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## Abstract

In this paper we solve the following problem in the affirmative: Let  $Z$  be a continuum in the plane  $\mathbb{C}$  and suppose that  $h : Z \times [0, 1] \rightarrow \mathbb{C}$  is an isotopy starting at the identity. Can  $h$  be extended to an isotopy of the plane? We will provide a new characterization of an accessible point in a planar continuum  $Z$  and use it to show that accessibility of a point is preserved during the isotopy. We show next that the isotopy can be extended over small hyperbolic crosscuts which are shown to remain small under the isotopy. The proof makes use of the notion of a metric external ray, which mimics the notion of a conformal external ray, but is easier to control during an isotopy. It also relies on the existence of a partition of a hyperbolic, simply connected domain  $U$  in the sphere, into hyperbolically convex subsets, which have limited distortion under conformal maps to the unit disk.

## 1. Introduction

Denote the complex plane by  $\mathbb{C}$ , the origin by  $O$ , the open unit disk by  $\mathbb{D}$  and the complex sphere by  $\mathbb{C}^* = \mathbb{C} \cup \{\infty\}$ . Suppose that  $h : Z \times [0, 1] \rightarrow \mathbb{C}$  is an isotopy of a continuum  $Z \subset \mathbb{C}$  such that if we denote  $h^t = h|_{Z \times \{t\}}$ , then  $h^0 = \text{id}_Z$ . We consider the old problem whether the isotopy  $h$  can be extended to an isotopy of the plane.<sup>1</sup>

A more restrictive form of an isotopy is the notion of a holomorphic motion. Given a set  $A \subset \mathbb{C}$ , a holomorphic motion is a function  $f : A \times \mathbb{D} \rightarrow \mathbb{C}^*$  such that:

- (1) for each  $\lambda \in \mathbb{D}$  the map  $f^\lambda = f|_{A \times \{\lambda\}} : A \rightarrow \mathbb{C}^*$  is one-to-one,
- (2)  $f^O = \text{id}_A$ ,
- (3) for each  $a \in A$ , the map  $f_a = f|_{\{a\} \times \mathbb{D}} : \mathbb{D} \rightarrow \mathbb{C}^*$  is holomorphic.

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<sup>1</sup>We are indebted to Professor R. D. Edwards who communicated this problem to us.

Note that the function  $f$  is not initially required to be continuous in  $a$  or in the pair  $(a, \lambda)$ . The remarkable  $\lambda$ -Lemma states that each holomorphic motion can be extended to a holomorphic motion  $F : \mathbb{C}^* \times \mathbb{D} \rightarrow \mathbb{C}^*$  of the entire sphere. Moreover, the map  $F$  is continuous and the maps  $F^\lambda$  are quasi-conformal for each  $\lambda$ . Partial results of this type are due to: Bers, Lyubich, Mañé, Royden, Sad, Sullivan and Thurston [BR86], [Lyu83], [MSS83], [ST86]. The full result is due to Ślodkowski [Slo91]; see [AM01] for a different proof of the above statement and some history of the problem.

Although the  $\lambda$ -Lemma holds for arbitrary (in particular not connected) sets  $A$ , easy examples show that an isotopy of a convergent sequence cannot necessarily be extended over the plane (see [Fab05, p. 991]). It is also easy to see that our main result cannot be generalized to higher dimensions because wild balls and spheres can be isotoped to tame balls and spheres. However, see [EK71] for related positive results in higher dimensions.

It follows from Rado's theorem [Wen92, Th. 4.2] that the isotopy  $h$  in the first paragraph can be extended to an isotopy of  $\mathbb{C}$  if  $Z$  is a simple closed curve (see [Bae27], [Bae28] for related results and [Eps66] for a generalization). Analytic techniques, in particular, boundary values of conformal maps have been powerful tools for studying plane continua. However, they appear insufficient to answer the general question. Suppose that  $U_n$  is a sequence of proper, simply connected domains in  $\mathbb{C}$  and  $w_0 \in U_n$  for all  $n$ . Then we say that the domains  $U_n$  converge to  $U$  in the sense of Carathéodory kernel convergence (with respect to  $w_0$ ) [Pom92, p. 13], denoted by  $(U_n, w_0) \rightarrow (U, w_0)$ , if:

- (1)  $U = \{w_0\}$ , or  $U$  is a simply connected, proper domain in  $\mathbb{C}$  such that for each  $w \in U$  there exists a neighborhood  $V_w$  of  $w$  with  $V_w \subset U_n$  for all  $n$  sufficiently large,
- (2) for each  $w \in \partial U$  there exist  $w_n \in \partial U_n$  such that  $\lim w_n = w$ .

Note that the limit of a sequence of domains depends on the choice of the point  $w_0$ . The following theorem is due to Carathéodory [Car12].

**THEOREM 1.1** ([Pom92, Th. 18.8]). *Let  $\varphi_n : \mathbb{D} \rightarrow U_n$  be conformal maps onto simply connected domains  $U_n$  such that  $\varphi_n(O) = w_0$  and  $\varphi'_n(O) > 0$ . If  $U = \{w_0\}$ , put  $\varphi(O) \equiv w_0$  and otherwise let  $\varphi : \mathbb{D} \rightarrow U$  be the conformal map onto the simply connected domain  $U$  with  $\varphi(O) = w_0$  and  $\varphi'(O) > 0$ . Then as  $n \rightarrow \infty$ ,*

$$\varphi_n \rightarrow \varphi \text{ locally uniformly in } \mathbb{D} \iff (U_n, w_0) \rightarrow (U, w_0).$$

Suppose that  $h : Z \times [0, 1] \rightarrow \mathbb{C}$  is an isotopy starting at the identity and  $U^t$  is the component of  $\mathbb{C}^* \setminus h^t(Z)$  which contains the point  $\infty$  at infinity. Then it follows easily that if  $t_n \rightarrow t$ , then  $(U^{t_n}, \infty) \rightarrow (U^t, \infty)$  in the sense of Carathéodory kernel convergence. One of the main complications addressed in this paper is that Carathéodory kernel convergence is insufficient to allow us control of the behavior of the conformal maps  $\varphi^t : \mathbb{D} \rightarrow U^t$  near the boundary of  $U^t$ . In particular it is not

clear if the conformal external rays of  $Z^t$  behave nicely under the isotopy. To this end we introduce metric external rays which depend only on distance and, hence, behave well under an isotopy. The existence of metric external rays was alluded to in [Bel67] and more fully developed in [Ili70], [Bel75]. We define a metric external ray as the projection of the equidistant set between two disjoint and closed sets in the covering space of  $\mathbb{C} \setminus \{O\}$  by the exponential map. Equidistant sets and metric external rays were studied in detail by G. Brouwer in [Bro05] and also in [ABO09]. We will use these metric external rays to show that the isotopy can be extended over conformal external rays.

We will always denote by  $Z$  a proper subcontinuum in the sphere  $\mathbb{C}^*$  (or equivalently in the plane  $\mathbb{C}$ ), by  $h : Z \times [0, 1] \rightarrow \mathbb{C}$  an isotopy such that  $h^0 = \text{id}_Z$  and by  $U$  a component of  $\mathbb{C}^* \setminus Z$  (or equivalently  $\mathbb{C} \setminus Z$ ). Given a fixed component  $U$  of  $\mathbb{C}^* \setminus Z$  we may assume, without loss of generality, that  $U$  contains the point at infinity (or is the unbounded component of  $\mathbb{C} \setminus Z$ ) and  $\infty \in \mathbb{C}^* \setminus h^t(Z)$  for all  $t \in [0, 1]$ . Denote by  $U^t$  the component of  $\mathbb{C}^* \setminus h^t(Z)$  containing the point at infinity (or the unbounded component of  $\mathbb{C} \setminus h^t(Z)$ ). Then  $U^t \cup \{\infty\}$  is simply connected. We always denote by  $\varphi^t : \mathbb{D} \rightarrow U^t \cup \{\infty\}$  the conformal map such that  $\varphi^t(O) = \infty$  and  $(\varphi^t)'(O) > 0$ . Then the maps  $\varphi^t$  are unique and, by Carathéodory kernel convergence, they are uniformly convergent in  $t$  on compact subsets of  $\mathbb{D}$ . By slightly abusing the language we will identify points in the boundary  $S^1$  of the disk  $\mathbb{D}$  with their arguments and call them *angles*.

