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Volumes of balls in large Riemannian manifolds

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Abstract

We prove two lower bounds for the volumes of balls in a Riemannian manifold. If (M^n, g) is a complete Riemannian manifold with filling radius at least R, then it contains a ball of radius R and volume at least $\delta(n)R^n$. If (M^n, hyp) is a closed hyperbolic manifold and if g is another metric on M with volume no greater than $\delta(n)\mathrm{Vol}(M, \mathrm{hyp})$, then the universal cover of (M, g) contains a unit ball with volume greater than the volume of a unit ball in hyperbolic n-space.

Let (M, g) be a Riemannian manifold of dimension n. Let $V(R)$ denote the largest volume of any metric ball of radius R in (M, g). In [6], Gromov made a number of conjectures relating the function $V(R)$ to other geometric invariants of (M, g). The spirit of these conjectures is that if (M, g) is "large", then $V(R)$ should also be large. In this paper, we prove one of Gromov's conjectures: $V(R)$ is large if the filling radius of (M, g) is large.

Gromov defined the filling radius in [5]. Roughly speaking, the filling radius describes how "thick" a Riemannian manifold is. For example, the standard product metric on the cylinder $S^1 \times \mathbb{R}^{n-1}$ has filling radius $\pi/3$, and the Euclidean metric on $\mathbb{R}^n$ has infinite filling radius.

Theorem 1. *For each dimension n, there is a number $\delta(n) > 0$ so that the following estimate holds. If (M^n, g) is a complete Riemannian n-manifold with filling radius at least R, then $V(R) \geq \delta(n)R^n$.*

Our second result involves a closed hyperbolic manifold (M, hyp) equipped with an auxiliary metric g. Slightly paradoxically, if the manifold (M, g) is small, then its universal cover $(\widetilde{M}, \tilde{g})$ tends to be large. For example, if we look at the universal cover of $(M, \lambda^2\mathrm{hyp})$, then we get the space form with constant curvature $-\lambda^{-2}$. As λ decreases, the strength of the curvature increases, which increases the volumes of balls. Our second theorem gives a large ball in the universal cover $(\widetilde{M}, \tilde{g})$ provided that the volume of (M, g) is sufficiently small.

Theorem 2. *For each dimension n, there is a number $\delta(n) > 0$ so that the following estimate holds. Suppose that (M^n, hyp) is a closed hyperbolic*

n-manifold and that g is another metric on M, and suppose that $\mathrm{Vol}(M, g) < \delta(n)\mathrm{Vol}(M, \mathrm{hyp})$. *Let $(\widetilde{M}, \tilde{g})$ denote the universal cover of M with the metric induced from g. Then there is a point $p \in \widetilde{M}$ so that the unit ball around p in $(\widetilde{M}, \tilde{g})$ has a larger volume than the unit ball in hyperbolic n-space. In other words, the following inequality holds*:

$$V_{(\widetilde{M},\tilde{g})}(1) > V_{\mathbb{H}^n}(1).$$

We spend most of this introduction giving a context for these two results. At the end, we give a quick overview of the proof.

Many readers may not be familiar with the filling radius. Before looking at its definition, we give some corollaries of Theorem 1 using more common vocabulary.

COROLLARY 1. *Let (M^n, g) be a closed Riemannian manifold. Suppose that there is a degree* 1 *map from (M^n, g) to the unit n-sphere with Lipschitz constant* 1. *Then $V(R) \geq \delta(n)R^n$ for all $R \leq 1$.*

COROLLARY 2 (systolic inequality). *Let (M^n, g) be a closed aspherical Riemannian manifold. Suppose that the shortest noncontractible curve in (M^n, g) has length at least S. Then $V(S) \geq \delta(n)S^n$.*

COROLLARY 3. *Let (M^n, g) be a closed aspherical Riemannian manifold, and let $V(R)$ measure the volumes of balls in the universal cover $(\widetilde{M}, \tilde{g})$. Then $V(R) \geq \delta(n)R^n$ for all R.*

Background on filling radius. We next review the definition of filling radius and some main facts about it. For much more information, see [5]. The filling radius of a Riemannian manifold is defined by analogy with an invariant for submanifolds of Euclidean space. Let $M^n \subset \mathbb{R}^N$ be a closed submanifold of Euclidean space. By a filling of M, we mean an (n+1)-chain C with boundary M. (If M is oriented, then the standard convention is to use a chain with integral coefficients, and if M is not oriented, then the standard convention is to use a chain with mod 2 coefficients.) The filling radius of M in $\mathbb{R}^N$ is the smallest number R so that M can be filled inside of its R-neighborhood. For example, the filling radius of an ellipse is its smallest principal axis. The main result about filling radius in Euclidean space is the following estimate.

THEOREM (Federer-Fleming, Michael-Simon, Bombieri-Simon). *If $M^n \subset \mathbb{R}^N$ is a closed submanifold, then its filling radius is bounded in terms of its volume by the following formula*:

$$\mathrm{Fill\,Rad}(M) \leq C_n \mathrm{Vol}(M)^{1/n}.$$

In [3], Federer and Fleming gave a direct construction to prove that the filling radius of M is bounded by $C_N \mathrm{Vol}(M)^{1/n}$. Their result is slightly weaker than the result above because their constant C_N depends on the ambient dimension N. In [13], Michael and Simon proved an isoperimetric inequality for minimal surfaces that implies the above theorem. In [2], Bombieri and Simon established the sharp constant, which occurs when M is a round sphere.

In [5], Gromov defined an analogous filling radius for a closed Riemannian manifold (M, g). The key observation is that (M, g) admits a canonical isometric embedding into the Banach space $L^\infty(M)$. The embedding, which goes back to Kuratowski, sends the point $x \in M$ to the function dist_x defined by $\mathrm{dist}_x(z) = \mathrm{dist}(x, z)$. This embedding depends only on the metric g, and it is isometric in the strong sense that $|\mathrm{dist}_x - \mathrm{dist}_y|_{L^\infty} = \mathrm{dist}(x, y)$. Gromov defined the filling radius of (M, g) to be the infimal R so that the image of M in L^∞ can be filled inside its R-neighborhood. There is a similar definition for any complete Riemannian manifold.

This definition may seem abstract at first, but [5] contains a number of estimates that make it a useful tool. Here are a few facts to give a flavor for it. The filling radius of the Euclidean metric on $\mathbb{R}^n$ is infinite, but the filling radius of the standard product metric on $S^n \times \mathbb{R}^q$ is finite for $n \geq 1$. If there is a degree 1 map from (M, g) to (N, h) with Lipschitz constant 1, then the filling radius of M is at least the filling radius of N. The most important result about filling radius is an analogue of the Euclidean estimate above.

THEOREM (Gromov [5]). *If (M^n, g) is a complete Riemannian manifold of dimension n, then its filling radius can be bounded in terms of its volume by the following formula*:

$$\mathrm{Fill\,Rad}(M, g) \leq C_n \mathrm{Vol}(M, g)^{1/n}.$$

In the Euclidean setting, Gromov found a stronger version of the filling radius estimate. (This result appears near the end of §F of Appendix 1 of [5].)

THEOREM (Gromov [5], local volume estimate). *Suppose that $M^n \subset \mathbb{R}^N$ is a closed manifold. If the filling radius of M in $\mathbb{R}^N$ is at least R, then there is some point $x \in \mathbb{R}^N$ so that the volume of $M \cap B(x, R)$ is at least $c_n R^n$.*

In addition to giving a lower bound for the total volume of M, this result also controls the way the volume is distributed. It rules out the possibility that M could have a large total volume distributed in a diffuse way. After explaining this result, Gromov raised the question whether this local volume estimate has an analogue for Riemannian manifolds. Our first theorem answers this question in the affirmative.

Relation to entropy estimates. Our second theorem is related to an inequality of Besson, Courtois, and Gallot. Their inequality bounds the entropy of a Riemannian manifold, which is a way of describing the asymptotic behavior of the volumes of large balls. In our language, their result goes as follows.

Theorem (Besson, Courtois, and Gallot [1]). *Let* (M^n, hyp) *be a closed hyperbolic manifold, and let* g *be another metric on* M *with* $\mathrm{Vol}(M,g) < \mathrm{Vol}(M,\mathrm{hyp})$. *Then there is some constant* R_0 *(depending on* g*), so that for every radius* $R > R_0$*, the following inequality holds*:

$$V_{(\widetilde{M},\tilde{g})}(R) > V_{\mathbb{H}^n}(R).$$

Our theorem is not as sharp as the theorem of Besson, Courtois, and Gallot. The sharp constant in their theorem is a major achievement. (The result was previously proven by Gromov with a nonsharp constant in [4].) To complement their theorem, it would be nice to estimate the value of R_0. It even looks plausible that the theorem remains true with $R_0 = 0$! We discuss this possibility more below.

