Annals of Mathematics **173** (2011), 127–167
doi: 10.4007/annals.2011.173.1.4

Hausdorff dimension of the set of singular pairs

By Yitwah Cheung

Abstract

In this paper we show that the Hausdorff dimension of the set of singular pairs is $\frac{4}{3}$. We also show that the action of $\mathrm{diag}(e^t, e^t, e^{-2t})$ on $\mathrm{SL}_3\mathbb{R}/\mathrm{SL}_3\mathbb{Z}$ admits divergent trajectories that exit to infinity at arbitrarily slow prescribed rates, answering a question of A. N. Starkov. As a by-product of the analysis, we obtain a higher-dimensional generalization of the basic inequalities satisfied by convergents of continued fractions. As an illustration of the technique used to compute Hausdorff dimension, we reprove a result of I. J. Good asserting that the Hausdorff dimension of the set of real numbers with divergent partial quotients is $\frac{1}{2}$.

1. Introduction

Let $\mathrm{Sing}\,(d)$ denote the set of all singular vectors in $\mathbb{R}^d$. Recall that $\mathbf{x} \in \mathbb{R}^d$ is said to be *singular* if for every $\delta > 0$ there exists T_0 such that for all $T > T_0$ the system of inequalities

$$\|q\mathbf{x} - \mathbf{p}\| < \frac{\delta}{T^{1/d}} \quad \text{and} \quad 0 < q < T \tag{1.1}$$

admits an integer solution $(\mathbf{p}, q) \in \mathbb{Z}^{d+1}$. Since $\mathrm{Sing}\,(d)$ contains every rational hyperplane in $\mathbb{R}^d$, its Hausdorff dimension is between $d-1$ and d. In this paper, we prove

Theorem 1.1. *The Hausdorff dimension of* $\mathrm{Sing}\,(2)$ *is* $\frac{4}{3}$.

Singular vectors that lie on a rational hyperplane are said to be *degenerate*. Implicit in this terminology is the expectation that the set $\mathrm{Sing}^*(d)$ of all *nondegenerate* singular vectors is somehow larger than the union of all rational hyperplanes in $\mathbb{R}^d$, which is a set of Hausdorff dimension $d-1$. The papers [1], [24], [19] and [2] give lower bounds on certain subsets of $\mathrm{Sing}^*(d)$ that,

This research is partially supported by NSF grant DMS-0701281 and a Spring 2007 Mini-Grant Award from San Francisco State University.

in particular, imply $\mathrm{H.dim}\,\mathrm{Sing}^*(d) \geq d-1$. Theorem 1.1 shows that strict inequality holds in the case $d=2$.

Divergent trajectories. There is a well-known dynamical interpretation of singular vectors. Let G/Γ be the space of oriented unimodular lattices in $\mathbb{R}^{d+1}$, where $G = \mathrm{SL}_{d+1}\mathbb{R}$ and $\Gamma = \mathrm{SL}_{d+1}\mathbb{Z}$. A path $(\Lambda_t)_{t\geq 0}$ in G/Γ is said to be *divergent* if for every compact subset $K \subset G/\Gamma$ there is a time T such that $\Lambda_t \notin K$ for all $t > T$. By Mahler's criterion, (Λ_t) is divergent if and only if the length of the shortest nonzero vector of Λ_t tends to zero as $t \to \infty$. It is not hard to see that $\mathbf{x} \in \mathbb{R}^d$ is singular if and only if $\ell(g_t h_{\mathbf{x}} \mathbb{Z}^{d+1}) \to 0$ as $t \to \infty$, where

$$g_t = \begin{pmatrix} e^t & & & \\ & \ddots & & \\ & & e^t & \\ & & & e^{-dt} \end{pmatrix}, \quad h_{\mathbf{x}} = \begin{pmatrix} 1 & & & -x_1 \\ & \ddots & & \vdots \\ & & 1 & -x_d \\ & & & 1 \end{pmatrix},$$

and $\ell(\cdot)$ denotes the length of the shortest nonzero vector. Thus, $\mathbf{x}$ is singular if and only if $(g_t h_{\mathbf{x}} \Gamma)_{t\geq 0}$ is a divergent trajectory of the homogeneous flow on G/Γ induced by the one-parameter subgroup (g_t) acting by left multiplication.

As a corollary of Theorem 1.1 we obtain

COROLLARY 1.2. *The set $D(g_t)$ of points in $\mathrm{SL}_3\mathbb{R}/\mathrm{SL}_3\mathbb{Z}$ whose forward trajectory under g_t is divergent has Hausdorff dimension $7\frac{1}{3}$.*

Proof. Let $P := \{p \in G | g_t p g_{-t} \text{ stays bounded as } t \to \infty\}$ and note that every $g \in G$ can be written as $p h_{\mathbf{x}} \gamma$ for some $p \in P$, $\mathbf{x} \in \mathbb{R}^d$ and $\gamma \in \Gamma$. Since $g_t p h$ and $g_t h$ differ by the action of an element from a bounded set, nondivergence of $g_t h$ is equivalent to that of $g_t p h$. Therefore,

$$D(g_t) = \cup_{\mathbf{x} \in \mathrm{Sing}\,(d)} P h_{\mathbf{x}} \Gamma.$$

Since P is a manifold and Γ is countable, the Hausdorff codimension of $D(g_t)$ in G/Γ coincides with that of $\mathrm{Sing}\,(d)$ in $\mathbb{R}^d$. □

Similarly, as a corollary to Theorem 1.6 below, we have

COROLLARY 1.3. *There is a one-parameter family of compact sets $K_\delta \subset \mathrm{SL}_3\mathbb{R}/\mathrm{SL}_3\mathbb{Z}$ such that the Hausdorff dimension of the set of points whose forward trajectory under g_t eventually stays outside of K_δ is a function that approaches $7\frac{1}{3}$ from above as $\delta \to 0$.*

Our techniques also allow us to answer a question of Starkov [21] concerning the existence of slowly divergent trajectories.

THEOREM 1.4. *Given any function $\varepsilon(t) \to 0$ as $t \to \infty$, there is a dense set of $\mathbf{x} \in \mathrm{Sing}^*(2)$ such that $\ell(g_t h_{\mathbf{x}} \mathbb{Z}^{d+1}) \geq \varepsilon(t)$ as $t \to \infty$.*

Further results about singular vectors and divergent trajectories can be found in the papers [11], [12], [22], and [23].

Diagonal flows. The notion of a singular vector is dual to that of a badly approximable vector. Recall that $\mathbf{x} \in \mathbb{R}^d$ is *badly approximable* if there is a $c > 0$ such that $\|q\mathbf{x} - \mathbf{p}\| > cq^{-1/d}$ for all $(\mathbf{p}, q) \in \mathbb{Z}^d \times \mathbb{Z}_{>0}$. As with $\mathrm{Sing}\,(d)$, the set $\mathrm{BA}(d)$ of badly approximable vectors in $\mathbb{R}^d$ admits a characterization in terms of the flow on G/Γ induced by (g_t): $\mathbf{x} \in \mathrm{BA}(d)$ if and only if $(g_t h_{\mathbf{x}} \Gamma)_{t \geq 0}$ is a *bounded trajectory*; i.e., its closure in G/Γ is compact. In [20] Schmidt showed that $\mathrm{H.dim}\,\mathrm{BA}(d) = d$, which implies that the set $B \subset G/\Gamma$ of points that lie on bounded trajectories of the flow induced by (g_t) has $\mathrm{H.dim}\, B = \dim G$. The latter statement was later generalised by Kleinbock and Margulis in [9] to the setting of nonquasi-unipotent homogeneous flows where G is a connected real semisimple Lie group, Γ a lattice in G, and (g_t) a one-parameter subgroup such that $\mathrm{Ad}\, g_1$ has an eigenvalue of absolute value $\neq 1$.

Divergent trajectories of these flows have also been investigated. In [6] Dani showed that if Γ is of "rank one" then the set $D \subset G/\Gamma$ of points that lie on divergent trajectories of the flow is a countable union of proper submanifolds, implying that its Hausdorff dimension is integral and strictly less than $\dim G$. In the case where $G = (\mathrm{SL}_2\, \mathbb{R})^n$ with $n \geq 2$ and Γ is the reducible lattice $(\mathrm{SL}_2\, \mathbb{Z})^n$ with g_t inducing the same diagonal flow in each factor, the Hausdorff dimension of D was shown to be $\dim G - \frac{1}{2}$ ([5]). Corollary 1.2 furnishes a higher rank example with irreducible Γ, providing further evidence for the following.

CONJECTURE. *For any nonquasi-unipotent flow* $(G/\Gamma, g_t)$ *on a finite-volume, noncompact homogeneous space, the set of points that lie on divergent trajectories of the flow has Hausdorff dimension strictly less than* $\dim G$.[1]

Results in a similar spirit are also known for the Teichmüller (geodesic) flow, further reinforcing the analogy with diagonal flows. In this setting, it is customary to restrict attention to a Teichmüller disk and consider sets of "directions" (as a subset of the unit circle) that determine bounded or divergent trajectories. For any Teichmüller disk, the Hausdorff dimension of the set of bounded directions is always one ([10]), while the Hausdorff dimension of the set of divergent directions is always at most $\frac{1}{2}$ ([15]), with equality in some cases ([4]), and nonzero generically ([16]).

We briefly sketch the main ideas in the proof of Theorem 1.1. The strategy is to encode $\mathrm{Sing}\,(d)$ via the sequence of shortest vectors that arise during

[1]It is a well-known result of G. A. Margulis that a quasi-unipotent flow on a finite-volume homogeneous space admits no divergent trajectories.

the evolution of the lattice $g_t h_{\mathbf{x}} \mathbb{Z}^{d+1}$ as $t \to \infty$, then recast the problem of computing Hausdorff dimension into the language of symbolic dynamics. It is well known that vectors in $h_{\mathbf{x}} \mathbb{Z}^{d+1}$ that become short under the action of g_t correspond to good rational approximations to $\mathbf{x}$. For an appropriate choice of the norm on $\mathbb{R}^{d+1}$, this correspondence becomes *exact*, with vectors that become shortest corresponding to rationals that are best approximations to $\mathbf{x}$. (Lemma 2.4)

We cover $\mathrm{Sing}\,(d)$ with sets of the following form

$$\Delta(v) = \{\mathbf{x} : v \text{ is a best approximation to } \mathbf{x}\}.$$

The diameters of these sets are roughly given by

$$\frac{\delta(v)}{|v|^{1+1/d}}$$

where $\delta(v)$ is a "distortion parameter" measuring the length of the shortest nonzero vector in some d-dimensional unimodular lattice $\mathcal{L}(v)$ that is naturally associated to v. (2.14)

In estimating these diameters, we are essentially led to the following generalization of the basic inequalities satisfied by the sequence of convergents of the continued fraction. (Theorem 2.15)

THEOREM 1.5. *Let* $\frac{\mathbf{p}_j}{q_j}, j = 0, 1, \ldots$ *be the sequence of best approximations to* $\mathbf{x}$ *relative to some given norm* $\|\cdot\|$ *on* $\mathbb{R}^d$. *Then*

$$\frac{\|\mathbf{m}_{j+1}\|}{q_j(q_{j+1}+q_j)} < \left\|\mathbf{x} - \frac{\mathbf{p}_j}{q_j}\right\| < \frac{2\|\mathbf{m}_{j+1}\|}{q_j q_{j+1}} \tag{1.2}$$

where $\mathbf{m}_{j+1} \in \mathbb{Z}^d$ *is given by* $m_{j,i} = p_{j,i} q_{j+1} - p_{j+1,i} q_j$.

From the sequence of best approximations, one can easily recover the local maxima of the shortest vector function. (Lemma 2.17)

Singular vectors are characterised as those $\mathbf{x} \in \mathbb{R}^d$ for which the sequence of best approximations satisfies $\lim_j \delta(v_j) = 0$. (Theorem 2.19)

For each $\delta > 0$ we now have a covering $\{\Delta(v)\}_{\delta(v)<\delta}$ of $\mathrm{Sing}\,(d)$ which admits a self-similar structure (see §3) defined by

$$\sigma(v) = \{v : v \vdash_{\mathbf{x}} v' \text{ for some } \mathbf{x} \in \mathrm{Sing}\,(d)\} \tag{1.3}$$

where $v \vdash_{\mathbf{x}} v'$ means that v and v' are successive elements in the sequence $\Sigma(\mathbf{x})$ of best approximations to $\mathbf{x}$, with v' immediately following v.[2]

[2]The sets in (1.3) are difficult to describe *exactly* and for practical reasons, we work with slightly larger sets that have simpler explicit descriptions.

Any real $s > 0$ for which the inequality

$$\sum_{v' \in \sigma(v)} (\operatorname{diam} \Delta(v'))^s \leq (\operatorname{diam} \Delta(v))^s \tag{1.4}$$

holds (for all v) is an upper bound for H.dim Sing (d), although the existence of such an s is not guaranteed. (Theorem 3.1)

It is well known that $\Sigma(\mathbf{x})$ can contain arbitrarily long segments of rationals that lie on the same affine line in $\mathbb{R}^d$. (See [14].) Along such segments, the distortion parameter is strictly decreasing. (Lemma 2.18)

By passing to the subsequence $\widehat{\Sigma}(\mathbf{x})$ formed by the initial elements of these segments (first *acceleration*), we obtain a self-similar structure $\widehat{\sigma}$ for which (1.4) is *nearly* satisfied. (Proposition 4.12)

By passing to a further subsequence (second acceleration) along which the distortion parameter is monotone, we arrive at a self-similar structure $\widehat{\sigma}'$ that permits nontrivial upper bounds satisfying (1.4) to be obtained. (Proposition 4.13)

For lower bounds, we construct a suitable subcover of the first acceleration that satisfies certain spacing conditions required by a general lower bound estimate. (Theorem 3.3)

The arguments used to prove Theorem 1.1 essentially also yield a proof of the following.

THEOREM 1.6. *Let* $\mathrm{DI}_\delta(d)$ *be the set of all* $\mathbf{x} \in \mathbb{R}^d$ *for which there exists* T_0 *such that for all* $T > T_0$ *the system of inequalities* (1.1) *admits an integer solution. There are positive constants* c_1, c_2 *such that*

$$\frac{4}{3} + \exp(-c_1\delta^{-4}) \leq \text{H.dim}\, \mathrm{DI}_\delta(2) \leq \frac{4}{3} + c_2\delta. \tag{1.5}$$

As a warm-up to the proof of Theorem 1.1, we include in Section 3 a proof of the following result of I. J. Good [7] that includes many of the basic features of the main calculation except for the use of accelerations.[3]

THEOREM 1.7. *Let* D_∞ *be the set of real numbers whose sequence* (a_k) *of partial quotients*[4] *tend to infinity as* $k \to \infty$. *Then* H.dim $D_\infty = \frac{1}{2}$.

Outline of the paper. In Section 2 we develop the properties of the sequence of best approximations as well as the characterization of the set of singular vectors in terms of them. In Section 3 we recall the notion of a self-similar covering and develop a general lower bound estimate for a certain class of well-spaced self-similar coverings (Theorem 3.3), ending the section with a proof

[3]The author would like to thank T. M. Jordan for bringing [7] to his attention.

[4]This terminology for the terms of the continued fraction is classical.

of Theorem 1.7. The upper and lower bound calculations for Theorems 1.1 and 1.6 are presented in Sections 4 and 5, respectively. The final section, Section 6, is devoted to the proof of Theorem 1.4.

Acknowledgments. I would like to thank Jayadev Athreya, Yann Bugeaud, S. G. Dani, Manfred Einsiedler, Alex Eskin, Anish Ghosh, Dmitry Kleinbock, Jeff Lagarias, Elon Lindenstrauss, Gregory Margulis, Howard Masur, Curt McMullen, Yair Minsky, Omri Sarig, Rich Schwartz, Nimish Shah, Yakov Sinai, Corinna Ulcigrai, and Barak Weiss for various helpful conversations about this work.

I would like in particular to thank Alex Eskin for introducing me to this problem, and Barak Weiss for many in-depth discussions and especially his suggestion to use the norm $\|\cdot\|'$, which in one fell swoop eliminated a number of annoying technicalities. I would also like to thank the referee for a very careful reading of various versions of the paper and many helpful suggestions for improving the exposition.

I am especially grateful to my wife, Ying Xu, for her understanding, encouragement and unwavering support during the entire time while this work was in progress.

2. **Best approximations**

Let $\|\cdot\|$ be any norm on $\mathbb{R}^d$ and let $\|\cdot\|'$ denote the norm on $\mathbb{R}^{d+1}$ given by $\|(\mathbf{x}, y)\|' := \max(\|\mathbf{x}\|, |y|)$. For any $S \subset \mathbb{R}^{d+1}$, let

$$\ell(S) = \inf\{\|v\|' : v \in S, v \neq 0\}$$

and for any $\mathbf{x} \in \mathbb{R}^d$, let $W_\mathbf{x} : \mathbb{R} \to \mathbb{R}$ be the function defined by

$$W_\mathbf{x}(t) = \log \ell(g_t h_\mathbf{x} \mathbb{Z}^{d+1}).$$

LEMMA 2.1. *The function $W_\mathbf{x}$ is continuous, piecewise linear with slopes $-d$ and $+1$; moreover, it has infinitely many local minima if and only if $\mathbf{x} \notin \mathbb{Q}^d$.*

Proof. For any $v \neq 0$ the function $t \to \log \|g_t v\|'$ is a continuous, piecewise linear function with at most one critical point. Its derivative is defined everywhere except at the critical point and is either equal to $-d$ or $+1$. For each $\tau \in \mathbb{R}$, there is a finite set $F_\tau \subset h_\mathbf{x}\mathbb{Z}^{d+1}$ such that $W_\mathbf{x}(t) = \log \ell(g_t F_\tau)$ for all t in some neighborhood of τ. Thus, $W_\mathbf{x}$ is continuous, piecewise linear with slopes $-d$ and $+1$, because it locally satisfies the same property.