We say that  $x \in Z$  is *accessible* from  $U$  if there exists an angle  $\theta \in [0, 2\pi)$  such that the (conformal) external ray  $R_\theta = \varphi(\{re^{i\theta} \mid r < 1\})$  lands on  $x$  (i.e.,  $\overline{R_\theta} \setminus R_\theta = \{x\}$ ). Similarly,  $R_\theta^t = \varphi^t(\{re^{i\theta} \mid r < 1\})$  is a conformal external ray of  $Z^t$  in  $U^t$ . It is well-known that a point  $x \in Z$  is accessible from  $U$  if and only if there exists a nondegenerate continuum  $Y \subset \overline{U}$  such that  $Y \cap Z = \{x\}$ . Moreover, in this case  $\overline{\varphi^{-1}(Y \setminus \{x\})} \cap S^1 = \{\theta\}$  is a single point and  $R_\theta$  lands on  $x$  in  $Z$  [Mil99, Cor. 17.10]. It is clearly necessary that the corresponding point  $x^t = h^t(x)$  remains accessible in  $h^t(Z)$  from  $U^t$ . However, Carathéodory kernel convergence is insufficient to show this and one of the first steps of the proof is to show that this is indeed the case. If we assume in addition that  $x$  is not a cut point of  $Z$ , then there exists for each  $t$  a unique angle  $\theta^t$  such that the external ray  $R_{\theta^t}^t$  of  $Z^t$  lands on  $x^t$ .

The next step of the proof is to show that this correspondence of angles is continuous in  $t$  and there exists an isotopy  $\alpha : S^1 \times [0, 1] \rightarrow S^1$  of the unit circle such that if  $R_\theta^0$  lands on  $x^0$  in  $Z^0$ , then  $R_{\alpha(\theta,t)}^t$  lands on  $x^t$  in  $Z^t$  for each  $t$ . Extending  $\alpha^t$  to an isotopy  $f^t : \mathbb{D} \rightarrow \mathbb{D}$ , defined by  $f^t(re^{i\theta}) = re^{i\alpha(\theta,t)}$  does not, however, provide a proper extension over  $\overline{U^0}$  since simple examples show that in general the isotopy  $H : U^0 \times [0, 1] \rightarrow \mathbb{C}^*$  defined by  $H(w, t) = \varphi^t \circ f^t \circ (\varphi^0)^{-1}(w)$  does not have a continuous extension over  $\text{Bd}(U^0)$ . The discontinuity at the boundary follows from the fact that conformal maps  $\varphi^t : \mathbb{D} \rightarrow U^t$  from the unit disk onto the simply connected domain  $U^t$  are not uniformly continuous if the boundary of  $U^t$  is not locally connected.

To address this concern we construct a lamination  $\mathcal{H}$  (a closed collection of noncrossing hyperbolic chords) in  $U$  such that  $\mathcal{L} = (\varphi)^{-1}(\mathcal{H})$  is a lamination in the disk  $\mathbb{D}$ . Following Thurston [Thu09], we will call each chord in  $\mathcal{H}$  (or in  $\mathcal{L}$ ) a *leaf* and every component of  $\mathbb{D} \setminus \bigcup \mathcal{L}$  a *gap*. Then the collection  $\mathcal{P} = \{G_\alpha\}$  of all leaves and gaps of  $\mathcal{L}$  partitions the disk into hyperbolically convex subsets. We say that the laminations  $\mathcal{H}$  and  $\mathcal{L}$  are a *canonic pair* if for each  $\varepsilon > 0$  there exists  $\delta > 0$  such that if  $d(x, y) < \delta$  and there exists  $\alpha$  such that  $x, y \in G_\alpha$ , then  $d(\varphi(x), \varphi(y)) < \varepsilon$ . In other words, the family of maps  $\{\varphi|_{\overline{G_\alpha}}\}$  is uniformly equicontinuous (here the map  $\varphi$  is naturally extended over angles corresponding to accessible points). To show the existence of a canonic pair of laminations we use a construction originally due to Bell and refined by Kulkarni-Pinkall [KP94]. This construction entails the consideration of closed round balls  $B$  such  $\text{Int}(B) \subset U$  and  $|B \cap \partial U| \geq 2$ . For each such  $B$  we consider the convex hull  $C(B)$  of  $B \cap \partial U$  in the hyperbolic metric on  $U$ . It can be shown that the collection  $\mathcal{H}$  of hyperbolic chords contained in the boundaries of all the sets  $C(B)$ , and its collection of preimages  $\mathcal{L}$  in  $\mathbb{D}$ , form the required pair of canonic laminations (see [FMOT08], [BO09, Th. 4.13] for more details).

Given the angle isotopy  $\alpha$  of the circle  $S^1$  it is now not difficult to construct an isotopy  $\Lambda : \mathbb{D} \times [0, 1] \rightarrow \mathbb{D}$  such that for each hyperbolic chord  $\beta\gamma \in \mathcal{L}$ , where  $\beta, \gamma \in S^1$ ,  $\Lambda(\beta\gamma, t)$  is the hyperbolic chord joining the points  $\alpha^t(\beta)$  and  $\alpha^t(\gamma)$ . Let  $\Lambda^t(\mathcal{L}) = \mathcal{L}^t$ , then  $\mathcal{L}^t$  is also a lamination in  $\mathbb{D}$ . Finally we will show in Theorem 6.1, that the laminations  $\mathcal{L}^t$  and  $\varphi^t(\mathcal{L}^t) = \mathcal{H}^t$  form a canonic pair as well. The existence of the canonic pairs of laminations will ensure the continuity of the extension  $H_U = \varphi^t \circ \Lambda^t \circ (\varphi^0)^{-1}$  over  $U$ .

Suppose that  $\{U_n\}$  is the collection of all complementary domains of  $X$ . For each  $n$  let  $\mathcal{L}_n$  and  $\mathcal{H}_n$  be a canonic pair of laminations as constructed above. If for each  $n$   $B_n$  is a round ball with  $\text{Int}(B_n) \subset U_n$  and  $|B \cap \partial U_n| \geq 2$ , then  $\text{diam}(C(B_n)) \rightarrow 0$  as  $n \rightarrow \infty$  [FMOT08, Lemma 4.2] and the extension of  $h$  is continuous over the union of all complementary domains.

We denote by  $\exp$  the covering map  $\exp : \mathbb{C} \rightarrow \mathbb{C} \setminus \{O\}$  defined by  $\exp(z) = e^z$ . Given a set  $X \subset \mathbb{C}$  we denote by  $\hat{\mathbf{X}} = \exp^{-1}(X \setminus \{O\})$  and we use bold face letters for subsets of  $\hat{\mathbf{X}}$ . However, for points  $x \in \mathbb{C} \setminus \{O\}$  we denote by  $\mathbf{x}$  a point in the set  $\exp^{-1}(x)$ . We also denote by  $\pi_j : \mathbb{C} \rightarrow \mathbb{R}$ ,  $j = 1, 2$ , the projections onto the  $x$ -axis and  $y$ -axis, respectively. The open ball with center  $x$  and radius  $r$  is denoted by  $B(x, r)$  and its boundary by  $S(x, r)$ . For a set  $A \subset \mathbb{C}$  we denote by  $B(A, \varepsilon) = \bigcup \{B(a, \varepsilon) \mid a \in A\}$ . By a *ray*  $R$  we mean a subset of  $\mathbb{C}$  homeomorphic to the real line  $\mathbb{R}$ . A ray is called a (*topological*) *line* if  $|\bar{R} \setminus R| \leq 1$  and  $\bar{R}$  is not a simple closed curve. If  $\bar{R} \setminus R = \emptyset$ , then we say that  $R$  is a *closed ray* or a *closed line*.

We will use the following notation throughout: for any set  $A \subset Z$  we denote by  $A^t$  the set  $h^t(A)$ . We are initially only interested in extending the isotopy over the unbounded component  $U$  of  $\mathbb{C} \setminus Z$ . Recall that  $U^t$  is the component of  $\mathbb{C}^* \setminus h^t(Z)$

containing  $\infty$  and denote by  $X^t$  the continuum  $\mathbb{C}^* \setminus U^t$ . Then  $X^t$  is a nonseparating plane continuum and, although the isotopy  $h$  is not necessarily defined on all of  $X^0$ , it is defined on  $\partial U^0 = \partial X^0 \subset Z$ . We may identify any particular point  $z \in \text{Bd}(U)$  with the origin  $O$ , assume that it is fixed under the isotopy and that  $X^t \subset B(O, 1)$  for all  $t \in [0, 1]$ . We will denote the Euclidean metric on  $\mathbb{C}$  by  $d$  and the spherical metric on  $\mathbb{C}^*$  by  $\rho$ . We will denote a point in the complex plane  $\mathbb{C}$  either as  $a + bi$  or, by its Euclidean coordinates,  $(a, b)$ . Finally, given two points  $x, y \in \mathbb{C}$ , we denote by  $xy$  the straight line segment joining them.