Our second theorem can be looked at as a step towards estimating R_0. According to Theorem 2, the stronger hypothesis $\mathrm{Vol}(M,g) < \delta(n)\mathrm{Vol}(M,\mathrm{hyp})$ implies the conclusion $V_{(\widetilde{M},\tilde{g})}(1) > V_{\mathbb{H}^n}(1)$. Our method can be modified to give a similar estimate for balls of radius R, but as R moves away from 1, the hypothesis gets stronger (and so the result gets weaker). For each R, there is a constant $\delta(n,R) > 0$, so that $\mathrm{Vol}(M,g) < \delta(n,R)\mathrm{Vol}(M,\mathrm{hyp})$ implies $V_{(\widetilde{M},\tilde{g})}(R) > V_{\mathbb{H}^n}(R)$. As R goes to infinity, the constant $\delta(n,R)$ falls off exponentially or faster. As R increases, the methods in this paper become less effective, whereas the methods in [4] and [1] are only effective asymptotically for very large R. Perhaps there is some way to combine the approaches to get a uniform estimate for $R \geq 1$.

Questions about the sharp constants. It would be interesting to know the sharp constants in Theorems 1 and 2. In [6], Gromov made the following sharp conjecture.

Conjecture (Gromov). *Let* (M^n, g) *be a complete Riemannian manifold with infinite filling radius. Let* ω_n *be the volume of the unit* n*-ball. Then* $V(R) \geq \omega_n R^n$ *for all* $R > 0$.

Imitating Gromov, we mention conjectures about the sharp constants in Theorems 1 and 2. I find the conjectures intriguing, but the evidence for them is not very strong.

Conjecture 1. *Let* (M^n, g) *be a complete Riemannian manifold, and let* (S^n, g_0) *be a round sphere. We choose the radius of the round sphere so that the filling radius of* (M^n, g) *is equal to that of* (S^n, g_0). *Then the following inequality holds for all* R:

$$V_{(M^n,g)}(R) \geq V_{(S^n,g_0)}(R).$$

Conjecture 2. *Let* (M^n, hyp) *be a closed hyperbolic manifold, and let* g *be another metric on* M *with* $\mathrm{Vol}(M,g) < \mathrm{Vol}(M, \mathrm{hyp})$. *Then the following inequality holds for all* R:

$$V_{(\widetilde{M},\tilde{g})}(R) > V_{\mathbb{H}^n}(R).$$

As Gromov pointed out in [6], estimates of $V(R)$ for small R are related to scalar curvature. If p is a point in a Riemannian manifold (M^n, g), then for small radii R, the volume of the ball $B(p,R)$ is equal to

$$\omega_n[R^n - (6n)^{-1}Sc(p)R^{n+2} + o(R^{n+2})].$$

Therefore, the above conjectures imply several important open conjectures about scalar curvature. The scalar curvature conjectures are known in some special cases, giving modest evidence in favor of the conjectures above.

Conjecture (Gromov [6]). *If* (M,g) *is a complete Riemannian manifold with infinite filling radius, then* $\inf_M Sc \leq 0$. *Therefore, if* N *is a closed aspherical manifold, then* N *does not admit a metric of positive scalar curvature.*

The last part of the conjecture is known for many particular aspherical manifolds, but not for all of them.

Conjecture 1A (Gromov [7]). *If* (M^n, g) *is a complete Riemannian manifold with scalar curvature at least* 1, *then the filling radius of* (M,g) *is bounded by a constant* C_n.

Gromov and Lawson proved this conjecture with a nonsharp constant for $n = 3$ when $H_1(M) = 0$. (See [5, p. 129] and [9].) Katz extended their method to 3-manifolds M with $\pi_1(M)$ finite or $\pi_1(M) = \mathbb{Z}$ [10]. In higher dimensions, the conjecture is open.

Conjecture 2A (Schoen [14]). *If* (M^n, hyp) *is a closed hyperbolic* n-*manifold and* g *is another metric on* M *with scalar curvature at least the scalar curvature of hyperbolic* n-*space, then the volume of* (M,g) *is at least the volume of* (M, hyp).

The conjecture is true for $n = 2$ by the Gauss-Bonnet theorem. For $n = 3$, it follows as a corollary of Perelman's proof of geometrization. For $n = 4$, the conjecture is unknown for hyperbolic manifolds, but LeBrun [11] proved a completely analogous result for certain other 4-manifolds, including the product of two hyperbolic surfaces. LeBrun's proof uses Seiberg-Witten theory. In higher dimensions, the conjecture is open.

Quick summary of the proof. The main idea of the proof — which is due to Gromov — is to cover (M,g) with balls, and look at the map from M to the nerve of the cover. This technique works well if the multiplicity of the covering is bounded, because then the dimension of the nerve and the Lipschitz constant of the map are both under control. If the Ricci curvature of g is bounded below, then Gromov constructed a cover by unit balls with bounded multiplicity in [4]. Using this method, he proved the following result on page 130 of [5].

THEOREM (Gromov 1983). *Suppose that (M^n, g) is a complete Riemannian manifold with filling radius at least R and with* $\mathrm{Ric} \geq -1$. *Then $V(R) \geq c(n)R^n$ for a dimensional constant $c(n) > 0$.*

According to Theorem 1, Gromov's result continues to hold without the hypothesis $\mathrm{Ric} \geq -1$. Our proof follows the outline of Gromov's proof, but it is more difficult because we work with no assumptions on the curvature of g. Without the Ricci curvature bound, we need to work a lot harder to produce a good cover of (M, g). We carefully construct a cover that has bounded multiplicity at most points and push Gromov's ideas to work on this cover.

For Sections 1–5, we assume that (M, g) is closed. In Section 6, we explain some minor technicalities to deal with the open case.

Throughout the paper, we use the following notation. If U is a region in (M, g), then we denote the volume of U by $|U|$. If z is a cycle or chain, then we denote the mass of z by $|z|$. If B is shorthand for a ball $B(p, r)$ — the ball around p of radius r — then $2B$ is shorthand for $B(p, 2r)$. Unless otherwise noted, the constants that appear depend only on the dimension n.

1. Good balls and good covers

In this section, we construct a covering of (M, g) by balls with certain good properties. All the material in this section is due to Gromov and appears in Sections 5 and 6 of [5].

Let $B(p, R) \subset M$ denote the ball around p of radius R. We say that the ball $B(p, R)$ is a good ball if it satisfies the following conditions:

A. Reasonable growth: $|B(p, 100R)| \leq 10^{4(n+3)}|B(p, 100^{-1}R)|$.
B. Volume bound: $|B(p, R)| \leq 10^{2n+6}V(1)R^{n+3}$.
C. Small radius: $R \leq (1/100)$.

The exact constants here are not important. Notice that in Euclidean space we would have $|B(p, 100R)| = 10^{4n}|B(p, 100^{-1}R)|$. The reasonable growth condition relaxes this bound by replacing 10^{4n} with $10^{4(n+3)}$. In the volume bound, in Euclidean space we would have $|B(p, R)| = V(1)R^n$. For small R, our bound $10^{2n+6}V(1)R^{n+3}$ is much stronger than the Euclidean bound. So good balls with small radii have tiny volumes. The only crucial point is that $n+3 > n$. The other constants were chosen to guarantee the following lemma.

LEMMA 1. *Let (M^n, g) be a complete Riemannian n-manifold, and let p be any point in M. Then there is a radius R so that $B(p, R)$ is a good ball.*

Proof. We define the density of a ball $B(p, R)$ to be the ratio $|B(p, R)|/R^n$. If $B(p, R)$ does not have reasonable growth, then the density falls off according to the following inequality:

$$(*) \qquad \text{Density}[B(p, 100^{-1}R)] < 10^{-12}\text{Density}[B(p, 100R)].$$

We consider the sequence of balls around p with radii $10^{-2}, 10^{-6}, 10^{-10}$, and so on. We first claim that one of these balls has reasonable growth. If the claim is false, then we can repeatedly use inequality $(*)$ to show that the density of $B(p, 10^{-4s})$ is at most $10^{-12s}V(1)$. Since (M, g) is a Riemannian manifold, the density of $B(p, \varepsilon)$ approaches the volume of the unit n-ball as ε goes to zero. This contradiction shows that one of the balls in our list has reasonable growth.

Now we define s so that $B(p, 10^{-4s-2})$ is the first ball in the list with reasonable growth. Applying inequality $(*)$ to the previous balls $B(p, 10^{-2}), \ldots$ $\ldots, B(p, 10^{-4s+2})$, we conclude that the density of $B(p, 10^{-4s})$ is at most $10^{-12s}V(1)$. In other words, $|B(p, 10^{-4s})| \leq 10^{-12s}V(1)[10^{-4s}]^n$. The ball $B(p, 10^{-4s-2})$ is contained in $B(p, 10^{-4s})$, so its volume obeys the following bound:

$$|B(p, 10^{-4s-2})| \leq 10^{-4s(n+3)}V(1) \leq 10^{2n+6}[10^{-4s-2}]^{n+3}V(1).$$

In other words, the ball $B(p, 10^{-4s-2})$ obeys condition B. It has radius 10^{-4s-2} $\leq 10^{-2}$, and so it obeys condition C. Therefore, it is a good ball. □

Because of Lemma 1, we can cover (M, g) with good balls. We now use the Vitali covering lemma to choose a convenient sub-covering with some control of the overlaps. More precisely, we call an open cover $\{B_i\}$ good if it obeys the following properties:

1. Each open set B_i is a good ball.
2. The concentric balls $(1/2)B_i$ cover M.
3. The concentric balls $(1/6)B_i$ are disjoint.

