in \mathcal{V}_0$ be the divisorial valuation associated to the blow-up of p. Then X' is tight if and only if $\nu \in \mathcal{V}_1$. First assume p is the intersection point between two primes E_1, E_2 of X with associated divisorial valuations $\nu_1, \nu_2 \in \mathcal{V}_0$. Then ν lies in the segment between ν_1 and ν_2. Since $\nu_i \in \mathcal{V}_1$ and $\mathcal{V}_1$ is a subtree, we get $\nu \in \mathcal{V}_1$, so that X' is tight.

Now assume p lies on a single prime E of X, with associated divisorial valuation $\nu_E \in \mathcal{V}_1$. In this case, $\nu > \nu_E$. Let $b_E = -\operatorname{ord}_E(L)$ as above. Then $A(\nu) - A(\nu_E) = 1/b_E$ and $\alpha(\nu) - \alpha(\nu_E) = -1/b_E^2$. On the other hand, $b_E^2\alpha(\nu_E), b_E A(\nu_E) \in \mathbf{Z}$. Hence $\nu \in \mathcal{V}_1$ if and only if $A(\nu_E) < 0$ and $\alpha(\nu_E) > 0$. □

COROLLARY 1.5. *Let $\nu \in \mathcal{V}_1$ and define a sequence of admissible compactifications $(X_m)_{m\geq 0}$ of $\mathbf{C}^2$ as follows: $X_0 = \mathbf{P}^2$ and X_{m+1} is obtained from X_m by blowing up the center p_m of ν on X_m. Then X_m is tight for all m.*

Proof. In view of Lemma 1.4, we only have to show that p_m never lies on a unique prime E of X_m whose associated normalized valuation $\nu_E \in \mathcal{V}_0$ has skewness zero or thinness zero. But if it did, the valuation $\nu_k \in \mathcal{V}_0$ associated to p_k would satisfy $\nu_k > \nu_E$ for all $k > m$. This would imply $\nu > \nu_E$ so that $\alpha(\nu) < 0$ or $A(\nu) > 0$, contradicting $\nu \in \mathcal{V}_1$. □

2. **Valuative dynamics**

Consider a dominant polynomial mapping $F : \mathbf{C}^2 \to \mathbf{C}^2$.

2.1. *Induced map on valuations.* Let $\nu \in \widehat{\mathcal{V}}_0$ be a valuation centered at infinity and set $d(F,\nu) := -\nu(F^*L) \geq 0$ for a generic affine function L on $\mathbf{C}^2$. Define a valuation $F_*\nu$ by $F_*\nu = 0$ if $d(F,\nu) = 0$ and $F_*\nu(P) = \nu(F^*P)$ if $d(F,\nu) > 0$. In the former case, note that $\nu(F^*P) = 0$ for a generic polynomial. In the latter case, $F_\bullet\nu := F_*\nu/d(F,\nu)$ is a well-defined normalized valuation in $\mathcal{V}_0$.

We have the following valuative criterion for properness of maps. It is a consequence of [30, Prop. 7.2].

PROPOSITION 2.1. *When F is not proper, one can find a divisorial valuation ν such that $d(F,\nu) = 0$. When F is proper, there exists a constant $c > 0$ such that $d(F,\nu) \geq c$ for all $\nu \in \mathcal{V}_0$.*

When $\nu \in \widehat{\mathcal{V}}_{\rm div}$ is divisorial and $d(F,\nu) > 0$, the valuation $\nu' := F_*\nu \in \widehat{\mathcal{V}}_{\rm div}$ is also divisorial. More precisely, given any two admissible compactifications X, X' of $\mathbf{C}^2$ such that the centers of ν and ν' are primes E and E' of X and X', respectively we can write $\nu = t\operatorname{ord}_E$ and $\nu' = t'\operatorname{ord}_{E'}$. Then t'/t is the coefficient of E in $\widetilde{F}^*E'$, where $\widetilde{F} : X \dashrightarrow X'$ is the lift of F.

The following result, proved in [30, §7.2] allows us to do dynamics on the subtree $\mathcal{V}_1$ of $\mathcal{V}_0$, even when F is not proper.

PROPOSITION 2.2. *We have $d(F,\cdot) > 0$ on $\mathcal{V}_1$ and $F_\bullet$ leaves $\mathcal{V}_1$ invariant.*

The map $F_\bullet$ preserves the tree structure on $\mathcal{V}_1$ in the sense that small segments are mapped homeomorphically onto small segments. See [30, Th. 7.4] for a precise statement.

2.2. *Eigenvaluations.* The *asymptotic degree* of F is defined by $\lambda_1 := \lim_{n\to\infty}(\deg F^n)^{1/n}$. The following result was proved in [30, §7.3].

PROPOSITION 2.3. *There exists a valuation ν_* belonging to the subtree $\mathcal{V}_1$ of $\mathcal{V}_0$ such that $F_*\nu_* = \lambda_1\nu_*$. In particular, $\alpha(\nu_*) \geq 0$ and $d(F,\nu_*) = \lambda_1$.*

Such a valuation is called an *eigenvaluation*. The proof is based on the fact that $F_\bullet$ preserves the tree structure on $\mathcal{V}_1$. We also proved that the eigenvaluation admits a small basin of attraction. Using the techniques of [8] as described in the appendix, we can strengthen these conclusions considerably under the assumption $\lambda_2 < \lambda_1^2$, where λ_2 is the topological degree of F.

THEOREM 2.4. *Assume $\lambda_2 < \lambda_1^2$. Then*

(a) *the valuation ν_* in Proposition 2.3 is the unique valuation $\nu \in \mathcal{V}_0$ with $\alpha(\nu) \geq 0$ and $F_*\nu = \lambda_1\nu$;*

(b) *if $\nu \in \mathcal{V}_0$ and $\alpha(\nu) > 0$, then $F_\bullet^n\nu \to \nu_*$ in $\mathcal{V}_0$ as $n \to \infty$;*

(c) *there exists at most one $\nu \in \mathcal{V}_0$ with $\alpha(\nu) = 0$ such that $F_\bullet^n\nu \not\to \nu_*$ as $n \to \infty$; this ν must satisfy $F_\bullet\nu = \nu$.*

Hence it makes sense to refer to ν_* as *the* eigenvaluation when $\lambda_2 < \lambda_1^2$.

Proof. The proof invokes the spectral properties of the operators F^* and F_* on the space $\mathrm{L}^2(\mathfrak{X})$ as discussed in Appendix A.

For $\nu \in \widehat{\mathcal{V}}_0$ let $Z_\nu \in W(\mathfrak{X})$ be the associated Weil class; see Section A.2. Then $F_*Z_\nu = Z_{F_*\nu}$. When $\nu \in \mathcal{V}_0$ and $\alpha(\nu) \geq 0$, Lemma A.3 shows that Z_ν is nef; hence in $\mathrm{L}^2(\mathfrak{X})$. Theorem A.8 then implies that $F_*\nu = \lambda_1\nu$ if and only if $Z_\nu = c\,\theta_*$ for some $c > 0$. In fact, $c = 1$ since $(\theta_* \cdot \mathcal{L}) = (Z_\nu \cdot \mathcal{L}) = 1$, and so $\nu = \nu_*$. This proves (a), and that $\theta_* = Z_{\nu_*}$.

To prove (b) and (c), pick any $\nu \in \mathcal{V}_0$ with $\alpha(\nu) \geq 0$. It follows from Theorem A.8 that $\lambda_1^{-n}F_*^nZ_\nu \to (Z_\nu \cdot \theta^*)\theta_* = (Z_\nu \cdot \theta^*)Z_{\nu_*}$ as $n \to \infty$. Thus $F_\bullet^n\nu \to \nu_*$ as long as $(Z_\nu \cdot \theta^*) > 0$. Since Z_ν is nef and $(\theta^* \cdot \theta^*) = 0$, the Hodge inequality implies that $(Z_\nu \cdot \theta^*) \geq 0$ with equality if and only if $Z_\nu = c\theta^*$ for some $c > 0$. In the latter case, ν is uniquely determined, $\alpha(\nu) = (Z_\nu \cdot Z_\nu) = 0$, and $F_\bullet\nu = \nu$ as $F_*\theta^* = (\lambda_2/\lambda_1)\theta^*$. □

PROPOSITION 2.5. *The asymptotic degree λ_1 is a quadratic integer. Moreover, when $\lambda_2 < \lambda_1^2$, we are in one of the following two cases*:

(i) $\lambda_1 \in \mathbf{N}$ *and ν_* is divisorial or infinitely singular*;

(ii) $\lambda_1 \notin \mathbf{Q}$ *and ν_* is irrational.*

Proof. The fact that λ_1 is a quadratic integer is contained in [30, Th. A′]. An outline of the proof goes as follows.

When ν_* is divisorial, it follows from the discussion above Proposition 2.1 that $\lambda_1 = d(F, \nu_*) \in \mathbf{N}$.

When ν_* is not divisorial, Theorem 7.7 in [30] provides local normal forms of F at some point at infinity of some suitable admissible compactification of $\mathbf{C}^2$. In the infinitely singular case, one sees by inspection that λ_1 is an integer.

If instead ν_* is irrational, then the local normal form at p is monomial, and λ_1 is the spectral radius of a 2×2 matrix M having nonnegative integer coefficients. Suppose $\lambda_1 \in \mathbf{Q}$. Then the other eigenvalue λ_1' of M is also rational. Now ν_* being irrational means that M has an eigenvector $(u, v) \in \mathbf{R}_+^2$ with u/v irrational. This is only possible if $\lambda_1' = \lambda_1$. But then the local topological degree of F at p equals $\det M = \lambda_1^2$, contradicting $\lambda_2 < \lambda_1^2$. □

PROPOSITION 2.6. *When $\lambda_2 < \lambda_1$, the eigenvaluation ν_* cannot be divisorial.*

Proof. Assume ν_* is divisorial. Pick an admissible compactification X of $\mathbf{C}^2$ such that ν_* is proportional to ord_E for some prime E of X. Then the rational lift $\widetilde{F} : X \dashrightarrow X$ maps E onto itself, and the eigenvalue λ_1 is the coefficient of E in $\widetilde{F}^* E$. This coefficient is dominated by the topological degree of F in a neighborhood of E. Hence $\lambda_1 \leq \lambda_2$. □

In the proof of Theorem A, the case when ν_* is divisorial and also an end in $\mathcal{V}_1$ needs special treatment. It occurs exactly when $\alpha(\nu_*) = 0 > A(\nu_*)$ (that is, ν_* is a rational pencil valuation), or when $\alpha(\nu_*) > 0 = A(\nu_*)$. As the next results show, the dynamics are then quite particular.

PROPOSITION 2.7. *Assume that $\nu \in \mathcal{V}_1$ is a rational pencil valuation such that $F_*\nu = \lambda\nu$ for some $\lambda > 0$. Then λ is an integer and F is conjugate by a polynomial automorphism of $\mathbf{C}^2$ to a skew product of the form $F(x, y) = (P(x), Q(x, y))$ with $\deg_y Q = \lambda$.*

Proof. As in [30, §7.4] this follows from the Line Embedding Theorem. □

PROPOSITION 2.8. *Assume $\lambda_1 > 1$. If there exists a valuation $\nu_* \in \mathcal{V}_0$ (not necessarily divisorial) with $F_*\nu_* = \lambda_1\nu_*$ and $\alpha(\nu_*) > 0 = A(\nu_*)$, then F is a counterexample to the Jacobian conjecture.*

Proof. The change of variables formula implies $A(F_*\nu) = A(\nu) + \nu(JF)$ for all $\nu \in \mathcal{V}_1$, where JF is the Jacobian determinant of F; see [30, Lemma 7.6]. Applying this to $\nu = \nu_*$ yields $\nu_*(JF) = 0$. As $\alpha(\nu_*) > 0$, this is only possible if JF is constant. But if F were a polynomial automorphism, we would have $\alpha(\nu_*) = (\theta_*(F) \cdot \theta_*(F)) = (\theta^*(F^{-1}) \cdot \theta^*(F^{-1})) = 0$. □

2.3. *Examples of eigenvaluations.* Curve valuations do not belong to $\mathcal{V}_1$, thus can never be eigenvaluations. As the following examples show, any other type of valuation can occur.

Example 2.9. Assume that the extension to $\mathbf{P}^2$ of $F : \mathbf{C}^2 \to \mathbf{C}^2$ does not contract the line at infinity. Then the divisorial valuation $-\deg$ is an eigenvaluation.

Example 2.10. Any rational pencil valuation appears as an eigenvaluation. Indeed, in suitable affine coordinates, the valuation is associated to the pencil $x = \text{const}$ and is the eigenvaluation of a suitable skew product $F(x, y) = (P(x), Q(x, y))$.

Example 2.11. Pick positive integers a, b, c, d such that

$$\Delta := (a + d)^2 - 4(ad - bc) > 0.$$

Then the 2×2 matrix M with entries a, b, c, d has two real eigenvalues $t > 1$ and $t' < t$. The eigenvalue t admits an eigenvector $(u, v) \in \mathbf{R}^2_-$ that we can normalize by the condition $\min\{u, v\} = -1$. The map $F(x, y) := (x^a y^b, x^c y^d)$ has topological degree $\lambda_2 = |ad - bc|$ and asymptotic degree $\lambda_1 = t$. It admits a unique eigenvaluation which is the monomial valuation with weights u on x and v on y. When Δ is not a square, the eigenvaluation is irrational. Otherwise it is divisorial.

One may also perturb F by adding terms of sufficiently low order. After doing so, λ_1, λ_2, and the eigenvaluation remain unchanged.

Example 2.12. The eigenvaluation ν_* of a polynomial automorphism F with $\lambda_1 > 1$ is always infinitely singular. Indeed, the eigenclass $\theta_* \in \mathrm{L}^2(\mathfrak{X})$ can be written $\theta_*(F) = \theta^*(F^{-1})$, hence $\alpha(\nu_*) = (\theta_*(F) \cdot \theta_*(F)) = 0$. Thus ν_* cannot be irrational, as that would imply $\alpha(\nu_*) \notin \mathbf{Q}$. It cannot be divisorial by Proposition 2.6. Hence it is infinitely singular. Proposition 2.5 now implies that the asymptotic degree λ_1 is an integer, a fact which also follows from the Friedland-Milnor classification [33].

It would be interesting to investigate the relation between the eigenvaluation ν_*, the solenoids constructed by Hubbard et al. [42], [40], and the singularity of the Green function of F at infinity; see also [29, Prop. 6.9].

Example 2.13. The argument above shows more generally that the eigenvaluation of any polynomial mapping $F : \mathbf{C}^2 \to \mathbf{C}^2$ with $\lambda_2 < \lambda_1 \in \mathbf{N}$ must be

infinitely singular. The condition of λ_1 being an integer is satisfied, for instance, if F defines an algebraically stable map on $\mathbf{P}^2$, in which case $\lambda_1 = \deg F$ [32].

3. Stability when $\lambda_2 < \lambda_1^2$: the nondivisorial case

Our aim is now to prove precise versions of Theorem A and B in the case when the eigenvaluation is nondivisorial. Recall that we are always in this situation when $\lambda_2 < \lambda_1$; see Proposition 2.6.