Let C be the set of critical points of $W_\mathbf{x}$ and note that F_τ can be chosen so that it is constant on each connected component of $\mathbb{R} \setminus C$. If $\mathbf{x} \notin \mathbb{Q}^d$ then $\ell(g_t F_\tau) \to \infty$ as $t \to \infty$ whereas Minkowski's theorem implies $\ell(g_t h_\mathbf{x} \mathbb{Z}^{d+1})$ is bounded above for all t. Hence, $F_t \neq F_\tau$ for some $t > \tau$ so that $C \cap [\tau, t] \neq \varnothing$ and since τ can be chosen arbitrarily large, it follows that $W_\mathbf{x}$ has infinitely many local minima. If $\mathbf{x} \in \mathbb{Q}^d$ then $\mathbf{x} = \frac{\mathbf{p}}{q}$ for some $\mathbf{p} \in \mathbb{Z}^d, q \in \mathbb{Z}$ such that $v = (\mathbf{p}, q)$ satisfies $\gcd(v) = 1$. Observe that we can take $F_\tau = \{h_{\mathbf{x}} v\}$ for all sufficiently large τ. Thus, C is bounded, hence finite, and in particular $W_{\mathbf{x}}$ has at most finitely many local minima. $\square$

Definition 2.2. Let v be a vector in

$$Q := \{(\mathbf{p}, q) \in \mathbb{Z}^{d+1} : \gcd(\mathbf{p}, q) = 1, q > 0\}$$

and τ a local minimum time of the function $W_{\mathbf{x}}$. We shall say "*v realises the local minimum of $W_{\mathbf{x}}$ at time τ*" if $W_{\mathbf{x}}(t) = \log \|g_t h_{\mathbf{x}} v\|'$ for all t in some neighborhood of τ. The set of vectors in Q that realise *some* local minimum of $W_{\mathbf{x}}$ will be denoted by

$$\Sigma(\mathbf{x}).$$

Convention: If $\mathbf{x} \in \mathbb{Q}^d$, then the vector $v = (\mathbf{p}, q) \in Q$ such that $\mathbf{x} = \frac{\mathbf{p}}{q}$ belongs to $\Sigma(\mathbf{x})$ because it realises the local minimum of $W_{\mathbf{x}}$ at the time $\tau = +\infty$.

Notation. Given $\mathbf{x} \in \mathbb{R}^d$ and $v = (\mathbf{p}, q) \in Q$ we let

$$\operatorname{hor}_{\mathbf{x}}(v) := \|q\mathbf{x} - \mathbf{p}\| \qquad \text{and} \qquad |v| := q. \tag{2.1}$$

Remark 2.3. For each $v \in \Sigma(\mathbf{x})$ there is an open interval of times t for which the minimum in

$$W_{\mathbf{x}}(t) = \min_{v \in Q} \log \max(e^{-dt}|v|, e^t \operatorname{hor}_{\mathbf{x}}(v))$$

is realised by v; the converse only holds if this open interval can be chosen to contain the "balance time" for v given by

$$\beta_{\mathbf{x}}(v) = -\frac{1}{d+1} \log \frac{\operatorname{hor}_{\mathbf{x}}(v)}{|v|}.$$

LEMMA 2.4. *Let $v \in Q$. Then $v \in \Sigma(\mathbf{x})$ if and only if for any $u \in Q$*

(i) $|u| < |v|$ *implies* $\operatorname{hor}_{\mathbf{x}}(u) > \operatorname{hor}_{\mathbf{x}}(v)$, *and*
(ii) $|u| = |v|$ *implies* $\operatorname{hor}_{\mathbf{x}}(u) \geq \operatorname{hor}_{\mathbf{x}}(v)$.

Proof. Suppose $v \in \Sigma(\mathbf{x})$, so that it realises a local minimum of $W_{\mathbf{x}}$. If $u \in Q$ does not satisfy (i) then $\|g_t h_{\mathbf{x}} u\|' \leq \|g_t h_{\mathbf{x}} v\|'$ for all t, with strict inequality for at least some t, which implies that v cannot realise a local minimum of $W_{\mathbf{x}}$, a contradiction. Therefore, (i) holds for any $u \in Q$. The argument that (ii) holds as well is similar. Conversely, suppose (i) and (ii) hold for all $u \in Q$. Let (τ, ε) be the unique local minimum of $t \to \|g_t h_{\mathbf{x}} v\|'$. Let B' be the closed $\|\cdot\|'$-ball of radius ε at the origin. Write it as $B \times I$ where $B \subset \mathbb{R}^d$ and $I \subset \mathbb{R}$. Let $Z = B' \cap g_t h_{\mathbf{x}} \mathbb{Z}^{d+1}$ and $Z^* = Z \setminus \{0\}$. Then (i) and (ii) imply $Z^* \subset \partial B \times \partial I$, which implies v realises the shortest nonzero vector at time τ. (By this we mean $g_\tau h_{\mathbf{x}} v$ is the shortest nonzero vector in $g_\tau h_{\mathbf{x}} \mathbb{Z}^{d+1}$.) Since there exists a slightly larger ball B'' containing B' such that $B'' \cap g_t h_{\mathbf{x}} \mathbb{Z}^{d+1} = Z$, it follows that v realises the shortest nonzero vector for an interval of t about τ. Thus, v realises a local minimum of $W_{\mathbf{x}}$ and $v \in \Sigma(\mathbf{x})$. $\square$

Recall that $\frac{\mathbf{p}}{q} \in \mathbb{Q}^d$ is a *best approximation* to $\mathbf{x}$ if

(i) $\|q\mathbf{x} - \mathbf{p}\| < \|n\mathbf{x} - \mathbf{m}\|$ for any $(\mathbf{m}, n) \in \mathbb{Z}^{d+1}$, $0 < n < q$,
(ii) $\|q\mathbf{x} - \mathbf{p}\| \leq \|q\mathbf{x} - \mathbf{p}'\|$ for any $\mathbf{p}' \in \mathbb{Z}^d$.

Lemma 2.4 gives a simple dynamical interpretation for the sequence of best approximations, ordered by increasing height: *they correspond precisely to the sequence of vectors that realise the local minima of* $W_{\mathbf{x}}$.

The study of best approximations has a long history going back to Lagrange, who showed that the sequence of best approximations in the case $d = 1$ are enumerated by the convergents of the continued fraction expansion. For further results on best approximations we refer the reader to the articles [13], [14] and the survey [17].

Notation. For any $v \in Q$ let

$$\dot{v} := \frac{\mathbf{p}}{q} \in \mathbb{Q}^d, \quad \text{where} \quad v = (\mathbf{p}, q) \in Q. \tag{2.2}$$

We shall often ignore the distinction between a vector $v \in Q$ and the rational $\dot{v}$ corresponding to it. Thus, we may refer to a sequence (v_j) in Q as the sequence of best approximations to $\mathbf{x}$, by which we mean that for every j the vector v_j realises the jth local minimum of $W_{\mathbf{x}}$. This raises the issue of the uniqueness of the vector realising a local minimum of $W_{\mathbf{x}}$, or equivalently, the existence of best approximations to $\mathbf{x}$ of the same height. In the case $d = 1$, this can only happen if $\mathbf{x}$ is a half integer. In general, there are at most finitely many local minima that can be realised by multiple vectors in Q. (See the remark following Theorem 2.11 below.) When referring to "the sequence of best approximations to $\mathbf{x}$" we really mean *any* sequence (v_j) such that the jth local minimum of $W_{\mathbf{x}}$ is realised by v_j.

The next lemma was proved in [14] for the case $d = 2$.

Lemma 2.5. *If $u, v \in \Sigma(\mathbf{x})$ realise a consecutive pair of local minima of the function $W_{\mathbf{x}}$, then they span a* primitive *two-dimensional sublattice of $\mathbb{Z}^{d+1}$; i.e., $\mathbb{Z}^{d+1} \cap (\mathbb{R}u + \mathbb{R}v) = \mathbb{Z}u + \mathbb{Z}v$.*

Proof. Let $F = \{\pm u, \pm v\}$ and denote its convex hull by $\operatorname{conv}(F)$. Let $\mathcal{C}(\mathbf{x})$ be the collection of subsets of $\mathbb{R}^{d+1}$ of the form

$$C(r, h) = \{(\mathbf{a}, b) : \|\mathbf{a}\| \leq r, |b| \leq h\}, \qquad r > 0,\ h > 0$$

that intersects $h_{\mathbf{x}}(\mathbb{Z}^{d+1} \setminus \{0\})$ on the boundary but not in the interior. Observe that the set of maximal (resp. minimal) elements of $\mathcal{C}(\mathbf{x})$, partially ordered by inclusion, is in one-to-one correspondence with the set of local maxima (resp. local minima) of the function $W_{\mathbf{x}}$. Since u and v realise distinct local minima,

$|u| \neq |v|$. Hence, without loss of generality, we may assume that $|u| < |v|$. The element of $\mathcal{C}(\mathbf{x})$ corresponding to the unique local maximum of $W_{\mathbf{x}}$ between the consecutive pair of local minima determined by u and v is given by the parameters $r = \mathrm{hor}_{\mathbf{x}}(u)$ and $h = |v|$. Note that $\mathrm{conv}(h_{\mathbf{x}}F)$ is a subset of $C(r,h)$ and intersects the boundary of $C(r,h)$ in the four points of $h_{\mathbf{x}}F$. Therefore,

$$\mathrm{conv}(F) \cap \mathbb{Z}^{d+1} = F \cup \{0\}. \tag{2.3}$$

Since u is primitive, there is a primitive $w \in \mathbb{Z}^{d+1}$ such that $\{u,w\}$ is an integral basis for $L = \mathbb{Z}^{d+1} \cap (\mathbb{R}u + \mathbb{R}v)$. Let $(c,d) = \varphi(v)$ where $\varphi : L \to \mathbb{Z}^2$ is the map that sends u to $(1,0)$ and w to $(0,1)$. Replacing w with $-w$, if necessary, we may assume $d > 0$, and by further replacing w with an integer multiple of u added to it, we may assume $0 \leq c < d$. Note that if $c > 0$ then $(1,1)$ lies in the triangle with vertices at (c,d), $(1,0)$ and the origin. This would imply that $\mathrm{conv}(F) \cap L$ contains $\varphi^{-1}(1,1)$, and since $(1,1) \notin \varphi(F \cup \{0\})$, this violates (2.3). Therefore, $c = 0$. And since v is primitive, we must have $d = 1$. Thus, $v = w$ and $L = \varphi(\mathbb{Z}^2) = \mathbb{Z}u + \mathbb{Z}v$. □

2.1. *Two-dimensional sublattices.*

Definition 2.6. For any $v \in Q$, we denote by

$$\mathcal{L}(v)$$

the set of primitive two-dimensional sublattices of $\mathbb{Z}^{d+1}$ *containing* v.

There is a natural way to view $\mathcal{L}(v)$ as the set of primitive elements in some d-dimensional lattice. Indeed, consider the exact sequence of real vector spaces

$$\mathbb{R} \longrightarrow \bigwedge^1 \mathbb{R}^{d+1} \xrightarrow{\varphi} \bigwedge^2 \mathbb{R}^{d+1} \longrightarrow \bigwedge^3 \mathbb{R}^{d+1} \tag{2.4}$$

where each map is exterior multiplication by v. Since $v \neq 0$, the kernel of φ is one-dimensional, from which it follows that the image of φ, denoted

$$\mathcal{L}_{\mathbb{R}}(v),$$

is a real vector space of dimension d. Similarly, we have an exact sequence of free $\mathbb{Z}$-modules

$$\mathbb{Z} \longrightarrow \bigwedge^1 \mathbb{Z}^{d+1} \longrightarrow \bigwedge^2 \mathbb{Z}^{d+1} \longrightarrow \bigwedge^3 \mathbb{Z}^{d+1}$$

where the image of the second map is a free $\mathbb{Z}$-module of rank d, denoted

$$\mathcal{L}_{\mathbb{Z}}(v).$$

It is a d-dimensional lattice embedded in $\mathcal{L}_{\mathbb{R}}(v)$. The set of primitive elements in $\mathcal{L}_{\mathbb{Z}}(v)$ is naturally identified with the set

$$\mathcal{L}_+(v) = \{u \wedge v : u, v \in Q, \mathbb{Z}u + \mathbb{Z}v \in \mathcal{L}(v)\}$$

of *oriented*, primitive, two-dimensional sublattices of $\mathbb{Z}^{d+1}$ that contain v, as the next lemma shows.

LEMMA 2.7. *Let $L = \mathbb{Z}u + \mathbb{Z}v$ be a two-dimensional sublattice of $\mathbb{Z}^{d+1}$. Then L is primitive if and only if $u \wedge v$ is primitive as an element of $\mathcal{L}_{\mathbb{Z}}(v)$; i.e., $u \wedge v \neq dw$ for any $d \geq 2$ and $w \in \mathcal{L}_{\mathbb{Z}}(v)$.*

Proof. Let $L' = \mathbb{Z}^{d+1} \cap \operatorname{span} L$ so that L is primitive if and only if $L' = L$. Suppose $u \wedge v = dw$ for some $d \geq 2$ and $w \in \mathcal{L}_{\mathbb{Z}}(v)$. Write $w = u' \wedge v$ for some $u' \in \mathbb{Z}^{d+1}$. Then $(du' - u) \wedge v = 0$ so that $du' = u + cv$ for some $c \in \mathbb{Z}$. Since $d \geq 2$, $u' \notin L$. Hence, $L' \neq L$. Conversely, suppose $L' \neq L$. Choose $u' \in Q$ so that $L' = \mathbb{Z}u' + \mathbb{Z}v$. Since $L \subset L'$, we may write $u = au' + bv$ for some $a, b \in \mathbb{Z}$. Then $u \wedge v = au' \wedge v$. Since the index of L in L' is given by $|a|$, we have $u \wedge v = dw$ where $d = |a|$ and $w = \pm u' \wedge v \in \mathcal{L}_{\mathbb{Z}}(v)$. Since $L \neq L'$, $d \geq 2$. $\square$

Identifying $\bigwedge^1 \mathbb{R}^{d+1}$ with $\mathbb{R}^{d+1}$, we note that the kernel of φ is given by the one-dimensional subspace $\mathbb{R}v$. Thus, φ induces an isomorphism of $\mathcal{L}_{\mathbb{R}}(v)$ with the space of cosets of $\mathbb{R}v$ in $\mathbb{R}^{d+1}$. The elements in $\mathcal{L}_{\mathbb{Z}}(v)$ correspond to cosets that have nonempty intersection with $\mathbb{Z}^{d+1}$.