(Recall that if B_i is shorthand for $B(p_i, r_i)$, then $(1/2)B_i$ is shorthand for $B(p_i, (1/2)r_i)$.)

LEMMA 2. *If (M^n, g) is a closed Riemannian manifold, then it has a good cover.*

Proof. This follows immediately from the Vitali covering lemma. For each point $p \in M$, pick a good ball $B(p)$. Then look at the set of balls $\{(1/6)B(p)\}_{p \in M}$. These balls cover M. Applying the Vitali covering lemma to this set of balls finishes the proof. □

We now fix a good cover for our manifold (M, g), which we will use for the rest of the paper. Our next goal is to control the amount of overlap between different balls in the cover. It would be convenient if we could prove a bound on the multiplicity of the cover. I do not see how to prove such a bound, and it may well be false. We will prove a weaker estimate in the next section, bounding the volume of the set where the multiplicity is high. We begin with an estimate that controls the number of balls of roughly equal radius which meet a given ball.

LEMMA 3. *If $s < 1$, and we look at any ball $B(s)$ of radius s, not necessarily in our cover, then the number of balls B_i from our cover, with radius in the range $(1/2)s \leq r_i \leq 2s$, intersecting $B(s)$, is bounded by a dimensional constant $C(n)$.*

Proof. Let $\{B_i\}$ be the set of balls in our cover that intersect $B(s)$ and have radii in the indicated range. We number them so that B_1 has the smallest volume. Now, all of the balls are contained in $B(5s)$, and $B(5s)$ is contained in the ball $20B_1$. On the other hand, all the $(1/6)B_i$ are disjoint. So we have $\sum_i |(1/6)B_i| < |20B_1|$. Because of the locally bounded growth of good balls, we have $\sum |B_i| < C|B_1|$. But since B_1 has the smallest volume of all the balls, we see that the number of balls is at most C. □

2. The volume of the high-multiplicity set

Let $m(x)$ be the multiplicity function of the cover. In other words $m(x)$ is defined to be the number of balls in our cover that contain the point x. Let $M(\lambda)$ be the set of points where the multiplicity is at least λ. We will not be able to prove an upper bound for the multiplicity of our covering, but we partly make up for that by bounding the size of the set $M(\lambda)$ for large λ.

We will prove a bound for the total volume $|M(\lambda)|$, but this bound by itself is not strong enough to prove our theorems. We also need bounds on the size of $M(\lambda) \cap B_i$ for balls B_i in our cover. For any open set $U \subset M$, we define $M_U(\lambda) = M(\lambda) \cap U$ to be the set of points in U with multiplicity at least λ. We write $N_w(U)$ to denote the w-neighborhood of U: the set of points $y \in M$ with $\operatorname{dist}(y, U) < w$.

LEMMA 4. *There are constants $\alpha(n), \gamma(n)$, depending only on n, that make the following estimate hold. For any open set $U \subset M$, any $\lambda \geq 0$, and any $w < (1/100)$,*

$$|M_U(\gamma \log(1/w) + \lambda)| \leq e^{-\alpha\lambda} |N_w(U)|.$$

Taking $U = M$, it follows that $|M(\lambda)| < Ce^{-\alpha\lambda}|M|$. If B is a good ball in our cover with radius r, then we have the following estimate:

$$|M_B(\gamma \log(1/r) + \lambda)| \leq e^{-\alpha\lambda} |B|.$$

According to this lemma, it may happen that every point in B has multiplicity $\gamma(n)\log(1/r)$. However, the set of points in B with multiplicity much higher than $\gamma(n)\log(1/r)$ constitutes only a small fraction of B.

Proof. Let $\{B_i\}_{i\in I}$ denote the subset of balls in our cover that intersect U. We divide this set of balls into layers, using the Vitali covering construction to choose each layer.

To choose Layer (1): Take the largest ball in the set. (More precisely, take the ball with the largest radius.) Then take the next largest ball disjoint from it. Then take the largest ball disjoint from the two already chosen. . . . When there are no more balls left, stop.

To choose Layer (2): Examine all the balls that are not part of Layer (1). Take the largest ball available. Then take the largest remaining ball disjoint from this one. . . .

To choose Layer (d): Examine all the balls that are not part of any previous layer. Take the largest ball available. . . .

In this way, our set of balls $\{B_i\}_{i\in I}$ is divided into layers. Each ball in the set belongs to exactly one layer. Each layer consists of disjoint balls. We define $L(d)$ to be the union of all the balls in Layer (d). We call Layer (1) the top layer, and if $d_2 > d_1$, we say that Layer (d_2) is lower than Layer (d_1). Now for each layer, we define a subset $\mathrm{Core}(d) \subset L(d)$ which intersects only a bounded number of balls from lower layers.

To define the core, we first introduce a partial ordering on the balls in a given layer Layer (d). If $B_i, B_j \in$ Layer (d), we say that $B_i < B_j$ if there is some ball B_k in a lower layer which meets both B_i and B_j, and if the radii obey the inequalities $2r_i \le r_k \le r_j$. We consider the minimal partial order that is generated by these relations. In other words, we say that $B_i < B_j$ if and only if there is a chain of balls $B_{l_1}, \ldots, B_{l_m}$ in Layer (d) and a chain of balls $B_{k_1}, \ldots, B_{k_{m+1}}$ in lower layers so that B_i meets B_{k_1} which meets B_{l_1} which meets B_{k_2} . . . which meets B_{l_m} which meets $B_{k_{m+1}}$ which meets B_j and so that the radii obey $2^{m+1} r_i \le 2^m r_{k_1} \le 2^m r_{l_1} \le 2^{m-1} r_{k_2} \le \cdots \le 2r_{l_m} \le r_{k_{m+1}} \le r_j$. Figure 1 illustrates the overlapping balls in case $m = 1$. The balls drawn in solid lines belong to Layer (d) and those in dotted lines belong to lower layers. The smallest ball on the left is B_i, and the largest ball on the right is B_j.

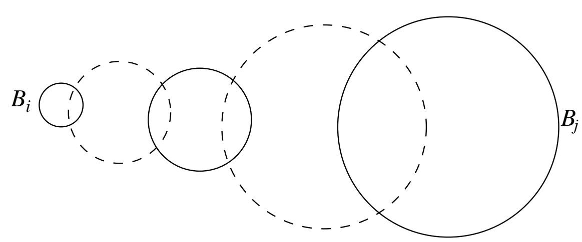

Figure 1.

We define $\mathrm{Max}\,(d) \subset \mathrm{Layer}\,(d)$ to be the maximal elements of this partial ordering. (A ball B_i is maximal if there is no other ball B_j with $B_i < B_j$.) For any ball B_i, we define the core of B_i to be the concentric ball $(1/10)B_i$. We define the core of $\mathrm{Layer}\,(d)$ to be the union of the cores of all the maximal balls in $\mathrm{Layer}\,(d)$:

$$\mathrm{Core}(d) = \cup_{B_i \in \mathrm{Max}\,(d)} \frac{1}{10} B_i.$$

(The balls in $\mathrm{Layer}\,(d)$ are disjoint, so the core is a union of disjoint balls.)

By looking at maximal balls, we buy the following inequality. Let B_i be a maximal ball in $\mathrm{Layer}\,(d)$ and let $x \in (1/10)B_i$ be a point in $\mathrm{Core}(d)$. Suppose that x also lies in a ball B_k from a lower layer. Then the radius r_k is pinched in the range $(1/15)r_i \le r_k \le 2r_i$. The upper bound depends on the maximality of B_i. Suppose that $r_k > 2r_i$. Since the ball B_k was not selected to join the layer $\mathrm{Layer}\,(d)$, there must be some larger ball B_j in $\mathrm{Layer}\,(d)$ intersecting B_k. But then it follows that $B_i < B_j$, contradicting the assumption that B_i is maximal. The lower bound for r_k does not depend on maximality. We know that the concentric balls $(1/6)B_i$ and $(1/6)B_k$ are disjoint. In particular, this fact implies that the center of B_k lies outside of $(1/6)B_i$. Now if $r_k < (1/15)r_i$, then the ball B_k lies outside of $(1/10)B_i$. On the other hand, x lies in $(1/10)B_i$, and we get a contradiction.

According to Lemma 3, the number of balls B_k containing x with radii in the range $(1/15)r_i \le r_k \le 2r_i$ is bounded by a dimensional constant $\eta(n)$. Therefore, the number of balls B_i so that $x \in B_i$ and so that $B_i \in \mathrm{Layer}\,(\lambda)$ with $\lambda \ge d$ is at most $\eta(n)$. Less formally, this estimate says that the core of $\mathrm{Layer}\,(d)$ is well-insulated from the balls in lower layers.