THEOREM 3.1. *Let $F : \mathbf{C}^2 \to \mathbf{C}^2$ be a dominant polynomial mapping with $\lambda_2 < \lambda_1^2$ such that the eigenvaluation ν_* is nondivisorial. Then there exists a tight compactification X of $\mathbf{C}^2$, a point $p \in X \setminus \mathbf{C}^2$ and local coordinates (z, w) at p such that the lift $\widetilde{F} : X \dashrightarrow X$ defines a superattracting fixed point germ at p taking one of the following forms*:

(a) $\widetilde{F}(z, w) = (z^a w^b, z^c w^d)$, *where $a, b, c, d \in \mathbf{N}$ and the 2×2 matrix M with entries a, b, c, d has spectral radius λ_1; locally at p we have $X \setminus \mathbf{C}^2 = \{zw = 0\}$;*

(b) $\widetilde{F}(z, w) = (z^{\lambda_1}, \mu z^c w + P(z))$, *where $c \geq 1$, $\mu \in \mathbf{C}^*$, and $P \not\equiv P(0) = 0$ is a polynomial; locally at p we have $X \setminus \mathbf{C}^2 = \{z = 0\}$.*

Moreover, if F is not conjugate to a skew product by a polynomial automorphism of $\mathbf{C}^2$, there exists $n \geq 1$ such that each prime of X is contracted to p by $\widetilde{F}^n$. When F is conjugate to a skew product, the same conclusion holds for all primes of X with the exception of a single prime invariant by $\widetilde{F}$.

Remark 3.2. The skew product case in Theorem 3.1 does not occur when $\lambda_2 < \lambda_1$. Indeed, if $F(x, y) = (P(x), Q(x, y))$, then

$$\lambda_2 = \deg P \cdot \deg_y Q \geq \max\{\deg P, \deg_y Q\} = \lambda_1.$$

COROLLARY 3.3. *Under the assumptions of Theorem* 3.1, *for any $\nu \in \mathcal{V}_1$, there exists $n = n(\nu)$ such that the sequence $(d(F^j, \nu))_{j \geq n}$ satisfies an integral linear recursion formula of order* 1 *or* 2. *In particular, $(\deg F^j)_{j \geq n}$ satisfies such a recursion formula for n large enough.*

Proof of Theorem 3.1. Since ν_* is nondivisorial, its center on any admissible compactification of $\mathbf{C}^2$ is a point. We may therefore define an infinite sequence $(X_m)_{m \geq 0}$ of admissible compactifications of $\mathbf{C}^2$ by setting $X_0 = \mathbf{P}^2$ and letting X_{m+1} be the blow-up of X_m at the center $p_m \in X_m$ of ν_*. Let $\nu_0, \dots, \nu_m$ be the normalized divisorial valuations associated to the primes of X_m. As the eigenvaluation ν_* lies in $\mathcal{V}_1$, the compactification X_m is tight by Corollary 1.5. Hence, for any j, either $\alpha(\nu_j) > 0$ or ν_j is a rational pencil valuation.

We claim that there exists $m \geq 0$ such that the lift $\widetilde{F}_m : X_m \dashrightarrow X_m$ defines a superattracting fixed point germ at p_m, taking one of the forms (a)

or (b) above. This follows from the proof of Theorem 7.7 in [30]. Let us briefly indicate how to proceed.

When ν_* is infinitely singular, there exists an infinite subsequence $(m_j)_{j\geq 1}$ such that $X_{m_j} \setminus \mathbf{C}^2$ is locally irreducible at p_{m_j}. The corresponding valuations ν_{m_j+1} increase to ν_* as $j \to \infty$. Consider the segment $I_j :=]\nu_{m_j+1}, \nu_*]$ in $\mathcal{V}_1$ and the corresponding open neighborhood U_j of ν_* in $\mathcal{V}_0$ consisting of valuations whose tree retraction to the closed segment $\overline{I_j}$ belongs to I_j. The attracting properties of ν_* imply that $F_\bullet I_j \subset I_j$ and $F_\bullet U_j \subset\subset U_j$ for large j. Now U_j is the set of valuations whose center on X_{m_j} equals p_{m_j}, so this means that $\widetilde{F}_{m_j}$ defines a holomorphic fixed point germ at p_{m_j} which in fact is rigid in the sense of [27]. The normal form in (b) follows from the classification in [27]. A direct inspection shows that $\widetilde{F}_{m_j}$ is superattracting at p_{m_j}.

If instead ν_* is irrational, then for all large m, p_m is the intersection of two primes of X_m. Now $F_\bullet$ preserves the tree structure on $\mathcal{V}_0$: if I_0 is any small open segment in $\mathcal{V}_0$ containing ν_* and I a small subsegment, then $F_\bullet$ is a homeomorphism of I onto its image $F_\bullet I \subset I_0$. Moreover, $F|_{I_0}$ is contracting at ν_* in the skewness metric. This means that if I is sufficiently symmetric around ν_*, then $F_\bullet I \subset I$. (The symmetry condition is only necessary when $F_\bullet$ is order-reversing on I_0.) Let $\mu_m, \mu'_m \in \mathcal{V}_0$ be the normalized divisorial valuations associated to the primes of X_m containing p_m and set $I_m =]\mu_m, \mu'_m[$. By repeating the arithmetic argument as in [30, Lemma 5.6], one shows that there exists an infinite subsequence $(m_j)_{j\geq 1}$ such that I_{m_j} is sufficiently symmetric so that $F_\bullet I_{m_j} \subset I_{m_j}$. For j large enough we will also have $F_\bullet U_{m_j} \subset U_{m_j}$, where U_m is the set of valuations whose center on X_m equals p_m. Thus $\widetilde{F}_{m_j}$ defines a holomorphic fixed point germ at p_{m_j}, which, by invoking [27], can be put in the monomial form (a) above. Finally $\widetilde{F}_{m_j}$ must be superattracting at p_{m_j} or else one of the primes of X_{m_j} containing p_{m_j} would be an eigenvaluation for F^2 with eigenvalue λ_1^2, contradicting Theorem 2.4(a).

In both the irrational and infinitely singular case we have found a tight compactification X of $\mathbf{C}^2$ and a point $p \in X \setminus \mathbf{C}^2$ such that the lift $\widetilde{F} : X \dashrightarrow X$ defines a superattracting fixed point germ at p. The point p defines an open subset $U(p)$ of $\mathcal{V}_0$: a valuation $\nu \in \mathcal{V}_0$ belongs to $U(p)$ if and only if the center of ν on X equals p.

Now pick any prime E of X and let $\nu_E \in \mathcal{V}_0$ be the associated divisorial valuation. We have $\alpha(\nu_E) \geq 0$ since X is tight. If $\alpha(\nu_E) > 0$, then Theorem 2.4(b) shows that $F_\bullet^n \nu_E \in U(p)$ for $n \gg 1$. This means that $\widetilde{F}^n$ contracts E to p. If instead $\alpha(\nu_E) = 0$, then ν_E must be a rational pencil valuation. Theorem 2.4(c) shows that $\widetilde{F}^n$ still contracts E to p for $n \gg 1$ unless $F_\bullet \nu_E = \nu_E$. In the latter case, F is conjugate to a skew product by Proposition 2.7. $\square$

Proof of Corollary 3.3. Write $\nu_j = F_\bullet^j(\nu)$ for $j \geq 0$. Then $d(F^j,\nu) = \prod_{i=0}^{j-1} d(F,\nu_i)$. We may assume $F_\bullet^n \nu \to \nu_*$ as $n \to \infty$, since otherwise $F_\bullet \nu = \nu$ and $d(F^j,\nu) = d(F,\nu)^j$.

If ν_* is infinitely singular, then $d(F,\cdot) \equiv \lambda_1$ in a neighborhood of ν_*. Since $\nu_j \to \nu_*$ as $j \to \infty$, we get $d(F^{j+1},\nu) = \lambda_1 d(F^j,\nu)$ for $j \gg 0$.

When instead ν_* is irrational, we use the local monomial form in Theorem 3.1(a). Let $E_z = \{z = 0\}$ and $E_w = \{w = 0\}$ be the primes of X containing p and write $b_z = -\operatorname{ord}_{E_z}(L)$ and $b_w = -\operatorname{ord}_{E_w}(L)$ for a generic affine function L on $\mathbf{C}^2$. For $j \geq n$, set $s_j := F_*^j\nu(z) > 0$ and $t_j := F_*^j\nu(w) > 0$. Then $(s_{j+1},t_{j+1}) = M(s_j,t_j)$. Now $d(F^j,\nu) = b_z s_j + b_w t_j$, so this easily implies that $(d(F^j,\nu))_{j\geq n}$ satisfies an integral linear recursion formula of order at most two. □

4. Stability when $\lambda_2 < \lambda_1^2$: the divisorial case

Next we prove Theorem A and B in the case when the eigenvaluation ν_* is divisorial. Recall that this implies $\lambda_2 \geq \lambda_1$. We distinguish between two subcases: ν_* may or may not be an end in the tree $\mathcal{V}_1$.

When ν_* is an end, either F is conjugate to a skew product or F is a counterexample to the Jacobian conjecture; see Propositions 2.7 and 2.8.

THEOREM 4.1. *Let $F : \mathbf{C}^2 \to \mathbf{C}^2$ be a dominant polynomial mapping with $\lambda_2 < \lambda_1^2$. Assume that the eigenvaluation ν_* is divisorial and an end in $\mathcal{V}_1$. Then there exist a tight compactification X of $\mathbf{C}^2$, a prime E_* of X, a point p on E_*, and an integer $n \geq 1$ such that the lift $\widetilde{F} : X \dashrightarrow X$ maps E_* onto E_* and defines a holomorphic fixed point germ at p. We are in one of the following situations*:

(i) *for each prime $E \neq E_*$ of X, either $\widetilde{F}^n(E) = E_*$, or $\widetilde{F}^n$ contracts E to p;*
(ii) *F is conjugate to a skew product by a polynomial automorphism of $\mathbf{C}^2$ and the properties in* (i) *hold for all primes E of X with the exception of a single prime invariant by $\widetilde{F}$.*

COROLLARY 4.2. *Under the assumptions of Theorem* 4.1, *for any $\nu \in \mathcal{V}_1$, there exists $n = n(\nu)$ such that the sequence $(d(F^j,\nu))_{j\geq n}$ satisfies an integral linear recursion formula of order* 1 *or* 2. *In particular, $(\deg F^j)_{j\geq n}$ satisfies such a recursion formula for n large enough.*

When ν_* is not an end, so that $\alpha(\nu_*) > 0 > A(\nu_*)$, the result is slightly less precise and the proof more subtle.

THEOREM 4.3. *Let $F : \mathbf{C}^2 \to \mathbf{C}^2$ be a dominant polynomial mapping with $\lambda_2 < \lambda_1^2$. Assume that the eigenvaluation $\nu_* \in \mathcal{V}_1$ is divisorial and not an end*

in $\mathcal{V}_1$. *Then there exist a tight compactification* X *of* $\mathbf{C}^2$, *a prime* E_* *of* X, *and an integer* $n \geq 1$ *such that the lift* $\widetilde{F} : X \dashrightarrow X$ *maps* E_* *onto* E_*, *and we are in one of the following situations*:

(i) *for each prime* $E \neq E_*$ *of* X, *either* $\widetilde{F}^n(E) = E_*$, *or* $\widetilde{F}^n$ *contracts* E *to a point on* E_* *at which all iterates of* $\widetilde{F}$ *are holomorphic*;

(ii) F *is conjugate to a skew product by a polynomial automorphism of* $\mathbf{C}^2$ *and the properties in* (i) *hold for all primes* E *of* X *with the exception of a single prime invariant by* $\widetilde{F}$.

COROLLARY 4.4. *Under the assumptions of Theorem* 4.3, *there exist* $l \geq 1$ *and, for any* $\nu \in \mathcal{V}_1$, $n = n(\nu)$ *such that the sequence* $(d(F^{lj+k}, \nu))_{j \geq 0}$ *satisfies an integral linear recursion formula of order* 1 *or* 2 *for any* $k \geq n$. *In particular,* $(\deg F^{lj+k})_{j \geq 0}$ *satisfies such a recursion formula for any sufficiently large* k.

Example 4.5. Define $F(x, y) = (x(x - y^2), x + y)$ and set $a_j = \deg(F^j)$. Then $a_0 = 1$, $a_1 = 3$, $a_2 = 6$ and $a_j = a_{j-1} + a_{j-2} + 2a_{j-3}$ for $j \geq 3$. One checks that $\lambda_1 = 2$, $\lambda_2 = 3$, and that $(a_j)_0^\infty$ does not satisfy any integral recursion formula of order smaller than three.

Proof of Theorem 4.1. The proof is similar to the infinitely singular case of Theorem 3.1. Let X_0 be the smallest admissible compactification such that the center of ν_* on X_0 is a prime E_* of X_0: it is obtained from $\mathbf{P}^2$ by successively blowing up the center of ν_*, so X_0 is tight by Corollary 1.5. Inductively define a sequence $(X_m)_{m \geq 0}$ of tight compactifications by letting X_{m+1} be the blow-up of X_m at the unique intersection point of (the strict transform of) E_* and the unique prime $E_m \neq E_*$ of X_m intersecting E_*.

Let $\nu_m \in \mathcal{V}_0$ be the divisorial valuation associated to E_m. Then $(\nu_m)_{m \geq 0}$ form a sequence of valuations increasing to ν_*. Set $I_m =]\nu_m, \nu_*[$. As in [30, §7.3] we see that $F_\bullet$ maps I_m into itself for large m. Moreover, we have $F_\bullet U_m \subset U_m$, where $U_m \subset \mathcal{V}_0$ is the set of valuations whose center on X_m equals p_m. Set $X = X_m$ and $p = p_m$. Then the lift $\widetilde{F} : X \dashrightarrow X$ maps E_* onto itself and defines a holomorphic fixed point germ at p.

Let U be the open neighborhood of ν_* in $\mathcal{V}$ consisting of valuations whose center on X is contained in E_*. Pick a prime $E \neq E_*$ of X. If $F_\bullet \nu_E = \nu_E$, then E is unique with this property, ν_E is a rational pencil valuation, and F is conjugate to a skew product. For all other primes E we have $F_\bullet^n \nu_E \in U$ for $n \gg 1$. If $F_\bullet^n \nu_E = \nu_*$, then $\widetilde{F}^n(E) = E_*$. Otherwise E is contracted by $\widetilde{F}^n$ to $p \in E_*$. This completes the proof. □

Proof of Corollary 4.2. We can use the same proof as that of Corollary 3.3 in the irrational case. Indeed, let E and E_* be the primes of X containing p and write $E = \{z = 0\}$, $E_* = \{w = 0\}$ for coordinates (z, w) at p. We may perhaps not arrange that F is monomial in (z, w), but if we set $s_j := F_*^j \nu(z) > 0$ and

$t_j := F_*^j \nu(w) > 0$, we will nevertheless have $(s_{j+1}, t_{j+1}) = M(s_j, t_j)$ for some 2×2 matrix M with nonnegative integer entries; see [30, Th. 7.4] □

Proof of Theorem 4.3. Since ν_* is not an end in $\mathcal{V}_1$, we have $\alpha(\nu_*) > 0 > A(\nu_*)$. Let X_0 be the smallest admissible compactification such that the center of ν_* on X_0 is a prime E_* of X_0: it is obtained from $\mathbf{P}^2$ by successively blowing up the center of ν_*, so X_0 is tight by Corollary 1.5.