After submission of this paper some of its ideas were further developed in subsequent papers. See [FMOT08, §§3 and 4] for more detailed description of the partition  $\mathcal{P}$  of a simply connected domain  $U$  into hyperbolically convex subsets and [BO09] where the notion of a canonic pair of laminations (called canonic foliations in that paper) was first introduced and used explicitly. The authors are also indebted to the referee for carefully reading the paper and for making many suggestions to improve the exposition. For the convenience of the reader we have added an index at the end of the paper.

## 2. Preliminaries

Crucial to our study is the notion of an equidistant set between two disjoint closed sets in  $\mathbb{C}$ . We start with the following definition from [Bro05] (see [ABO09] for a more accessible reference and related results). Suppose that  $A$  and  $B$  are two disjoint closed subsets of the plane. For  $z \in \mathbb{C} \setminus [A \cup B]$ , let  $r(z) = d(z, A \cup B)$ . Then we say that  $A$  and  $B$  are *noninterlaced* if for each  $z \in \mathbb{C} \setminus [A \cup B]$ , the sets  $A \cap S(z, r(z))$  and  $B \cap S(z, r(z))$  are contained in two disjoint closed and connected subsets of  $S(z, r(z))$  (one may be empty). Let  $E(A, B) = \{z \in \mathbb{C} \mid d(z, A) = d(z, B)\}$  be the *equidistant set* between  $A$  and  $B$ .

Let  $A$  and  $B$  be two disjoint, closed and noninterlaced sets in  $\mathbb{C}$ . By Gaston Brouwer [Bro05, Th. 3.4.4]  $E(A, B)$  is a 1-manifold (see also [ABO09, Cor. 2.2]). Moreover, if  $A$  and  $B$  are connected, then  $E(A, B)$  is connected and, hence, it is either a closed ray in the plane or a simple closed curve. In particular if  $A$  and  $B$  are both connected and unbounded, then  $E(A, B)$  is a closed ray which separates  $\mathbb{C}$  into two disjoint open and connected sets one containing  $A$  and the other containing  $B$ . We will slightly generalize this case by replacing the condition that  $A$  and  $B$  are unbounded and connected by the weaker condition that every component of  $A \cup B$  is unbounded and that  $A$  lies above  $B$  (see Definition 2.1).

Since  $O \in X^t \subset B(O, 1)$  for all  $t$ ,  $\max\{\pi_1(\widehat{X}^t)\} < 0$ . It follows from this and the fact that  $X$  is a continuum that for any component  $\mathbf{C}$  of  $\widehat{\mathbf{X}}$ ,  $\pi_1(\mathbf{C}) = (-\infty, m_{\mathbf{C}}]$  with  $m_{\mathbf{C}} < 0$ . Moreover, since  $X$  is nonseparating,  $\widehat{\mathbf{X}}$  is also nonseparating and, hence, each component  $\mathbf{C}$  of  $\widehat{\mathbf{X}}$  is also nonseparating. To see this note that  $\widehat{\mathbf{X}}$  has a unique complementary domain  $W$  such that  $\pi_1^{-1}([0, \infty)) \subset W$ . If  $V$  is any complementary domain of  $\widehat{\mathbf{X}}$ , then  $V$  must contain a point  $\mathbf{v} \in V \setminus \widehat{\mathbf{X}}$  and

$\exp(\mathbf{v}) = v \in \mathbb{C} \setminus X$ . Hence there exists a ray  $R \subset \mathbb{C} \setminus X$  joining  $v$  to infinity. Then the lift  $\mathbf{R}$  of  $R$  with initial point  $\mathbf{v} \in \mathbf{R}$  is a ray in  $\mathbb{C} \setminus \hat{\mathbf{X}}$  which intersects  $W$  and  $V = W$  as required. These facts will allow us to define what it means for one component of  $\hat{\mathbf{X}}$  to lie above another component.

*Definition 2.1.* Let  $\mathbf{C}$  and  $\mathbf{D}$  be two distinct components of  $\hat{\mathbf{X}}$ . We say that  $\mathbf{C}$  *lies above*  $\mathbf{D}$  if there is a path  $s : [0, 1] \rightarrow \pi_1^{-1}((-\infty, 0]) \setminus \mathbf{C}$  such that the initial point  $s(0)$  is in  $\mathbf{D}$ ,  $s(1) = O$  and if  $R = s([0, 1]) \cup ([0, \infty) \times \{0\})$ , then  $\mathbf{C}$  lies in the unique unbounded component of  $\mathbb{C} \setminus [\mathbf{D} \cup R]$  which contains the point  $1 + 2\pi i$ .

Moreover, if  $\hat{\mathbf{X}} = \mathbf{A} \cup \mathbf{B}$ , where  $\mathbf{A}$  and  $\mathbf{B}$  are disjoint closed sets, such that every component of  $\mathbf{A}$  lies above every component of  $\mathbf{B}$ , then we say that  $\mathbf{A}$  *lies above*  $\mathbf{B}$ . Note that the image of the path  $s([0, 1])$  in Definition 2.1 is contained in the complement of  $\mathbf{C}$  and not necessarily in the complement of  $\hat{\mathbf{X}}$ . This will allow us to consider otherwise inaccessible components of  $\hat{\mathbf{X}}$ .

*LEMMA 2.2.* *If  $\mathbf{C}$  and  $\mathbf{D}$  are two components of  $\hat{\mathbf{X}}$ , then exactly one of the following holds:*

- (1)  $\mathbf{C}$  lies above  $\mathbf{D}$ ,
- (2)  $\mathbf{D}$  lies above  $\mathbf{C}$ .

*Proof.* We show that the notion that  $\mathbf{C}$  lies above  $\mathbf{D}$  is independent of the choice of the path  $s$  in Definition 2.1. Hence, let  $s_1, s_2 : [0, 1] \rightarrow \pi_1^{-1}((-\infty, 0]) \setminus \mathbf{C}$  be two paths such that  $s_1(0), s_2(0) \in \mathbf{D}$  and  $s_1(1) = O = s_2(1)$ . Put  $R_1 = s_1([0, 1]) \cup ([0, \infty) \times \{0\})$ ,  $R_2 = s_2([0, 1]) \cup ([0, \infty) \times \{0\})$  and suppose that  $\mathbf{C}$  lies in the unbounded component of  $\mathbf{D} \cup R_1$  which contains the point  $1 + 2\pi i$ . Suppose first that  $R_1$  and  $R_2$  have the same initial point  $s_1(0) = s_2(0)$ . Since  $\pi_1^{-1}((-\infty, 0]) \setminus \mathbf{C}$  is simply connected, there exists a homotopy  $j : [0, 1] \times [0, 1] \rightarrow \pi_1^{-1}((-\infty, 0]) \setminus \mathbf{C}$ , with endpoints fixed, between  $s_1$  and  $s_2$ . Since  $j^t$  misses the connected set  $\mathbf{C}$  for each  $t$ , it follows that  $\mathbf{C}$  lies in the component of  $\mathbb{C} \setminus [R_2 \cup \mathbf{D}]$  containing  $1 + 2\pi i$ .