The next main point is that $\mathrm{Core}(d)$ contains a substantial fraction of the volume of $L(d)$. We first claim that $L(d) \subset \cup_{B_j \in \mathrm{Max}\,(d)} 10B_j$. Suppose that B_i is any ball in $\mathrm{Layer}\,(d)$. If B_i is itself maximal, then it is contained in the union $\cup_{B_j \in \mathrm{Max}\,(d)} 10B_j$. If B_i is not maximal, then there is some chain of overlapping balls $B_i, B_{k_1}, B_{l_1}, \ldots, B_{k_m}, B_{l_m}, B_{k_{m+1}}, B_j$, where B_j is maximal, and the radii obey $2^{m+1} r_i \le 2^m r_{k_1} \le 2^m r_{l_1} \le \cdots \le 2r_{k_m} \le 2r_{l_m} \le r_{k_{m+1}} \le r_j$. Because the balls overlap, B_i is contained in the ball with the same center as B_j and with radius $R = r_j + 2r_{k_{m+1}} + 2r_{l_m} + 2r_{k_m} + \cdots + 2r_{l_1} + 2r_{k_1} + 2r_i$. According to our bounds for the radii of the balls, $R \le r_j + 2r_j + 4 \cdot 2^{-1} r_j + 4 \cdot 2^{-2} r_j + 4 \cdot 2^{-3} r_j + \cdots \le 7r_j$. Therefore, $B_i \subset 10B_j$. Since B_j is maximal, $B_i \subset \cup_{B_j \in \mathrm{Max}\,(d)} 10B_j$. Therefore, $L(d) \subset \cup_{B_j \in \mathrm{Max}\,(d)} 10B_j$. Using this inclusion and the reasonable growth estimates for good balls, we can estimate the volume of $\mathrm{Core}(d)$:

$$\begin{aligned} |L(d)| &\le |\cup_{B_j \in \mathrm{Max}\,(d)} 10B_j| \\ &\le \sum_{B_j \in \mathrm{Max}\,(d)} |10B_j| \le C \sum_{B_j \in \mathrm{Max}\,(d)} |\frac{1}{10} B_j| = C |\mathrm{Core}(d)|. \end{aligned}$$

We will use these estimates about the core to prove the exponential decay of the high-multiplicity set. We now introduce some vocabulary that describes how many times a point is contained in balls from different layers:

$$L^\mu(\lambda) := \{x | x \in L(d) \text{ for at least } \mu \text{ different values of } d \text{ in the range } d \geq \lambda\}.$$

The sets $L^\mu(\lambda)$ are nested: $L^1(\lambda) \supset L^2(\lambda) \supset \dots$. Because of the construction of the layers, $\cup_{B_i \in \mathrm{Layer}\,(\lambda)} 3B_i$ contains $\cup_{d \geq \lambda} \cup_{B_i \in \mathrm{Layer}\,(d)} B_i = L^1(\lambda)$. Because the balls in each layer are disjoint and because of the reasonable growth bound, we get the following upper bound for $|L^1(\lambda)|$:

$$\begin{aligned} |L^1(\lambda)| &= | \cup_{d \geq \lambda} \cup_{B_i \in \mathrm{Layer}\,(d)} B_i | \leq | \cup_{B_i \in \mathrm{Layer}\,(\lambda)} 3B_i | \\ &\leq \sum_{B_i \in \mathrm{Layer}\,(\lambda)} |3B_i| \leq C \sum_{B_i \in \mathrm{Layer}\,(\lambda)} |B_i| = C|L(\lambda)|. \end{aligned}$$

Now we define a function $F(\lambda)$ which is an average of the volumes of $L^\mu(\lambda)$:

$$F(\lambda) := \frac{1}{\eta(n)} \sum_{\mu=1}^{\eta(n)} |L^\mu(\lambda)|.$$

Our estimates about the core imply that $F(\lambda)$ decays exponentially. We proved that each point x in $\mathrm{Core}(\lambda)$ lies in at most $\eta(n)$ balls from layers $\mathrm{Layer}\,(d)$ with $d \geq \lambda$. We know that $\mathrm{Core}(\lambda) \subset L(\lambda)$. Therefore, we get the following estimate:

$$\sum_{\mu=1}^{\eta(n)} |L^\mu(\lambda)| - |L^\mu(\lambda+1)| \geq |\operatorname{Core}(\lambda)|.$$

Using the formula for $F(\lambda)$, we see that $F(\lambda) - F(\lambda+1) \geq (1/\eta)|\operatorname{Core}(\lambda)|$. Now we plug in our volume estimates for $|\operatorname{Core}(\lambda)|$ and $|L^1(\lambda)|$:

$$F(\lambda) - F(\lambda+1) \geq (1/\eta)|\operatorname{Core}(\lambda)| \geq c|L(\lambda)| \geq c'|L^1(\lambda)| \geq c' F(\lambda).$$

Rearranging the equation, we can deduce the exponential decay of $F(\lambda)$. For a small constant $c' > 0$, depending only on n,

$$F(\lambda+1) \leq (1-c')F(\lambda). \tag{$*$}$$

To control $F(\lambda)$ for large values of λ, we combine equation $(*)$ with some simple estimates for $F(\lambda)$ for small values of λ. To control $F(\lambda)$ for small λ, we need one more inequality, which says that the large balls are put into the top layers. More precisely, if a ball B_i of radius r_i belongs to $\mathrm{Layer}\,(d)$, then $d \leq \gamma \log(1/r_i)$ (for a constant γ depending only on n). Because B_i does not belong to any of the first $d-1$ layers, there must be a larger ball in each of those layers intersecting B_i. All of the balls in our cover have radius at most $(1/100)$, so the number of different scales of the radii of these balls is $\log(1/r_i)$. For each scale, Lemma 3 tells us that there are at most C balls of that scale intersecting B_i. Therefore, we can choose a constant $\gamma(n)$ so that the total number of larger balls intersecting B_i is at most $\gamma \log(1/r_i)$, and $d \le \gamma \log(1/r_i)$ as claimed. Thus, if B_i belongs to Layer(d) with $d \ge \gamma \log(1/w)$, then $r_i \le w$. In other words, all the balls in layers lower than $\gamma \log(1/w)$ have radius at most w. Since all of our balls intersect U, the union of all the balls in the lower layers is contained in $N_{2w}(U)$. This argument gives us the following estimate for $F(\gamma \log(1/w))$:

$$F(\gamma \log(1/w)) \le |L^1(\gamma \log(1/w))| \le |N_{2w}(U)|.$$

Combining this inequality with the exponential decay $(*)$ we get the following bound:

$$F(\gamma \log(1/w) + \lambda) \le e^{-\alpha\lambda} |N_{2w}(U)|.$$

Finally, we bound the size of $M_U(\lambda)$ in terms of F. Since each layer consists of disjoint balls, $M_U(\lambda + \eta(n)) \subset L^{\eta(n)}(\lambda)$, and so $|M_U(\lambda + \eta(n))| \le |L^{\eta(n)}(\lambda)| \le F(\lambda)$. Combining this observation with our last inequality gives the following:

$$|M_U(\gamma \log(1/w) + \lambda + \eta(n))| \le e^{-\alpha\lambda} |N_{2w}(U)|.$$

This inequality is equivalent to the one we wanted to prove.

As a special case, we can take $U = M$. Applying our inequality with $w = (1/100)$ yields $|M(\lambda)| \le C e^{-\alpha\lambda}|M|$. As another special case, we can take $U = B$ for a ball B in our cover of radius r. Applying our inequality with $w = r$ yields

$$|M_B(\gamma \log(1/r) + \lambda + \eta(n))| \le e^{-\alpha\lambda} |2B|.$$

Since B is a good ball, $|2B| < C|B|$, and so we get the following inequality:

$$|M_B(\gamma \log(1/r) + \lambda + \eta(n))| \le C e^{-\alpha\lambda} |B|.$$

This inequality is equivalent to the one we wanted to prove. □

3. **The rectangular nerve**

Gromov had the idea to prove estimates about a Riemannian manifold (M, g) by covering it with balls and considering the induced map from M to the nerve of the covering. I believe that this idea first appeared in [4]. It is also discussed in [8, §5.32]. In our case, there is an added wrinkle because the balls in our covering have a wide range of radii, and we need to choose a metric on the nerve that reflects the radii of the balls in the covering. In order to accomplish that, we slightly modify the idea of the nerve, introducing a "rectangular nerve".

For each ball B_i of radius r_i in our good cover, define $\phi_i : M \to [0, r_i]$ as follows. Outside of B_i, $\phi_i = 0$. Let d be the distance from $x \in B_i$ to the center of B_i. If $d \le (1/2)r_i$, then $\phi(x) = r_i$. If $(1/2)r_i \le d \le r_i$, then $\phi(x) = 2(r_i - d)$. The Lipschitz constant of ϕ_i is 2.

All the ϕ_i together map M into the high-dimensional rectangle R with dimensions $r_1 \times \cdots \times r_D$. Because $(1/2)B_i$ covers M, we know that for each $x \in M$, there is some ϕ_i so that $\phi_i(x) = r_i$. Therefore, the image of M lies in the union of certain hyperfaces of the rectangle R: namely those closed hyperfaces that do not contain 0. As in the simplicial world, we define a nerve N which will be a subcomplex of the rectangle R. An open face F of R is determined by dividing the dimensions $1, 2, \ldots, D$ into three categories: I_0, I_1, and $I_{(0,1)}$. Then F is given by the equalities and inequalities $\phi_i = 0$ for $i \in I_0$, $\phi_i = r_i$ for $i \in I_1$, and $0 < \phi_i < r_i$ for $i \in I_{(0,1)}$. We denote $I_+ = I_1 \cup I_{(0,1)}$. A face F is contained in the nerve if and only if $\cap_{i \in I_+(F)} B_i \neq \emptyset$, and if $I_1(F) \neq \emptyset$. We see that ϕ maps M into the rectangular nerve N.