Let E_+ be the expanding eigenspace for the action of g_1. Then

$$\mathbb{R}^{d+1} = E_+ \oplus \mathbb{R}v$$

and the map that sends a coset of $\mathbb{R}v$ to the point of intersection with E_+ induces an isomorphism of $\mathcal{L}_{\mathbb{R}}(v)$ with E_+. The norm $\|\cdot\|$ on $\mathbb{R}^d$, which is naturally identified with E_+, induces a norm on $\mathcal{L}_{\mathbb{R}}(v)$, which we shall denote by

$$\|\cdot\|_{\mathcal{L}(v)}.$$

Let E_- be the contracting eigenspace for g_1. Since $\dim E_- = 1$, the kth exterior power decomposes into two eigenspaces for g_t

$$\textstyle\bigwedge^k \mathbb{R}^{d+1} = E_+^k \oplus E_-^k,$$

where E_+^k and E_-^k are naturally identified with $\bigwedge^k E_+$ and $\bigwedge^{k-1} E_+$, respectively. Let $e_1, \ldots, e_{d+1}$ be the standard basis vectors for $\mathbb{R}^{d+1}$. The operation of wedging with e_{d+1} induces an isomorphism between E_+^k and E_-^{k+1}; in particular, we have an isomorphism of E_-^2 with E_+^1, which is naturally identified with $\mathbb{R}^d$ through the isomorphisms with E_+. The norm $\|\cdot\|$ on $\mathbb{R}^d$ induces a norm on E_-^2, which may be extended to a seminorm on all of $\bigwedge^2 \mathbb{R}^{d+1}$ by defining the (semi)norm of an element to be the norm of the component in E_-^2. This seminorm will be denoted by

$$|\cdot|.$$

Given $L \in \mathcal{L}(v)$ we may form an element $u \wedge v \in \mathcal{L}_{\mathbb{Z}}(v)$ by choosing any pair in Q such that $L = \mathbb{Z}u + \mathbb{Z}v$. This element is well defined up to sign, so it makes sense to talk about the norm of L as an element in $\mathcal{L}_{\mathbb{R}}(v)$ and also as an element of $\bigwedge^2 \mathbb{R}^{d+1}$; denote these, respectively, by

$$\|L\|_{\mathcal{L}(v)} \quad \text{and} \quad |L|.$$

There is a simple relation between these norms. Note that the action of $h_{\mathbf{x}}$ on an element in $\bigwedge^2 \mathbb{R}^{d+1}$ preserves the component in E_-^2. Now let u', v' be the respective images of u, v under $h_{\dot{v}}$. Then $u' \wedge v' = |v| u'_+ \wedge e_{d+1}$, where u'_+ is the component of u' in E_+. Note that u'_+ is precisely the point where the coset of $\mathbb{R}v$ corresponding to L intersects E_+. Note also that its norm is given by $\mathrm{hor}_{\dot{v}}(u)$. It follows that

$$\|L\|_{\mathcal{L}(v)} = \mathrm{hor}_{\dot{v}}(u) = \frac{|L|}{|v|}. \tag{2.5}$$

The image of $\mathcal{L}_{\mathbb{Z}}(v)$ under the isomorphism of $\mathcal{L}_{\mathbb{R}}(v)$ with E_+ is simply the image of $\mathbb{Z}^{d+1}$ under the projection of $\mathbb{R}^{d+1} = E_+ \oplus \mathbb{R}v$ onto E_+, i.e. the projection along lines parallel to v. Alternatively, it can be described as the set of all components in E_+ of vectors in $h_{\dot{v}}\mathbb{Z}^{d+1}$. Its volume is given by

$$\mathrm{vol}\left(\mathcal{L}_{\mathbb{Z}}(v)\right) = \frac{1}{|v|}.$$

Since $\mathcal{L}_+(v)$ is a discrete subset of a normed vector space, there exists an element of minimal positive norm. While this element may not be unique, we shall choose one for each $v \in Q$, once and for all, and denote it by

$$L(v).$$

The corresponding element in $\mathcal{L}(v)$ will be denoted by the same symbol.[5] Since the norm of the shortest nonzero vector in a unimodular lattice in $\mathbb{R}^d$ is bounded above by some constant μ_0 (depending on $\|\cdot\|$) we have for all $v \in Q$,

$$\frac{|L(v)|}{|v|^{1-1/d}} \leq \mu_0. \tag{2.6}$$

Let us mention a form of (2.5) that is symmetric with respect to u and v. Let $\mathrm{dist}\,(\cdot,\cdot)$ denote the metric on $\mathbb{R}^d$ induced by the norm $\|\cdot\|$. Then $\mathrm{hor}_{\dot{v}}(u) = |u|\,\mathrm{dist}\,(\dot{u},\dot{v})$ so that

$$\mathrm{dist}\,(\dot{u},\dot{v}) = \frac{|u \wedge v|}{|u||v|}. \tag{2.7}$$

Let us extend the notation $\dot{v}, |v|$ and $\mathrm{hor}_{\mathbf{x}}(v)$ introduced in (2.2) and (2.1) to the set $E_+^c = \mathbb{R}^{d+1} \setminus E_+$. Then (2.7) holds for all $u, v \in E_+^c$. It will be

[5] We shall often blur the distinction between elements in $\mathcal{L}_+(v)$ and $\mathcal{L}(v)$, leaving it to the context to determine which meaning is intended.

convenient to allow overscripts on the arguments of $\operatorname{dist}(\cdot,\cdot)$ to be dropped; formally, we are extending $\operatorname{dist} : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ to a function that is defined on $E \times E$ where E is the disjoint union of $\mathbb{R}^d$ and E_+^c. These conventions allow for more appealing formulas such as

$$|u \wedge v| = |u||v| \operatorname{dist}(u, v), \quad \operatorname{hor}_{\mathbf{x}}(v) = |v| \operatorname{dist}(v, \mathbf{x}).$$

2.2. *Domains of approximation.* We now investigate the sets

$$\Delta(v) := \{\mathbf{x} \in \mathbb{R}^d : v \in \Sigma(\mathbf{x})\}$$

for $v \in Q$. By Lemma 2.4,

$$\Delta(v) = \big(\cap_{|u|<|v|} \Delta_u(v)\big) \cap \big(\cap_{|u|=|v|} \overline{\Delta_u(v)}\big),$$

where the sets $\Delta_u(v)$, defined only for $u \in Q \setminus \{v\}$, are given by

$$\Delta_u(v) = \{\mathbf{x} \in \mathbb{R}^d : \operatorname{hor}_{\mathbf{x}}(u) > \operatorname{hor}_{\mathbf{x}}(v)\}.$$

We note that $\Delta_u(v)$ is bounded if and only if $|u| < |v|$.

LEMMA 2.8. *For any (distinct) $u, v \in Q$ with $|u| \le |v|$ we have*

$$\operatorname{dist}(\mathbf{x}, u) > \operatorname{dist}(u + v, u) \quad \forall \mathbf{x} \in \Delta_u(v).$$

Here, $\operatorname{dist}(u + v, \cdot)$ *means* $\operatorname{dist}(\dot{w}, \cdot)$ *where* $w = u + v$.

Proof. By definition, $\mathbf{x} \in \Delta_u(v)$ if and only if

$$\operatorname{dist}(v, \mathbf{x}) < \lambda \operatorname{dist}(u, \mathbf{x}) \quad \text{where} \quad \lambda = \frac{|u|}{|v|} \le 1.$$

Let $w = u + v$. Since $\operatorname{dist}(u, v) = \operatorname{dist}(u, w) + \operatorname{dist}(w, v)$ and

$$\operatorname{dist}(v, w) = \frac{|v \wedge w|}{|v||w|} = \frac{|u|}{|v|} \left(\frac{|u \wedge w|}{|u||w|} \right) = \lambda \operatorname{dist}(u, w),$$

the triangle inequality implies

$$(1 + \lambda) \operatorname{dist}(u, w) \le \operatorname{dist}(u, \mathbf{x}) + \operatorname{dist}(v, \mathbf{x}) < (1 + \lambda) \operatorname{dist}(u, \mathbf{x})$$

and the lemma follows. □

Remark 2.9. It follows easily from Lemma 2.8 that the infimum of

$$\{\operatorname{dist}(u, \mathbf{x}) : \mathbf{x} \in \overline{\Delta_u(v)}\}$$

is realised at the rational point corresponding to $u + v$. It can similarly be shown that the supremum is realised by the rational point corresponding to $v - u$. In the case when $\|\cdot\|$ is the Euclidean norm, the set $\Delta_u(v)$ is the open Euclidean ball having these points as antipodal points.

LEMMA 2.10. *For any $L \in \mathcal{L}_+(v)$ there are unique vectors $u_\pm \in Q$ satisfying $L = u_\pm \wedge v$ and $|u_\pm| \le |v|$. These vectors satisfy $|u_+| < |v|$ if and only if $v = u_+ + u_-$, if and only if $|u_-| < |v|$. Similarly, $|u_+| = |v|$ if and only if $2v = u_+ + u_-$, if and only if $|u_-| = |v|$. Furthermore, if $L = L(v)$ then $|u_+| < |v|$ provided*

$$|v| > \frac{\mu_0^d}{\lambda_0^d}, \tag{2.8}$$

where λ_0 is the length of the shortest nonzero vector in $\mathbb{Z}^d$ with respect to the norm $\|\cdot\|$.

Proof. Existence and uniqueness of $u_\pm$ are clear. If $|u_+| < |v|$ then $v - u_+$ satisfies the conditions defining u_- so that $v - u_+ = u_-$. Similarly, if $|u_+| = |v|$ then $2v - u_+ = u_-$. Finally, if $L = L(v)$ and $|u_+| = |v|$ then $v - u_+ \in \mathbb{Z}^d$ and is also the shortest nonzero vector in $\mathcal{L}_{\mathbb{Z}}(v)$, represented as a subset of E_+ via the projection of $\mathbb{Z}^{d+1}$ along lines parallel to v. Since $\mathcal{L}_{\mathbb{Z}}(v)$ contains $\mathbb{Z}^d$, we have

$$\lambda_0 = \|v - u_+\| = \frac{|L(v)|}{|v|}.$$

From (2.6) it follows that $\lambda_0 \le \frac{\mu_0}{|v|^{1/d}}$, which is precluded by (2.8). □

Let $B(\mathbf{x}, r) \subset \mathbb{R}^d$ denote the open ball at $\mathbf{x}$ of radius r.

THEOREM 2.11. *For any $v \in Q$ satisfying* (2.8),

$$B\Big(\dot{v}, \frac{r}{2}\Big) \subset \Delta(v) \subset B(\dot{v}, 2r) \quad \textit{where} \quad r = \frac{|L(v)|}{|v|^2}. \tag{2.9}$$

Proof. Consider $u \in Q$ with $|u| \le |v|$ and $u \ne v$. Let $\lambda = \frac{|u|}{|v|}$ and $\mu > 0$. For any $\mathbf{x} \in B(\dot{v}, \mu r)$,

$$\operatorname{dist}(\mathbf{x}, v) < \frac{\mu |L(v)|}{|v|^2} \le \lambda\mu \frac{|u \wedge v|}{|u||v|} = \lambda\mu \operatorname{dist}(u, v)$$

so that

$$\operatorname{dist}(\mathbf{x}, u) > (1 - \lambda\mu)\frac{|u \wedge v|}{|u||v|}.$$

Thus, $\mathbf{x} \in \Delta_u(v)$ provided

$$\mu \le 1 - \lambda\mu.$$

Since $\lambda \le 1$, this holds for $\mu = \frac{1}{2}$. This establishes the first inequality in (2.9).

Let $L \in \mathcal{L}(v)$ and choose $u_\pm$ as in Lemma 2.10 so that $|u_-| \le |u_+|$. For any $\mathbf{x} \in \Delta_{u_-}(v)$,

$$\operatorname{dist}(\mathbf{x}, v) < \frac{|u_-|}{|v|} \operatorname{dist}(\mathbf{x}, u_-) \le \frac{|u_-|}{|v|}\left(\operatorname{dist}(\mathbf{x}, v) + \frac{|L|}{|u_-||v|}\right)$$

so that

$$\left(1-\frac{|u_-|}{|v|}\right)\operatorname{dist}(\mathbf{x},v)<\frac{|L|}{|v|^2}.$$

If $|u_+|<|v|$ then it follows that

$$\operatorname{dist}(\mathbf{x},v)<\frac{|L|}{|u_+||v|}\leq\frac{2|L|}{|v|^2}. \tag{2.10}$$

The second inequality in (2.9) now follows by setting $L=L(v)$ and application of Lemma 2.10. $\square$

As a corollary of Theorem 2.11 we get

$$\operatorname{diam}\Delta(v)\asymp\frac{|L(v)|}{|v|^2}\leq\frac{\mu_0}{|v|^{1+1/d}}, \tag{2.11}$$

where μ_0 is the constant satisfying (2.6). Here, $A\asymp B$ means $C^{-1}B\leq A\leq CB$ for some constant C.

Remark 2.12. For ℓ_p-norms $(1\leq p\leq\infty)$, $\Delta(v)\not\subset B(v,2r)$ implies $\mathcal{L}(v)$ is a well-rounded lattice with the standard bases as minimal vectors; consequently $\Delta(v)\subset\bar{B}_\infty(v,\frac{r}{2})$, where $\bar{B}_\infty$ denotes a (closed) ball with respect to the sup norm.

Remark 2.13. If $u,v\in Q$ are distinct vectors that realise the *same* local minimum of $W_{\mathbf{x}}$, then Theorem 2.11 and (2.6) imply

$$\operatorname{dist}(u,v)\leq\frac{2|L(u)|}{|u|^2}+\frac{2|L(v)|}{|v|^2}\leq\frac{4\mu_0}{|v|^{1+1/d}},$$

provided $|u|=|v|>\frac{\mu_0^d}{\lambda_0^d}$. Since $u-v\in\mathbb{Z}^d$,

$$\lambda_0\leq\|u-v\|=|v|\operatorname{dist}(u,v)\leq\frac{4\mu_0}{|v|^{1/d}}$$

so that the condition

$$|v|>\frac{4^d\mu_0^d}{\lambda_0^d}$$

implies v is uniquely determined by the local minimum that it realises. Thus, the tail of the sequence of best approximations is uniquely determined, a fact already observed in [13].

Theorem 2.14. *Let $v\in\Sigma(\mathbf{x})$ and suppose $u\in Q$ is such that $|u|<|v|$. Then*

$$\frac{1}{2}\operatorname{dist}(u,v)<\operatorname{dist}(u,\mathbf{x})<2\operatorname{dist}(u,v). \tag{2.12}$$

Proof. Let $L \in \mathcal{L}(v)$ contain u. Then $|u \wedge v| = b|L|$ for some positive integer b. By Lemma 2.8,

$$\operatorname{dist}(\mathbf{x}, u) > \operatorname{dist}(u+v, u) = \frac{b|L|}{|u+v||u|} \geq \frac{b|L|}{2|u||v|} \geq \frac{1}{2}\operatorname{dist}(u, v).$$

Let $u_\pm$ be as in Lemma 2.10 and note that $|u_+| < |v|$ as a consequence of $|u| < |v|$. From the *first* inequality in (2.10) we have

$$\begin{aligned}\operatorname{dist}(\mathbf{x}, u) &\leq \operatorname{dist}(u, v) + \operatorname{dist}(v, \mathbf{x}) \\ &< \frac{b|L|}{|u||v|} + \frac{|L|}{|u_+||v|} = \left(1 + \frac{|u|}{b|u_+|}\right)\operatorname{dist}(u, v).\end{aligned}$$

If $b \geq 2$, then the expression in parentheses is at most 2. If $b = 1$, then $u = u_\pm$ and the same is true again. Thus, $\operatorname{dist}(\mathbf{x}, u) < 2\operatorname{dist}(u, v)$. □

Theorem 1.5 is a consequence of the following:

THEOREM 2.15. *Let* $\frac{\mathbf{p}_j}{q_j}$ *be the sequence of best approximations to* $\mathbf{x}$ *and set* $v_j = (\mathbf{p}_j, q_j)$ *and* $L_{j+1} = \mathbb{Z}v_{j+1} + \mathbb{Z}v_j$. *Then*

$$\frac{|L_{j+1}|}{q_j(q_{j+1} + q_j)} < \left\|\mathbf{x} - \frac{\mathbf{p}_j}{q_j}\right\| < \frac{2|L_{j+1}|}{q_j q_{j+1}}. \tag{2.13}$$

Proof. The first inequality follows by an application of Lemma 2.8 with $u = v_j$ and $v = v_{j+1}$, while the second inequality follows by a similar application of Theorem 2.14. □

It can be shown that $|L_j| \leq C|L(v_j)|$ for all j (and $\mathbf{x}$), where C is a constant that depends only on the norm $\|\cdot\|$. Using this, we may rewrite (2.13) as

$$\|q_j\mathbf{x} - \mathbf{p}_j\| \asymp \frac{\delta(v_{j+1})}{q_{j+1}^{1/d}},$$

where

$$\delta(v) = |v|^{1/d}\|L(v)\|_{\mathcal{L}(v)} \leq \mu_0. \tag{2.14}$$

As we shall see, $\mathbf{x} \in \operatorname{Sing}(d)$ if and only if $\delta(v_j) \to 0$ as $j \to \infty$. See Theorem 2.19 below.

2.3. *Characterisation of singular vectors.* The sequence of critical times of the function $W_\mathbf{x}$ are ordered by

$$\tau_0 < t_0 < \tau_1 < t_1 < \dots,$$

where τ_j (resp. t_j) is the jth local maximum (resp. minimum) time. Note that the first critical point τ_0 is a local maximum because as $t \to -\infty$ we have $W_\mathbf{x}(t) = t + \log\|v_{-1}\|$, where v_{-1} is any nonzero vector in $\mathbb{Z}^d$ of minimal $\|\cdot\|$-norm.