Next suppose that  $R_1$  and  $R_2$  have initial points  $\mathbf{z}_1$  and  $\mathbf{z}_2$ , respectively. Let  $\mathcal{U} = \{B(y, \varepsilon) \mid y \in \mathbf{D} \text{ and } \varepsilon = (1/3)d(y, [C \cup \pi_1^{-1}([0, \infty)])\}$ . Then  $\mathcal{U}$  is an open cover of  $\mathbf{D}$ . Since  $\mathbf{D}$  is connected, there exists a chain  $\{B_1, \dots, B_n\}$  of balls in  $\mathcal{U}$  such that  $\mathbf{z}_1 \in B_1$ ,  $\mathbf{z}_2 \in B_n$  and  $B_j \cap B_{j+1} \neq \emptyset$  for  $j = 1, \dots, n-1$ . Let  $J$  be a piecewise linear arc in  $\cup B_j$  from  $\mathbf{z}_1$  to  $\mathbf{z}_2$ . Then there exists a path  $s_3$  such that  $s_3([0, 1]) = J \cup s_2([0, 1])$  is a path with initial point  $\mathbf{z}_1$  and terminal point  $O$ . Put  $R_3 = s_3([0, 1]) \cup ([0, \infty) \times \{0\})$ . Then  $\mathbf{C}$  lies in the unbounded component of  $\mathbb{C} \setminus [\mathbf{D} \cup R_3]$  which contains the point  $1 + 2\pi i$ . Hence,  $\mathbf{C}$  lies in the unbounded component of  $\mathbb{C} \setminus [\mathbf{D} \cup R_2]$  which contains the point  $1 + 2\pi i$ .

Suppose that  $\mathbf{C}$  and  $\mathbf{D}$  are any two components of  $\hat{\mathbf{X}}$ . Then  $U_{\mathbf{C}} = \mathbb{C} \setminus \mathbf{C}$  and  $U_{\mathbf{D}} = \mathbb{C} \setminus \mathbf{D}$  are open and connected sets homeomorphic to  $\mathbb{C}$ . Hence there exist two arcs  $J_{\mathbf{C}} \subset U_{\mathbf{D}} \cap \pi_1^{-1}((-\infty, 0])$  and  $J_{\mathbf{D}} \subset U_{\mathbf{C}} \cap \pi_1^{-1}((-\infty, 0])$  joining points  $\mathbf{c} \in \mathbf{C}$  and  $\mathbf{d} \in \mathbf{D}$  to  $O$ , respectively. In addition we may assume that  $J_{\mathbf{C}} \cap J_{\mathbf{D}} = \{O\}$ . If  $\mathbf{D}$  is not contained in the component of  $\mathbb{C} \setminus J_{\mathbf{C}} \cup ([0, \infty) \times \{0\})$  containing  $1 + 2\pi i$ ,

then  $\mathbf{C}$  is contained in the component of  $\mathbb{C} \setminus J_{\mathbf{D}} \cup ([0, \infty) \times \{0\})$  containing  $1 + 2\pi i$  and  $\mathbf{C}$  lies above  $\mathbf{D}$ .  $\square$

Our goal is to show that the condition that  $\mathbf{A}$  lies above  $\mathbf{B}$  is preserved under the lift of the isotopy  $h^t$ .

LEMMA 2.3. *Suppose  $h^t : \text{Bd}(X) \rightarrow \mathbb{C}$  is an isotopy such that  $h^0 = \text{id}_{\text{Bd}(X)}$ ,  $O \in \text{Bd}(X)$  and  $h^t(O) = O$  for all  $t$ . Then there exists an isotopy  $\mathbf{h}^t : \text{Bd}(\widehat{\mathbf{X}}) \rightarrow \mathbb{C}$  which lifts  $h^t$  such that  $\mathbf{h}^0 = \text{id}_{\text{Bd}(\widehat{\mathbf{X}})}$ .*

*Proof.* For each  $x \in \text{Bd}(X) \setminus \{O\}$  and each  $\mathbf{x} \in \exp^{-1}(x)$  the path  $h|_{\{x\} \times [0,1]}$  has a unique lift to a path  $\mathbf{h}_{\mathbf{x}} : [0, 1] \rightarrow \mathbb{C}$  with initial point  $\mathbf{x}$ . Define  $\mathbf{h}^t(\mathbf{x}) = \mathbf{h}_{\mathbf{x}}(t)$ . By uniqueness of lifts,  $\mathbf{h}^t$  is one-to-one. It now follows easily that  $\mathbf{h}^t$  is an isotopy of  $\text{Bd}(\widehat{\mathbf{X}})$  lifting  $h^t$  with  $\mathbf{h}^0 = \text{id}_{\text{Bd}(\widehat{\mathbf{X}})}$ .  $\square$

The following easy lemma follows immediately from the fact that  $h^t(O) = O$  for all  $t$  and that  $h^t$  is uniformly continuous.

LEMMA 2.4. *Suppose that  $h^t(O) = O$  for all  $t$  and let  $\mathbf{h}^t : \text{Bd}(\widehat{\mathbf{X}}) \rightarrow \mathbb{C}$  be the isotopy which is the lift of  $h^t$  to  $\text{Bd}(\widehat{\mathbf{X}}) = \exp^{-1}(\text{Bd}(X) \setminus \{O\})$  such that  $\mathbf{h}^0 = \text{id}_{\text{Bd}(\widehat{\mathbf{X}})}$ . Denote  $\mathbf{h}^t(\mathbf{x})$  by  $\mathbf{x}^t$ . For all  $\varepsilon \in \mathbb{R}$  there exists  $\delta \in \mathbb{R}$  such that if there exists  $t_0 \in [0, 1]$  such that  $\mathbf{x}^{t_0} \in \widehat{\mathbf{X}}^{t_0}$  and  $\pi_1(\mathbf{x}^{t_0}) \leq \delta$ , then  $\pi_1(\mathbf{x}^t) < \varepsilon$  for all  $t \in [0, 1]$ . In other words, if there exists  $t_0$  such that  $\pi_1(\mathbf{x}^{t_0}) \geq \varepsilon$ , then  $\pi_1(\mathbf{x}^t) > \delta$  for all  $t \in [0, 1]$ .*

Given the existence of the lifted isotopy  $\mathbf{h}^t$  we will use similar notation as for  $h^t$ : for any set  $\mathbf{A} \subset \text{Bd}(\widehat{\mathbf{X}})$  we denote by  $\mathbf{A}^t$  the set  $\mathbf{h}^t(\mathbf{A})$ . Recall that  $U^t$  is the unbounded component of  $\mathbb{C} \setminus h^t(\text{Bd}(X))$ ,  $X^t = \mathbb{C} \setminus U^t$  and  $\widehat{\mathbf{X}}^t = \exp^{-1}(X^t \setminus \{O\})$ . Also, if  $\mathbf{C}^0$  is a component of  $\widehat{\mathbf{X}}^0$  choose a point  $\mathbf{x}^0 \in \text{Bd}(\widehat{\mathbf{X}}^0) \cap \mathbf{C}^0$ . Then we denote by  $\mathbf{C}^t$  the component of  $\widehat{\mathbf{X}}^t$  containing the point  $\mathbf{x}^t = \mathbf{h}^t(\mathbf{x})$ . Next we show that the notion of the component  $\mathbf{C}$  being above  $\mathbf{D}$  in  $\widehat{\mathbf{X}}$  is preserved throughout the isotopy  $\mathbf{h}$ .

LEMMA 2.5. *Let  $\mathbf{C} = \mathbf{C}^0$  and  $\mathbf{D} = \mathbf{D}^0$  be components of  $\widehat{\mathbf{X}}^0$  such that  $\mathbf{C}$  lies above  $\mathbf{D}$ . Then  $\mathbf{C}^t$  lies above  $\mathbf{D}^t$  for each  $t \in [0, 1]$ .*

*Proof.* It suffices to show that there exists  $0 < t_0$  such that for all  $t \leq t_0$   $\mathbf{C}^t$  lies above  $\mathbf{D}^t$ . Let  $R = s([0, 1]) \cup ([0, \infty) \times \{0\})$  be a piecewise linear ray landing on  $\mathbf{d}^0 \in \mathbf{D}^0$  which satisfies the conditions of Definition 2.1 and such that  $R \cap \mathbf{C}^0 = \emptyset$  and  $R \cap \mathbf{D}^0 = \{\mathbf{d}^0\}$ . Then  $R \cup \mathbf{D}^0$  has exactly two complementary domains and each is homeomorphic to  $\mathbb{C}$ . Hence there exists an arc  $A \subset \mathbb{C} \setminus [\mathbf{D}^0 \cup R]$  joining a point  $\mathbf{c}^0 \in \mathbf{C}^0$  to the point  $1 + 2\pi i$ . Choose  $a < 0$  such that  $A \cup R \subset \pi^{-1}([a, \infty))$ . Choose  $\varepsilon < (1/3) d(A \cup \pi_1^{-1}([2a, \infty)) \cap \mathbf{C}^0, R \cup \pi_1^{-1}([2a, \infty)) \cap \mathbf{D}^0)$ . Let  $0 < t_0$  such that for each  $\mathbf{x} \in \text{Bd}(\widehat{\mathbf{X}}) \cap \pi_1^{-1}([2a, \infty))$ ,  $|\mathbf{h}^t(\mathbf{x}) - \mathbf{h}^0(\mathbf{x})| < \varepsilon/2$  and  $\pi_1(\mathbf{h}^t(\mathbf{x})) < a$  for all  $\mathbf{x} \in \pi_1^{-1}((-\infty, 2a]) \cap \text{Bd}(\widehat{\mathbf{X}})$  for all  $t$ . Then for all  $t \leq t_0$ ,  $\mathbf{C}^t \cup \mathbf{c}^0 \mathbf{c}^t \cup A$  is connected, closed and disjoint from  $\mathbf{D}^t \cup \mathbf{d}^0 \mathbf{d}^t \cup R$ . The first set contains an arc  $A^*$  from  $\mathbf{c}^t$  to the point  $1 + 2\pi i$  and the latter set contains a half ray