LEMMA 4.6. *There exists a tight compactification X of $\mathbf{C}^2$ dominating X_0 such that the lift $\widetilde{F} : X \dashrightarrow X$ is holomorphic at all periodic points of $\widetilde{F}|_{E_*}$, where E_* denotes the center of ν_* on X.*

Using this lemma we now conclude the proof. Let U be the open neighborhood of ν_* in $\mathcal{V}$ consisting of valuations whose center on X is contained in E_*. Pick a prime $E \neq E_*$ of X. If $F_\bullet \nu_E = \nu_E$, then E is unique with this property, ν_E is a rational pencil valuation, and F is conjugate to a skew product. For all other primes E we have $F_\bullet^n \nu_E \in U$ for $n \gg 1$. If $F_\bullet^n \nu_E = \nu_*$, then $\widetilde{F}^n(E) = E_*$. Otherwise E is contracted by $\widetilde{F}^n$ to a point $p \in E_*$. By increasing n we can assume that the orbit of p under $\widetilde{F}|_{E_*}$ is either periodic or infinite. In the first case, $\widetilde{F}$ is holomorphic at p by Lemma 4.6. In the second case, we can by increasing n assume that the orbit does not intersect the indeterminacy set of $\widetilde{F}$. This completes the proof. □

Proof of Lemma 4.6. Write $\widetilde{F_0}$ for the lift of F_0 to X_0. Let $Z \subset E_*$ be the (finite) set of periodic points of $\widetilde{F_0}|_{E_*}$ whose orbit contains an indeterminacy point of $\widetilde{F_0}$. First assume for simplicity that Z consists of a single periodic orbit $p_0, p_1, \dots, p_l = p_0$ of length $l \geq 1$ for $\widetilde{F_0}|_{E_*}$.

Let $\mu_k \in \mathcal{V}_0$ be the divisorial valuation associated to the blow-up of X_0 at p_k, $0 \leq k \leq l$. The segment $J_k :=]\mu_k, \nu_*[$ in $\mathcal{V}_1$ has length $(b_* b_k)^{-1}$ in the skewness metric, where $b_* = -\operatorname{ord}_{E_*}(L)$, $b_k = -\operatorname{ord}_{p_k}(L)$ for a generic affine function L on $\mathbf{C}^2$.

For large positive integers $m_0, m_1, \dots, m_l = m_0$ to be determined shortly, define valuations ν_k, $0 \leq k < l$ as follows: blow up p_k, then successively m_k times blow up the intersection point between the (strict transform of) E_* and the previously obtained exceptional divisor. The segment $I_k :=]\nu_k, \nu_*[\subset J_k$ then has length $(b_*(b_k + m_k b_*))^{-1}$ in the skewness metric.

For m_k large, the segment I_k is small enough so that $F_\bullet$ maps I_k homeomorphically onto a subsegment of J_{k+1}. Moreover, when the segments are parametrized by skewness, $F_\bullet$ is given by a Möbius map with nonnegative integer coefficients; see [30, Th. 7.4]. Thus the one-sided derivative of $F_\bullet$ on I_k at ν_* is a well-defined rational number $s_k > 0$.

The key fact is now that the Möbius property above implies that the iterate $F_\bullet^l$ maps the segment I_0 into itself and that either $F_\bullet^l \equiv \mathrm{id}$ or $F_\bullet^l$ is

a contraction on I_0; see [30, Lemma 5.5]. The former case is impossible by Theorem 2.4. Hence we conclude that $\prod_{k=0}^{l-1} s_k < 1$.

Pick $\varepsilon > 0$ such that $\prod_{k=0}^{l-1} s_k \le (1-2\varepsilon)^l$. We may then pick the integers m_k (with $m_0 = m_l$) above arbitrarily large so that

$$\frac{s_k}{m_k b_* + b_k} \le (1-\varepsilon)\frac{1}{m_{k+1} b_* + b_{k+1}} \quad \text{for } 0 \le k < l; \tag{4.1}$$

we just need to make m_{k+1}/m_k slightly smaller than $1/s_k$. By the definition of s_k, (4.1) implies that $F_\bullet$ maps I_k into I_{k+1}. Let U_k be the open subset of $\mathcal{V}_0$ consisting of valuations whose tree retraction to the closed segment $\overline{I_k}$ is contained in I_k. Then $F_\bullet$ maps U_k into U_{k+1} for $0 \le k < l$, assuming the m_k's are large enough (again, by convention, $U_l = U_0$).

Let X be the smallest admissible compactification of $\mathbf{C}^2$ dominating X_0 such that the center of ν_k is one-dimensional for $0 \le k < l$. Then X is obtained from X_0 by performing all the blow-ups mentioned above, so X is tight and the morphism $X \to X_0$ induced by the identity on $\mathbf{C}^2$ is an isomorphism above $X_0 \setminus Z$.

The center of ν_k on X intersects (the strict transform of) E_* at some point q_k and the open set U_k above exactly consists of the valuations in $\mathcal{V}_0$ whose center on X equals q_k. Thus $F_\bullet U_k \subset U_{k+1}$ translates into the lift $\widetilde{F} : X \dashrightarrow X$ of F being holomorphic at q_k.

This completes the proof when Z consists of a single periodic orbit. In general, there are several orbits, but we can handle them one at a time. □

Proof of Corollary 4.4. There is a finite subset $Z \subset E_*$ such that if $p \in E_* \setminus Z$, we have $d(F,\cdot) \equiv \lambda_1$ on the open set $U(p) \subset \mathcal{V}_0$ of valuations whose center on X is p. We may pick $l \ge 1$ such that any periodic orbit of $\widetilde{F}|E_*$ intersecting Z has order dividing l.

As in the proof of Corollary 3.3, we may assume $F_\bullet^n \nu \to \nu_*$ as $n \to \infty$. Similarly, we may assume $F_\bullet^n \nu \ne \nu_*$ for all n.

Pick $n = n(\nu)$ so that the center of $F_\bullet^k \nu$ on X is a point $p_k \in E_*$ for $k \ge n$. After increasing n, we may assume that the orbit $p_n, p_{n+1}, \dots$ is either disjoint from Z or periodic of order (dividing) l. In the first case, $d(F^{k+j+1}, \nu) = \lambda_1 d(F^{k+j}, \nu)$ for any $j \ge 0$. In the second case, we conclude the proof as in Corollary 3.3. □

5. **Maximum topological degree:** $\lambda_2 = \lambda_1^2$

Next we turn to maps with maximum topological degree $\lambda_2 = \lambda_1^2$. As we do not have an analog of Theorem A.8 at our disposal, we base our analysis on a detailed description of the dynamics of $F_\bullet$ on $\mathcal{V}_1$ using tree arguments.

5.1. *Dynamics on* $\mathcal{V}_1$. The results of this section form the basis for the proof of Theorems A, B, and C in the case $\lambda_2 = \lambda_1^2$. Define

$$\mathcal{T}_F := \{\nu \in \mathcal{V}_1 \mid F_\bullet \nu = \nu\}. \tag{5.1}$$

This set is nonempty by Proposition 2.3. The following three results summarize the structure of $\mathcal{T}_F$ and its dynamical significance.

PROPOSITION 5.1. *Suppose* $\deg F^n/\lambda_1^n$ *is unbounded. Then* $\mathcal{T}_F = \{\nu_*\}$ *is a singleton, where* ν_* *is a rational pencil valuation, and* $F_\bullet^n \to \nu_*$ *on* $\mathcal{V}_1$ *as* $n \to \infty$. *Moreover,* F *is not proper,* $\deg F^n \sim n\lambda_1^n$, *and there exist affine coordinates in which* $F(x,y) = (P(x), A(x)y^{\lambda_1} + O_x(y^{\lambda_1 - 1}))$, *where* $\deg P = \lambda_1$ *and* $\deg A \geq 1$.

PROPOSITION 5.2. *Suppose* $\deg F^n$ *is bounded. Then* F *is a polynomial automorphism of* $\mathbf{C}^2$. *In suitable affine coordinates, either*

(a) F *is an affine map and* $-\deg \in \mathcal{T}_F$, *or*

(b) F *is a skew product of the form* $F(x) = (ax + b, cy + P(x))$, *where* $a, c \in \mathbf{C}^*$, $b \in \mathbf{C}$; *we may then assume* $\mathcal{T}_F = [\nu_0, \nu_1]$, *where* ν_1 *is associated to the pencil* $x = \text{const}$ *and* ν_0 *is a monomial valuation satisfying* $\nu_0(y) = -1$, $\nu_0(x) = -1/q$, *where* $q = \deg P > 1$.

PROPOSITION 5.3. *Suppose* $\deg F^n/\lambda_1^n$ *is bounded and* $\lambda_1 > 1$. *Then* F *is proper. Moreover*:

(a) *either* $\mathcal{T}_F$ *consists of a single quasimonomial valuation* $\nu_* \in \mathcal{V}_1$ *with* $\alpha(\nu_*) > 0$, *or* $\mathcal{T}_F$ *is a closed segment in* $\mathcal{V}_1$ *whose endpoints are divisorial valuations*;

(b) $\mathcal{T}_{F^n} = \mathcal{T}_{F^2}$ *for* $n \geq 2$, *and either* $\mathcal{T}_F = \mathcal{T}_{F^2}$ *or* $\mathcal{T}_F$ *is a singleton, lying in the interior of* $\mathcal{T}_{F^2}$;

(c) *for* $\nu \in \mathcal{V}_1$, $F_\bullet^{2n}\nu \to r(\nu)$ *as* $n \to \infty$, *where* $r : \mathcal{V}_1 \to \mathcal{T}_{F^2}$ *is the natural retraction*;

(d) *in suitable affine coordinates, all the valuations in* $\mathcal{T}_{F^2}$ *are monomial.*

The convergence in (c) holds in a strong sense: $F_\bullet^{2n}\nu \to r(\nu)$ weakly and $A(F_\bullet^{2n}\nu) \to A(r(\nu))$.

The proofs of these results are given in Section 5.4.

5.2. *Proof of Theorem* C. If $\deg(F^n)/\lambda_1^n$ is unbounded, then we are in case (1) by Proposition 5.1.

If $\deg(F^n)$ is bounded, then we pick suitable affine coordinates as in Proposition 5.2. When F is affine, it extends holomorphically to $X = \mathbf{P}^2$. When F is a skew product as in (b), it extends holomorphically to the Hirzebruch surface $X = \mathbf{F}_q$. Indeed, we can view $\mathbf{F}_q$ as a toric surface associated to the complete

fan generated by the vectors $(1,0)$, $(0,1)$, $(0,-1)$ and $(-q,-1)$ in $\mathbf{R}^2$; see [34, pp. 6–8]. Then X is a compactification of $\mathbf{C}^2$ and $X \setminus \mathbf{C}^2$ is a union of two rational curves, corresponding to the centers of ν_0 and ν_1 on X. As ν_0 and ν_1 are totally invariant under $F_\bullet$, the lift of F to $\mathbf{F}_q$ is holomorphic.

Finally, when $\deg(F^n)/\lambda_1^n$ is bounded but $\lambda_1 > 1$, we apply Proposition 5.3. Hence we may assume that $\mathcal{T}_{F^2}$ consists of monomial valuations. If $\mathcal{T}_F$ contains a divisorial valuation ν_*, then $\nu_*(x) = -p/q$, $\nu_*(y) = -1$, where $q \geq p \geq 1$ and $\gcd(p,q) = 1$. Let $X := X_{p,q}$ be the toric surface associated to the complete fan generated by the vectors $(1,0)$, $(0,1)$, and $(-p,-q)$ in $\mathbf{R}^2$. Then X has at worst quotient singularities and is in fact a weighted projective plane; see below. Note that $X \setminus \mathbf{C}^2$ is a single, totally invariant, rational curve. As ν_* is totally invariant under $F_\bullet$, F lifts to a holomorphic self-map of X.

If $\mathcal{T}_F$ contains no divisorial valuation, then $\mathcal{T}_F = \{\nu_*\}$, where ν_* is an irrational valuation belonging to the interior of $\mathcal{T}_{F^2}$. Note that $d(F,\cdot)$ cannot be locally constant at ν_*, or else the tangent map of F at ν_* (see below) would be the identity and $\mathcal{T}_F = \mathcal{T}_{F^2}$. Thus $\lambda_1 = d(F,\nu_*)$ is irrational.

Pick any divisorial valuation $\nu \in \mathcal{T}_{F^2}$ with $\nu < \nu_*$, and set $\nu' = F_\bullet \nu$. Then $\nu' > \nu_*$, and ν and ν' are both totally invariant under $F_\bullet^2$. We have $\nu(y) = \nu'(y) = -1$, $\nu(x) = -p/q$, $\nu'(x) = -p'/q'$ for some integers with $q \geq p \geq 1$, $q' > p' \geq 1$, and $\gcd(p,q) = \gcd(p',q') = 1$. Define X to be the toric surface associated to the complete fan generated by $(1,0)$, $(0,1)$, $(-p,-q)$, $(-p',-q')$ in $\mathbf{R}^2$. Then X has at worst quotient singularities, and $X \setminus \mathbf{C}^2$ consists of two irreducible rational curves E and E' corresponding to the centers of ν and ν' on X. As these valuations are permuted by $F_\bullet$ and totally invariant by $F_\bullet^2$, F lifts to a holomorphic self-map of X which permutes E and E'. This completes the proof of Theorem C.

5.3. *Weighted projective spaces and normal forms.* Suppose $\lambda_2 = \lambda_1^2$, $\deg F^n/\lambda_1^n$ is bounded, and $\lambda_1 > 1$.

Assume that $\mathcal{T}_F$ contains a divisorial valuation ν_*, with $\nu_*(x) = -p/q$, $\nu_*(y) = -1$, where $q \geq p \geq 1$ are relatively prime integers. We saw that F is holomorphic on the toric surface $X_{p,q}$. Conversely any polynomial map of $\mathbf{C}^2$ which extends as a holomorphic map to $X_{p,q}$ satisfies $\lambda_2 = \lambda_1^2$, and $\deg F^n/\lambda_1^n$ is bounded. The surface $X_{p,q}$ is the weighted projective space with homogeneous coordinates $[x : y : z] \sim [\lambda^p x : \lambda^q y : \lambda z]$ for all $\lambda \in \mathbf{C}^*$; see [34, p. 35], [24] or [12]. For any polynomial P, let P_+ be its ν_*-leading homogeneous part, i.e., the sum of all monomials $a_{ij}x^i y^j$ in P such that $-(pi/q + j) = \nu_*(P)$. Then a polynomial map $F = (P,Q)$ is holomorphic on $X_{p,q}$ if and only if P_+ and Q_+ have no common zeroes on the weighted projective line $[x : y] \sim [\lambda^p x : \lambda^q y]$, i.e., if and only if $P_+(x^q, y^p)$ and $Q_+(x^q, y^p)$ have no common zeroes in $\mathbf{C}^2 \setminus \{0\}$.

Note that there exist polynomial maps of $\mathbf{C}^2$ which extend to a unique $X_{p,q}$. For such an example, pick any λ_1 divisible by p and q, write $\lambda_1 = pa = qb$

and take $F(x,y)=(P,Q)$ with $P_+=\alpha x^{\lambda_1}+\beta y^{pb}$, $Q_+=\gamma x^{qa}+\delta y^{\lambda_1}$, where $\alpha\beta\gamma\delta\neq 0$. When λ_1 is not divisible by $\mathrm{lcm}(p,q)$, $\mathcal{T}_F$ is not reduced to a singleton.