Definition 2.16. For $u, v \in Q$ and $\mathbf{x} \in \mathbb{R}^d$, let $\varepsilon_{\mathbf{x}}(u,v)$ and $\tau_{\mathbf{x}}(u,v)$ be defined by

$$\varepsilon_{\mathbf{x}}(u,v)^{1+1/d} = |v|^{1/d}\,\mathrm{hor}_{\mathbf{x}}(u),$$
$$\tau_{\mathbf{x}}(u,v) = -\frac{1}{d+1}\log\frac{\mathrm{hor}_{\mathbf{x}}(u)}{|v|} = -\frac{1}{d}\log\frac{\varepsilon_{\mathbf{x}}(u,v)}{|v|}.$$

LEMMA 2.17. *For any $u, v \in Q$ with $|u| < |v|$ and for any $\mathbf{x} \in \Delta(v)$ there is a unique time τ when $\|g_\tau h_{\mathbf{x}} u\|' = \|g_\tau h_{\mathbf{x}} v\|'$, given by $\tau = \tau_{\mathbf{x}}(u,v)$. Moreover, the common length is given by $\varepsilon_{\mathbf{x}}(u,v)$.*

Proof. Since $|u| < |v|$ and $\mathbf{x} \in \Delta(v)$ we have $\mathrm{hor}_{\mathbf{x}}(u) > \mathrm{hor}_{\mathbf{x}}(v)$. This implies the existence of τ. Let $\varepsilon = \varepsilon_{\mathbf{x}}(u,v)$. The point $(\tau, \log\varepsilon)$ is where the lines $y = -d(t-a)$ and $y = (t-b)$ meet, where $a = \frac{1}{d}\log|v|$ and $b = -\log\mathrm{hor}_{\mathbf{x}}(u)$. Since

$$(t,y) = \left(\frac{ad+b}{d+1}, \frac{ad-bd}{d+1}\right),$$

we have

$$\log\varepsilon = \frac{1}{d+1}\log|v| + \frac{d}{d+1}\log\mathrm{hor}_{\mathbf{x}}(u),$$
$$\tau = \frac{1}{d+1}\log|v| - \frac{1}{d+1}\log\mathrm{hor}_{\mathbf{x}}(u)$$

from which it follows that

$$\varepsilon^{1+1/d} = |v|^{1/d}\,\mathrm{hor}_{\mathbf{x}}(u) \quad \text{and} \quad e^{-(d+1)\tau} = \frac{\mathrm{hor}_{\mathbf{x}}(u)}{|v|}. \qquad \square$$

LEMMA 2.18. *Assume $\mathbf{x} \notin \mathbb{Q}^d$ and set $\delta_j = \varepsilon_{\mathbf{x}}(v_{j-1}, v_j)^{1+1/d}$, where (v_j) is the sequence of best approximations to $\mathbf{x}$. Then $\mathbf{x} \in \mathrm{DI}_\delta(d)$ if and only if $\delta_j < \delta$ for all sufficiently large j.*

Proof. Write $v_j = (\mathbf{p}_j, q_j)$ so that Lemma 2.17 implies

$$\|q_j\mathbf{x} - \mathbf{p}_j\| = \mathrm{hor}_{\mathbf{x}}(v_j) = \frac{\delta_{j+1}}{q_{j+1}^{1/d}}.$$

If $\delta_j < \delta$ then $(\mathbf{p}_j, q_j)$ solves (1.1) for all $q_j < T \le q_{j+1}$. It follows that $\delta_j < \delta$ for all large enough j implies $\mathbf{x} \in \mathrm{DI}_\delta(d)$. Conversely, suppose $\mathbf{x} \in \mathrm{DI}_\delta(d)$ so that there exists T_0 such that (1.1) admits a solution for all $T > T_0$. Suppose j is large enough so that $q_{j+1} > T_0$. Let $(\mathbf{p}, q)$ be a solution to (1.1) for $T = q_{j+1}$. Then $q < q_{j+1}$, implying that

$$\|q_j\mathbf{x} - \mathbf{p}_j\| \le \|q\mathbf{x} - \mathbf{p}\| < \frac{\delta}{q_{j+1}^{1/d}}$$

from which it follows that $\delta_j < \delta$. $\square$

The next theorem gives a characterization purely in terms of the sequence of best approximations, without explicit reference to $\mathbf{x}$. It allows us to translate the problem of computing Hausdorff dimension into the language of symbolic dynamics.

THEOREM 2.19. *Assume $\mathbf{x} \notin \mathbb{Q}^d$ and for each $j \geq 1$ set*

$$\varepsilon_j^{1+1/d} = \frac{|v_{j-1} \wedge v_j|}{|v_j|^{1-1/d}},$$

where $(v_j)_{j\geq 0}$ is the sequence of best approximations to $\mathbf{x}$. Assume $\delta > 0$ and $\varepsilon > 0$ are related by $\delta = \varepsilon^{1+1/d}$. Then $\mathbf{x} \in \mathrm{DI}_{\delta/2}(d)$ implies $\varepsilon_j < \varepsilon$ for all sufficiently large j, which in turn implies $\mathbf{x} \in \mathrm{DI}_{2\delta}(d)$. In particular, $\mathbf{x} \in \mathrm{Sing}\,(d)$ if and only if $\varepsilon_j \to 0$ as $j \to \infty$.

Proof. By Lemma 2.18 it suffices to show

$$\frac{1}{2}\varepsilon_j^{1+1/d} \leq \varepsilon_{\mathbf{x}}(v_{j-1}, v_j) \leq 2\varepsilon_j^{1+1/d}$$

which holds by Theorem 2.14. □

3. Self-similar coverings

Let X be a metric space and J a countable set. Given $\sigma \subset J \times J$ and $\alpha \in J$ we let $\sigma(\alpha)$ denote the set of all $\alpha' \in J$ such that $(\alpha, \alpha') \in \sigma$. We say a sequence (α_j) of elements in J is *σ-admissible* if $\alpha_{j+1} \in \sigma(\alpha_j)$ for all j; also we let J^σ denote the set of all σ-admissible sequences in J. By a *self-similar covering* of X we mean a triple $(\mathcal{B}, J, \sigma)$ where $\mathcal{B}$ is a collection of bounded subsets of X, J a countable index set for $\mathcal{B}$, and $\sigma \subset J \times J$ such that there is a map $\mathcal{E} : X \to J^\sigma$ that assigns to each $x \in X$ a σ-admissible sequence $(\alpha_j^{\mathbf{x}})$ such that for all $x \in X$

(i) $\cap B(\alpha_j^{\mathbf{x}}) = \{x\}$, and
(ii) $\mathrm{diam}\, B(\alpha_j^{\mathbf{x}}) \to 0$ as $j \to \infty$,

where $B(\alpha)$ denotes the element of $\mathcal{B}$ indexed by α.

THEOREM 3.1 ([5][Th. 5.3]). *If X is a metric space that admits a self-similar covering $(\mathcal{B}, J, \sigma)$, then* $\mathrm{H.dim}\, X \leq s(\mathcal{B}, J, \sigma)$ *where*

$$s(\mathcal{B}, J, \sigma) = \sup_{\alpha\in J} \inf \left\{ s > 0 : \sum_{\alpha' \in \sigma(\alpha)} \left(\frac{\mathrm{diam}\, B(\alpha')}{\mathrm{diam}\, B(\alpha)} \right)^s \leq 1 \right\}. \tag{3.1}$$

In many applications, X is a subset of some ambient metric space Y and we are given a self-similar covering $(\mathcal{B}, J, \sigma)$ of X by bounded subsets of Y rather than X. For any bounded subset $B \subset Y$ we have $\mathrm{diam}_X\, B \cap X \leq \mathrm{diam}_Y\, B$ but equality need not hold in general. To compute $s(cB, J, \sigma)$ one would also need an inequality going in the other direction. While such an inequality may not be difficult to obtain, it is both awkward and unnecessary: *Theorem* 3.1 *remains valid in this more general situation if the diameters in* (3.1) *are taken with respect to the metric of* Y. The proof of this more general statement does not follow directly from Theorem 3.1, but the argument given in [5] applies with essentially no change and will not be repeated here.

For lower bounds, we shall use

THEOREM 3.2. *Let* $\mathcal{B}$ *be a collection of nonempty compact subsets of a metric space* Y *indexed by a countable set* J. *Suppose* $X \subset Y$ *and* $\sigma \subset J \times J$ *are such that*

(i) *For each* $\alpha \in J$, $\sigma(\alpha)$ *is a finite subset of* J *with at least* 2 *elements and for each* $\alpha' \in \sigma(\alpha)$ *we have* $B(\alpha') \subset B(\alpha)$.

(ii) *For each* $(\alpha_j) \in J^\sigma$, *we have* $\operatorname{diam} B(\alpha_j) \to 0$ *and the unique point in* $\cap B(\alpha_j)$ *belongs to* X.

(iii) *There exists* $\rho > 0$ *such that for any* $\alpha \in J$ *and for any pair of distinct* $\alpha', \alpha'' \in \sigma(\alpha)$

$$\operatorname{dist}(B(\alpha'), B(\alpha'')) \geq \rho \operatorname{diam} B(\alpha). \tag{3.2}$$

(iv) *There exists* $s > 0$ *such that for every* $\alpha \in J$,

$$\sum_{\alpha' \in \sigma(\alpha)} [\operatorname{diam} B(\alpha')]^s \geq [\operatorname{diam} B(\alpha)]^s. \tag{3.3}$$

Then $\text{H.dim}\, X \geq s$.

Theorem 3.2 is a special case of the next theorem, which allows for a spacing condition with *weights*.

THEOREM 3.3. *Suppose* (i) *and* (ii) *of Theorem* 3.2 *holds*, *and there exists a function* $\rho : J \to (0,1)$ *such that*

(iii′) *For any* $\alpha \in J$ *and for any pair of distinct* $\alpha', \alpha'' \in \sigma(\alpha)$,

$$\operatorname{dist}(B(\alpha'), B(\alpha'')) > \rho(\alpha) \operatorname{diam} B(\alpha). \tag{3.4}$$

(iv′) *There exists* $s > 0$ *such that for every* $\alpha \in J$,

$$\sum_{\alpha' \in \sigma(\alpha)} [\rho(\alpha') \operatorname{diam} B(\alpha')]^s \geq [\rho(\alpha) \operatorname{diam} B(\alpha)]^s.$$

Then the s*-dimensional Hausdorff measure of* X *is positive. In particular*, $\text{H.dim}\, X \geq s$.

Proof. Fix any $\alpha_0 \in J$ and let

$$E = E(\alpha_0)$$

be the set of all $x \in X$ such that $\cap B(\alpha_j) = \{x\}$ for some σ-admissible sequence (α_j) starting with α_0. Let $J_0 = \{\alpha_0\}$ and $J_k = \cup_{\alpha \in J_{k-1}} \sigma(\alpha)$ for $k > 0$. Note that

$$E = \cap_{k\geq 0} E_k \quad \text{where} \quad E_k = \cup_{\alpha\in J_k} B(\alpha).$$

Since each J_k is finite, E is compact. Let $J' = \cup_{k\geq 0} J_k$.

CLAIM. *For any finite subset $F \subset J'$ such that $\mathcal{B}_F = \{B(\alpha)\}_{\alpha\in F}$ covers E,*

$$\sum_{\alpha\in F} [\rho(\alpha)\operatorname{diam} B(\alpha)]^s \geq [\rho(\alpha_0)\operatorname{diam} B(\alpha_0)]^s. \tag{3.5}$$

To prove the claim, it is enough to consider the case where $\mathcal{B}_F$ has no redundant elements; i.e., $B(\alpha)\cap E \neq \varnothing$ for all $\alpha \in F$, and $B(\alpha) \not\subset B(\alpha')$ for any distinct pair $\alpha, \alpha' \in F$. It follows by (i) that the elements of $\mathcal{B}_F$ form a disjoint collection.

Proceed by induction on the smallest k such that $F \subset J_0 \cup\cdots\cup J_k$. If $k = 0$, then $F = \{\alpha_0\}$ and (3.5) holds with equality. For $k > 0$, first note that for any $\alpha' \in F \cap J_k$ we have $\sigma(\alpha) \subset F$ where α is the unique element of J_{k-1} such that $\alpha' \in \sigma(\alpha)$. Indeed, given $\alpha'' \in \sigma(\alpha)$, $B(\alpha'')\cap E \neq \varnothing$ implies that $B(\alpha'')$ intersects $B(\alpha''')$ for some $\alpha''' \in F$. We cannot have $B(\alpha'') \subset B(\alpha''')$ because otherwise $B(\alpha')$ would be a redundant element in $\mathcal{B}_F$; therefore, $\alpha''' \notin J_i$ for any $i < k$. Since $\alpha''' \in F$, we have $\alpha''' \in J_k$ so that $\alpha'' = \alpha''' \in F$.

Let $F' = F \cap (J_0 \cup \cdots \cup J_{k-1})$ and let $\widetilde{F}$ be the subset of J_{k-1} such that $F \cap J_k$ is the disjoint union of $\sigma(\alpha)$ as α ranges over the elements of $\widetilde{F}$. Then (iv) implies

$$\sum_{\alpha\in F} [\rho(\alpha)\operatorname{diam} B(\alpha)]^s \geq \sum_{\alpha\in F'\cup\widetilde{F}} [\rho(\alpha)\operatorname{diam} B(\alpha)]^s$$

and the claim follows by the induction hypothesis applied to $F' \cup \widetilde{F}$.

Now suppose $\mathcal{U}$ is a covering of E by open balls of radius at most ε. Since E is compact, there is a finite subcover $\mathcal{U}_0$ and without loss of generality we may assume each element of $\mathcal{U}_0$ contains some point of E. For each $U \in \mathcal{U}_0$ let (α_k) be the sequence determined by a choice of $x \in U \cap E$ and requiring $x \in B(\alpha_k), \alpha_k \in J_k$ for all k. Let k_0 be the largest index k such that $U \cap E \subset B(\alpha_k)$. Then there are distinct elements $\alpha', \alpha'' \in \sigma(\alpha_k)$ such that U intersects both $B(\alpha')$ and $B(\alpha'')$ so that (iii) implies

$$\operatorname{diam} U \geq \operatorname{dist}(B(\alpha'), B(\alpha'')) \geq \rho(\alpha_k)\operatorname{diam} B(\alpha_k).$$

Let F be the collection of α_k associated to $U \in \mathcal{U}_0$. Then

$$\sum_{U\in\mathcal{U}} (\operatorname{diam} U)^s \geq \sum_{\alpha\in F} [\rho(\alpha)\operatorname{diam} B(\alpha)]^s \geq [\rho(\alpha_0)\operatorname{diam} B(\alpha_0)]^s.$$

Since $\varepsilon > 0$ was arbitrary, it follows that E (and therefore also X) has positive s-dimensional Hausdorff measure. $\square$

Divergent partial quotients. In this section, we prove Theorem 1.7 as an illustration of the basic technique. The rest of this section is independent of the other parts of the paper and may be skipped without loss of continuity.

Let D_N be the set of all irrational numbers whose sequence of partial quotients satisfies $a_k > N$ for all sufficiently large k. Given $p/q \in \mathbb{Q}$ with $q \geq 2$ let $p_-/q_- < p_+/q_+$ be the convergents that precede p/q in the two possible continued fraction expansions for p/q. They are determined by the conditions

$$p_\pm q - pq_\pm = \pm 1, \quad 0 < q_\pm < q$$

and we note that $q = q_+ + q_-$. Let $v = (p, q)$ and set

$$I_N(v) = \left[\frac{Np + p_-}{Nq + q_-}, \frac{Np + p_+}{Nq + q_+}\right].$$

This interval consists of all real numbers that have p/q as a convergent and such that the next partial quotient is at least N. (In particular, we note that $I_1(v) = \overline{\Delta(v)}$.) The length of the interval is

$$\begin{aligned} \left|I_N(v)\right| &= \left|\frac{Np + p_-}{Nq + q_-} - \frac{p}{q}\right| + \left|\frac{p}{q} - \frac{Np + p_+}{Nq + q_+}\right| \\ &= \frac{1}{(Nq + q_-)q} + \frac{1}{(Nq + q_+)q} = \frac{2N + 1}{(Nq + q_-)(Nq + q_+)} \end{aligned}$$

so that

$$\frac{2}{(N+1)q^2} \leq \left|I_N(v)\right| \leq \frac{2}{Nq^2}. \tag{3.6}$$

Let $\mathcal{B}_N$ be the collection of intervals $I_N(v)$ for $v \in Q$, $|v| \geq 2$. Let $\sigma_N(v)$ be the set of all $v' \in Q$ of the form $av + v_\pm$, where $v_\pm = (p_\pm, q_\pm)$ and $a > N$. Then $(\mathcal{B}_N, Q, \sigma_N)$ is a self-similar covering of D_N. The map $\mathcal{E}$ is realised by sending $x \in D_N$ to a tail of the sequence of convergents of x. For any $v' \in \sigma_N(v)$ we have

$$\frac{N}{(a+1)^2(N+1)} \leq \frac{\left|I_N(v')\right|}{\left|I_N(v)\right|} \leq \frac{N+1}{a^2 N},$$

where a is the greatest integer less than $\frac{|v'|}{|v|}$. Note that there are two elements of $\sigma(v)$ associated with each $a > N$.

To estimate Hausdorff dimension we need to consider the expression

$$\sum_{v' \in \sigma(v)} \frac{\left|I_N(v')\right|^s}{\left|I_N(v)\right|^s}.$$

For any $0 < s < 1$,

$$\sum_{a > N} \frac{2(N+1)^s}{a^{2s} N^s} \leq \frac{4}{(2s-1)N^{2s-1}},$$

which is ≤ 1 provided

$$\log N \geq \frac{1}{2s-1} \log \frac{4}{2s-1}. \tag{3.7}$$

Let $s_+ = s_+(N)$ be the unique s such that (3.7) holds with equality. Note that if $y = x \log x$ then in the limit as $x \to \infty$ we have $\log y \simeq \log x$ so that $x \simeq \frac{y}{\log y}$. Here, the notation $A \simeq B$ means the ratio tends to one. It follows that, in the limit as $N \to \infty$ we have

$$\frac{1}{2s_+ - 1} \simeq \frac{\log N}{\log \log N}$$

so that applying Theorem 3.1 we now get

$$\text{H.dim}\, D_N \leq \frac{1}{2} + \frac{c \log \log N}{\log N}$$

for some constant $c > 0$.