$R^*$  satisfying the conditions in Definition 2.1 from the point  $\mathbf{d}^t \in \mathbf{D}^t$  to  $\infty$ . Since  $1 + 2\pi i$  is above  $R^*$  and  $A^* \cap [\mathbf{D}^t \cup R^*] = \emptyset$ ,  $\mathbf{C}^t$  is above  $\mathbf{D}^t$  for all  $t \in [0, t_0]$ .  $\square$

LEMMA 2.6. *Suppose that  $\hat{\mathbf{X}}^0 = \mathbf{A}^0 \cup \mathbf{B}^0$ , where  $\mathbf{A}^0$  and  $\mathbf{B}^0$  are disjoint closed sets such that  $\mathbf{A}^0$  lies above  $\mathbf{B}^0$ . Then for each  $t$ ,  $\hat{\mathbf{X}}^t = \mathbf{A}^t \cup \mathbf{B}^t$  and  $\mathbf{A}^t$  and  $\mathbf{B}^t$  are disjoint, closed and noninterlaced sets.*

*Proof.* By Lemma 2.5, every component of  $\mathbf{A}^t$  lies above every component of  $\mathbf{B}^t$  for all  $t$ . Since  $\mathbf{h}^t$  is an isotopy it only remains to show that  $\mathbf{A}^t$  and  $\mathbf{B}^t$  are noninterlaced. To see this fix  $t$ , let  $\mathbf{w} \in E(\mathbf{A}^t, \mathbf{B}^t)$  and let  $K \subset \mathbb{C} \setminus \mathbf{A}^t \cup \mathbf{B}^t$  be the minimal open ball with center  $\mathbf{w}$  whose boundary  $S$  meets both  $\mathbf{A}^t$  and  $\mathbf{B}^t$ . Suppose that there exist  $\mathbf{x}, \mathbf{x}' \in S \cap \mathbf{A}^t$  and  $\mathbf{y}, \mathbf{y}' \in S \cap \mathbf{B}^t$  such that  $\{\mathbf{y}, \mathbf{y}'\}$  separates  $\mathbf{x}$  and  $\mathbf{x}'$  in  $S$ . For  $\mathbf{z} \in \hat{\mathbf{X}}^t$ , let  $\mathbf{C}_z$  denote the component of  $\hat{\mathbf{X}}^t$  which contains the point  $\mathbf{z}$ . Suppose, without loss of generality, that  $\mathbf{C}_y$  lies above  $\mathbf{C}_{y'}$ . We may suppose that  $\mathbf{C}_y \cup \mathbf{C}_{y'} \cup \mathbf{y}\mathbf{y}'$  separates  $\mathbf{x}$  from  $1 + 2\pi i$  in  $\mathbb{C}$ . Since  $\mathbf{C}_y \cup \mathbf{C}_{y'}$  does not separate  $\mathbb{C}$  by unicoherence, we can choose an arc  $D$  in  $\mathbb{C} \setminus [\mathbf{C}_y \cup \mathbf{C}_{y'}]$  irreducible from  $O$  to  $\mathbf{y}\mathbf{y}'$  such that  $\pi_1(D) \subset (-\infty, 0]$ . Let  $\{d\} = D \cap \mathbf{y}\mathbf{y}'$ . Then  $\mathbf{C}_y \cup \mathbf{y}d \cup D \cup [0, \infty) \times \{0\}$  separates  $\mathbf{y}'$ , and hence also  $\mathbf{x}$ , from  $1 + 2\pi i$  and  $\mathbf{C}_x$  is below  $\mathbf{C}_y$ . This contradicts Lemma 2.5 and completes the proof.  $\square$

LEMMA 2.7. *Suppose  $\hat{\mathbf{X}} = \mathbf{A} \cup \mathbf{B}$ , where  $\mathbf{A}$  and  $\mathbf{B}$  are disjoint closed subsets of  $\mathbb{C}$  such that  $\mathbf{A}$  lies above  $\mathbf{B}$ . Let  $E$  be a component of  $E(\mathbf{A}, \mathbf{B})$ . Then  $E$  is a closed ray. If  $e \in E$  and  $r = d(e, \mathbf{A} \cup \mathbf{B})$ , then there exist disjoint irreducible arcs or points  $J_A$  and  $J_B$  in  $S(e, r)$  such that  $\mathbf{A} \cap S(e, r) \subset J_A$  and  $\mathbf{B} \cap S(e, r) \subset J_B$ , and  $E$  separates  $J_A$  from  $J_B$  in  $\mathbb{C}$ .*

*Proof.* By Lemma 2.6,  $\mathbf{A}$  and  $\mathbf{B}$  are noninterlaced. By [Bro05, Th. 3.4.4],  $E$  is a 1-manifold. Let  $E$  be a component of  $E(\mathbf{A}, \mathbf{B})$ ,  $e \in E$  and  $d(e, \mathbf{A} \cup \mathbf{B}) = r$ . Since  $\mathbf{A}$  and  $\mathbf{B}$  are noninterlaced, there exist disjoint irreducible arcs or points  $J_A$  and  $J_B$  in  $S(e, r)$  such that  $\mathbf{A} \cap S(e, r) \subset J_A$  and  $\mathbf{B} \cap S(e, r) \subset J_B$ . Let  $\mathbf{a}_1$  and  $\mathbf{a}_2$  be the endpoints of  $J_A$ . For  $\mathbf{z} \in \hat{\mathbf{X}}$ , let  $\mathbf{C}_z$  be the component of  $\mathbf{z}$  in  $\hat{\mathbf{X}}$ . Let  $V$  be the component of  $\mathbb{C} \setminus [\mathbf{C}_{a_1} \cup J_A \cup \mathbf{C}_{a_2}]$  containing  $e$  and let  $W = \mathbb{C} \setminus V$ . It follows from the proof of Lemma 2.6 that  $W \cap \mathbf{B} = \emptyset$ .

We prove that  $E(\mathbf{A}, \mathbf{B}) \cap J_A = \emptyset$  as follows. Suppose that  $\mathbf{z} \in J_A \setminus A$ . We will show that  $d(\mathbf{z}, \mathbf{A}) < d(\mathbf{z}, \mathbf{B})$ . Choose  $\mathbf{w} \in \mathbf{B}$ . If  $\mathbf{z}\mathbf{w} \cap \mathbf{A} \neq \emptyset$ , then  $d(\mathbf{z}, \mathbf{A}) < d(\mathbf{z}, \mathbf{w})$ . If  $\mathbf{z}\mathbf{w} \cap \mathbf{A} = \emptyset$ , then it follows easily that  $d(\mathbf{z}, \mathbf{w}) > \min\{d(\mathbf{z}, \mathbf{a}_1), d(\mathbf{z}, \mathbf{a}_2)\}$ . Hence for all  $\mathbf{w} \in \mathbf{B}$ ,  $d(\mathbf{z}, \mathbf{A}) < d(\mathbf{z}, \mathbf{w})$  and  $E(\mathbf{A}, \mathbf{B}) \cap J_A = \emptyset$ . Choose  $\mathbf{b} \in J_B \cap \mathbf{B}$ . Note that  $E(\mathbf{A}, \mathbf{B}) \cap \mathbf{a}_1 e \setminus \{e\} = \emptyset = E(\mathbf{A}, \mathbf{B}) \cap e\mathbf{b} \setminus \{e\}$ . Now  $E(\mathbf{A}, \mathbf{B})$  separates  $\mathbf{a}_1$  and  $\mathbf{b}$ . By unicoherence of  $\mathbb{C}$  (see [Wil63, p. 47]) a component of  $E(\mathbf{A}, \mathbf{B})$  separates  $\mathbf{a}_1$  and  $\mathbf{b}$ . Since this component must contain  $e$ ,  $E$  separates  $\mathbf{a}_1$  and  $\mathbf{b}$  in  $\mathbb{C}$ . Hence  $E$  separates  $\mathbf{C}_{a_1} \cup J_A$  and  $\mathbf{C}_b \cup J_B$  which both are unbounded sets. Hence,  $E$  is an unbounded closed 1-manifold, i.e.,  $E$  is a closed ray.  $\square$

In Lemma 2.7,  $\pi_1(\mathbf{A}) = \pi_1(\mathbf{B}) = (-\infty, c]$  for some  $c < 0$ . In the next theorem we shall choose  $\mathbf{a} \in \mathbf{A}$  and  $\mathbf{b} \in \mathbf{B}$  with  $\pi_1(\mathbf{a}) = c = \pi_1(\mathbf{b})$  and  $0 < \pi_2(\mathbf{a}) - \pi_2(\mathbf{b}) \leq 2\pi$ .