Pick p,q and p',q' any two pairs of relatively prime integers with associated monomial valuations ν and ν' and $p'/q'>p/q$. Then there exists a polynomial map of $\mathbf{C}^2$ for which $\mathcal{T}_F$ is precisely the segment of monomial valuations $[\nu,\nu']$. Take λ_1 divisible by $\mathrm{lcm}(p,q,p',q')$, write $\lambda_1=pa=qb=p'a'=q'b'$ and define $F(x,y)=(\alpha x^{\lambda_1}+\beta y^{pb}+C_0,\delta y^{\lambda_1}+\gamma x^{q'a'}+C_1)$ with $\alpha\beta\gamma\delta\neq 0$, and $C_0,C_1\in\mathbf{C}$.

Finally if $\mathcal{T}_F$ contains no divisorial valuation, then $\lambda_1\notin\mathbf{N}$ and $\mathcal{T}_F$ consists of a single irrational monomial valuation ν_*. We may assume $\nu_*(x)=-t$, $\nu_*(y)=-1$, where $t\in(0,1)$ is irrational. This leads to

$$P_+=\alpha y^b \quad\text{and}\quad Q_+=\beta x^c,$$

where $b,c\in\mathbf{N}$, $0<b<c$, $bc=\lambda_2=\lambda_1^2$, $t=\sqrt{b/c}$, and $\alpha,\beta\in\mathbf{C}^*$.

5.4. *Proofs of Propositions* 5.1– 5.3. The arguments utilize the tree structure of $\mathcal{V}_1$ in much more detail than other parts of this paper. In particular, we need to exploit the relationship between the parametrizations α and A on the tree $\mathcal{V}_0$ as explained in [30, App. A]. There is an increasing, lower semi-continuous *multiplicity* function $m:\mathcal{V}_0\to\mathbf{N}^*\cup\{+\infty\}$ such that $A(\nu)=-2-\int_{-\deg}^{\nu}m(\mu)d\alpha(\mu)$ for all $\nu\in\mathcal{V}_0$; see [30, Th. A.4]. The multiplicity of any quasimonomial valuation is finite, whereas infinitely singular valuations have infinite multiplicity.

Write JF for the Jacobian determinant of F. The multiplicity function will be primarily exploited through the following *Jacobian formula* (see [30, Lemma 7.6]):

$$A(\nu)+\nu(JF)=d(F,\nu)A(F_\bullet\nu). \tag{5.2}$$

We start by proving some general facts about the set $\mathcal{T}_F$ defined in (5.1).

LEMMA 5.4. *The set $\mathcal{T}_F$ is nonempty. For every $\nu\in\mathcal{T}_F$, $F_*\nu=\lambda_1\nu$ and $F^*Z_\nu=F_*Z_\nu=\lambda_1 Z_\nu$. If F is proper, every $\nu\in\mathcal{T}_F$ is totally invariant under $F_\bullet$.*

Proof. By Proposition 2.3 there exists $\nu\in\mathcal{V}_1$ with $F_*\nu=\lambda_1\nu$, hence $\mathcal{T}_F$ is nonempty. For any such ν, we have $F_*Z_\nu=\lambda_1 Z_\nu$ by Lemma A.6. The condition $\lambda_2=\lambda_1^2$ and the Hodge Index Theorem then imply $F^*Z_\nu=\lambda_1 Z_\nu$. When F is proper, the latter equation implies that ν is totally invariant by Proposition A.7.

Now pick any $\mu\in\mathcal{T}_F$. We must prove that $F_*\mu=\lambda_1\mu$. In any case, $F_*\mu=\lambda\mu$ for some $\lambda>0$. Pick $\nu\in\mathcal{V}_1$ such that $F_*\nu=\lambda_1\nu$. We may assume that $\nu\neq\mu$. By what precedes and by (A.1), $\lambda_1\alpha(\mu\wedge\nu)=(Z_\mu\cdot F^*Z_\nu)=(F_*Z_\mu\cdot Z_\nu)=\lambda\alpha(\mu\wedge\nu)$. Since $\mu\neq\nu$, $\alpha(\mu\wedge\nu)>0$ and so $\lambda=\lambda_1$. □

Proof of Proposition 5.1. Every valuation $\nu \in \mathcal{T}_F$ must have $\alpha(\nu) = 0$, or else $\deg(F^n)/\lambda_1^n$ would be bounded by $1/\alpha(\nu)$. By Theorem B′ and Section 7.4 in [30], there exists a rational pencil valuation $\nu_* \in \mathcal{T}_F$. Moreover, in suitable affine coordinates, ν_* corresponds to the pencil $x = \text{const}$ and F takes the required form. Hence $\deg F^n \sim n\lambda_1^n$.

Now note that since $F^* Z_{\nu_*} = \lambda_1 Z_{\nu_*}$, we have

$$\alpha(F_\bullet^n \nu \wedge \nu_*) = \frac{\lambda_1^n}{d(F^n, \nu)} \alpha(\nu \wedge \nu_*) \tag{5.3}$$

for any $\nu \in \mathcal{V}_1$ and $n \geq 1$.

Apply this to $n = 1$ and $\nu < \nu_*$ close to ν_*. Then $F_\bullet \nu < \nu_*$, so $d(F, \nu) < \lambda_1$ or else $\nu \in \mathcal{T}_F$. Hence $d(F, \cdot)$ is nonconstant near ν_*. By [30, Prop. 7.2] there exists $\nu_0 \in \mathcal{V}_0$, $\nu_0 > \nu_*$ such that $F_* Z_{\nu_0} = c\mathcal{L}$, where $c \geq 0$ and $\mathcal{L}$ is the class of a line. Then $c = (F_* Z_{\nu_0} \cdot Z_{\nu_*}) = \lambda_1 \alpha(\nu_0 \wedge \nu_*) = 0$. In particular, F is not proper; see Proposition 2.1.

It remains to prove that $F_\bullet^n \nu \to \nu_*$ as $n \to \infty$ for every $\nu \in \mathcal{V}_1$. This will in particular imply $\mathcal{T}_{F^n} = \{\nu_*\}$ for all $n \geq 1$. When $\alpha(\nu) > 0$, this follows from (5.3), since $d(F^n, \nu) \geq \deg(F^n)\alpha(\nu)$, so suppose $\alpha(\nu) = 0$ and set $\nu' = F_\bullet \nu$. If $\alpha(\nu') > 0$, then again $F_\bullet^n \nu \to \nu_*$, so suppose $\alpha(\nu') = 0$. Consider the nef Weil class $Z_{\nu'}$ and note that $(F^* Z_{\nu'} \cdot Z_\nu) = 0$. By the Hodge Index Theorem, $F^* Z_{\nu'} = c Z_\nu$, where $c = \lambda_2 / d(F, \nu) > 0$. Let $\nu_0 > \nu_*$ be the valuation with $F_* Z_{\nu_0} = 0$ considered above. Then $0 = (F_* Z_{\nu_0} \cdot Z_{\nu'}) = (Z_{\nu_0} \cdot F^* Z_{\nu'}) = c\alpha(\nu_0 \wedge \nu')$, which implies $\nu' = \nu_*$, completing the proof. □

Proof of Proposition 5.2. Note that $\lambda_1 = \lambda_2 = 1$. It suffices to prove that F is proper. Indeed, then F is a polynomial automorphism, and all the assertions follow from the Friedland-Milnor classification [33].

Now, if F were not proper, by Proposition 2.1 we could find a divisorial valuation $\nu_0 \in \mathcal{V}_0$ such that $F_* Z_{\nu_0} = 0$. Hence, for any $\nu \in \mathcal{T}_F$, $0 = (F_* Z_{\nu_0} \cdot Z_\nu) = \lambda_1 \alpha(\nu_0 \wedge \nu)$, so that $\alpha(\nu) = 0$ and $\nu_0 \geq \nu$. Thus F would be of the form $(ax + b, C(x)y + D(x))$ in suitable coordinates. As F is nonproper, $\deg C \geq 1$, contradicting that $\deg F^n$ is bounded. □

Next we turn to Proposition 5.3, which is significantly harder to prove than the previous two propositions. Assume therefore, for the rest of Section 5.4 that $\lambda_2 = \lambda_1^2 > 1$ and that $\deg F^n / \lambda_1^n$ is bounded. Then F is proper as follows from the proof of Proposition 5.2. Hence, by Lemma 5.4, every $\nu \in \mathcal{T}_F$ is totally invariant under $F_\bullet$.

To continue the proof, we recall the definition of the (tree) tangent map of F at any valuation $\nu \in \mathcal{V}_0$; see [30, §3]. Declare two segments of the form $[\nu_1, \nu[$ and $[\nu_2, \nu[$ to be equivalent if and only if they have nonempty intersection. An equivalence class is called a *tangent vector* at ν and the set $T\nu$ of tangent vectors the *tangent space* at ν. If $\vec{v}$ is a tangent vector, we denote by $U(\vec{v})$

the open set of all valuations determining $\vec{v}$. These open sets form a basis for the weak topology on $\mathcal{V}_0$ [28, Th. 5.1]. As $F_\bullet$ preserves the tree structure it naturally induces a surjective self-map $\mathsf{F} : T\nu \to T\nu$, the *tangent map*, for any eigenvaluation $\nu \in \mathcal{T}_F$.

When $\nu \in \mathcal{T}_F$ is infinitely singular, $T\nu$ is a singleton, so $\mathsf{F} \equiv \mathrm{id}$. When $\nu \in \mathcal{T}_F$ is irrational, $T\nu$ consists of two tangent vectors and $\mathsf{F}^2 \equiv \mathrm{id}$. If instead $\nu \in \mathcal{T}_F$ is divisorial and X is an admissible compactification on which the center of ν is a prime E of X, then there exists a canonical identification of E with the tangent space $T\nu$ at ν as follows. For any point $p \in E$, all valuations centered at p determine the same tangent vector $\vec{v}_p$ at ν. Conversely, all valuations in $U(\vec{v})$ are centered along a connected subspace intersecting E in a single point $p(\vec{v})$; see [28, Th. B.1]. With this identification, F can be viewed as a rational self-map of $E \simeq \mathbf{P}^1$.

LEMMA 5.5. *Assume $\nu_* \in \mathcal{T}_F$ and consider a tangent vector $\vec{v}$ at ν_* represented by a valuation with $\alpha > 0$. If $\vec{v}$ is totally invariant by F, then $F_\bullet \equiv \mathrm{id}$ on a small segment representing $\vec{v}$.*

LEMMA 5.6. *Assume $\nu_* \in \mathcal{T}_F$ is divisorial but not a rational pencil valuation. Then $F_\bullet U(\vec{v}) = U(\mathsf{F}\vec{v})$ for any tangent vector $\vec{v}$ at ν_*.*

With these two lemmas at hand, we continue the proof of Proposition 5.3.

First suppose $\mathcal{T}_F = \{\nu_*\}$ is a singleton. Then ν_* cannot be infinitely singular by Lemma 5.5. Neither can it be a rational pencil valuation, since then $\deg F^n \sim n\lambda_1^n$. Hence ν_* is quasimonomial with $\alpha(\nu_*) > 0$.

If $\mathcal{T}_F$ is not a singleton, pick two distinct valuations $\nu_1, \nu_2 \in \mathcal{T}_F$. The set $[\nu_1, \nu_2] \cap \mathcal{T}_F$ is clearly closed, and Lemmas 5.5 and 5.6 show that it is open. Hence $[\nu_1, \nu_2] \subset \mathcal{T}_F$, so $\mathcal{T}_F$ is a subtree of $\mathcal{V}_1$. Suppose ν is a branch point of $\mathcal{T}_F$. Then ν is divisorial and $\alpha(\nu) > 0$. Each branch of $\mathcal{T}_F$ emanating from ν corresponds to a tangent vector $\vec{v}$ which is totally invariant by the tangent map F at ν, since a valuation in $\mathcal{T}_F$ is totally invariant. Now F can be identified with a rational map on $\mathbf{P}^1$ whose degree equals $\lambda_2/d(F,\nu) = \lambda_1 > 1$. Hence F admits at most two totally invariant tangent vectors. This gives a contradiction. We have shown that $\mathcal{T}_F$ is a nonempty closed segment of $\mathcal{V}_1$.

Suppose that $\mathcal{T}_F \subsetneq \mathcal{T}_{F^2}$ and that $\mathcal{T}_F$ is not a singleton. We can then find a valuation ν which is an interior point of $\mathcal{T}_{F^2}$ but an endpoint of $\mathcal{T}_F$. Consider the tangent map F at ν. We see that F^2 admits two totally invariant tangent vectors, exactly one of which is (totally) invariant by F. This is a contradiction. Similarly, $\mathcal{T}_F$ cannot be a singleton consisting of an endpoint of $\mathcal{T}_{F^2}$. An analogous argument shows that $\mathcal{T}_{F^n} = \mathcal{T}_F$ for all $n \geq 3$, establishing (b).

Now suppose that $\mathcal{T}_F$ is a nontrivial segment, and consider a subsegment $I =]\nu_1, \nu_2[\subset \mathcal{T}_F$, that is totally ordered, i.e., $\nu_1 < \nu_2$. We claim that the

multiplicity function m is constant on I. This will imply that the endpoints of $\mathcal{T}_F$ are divisorial (rather than infinitely singular), and hence complete the proof of (a). The Jacobian formula (5.2) gives $\nu(JF) = (\lambda_1 - 1)A(\nu)$, so the function $\nu \to \nu(JF)$ is piecewise affine on I (with respect to skewness) with slope $m(\nu)(\lambda_1 - 1)$. Now $\nu \to \nu(JF)$ is concave on I [30, §A.4], whereas $\nu \mapsto m(\nu)$ is nondecreasing on I. Thus m is constant on I, as required.

Next we turn to (c). We may replace F by F^2 so that $\mathcal{T}_F = \mathcal{T}_{F^2}$. Pick $\nu \in \mathcal{V}_1 \setminus \mathcal{T}_F$ and write $\nu_* := r(\nu) \in \mathcal{T}_F$. Then ν_* is divisorial and $\alpha(\nu_*) > 0$. We need to show that $F_\bullet^n \nu \to \nu_*$ as $n \to \infty$. Denote by $\vec{v}$ the tangent vector at ν_* represented by ν. Note that $U(\vec{v}) \cap \mathcal{T}_F = \emptyset$. If $\vec{v}$ is not preperiodic, then for n large enough, the functions $\mu \mapsto \mu(JF)$ and $\mu \mapsto d(F, \mu)$ are both constant on $U(\vec{v}_n)$, where $\vec{v}_n := \mathsf{F}^n \vec{v}$; see [30, Prop. 3.4]. By Lemma 5.6, $F_\bullet^n \nu \in U(\vec{v}_n)$ for all n, and the Jacobian formula (5.2) implies that $|A(F_\bullet^n \nu) - A(\nu_*)| \sim \lambda_1^{-n} \to 0$. This implies that $F_\bullet^n \nu \to \nu_*$ in the weak topology.

When $\vec{v}$ is preperiodic, we may assume it is fixed but not totally invariant by F. Let $I := [\nu_*, \nu]$ and $\Omega := \{\mu \in I \mid F_\bullet^n \mu \to \nu_*\}$. As in the proof of Lemma 4.6, $F_\bullet$ is given on I by a (piecewise) Möbius transformation with non-negative integer coefficient fixing ν_*. Hence Ω contains a small neighborhood of ν_* in I, and is therefore open.