Using our bounds for the high-multiplicity set in M, we can bound the volume of the image $\phi(M)$, and also the volume of $\phi(M)$ contained in certain regions of N. If F is an open face in N, then we define $\mathrm{Star}(F)$ to be the union of all open faces of N which contain F in their closures. In other words, $\mathrm{Star}(F)$ is the union of F itself together with each higher-dimensional open face that contains F in its boundary. No lower dimensional faces are contained in $\mathrm{Star}(F)$. We let $d(F)$ denote the dimension of F. Each face F is itself a rectangle with dimensions $r_1(F) \leq \cdots \leq r_{d(F)}(F)$.

Lemma 5. *There are constants $C(n)$ and $\beta(n) > 0$, depending only on n, so that the volume of $\phi(M) \cap \mathrm{Star}(F)$ obeys the following inequality*:

$$|\phi(M) \cap \mathrm{Star}(F)| < CV(1) r_1(F)^{n+1} e^{-\beta d(F)}.$$

Also, $|\phi(M)| < C|M|$.

Proof. The set $\phi^{-1}[\mathrm{Star}(F)]$ is contained in B_1, the ball with radius $r_1 = r_1(F)$. By the good ball estimate, B_1 has volume at most $CV(1)r_1^{n+3}$. The Lipschitz constant of ϕ at a point $x \in B_1$ is bounded by $2m(x)^{1/2}$. According to Lemma 4, the set of points in B_1 with multiplicity more than $\gamma \log(r_1^{-1}) + \lambda$ has volume bounded by $e^{-\alpha\lambda}|B_1|$. Adding up the contributions from the regions of different multiplicity, we see that $|\phi(B_1)| < CV(1)r_1^{n+2}$. Now, by Lemma 3, it follows that $r_1(F) < Ce^{-\beta d(F)}$. Plugging in, we get the lemma.

To prove the last claim, we apply the same argument, using the bound for $|M(\lambda)|$ in Lemma 4. □

We should make some remarks about this inequality. In our paper, it turns out to be natural to compare $|\phi(M) \cap \mathrm{Star}(F)|$ with $r_1(F)^n$. According to our inequality, the ratio $|\phi(M) \cap \mathrm{Star}(F)|/r_1(F)^n$ is at most $CV(1)r_1(F)e^{-\beta d(F)}$. In other words, the ratio becomes favorable if $V(1)$ is small, or if $r_1(F)$ is small, or if $d(F)$ is large.

To finish this section, we explain the connections between the rectangular nerve and the filling radius and simplicial volume of M.

Lemma 6. *If* $\phi_*([M]) = 0$ *in* N, *then the filling radius of* (M^n, g) *is at most* 1.

Proof. Let A be a chain in N filling $\phi(M)$. The filling A consists of the following data: an abstract chain A with $\partial A = M$, together with a map $f : A \to N$ with $f|_{\partial A} = \phi$. By abuse of notation, we identify M with its image in $L^\infty(M)$ under the Kuratowski embedding. Using the above data, we will construct a filling of M in its 1-neighborhood.

First, we construct a map ψ from A to $L^\infty(M)$. We pick a fine triangulation of A, subordinate to the faces of N. We begin by defining ψ on the vertices of the triangulation. Each vertex v lies in some face F of the nerve N. We pick a ball B_i so that the index i lies in $I_+(F)$. Then we define $\psi(v)$ to be the point p_i which is the center of the ball B_i. Next we define ψ on each simplex by extending it linearly.

Because our triangulation of A is subordinate to the faces of N, each edge of A joins two points that lie in a common closed face of N. Therefore, ψ maps the endpoints of any edge to the centers of two overlapping good balls. The distance between the centers is at most $(2/100)$. Each simplex of A is mapped to a simplex in $L^\infty(M)$ whose edges have length at most $2/100$. Also, each vertex is mapped into $M \subset L^\infty(M)$. Therefore, the image $\psi(A)$ lies in the $2/100$ neighborhood of M.

We are not finished, because the map ψ restricted to M is not the Kuratowksi embedding. To finish the proof, we will homotope M to $\psi(M)$ inside of the $2/100$-neighborhood of M. Let p be a point of M. Let Δ be the smallest simplex of our triangulation that contains p, and let $y_1, \ldots, y_m$ be the vertices of Δ. The map ψ sends each y_i to the center p_i of some ball B_i containing y_i. The map ψ sends p to a point on the linear simplex in L^∞ spanned by the points p_i. The distance from p to $\psi(p)$ is at most the largest distance from p to any of the p_i. Since each triangle is small, we can assume that the distance from p to y_i is less than $1/100$. Also, since each good ball has radius at most $1/100$, the distance from y_i to p_i is at most $1/100$. Combining these inequalities, the distance from p to $\psi(p)$ is at most $2/100$. Therefore, the Kuratowski embedding can be homotoped to the map ψ inside the $2/100$ neighborhood of M in $L^\infty(M)$.

Combining this homotopy with the chain $\psi(A)$, it follows that M bounds inside its $2/100$ neighborhood. In other words, the filling radius of (M, g) is at most $2/100 < 1$. □

Lemma 7. *If* (M, g) *is a closed aspherical manifold with systole at least* 1, *then there is a map* $\psi : N \to M$ *so that the composition* $\psi \circ \phi : M \to M$ *is homotopic to the identity.*

Proof. This proof is essentially the same as Gromov's from [4, pp. 85–86], which uses the standard simplicial nerve instead of the rectangular nerve.

We slightly homotope ϕ to a map ϕ' which is simplicial with respect to some fine triangulations of M and N. We can assume that the triangulation of N is subordinate to the faces of N.

We begin by defining the map ψ from N to M. We define the map one skeleton at a time. For each vertex v of N, we consider the smallest face $F \supset v$, and we pick an index in $I_+(F)$. Then we map v to the center of B_i. Now each edge E of N joins two vertices lying in the same closed face. If the boundary of E is $v_1 \cup v_2$, then it follows that we have mapped v_1 and v_2 to two overlapping balls from our covering. Since each ball has radius at most $(1/100)$, the distance between the centers is at most $(2/100)$, and we may map E to an arc of length at most $(2/100)$. Now the boundary of each 2-simplex has been mapped to an arc of length at most $(6/100)$. Since the 1-systole of (M, g) is at least 1, the image curve is contractible, and so we can extend our map to each 2-simplex. Since M is aspherical, we can then extend the map to each higher-dimensional simplex. This completes the construction of ψ.

Next we have to show that $\psi \circ \phi'$ is homotopic to the identity. We have to define a map H on $M \times [0, 1]$ with $H(m, 0) = \psi \circ \phi'(m)$ and $H(m, 1) = m$. We define H one skeleton at a time. For each vertex v, $\phi'(v)$ is a vertex of the triangulation of N lying very near to $\phi(v)$. We let $F(v)$ denote the smallest face of N containing $\phi'(v)$. It may not be the case that $\phi(v)$ lies in $F(v)$, but at least $\phi(v)$ lies in a face bordering $F(v)$. Therefore, $\psi \circ \phi'(v)$ is the center of some ball B_i overlapping some other ball B_j containing v. Since each ball in our cover has radius at most $(1/100)$, the distance from $\psi \circ \phi'(v)$ to v is at most $(3/100)$. We define H on $v \times (0, 1)$ by mapping the interval to a curve from $\psi \circ \phi'(v)$ to v, with length at most $(3/100)$.

Next we look at an edge E of the triangulation of M. The map ϕ' either collapses E to a point or maps it onto an edge of the triangulation of N. Therefore, $\psi \circ \phi'(E)$ is either a point or an arc of length at most $(2/100)$. We have already defined H on the boundary of $E \times (0, 1)$. The restriction of H to the boundary is a curve of length at most $(8/100)$ plus the length of E. We can assume the length of E is at most $(1/100)$. Since the 1-systole of (M, g) is at least 1, this curve is contractible, and so we can extend H to $E \times (0, 1)$ for every edge E.

Finally, since M is aspherical, we can extend H to $\Delta \times (0, 1)$ for each 2-simplex Δ of M, and then for each higher-dimensional simplex. Therefore, $\psi \circ \phi'$ is homotopic to the identity. Since ϕ' is homotopic to ϕ, $\psi \circ \phi$ is homotopic to the identity. □

4. **Filling cycles in the rectangular nerve**

Using the bounds proved in the last section, we will now show that if $V(1)$ is sufficiently small, then $\phi_*([M]) = 0$ in the rectangular nerve N.

Lemma 8. *For any $\beta > 0$ and any integer $n > 0$, there is a small positive $\varepsilon(\beta, n)$ that makes the following statement true. Let z be an n-cycle in the rectangular complex N. Suppose that for every face F in N,*

$$|z \cap \mathrm{Star}(F)| < \varepsilon r_1(F)^n e^{-\beta d(F)}.$$

Then $[z] = 0$ in N.