By unicoherence of the plane a component  $E$  of  $E(\mathbf{A}, \mathbf{B})$  separates  $\mathbf{a}$  and  $\mathbf{b}$  in  $\mathbb{C}$ . Next we show that  $E$  separates  $\mathbf{A}$  and  $\mathbf{B}$  in  $\mathbb{C}$ . Then we prove that  $E = E(\mathbf{A}, \mathbf{B})$ . Finally we prove that  $E$  is  $\pi_1$ -monotone for  $x \geq 0$ .

**THEOREM 2.8.** *Suppose that  $\hat{\mathbf{X}} = \mathbf{A} \cup \mathbf{B}$ , where  $\mathbf{A}$  and  $\mathbf{B}$  are disjoint, closed, nonempty sets such that  $\mathbf{A}$  lies above  $\mathbf{B}$ . Then  $E(\mathbf{A}, \mathbf{B})$  is a closed ray such that  $\pi_1(E(\mathbf{A}, \mathbf{B})) = (-\infty, \infty)$  and for  $x > 0$ ,  $|\pi_1^{-1}(x) \cap E(\mathbf{A}, \mathbf{B})| = 1$ .*

*Proof.* By Lemma 2.7, each component of  $E(\mathbf{A}, \mathbf{B})$  is a closed ray which stretches to  $-\infty$ . For  $\mathbf{z} \in \hat{\mathbf{X}}$ , let  $\mathbf{C}_z$  be the component of  $\mathbf{z}$  in  $\hat{\mathbf{X}}$ .

Let  $\mathbf{a} \in \hat{\mathbf{X}}$  be such that  $\pi_1(\mathbf{a}) = \max(\pi_1(\hat{\mathbf{X}})) < 0$ . Without loss of generality,  $\mathbf{a} \in \mathbf{A}$ . Let  $R = \mathbf{a}O \cup ([0, \infty) \times \{0\})$ , then  $R \setminus \{\mathbf{a}\}$  is a ray disjoint from  $\hat{\mathbf{X}}$  which lands on  $\mathbf{a}$ . Note that  $\mathbb{C} \setminus [R \cup \mathbf{C}_a] = W \cup V$ , where  $W$  and  $V$  are disjoint, connected, open and nonempty sets. Without loss of generality,  $1 + 2\pi i \in W$ . Then every component of  $\hat{\mathbf{X}} \cap W$  is above  $\mathbf{C}_a \subset \mathbf{A}$ . Hence  $\mathbf{B} \subset V$ . Since  $\mathbf{B} \neq \emptyset$ , there exists  $\mathbf{b} \in \mathbf{B}$  such that  $\pi_1(\mathbf{b}) = \pi_1(\mathbf{a})$  since  $\hat{\mathbf{X}}$  is invariant under upward translation by  $2\pi$ . By compactness of  $X \cap S(O, e^{\pi_1(\mathbf{a})})$ , we may assume that

$$\pi_2(\mathbf{a}) = \min(\pi_2(\mathbf{A} \cap \pi_1^{-1}(\pi_1(\mathbf{a})))) \quad \text{and} \quad \pi_2(\mathbf{b}) = \max(\pi_2(\mathbf{B} \cap \pi_1^{-1}(\pi_1(\mathbf{a})))).$$

Then  $0 < \pi_2(\mathbf{a}) - \pi_2(\mathbf{b}) \leq 2\pi$  and we may assume that  $0 < \pi_2(\mathbf{a}) \leq \pi$ . For

$$z \in [\pi_1(\mathbf{a}), \infty) \times i[\pi_2(\mathbf{a}), \infty), \quad d(z, \mathbf{A}) < d(z, \mathbf{B})$$

and for

$$z \in [\pi_1(\mathbf{a}), \infty) \times i(-\infty, \pi_2(\mathbf{b})], \quad d(z, \mathbf{B}) < d(z, \mathbf{A}).$$

By unicoherence of the plane there exists a component  $E$  of  $E(\mathbf{A}, \mathbf{B})$  which separates  $\mathbf{a}$  and  $\mathbf{b}$ . Then  $E$  separates  $\mathbf{C}_a \cup ([\pi_1(\mathbf{a}), \infty) \times i\pi_2(\mathbf{a}))$  from

$$\mathbf{C}_b \cup ([\pi_1(\mathbf{a}), \infty) \times i\pi_2(\mathbf{b})).$$

So  $\pi_1(E) = (-\infty, \infty)$  and

$$E(\mathbf{A}, \mathbf{B}) \cap ([\pi_1(\mathbf{a}), \infty) \times i\mathbb{R}) \subset [\pi_1(\mathbf{a}), \infty) \times i[\pi_2(\mathbf{b}), \pi_2(\mathbf{a})].$$

In particular,  $E(\mathbf{A}, \mathbf{B}) \cap \pi_1^{-1}(x)$  is compact for  $x > 0$ .

In order to show that  $E(\mathbf{A}, \mathbf{B})$  is connected we must first prove that  $E$  separates  $\mathbf{A}$  and  $\mathbf{B}$ . Suppose that  $\mathbb{C} \setminus E = W' \cup V'$ , where  $W'$  and  $V'$  are disjoint, nonempty, open and connected sets, and  $1 + 2\pi i \in W'$ . Just suppose there exist  $\mathbf{y} \in \mathbf{A} \cap V'$ . Since neither of the disjoint closed sets  $E$  nor  $\mathbf{B}$  separates  $\mathbf{y}$  from  $1 - 2\pi i$ , neither does their union. Let  $D \subset \mathbb{C} \setminus [E \cup \mathbf{B}]$  be an arc from  $\mathbf{y}$  to  $1 - 2\pi i \in V'$ . Choose  $e \in E$  such that if  $r = d(e, \hat{\mathbf{X}})$ , then  $\pi_1(e) + r < \min\{\pi_1(D)\}$ . Let  $\mathbf{w} \in S(e, r) \cap \mathbf{B}$ . Then  $\mathbf{C}_w \cup \mathbf{w}e \cup E'$ , where  $E'$  is the component of  $E \setminus \{e\}$  which projects under  $\pi_1$  over  $[\pi_1(\mathbf{y}), \infty)$ , does not separate  $\mathbf{y}$  from  $1 - 2\pi i$ . It now follows easily that  $\mathbf{C}_y$  lies below  $\mathbf{C}_w$ , a contradiction. Hence, we can conclude that  $\mathbf{A} \subset W'$  and  $\mathbf{B} \subset V'$ .

We prove next that  $E(\mathbf{A}, \mathbf{B}) = E$ . For suppose  $e' \in E(\mathbf{A}, \mathbf{B}) \setminus E$ . Let  $x \in S(e', r(e')) \cap \mathbf{A}$  and let  $y \in S(e', r(e')) \cap \mathbf{B}$ . Since  $E$  separates  $x$  and  $y$  in  $\mathbb{C}$ ,

$E \cap (e'x \cup e'y) \neq \emptyset$ . Without loss of generality  $E \cap e'x \neq \emptyset$ . Let  $e \in E \cap e'x$ . Then  $d(x, e) = d(e, \mathbf{A}) < d(e, \mathbf{B})$  since  $(\mathbf{A} \cup \mathbf{B}) \cap B(e', r(e')) = \emptyset$ . This contradiction proves that  $E(\mathbf{A}, \mathbf{B}) \subset E$  and, hence,  $E(\mathbf{A}, \mathbf{B}) = E$ . It follows that for all  $z \in W'$ ,  $d(z, \mathbf{A}) < d(z, \mathbf{B})$  and for all  $z \in V'$ ,  $d(z, \mathbf{B}) < d(z, \mathbf{A})$ .