Let $\sigma'_N(v)$ be the subset of $\sigma_N(v)$ consisting of those v' for which $\lfloor \frac{|v'|}{|v|} \rfloor \leq 2N$. For distinct $v', v'' \in \sigma'_N(v)$ with $v' = (p', q'), v'' = (p'', q'')$ we have

$$\left| \frac{p'}{q'} - \frac{p''}{q''} \right| \geq \frac{1}{q'q''} \geq \frac{1}{(2N+1)^2 q^2}, \tag{3.8}$$

whereas

$$\left| I(v') \right| \leq \frac{2}{N(q')^2} \leq \frac{2}{N^3 q^2}$$

from which we see that the gap between $I(v')$ and $I(v'')$ is at least (assuming $N \geq 72$)

$$\frac{1}{9N^2q^2} - \frac{4}{N^3q^2} \geq \frac{1}{18N^2q^2} \geq \frac{1}{36N} \left| I_N(v) \right|.$$

Thus, (3.2) holds with $\rho = \frac{1}{36N}$. For any $0 < s < 1$ we have (using $N \geq 2$)

$$\begin{aligned} \sum_{N < a \leq 2N} \frac{2N^s}{(a+1)^{2s}(N+1)^s} &\geq \frac{1}{2s-1} \left(\frac{1}{(N+2)^{2s-1}} - \frac{1}{(2N+2)^{2s-1}} \right) \\ &\geq \frac{1}{3(2s-1)(N+2)^{2s-1}} \geq \frac{1}{6(2s-1)N^{2s-1}} \end{aligned}$$

which is ≥ 1 provided

$$\log N \leq \frac{1}{2s-1} \log \frac{1/6}{2s-1}. \tag{3.9}$$

Let $s_- = s_-(N)$ be the unique s such that (3.9) holds with equality. It follows that in the limit as $N \to \infty$ we have

$$\frac{1}{2s_- - 1} \simeq \frac{\log N}{\log \log N}$$

so that applying Theorem 3.2 we now get

$$\mathrm{H.dim}\, D_N \geq \frac{1}{2} + \frac{c \log\log N}{\log N}$$

for some constant $c > 0$. This establishes the following (cf. [7]).

THEOREM 3.4. *There are $c_2 > c_1 > 0$ such that for all $N \geq 72$*

$$\frac{1}{2} + \frac{c_1 \log\log N}{\log N} \leq \mathrm{H.dim}\, D_N \leq \frac{1}{2} + \frac{c_2 \log\log N}{\log N}.$$

In particular, $\mathrm{H.dim}\, D_\infty \leq \frac{1}{2}$.

For the opposite inequality, consider the set $D'_M \subset D_\infty$ of real numbers whose partial quotients satisfy

$$\lceil \log q_k \rceil \leq a_{k+1} < M \lceil \log q_k \rceil,$$

where q_k is the height of the kth convergent. Let $\mathcal{B}$ be the collection of intervals $I_{N(v)}(v)$ for $v \in Q$ where

$$N(v) = \lceil \log |v| \rceil.$$

It will be convenient to simply write $I(v)$ for $I_{N(v)}(v)$ and N for $N(v)$.

Let $\sigma'_M(v)$ be the set of all $v' \in Q$ of the form $av + v_\pm$ with

$$N \leq a < MN. \tag{3.10}$$

Then $(\mathcal{B}, Q, \sigma'_M)$ is a self-similar covering of D'_M. From (3.6) we have

$$\frac{1}{N|v|^2} \leq |I(v)| \leq \frac{2}{N|v|^2}$$

so that for any $v' \in \sigma'_M(v)$,

$$\frac{N|v|^2}{2N'|v'|^2} \leq \frac{|I(v')|}{|I(v)|} \leq \frac{2N|v|^2}{N'|v'|^2}$$

where $N' = N'(v') = \lceil \log |v'| \rceil$. Note that for any $v' \in \sigma'_M(v)$,

$$N|v| \leq a|v| \leq |v'| \leq (a+1)|v| \leq MN|v|.$$

For distinct $v', v'' \in \sigma'_M(v)$ (with $v' = (p', q')$ and $v'' = (p'', q'')$),

$$\left| \frac{p'}{q'} - \frac{p''}{q''} \right| \geq \frac{1}{q'q''} \geq \frac{1}{M^2N^2|v|^2},$$

whereas

$$|I(v')| \leq \frac{2}{N'|v'|^2} \leq \frac{2}{N^3|v|^2} \leq \frac{1}{4M^2N^2|v|^2}$$

provided $|v| > e^{8M^2}$. This implies the gap between $I(v')$ and $I(v'')$ is at least

$$\frac{1}{2M^2N^2|v|^2} \geq \frac{1}{4M^2N}|I(v)|$$

which leads us to define

$$\rho(v) = \frac{1}{4M^2N}.$$

For $|v|$ large enough, $N' \leq \log|v'| + 1 \leq \log|v| + \log(MN) + 1 \leq 2N$. For such v, we now compute, using $s = \frac{1}{2}$ and noting that there are two v''s for each a in the range (3.10),

$$\begin{aligned}\sum_{v'} \frac{\rho(v')^s |I(v')|^s}{\rho(v)^s |I(v)|^s} &\geq \frac{1}{\sqrt{2}} \sum_{v'} \frac{N|v|}{N'|v'|} \geq \frac{1}{\sqrt{2}} \sum_{a} \frac{1}{a+1} \\ &\geq \frac{1}{2\sqrt{2}} \sum_{a=N}^{MN-1} \frac{1}{a} \geq \frac{\log M}{2\sqrt{2}} \geq 1,\end{aligned}$$

provided $M \geq e^{2\sqrt{2}}$. Theorem 3.3 now implies that $\mathrm{H.dim}\, D_\infty \geq \frac{1}{2}$.

This completes the proof of Theorem 1.7.

The lower bound on $\mathrm{H.dim}\, D_\infty$ can also be obtained using the following result of H. Reeve.

THEOREM 3.5 ([18]). *Let A_δ be the set of all real numbers whose continued fraction satisfies*

$$a_{k+1} > q_k^\delta \tag{3.11}$$

for all sufficiently large k. Then $\mathrm{H.dim}\, A_\delta = \frac{1}{2+\delta}$.

In contrast, the classical theorem of Jarnik [8] and Besicovitch [3] asserts that the set B_δ of all real numbers whose continued fraction satisfies (3.11) for infinitely many k has $\mathrm{H.dim}\, B_\delta = \frac{2}{2+\delta}$. Thus, the Hausdorff codimension of A_δ as a subset of B_δ is strictly positive, which is in the spirit of the conjecture mentioned in the introduction.

4. **Upper bound calculation**

In the rest of the paper, we assume $d = 2$.

Definition 4.1. For each $v \in Q$, let

$$\varepsilon(v)^{3/2} := \frac{|L(v)|}{|v|^{1/2}}$$

and define

$$Q_\varepsilon := \{v \in Q : \varepsilon(v) < \varepsilon\}.$$

Definition 4.2. For each $v \in Q$, let

$$\mathcal{L}^*(v) := \mathcal{L}(v) \setminus \{L(v)\}.$$

Fix, once and for all, an element $\widehat{L} \in \mathcal{L}^*(v)$ such that $|\widehat{L}|$ is minimal, and denote this element by

$$\widehat{L}(v).$$

An important consequence of the assumption $d = 2$ is the following.

LEMMA 4.3. *For any* $v \in Q_\varepsilon$,

$$\frac{|v|}{|L(v)|} \le |\widehat{L}(v)| \le (1+\varepsilon^3)\frac{|v|}{|L(v)|} \tag{4.1}$$

and

$$|\widehat{L}(v)| > \varepsilon^{-3/2}|v|^{1/2} \quad \textit{and} \quad |\widehat{L}(v)| > \varepsilon^{-3}|L(v)|. \tag{4.2}$$

Proof. Let $L = L(v)$ and $\widehat{L} = \widehat{L}(v)$. We may think of them as vectors in the plane of lengths $|L|$ and $|\widehat{L}|$, respectively, such that the area of the lattice they span is $|v|$. Without loss of generality we may assume L is horizontal. The vertical component of $\widehat{L}$ is then $\frac{|v|}{|L|}$ so that

$$\frac{|v|}{|L|} \le |\widehat{L}| \le \frac{|v|}{|L|} + |L|,$$

giving (4.1). If $|L| < \varepsilon^{3/2}|v|^{1/2}$, then the vertical component of $\widehat{L}$ is greater than $\varepsilon^{-3/2}|v|^{1/2}$, giving the first inequality in (4.2). From this and $v \in Q_\varepsilon$, the second inequality in (4.2) follows. $\square$

By Theorem 2.19, for any $\mathbf{x} \in \mathrm{Sing}^*(d)$ and any $\varepsilon > 0$ the elements $v \in \Sigma(\mathbf{x})$ belong to Q_ε if $|v|$ is large enough. Let $\mathcal{B}_\varepsilon = \{\Delta(v)\}_{v \in Q_\varepsilon}$ and define

$$\sigma_\varepsilon \subset Q_\varepsilon \times Q_\varepsilon$$

to consist of all pairs (v, v') such that $|v| < |v'|$ and v and v' realise a consecutive pair of local minima of the function $W_\mathbf{x}$ for some $\mathbf{x} \in \mathrm{Sing}^*(d)$. For each $\mathbf{x} \in \mathbb{R}^d$, we fix, once and for all, a sequence (v_j) in $\Sigma(\mathbf{x})$ such that each local minimum of $W_\mathbf{x}$ is realised by exactly one v_j, and, by an abuse of notation, we shall denote this sequence by the same symbol

$$\Sigma(\mathbf{x}).$$

Then it is easy to see that $(\mathcal{B}_\varepsilon, Q_\varepsilon, \sigma_\varepsilon)$ is a self-similar covering of $\mathrm{Sing}^*(d)$ for any $\varepsilon > 0$: the map $\mathcal{E}$ can be realised by sending $\mathbf{x}$ to a tail of $\Sigma(\mathbf{x})$. However, Theorem 3.1 does not yield any upper bound because it happens that $s(\mathcal{B}_\varepsilon, Q_\varepsilon, \sigma_\varepsilon) = \infty$ for all $\varepsilon > 0$, the main reason being that there is no way to bound the ratios

$$\frac{\operatorname{diam}\Delta(v')}{\operatorname{diam}\Delta(v)}$$

away from one.

4.1. *First acceleration.*

Definition 4.4. Suppose that $\Sigma(\mathbf{x}) = (v_j)$. We define

$$\widehat{\Sigma}(\mathbf{x})$$

to be the subsequence of $\Sigma(\mathbf{x})$ consisting of those v_{j+1} such that

$$v_{j+1} \notin \mathbb{Z}v_j + \mathbb{Z}v_{j-1}.$$

LEMMA 4.5. *The sequence $\widehat{\Sigma}(\mathbf{x})$ has infinite length if and only if $\mathbf{x}$ does not lie on a rational affine line in $\mathbb{R}^d$.*

Proof. Suppose ℓ is a rational affine line that contains $\mathbf{x}$. Let $L \subset \mathbb{Z}^{d+1}$ be the (primitive) two-dimensional sublattice that contains all $v \in Q$ such that $\dot{v} \in \ell$. The shortest nonzero vector in $g_t h_{\mathbf{x}} \mathbb{Z}^{d+1}$ is necessarily realised by a vector in L for all large enough t. This means $v_j \in L$ for all large enough j, which in turn implies that $\widehat{\Sigma}(\mathbf{x})$ is a finite sequence. Conversely, if $\widehat{\Sigma}(\mathbf{x})$ is finite, then there is a two-dimensional lattice $L \subset \mathbb{Z}^{d+1}$ that contains v_j for all large enough j. Let ℓ be the affine line that contains $\dot{v}_j$ for j large. Then $\mathbf{x} = \lim \dot{v}_j \in \ell$. □

LEMMA 4.6. *Let $\widehat{\Sigma}(\mathbf{x}) = (u_k)$ where $\mathbf{x}$ does not lie on a rational line. Let $\delta > 0$ and $\varepsilon > 0$ be related by $\delta = \varepsilon^{1+1/d}$. Then $\mathbf{x} \in \mathrm{DI}_{\delta/2}(d)$ implies $\varepsilon(u_k) < \varepsilon$ for all sufficiently large k, which in turn implies $\mathbf{x} \in \mathrm{DI}_{2\delta}(d)$. In particular, $\mathbf{x} \in \mathrm{Sing}\,(d)$ if and only if $\varepsilon(u_k) \to 0$ as $k \to \infty$.*

Proof. Let $\Sigma(\mathbf{x}) = (v_j)$ and for each j, let $L_j = \mathbb{Z}v_j + \mathbb{Z}v_{j-1}$ and

$$\varepsilon_j = \frac{|L_j|}{|v_j|^{1-1/d}}.$$

Given u_k there are indices $i < j$ such that $u_k = v_i, \ldots, v_j = u_{k+1}$. Suppose that $j > i + 1$. Then $v_{i+1} \in L_i$ so that $L_{i+1} \subset L_i$ and we have equality by Lemma 2.5. It follows by induction that $L_{j-1} = \cdots = L_i$, from which it follows that $\varepsilon_i > \cdots > \varepsilon_{j-1}$. Therefore, $\varepsilon(u_k) < \varepsilon$ for all sufficiently large k if and only if $\varepsilon(v_j) < \varepsilon$ for all sufficiently large j. The lemma now follows from Theorem 2.19. □

Definition 4.7. For any $u \in Q$, let

$$\mathcal{V}(u)$$

be the set of vectors in Q of the form $au + b\tilde{v}$ where a, b are relatively prime integers such that $|b| \le a$ and $\tilde{v} \in Q$ is a vector that satisfies $L(u) = \mathbb{Z}u + \mathbb{Z}\tilde{v}$, $|\tilde{v}| < |u|$ and $L(\tilde{v}) \ne L(u)$. The set

$$\mathcal{V}_\varepsilon(u)$$

is defined similarly, except that we additionally require $\tilde{v} \in Q_\varepsilon$ and $|\tilde{v}| < \varepsilon|u|$.

LEMMA 4.8. *Let $u, u' \in \widehat{\Sigma}(\mathbf{x})$ be consecutive elements with $|u| < |u'|$. Let $\tilde{v}, v \in \Sigma(\mathbf{x})$ be the elements that immediately precede u and u', respectively. Then $v \in \mathcal{V}(u)$, provided $L(u) = \mathbb{Z}u + \mathbb{Z}\tilde{v} \neq L(\tilde{v})$.*

Proof. Suppose $\Sigma(\mathbf{x}) = (v_j)$ so that $u = v_i$, $v = v_j$, $\tilde{v} = v_{i-1}$ and $u' = v_{j+1}$ for some indices $i \leq j$. We shall argue by induction on $j \geq i$ to show that

(i) $v_j = au + b\tilde{v}$ for some integers $-a < b \leq a$, and
(ii) $v_j - v_{j-1} = a'u + b'\tilde{v}$ for some integers $-a' \leq b' \leq a'$.

This is clear if $j = i$. For $j > i$, since $v_j \in \mathbb{Z}v_{j-1} + \mathbb{Z}v_{j-2}$, Lemma 2.5 implies that (v_j, v_{j-1}) is an integral basis for $\mathbb{Z}v_{j-1} + \mathbb{Z}v_{j-2}$. Since $|v_j| > |v_{j-1}| > |v_{j-2}|$, there is a positive integer c such that

$$v_j = cv_{j-1} + v_{j-2}, \quad \text{or} \quad v_j = cv_{j-1} + (v_{j-1} - v_{j-2}).$$

In either case $v_j = mu + n\tilde{v}$, where $(m, n) = (ca + a', cb + b')$ satisfies $-m < n \leq m$ by the induction hypothesis. Similarly, $v_j - v_{j-1} = m'u + n'\tilde{v}$ where $(m', n') = (ca - a', cb - b')$ satisfies $-m \leq n \leq m$, again, by the induction hypothesis. The lemma now follows from (i). $\square$

Definition 4.9. For any $\varepsilon > 0$, define

$$\widehat{\sigma}_\varepsilon \subset Q_\varepsilon \times Q_\varepsilon$$

to be the set consisting of pairs (u, u') for which there exists $v \in \mathcal{V}_\varepsilon(u)$ such that $L(u') = \mathbb{Z}u' + \mathbb{Z}v \in \mathcal{L}^*(v)$.

COROLLARY 4.10. *Let $\delta = \varepsilon^{3/2}$ where $0 < \varepsilon < 1$. Then $(\mathcal{B}_\varepsilon, Q_\varepsilon, \widehat{\sigma}_\varepsilon)$ is a self-similar covering of $\mathrm{DI}_{\delta/2}(2)$.*

Proof. Let $\Sigma(\mathbf{x}) = (v_j)$ for a given $\mathbf{x} \in \mathrm{DI}_{\delta/2}(2)$. Lemma 4.6 implies that for all large enough j,

$$\|\mathbb{Z}v_j + \mathbb{Z}v_{j-1}\|_{\mathcal{L}(v_j)} = \frac{|v_{j-1} \wedge v_j|}{|v_j|^{1/2}} < \varepsilon^{3/2} < 1$$

so that $L(v_j) = \mathbb{Z}v_j + \mathbb{Z}v_{j-1}$ and $\varepsilon(v_j) < \varepsilon$. Lemma 4.8 now implies that $\widehat{\Sigma}(\mathbf{x})$ is eventually $\widehat{\sigma}_\varepsilon$-admissible. $\square$

Our next task is to estimate $s(\mathcal{B}_\varepsilon, Q_\varepsilon, \widehat{\sigma}_\varepsilon)$. For this, we need to enumerate the elements of $\widehat{\sigma}_\varepsilon(u)$ for any given $u \in Q_\varepsilon$.

Definition 4.11. For any $v \in Q$ and $L' \in \mathcal{L}^*(v)$, let

$$\mathcal{U}_\varepsilon(v, L')$$

be the set of $u' \in Q_\varepsilon$ such that $L' = L(u') = \mathbb{Z}u' + \mathbb{Z}v$.