LEMMA 5.7. *Let $\nu_1, \nu_2 \in \mathcal{V}_1$ be comparable (i.e., $\nu_1 \le \nu_2$ or $\nu_2 \le \nu_1$) valuations with $\alpha(\nu_i) > 0$, $i = 1, 2$. Then*

$$|A(F_\bullet \nu_2) - A(F_\bullet \nu_1)| \le \frac{4}{\alpha(\nu_1)\alpha(\nu_2)} |A(\nu_2) - A(\nu_1)| \tag{5.4}$$

for any dominant polynomial mapping $F : \mathbf{C}^2 \to \mathbf{C}^2$.

This result implies the (local) equicontinuity of the family $(F_\bullet^n)$ on $\Omega \cap \{\alpha > 0\}$, hence $\Omega \cap \{\alpha > 0\}$ is closed. When $\alpha(\nu) > 0$, we conclude that $\nu \in \Omega$. Otherwise, $\alpha(\nu) = 0$. To complete the proof, we only need to show that $\alpha(\nu_n) > 0$ for some $n \ge 1$, where $\nu_n = F_\bullet^n \nu$.

Suppose to the contrary that $\alpha(\nu_n) = 0$ for all $n \ge 1$. As in the proof of Proposition 5.1 we get $F^{n*} Z_{\nu_n} = c_n Z_\nu$, where $c_n > 0$. By Proposition A.7, this shows that ν is the only preimage of ν_n under $F_\bullet^n$. Let $\vec{v}_n$ be the tangent vector at ν_* represented by ν_n. By Lemma 5.6, $\vec{v}$ is the only preimage of $\vec{v}_n$ under F^n. If $n \ge 2$, this implies that $\vec{v}$ is totally invariant under F^2, contradicting Lemma 5.5 since $U(\vec{v}) \cap \mathcal{T}_{F^2} = \emptyset$. This proves (c).

Finally we consider the monomialization statement in (d). The starting point is

LEMMA 5.8. *A quasimonomial valuation $\nu \in \mathcal{V}_1$ is monomial in some affine coordinates if and only if $A(\nu) + m(\nu)\alpha(\nu) < 0$.*

Define ν_* as the minimal element in $\mathcal{T}_{F^2}$. We claim that, in suitable affine coordinates, ν_* becomes a monomial valuation. In view of Lemma 5.8 it suffices to prove that $A(\nu_*)+m(\nu_*)\alpha(\nu_*)<0$. We assume that $\nu_*\neq -\deg$. By Lemma 5.5 and the minimality of ν_*, the tangent vector at ν_* represented by $-\deg$ is not totally invariant under the tangent map. Hence we can find a small segment $I'\subset[-\deg,\nu_*]$ and another small segment I such that $I\cap I'=\{\nu_*\}$ and $F_\bullet$ maps I homeomorphically onto I'.

We use the Jacobian formula (5.2). If the segments are chosen small enough, we have $A=-m_*\alpha+\delta$ on I', where $m_*=m(\nu_*)$ and $\delta\in\mathbf{R}$. We must prove that $\delta<0$. The right-hand side of (5.2) can be written

$$\begin{aligned}-m_*d(F,\nu)\alpha(F_\bullet\nu)+\delta\,d(F,\nu)&=-m_*(Z_{F_*\nu}\cdot Z_{\nu_*})+\delta\,d(F,\nu)\\&=-m_*(Z_\nu\cdot F^*Z_{\nu_*})+\delta\,d(F,\nu)=-m_*\lambda_1\alpha(\nu_*)+\delta\,d(F,\nu).\end{aligned}$$

Since the left-hand side in (5.2) is strictly increasing in ν, we get $\delta<0$. Hence $\mathcal{T}_{F^2}$ contains a monomial valuation. If $\mathcal{T}_{F^2}=\{\nu_*\}$ is a singleton, then $\mathcal{T}_F=\{\nu_*\}$, ν_* is divisorial and $\alpha(\nu_*)>0$, and there is nothing left to do. Assume therefore that $\mathcal{T}_{F^2}$ is a nontrivial segment that contains at least one monomial valuation.

First suppose that $\mathcal{T}_{F^2}$ contains a unique monomial valuation ν_*; this is then necessarily divisorial, given by $\nu(x)=-p/q$, $\nu(y)=-1$, where $q\geq p\geq 1$ and $\gcd(p,q)=1$. We wish to change coordinates so that $\mathcal{T}_{F^2}$ contains a nontrivial segment of monomial valuations. For this, it suffices to prove that $p=1$. Indeed, there then exists a valuation $\nu_1\in\mathcal{T}_{F^2}$ with $\nu_1>\nu_*$, $\nu_1(x)=-1/q$, $\nu_1(y)=-1$, and $\nu_1(y+ax^q)>-1$ for some $a\in\mathbf{C}^*$. After a change of coordinates by a shear of the form $(x,y)\mapsto(x,y+ax^q)$, $\mathcal{T}_{F^2}$ will then contain a monomial subsegment.

To prove $p=1$, pick $\nu_1>\nu_*$ such that $\nu_1\in\mathcal{T}_{F^2}$ and such that the multiplicity is constant, equal to q, on $I=]\nu_*,\nu_1]$. The Jacobian formula (5.2) applied to F^{2n} on I yields $\nu(JF^{2n})=(\lambda_1^{2n}-1)A(\nu)$, so $\nu\mapsto\nu(JF^{2n})$ is affine on I with slope $q(\lambda_1^{2n}-1)$. This implies $\deg(JF^{2n})\geq q(\lambda_1^{2n}-1)$. Applying the Jacobian formula to $-\deg$ gives $\deg(JF^{2n})=-\deg(F^{2n})A(F_\bullet^{2n}(-\deg))-2$. The retraction of $-\deg$ on $\mathcal{T}_{F^2}$ equals ν_*, so $A(F_\bullet^{2n}(-\deg))\to A(\nu_*)=-1-p/q$ as $n\to\infty$. On the other hand, $\deg(F^{2n})\leq d(F^{2n},\nu_*)/\alpha(\nu_*)=\lambda_1^{2n}q/p$. Altogether this gives $q/p+1\geq q$, so $p=1$.

We may therefore assume that $\mathcal{T}_{F^2}$ contains a nontrivial subsegment consisting of monomial valuations. We saw that the multiplicity on any totally ordered open subsegment of $\mathcal{T}_{F^2}$ is constant; hence $\mathcal{T}_{F^2}$ is a segment which is the union of a segment of monomial valuations $[\nu_0,\nu_1]$, $\nu_0\leq\nu_1$, and a totally ordered segment whose minimum is ν_0. We may assume $\nu_0(x)=-p/q$, $\nu_0(y)=-1$, where $q\geq p\geq 1$ and $\gcd(p,q)=1$. If $p>1$, then the above argument shows that ν_0 is an endpoint in $\mathcal{T}_{F^2}$, yielding $\mathcal{T}_{F^2}=[\nu_0,\nu_1]$. Hence

assume $p = 1$ and that $\mathcal{T}_{F^2} \neq [\nu_0, \nu_1]$. We proceed by induction on q. If $q = 1$, then $\nu_0 = -\deg$ and we may change coordinate by an affine map of the form $(x, y) \mapsto (x, y + ax)$. The valuations in $[\nu_0, \nu_1]$ remain monomial, but $\mathcal{T}_{F^2}$ contains a new monomial valuation ν_1', which we may assume maximal. Thus $\mathcal{T}_{F^2} = [\nu_1', \nu_1]$ and we are done. If $q > 1$, then we change coordinates by a shear of the form $(x, y) \mapsto (x, y + ax^q)$. The valuations in $[\nu_0, \nu_1]$ remain monomial, but ν_0 is no longer the minimal monomial valuation in $\mathcal{T}_{F^2}$. By the inductive hypothesis we may now make $\mathcal{T}_{F^2}$ monomial. This completes the proof of Proposition 5.3.

5.5. *Proofs of Lemmas.* Finally we prove the lemmas used above.

Proof of Lemma 5.5. Pick a segment I in $\mathcal{V}_0$ representing the totally invariant tangent vector at ν_*. If $\nu \in I$ is close enough to ν_*, then $\nu' := F_\bullet \nu \in I$. As the tangent vector is totally invariant, ν is then the only preimage of ν' under $F_\bullet$. Write $d = d(F, \nu)$, $\alpha = \alpha(\nu)$, $\alpha' = \alpha(\nu')$. Then $F_* Z_\nu = dZ_{\nu'}$ and, since F is proper, $F^* Z_{\nu'} = \frac{\lambda_2}{d} Z_\nu$. On the one hand, this gives

$$\lambda_2 \alpha' = \lambda_2 (Z_{\nu'} \cdot Z_{\nu'}) = (F^* Z_{\nu'} \cdot F^* Z_{\nu'}) = \frac{\lambda_2^2}{d^2} (Z_\nu \cdot Z_\nu) = \frac{\lambda_2^2}{d^2} \alpha.$$

On the other hand, we get

$$\frac{\lambda_2}{d} \alpha(\nu \wedge \nu_*) = (F^* Z_{\nu'} \cdot Z_*) = (Z_{\nu'} \cdot F_* Z_*) = \lambda_1 \alpha(\nu' \wedge \nu_*).$$

Now either $\nu, \nu' > \nu_*$ or $\nu, \nu' < \nu_*$. In both cases, we easily deduce, using $\lambda_2 = \lambda_1^2$ and $\alpha(\nu_*) > 0$, that $\lambda_1 = d$ and $\alpha = \alpha'$. Hence $\nu' = F_\bullet \nu = \nu$, which completes the proof. □

Proof of Lemma 5.6. Pick a tight compactification X_0 of $\mathbf{C}^2$ on which the center of ν_* is a prime E_0 of X. Let $V \subset \mathrm{Pic}(X_0)$ be the subspace spanned by all primes $E \neq E_0$. Then V lies in the orthogonal complement of the class Z_0 determined by the condition $Z_0 \cdot Z = \mathrm{ord}_{E_0}(Z)$ for all Z. But $\alpha(\nu_*) > 0$, hence $Z_0^2 > 0$; see Lemma A.2. Thus the intersection form is negative definite on V. We may therefore contract all primes $E \neq E_0$: we get a normal (singular) surface X_1 on which the center of ν_* is a prime E_1, and the lift $\widetilde{F} : X_1 \to X_1$ is holomorphic. Pick a tangent vector $\vec{v}$ at ν_*. It corresponds to a point $p \in E_1$: the set $U(\vec{v})$ is exactly the set of valuations $\nu \in \mathcal{V}_0$ centered at p. The map $\widetilde{F}$ induces a finite germ $(X_1, p) \to (X_1, \widetilde{F}(p))$ hence $F_\bullet$ maps $U(\vec{v})$ onto $U(\mathsf{F}(\vec{v}))$. □

Proof of Lemma 5.7. The proof is based on the Jacobian formula (5.2). Write $\alpha_i = \alpha(\nu_i) > 0$, $A_i = A(\nu_i) < 0$, $A_i' = A(F_\bullet \nu_i) < 0$, $d_i = d(F, \nu_i) > 0$, and $J_i = \nu_i(JF) < 0$ for $i = 1, 2$ and write $d = \deg F$. The functions $\nu \mapsto -d(F, \nu)$ and $\nu \mapsto \nu(JF)$ define tree potentials on $\mathcal{V}_0$ in the sense of [30, §A.4].

Hence $|d_2 - d_1| \le d|\alpha_2 - \alpha_1| \le d|A_2 - A_1|$, $|J_2 - J_1| \le (\deg JF)|\alpha_2 - \alpha_1| \le (2d-2)|A_2 - A_1|$, and $d_i \ge d\alpha_i$. Moreover, $|A_i + J_i| \le 2d$. Thus (5.2) gives

$$\begin{aligned} |A_2' - A_1'| &= \left| \frac{A_2 + J_2}{d_2} - \frac{A_1 + J_1}{d_1} \right| \\ &= \left| \frac{(A_2 + J_2)(d_1 - d_2)}{d_1 d_2} + \frac{(A_2 - A_1) + (J_2 - J_1)}{d_1} \right| \\ &\le \left(\frac{2}{\alpha_1 \alpha_2} + \frac{2}{\alpha_1} \right) |A_2 - A_1|, \end{aligned}$$

which completes the proof since $0 < \alpha_2 \le 1$. □

Proof of Lemma 5.8. The proof is based on [30, App. A]. By the Line Embedding Theorem it is sufficient to prove that $A(\nu) + m(\nu)\alpha(\nu) < 0$ if and only if $\nu \in \mathcal{V}_1$ is dominated by a rational pencil valuation.

Suppose C is a curve with one place at infinity whose associated pencil valuation $\nu_{|C|}$ dominates ν. Then $\alpha(\nu_{|C|}) = 0$ and $\alpha \le 0$ on $[\nu, \nu_{|C|}]$, so

$$A(\nu_{|C|}) - A(\nu) = -\int_\nu^{\nu_{|C|}} m(\mu)\, d\alpha(\mu) \ge -\int_\nu^{\nu_{|C|}} m(\nu)\, d\alpha(\mu) = m(\nu)\alpha(\nu).$$

By [30, Prop. A.4], the pencil $|C|$ is rational if and only if $A(\nu_{|C|}) < 0$. Hence, $A(\nu) + m(\nu)\alpha(\nu) < 0$ if $|C|$ is rational. On the other hand, we may always pick the curve C with $\deg(C) = m(\nu)$. Then the multiplicity m is constant equal to $m(\nu)$ on $[\nu, \nu_{|C|}]$. Thus equality holds above, and we get $A(\nu_{|C|}) = A(\nu) + m(\nu)\alpha(\nu)$. So when $A(\nu) + m(\nu)\alpha(\nu) < 0$, $|C|$ is rational. □

6. **Proofs of Theorems A and B**

Fix an embedding $\mathbf{C}^2 \subset \mathbf{P}^2$ and consider an arbitrary polynomial mapping $F : \mathbf{C}^2 \to \mathbf{C}^2$. We start by making a few general remarks.

First, if X is an admissible compactification of $\mathbf{C}^2$ and the lift $\widetilde{F} : X \dashrightarrow X$ satisfies $\widetilde{F}^{(n+j)*} = \widetilde{F}^{*n}\widetilde{F}^{j*}$ on $\mathrm{Pic}(X)$ for $j \ge 1$, then

$$\deg(F^{n+j}) = (\widetilde{F}^{(n+j)*}\mathcal{L} \cdot \mathcal{L}) = (\widetilde{F}^{n*}\widetilde{F}^{*j}\mathcal{L} \cdot \mathcal{L})$$

with $\mathcal{L} \in \mathrm{Pic}(X)$ the (pullback of the) class of a line in $\mathbf{P}^2$. Hence $(\deg(F^j))_{j \ge n}$ satisfies the linear recurrence relation determined by the characteristic polynomial of the linear map $\widetilde{F}^* : \mathrm{Pic}(X) \to \mathrm{Pic}(X)$. In the basis given by the primes of X, $\widetilde{F}^*$ can be expressed with integer coefficients. Hence Theorem B follows from Theorem A in this case.