Proof. Let D be the dimension of N. We will construct a sequence of homologous cycles $z = z_D \sim z_{D-1} \sim \cdots \sim z_n$. The cycle z_k will be contained in the k-skeleton of N. Moreover, every cycle will obey the following estimate, slightly weaker than the estimate that z obeys:

$$|z_k \cap \mathrm{Star}(F)| < 2\varepsilon r_1(F)^n e^{-\beta d(F)}.$$

In particular, for each n-face F, the cycle z_n obeys the following estimate:

$$|z_n \cap F| < 2\varepsilon r_1(F)^n.$$

Our constant ε will be less than $1/2$, so we conclude that $|z_n \cap F| < |F|$. Therefore, z_n is homologous to a cycle lying in the $(n-1)$-skeleton of N, and hence $[z_n] = 0$.

Now we describe the inductive step, getting from z_k to z_{k-1}. Let F be a k-dimensional face of N. Consider $z_k \cap F$, which defines a relative cycle in F, and we replace it with the minimal relative cycle with the same boundary. Performing this surgery on each k-face F, we get a new n-cycle z_k', homologous to z_k, and still contained in the k-skeleton of N.

We examine the intersection $z_k' \cap F$ for a k-dimensional face F. Since z_k' was chosen to minimize volume, it follows that $|z_k' \cap F| \leq |z_k \cap F|$. By the inductive hypothesis, $|z_k \cap F| \leq 2\varepsilon r_1(F)^n e^{-\beta k}$. Using this volume estimate, we can show that z_k' lies near to the boundary of F. Suppose that $x \in z_k'$ and that the distance from x to ∂F is s. By the monotonicity formula, it follows that $\omega_n s^n \leq |z_k' \cap F| \leq 2\varepsilon r_1(F)^n e^{-\beta k}$. Rearranging this formula, we get the following inequality, bounding the distance from any point in z_k' to the boundary ∂F:

$$s/r_1(F) \leq [2\omega_n^{-1} \varepsilon e^{-\beta k}]^{1/n}. \tag{1}$$

Following Gromov in [5], we define a map that pulls a small neighborhood of the $(k-1)$-skeleton of N into the $(k-1)$-skeleton. Our map will be called R_δ, and it depends on a number δ in the range $0 < \delta < 1/2$. The basic map is a map from an interval $[0, r]$ to itself, which takes the set $[0, \delta r]$ to 0, and the set $[r - \delta r, r]$ to r and linearly stretches the set $[\delta r, r - \delta r]$ to cover $[0, r]$. The Lipschitz constant of this map is $(1-2\delta)^{-1}$. Now we apply this map separately to each coordinate ϕ_i of the big rectangle R. The resulting map is R_δ.

The map R_δ has the following nice properties. It maps the nerve N into itself. The map R_0 is the identity, and so each R_δ is homotopic to the identity (in the space of self-maps of N). Therefore, the map R_δ moves any cycle to a homologous cycle. The preimage $R_\delta^{-1}[\mathrm{Star}(F)] = \mathrm{Star}(F)$ for any face F. Since the Lipschitz constant of R_δ is $[1-2\delta]^{-1}$, the following estimate holds for any n-cycle y:

$$|R_\delta(y) \cap \mathrm{Star}(F)| \leq (1-2\delta)^{-n}|y \cap \mathrm{Star}(F)|. \tag{2}$$

Also, for sufficiently big δ, the map R_δ takes z'_k into the $(k-1)$-skeleton of N. In particular, if $\delta \geq [2\omega_n^{-1}\varepsilon e^{-\beta k}]^{1/n}$, then inequality (1) guarantees that $R_\delta(z'_k)$ lies in the $(k-1)$-skeleton of N. We define $\delta(k) = [2\omega_n^{-1}\varepsilon e^{-\beta k}]^{1/n}$, and then we define $z_{k-1} = R_{\delta(k)}(z'_k)$.

We have to check that z_{k-1} obeys the volume estimate in the inductive hypothesis. Let F be a face of N with any dimension. First we claim that it obeys the following estimate:

$$|z_{k-1} \cap \mathrm{Star}(F)| \leq \prod_{l=k}^{D}(1-2\delta(l))^{-n}\varepsilon r_1(F)^n e^{-\beta d(F)}.$$

This estimate follows from three observations. First, by hypothesis,

$$|z_D \cap \mathrm{Star}(F)| \leq \varepsilon r_1(F)^n e^{-\beta d(F)}.$$

Second, $|z'_k \cap \mathrm{Star}(F)| \leq |z_k \cap \mathrm{Star}(F)|$. Third, by equation (2),

$$|z_{k-1} \cap \mathrm{Star}(F)| \leq (1-2\delta(k))^{-n}|z'_k \cap \mathrm{Star}(F)|.$$

So to make the induction work, we have to choose ε sufficiently small that the following estimate holds:

$$\prod_{l=n+1}^{\infty}(1-2\delta(l))^{-n} = \prod_{l=n+1}^{\infty}(1-2[2\omega_n^{-1}\varepsilon e^{-\beta l}]^{1/n})^{-n} < 2.$$

The product converges because of the exponential decay in the term $e^{-\beta l}$, and by taking $\varepsilon > 0$ sufficiently small, we can guarantee that it is less than 2. The value of ε here depends on n and β. □

We now have enough ammunition to prove Theorem 1 for closed manifolds.

THEOREM 1 (closed case). *For each dimension n, there is a number $\delta(n) > 0$ so that the following estimate holds. If (M^n, g) is a closed Riemannian n-manifold with filling radius greater than R, then $V(R) \geq \delta(n)R^n$.*

Proof. By scaling, it suffices to prove the theorem when $R = 1$. We consider the map ϕ from M to the rectangular nerve N of a good cover. According to Lemma 5, the image obeys the following estimate for each face F of N:

$$|\phi(M) \cap \mathrm{Star}(F)| < CV(1)r_1(F)^{n+1}e^{-\beta d(F)}.$$

The constants C and β in this equation depend only on n. Let $\varepsilon(\beta, n)$ be the number defined in Lemma 8. Hence there is some number $\delta(n)$ so that if $V(1) < \delta$, then we get the following estimate for each face F of N:

$$|\phi(M) \cap \mathrm{Star}(F)| < \varepsilon r_1(F)^n e^{-\beta d(F)}.$$

According to Lemma 8, this estimate implies that the cycle $\phi(M)$ is homologous to zero in N. Now, according to Lemma 6, the filling radius of (M, g) is at most 1. □

We now prove two of the corollaries from the introduction.

Corollary 1. *Let (M^n, g) be a closed Riemannian manifold. Suppose that there is a degree nonzero map F from (M^n, g) to the unit n-sphere with Lipschitz constant* 1. *Then $V(R) \geq \delta(n)R^n$ for $R \leq 1$.*

Proof. In [5, p. 8], Gromov proved that the filling radius of (M, g) is at least the filling radius of the unit n-sphere. Theorem 1 then implies that $V(R) > \delta(n)R^n$ for $R \leq 1$. □

Corollary 2 (systolic inequality). *Let (M^n, g) be a closed aspherical Riemannian manifold. Suppose the shortest noncontractible curve in (M^n, g) has length at least S. Then $V(S) \geq \delta(n)S^n$.*

Proof. In [5, §1], Gromov proved that the filling radius of (M, g) is at least $S/6$. According to Theorem 1, $V(S/6) \geq \delta(n)(S/6)^n$. For a smaller constant $\delta(n)$, $V(S) \geq V(S/6) \geq \delta(n)S^n$. □

5. Estimates for simplicial norms

In this section, we consider the consequences of a weaker upper bound on $V(1)$, such as $V(1) < 10\omega_n$. In this case, it does not follow that $\phi_*([M]) = 0$ in N. Instead, we get an upper bound for the simplicial norm of $\phi_*([M])$ in $H_n(N)$.

At this point, we recall the relevant facts about simplicial norms. For more information, see [4]. Suppose that C is a rational k-cycle in M. We can write C as a finite sum $\sum a_i \Delta_i$, where a_i is a rational number and Δ_i is a map from the k-simplex to M. We say that the size of C is equal to the sum $\sum |a_i|$. The size of C is just the number of simplices counted with multiplicity. Then we define the simplicial norm of a homology class $h \in H(M, \mathbb{Q})$ to be the infimal size of any rational cycle C in the class h. We will write the simplicial norm of h as $\|h\|$. For a closed oriented manifold M, the simplicial volume of M is defined to be the simplicial norm of the fundamental class $[M]$.

We will use two facts about the simplicial norm. The first fact is that it decreases under any mapping. In other words, if $\phi : M \to N$ is a continuous map between spaces, then $\|\phi_*(h)\| \leq \|h\|$. This property follows immediately from the definition. The second fact is that the simplicial volume of a closed oriented hyperbolic manifold is bounded below by the volume of the manifold. We state this result as a theorem.

THEOREM (Thurston; see [4]). *Suppose that* (M, hyp) *is a closed hyperbolic n-manifold. Then the simplicial volume of M is at least* $c(n)\mathrm{Vol}(M, \mathrm{hyp})$.

(In fact, the simplicial volume of M is equal to $c(n)\mathrm{Vol}(M, \mathrm{hyp})$ for an appropriate constant $c(n)$, but we do not need this fact.)

Now we suppose that (M^n, g) is a closed orientable Riemannian manifold. Let N be the rectangular nerve constructed from a good cover of (M, g), and let $\phi : M \to N$ be the map to the rectangular nerve.