Note that, by definition, for any $u' \in \widehat{\sigma}_\varepsilon(u)$ there exists a $v \in \mathcal{V}_\varepsilon(u)$ and an $L' \in \mathcal{L}^*(v)$ such that $u' \in \mathcal{U}_\varepsilon(v, L')$. Hence, for any $f : Q_\varepsilon \to \mathbb{R}_+$

$$\sum_{u' \in \widehat{\sigma}_\varepsilon(u)} f(u') \leq \sum_{v \in \mathcal{V}_\varepsilon(u)} \sum_{L' \in \mathcal{L}^*(v)} \sum_{u' \in \mathcal{U}_\varepsilon(v,L')} f(u').$$

Notation. We write $A \preceq B$ to mean $A \leq CB$ for some universal constant C. The notation $A \asymp B$ used earlier in (2.11) is equivalent to $A \preceq B$ and $B \preceq A$. We write $A \succeq B$ to mean the same thing as $B \preceq A$.

PROPOSITION 4.12. *There is a constant C such that for any $u \in Q_\varepsilon$, and for any $s > 4/3$ and any $r < 6s - 3$, we have*

$$\sum_{u' \in \widehat{\sigma}_\varepsilon(u)} \left(\frac{\varepsilon(u)}{\varepsilon(u')}\right)^r \left(\frac{\operatorname{diam} \Delta(u')}{\operatorname{diam} \Delta(u)}\right)^s \leq \frac{C(6s-3-r)^{-1} \varepsilon^{6s-3-r}}{(3s-4)^2 (\varepsilon(u))^{6-3s-r}}. \tag{4.3}$$

Proof. Given $v \in \mathcal{V}_\varepsilon(u)$ there are $a, b \in \mathbb{Z}$ with $|b| < a$ and $\tilde{v} \in Q_\varepsilon$ such that $v = au + b\tilde{v}$, $|\tilde{v}| < \varepsilon|u|$, $L(u) = \mathbb{Z}u + \mathbb{Z}\tilde{v}$ and $L(u) \neq L(\tilde{v})$. Note that $\tilde{v} \in Q_\varepsilon$ implies $|\widehat{L}(\tilde{v})| > \varepsilon^{-3/2} |\tilde{v}|^{1/2}$ and since $L(u) \neq L(\tilde{v})$, we have $|L(u)| \geq |\widehat{L}(\tilde{v})|$, so that $u \in Q_\varepsilon$ now implies

$$|u| > \varepsilon^{-3} |L(u)|^2 > \varepsilon^{-6} |v|,$$

so that

$$|v| \asymp a|u|.$$

Since $|\tilde{v}| < |u|$ and $(u, \tilde{v})$ is an integral basis for $L(u)$, there are at most two possibilities for $\tilde{v}$, so that given u and the positive integer a, there are at most $O(a)$ possibilities for v. It follows that for any $q > 2$,

$$\sum_{v \in \mathcal{V}_\varepsilon(u)} \frac{|u|^q}{|v|^q} \preceq \sum_a \frac{1}{a^{q-1}} \asymp \frac{1}{q-2}. \tag{4.4}$$

Let $\mathcal{L}_+(v)$ be the set of elements in $\mathcal{L}(v)$ considered with orientations, and think of it as a subset of $\bigwedge^2 \mathbb{Z}^3$. Note that addition is defined for those pairs $L, L' \in \mathcal{L}_+(v)$ whose $\mathbb{Z}$-span contains all of $\mathcal{L}_+(v)$. Let L and $\widehat{L}$ be the elements in $\mathcal{L}_+(v)$ corresponding to a fixed choice of orientation for $L(v)$ and $\widehat{L}(v)$, respectively. Each $L' \in \mathcal{L}^*(v)$ can be oriented so that, as an element in $\mathcal{L}_+(v)$ we have $L' = \tilde{a}\widehat{L} + \tilde{b}L$ for some (relatively prime) integers $\tilde{a}, \tilde{b}$ with $\tilde{a} > 0$. Let

$$\widehat{L}_m = \widehat{L} + mL, \quad m \in \mathbb{Z}.$$

There is a unique integer m such that $L' = L_m$ or L' is a postive linear combination of $\widehat{L}_m$ and $\widehat{L}_{m+1}$. In any case, for each $L' \in \mathcal{L}^*(v)$ there is an integer m (and an orientation for L') such that

$$L' = a'\widehat{L}_m + b'L, \quad a' > b' \geq 0.$$

Note that $v \in Q_\varepsilon$ implies
$$|\widehat{L}_m| \geq |\widehat{L}| > \varepsilon^{-3/2}|v|^{1/2} > \varepsilon^{-3}|L|$$
so that
$$|L'| \asymp a'|\hat{L}_m|.$$
Let $N = \left\lfloor \frac{|\widehat{L}|}{|L|} \right\rfloor$ so that $|\widehat{L}_m| \asymp |L|(N+|m|)$ and
$$\begin{aligned}\sum_{m\in\mathbb{Z}} \frac{|\widehat{L}|^p}{|\widehat{L}_m|^p} &\asymp \frac{|\widehat{L}|^p}{|L|^p} \sum_{m\geq 1} \frac{1}{(N+m)^p} \\ &\asymp \frac{|\widehat{L}|^p}{|L|^p} \left(\sum_{m=1}^{N} \frac{1}{N^p} + \sum_{m>N} \frac{1}{m^p} \right) \\ &\asymp \frac{|\widehat{L}|^p}{|L|^p} \left(\frac{1}{N^{p-1}} + \frac{1}{(p-1)N^{p-1}} \right) \asymp \frac{p}{p-1} \frac{|\widehat{L}|}{|L|}.\end{aligned}$$
Since there are at most $O(a')$ possibilities for L' given v, m and a', and since $|L||\widehat{L}| \asymp |v|$, it follows that for any $q > 2$,
$$\sum_{L'\in\mathcal{L}^*(v)} \frac{1}{|L'|^q} \preceq \sum_{m\in\mathbb{Z}} \frac{1}{|\widehat{L}_m|^q} \sum_{a'} \frac{1}{(a')^{q-1}} \asymp \frac{1}{q-2} \frac{|L|^{q-2}}{|v|^{q-1}}. \tag{4.5}$$

Associate to each $u' \in \mathcal{U}_\varepsilon(v, L')$ the positive integer
$$c = \left\lceil \frac{|u'|}{|v|} \right\rceil > \frac{\varepsilon^{-3}|L'|^2}{|v|} \geq \varepsilon^{-3} \frac{|\widehat{L}|^2}{|v|} \geq \varepsilon^{-3} \frac{|v|}{|L|^2} > \varepsilon^{-6}.$$
Since $|u'| \asymp c|v|$ and there are 2 possibilities for u', given v, L' and c, and since $c > \frac{\varepsilon^{-3}|L'|^2}{|v|}$, it follows that for any $p > 1$,
$$\sum_{u'\in\mathcal{U}_\varepsilon(v,L')} \frac{1}{|u'|^p} \asymp \frac{1}{|v|^p} \sum_{c} \frac{1}{c^p} \asymp \frac{\varepsilon^{3p-3}}{(p-1)|v||L'|^{2p-2}}. \tag{4.6}$$

Using (4.6), (4.5), and (4.4) with $p = 2s - \frac{r}{3}$ and $q = 3s - 2$, we obtain
$$\begin{aligned}&\sum_{v\in\mathcal{V}_\varepsilon(u)} \sum_{L'\in\mathcal{L}^*(v)} \sum_{u'\in\mathcal{U}_\varepsilon(v,L')} \left(\frac{|L'|}{|L|}\right)^{s-\frac{2r}{3}} \left(\frac{|u|}{|u'|}\right)^{2s-\frac{r}{3}} \\ &\preceq \sum_{v\in\mathcal{V}_\varepsilon(u)} \sum_{L'\in\mathcal{L}^*(v)} \frac{(6s-3-r)^{-1}\varepsilon^{6s-3-r}|u|^{2s-\frac{r}{3}}}{|L|^{s-\frac{2r}{3}}|v||L'|^{3s-2}} \\ &\preceq \sum_{v\in\mathcal{V}_\varepsilon(u)} \frac{(6s-3-r)^{-1}\varepsilon^{6s-3-r}|u|^{2s-\frac{r}{3}}|L|^{2s-4+\frac{2r}{3}}}{(3s-4)|v|^{3s-2}} \\ &\preceq \frac{(6s-3-r)^{-1}\varepsilon^{6s-3-r}|L|^{2s-4+\frac{2r}{3}}}{(3s-4)^2|u|^{s-2+\frac{r}{3}}}.\end{aligned}$$
This completes the proof of the proposition. □

Proposition 4.12 implies there exists $C > 0$ such that for any $s > \frac{4}{3}$,

$$\sum_{u' \in \widehat{\sigma}_\varepsilon(u)} \left(\frac{\operatorname{diam} \Delta(u')}{\operatorname{diam} \Delta(u)} \right)^s \leq \frac{C\varepsilon^{6s-3}}{(3s-4)^2(\varepsilon(u))^{6-3s}}. \tag{4.7}$$

However, this still does not imply $s(\mathcal{B}_\varepsilon, Q_\varepsilon, \widehat{\sigma}_\varepsilon) < \infty$ for any $\varepsilon > 0$.

4.2. *Second acceleration.* Let $\widehat{\sigma}'_\varepsilon \subset Q \times Q$ be the set of all pairs (u, u') satisfying $u \in Q_\varepsilon$ and $u' \in \bigcup_{j \geq 1} \sigma''_j(u)$, where

$$\sigma''_1(u) := \widehat{\sigma}_{\varepsilon(u)}(u), \qquad \sigma''_j(u) := \bigcup \{\widehat{\sigma}_{\varepsilon(u)}(u') : u' \in \sigma'_{j-1}(u)\},$$
$$\sigma'_1(u) := \widehat{\sigma}_\varepsilon(u), \qquad \sigma'_j(u) := \bigcup \{\widehat{\sigma}_\varepsilon(u') : u' \in \sigma'_{j-1}(u)\}.$$

PROPOSITION 4.13. $s(\mathcal{B}_\varepsilon, Q_\varepsilon, \widehat{\sigma}'_\varepsilon) = \frac{4}{3} + O(\varepsilon^{3/2})$.

Proof. To simplify notation, we denote the diameter of a set by $|\cdot|$. Given $s > \frac{4}{3}$, we apply Proposition 4.12 to ensure that if $\varepsilon > 0$ is sufficiently small, then by (4.7), for any $u \in Q_\varepsilon$,

$$\sum_{u' \in \widehat{\sigma}_\varepsilon(u)} \frac{|\Delta(u')|^s}{|\Delta(u)|^s} \leq \frac{1}{2} \left(\frac{\varepsilon}{\varepsilon(u)} \right)^{6-3s} \quad \text{and} \tag{4.8}$$

$$\sum_{u' \in \widehat{\sigma}_\varepsilon(u)} \left(\frac{\varepsilon(u)}{\varepsilon(u')} \right)^{6-3s} \frac{|\Delta(u')|^s}{|\Delta(u)|^s} \leq \frac{C\varepsilon^{9s-9}}{(3s-4)^2} \leq \frac{1}{2}.$$

We may choose ε so that $\varepsilon^{9s-9} \asymp (3s-4)^2$; hence $s = \frac{4}{3} + O(\varepsilon^{3/2})$.

For each $u' \in \sigma''_j(u)$ there are $u_1, \ldots, u_{j-1} \in Q_\varepsilon$ such that

$$u_1 \in \widehat{\sigma}_\varepsilon(u), \;\; \ldots, \;\; u_{j-1} \in \widehat{\sigma}_\varepsilon(u_{j-2}), \quad \text{and} \quad u' \in \widehat{\sigma}_{\varepsilon(u)}(u_{j-1}).$$

Also,

$$\begin{aligned} \sum_{u' \in \sigma''_j(u)} \frac{|\Delta(u')|^s}{|\Delta(u)|^s} &\leq \sum_{u_1} \frac{|\Delta(u_1)|^s}{|\Delta(u)|^s} \cdots \sum_{u_{j-1}} \frac{|\Delta(u_{j-1})|^s}{|\Delta(u_{j-2})|^s} \sum_{u'} \frac{|\Delta(u')|^s}{|\Delta(u_{j-1})|^s} \\ &\leq \frac{1}{2} \sum_{u_1} \frac{|\Delta(u_1)|^s}{|\Delta(u)|^s} \cdots \sum_{u_{j-1}} \frac{|\Delta(u_{j-1})|^s}{|\Delta(u_{j-2})|^s} \left(\frac{\varepsilon(u)}{\varepsilon(u_{j-1})} \right)^{6-3s} \\ &\leq \frac{1}{2^2} \sum_{u_1} \frac{|\Delta(u_1)|^s}{|\Delta(u)|^s} \cdots \sum_{u_{j-2}} \frac{|\Delta(u_{j-2})|^s}{|\Delta(u_{j-3})|^s} \left(\frac{\varepsilon(u)}{\varepsilon(u_{j-2})} \right)^{6-3s} \\ &\leq \cdots \leq \frac{1}{2^j}, \end{aligned} \tag{4.9}$$

so that

$$\sum_{u' \in \widehat{\sigma}'_\varepsilon(u)} \frac{|\Delta(u')|^s}{|\Delta(u)|^s} \leq \sum_{j \geq 1} \sum_{u' \in \sigma''_j(u)} \frac{|\Delta(u')|^s}{|\Delta(u)|^s} \leq \sum_{j \geq 1} \frac{1}{2^j} \leq 1$$

and therefore $s(\mathcal{B}_\varepsilon, Q_\varepsilon, \widehat{\sigma}_\varepsilon) \leq s$. $\square$

Now we verify that $(\mathcal{B}_\varepsilon, Q_\varepsilon, \widehat{\sigma}'_\varepsilon)$ is a self-similar covering of $\mathrm{Sing}^*(2)$. For any $\mathbf{x} \in \mathrm{Sing}^*(2)$, let $\mathcal{E}(\mathbf{x})$ be any choice of a subsequence (w_i) of $\widehat{\Sigma}(\mathbf{x})$ such that $\varepsilon(w_i)$ is strictly decreasing to zero as $i \to \infty$, and such that the initial element w_0 is chosen so that for all v that occur after w_0 in the sequence $\Sigma(\mathbf{x})$, we have $\varepsilon(v) < \varepsilon$. The existence of this subsequence is implied by Lemma 4.6. The sequence $\mathcal{E}(\mathbf{x})$ is $\widehat{\sigma}'_\varepsilon$-admissible by construction. It follows that $(\mathcal{B}_\varepsilon, Q_\varepsilon, \widehat{\sigma}'_\varepsilon)$ is a self-similar covering of $\mathrm{Sing}^*(2)$, and by Theorem 3.1 we can conclude that

$$\mathrm{H.dim}\,\mathrm{Sing}^*(2) \leq \frac{4}{3}.$$

We now describe how the preceding argument can be modified to give an upper bound estimate on $\mathrm{H.dim}\,\mathrm{DI}_\delta(2)$. First, modify the definition of $\widehat{\sigma}'_\varepsilon$ by replacing the subscript $\varepsilon(u)$ (in the formula of σ''_j) with $2\varepsilon(u)$. This affects the second step (4.9) of the main calculation by introducing a factor $2^{6-3s} < 4$, which can be offset by choosing $\varepsilon > 0$ in the proof of Proposition 4.13 so that (4.8) holds with the constant 2 replaced by 8. The statement of Proposition 4.13 remains true, albeit with different implicit constants.

Let $\mathrm{DI}^*_\delta(2)$ denote the set $\mathrm{DI}_\delta(2)$ with all rational affine lines removed. Set $\delta = \frac{\varepsilon^{3/2}}{2}$. For any $\mathbf{x} \in \mathrm{DI}^*_\delta(2)$, let $\mathcal{E}(\mathbf{x})$ be any choice of a subsequence (w_i) of $\widehat{\Sigma}(\mathbf{x})$ such that $\varepsilon(w_i)$ is monotone (increasing or decreasing), and such that the initial element w_0 is chosen so that for all v that occur after w_0 in the sequence $\Sigma(\mathbf{x})$ we have $\varepsilon(v) < \varepsilon$, and for any i we have $\varepsilon(w_i) < 2\varepsilon(w_0)$. Again, the existence of $\mathcal{E}(\mathbf{x})$ is ensured by Lemma 4.6, and it is $\widehat{\sigma}'_\varepsilon$-admissible by construction. We have thus shown that $(\mathcal{B}_\varepsilon, Q_\varepsilon, \widehat{\sigma}'_\varepsilon)$ is a self-similar covering of $\mathrm{DI}^*_\delta(2)$. Theorem 3.1 and Proposition 4.13 now imply that there is a constant $C > 0$ such that for any $0 < \delta < 1$,

$$\mathrm{H.dim}\,\mathrm{DI}_\delta(2) \leq \frac{4}{3} + C\delta.$$

5. **Lower bound calculation**

LEMMA 5.1. *Let $0 < \varepsilon < \frac{1}{2}$. Suppose $u \in Q$, $L' \in \mathcal{L}^*(u)$, and $u' \in L'$ is such that $L' = \mathbb{Z}u' + \mathbb{Z}u$ and $|u'| > \varepsilon^{-3}|L'|^2$. Then $L' = L(u')$, $u' \in Q_\varepsilon$, and*

$$\overline{\Delta(u')} \subset \Delta(u).$$

Moreover, if $u \in Q_\varepsilon$ then $|u'| > \varepsilon^{-6}|u|$.