Second, if $F : \mathbf{C}^2 \to \mathbf{C}^2$ is not dominant, its image is a point or a curve. In either case, we pick an admissible compactification X of $\mathbf{C}^2$ such that the map $X \to \mathbf{P}^2$ induced by F is holomorphic. One can then check that the lift $\widetilde{F} : X \to X$ is also holomorphic. This establishes Theorems A and B in the nondominant case.

Third, if $F : X \dashrightarrow Y$, $G : Y \dashrightarrow Z$ are dominant rational maps between surfaces, and F^*, G^* denotes the action on the respective Picard groups of these surfaces, then $F^* \circ G^* = (G \circ F)^*$ if and only if no curve in X is contracted by F to an indeterminacy point of G.

Using this, it is not difficult to see that in the case $\lambda_2 < \lambda_1^2$, Theorem A follows directly from Theorems 3.1, 4.1, and 4.3. We obtain Theorem B as a consequence, in view of the argument above, but we have in any case established stronger versions in Corollaries 3.3, 4.2, and 4.4.

From now on, assume F is dominant and $\lambda_2 = \lambda_1^2$. We shall freely use the results in Section 5.

If $\deg F^n/\lambda_1^n$ is unbounded, then by Proposition 5.1 there is a unique rational pencil valuation ν_* such that $F_\bullet^n \nu \to \nu_*$ for every $\nu \in \mathcal{V}_1$. We may then proceed exactly as in Section 4 and prove analogs of Theorem 4.1 and Corollary 4.2.

If instead $\deg F^n/\lambda_1^n$ is bounded, then we have already seen in the proof of Theorem C that F lifts to a holomorphic self-map of a suitable compactification X of $\mathbf{C}^2$, proving Theorem A. However, this compactification need not be smooth or dominate the given compactification $\mathbf{P}^2 \supset \mathbf{C}^2$, so Theorem B does not immediately follow.

First assume $\lambda_1 = 1$ so that $\deg F^n$ is bounded. Then F is in particular birational. The argument by Diller-Favre [17, Th. 0.1] gives us an admissible compactification X of $\mathbf{C}^2$ such that the lift $\widetilde{F} : X \dashrightarrow X$ is algebraically stable. Thus Theorems A and B hold in this case.

Finally assume $\lambda_1 > 1$ and $\deg F^n/\lambda_1^n$ is bounded and consider the set $\mathcal{T}_{F^2}$ of eigenvaluations for F^2. We consider three cases. In the first case, $\mathcal{T}_{F^2} = \mathcal{T}_F = \{\nu_*\}$ is a singleton. Then Proposition 5.3 shows that $F_\bullet^n \nu \to \nu_*$ for any $\nu \in \mathcal{V}_1$. We may then proceed exactly as in Section 4 and prove analogs of Theorem 4.3 and Corollary 4.4. This gives a precise version of Theorem A (with X a tight — hence smooth — compactification of $\mathbf{P}^2$) and Theorem B.

In the second case, $\mathcal{T}_{F^2} = \mathcal{T}_F = [\nu_1, \nu_2]$ is a segment, where ν_1 and ν_2 are divisorial. We can then proceed essentially as in Section 4. Namely, let X_0 be the minimal admissible compactification of $\mathbf{C}^2$ such that the centers of ν_1 and ν_2 are one-dimensional. We can then make further blow-ups to arrive at a tight compactification X such that for any prime E of X that is the center of a divisorial valuation in $\mathcal{T}_F$, the lift $\widetilde{F} : X \dashrightarrow X$ is holomorphic at any periodic point of $\widetilde{F}|_E$. In view of Proposition 5.3(c) this shows that there exists $n \geq 1$ such that the following holds for any prime E of X: either E is the center of a valuation in $\mathcal{T}_F$ and then $\widetilde{F}E = E$, or $\widetilde{F}^n$ contracts E onto a point at which all iterates of X are holomorphic. Thus a precise form of Theorem A holds. We can also prove a precise version of Theorem B as in Corollary 4.4 in this

setting; in particular, the sequence $(d(F^j,\nu)_{j\geq n(\nu)})$ satisfies an integral linear recursion formula for all $\nu\in\mathcal{V}_1$.

In the third and final case, $\mathcal{T}_{F^2}=[\nu_1,\nu_2]$ is a nontrivial segment and $\mathcal{T}_F=\{\nu_*\}$ is a singleton. Again we can prove a precise version of Theorem B as in Corollary 4.4; it suffices to apply the result just proved to F^2.

Remark 6.1. One can check that for $F(x,y)=(y^3,x^2)$ there is no toric admissible compactification of $X\supset\mathbf{C}^2$ such that the lift of F to X satisfies the properties in Theorem A. However, F is holomorphic on $\mathbf{P}^1\times\mathbf{P}^1$. We conjecture that by adding suitable lower degree terms to F, no smooth compactification of $\mathbf{C}^2$ (admissible or not) will do.

7. **Small topological degrees:** $\lambda_2\leq\lambda_1$

Here we prove Theorem D and provide examples of maps with $\lambda_2=\lambda_1$.

7.1. *Proof of Theorem* D. Let $F:\mathbf{C}^2\to\mathbf{C}^2$ be a dominant polynomial mapping with $\lambda_2<\lambda_1$. Define

$$G^+:=\limsup_{n\to\infty}\lambda_1^{-n}\log^+\|F^n\|.$$

We shall see momentarily that the lim sup is in fact a limit (and ultimately even a locally uniform one), but let us first establish

LEMMA 7.1. *We have* $G^+\leq C_1\log^+\|\cdot\|+C_2$ *on* $\mathbf{C}^2$ *for some constants* $C_1,C_2>0$.

Proof. By Theorem B′ in [30] there exists a psh function U on $\mathbf{C}^2$ and a constant $C>0$ such that $U\circ F\leq\lambda_1U$ on $\mathbf{C}^2$ and such that $C^{-1}\log\|\cdot\|\leq U\leq C\log\|\cdot\|$ outside a compact subset of $\mathbf{C}^2$. This easily implies the lemma. □

The key step in the proof of Theorem D is to find a sort of filtration for the dynamics, similar to the one in the case of polynomial automorphisms; see e.g. [5], [33], [41].

LEMMA 7.2. *For every* $\varepsilon>0$ *there exists an integer* $n_0\geq1$, *a constant* $C>0$ *and a partition* $\mathbf{C}^2=V\cup V^+$ *with* V^+ *open,* $FV^+\subset V^+$ *and such that*:

(i) *the* lim sup *defining* G^+ *is a locally uniform limit on* V^+; G^+ *is pluriharmonic and strictly positive there*;
(ii) *for* $p\in V$ *we have* $\log^+\|F^{n_0}p\|\leq(\lambda_2+\varepsilon)^{n_0}\log^+\|p\|+C$.

Proof of Theorem D. Apply Lemma 7.2 with $0<\varepsilon<\lambda_1-\lambda_2$. Set $U^+=\bigcup_{n\geq0}F^{-n}V^+$ and $K^+:=\mathbf{C}^2\setminus U^+$. Then U^+ is open and G^+ is pluriharmonic and strictly positive there. On the other hand, the estimate in (ii) implies $G^+\equiv0$ on K^+. Thus G^+ is everywhere defined and satisfies $G^+\circ F=\lambda_1G^+$.

Let G^{+*} be the upper semicontinuous regularization of G^+. We shall prove that $G^{+*} = 0$ on K^+. Lemma 7.1 implies $G^{+*} \le C_1 \log^+ \|\cdot\| + C_2$ on $\mathbf{C}^2$ for some constants $C_1, C_2 > 0$. Now pick $p \in K^+$. Then $F^{kn_0} p \in V$ for all $k \ge 0$, so using the estimate in Lemma 7.2(ii) we get

$$G^{+*} \circ F^{kn_0}(p) \le C_3(\lambda_2 + \varepsilon)^{kn_0}(\log^+ \|p\| + C_4) \tag{7.1}$$

for suitable constants $C_3, C_4 > 0$, and all $k \ge 1$. Now the equality $G^+ \circ F = \lambda_1 G^+$ yields the inequality $G^{+*} \circ F \ge \lambda_1 G^{+*}$ (we have equality outside the curves contracted by F), so (7.1) gives

$$G^{+*}(p) \le C_3 \left(\frac{\lambda_2 + \varepsilon}{\lambda_1}\right)^{kn_0} (\log^+ \|p\| + C_4) \to 0 \quad \text{as } k \to \infty.$$

Hence $G^{+*} \equiv G^+ \equiv 0$ on K^+. We may now argue as in the proof of [5, Prop. 3.4] to prove that G^+ is continuous and psh on $\mathbf{C}^2$, that the limit defining G^+ is locally uniform on $\mathbf{C}^2$, and that the support of the current $dd^c G^+$ equals ∂K^+.

The estimate in Theorem D follows from the corresponding estimate in Lemma 7.2(ii). This completes the proof. □

The filtration in Lemma 7.2 is constructed using a well-chosen compactification of $\mathbf{C}^2$.

LEMMA 7.3. *For every $\varepsilon > 0$ there exists an integer $n_0 \ge 1$, an admissible compactification X of $\mathbf{C}^2$, and a decomposition $X \setminus \mathbf{C}^2 = Z^+ \cup Z^-$ into (reducible) curves Z^+ and Z^- without common components such that*:

(i) *if E is any irreducible component of Z^- and L is a generic affine function on $\mathbf{C}^2$, then*

$$\operatorname{ord}_E(F^{n_0 *} L) \le (\lambda_2 + \varepsilon)^{n_0} \operatorname{ord}_E(L); \tag{7.2}$$

(ii) *there exists a point $p \in Z^+ \setminus Z^-$ such that $\widetilde{F}^{n_0}$ is holomorphic in a neighborhood of Z^+, $\widetilde{F}^{n_0}(Z^+) = \{p\}$, $\widetilde{F}$ is holomorphic at p, $\widetilde{F}(p) = p$; and there exist local coordinates (z, w) at p in which $\widetilde{F}$ takes a simple normal form as in Theorem* 3.1:

(a) *if Z^+ is locally reducible at p, then $Z^+ = \{zw = 0\}$ and $\widetilde{F}(z, w) = (z^a w^b, z^c w^d)$, where $a, b, c, d \in \mathbf{N}$ and the 2×2 matrix M with entries a, b, c, d has spectral radius λ_1;*

(b) *if Z^+ is locally irreducible at p, then $Z^+ = \{z = 0\}$ and $\widetilde{F}(z, w) = (z^{\lambda_1}, \mu z^c w + P(z))$, where $c \ge 1$, $\mu \in \mathbf{C}^*$, and P is a nonconstant polynomial with $P(0) = 0$.*

The admissible compactification X will not be tight in general. We first show how to deduce Lemma 7.2 from Lemma 7.3, then prove the latter lemma.

Proof of Lemma 7.2. Apply Lemma 7.3 after having decreased ε slightly. We can find a small neighborhood Ω_p of p of the form $\{|z| < \delta_1, |w| < \delta_2\}$ such

that $\widetilde{F}\Omega_p \subset \Omega_p$. Set $V_p = \Omega_p \cap \mathbf{C}^2$, $V^+ := \widetilde{F}^{-n_0}(\Omega_p) \cap \mathbf{C}^2 = F^{-n_0}V_p$, and $V = \mathbf{C}^2 \setminus V^+$. Then $V^+ \subset \mathbf{C}^2$ is open and $FV^+ \subset V^+$.

In local coordinates (z,w) near a prime $E = \{z = 0\}$ of X, the function $\log^+ \|\cdot\|$ in $\mathbf{C}^2$ equals $-\operatorname{ord}_E(L)\log|z| + O(1)$. Similarly, in local coordinates (z,w) at the intersection point between two primes $E = \{z = 0\}$ and $E' = \{w = 0\}$ we have $\log^+ \|(z,w)\| = -\operatorname{ord}_E(L)\log|z| - \operatorname{ord}_{E'}(L)\log|w| + O(1)$.

Note that $\widetilde{F}^{-n_0}(\Omega_p)$ contains a neighborhood of Z^+ in X. Estimate (ii) in Lemma 7.2 is therefore a consequence of (7.2).

It remains to prove (i). For this we use the normal forms in Lemma 7.3. Suppose we are in case (a). Then

$$\log^+ \|(z,w)\| = -s\log|z| - t\log|w| + \varphi(z,w)$$

in V_p for some constants $s,t > 0$ and a bounded function φ. It then follows easily that $\lambda_1^{-n}\log^+ \|F^n\|$ converges locally uniformly on V_p to

$$G^+ = -s'\log|z| - t'\log|w|,$$

where $s',t' > 0$. (The vector (s',t') is proportional to the eigenvector with eigenvalue λ_1 of M^t.) Hence G^+ is pluriharmonic and strictly positive in V_p. Since $F^{n_0}V^+ \subset V_p$, the same properties must hold in V^+. This completes the proof in case (a). Case (b) is similar and left to the reader. □

Proof of Lemma 7.3. We apply Theorem 3.1. We shall only treat case (a) of that theorem, case (b) being similar.

Thus we have an admissible (tight) compactification X_0 of $\mathbf{C}^2$, two primes E_1, E_2 of X_0, intersecting in a point p, such that the lift $\widetilde{F}_0 : X_0 \dashrightarrow X_0$ defines an attracting holomorphic fixed point germ at p given by $\widetilde{F}_0(z,w) = (z^a w^b, z^c w^d)$ in suitable local coordinates (z,w). Write $Z_0 = E_1 \cup E_2$. Set $\gamma := \max_{i=1,2}\operatorname{ord}_{E_i}(L)$, where L is a generic affine function on $\mathbf{C}^2$. Pick $n_0 \geq 1$ large enough so that $\gamma\lambda_2^{n_0} < (\lambda_2 + \varepsilon)^{n_0}$.

By increasing n_0 if necessary, we can also assume that all valuations in the segment $I = [-\deg, \nu_*]$ are mapped by $F_\bullet^{n_0}$ into the open set $U(p) \subset \mathcal{V}_0$. Here, as before, $U(p)$ consists of valuations whose center in X_0 is the point p. Indeed, $U(p)$ is $F_\bullet$-invariant as F is holomorphic at p, and by Theorem 2.4(b), $F_\bullet^n\nu$ eventually falls into $U(p)$ for n large enough for any $\nu \in I$.

We can now find an admissible compactification X of $\mathbf{C}^2$ dominating X_0 such that F^{n_0} lifts to a holomorphic map $G : X \to X_0$. Since $\widetilde{F}_0$ was holomorphic at p, we may assume that the strict transforms of E_1 and E_2 in X still intersect in a point p. We use the notation E_1, E_2, and p also for these objects on X.

Let $Z^+ \subset X \setminus \mathbf{C}^2$ be the connected component of $G^{-1}(p)$ containing p. Our choice of n_0 implies that all primes whose associated normalized divisorial valuation in $\mathcal{V}_0$ is dominated by the eigenvaluation lie in Z^+. In particular, the

center L_∞ of $-\deg$ in X belongs to Z^+. Let Z^- be the union of the primes of X not in Z^+. Let $\widetilde{F} : X \dashrightarrow X$ be the rational lift of F. It is easy to see that (ii) holds. The key point is to establish (i).