LEMMA 9. *For each number* $V_0 > 0$ *and each dimension* n, *there is a constant* $C(V_0, n)$ *so that the following estimate holds. If* (M, g) *has* $V(1) < V_0$, *then the simplicial norm* $\|\phi_*([M])\| \leq C(V_0, n)\mathrm{Vol}(M, g)$.

Proof. The proof of this lemma is a modification of the proof of Lemma 8. According to Lemma 5, we have the following bound for the volume of $\phi(M)$ in various regions of N:

$$\text{(A)} \qquad |\phi(M) \cap \mathrm{Star}(F)| < C_1 V(1) r_1(F)^{n+1} e^{-\beta d(F)}.$$

In this formula, C_1 and β are dimensional constants, and F can be any face of the rectangular nerve N. We let $\varepsilon = \varepsilon(\beta, n)$ be the same constant as in the proof of Lemma 8. We now divide the faces of N into thick and thin faces as follows. If $C_1 V_0 r_1(F) < \varepsilon$, then we say that F is thin, and otherwise we say that F is thick.

We begin with some simple estimates about the thick and thin simplices. If F is thin, then any higher-dimensional face containing F in its boundary is also thin. Also, the dimension of a thick face is bounded by $d(V_0, n)$, a constant depending only on n and V_0. This estimate on the dimension follows from Lemma 3, because if $I_{(0,1)}(F)$ contains $d(F)$ indices, then the corresponding $d(F)$ balls contain a common intersection. If B_1 is the smallest of these balls, Lemma 3 guarantees that $r_1 < C[\log d(F)]^{-1}$. But by definition of a thick face, $r_1(F) \geq c\varepsilon/V_0$.

As before, we let D be the dimension of N, and we define z_D to be the n-cycle $\phi(M)$. We will construct a sequence of homologous n-cycles $z_D \sim z_{D-1} \sim \cdots \sim z_n$, with z_k lying in the k-skeleton of N. For any thin face F, the cycle z_k will obey the same estimate as in Lemma 8:

$$|z_k \cap \mathrm{Star}(F)| < 2\varepsilon r_1(F)^n e^{-\beta d(F)}.$$

The construction is similar to the one in Lemma 8, but there is an extra wrinkle having to do with the thick simplices. First, we show that z_D obeys the estimate that we want. Because of the definition of thin faces and the estimate in equation (A), $|z_D \cap \mathrm{Star}(F)| < \varepsilon r_1(F)^n e^{-\beta d(F)}$ for each thin face F.

Now we suppose we have constructed z_k for some $k > n$, and we describe the construction of z_{k-1}. Pick a k-dimensional face F. If F is thin, then we define $z'_k \cap F$ to be a minimal cycle with boundary $\partial(z_k \cap F) \subset \partial F$. As in the proof of Lemma 8, $z'_k \cap F$ lies within the $s(F)$-neighborhood of ∂F, where $s(F) = r_1(F)[2\omega_n^{-1}\varepsilon e^{-\beta k}]^{1/n}$. If F is a thick face, then we define z'_k by removing $z_k \cap F$ and replacing it by a chain in ∂F with the same boundary as $z_k \cap F$. According to the deformation theorem of Federer-Fleming, we can choose a chain with volume bounded by $G(V_0, n)|z_k \cap F|$. (The constant in the Federer-Fleming construction depends on the dimension $d(F)$. We noted above that for a thick face $d(F)$ is bounded by $d(V_0, n)$. Also, the Federer-Fleming construction gives a certain constant if we do it in a cube. We have to do it in a rectangular face. If the dimensions of the rectangular face are very uneven, then the constant can blow up. In our case, though, r_1 is bounded below by a constant depending only on n and V_0, and all the dimensions are bounded above by $(1/100)$. Therefore, the stretching factor $G(V_0, n)$ depends only on the dimension n and V_0.)

We proceed as in the proof of Lemma 8. We define $\delta(k)$ as $[2\omega_n^{-1}\varepsilon e^{-\beta k}]^{1/n}$, and we define z_{k-1} as $R_{\delta(k)}(z'_k)$. The cycle z_{k-1} is homologous to z_k and lies in the $(k-1)$-skeleton of N. By the same calculation as in Lemma 8, it follows that z_{k-1} obeys the volume estimate for thin faces F: $|z_{k-1} \cap \mathrm{Star}(F)| < 2\varepsilon r_1(F)^n e^{-\beta d(F)}$.

To get at the simplicial norm, we consider the cycle z_n. It lies in the n-skeleton of N. The cycle z_n is homologous to a sum of n-faces of F, $\sum_i c_i F_i$, where $|c_i| \le |z_n \cap F_i|/|F_i|$. Taking the barycentric triangulation of each face, it follows that the simplicial norm of z_n is bounded by $C \sum_i |c_i|$. If F_i is a thin n-face, then the bound $|z_n \cap \mathrm{Star}(F_i)| < 2\varepsilon r_1(F_i)^n$ guarantees that $c_i = 0$. For thick faces, c_i may be nonzero. If F_i is a thick n-face, then the volume $|F_i|$ is bounded below, and so it follows that $\sum |c_i| < C(V_0, n)|z_n|$.

To bound $|z_n|$, we consider the increase of volume $|z_k|/|z_{k+1}|$. We form z_k from z_{k+1} by a surgery in each (k+1)-face — yielding z'_{k+1}, followed by applying $R_{\delta(k)}$. For $k > d(V_0, n)$, all the surgeries occur in thin faces, and therefore each surgery decreases the volume of the cycle. The application of $R_{\delta(k)}$ increases volume by at most a factor $[1 - 2\delta(k)]^{-n}$. Therefore,

$$|z_{d(V_0,n)}| < \prod_{k=d(V_0,n)}^{\infty} [1 - 2\delta(k)]^{-n} |z_D|.$$

By the same calculation as in Lemma 8, the last expression is bounded by $2|z_D|$.

If $k < d(V_0, n)$, then we have the following much weaker volume bound:

$$|z_{k-1}| \leq G(V_0, n)[1 - 2\delta(k)]^{-n}|z_k|.$$

Therefore, the volume $|z_n| < 4G(V_0, n)^{d(V_0,n)}|z_D|$. Finally, by Lemma 4, the volume $|z_D|$ is bounded by $C(n)|(M, g)|$. Assembling all the inequalities, we see that the simplicial norm of $\phi_*([M])$ is bounded by $C(V_0, n)|(M, g)|$. $\square$

Now we have enough ammunition to prove our second theorem.

THEOREM 2. *For each dimension n, there is a number $\delta(n) > 0$ so that the following estimate holds. Suppose that (M^n, hyp) is a closed hyperbolic n-manifold and that g is another metric on M, and suppose that* $\text{Vol}(M, g) < \delta(n)\text{Vol}(M, \text{hyp})$. *Let $(\widetilde{M}, \tilde{g})$ denote the universal cover of M with the metric induced from g. Then the following inequality holds*:

$$V_{(\widetilde{M},\tilde{g})}(1) > V_{\mathbb{H}^n}(1).$$

Proof. First we consider the special case that M is oriented and that each noncontractible curve in (M, g) has length at least 1. In this case, $V_{(\widetilde{M},\tilde{g})}(1) = V_{(M,g)}(1)$. We will assume that $V_{(M,g)}(1)$ is at most $V_{\mathbb{H}^n}(1) = V_0$, and we need to prove that $\text{Vol}(M, g) \geq \delta(n)\text{Vol}(M, \text{hyp})$.

As usual, we choose a good cover of (M, g). We let N be the rectangular nerve of the cover and let $\phi : M \to N$ be the map to the rectangular nerve constructed in Section 3. According to Lemma 9, $C(n, V_0)\text{Vol}(M, g) \geq \|\phi_*([M])\|$. We have assumed that the shortest noncontractible curve in (M, g) has length at least 1. According to Lemma 7, there is a map $\psi : N \to M$ so that $\psi \circ \phi : M \to M$ is homotopic to the identity. In particular, $\psi_*(\phi_*([M])) = [M]$. Since the simplicial volume decreases under maps, it follows that $\|\phi_*([M])\|$ is equal to $\|[M]\|$, the simplicial volume of M. Finally, since M is closed and oriented, Thurston's theorem guarantees that the simplicial volume of M is at least $c(n)\text{Vol}(M, \text{hyp})$. Putting together these inequalities, we see that $\text{Vol}(M, g) \geq C(n, V_0)^{-1}c(n)\text{Vol}(M, \text{hyp})$. Since V_0, the volume of the unit ball in $\mathbb{H}^n$ is itself a dimensional constant, we see that $\text{Vol}(M, g) \geq \delta(n)\text{Vol}(M, \text{hyp})$ as desired.