Proof. Since the norm of $L' \in \mathcal{L}(u')$ is

$$\|L'\|_{\mathcal{L}(u')} = \frac{|L'|}{|u'|^{1/2}} < \varepsilon^{3/2} < 1,$$

we have $L' = L(u')$ and $\varepsilon(u') < \varepsilon$ so that $u' \in Q_\varepsilon$. Let $L = L(u)$ and note that since $L' \neq L$, we have

$$|L'| \geq |\widehat{L}(u)| \geq \frac{|u|}{|L|}.$$

Then

$$\operatorname{dist}(\dot{u}, \dot{u}') = \frac{|u \wedge u'|}{|u||u'|} < \frac{\varepsilon^3}{|u||L'|} \leq \frac{\varepsilon^3 |L|}{|u|^2}.$$

The fact that the Euclidean length of the shortest nonzero vector in any two-dimensional unimodular lattice is universally bounded above by $\sqrt{2}$ implies that

$$\varepsilon(u)^3 < 2 \quad \text{for any} \quad u \in Q. \tag{5.1}$$

Therefore,

$$\frac{|L'||u|^2}{|L||u'|^2} \leq \frac{\varepsilon^6 |u|^2}{|L||L'|^3} \leq \frac{\varepsilon^6 |L|^2}{|u|} < 2\varepsilon^6$$

so that

$$\frac{\varepsilon|L|}{|u|^2} + \frac{2|L'|}{|u'|^2} < (\varepsilon^3 + 4\varepsilon^6)\frac{|L|}{|u|^2} < \frac{|L|}{2|u|^2}.$$

Theorem 2.11 now implies $\overline{\Delta(u')} \subset \Delta(u)$. If $u \in Q_\varepsilon$, then

$$|u'| > \varepsilon^{-3}|L'|^2 > \frac{\varepsilon^{-3}|u|^2}{|L|^2} > \varepsilon^{-6}|u|. \qquad \square$$

Definition 5.2. For each $u \in Q$, let

$$\mathcal{N}_\varepsilon(u)$$

be the set of $u' \in Q$ such that $\mathbb{Z}u' + \mathbb{Z}u \in \mathcal{L}^*(u)$ and $|u'| > \varepsilon^{-3}|u \wedge u'|^2$.

Note that for any $u' \in \mathcal{N}_\varepsilon(u)$ we have

$$\frac{|L(u')|}{|u'|^{1/2}} \leq \frac{|u \wedge u'|}{|u'|^{1/2}} < \varepsilon^{3/2}$$

from which it follows that $\mathcal{N}_\varepsilon(u) \subset Q_\varepsilon$.

THEOREM 5.3. *Let* $0 < \varepsilon < \frac{1}{3}$. *Suppose* (u_k) *is a sequence in* Q *satisfying* $u_{k+1} \in \mathcal{N}_\varepsilon(u_k)$ *for all* $k \geq 0$. *Then*:

(a) *The limit* $\mathbf{x} := \lim_k \dot{u}_k$ *exists and* $u_k \in \Sigma(\mathbf{x})$ *for all* k.
(b) $\mathbf{x} \in \mathrm{DI}_\delta(2)$, *where* $\delta = 2\varepsilon^{3/2}$.
(c) *If* $\varepsilon(u_k) \to 0$ *as* $k \to \infty$ *then* $\mathbf{x} \in \mathrm{Sing}\,(2)$.
(d) *For all sufficiently large* t,

$$W(t) + \log(1 - \varepsilon^6) \leq W_{\mathbf{x}}(t) \leq W(t), \tag{5.2}$$

where

$$W(t) = \log \ell(g_t h_{\mathbf{x}}\{u_k\}) = \log \min_{k \geq 0} \|g_t h_{\mathbf{x}} u_k\|'.$$

Proof. Apply Lemma 5.1 with $L' = \mathbb{Z}u_{k+1} + \mathbb{Z}u_k$ to conclude that $\cap_k \Delta(u_k)$ is nonempty, and $u_k \in Q_\varepsilon$ for all $k \geq 1$. Moreover, $|u_k| \to \infty$ so that $\operatorname{diam} \Delta(u_k) \to 0$ and (a) follows. Lemma 5.1 also implies $L(u_{k+1}) = \mathbb{Z}u_{k+1} + \mathbb{Z}u_k$, so that by Theorem 2.14, we have

$$\varepsilon_{\mathbf{x}}(u_k, u_{k+1})^{3/2} \leq 2\frac{|u_k \wedge u_{k+1}|}{|u_{k+1}|^{1/2}} = 2\varepsilon(u_{k+1})^{3/2} < \delta. \tag{5.3}$$

Hence, Lemma 2.17 implies the local maxima of the piecewise linear function W are all bounded above by $\log \delta$. Since $W_{\mathbf{x}} \leq W$ (by monotonicity of ℓ), it follows that the local maxima of $W_{\mathbf{x}}$ are bounded by $\log \delta$, eventually. By Theorem 2.19, this means $\mathbf{x} \in \mathrm{DI}_\delta(2)$, giving (b). If $\varepsilon(u_k) \to 0$ then $W_{\mathbf{x}}(t) \to -\infty$, so that $\mathbf{x} \in \operatorname{Sing}(2)$. This proves (c).

Since $W_{\mathbf{x}} \leq W$, the second inequality in (5.2) actually holds for all t. For the first inequality, we consider a local maximum time t for W. Thus for some index k, if we set $u = g_t h_{\mathbf{x}} u_k$ and $u' = g_t h_{\mathbf{x}} u_{k+1}$, then $\|u\|' = \|u'\|'$. The corresponding local maximum value is $\log \varepsilon'$, where ε' is the common $\|\cdot\|'$-length of u and u'. To prove the first inequality in (5.2) we need to show that for any $w \in g_t h_{\mathbf{x}} \mathbb{Z}^3$,

$$\|w\|' \geq (1 - \varepsilon^6)\varepsilon'. \tag{5.4}$$

Lemma 5.1 implies (for $k \geq 1$) $|u'| > \varepsilon^{-6}|u|$. Hence, $\|u' \pm u\|' \geq |u' \pm u| \geq (1-\varepsilon^6)\varepsilon'$. Note that $\|au' + bu\|' \geq \varepsilon'$ for any pair of integers with $|a| \neq |b|$. (If $|a| < |b|$ then $\|au' + bu\|' \geq |b|\|u\|' - |a|\|u'\|' \geq \varepsilon'$; the case $|b| < |a|$ is similar.) This establishes (5.4) for $w \in \mathbb{Z}u + \mathbb{Z}u'$. Let $\|\cdot\|_e$ denote the Euclidean norm on $\mathbb{R}^3$. For any $v \in \mathbb{R}^3$ we have

$$\|v\|' \leq \|v\|_e \leq \sqrt{2}\|v\|'.$$

The Euclidean area of a fundamental parallelogram for $\mathbb{Z}u + \mathbb{Z}u'$ is

$$\|u \wedge u'\|_e \leq \|u\|_e \|u'\|_e \leq 2(\varepsilon')^2$$

so that for any $w \in g_t h_{\mathbf{x}} \mathbb{Z}^3 \setminus (\mathbb{Z}u + \mathbb{Z}u')$ we have

$$\|w\|' \geq \frac{\|w\|_e}{\sqrt{2}} \geq \frac{1}{2\sqrt{2}(\varepsilon')^2}$$

which is $> \varepsilon'$ since

$$\varepsilon' \leq 3^{2/3}\varepsilon(u_{k+1}) < 3^{2/3}\varepsilon < \frac{1}{\sqrt[3]{3}} < \frac{1}{\sqrt{2}}.$$

Thus (5.4) holds for all $w \in g_t h_{\mathbf{x}} \mathbb{Z}^3$ and this establishes (5.2) for any local maximum time of W.

Finally, suppose $t_0 < t_2$ are consecutive local maximum times and let t_1 be the unique local minimum time between them. By the preceding, we know (5.2) holds at t_0 and t_2. Since $\dot{W}(t) = -d$ for all $t \in (t_0, t_1)$ while all one-sided derivatives of $W_{\mathbf{x}}$ are bounded below by $-d$ on the same interval, we conclude that (5.2) holds for $t \in [t_0, t_1]$. Here, we are using the fact that (5.2) holds for $t = t_0$. Similarly, using the fact that (5.2) holds for $t = t_2$, and the fact that $\dot{W}(t) = +1$ on (t_1, t_2) while all one-sided derivatives of $W_{\mathbf{x}}$ are bounded above by $+1$, we conclude that (5.2) holds for all $t \in [t_1, t_2]$. From (5.3) we see that

$$\tau_{\mathbf{x}}(u_k, u_{k+1}) = -\frac{1}{2}\log\frac{\varepsilon_{\mathbf{x}}(u_k, u_{k+1})}{|u_{k+1}|} \geq \frac{1}{2}\log|u_{k+1}| + \frac{1}{3}\log\delta$$

from which it follows easily that W has infinitely many local minima. Therefore, (5.2) holds for all t beyond the first local maximum time, proving (d). $\square$

We assume, for each $u \in Q$, orientations for $L(u)$ and $\widehat{L}(u)$ have been chosen so that we may think of them as elements of $\bigwedge^2 \mathbb{Z}^3$.

Definition 5.4. Given $u \in Q$ and integers $a \geq b \geq 0$ such that $\gcd(a,b) = 1$, we set

$$L' = a'\widehat{L}(u) + b'L(u).$$

Additionally, given $0 < \varepsilon < 1$ and an integer $c \geq 1$ satisfying

$$M_\varepsilon < c < 2M_\varepsilon - 1, \quad \text{where} \quad M_\varepsilon = \frac{\varepsilon^{-3}|L'|^2}{|u|},$$

we define $\psi_\varepsilon(u, a, b, c)$ to be the unique $u' \in Q$ such that

$$L' = u' \wedge u \quad \text{and} \quad \left\lfloor \frac{|u'|}{|u|} \right\rfloor = c.$$

Note that $\psi_\varepsilon(u, a, b, c) \in \mathcal{N}_\varepsilon(u)$ because $L' = L(u')$. Note also that $c > M_\varepsilon$ implies $\psi_\varepsilon(u, a, b, c) \in Q_\varepsilon$ while $c < 2M_\varepsilon - 1$ implies $\psi_\varepsilon(u, a, b, c) \notin Q_{\varepsilon/2}$. Therefore, we always have

$$\psi_\varepsilon(u, a, b, c) \in \mathcal{N}_\varepsilon(u) \cap Q'_\varepsilon \quad \text{where} \quad Q'_\varepsilon := Q_\varepsilon \setminus Q_{\varepsilon/2}.$$

Lemma 5.5. *Let $0 < \varepsilon < 2^{-7}$. If $u' = \psi_\varepsilon(u, a, b, c)$ and $u'' = \psi_\varepsilon(u, a', b', c')$ are such that $(a, b) \neq (a', b')$ or $|c - c'| \geq 20$, then*

$$\operatorname{dist}(\Delta(u'), \Delta(u'')) \geq \frac{\varepsilon^9}{2^{11}N^3}\operatorname{diam}\Delta(u), \tag{5.5}$$

where $N = \max(a, a')$.

Proof. Let $L = L(u)$, $\widehat{L} = \widehat{L}(u)$ and $L' = u' \wedge u$. Note that by (4.1),

$$|L'| \leq 2a|\widehat{L}| \leq \frac{4N|u|}{|L|}.$$

Theorem 2.11 implies

$$\operatorname{diam}\Delta(u) \leq \frac{4|L|}{|u|^2}$$

and also $\Delta(u') \subset B(\dot{u}', 2r')$, where

$$r' = \frac{|L'|}{|u'|^2} \le \frac{|L'|}{c^2|u|^2} < \frac{|L'|}{M_\varepsilon^2|u|^2} = \frac{\varepsilon^6}{|L'|^3}.$$

If $(a,b) = (a',b')$, then

$$\mathrm{dist}\,(\dot{u}', \dot{u}'') = \frac{|u' \wedge u''|}{|u'||u''|} > \frac{20|L'|}{(c+1)^2|u|^2} > \frac{5|L'|}{M_\varepsilon^2|u|^2} = \frac{5\varepsilon^6}{|L'|^3},$$

so that

$$\mathrm{dist}\,(\Delta(u'), \Delta(u'')) \ge \frac{\varepsilon^6}{|L'|^3} \ge \frac{\varepsilon^6|L|^3}{2^6N^3|u|^3} \ge \frac{\varepsilon^9|L|}{2^9N^3|u|^2},$$

giving (5.5) in the case $(a,b) = (a',b')$. If $(a,b) \ne (a',b')$, let $L'' = u'' \wedge u$ and note that

$$\sin \angle \pi_u(L')\pi_u(L'') = \frac{|u|}{|L'||L''|} \ge \frac{|L|^2}{16N^2|u|} \ge \frac{\varepsilon^3}{2^7N^2}$$

and

$$\mathrm{dist}\,(\dot{u}, \dot{u}') = \frac{|u \wedge u'|}{|u||u'|} \ge \frac{|L'|}{2M_\varepsilon|u|^2} = \frac{\varepsilon^3}{2|u||L'|} \ge \frac{\varepsilon^3|L|}{8N|u|^2}$$

so that

$$\mathrm{dist}\,(\dot{u}', \dot{u}'') \ge \frac{\varepsilon^6|L|}{2^{10}N^3|u|^2}.$$

Considering the component of L' perpendicular to L, as in the proof of Lemma 4.3, we get

$$|L'| \ge \frac{a|u|}{|L|} \ge \frac{N|u|}{|L|},$$

so that

$$2r' < \frac{2\varepsilon^6}{|L'|^3} \le \frac{2^7\varepsilon^6|L|^3}{N^3|u|^3} < \frac{2^8\varepsilon^6|L|}{|u|^2}$$

by (5.1). Since $\varepsilon < 2^{-7}$, it follows that

$$\mathrm{dist}\,(\Delta(u'), \Delta(u'')) \ge (\frac{1}{2^{10}} - 2^9\varepsilon^3)\frac{\varepsilon^6|L|}{N^3|u|^2} \ge \frac{\varepsilon^6}{2^{13}N^3}\,\mathrm{diam}\,\Delta(u)$$

which easily implies (5.5). $\square$

PROPOSITION 5.6. *There is a constant $c > 0$ such that for $0 < \delta < 2^{-10}$,*

$$\mathrm{H.dim}\,\mathrm{DI}_\delta(2) \ge \frac{4}{3} + \exp(-c\delta^{-4}).$$

Proof. Fix a parameter N to be determined later and set

$$\sigma_\varepsilon(u) = \{\psi_\varepsilon(u,a,b,c) : a \le N,\ 20|c\}.$$

Fix $u_0 \in Q$, let $U_0 = \{u_0\}$, and recursively define

$$U_{k+1} = \bigcup_{u \in U_k} \sigma_\varepsilon(u)$$

where ε is defined by $\delta = 3\varepsilon^{3/2}$. Note that

$$E_k = \bigcup_{u \in U_k} \overline{\Delta(u)}$$

is a disjoint union, by Lemma 5.5. We have $E_{k+1} \subset E_k$ by Lemma 5.1, and by Theorem 5.3(a), there is a one-to-one correspondence between the points of $E = \cap E_k$ and the sequences (u_k) starting with u_0 and satisfying $u_{k+1} \in \sigma_\varepsilon(u_k)$ for all k. Theorem 5.3(b) implies $E \subset \mathrm{DI}_\delta(2)$. The hypotheses (i)–(iii) of Theorem 3.2 now hold with

$$\rho = \frac{\varepsilon^9}{2^{11} N^3}.$$

Before checking (iv), we note that given $1 \leq a \leq N$ we have $\phi(a)$ choices for b such that $a \geq b \geq 0$ and $\gcd(a, b) = 1$, where ϕ is the Euler totient function. It is well known that

$$\liminf_{n \to \infty} \frac{\phi(n) \log \log n}{n} > 0.$$

Now, for (iv) we compute (assuming $s > \frac{4}{3}$)

$$\begin{aligned}
\sum_{u' \in \sigma_\varepsilon(u)} \frac{|L'|^s |u|^{2s}}{|L|^s |u'|^{2s}} &\asymp \sum_{L'} \frac{|L'|^s}{|L|^s} \sum_c \frac{1}{c^{2s}} \\
&\asymp \sum_{L'} \frac{|L'|^s}{|L|^s} \left(\frac{\varepsilon^3 |u|}{|L'|^2} \right)^{2s-1} \\
&\asymp \frac{\varepsilon^{6s-3} |u|^{2s-1}}{|L|^s |\widehat{L}|^{3s-2}} \sum_{a'} \frac{\phi(a')}{(a')^{3s-2}} \\
&\succeq \varepsilon^{9s-6} \int_e^N \frac{dx}{x^{3s-3} \log x}.
\end{aligned}$$

Note that as $p \to 1^+$,

$$\begin{aligned}
\int_e^\infty \frac{dx}{x^p \log x} &\asymp \sum_{k \geq 1} \int_{e^k}^{e^{k+1}} \frac{dx}{x^p k} = \sum_{k \geq 1} \frac{e^{-k(p-1)}}{(p-1)k} (1 - e^{-(p-1)}) \\
&= \frac{1 - e^{-(p-1)}}{p-1} \log \frac{1}{1 - e^{-(p-1)}} \asymp \log \frac{1}{p-1}.
\end{aligned}$$

Thus, we conclude that there is a constant $C > 1$ such that for any $s > \frac{4}{3}$ satisfying

$$\varepsilon^{9s-6} \left| \log \left(s - \frac{4}{3} \right) \right| > C$$

the condition (iv) of Theorem 3.2 holds by choosing N large enough (depending on ε). Since $\delta = 3\varepsilon^{3/2}$, the proposition follows. $\square$

This completes the proof of Theorem 1.6.