Finally we prove that  $E$  is  $\pi_1$ -monotone for  $x \geq 0$ . Suppose  $e_1, e_2 \in E$  with  $\pi_1(e_1) = \pi_1(e_2) \geq 0$  and  $\pi_2(e_2) > \pi_2(e_1)$ . Let  $z_1 \in \mathbf{A} \cap S(e_1, r(e_1))$  and let  $z_2 \in \mathbf{B} \cap S(e_2, r(e_2))$ . By [ABO09, Lemma 1.11]  $e_2 z_2 \cap e_1 z_1 = \emptyset$ . (The proof in [ABO09] is for the spherical metric on the sphere but is also valid for the Euclidean metric on the plane.) Hence  $C_{z_1} \cup e_1 z_1 \cup ([\pi_1(e_1), \infty) \times \{\pi_2(e_1)\}i)$  separates  $e_2$  (and, hence,  $C_{z_2}$ ) from  $e_1 - \pi i$ . It now follows easily that  $C_{z_2} \subset \mathbf{B}$  lies above  $C_{z_1} \subset \mathbf{A}$ , a contradiction. This completes the proof of the theorem.  $\square$

### 3. Characterizing accessibility

In this section we provide a characterization of accessibility for points in  $Z$  and show that accessibility is preserved under the isotopy  $h$ . Recall that the isotopy  $h$  is defined on  $Z$ , that  $U$  is the unbounded component of  $\mathbb{C} \setminus Z$  and that  $X = \mathbb{C} \setminus U$ . In this section we will always assume that  $O \in \text{Bd}(X) \subset Z$  is fixed under the isotopy  $h$ . Easy examples (e.g., a half ray spiraling around the closed interval  $[-1, 1] \times \{0\}$ ) show that accessibility of  $O$  from  $U$  is not equivalent to  $\hat{\mathbf{X}}$  being not connected. Nevertheless the spirit of this idea is correct:

LEMMA 3.1. *Suppose that  $O \in \text{Bd}(X)$ . Then, the fact that  $O$  is accessible from  $U$  is equivalent to the following two conditions:*

- (1)  $\hat{\mathbf{X}} = \mathbf{A} \cup \mathbf{B}$ , where  $\mathbf{A}$  and  $\mathbf{B}$  are nonempty, disjoint and closed such that:

$\mathbf{A}$  lies above  $\mathbf{B}$

and

- (2) for all  $x \in \mathbb{R}$  there exists  $y_1 < y_2$  in  $\mathbb{R}$  such that

$$(2a) \quad \pi_1^{-1}([x, \infty)) \cap \pi_2^{-1}([y_2, \infty) i) \cap \mathbf{B} = \emptyset \text{ and}$$

$$(2b) \quad \pi_1^{-1}([x, \infty)) \cap \pi_2^{-1}((-\infty, y_1] i) \cap \mathbf{A} = \emptyset.$$

*Proof.* Suppose first that  $O$  is accessible, let  $R$  be a conformal external ray in  $U$  landing on  $O$  and let  $\mathbf{J}$  be a component of  $\exp^{-1}(R)$ . Then  $\mathbf{J}$  is a closed ray in  $\mathbb{C} \setminus \hat{\mathbf{X}}$  such that  $\pi_1(\mathbf{J}) = (-\infty, \infty)$  and for every vertical line  $\ell$ ,  $\mathbf{J} \cap \ell$  is compact. Note that  $\mathbb{C} \setminus \mathbf{J} = U \cup V$ , where  $U$  and  $V$  are disjoint open and connected sets. We may assume that for some vertical line  $\ell$ ,  $\pi_2(\ell \cap U)$  has no upper bound and, since  $\hat{\mathbf{X}}$  is invariant under vertical translations by  $2\pi$ , that  $\{1 + 2\pi i\} \subset U$ . Put  $\mathbf{A} = \hat{\mathbf{X}} \cap U$  and  $\mathbf{B} = \hat{\mathbf{X}} \cap V$  then condition (1) holds. The fact every component of  $\mathbf{A}$  lies above every component of  $\mathbf{B}$  follows from the fact that  $U$  and  $V$  are open and connected, and  $U$  is “above”  $V$ . To see that (2a) and (2b) hold note that close to infinity  $R$  behaves like a radial line segment in the plane and, hence,  $\mathbf{J}$  behaves

like a horizontal line segment near  $+\infty$  so  $\pi_2(\mathbf{J} \cap \pi_1^{-1}([a, \infty)))$  is bounded for each  $a \in \mathbb{R}$ .

Suppose next that conditions (1), (2a), and (2b) hold. By Theorem 2.8,  $E = E(\mathbf{A}, \mathbf{B})$  is a closed ray which runs from  $-\infty$  to  $\infty$  and separates  $\mathbf{A}$  from  $\mathbf{B}$ .

CLAIM. For every compact arc  $C \subset x$ -axis,  $\pi_1^{-1}(C) \cap E$  is compact.

*Proof of the claim.* Let  $C \subset x$ -axis be a compact interval. Suppose without loss of generality that  $\pi_1^{-1}(C) \cap E$  contains points  $e_i$  with  $\lim \pi_2(e_i) = +\infty$ . Note that there exists  $K > 0$  such that for each  $z \in \pi_1^{-1}(C)$ ,  $d(z, \hat{\mathbf{X}}) \leq K$ . Hence for each  $i$  there exists  $\mathbf{b}_i \in \mathbf{B}$  such that  $d(e_i, \mathbf{b}_i) \leq K$  and  $\lim \pi_2(\mathbf{b}_i) = +\infty$ . This contradicts (2a) and completes the proof of the claim.

Now let  $\mathcal{R} = \overline{\exp(E)}$ . Then  $\mathcal{R}$  is a closed and connected set in  $\mathbb{C}$  and it suffices to show that  $\mathcal{R} \cap X = \{O\}$ . Since  $\pi_1(E) = (-\infty, \infty)$ ,  $O \in \mathcal{R}$ . Suppose that  $x \in X \setminus \{O\}$  is also a limit point of  $\mathcal{R}$ . Choose  $z_i \in \exp(E)$  such that  $\lim z_i = x$  and  $\mathbf{z}_i \in E$  such that  $\exp(\mathbf{z}_i) = z_i$ . Then  $d(\mathbf{z}_i, \exp^{-1}(x)) \rightarrow 0$ . Since  $E$  is closed and disjoint from  $\hat{\mathbf{X}}$ , the sequence  $\mathbf{z}_i$  cannot be convergent and so  $\lim |\pi_2(\mathbf{z}_i)| = \infty$ . By choosing a compact arc  $C$  in  $\mathbb{R}$  which contains  $\pi_1(\exp^{-1}(x))$  in its interior, we see that  $E \cap \pi_1^{-1}(C)$  is not compact. This contradiction completes the proof.  $\square$

The characterization Lemma 3.1 allows us to show that an accessible point remains accessible throughout the isotopy.

**THEOREM 3.2.** *If  $x \in \text{Bd}(X)$  is accessible from  $U = \mathbb{C} \setminus X$ , then  $h^t(x)$  is accessible from  $U^t$ , where  $U^t$  is the unbounded component of  $\mathbb{C} \setminus h^t(\text{Bd}(X))$ , for each  $t \in [0, 1]$ .*

*Proof.* Suppose that  $x^0$  is an accessible point of  $X^0$ . We may assume that  $x^0 = O$ ,  $h^t(O) = O$  and  $X^t \subset B(O, 1)$  for all  $t$ .