The dual graph of $X \setminus \mathbf{C}^2$ being a tree and Z^+ being connected imply that each connected component W of Z^- contains a unique irreducible component $E = E_W$ intersecting Z^+. Moreover, since $L_\infty \subset Z^+$, the normalized divisorial valuations in $\mathcal{V}_0$ associated to the irreducible components of W all dominate (in the partial ordering on $\mathcal{V}_0$) the normalized divisorial valuation associated to E_W. Thus it suffices to verify (i) for $E = E_W$.

Now G must map E onto one of E_1 or E_2, say $G(E) = E_1$. The restriction of G to a neighborhood of E has topological degree at most $\lambda_2^{n_0}$. This implies in particular that the coefficient of E in the divisor G^*E_1 is at most $\lambda_2^{n_0}$. But this coefficient is easily seen to be

$$\frac{\operatorname{ord}_E(F^{n_0*}L)}{\operatorname{ord}_{E_1}(L)} \geq \gamma^{-1} \operatorname{ord}_E(F^{n_0*}L)$$

for a generic affine function L on $\mathbf{C}^2$. This proves (7.2) since $\operatorname{ord}_E(L) \geq 1$ and $\gamma\lambda_2^{n_0} < (\lambda_2 + \varepsilon)^{n_0}$. □

7.2. *Examples with* $\lambda_2 = \lambda_1$. Very few surface maps with $\lambda_2 = \lambda_1$ have been described in the literature. We provide here a (presumably incomplete) list of examples. The mappings in (6) below appear in [20, §4]. A classification of quadratic polynomial maps with $\lambda_2 = \lambda_1$ is given in [37]. Note that Proposition 2.5 implies that the eigenvaluation of a map with $\lambda_1 = \lambda_2 > 1$ is always divisorial or infinitely singular.

(1) $F = (A(x), Q(x, y))$ with A affine and $\deg_y Q \geq 1$. Then $\lambda_1 = \lambda_2 = \deg_y Q$, and F is proper if and only if for any fixed x, $\deg_y Q(x, y) = \deg_y Q$. One can show that any proper polynomial map with $\lambda_1 = \lambda_2 > 1$ whose eigenvaluation is divisorial takes this form in suitable affine coordinates.

(2) $F = (P(x), A(x, y))$ with $\deg_y(A) = 1$. Then $\lambda_1 = \lambda_2 = \deg(P)$, and F is proper if and only if $A(x, y) = ay + B(x)$, $a \neq 0$.

(3) $F = (\lambda xP+a, \mu yP+b)$, or $F = (xP+a, (x+y)P+b)$ with $P = P(x, y)$ of degree $d - 1$, and $a, b \in \mathbf{C}$, $\lambda, \mu \in \mathbf{C}^*$. Then $\lambda_1 = \lambda_2 = d$, the eigenvaluation is $-\deg$ which is divisorial, and F is not proper.

(4) $F = (ay^p + P(x), x^q)$ with $\deg P = pq$, $a \in \mathbf{C}$. Then $\lambda_1 = \lambda_2 = pq$, F is proper and has an infinitely singular eigenvaluation.

(5) $F = (x + aP, y + bP)$ with $P = P_d + \text{l.o.t.}$, $d = \deg(P) \geq 2$, and $P_d(a, b) \neq 0$. Then F is proper and $\lambda_1 = \lambda_2 = \deg P$.

(6) $F = (ax + by + c, P(x, y))$ with $\deg P = \deg_y P \geq 2$. Then $\lambda_1 = \lambda_2 = \deg P$.

(7) $F = (P(y), ax + b + Q(y))$ with $d = \deg P = \deg Q \geq 2$. Then $\lambda_1 = \lambda_2 = d$.

There seems to be no general conjecture or approach for studying the ergodic properties of these maps.

Appendix A. The Riemann-Zariski space at infinity

In this appendix we briefly develop the necessary material needed for the proof of Theorem 2.4. Most of the discussion is completely analogous to the one in the paper [8], to which we refer for details. See also [11], [43].

The main new consideration is the construction and study of the Weil class Z_ν associated to a valuation ν centered at infinity.

A.1. *Weil and Cartier classes.* The set of admissible compactifications of $\mathbf{C}^2$ defines an inverse system: X' dominates X if the birational map $X' \dashrightarrow X$ induced by the identity on $\mathbf{C}^2$ is a morphism. Typically we then identify the primes of X with their strict transforms in X'. Formally, the *Riemann-Zariski space* (of $\mathbf{P}^2$ at infinity) is defined as $\mathfrak{X} := \varprojlim X$. (The only difference to [8] is that here we never blow up points in $\mathbf{C}^2$.)

Our concern is with classes on $\mathfrak{X}$ rather than $\mathfrak{X}$ itself. Given X we let $\mathrm{NS}(X)$ be the vector space of $\mathbf{R}$-divisors on X modulo numerical equivalence. Then $\mathrm{NS}(X) \simeq \mathrm{Pic}(X) \otimes_{\mathbf{Z}} \mathbf{R}$. When X' dominates X, the associated morphism $\mu : X' \to X$ induces linear maps $\mu_* : \mathrm{NS}(X') \to \mathrm{NS}(X)$ and $\mu^* : \mathrm{NS}(X) \to \mathrm{NS}(X')$ satisfying $\mu_*\mu^* = \mathrm{id}$. The space of *Weil classes* on $\mathfrak{X}$ is $W(\mathfrak{X}) := \varprojlim \mathrm{NS}(X)$. We equip it with the projective limit topology. Concretely, a Weil class $\beta \in W(\mathfrak{X})$ is given by a collection of classes $\beta_X \in \mathrm{NS}(X)$, the *incarnation* of β on X, compatible under pushforward.

A class $\beta \in \mathrm{NS}(X)$ for a fixed X defines a Weil class whose incarnation in any compactification X' dominating X is the pullback of β to X'. Such a Weil class is called a *Cartier class*. It is *determined in* X. Formally, the set of Cartier classes is $C(\mathfrak{X}) := \varinjlim \mathrm{NS}(X)$.

The intersection pairing on each X extends to a nondegenerate pairing $W(\mathfrak{X}) \times C(\mathfrak{X}) \to \mathbf{R}$. In particular, we have an inner product on $C(\mathfrak{X})$. By the Hodge Index Theorem, this is of Minkowski type, allowing us to define the completion $\mathrm{L}^2(\mathfrak{X})$.

A Weil class is *nef* if all its incarnations are nef. The class of a line in $\mathbf{P}^2$ defines a nef Cartier class $\mathcal{L}$ on $\mathfrak{X}$. The set of nef classes forms a closed convex cone in $W(\mathfrak{X})$. Any nef Weil class belongs to $\mathrm{L}^2(\mathfrak{X})$.

A.2. *Classes and valuations.* Every class in $\mathrm{NS}(X)$ admits a unique representation as a divisor with support at infinity, i.e., a real-valued function on the set of primes of X. Hence a Weil class Z on $\mathfrak{X}$ can be identified with a homogeneous function on $\widehat{\mathcal{V}}_{\mathrm{div}}$: its value at ν will be denoted $\nu(Z)$. For example, $\mathrm{ord}_E(\mathcal{L}) = b_E$, where $\mathcal{L} \in W(\mathfrak{X})$ is the class of a line on $\mathbf{P}^2$.

Given a valuation $\nu \in \widehat{\mathcal{V}}_0$ we define a Weil divisor $Z_\nu \in W(\mathfrak{X})$ as follows: $\mathrm{ord}_E(Z_\nu) = tb_E\alpha(\tilde{\nu} \wedge \nu_E)$, where α denotes skewness and $\nu = t\tilde{\nu}$, $\tilde{\nu} \in \mathcal{V}_0$. When ν is divisorial, Z_ν is Cartier; see Lemma A.2 below.

LEMMA A.1. *The assignment $\nu \mapsto Z_\nu$ defines a continuous embedding of $\widehat{\mathcal{V}}_0$ onto a closed subset of $W(\mathfrak{X})$.*

Proof. After unwinding definitions, the statement boils down to the topology on $\mathcal{V}_0$ being the weakest topology such that $\nu \mapsto \alpha(\nu \wedge \nu_E)$ is continuous for all $\nu_E \in \mathcal{V}_{\mathrm{div}}$. This in turn follows from the characterization of the topology on $\mathcal{V}_0$ in terms of the tree structure. □

LEMMA A.2. *We have $(Z_\nu \cdot W) = \nu(W)$ for $\nu \in \widehat{\mathcal{V}}_{\mathrm{div}}$ and $W \in C(\mathfrak{X})$. In particular, for any two valuations $\mu, \nu \in \mathcal{V}_0$ one has*

$$(Z_\nu \cdot Z_\mu) = \alpha(\nu \wedge \mu) \in [-\infty, 1]. \tag{A.1}$$

If E is a prime of some admissible compactification X, then Z_{ord_E} is a Cartier class on $\mathfrak{X}$ determined in X by an integral class. Thus $b_E^2\alpha(\nu_E) \in \mathbf{Z}$.

Proof. To prove $(Z_\nu \cdot W) = \nu(W)$, we pick X such that $W \in C(\mathfrak{X})$ is determined in X. By linearity we may assume $W = E$ is a prime of X. What we seek to prove is then a special case of a more general formula $(Z \cdot E) = b_E\Delta g\{\nu_E\}$ for any $Z \in \mathrm{NS}(X)$, where $g = g_Z$ is the function on $\mathcal{V}_0$ defined by Z and Δg is its tree Laplacian as defined (up to a sign) in [30, §A.4]. Indeed, $Z = Z_\nu$ as above is chosen so that $\Delta g_Z = \delta_\nu$.

Let $E_1, \dots, E_n$ be the primes of X intersecting E properly and write $b_i = b_{E_i}$, $\nu_i = \nu_{E_i}$. Assume $E \neq L_\infty$ for simplicity. Then $0 = (\mathcal{L} \cdot E) = b_E(E \cdot E) + \sum_i b_i$ and $(Z \cdot E) = b_E g(\nu_E)(E \cdot E) + \sum_i b_i g(\nu_i)$. Subtracting and rearranging using $|\alpha(\nu_i) - \alpha(\nu_E)| = (b_i b_E)^{-1}$ yields $(Z \cdot E) = b_E \sum_i (g(\nu_i) - g(\nu_E))/|\alpha(\nu_i) - \alpha(\nu_E)|$, which equals $b_E\Delta g\{\nu_E\}$. A similar computation works in the case $E = L_\infty$ and completes the proof of the relation $(Z_\nu \cdot W) = \nu(W)$. This relation and the definition of Z_ν and Z_μ imply (A.1).

For the last statement it suffices to observe that by the nondegeneracy and unimodularity of the intersection form on $\mathrm{Pic}(X)$, there exists a unique integral class Z satisfying $(Z \cdot W) = \mathrm{ord}_E(W)$ for any $W \in \mathrm{Pic}(X)$. □

LEMMA A.3. *If $\nu \in \mathcal{V}_0$, then Z_ν is nef if and only if $\alpha(\nu) \geq 0$.*

Proof. If Z_ν is nef, then (A.1) shows that $\alpha(\nu) = (Z_\nu \cdot Z_\nu) \geq 0$. Conversely, if $\alpha(\nu) \geq 0$, then the definition of Z_ν shows that $Z_\nu \geq 0$ as a function on $\widehat{\mathcal{V}}_{\rm div}$ and that $\nu \mapsto Z_\nu$ is decreasing. The nef cone in $W(\mathfrak{X})$ being closed, it suffices by Lemma A.1 to consider the case when ν is divisorial.

Hence assume $\nu = b_E^{-1} \operatorname{ord}_E$ is divisorial, where E is a prime in some admissible compactification X. We must show that $(Z_\nu \cdot C) \geq 0$ for every irreducible curve C in X. If C is a prime of X, then this is clear by the above. If instead C is the closure of a curve $\{P = 0\}$ in $\mathbf{C}^2$, then $(Z_\nu \cdot C) = -\nu(P)$ which is nonnegative since $\alpha(\nu) \geq 0$. □

Remark A.4. One can also show that $Z_\nu \in \mathrm{L}^2(\mathfrak{X})$ if and only if $\alpha(\nu) > -\infty$.

A.3. *Functoriality.* Let $F : \mathbf{C}^2 \to \mathbf{C}^2$ be a dominant polynomial mapping. Following [8] we define actions of F by pushforward and pullback on classes on the Riemann-Zariski space $\mathfrak{X}$. The two key facts are 1) given any admissible compactification X', there exists another admissible compactification X such that the lift $X \to X'$ of F is holomorphic, and 2) given any admissible compactification X and any prime E of X, either F maps E onto a point or a curve in $\mathbf{C}^2$, or there exists another compactification X' and a prime E' of X' such that the lift $X \dashrightarrow X'$ of F maps E onto E'.

To begin with, we have natural actions $F^* : C(\mathfrak{X}) \to C(\mathfrak{X})$ and $F_* : W(\mathfrak{X}) \to W(\mathfrak{X})$. For example, if $\beta \in W(\mathfrak{X})$ is a Weil class, the incarnation of $F_*\beta \in W(\mathfrak{X})$ on a given admissible compactification X' is the pushforward of $\beta_X \in \mathrm{NS}(X)$ by the map $X \to X'$ induced by F for any X such that this is holomorphic.

As in [8], the pushforward (resp. pullback) preserves (resp. extends to) L^2-classes. We obtain bounded operators $F_*, F^* : \mathrm{L}^2(\mathfrak{X}) \to \mathrm{L}^2(\mathfrak{X})$, and $(F_*\beta \cdot \gamma) = (\beta \cdot F^*\gamma)$ for $\beta, \gamma \in \mathrm{L}^2(\mathfrak{X})$. These operators preserve nef classes. We have $F_*F^* = \lambda_2 \cdot \mathrm{id}$ on $\mathrm{L}^2(\mathfrak{X})$ where λ_2 is the topological degree of F.

Remark A.5. Note that F defines a dominant rational self-map of $\mathbf{P}^2$, so the constructions in [8] apply, but they may not yield the same result as above. For example, consider the map $F(x, y) = (x^2, xy)$, which contracts the line $x = 0$ to the origin. If $\mathcal{L}$ is the class of a line in $\mathbf{P}^2$, then $F_*\mathcal{L} = 2\mathcal{L}$ with F_* as above, whereas the image of $\mathcal{L}$ under the pushforward considered in [8] is a Cartier class determined in the blow-up of $\mathbf{P}^2$ at the origin.

Lemma A.6. *We have $F_*Z_\nu = Z_{F_*\nu}$ for any $\nu \in \widehat{\mathcal{V}}_0$.*

Proof. By continuity and homogeneity it suffices to prove this when $\nu = \operatorname{ord}_E$ is divisorial, associated to a prime E of some admissible compactification X of $\mathbf{C}^2$. Then we can find another admissible compactification X' such that

the lift $\widetilde{F} : X \dashrightarrow X'$ is holomorphic and such that $\widetilde{F}(E)$ is either 1) a prime E' of X' or 2) a point or a curve in $\mathbf{C}^2$.

In the first case, $F_*\nu = k \operatorname{ord}_{E'}$, where k is the coefficient of $\widetilde{F}^*E'$ in E. For any Cartier class $W \in C(\mathfrak{X})$ we then have

$$(F_*Z_\nu \cdot W) = (Z_\nu \cdot F^*W) = \operatorname{ord}_E(F^*W) = k \operatorname{ord}_{E'}(W) = (Z_{F_*\nu} \cdot W).$$

In the second case, we have $F_*\nu = 0$ by definition and the same computation above shows that $(F_*Z_\nu \cdot W) = 0$ for all $W \in C(\mathfrak{X})$. Indeed, W can be represented as a divisor supported at infinity. Hence $F_*Z_\nu = 0$. □

The situation for the pullback is more delicate. For simplicity we state a result only in the proper case.