We now describe how to modify the preceding argument to obtain the lower bound in Theorem 1.1. Fix a parameter $C > 1$ to be determined later and choose positive sequences $\varepsilon_k \to 0$ and $N_k \to \infty$ such that for all k,

$$\varepsilon_k < 2^{-7}, \quad N_k \geq 1, \quad \text{and} \quad \varepsilon_k^6 \log\log N_k > C.$$

We shall also assume that the sequences are slowly varying and that the ratio of consecutive terms is bounded above and below by positive constants, say 2 and $\frac{1}{2}$. For example,

$$N_k = k+1, \qquad \varepsilon_k = \frac{1}{2^7 \log\log\log(k+C')},$$

where $C' > 1$ is chosen large enough depending only on C. The definitions of the sets U_k are modified by the formula

$$U_{k+1} = \bigcup_{u \in U_k} \sigma_{\varepsilon_{k+1}}(u).$$

With E defined the same way as before, Theorem 5.3(c) now implies $E \subset \operatorname{Sing}(2)$. For each $u \in U_k$ set

$$\rho(u) = \frac{\varepsilon_{k+1}^9}{2^{11} N_k^3}$$

so that (i)–(iii) of Theorem 3.3 hold. The main calculation in the proof of Proposition 5.6 with $s = \frac{4}{3}$ now yields

$$\sum_{u' \in \sigma_{\varepsilon_{k+1}}(u)} \frac{|L'|^s |u|^{2s}}{|L|^s |u'|^{2s}} \succeq \varepsilon_{k+1}^6 \log\log N_k$$

so that (iv) of Theorem 3.3 holds provided C was chosen large enough at the beginning. It follows that

$$\text{H.dim} \operatorname{Sing}(2) \geq \frac{4}{3}.$$

This completes the proof of Theorem 1.1.

6. **Slowly divergent trajectories**

In this section, we prove

THEOREM 6.1. *Given any function $W(t) \to -\infty$ as $t \to \infty$ there exists a dense set of $\mathbf{x} \in \operatorname{Sing}^*(2)$ with the property $W_{\mathbf{x}}(t) \geq W(t)$ for all sufficiently large t.*

This affirmatively answers a question of A.N. Starkov [21] concerning the existence of slowly divergent trajectories for the flow on $\mathrm{SL}_3 \mathbb{R} / \mathrm{SL}_3 \mathbb{Z}$ induced by g_t.

LEMMA 6.2. *Given $\delta > 0$ and a function $F(t) \to \infty$ as $t \to \infty$, there exists $t_0 > 0$ and a monotone function $f(t) \to \infty$ as $t \to \infty$ such that*

(i) *$f(t) \le F(t)$ for all $t > t_0$, and*
(ii) *$f(t + f(t)) \le f(t) + \delta$ for all $t > t_0$.*

Proof. We may reduce to the case where $F(t)$ is a nondecreasing function. Let t_0 be large enough so that $y_0 = F(t_0) > 0$ and for $k > 0$ set $t_k = t_{k-1} + y_{k-1}$ and $y_k = \min(F(t_k), y_{k-1} + \delta)$. Since $y_k \ge y_0 > 0$ for all k, we have $t_k \to \infty$ and therefore also $y_k \to \infty$. Let $f(t) = y_k$ for $t_k \le t < t_{k+1}$ so that $f(t) = y_k \le F(t_k) \le F(t)$ since $t_k \le t$ and $F(t)$ is nondecreasing. Moreover, $t_{k+1} = t_k + y_k \le t + f(t) < t_{k+1} + y_k \le t_{k+1} + y_{k+1} = t_{k+2}$ so that $f(t + f(t)) = y_{k+1} \le y_k + \delta = f(t) + \delta$. □

Definition 6.3. For any $v \in Q$,

$$\tau(v) := -\frac{1}{3} \log \frac{|L(v)|}{|v|^2} = -\frac{1}{2} \log \frac{\varepsilon(v)}{|v|}.$$

LEMMA 6.4. *There exists $C > 0$ such that for any $0 < \varepsilon' < 1$ and any $u \in Q$, there exists $u' \in \mathcal{N}_{\varepsilon'}(u)$ such that*

(i) *$|\log \varepsilon(u') - \log \varepsilon'| \le C$, and*
(ii) *$\Big|\tau(u') - \tau(u) + 2\log\varepsilon' - |\log\varepsilon(u)|\Big| \le C$.*

Proof. Let $L' = \widehat{L}(u)$ and let $u' \in Q$ be determined by $L' = u \wedge u'$ and

$$|u'| > (\varepsilon')^{-3}|L'|^2 \ge |u'| - |u|.$$

Then $u' \in \mathcal{N}_{\varepsilon'}(u)$ by the first inequality. By Lemma 4.3 and (5.1)

$$|u'| > |L'|^2 \ge \frac{|u|^2}{|L(u)|^2} > 2|u|$$

so that $\varepsilon(u')^3 \asymp \frac{|L'|^2}{|u'|} \asymp (\varepsilon')^3$, giving (i). Since

$$\begin{aligned}\tau(u') - \tau(u) &= \frac{1}{2}\log\frac{|u'|}{|u|} - \frac{1}{2}\log\frac{\varepsilon'}{\varepsilon(u)} + O(1)\\ &= 2|\log\varepsilon'| + |\log\varepsilon(u)| + O(1)\end{aligned}$$

(ii) follows. □

Proof of Theorem 6.1. Let $\tilde{f}$ be the function obtained when Lemma 6.2 is applied to $F = -W(t)$ and some given $\delta > 0$ to be determined later. Set $f = 3^{-1}\tilde{f}$ and note that f satisfies

(i) $3f(t) \le -W(t)$ for all $t > t_0$, and
(ii) $f(t + 3f(t)) \le f(t) + \delta$ for all $t > t_0$.

Since $f(t) \to \infty$, given any $A > 0$ we can choose t_0, perhaps even larger, so that, in addition to (i) and (ii), f also satisfies

(iii) $f(t + 3f(t) + A) \le f(t) + 2\delta$ for all $t > t_0$.

We claim there is a constant B such that for any $u \in Q_1$ satisfying

$$|f(\tau(u)) + \log \varepsilon(u)| \leq B \tag{6.1}$$

and such that $|u|$ is larger than some constant depending only on f, there exists $u' \in \widehat{\sigma}_1(u)$ such that

$$|f(\tau(u')) + \log \varepsilon(u')| \leq B.$$

Indeed, given u satisfying (6.1), we let u' be obtained by applying Lemma 6.4 with $\varepsilon' < 1$ determined by

$$|\log \varepsilon'| = f\Big(\tau(u) + |\log \varepsilon(u)|\Big).$$

Then, if $A \geq 3B$ we have

$$\begin{aligned} |\log \varepsilon'| &\leq f(\tau(u)) + 3f(\tau(u)) + 3B \\ &\leq f(\tau(u)) + 2\delta \\ &\leq |\log \varepsilon(u)| + B + 2\delta. \end{aligned}$$

By Lemma 6.4,

$$\begin{aligned} \tau(u') &\leq \tau(u) + 2|\log \varepsilon'| + |\log \varepsilon(u)| + C \\ &\leq \tau(u) + 3|\log \varepsilon(u')| + 2B + C + 4\delta \\ &\leq \tau(u) + 3f(\tau(u)) + 5B + C + 4\delta \end{aligned}$$

so that if $A \geq 5B + C + 4\delta$,

$$\begin{aligned} f(\tau(u')) &\leq f(\tau(u)) + 2\delta \\ &\leq |\log \varepsilon'| + 2\delta \\ &\leq |\log \varepsilon(u')| + C + 2\delta. \end{aligned}$$

Now,

$$|\log \varepsilon'| \geq f(\tau(u)) \geq |\log \varepsilon(u)| - B$$

so that

$$\tau(u') \geq \tau(u) + 3|\log \varepsilon(u)| - 2B - C.$$

Assuming $|u|$ large enough so that $3|\log \varepsilon(u)| \geq 2B + C$, we have

$$\begin{aligned} f(\tau(u')) &\geq f(\tau(u)) \\ &\geq f\Big(\tau(u) + 3f(\tau(u)) + A\Big) - 2\delta \\ &\geq f\Big(\tau(u) + 3|\log \varepsilon(u)| + A - 3B\Big) - 2\delta \\ &\geq |\log \varepsilon'| - 2\delta \\ &\geq |\log \varepsilon(u')| - C - 2\delta. \end{aligned}$$

Setting $A = 6C + 14\delta$, we see that the claim follows with $B = C + 2\delta$.

Given any nonempty open set $U \subset \mathbb{R}^2$, we can choose $u_0 \in Q$ such that $\Delta(u_0) \subset U$. Indeed, choose any $\mathbf{x}_0 \in U \setminus \mathbb{Q}^2$ and let $u_0 \in \Sigma(\mathbf{x}_0)$ be such that $|u_0|$ is large enough, so that $\Delta(u_0) \subset U$. Let δ be chosen large enough at the beginning so that (6.1) holds for $u = u_0$. Let $\Sigma_0 = (u_k)$ be a sequence constructed by recursive definition and use of the claim. Since

$$\tau(u_{k+1}) = \tau(u_k) + 3|\log \varepsilon(u_k)| + O(1) \tag{6.2}$$

and $\varepsilon(u_k) \asymp \exp(-f(\tau(u_k)))$ by construction, by choosing $|u_0|$ large enough initially we can ensure that $\varepsilon(u_k) < \frac{1}{3}$ for all k so that $\tau(u_k)$ increases to infinity as $k \to \infty$. Since $f(t) \to \infty$ as $t \to \infty$, this implies $\varepsilon(u_k) \to 0$ as $k \to \infty$. By construction, $u_{k+1} \in \mathcal{N}_{\varepsilon_k}(u_k)$ so that Theorem 5.3(c) implies

$$\mathbf{x} := \lim_k \dot{u}_k \in \mathrm{Sing}\,(2).$$

If $\mathbf{x}$ lies on a rational line, then $W_{\mathbf{x}}(t) \leq -\frac{t}{2} + C$ for some constant C and all large enough t. It is clear that we could have, at the start, reduced to the case where, say, $W(t) > -\log t$ for all t, so that $\mathbf{x} \in \mathrm{Sing}^*(2)$.

Let $D = |\log(1 - 3^{-6})|$. Theorem 5.3(d) implies that

$$\begin{aligned} -W_{\mathbf{x}}(t) &\leq 3|\log \varepsilon(u_k)| + D \\ &\leq 3f(\tau(u_k)) + 3B + D \\ &\leq -W(\tau(u_k)) + 3B + D \\ &\leq -W(t) + 3B + D \end{aligned}$$

for all $t \in [\tau(u_k), \tau(u_{k+1})]$. It is clear that we could have chosen f initially to satisfy (i′) $3f(t) \leq -W(t) - 3B - D$ for all $t > t_0$ instead of (i). With this choice, we conclude $W_{\mathbf{x}}(t) \geq W(t)$ for all $t > t_0$. □

References

[1] R. C. Baker, Singular n-tuples and Hausdorff dimension, *Math. Proc. Cambridge Philos. Soc.* **81** (1977), 377–385. MR 0435010. Zbl 0351.10041. doi: 10.1017/S0305004100053457.

[2] ———, Singular n-tuples and Hausdorff dimension. II, *Math. Proc. Cambridge Philos. Soc.* **111** (1992), 577–584. MR 1151334. Zbl 0755.11021. doi: 10.1017/S030500410002171X.

[3] A. S. Besicovitch, Sets of fractional dimensions IV: On rational approximation to real numbers, *J. London Math. Soc.* **9** (1934), 126–131. Zbl 0009.05301. doi: 10.1112/jlms/s1-9.2.126.

[4] Y. Cheung, Hausdorff dimension of the set of nonergodic directions, *Ann. of Math.* **158** (2003), 661–678, with an appendix by M. Boshernitzan. MR 2018932. Zbl 1037.37018. doi: 10.4007/annals.2003.158.661.

[5] ———, Hausdorff dimension of the set of points on divergent trajectories of a homogeneous flow on a product space, *Ergodic Theory Dynam. Systems* **27** (2007), 65–85. MR 2297087. Zbl 1114.22013. doi: 10.1017/S0143385706000678.

[6] S. G. Dani, Divergent trajectories of flows on homogeneous spaces and Diophantine approximation, *J. Reine Angew. Math.* **359** (1985), 55–89. MR 794799. Zbl 0578.22012. doi: 10.1515/crll.1985.359.55.

[7] I. J. Good, The fractional dimensional theory of continued fractions, *Proc. Cambridge Philos. Soc.* **37** (1941), 199–228. MR 0004878. Zbl 0061.09408.

[8] V. Jarnìk, Diophantischen approximationen und hausdorffsches mass, *Mat. Sbornik* **36** (1929), 371–382. JFM 55.0719.01.

[9] D. Y. Kleinbock and G. A. Margulis, Bounded orbits of nonquasiunipotent flows on homogeneous spaces, in *Sinaĭ's Moscow Seminar on Dynamical Systems*, *Amer. Math. Soc. Transl. Ser.* 2 **171**, Amer. Math. Soc., Providence, RI, 1996, pp. 141–172. MR 1359098. Zbl 0843.22027.

[10] D. Kleinbock and B. Weiss, Bounded geodesics in moduli space, *Int. Math. Res. Not.* **2004** (2004), 1551–1560. MR 2049831. Zbl 1075.37008. doi: 10.1155/S1073792804133412.

[11] ———, Friendly measures, homogeneous flows and singular vectors, in *Algebraic and Topological Dynamics*, *Contemp. Math.* **385**, Amer. Math. Soc., Providence, RI, 2005, pp. 281–292. MR 218024. Zbl 1130.11040.

[12] ———, Dirichlet's theorem on Diophantine approximation and homogeneous flows, *J. Mod. Dyn.* **2** (2008), 43–62. MR 2366229. Zbl 1143.11022.

[13] J. C. Lagarias, Best simultaneous Diophantine approximations. I. Growth rates of best approximation denominators, *Trans. Amer. Math. Soc.* **272** (1982), 545–554. MR 0662052. doi: 10.2307/1998713.

[14] ———, Best simultaneous Diophantine approximations. II. Behavior of consecutive best approximations, *Pacific J. Math.* **102** (1982), 61–88. MR 682045. Zbl 0497.10025. doi: 10.1090/S0002-9947-1982-0662052-7.

[15] H. Masur, Hausdorff dimension of the set of nonergodic foliations of a quadratic differential, *Duke Math. J.* **66** (1992), 387–442. MR 1167101. Zbl 0780.30032. doi: 10.1215/S0012-7094-92-06613-0.

[16] H. Masur and J. Smillie, Hausdorff dimension of sets of nonergodic measured foliations, *Ann. of Math.* **134** (1991), 455–543. MR 1135877. Zbl 0774.58024. doi: 10.2307/2944356.

[17] N. G. Moshchevitin, Best Diophantine approximations: the phenomenon of degenerate dimension, in *Surveys in Geometry and Number Theory*: *Reports on Contemporary Russian Mathematics*, *London Math. Soc. Lecture Note Ser.* **338**, Cambridge Univ. Press, Cambridge, 2007, pp. 158–182. MR 2306143. Zbl 1151.11031. doi: 10.1017/CBO9780511721472.006.

[18] H. Reeve, On the Hausdorff dimension of sets of uniformly approximable numbers, preprint.

[19] B. P. Rynne, A lower bound for the Hausdorff dimension of sets of singular n-tuples, *Math. Proc. Cambridge Philos. Soc.* **107** (1990), 387–394. MR 1027791. Zbl 0707.11056. doi: 10.1017/S0305004100068651.

[20] W. M. Schmidt, On badly approximable numbers and certain games, *Trans. Amer. Math. Soc.* **123** (1966), 178–199. MR 0195595. Zbl 0232.10029. doi: 10.2307/1994619.

[21] A. N. Starkov, *Dynamical Systems on Momogeneous Spaces*, *Transl. Math. Monogr.* **190**, Amer. Math. Soc., Providence, RI, 2000, translated from the 1999 Russian original by the author. MR 1746847. Zbl 1143.37300.

[22] B. Weiss, Divergent trajectories on noncompact parameter spaces, *Geom. Funct. Anal.* **14** (2004), 94–149. MR 2053601. Zbl 1074.37004. doi: 10.1007/s00039-004-0453-z.

[23] ______, Divergent trajectories and $\mathbb{Q}$-rank, *Israel J. Math.* **152** (2006), 221–227. MR 2214461. Zbl 1126.22007. doi: 10.1007/BF02771984.

[24] K. Y. Yavid, An estimate for the Hausdorff dimension of a set of singular vectors, *Dokl. Akad. Nauk BSSR* **31** (1987), 777–780, 859. MR 912957. Zbl 0634.10044.

(Received: September 27, 2007)
(Revised: October 21, 2008)

San Francisco State University, San Francisco, CA
E-mail: cheung@math.sfsu.edu
http://math.sfsu.edu/cheung/index.html