Next we consider the general case, with no restriction on the lengths of noncontractible curves in (M, g). Again, we assume that $V_{(\widetilde{M},\tilde{g})}(1) \leq V_0$, and we have to prove that $\text{Vol}(M, g) \geq \delta(n)\text{Vol}(M, \text{hyp})$. Since M admits a hyperbolic metric, the fundamental group of M is residually finite. (The group of isometries of hyperbolic n-space is a subgroup of $\text{SL}(N, \mathbb{C})$ for sufficiently large N, and any finitely generated subgroup of $\text{SL}(N, \mathbb{C})$ is residually finite according to [12].) Therefore, we can choose a finite cover $(\widehat{M}, \hat{g})$ so that $\widehat{M}$ is oriented and so that every noncontractible closed curve in $\widehat{M}$ has length at least 1. Let $\widehat{\text{hyp}}$ be the pullback of the hyperbolic metric on M to $\widehat{M}$. By assumption, $V_{(\widetilde{M},\tilde{g})}(1) \le V_0$. Since the universal cover of $(\widehat{M},\hat{g})$ is the same as that of (M,g), it follows that $V_{(\widehat{M},\hat{g})}(1) \le V_0$. Now by the first case, we can conclude that $\mathrm{Vol}(\hat{M},\hat{g}) \ge \delta(n)\mathrm{Vol}(\widehat{M},\widehat{\mathrm{hyp}})$.

Now if the covering map, $\pi : \widehat{M} \to M$ has degree D, then

$$\mathrm{Vol}(M,g) = (1/D)\mathrm{Vol}(\widehat{M},\hat{g})$$

and $\mathrm{Vol}(M,\mathrm{hyp}) = (1/D)\mathrm{Vol}(\widehat{M},\widehat{\mathrm{hyp}})$. Therefore, the last inequality implies that $\mathrm{Vol}(M,g) \ge \delta(n)\mathrm{Vol}(M,\mathrm{hyp})$. $\square$

Remark. There is a slightly more general result that holds with the same proof. It applies to products of hyperbolic manifolds. Suppose that M^n is a product of closed manifolds, $M = M_1 \times \cdots \times M_d$, and that each manifold M_i admits a hyperbolic metric hyp_i. Let prod denote the product metric $\mathrm{hyp}_1 \times \cdots \times \mathrm{hyp}_d$. If (M,g) has volume less than $\delta(n)\mathrm{Vol}(M,\mathrm{prod})$, then $V_{(\widetilde{M},\tilde{g})}(1) \ge V_{(\widetilde{M},\widetilde{\mathrm{prod}}))}(1)$.

6. **Open manifolds**

So far, we have proved Theorem 1 for closed manifolds. Theorem 1 also holds for all complete Riemannian manifolds. In this section, we deal with the general case. It requires only minor technical modifications from the closed case. We encourage the reader not to take this section too seriously.

First, we review the definition of the filling radius of a complete manifold. The original definition appears on page 41 of [5]. Let (M^n,g) be a complete Riemannian manifold. The Kuratowski embedding maps a point $x \in M$ to the function dist_x. Since M may not be compact, this function is unbounded. Nevertheless, it defines a measurable function, and the triangle inequality implies that $|\mathrm{dist}_x - \mathrm{dist}_y|_\infty = \mathrm{dist}(x,y)$. The image of the Kuratowski embedding lies in an affine copy of $L^\infty(M)$, namely all functions of the form $\mathrm{dist}_x + f$, where $f \in L^\infty(M)$.

Since (M,g) is complete, any ball of finite radius is compact. The Kuratowski embedding is an isometry, and so the inverse image of any compact set lies in a ball of finite radius and is compact. In other words, the Kuratowski embedding is proper. Therefore, the image of M defines a cycle in the sense of locally-finite homology theory. The filling radius of (M,g) is the infimal R so that this cycle bounds a locally finite chain inside its R-neighborhood. (If the cycle does not bound within its R-neighborhood for every finite R, then the filling radius is infinite.)

Most of the lemmas apply smoothly to complete manifolds with this definition, but a couple of them require some minor discussion.

Lemma 1 is local and applies immediately on a complete manifold.

Lemma 2 also holds on a complete manifold, but the proof requires a minor trick. Let $K_1 \subset K_2 \subset \dots$ be an exhaustion of M. Consider the set of all balls $B(p, R/6)$ such that $p \in K_i$ and $B(p, R)$ is a good ball. The balls $B(p, R/6)$ cover K_i, and we can find a finite subset of them that covers K_i. Applying the Vitali covering lemma, we get a good cover of K_i. Repeating this procedure for each i, we get a sequence of covers including more and more of M. If we restrict attention to the balls meeting a given compact set $K \subset M$ the set of possible covers is compact, which we can check as follows. Clearly, the set of possible centers $p \in K$ is compact. We can find a radius r so that any ball of radius less than r centered in K has volume at least $(1/2)\omega_n r^n$. Therefore, every good ball has radius at least r and at most $1/100$. Also, every good ball has volume at least $(1/2)\omega_n r^n$. Therefore, the number of balls meeting K is bounded. We can therefore choose a subsequence so that the ball coverings restricted to K converge. Now, we again consider an exhaustion of M by compact sets and diagonalize to give a sequence of good coverings of K_i that converge on all of M. Their limit is our good covering.

Lemma 3 is local and applies immediately on a complete manifold. As long as U is bounded, Lemma 4 is local, and in this case it applies on a complete manifold. Lemma 5 follows immediately from Lemma 4. It also holds on a complete manifold, except possibly for the last estimate of the volume of $\phi(M)$. This estimate is not used in the proof of Theorem 1 anyway. Lemma 6 follows for complete manifolds with the same proof. Lemma 7 is not part of the proof of Theorem 1.

The most annoying technical problem occurs in Lemma 8. The problem occurs because the nerve N may contain rectangles of every dimension. Any given point will lie in only finitely many balls, but as the point goes to infinity this number may blow up.

In the original proof of Lemma 8, we had a cycle z in N, and we built a sequence of cycles $z = z_D \sim z_{D-1} \sim \dots \sim z_n$, where D was the dimension of N and z_k lay in the k-skeleton of N. In general, the dimension of N is not finite, but is only locally finite, and we must proceed a little differently. Instead, we construct an infinite sequence of cycles, $\dots \sim z_{k+1} \sim z_k \sim z_{k-1} \sim \dots \sim z_n$, with z_k lying in the k-skeleton of N, so that the sequence z_k converges to z as k tends to infinity.

In a region of N where the dimension is less than k, we define z_k to be the infinite composition $R_{\delta(k+1)} \circ R_{\delta(k+2)} \circ \dots$ applied to z. (This infinite composition is defined to be the limit of the maps $R_{\delta(k+1)} \circ \dots \circ R_{\delta(N)}$ as N goes to infinity. The sequence of maps converges uniformly on compact sets.) In a region where the dimension of N is at least k, we define z_k from z_{k+1} as in the proof of Lemma 8. Every cycle z_k is homologous to z by a locally finite chain in N, and the rest of the argument in Lemma 8 applies as before.

With these modifications, we get the proof of Theorem 1 in the general case.

THEOREM 1 (general case). *For each dimension n, there is a number $\delta(n) > 0$ so that the following estimate holds: If (M^n, g) is a complete Riemannian n-manifold with filling radius at least R, then $V(R) \geq \delta(n)R^n$.*

Finally, we prove a corollary about universal covers.

COROLLARY 3. *Let (M^n, g) be a closed aspherical Riemannian manifold, and let $V(R)$ measure the volumes of balls in the universal cover $(\widetilde{M}, \tilde{g})$. Then $V(R) \geq \delta(n)R^n$ for all R.*

Proof. In [5, p. 43], Gromov proved that the universal cover of M has infinite filling radius. Applying Theorem 1, we get the corollary. □

7. A question about Uryson width

We say that the Ursyon k-width of a metric space X is at most W if there is a continuous map π from X to a k-dimensional polyhedron whose fibers have diameter at most W. The Uryson width is another way of measuring how "thick" a manifold is, in a similar spirit to the filling radius. Gromov proved in [5] that Uryson $(n-1)$-width of a Riemannian manifold (M^n, g) controls its filling radius. It is an open problem to understand how the volume of a Riemannian manifold constrains its Uryson width.

Question. Is there a dimensional constant $C(n)$ so that every closed Riemannian manifold (M^n, g) has Uryson $(n-1)$-width at most $C(n)\mathrm{Vol}(M, g)^{1/n}$?

This question is analogous to Gromov's estimate for the filling radius. There is another question, analogous to Theorem 1.

Question. Is there a dimensional constant $c(n)$ so that every closed Riemannian manifold (M^n, g) with Uryson $(n-1)$-width at least W has $V(W) \geq c(n)W^n$?

An affirmative answer to the second question is stronger than an affirmative answer to the first question.

It looks plausibe that our proof of Theorem 1 can be modified to bound the Uryson $(n-1)$-width of M instead of its filling radius, giving an affirmative answer to the second question. To bound the Uryson width, we would modify the proof of Lemma 8. For each k-face F of N, we make $z_k' \cap F$ the image of z_k under a MAP that fixes the boundary $z_k \cap \partial F$ and minimizes volume subject to the boundary restriction. I believe that z_k' should be a minimal cycle plus a measure 0 region that can be pushed as close to ∂F as the minimal cycle piece. Then our argument gives a sequence of homotopies of the map ϕ, ending with a map π into the $(n-1)$-skeleton of N. If $\pi(x)$ belongs to a face F, then $\phi(x)$ must have belonged to $\mathrm{Star}(F)$. Therefore, if $I_+(F)$ contains an index i, then $\pi^{-1}(F) \subset \phi^{-1}[\mathrm{Star}(F)]$ lies in B_i. Since each ball in our cover has radius at most $(1/100)$, it would follow that the Uryson $(n-1)$-width of (M, g) is at most $(2/100)$.

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