PROPOSITION A.7. *Suppose F is a* proper *polynomial map. Then any $\nu \in \mathcal{V}_0$ admits at most λ_2 preimages in $\mathcal{V}_0$ under $F_\bullet$, and one can write*:

$$F^*Z_\nu = \sum_{F_\bullet \mu = \nu} a(\mu) Z_\mu, \tag{A.2}$$

for some positive constants $a(\mu)$.

Proof. Let K be the function field of $\mathbf{C}^2$. The field extension $[K : F^*K]$ has degree λ_2. By standard valuation theory (see [49]) for any valuation ν on K there exist at most λ_2 valuations μ such that $F_*\mu = \nu$. This implies the first assertion.

As F is proper, $F_\bullet$ is an open surjective map for the weak topology on $\mathcal{V}_0$. Note also that if (A.2) is true, then one has the uniform bound $\sum_{F_\bullet^{-1}\{\nu\}} a(\mu) = (Z_\nu \cdot F_*\mathcal{L}) \le \deg(F)\alpha(F_\bullet(-\deg)) \le \deg F$. Hence we can reduce the proof to the divisorial case by continuity. The proof is then of the same flavor as the proof of Lemma A.6 and left to the reader. □

A.4. *Dynamics.* Let $F : \mathbf{C}^2 \to \mathbf{C}^2$ be a dominant polynomial mapping. Denote its topological degree by $\lambda_2 \ge 1$ and set $\lambda_1 := \lim_{n\to\infty}(\deg F^n)^{1/n}$. The techniques of [8] can be easily adapted to prove the following result, which corresponds to parts of Theorems 3.2 and 3.5 in that paper.

THEOREM A.8. *Assume $\lambda_2 < \lambda_1^2$. Then there exist nonzero nef Weil classes $\theta_*, \theta^* \in \mathrm{L}^2(\mathfrak{X})$ such that $F^*\theta^* = \lambda_1\theta^*$ and $F_*\theta_* = \lambda_1\theta_*$. They are unique up to scaling and we may normalize them by $(\theta_* \cdot \mathcal{L}) = (\theta_* \cdot \theta^*) = 1$. Then $(\theta^* \cdot \theta^*) = 0$, $F_*\theta^* = (\lambda_2/\lambda_1)\theta^*$ and for any Weil class $\theta \in \mathrm{L}^2(\mathfrak{X})$, we have*

$$\frac{1}{\lambda_1^n} F^{n*}\theta \to (\theta \cdot \theta_*)\theta^* \quad \text{and} \quad \frac{1}{\lambda_1^n} F_*^n\theta \to (\theta \cdot \theta^*)\theta_* \quad \text{as } n \to \infty.$$

References

[1] E. Bedford and J. Diller, Energy and invariant measures for birational surface maps, *Duke Math. J.* **128** (2005), 331–368. MR 2140266. Zbl 1076.37031. doi: 10.1215/S0012-7094-04-12824-6.

[2] E. Bedford and K. Kim, On the degree growth of birational mappings in higher dimension, *J. Geom. Anal.* **14** (2004), 567–596. MR 2111418. Zbl 1067.37054.

[3] ———, Periodicities in linear fractional recurrences: degree growth of birational surface maps, *Michigan Math. J.* **54** (2006), 647–670. MR 2280499. Zbl 1171.37023. doi: 10.1307/mmj/1163789919.

[4] ———, Linear recurrences in the degree sequences of monomial mappings, *Ergodic Theory Dynam. Systems* **28** (2008), 1369–1375. MR 2449533. Zbl 1161.37032. doi: 10.1017/S0143385708000242.

[5] E. Bedford and J. Smillie, Polynomial diffeomorphisms of $\mathbf{C}^2$: currents, equilibrium measure and hyperbolicity, *Invent. Math.* **103** (1991), 69–99. MR 1079840. Zbl 0721.58037. doi: 10.1007/BF01239509.

[6] M. P. Bellon and C.-M. Viallet, Algebraic entropy, *Comm. Math. Phys.* **204** (1999), 425–437. MR 1704282. Zbl 0987.37007. doi: 10.1007/s002200050652.

[7] A. M. Bonifant and J. E. Fornæss, Growth of degree for iterates of rational maps in several variables, *Indiana Univ. Math. J.* **49** (2000), 751–778. MR 1793690. Zbl 0987.37033. doi: 10.1512/iumj.2000.49.1813.

[8] S. Boucksom, C. Favre, and M. Jonsson, Degree growth of meromorphic surface maps, *Duke Math. J.* **141** (2008), 519–538. MR 2387430. Zbl 1185.32009. doi: 10.1215/00127094-2007-004.

[9] J.-Y. Briend and J. Duval, Deux caractérisations de la mesure d'équilibre d'un endomorphisme de $\mathrm{P}^k(\mathbf{C})$, *Publ. Math. Inst. Hautes Études Sci.* **93** (2001), 145–159. MR 186373. Zbl 1010.37004. doi: 10.1007/s10240-001-8190-4.

[10] A. Campillo, O. Piltant, and A. J. Reguera-López, Cones of curves and of line bundles on surfaces associated with curves having one place at infinity, *Proc. London Math. Soc.* **84** (2002), 559–580. MR 1888423. Zbl 1074.14505. doi: 10.1112/S0024611502013394.

[11] S. Cantat, Sur les groupes de transformations birationnelles des surfaces, *Ann. of Math.* **173** (2011), to appear; available at http://perso.univ-rennes1.fr/serge.cantat/publications.html.

[12] D. A. Cox, The homogeneous coordinate ring of a toric variety, *J. Algebraic Geom.* **4** (1995), 17–50. MR 1299003. Zbl 0846.14032.

[13] H. de Thélin and G. Vigny, Entropy of meromorphic maps and dynamics of birational maps, *Mem. de la SMF* **122** (2010), to appear. arXiV 0806.4874.

[14] J. Diller, R. Dujardin, and V. Guedj, Dynamics of meromorphic maps with small topological degree I: From cohomology to currents, *Indiana Univ. Math. J.* **59** (2010), 521–562.

[15] ———, Dynamics of meromorphic maps with small topological degree II: Energy and invariant measure, *Comment. Math. Helv.*, to appear.

[16] J. Diller, R. Dujardin, and V. Guedj, Dynamics of meromorphic maps with small topological degree III: Geometric currents and ergodic theory, *Ann. Sci. Éc. Norm. Supér.* **43** (2010), 235–278. MR 2662665.

[17] J. Diller and C. Favre, Dynamics of bimeromorphic maps of surfaces, *Amer. J. Math.* **123** (2001), 1135–1169. MR 1867314. Zbl 1112.37308. Available at http://muse.jhu.edu/journals/american_journal_of_mathematics/v123/123.6diller.pdf.

[18] T. C. Dinh, R. Dujardin, and N. Sibony, On the dynamics near infinity of some polynomial mappings in $\mathbb{C}^2$, *Math. Ann.* **333** (2005), 703–739. MR 2195140. Zbl 1079.37040. doi: 10.1007/s00208-005-0661-3.

[19] T.-C. Dinh and N. Sibony, Dynamique des applications d'allure polynomiale, *J. Math. Pures Appl.* **82** (2003), 367–423. MR 1992375. Zbl 1033.37023. doi: 10.1016/S0021-7824(03)00026-6.

[20] ———, Dynamique des applications polynomiales semi-régulières, *Ark. Mat.* **42** (2004), 61–85. MR 2172806. Zbl 1059.37033.

[21] ———, Decay of correlations and the central limit theorem for meromorphic maps, *Comm. Pure Appl. Math.* **59** (2006), 754–768. MR 2172806. Zbl 1137.37023. doi: 10.1002/cpa.20119.

[22] ———, Geometry of currents, intersection theory and dynamics of horizontal-like maps, *Ann. Inst. Fourier* (*Grenoble*) **56** (2006), 423–457. MR 2226022. Zbl 1089.37036. Available at http://aif.cedram.org/item?id=AIF_2006__56_2_423_0.

[23] ———, Super-potentials of positive closed currents, intersection theory and dynamics, *Acta Math.* **203** (2009), 1–82. MR 2545825. doi: 10.1007/s11511-009-0038-7.

[24] I. Dolgachev, Weighted projective varieties, in *Group Actions and Vector Fields* (Vancouver, B.C., 1981), *Lecture Notes in Math.* **956**, Springer-Verlag, New York, 1982, pp. 34–71. MR 704986. Zbl 0516.14014. doi: 10.1007/BFb0101508.

[25] R. Dujardin, Hénon-like mappings in $\mathbb{C}^2$, *Amer. J. Math.* **126** (2004), 439–472. MR 204550. Zbl 1064.37035. doi: 10.1353/ajm.2004.0010. Available at http://muse.jhu.edu/journals/american_journal_of_mathematics/v126/126.2dujardin.pdf.

[26] ———, Laminar currents and birational dynamics, *Duke Math. J.* **131** (2006), 219–247. MR 2219241. Zbl 1099.37037. doi: 10.1215/S0012-7094-06-13122-8.

[27] C. Favre, Classification of 2-dimensional contracting rigid germs and Kato surfaces. I, *J. Math. Pures Appl.* **79** (2000), 475–514. MR 1759437. Zbl 0983.32023. doi: 10.1016/S0021-7824(00)00162-8.

[28] C. Favre and M. Jonsson, *The Valuative Tree*, *Lecture Notes in Math.* **1853**, Springer-Verlag, New York, 2004. MR 2097722. Zbl 1064.14024.

[29] ———, Valuative analysis of planar plurisubharmonic functions, *Invent. Math.* **162** (2005), 271–311. MR 2199007. Zbl 1089.32032. doi: 10.1007/s00222-005-0443-2.

[30] ———, Eigenvaluations, *Ann. Sci. École Norm. Sup.* **40** (2007), 309–349. MR 2339287. Zbl 1135.37018. doi: 10.1016/j.ansens.2007.01.002.

[31] J. E. Fornæss and N. Sibony, Complex Hénon mappings in $\mathbf{C}^2$ and Fatou-Bieberbach domains, *Duke Math. J.* **65** (1992), 345–380. MR 1150591. Zbl 0761.32015. doi: 10.1215/S0012-7094-92-06515-X.

[32] ———, Complex dynamics in higher dimension. II, in *Modern Methods in Complex Analysis*, *Ann. of Math. Stud.* **137**, Princeton Univ. Press, Princeton, NJ, 1995, pp. 135–182. MR 1369137. Zbl 0847.58059.

[33] S. Friedland and J. Milnor, Dynamical properties of plane polynomial automorphisms, *Ergodic Theory Dynam. Systems* **9** (1989), 67–99. MR 991490. Zbl 0651.58027. doi: 10.1017/S014338570000482X.

[34] W. Fulton, *Introduction to Toric Varieties*, *Annals of Math. Studies* **131**, Princeton Univ. Press, Princeton, NJ, 1993. MR 1234037. Zbl 0813.14039.

[35] M. H. Gizatullin, Rational G-surfaces, *Izv. Akad. Nauk SSSR Ser. Mat.* **44** (1980), 110–144, 239. MR 563788. Zbl 0428.14022.

[36] V. Guedj, Dynamics of polynomial mappings of $\mathbb{C}^2$, *Amer. J. Math.* **124** (2002), 75–106. MR 1879000. Zbl 01731124. Available at http://muse.jhu.edu/journals/american_journal_of_mathematics/v124/124.1guedj.pdf.

[37] ———, Dynamics of quadratic polynomial mappings of $\mathbb{C}^2$, *Michigan Math. J.* **52** (2004), 627–648. MR 2097402. Zbl 1073.37053. doi: 10.1307/mmj/1100623417.

[38] ———, Ergodic properties of rational mappings with large topological degree, *Ann. of Math.* **161** (2005), 1589–1607. MR 2179389. Zbl 1088.37020. doi: 10.4007/annals.2005.161.1589.

[39] B. Hasselblatt and J. Propp, Degree-growth of monomial maps, *Ergodic Theory Dynam. Systems* **27** (2007), 1375–1397. MR 2358970. Zbl 2358970. doi: 10.1017/S0143385707000168.

[40] J. Hubbard, P. Papadopol, and V. Veselov, A compactification of Hénon mappings in $\mathbf{C}^2$ as dynamical systems, *Acta Math.* **184** (2000), 203–270. MR 1768111. Zbl 0987.37035. doi: 10.1007/BF02392629.

[41] J. H. Hubbard and R. W. Oberste-Vorth, Hénon mappings in the complex domain. I. The global topology of dynamical space, *Inst. Hautes Études Sci. Publ. Math.* (1994), 5–46. MR 1307296. Zbl 0839.54029. doi: 10.1007/BF02698886. Available at http://www.numdam.org/item?id=PMIHES_1994__79__5_0.

[42] J. H. Hubbard and P. Papadopol, Newton's method applied to two quadratic equations in $\mathbb{C}^2$ viewed as a global dynamical system, *Mem. Amer. Math. Soc.* **891** (2008), vi+146. MR 2376103. Zbl 1133.37018.

[43] Y. I. Manin, *Cubic Forms. Algebra*, *Geometry*, *Arithmetic*, second ed., *North-Holland Math. Library* **4**, North-Holland Publishing Co., Amsterdam, 1986. MR 833513. Zbl 0582.14010.

[44] T. T. Moh, On analytic irreducibility at ∞ of a pencil of curves, *Proc. Amer. Math. Soc.* **44** (1974), 22–24. MR 0357409. Zbl 0309.14011. doi: 10.2307/2039219.

[45] V.-A. Nguyên, Algebraic degrees for iterates of meromorphic self-maps of $\mathbb{P}^k$, *Publ. Mat.* **50** (2006), 457–473. MR 2273670. Zbl 1112.37035.

[46] A. Russakovskii and B. Shiffman, Value distribution for sequences of rational mappings and complex dynamics, *Indiana Univ. Math. J.* **46** (1997), 897–932. MR 1488341. Zbl 0901.58023. doi: 10.1512/iumj.1997.46.1441.

[47] N. Sibony, Dynamique des applications rationnelles de $\mathbf{P}^k$, in *Dynamique et Géométrie Complexes* (Lyon, 1997), *Panor. Synthèses* **8**, Soc. Math. France, Paris, 1999, pp. ix–x, xi–xii, 97–185. MR 1760844. Zbl 1020.37026.

[48] G. Vigny, Dynamics semi-conjugated to a subshift for some polynomial mappings in $\mathbb{C}^2$, *Publ. Mat.* **51** (2007), 201–222. MR 2307152. Zbl 1125.37035.

[49] O. Zariski and P. Samuel, *Commutative Algebra*, *Vol.* 2, *Grad. Texts Math.* **29**, Springer-Verlag, New York, 1975. MR 0389876. Zbl 0322.13001.

(Received: November 15, 2007)
(Revised: September 24, 2008)

CNRS-Université Paris 7, Institut de Mathématiques, Paris, France
Current address: Centre de Mathématiques Laurent Schwartz,
École Polytechnique, Palaiseau, France
E-mail: charles.favre@math.polytechnique.fr

University of Michigan, Ann Arbor, MI
E-mail: mattiasj@umich.edu