By Lemma 3.1  $\hat{\mathbf{X}}^0 = \mathbf{A}^0 \cup \mathbf{B}^0$  such that conditions (1), (2a) and (2b) of Lemma 3.1 hold. By Lemma 2.3 we can lift the isotopy  $h^t$  to an isotopy  $\mathbf{h}^t : \text{Bd}(\hat{\mathbf{X}}^0) \rightarrow \mathbb{C}$  such that  $\mathbf{h}^0 = \text{id}_{\text{Bd}(\hat{\mathbf{X}}^0)}$ . By Lemma 2.5,  $\mathbf{A}^t$  lies above  $\mathbf{B}^t$  for all  $t$ . It remains to be shown that conditions (2a) and (2b) are satisfied for all  $t$ . By symmetry it suffices to show that (2a) holds. Suppose  $x \in \mathbb{R}$ . By Lemma 2.4 there exists  $x' \in \mathbb{R}$  such that for all  $t$ ,

$$\max \pi_1 \circ \mathbf{h}^t[\pi_1^{-1}((-\infty, x']) \cap \hat{\mathbf{X}}^0)] < x.$$

By (2a) for  $t = 0$ , there exists  $y_2$  such that

$$\pi_1^{-1}([x', \infty)) \cap \pi_2^{-1}([y_2, \infty)) \cap B^0 = \emptyset.$$

Choose  $y_3$  such that for all  $t$ ,

$$\max \pi_2 \circ \mathbf{h}^t[\pi_1^{-1}([x', \infty)) \cap \pi_2^{-1}((-\infty, y_2]) \cap \hat{\mathbf{X}}^0)] < y_3.$$

Then

$$\pi_1^{-1}([x, \infty)) \cap \pi_2^{-1}([y_3, \infty)) \cap B^t = \emptyset$$

and (2a) holds for all  $t$ . Hence by Lemma 3.1,  $O$  is accessible for all  $t$ .  $\square$

#### 4. Continuity of external angles

Recall that  $h$  is an isotopy of a plane continuum  $Z$ ,  $U$  is the unbounded component of  $\mathbb{C} \setminus Z$  and  $X = \mathbb{C} \setminus U$ . Let  $U^t$  be the unbounded component of  $\mathbb{C} \setminus h^t(\text{Bd}(X))$  and  $X^t = \mathbb{C} \setminus U^t$ . Moreover,  $\varphi^t : \mathbb{D} \rightarrow U^t \cup \{\infty\}$  is the normalized Riemann map such that  $\varphi^t(O) = \infty$ ,  $(\varphi^t)'(O) > 0$  and  $R_\theta^t = \varphi^t(\{re^{i\theta} \mid 0 \leq r < 1\})$  is a conformal external ray of  $X^t$ . We construct an isotopy  $\alpha : S^1 \times [0, 1] \rightarrow S^1$  such that if the conformal ray  $R_\theta \subset U^0$  lands on  $x$ , then  $R_{\alpha(\theta, t)}^t \subset U^t$  lands on  $x^t$  in  $X^t$  for each  $t$ . This is accomplished in two steps. We first construct in Lemma 4.1 for each  $t$  a continuously (in the sense of Hausdorff metric) varying arc  $L^t \subset U^t$  landing on  $x^t$ . This arc is contained in the image under the exponential map of the equidistant set constructed in Section 2. The arc  $L^t$  can be extended naturally to a *metric external ray* of  $X^t$  but, for the purpose of this paper, the subarc  $L^t$  is sufficient. Then we show in Theorem 4.2 how the arc  $L^t$  defines a continuous function  $\alpha : [0, 1] \rightarrow S^1$  such that if  $R_{\alpha(0)}^0 \subset U^0$  lands on  $x^0$  in  $Z^0$ , then  $R_{\alpha(t)}^t \subset U^t$  lands on  $x^t$  in  $Z^t$  for each  $t$ .

**LEMMA 4.1.** *Let  $O$  be an accessible point of  $X$ . Then there exists for each  $t$  an arc  $L^t$  such that  $X^t \cap L^t = \{O^t\}$  is an endpoint of  $L^t$  and the function  $\beta : [0, 1] \rightarrow C(\mathbb{C})$  defined by  $\beta(t) = L^t$  is a continuous function to the space  $C(\mathbb{C})$  of compact subsets of  $\mathbb{C}$  with the Hausdorff metric.*

*Proof.* We assume as usual that  $h^t(O) = O$  and  $X^t \subset B(O, 1)$  for all  $t$ . Since every half ray in the plane is tame, we may assume that the positive  $x$ -axis is contained in  $\mathbb{C} \setminus X^0$ . Then  $\hat{\mathbf{X}} \subset \mathbb{C} \setminus \pi_2^{-1}(\{0\})$ . Let  $\mathbf{A}^0 = \hat{\mathbf{X}} \cap \pi_2^{-1}((0, \infty))$  and  $\mathbf{B}^0 = \hat{\mathbf{X}} \cap \pi_2^{-1}((-\infty, 0))$ . Then  $\hat{\mathbf{X}}$  is the union of these two disjoint closed sets and  $\mathbf{A}^0$  lies above  $\mathbf{B}^0$ . Since  $\hat{\mathbf{X}}$  is invariant under vertical translation by  $2\pi$ , it follows that  $E(\mathbf{A}^0, \mathbf{B}^0)$  is contained in  $\pi_2^{-1}([-2\pi, 2\pi])$ . By Lemma 2.5,  $\mathbf{A}^t$  lies above  $\mathbf{B}^t$  for each  $t$ . By Theorem 2.8,  $E(\mathbf{A}^t, \mathbf{B}^t)$  is a ray which separates  $\mathbf{A}^t$  and  $\mathbf{B}^t$  in  $\mathbb{C}$  and  $\pi_1(E(\mathbf{A}^t, \mathbf{B}^t)) = (-\infty, \infty)$ .

Let  $t_i \rightarrow t_0 \in [0, 1]$ . Then  $\mathbf{A}^{t_i} \rightarrow \mathbf{A}^{t_0}$  and  $\mathbf{B}^{t_i} \rightarrow \mathbf{B}^{t_0}$  on compact sets (i.e.,  $K$  compact in  $\mathbf{A}^0$  implies  $K^{t_i} \rightarrow K^{t_0}$ ). It is easy to check that if  $e_i \in E(\mathbf{A}^{t_i}, \mathbf{B}^{t_i})$  and  $e_i \rightarrow e$ , then  $e \in E(\mathbf{A}^{t_0}, \mathbf{B}^{t_0})$ .

By Theorem 2.8  $|E(\mathbf{A}^t, \mathbf{B}^t) \cap \pi_1^{-1}(1)| = 1$ . For each  $t$ , let  $M^t = E(\mathbf{A}^t, \mathbf{B}^t) \cap \pi_1^{-1}((-\infty, 1])$ . Then  $M^t$  is connected and  $\lim M^{t_i} = M^{t_0}$ . Hence  $L^t = \overline{\exp(M^t)}$  is the required arc.  $\square$

By a *crosscut*  $C$  of a nonseparating continuum  $X \subset \mathbb{C}$  we mean an open arc  $C \subset \mathbb{C} \setminus X$  whose closure is a closed arc with distinct endpoints  $a$  and  $b$  which are in  $X$ . In this case we say that the crosscut  $C$  *joins* the points  $a$  and  $b$  of  $X$ . By the *shadow* of  $C$ , denoted by  $\text{Sh}(C)$ , we mean the closure of the bounded complementary domain of  $\mathbb{C} \setminus [X \cup C]$ . Recall that  $U^t$  is the unbounded component of  $\mathbb{C} \setminus Z^t$ ,  $U^0 = U$ ,  $X^t = \mathbb{C} \setminus U^t$  and  $X^0 = X$ . Hence the isotopy  $h$  is defined on  $\text{Bd}(X) \subset Z$ .

**THEOREM 4.2.** *Suppose that  $O \in \text{Bd}(X)$ ,  $h^t : \text{Bd}(X) \rightarrow \mathbb{C}$  is an isotopy such that  $h^0 = \text{id}_{\text{Bd}(X)}$ ,  $h^t(O) = O$  and  $\text{diam}(X^t) < 1$  for all  $t$ . Let  $U^t$  be the component of  $\mathbb{C}^* \setminus h^t(\text{Bd}(X))$  containing  $\infty$ , let  $\varphi^t : \mathbb{D} \rightarrow U^t$  be the normalized Riemann map and let  $L^t$  be an arc with one endpoint at  $O$  such that  $L^t$  varies continuously with  $t$  in the Hausdorff metric and  $L^t \cap X^t = \{O\}$  for all  $t$ . Then the function  $\alpha : [0, 1] \rightarrow S^1$  defined by  $\alpha(t) = S^1 \cap (\varphi^t)^{-1}(L^t)$  is continuous and the external ray  $\varphi^t(\{re^{2\pi i \alpha(t)} \mid r < 1\}) = R_{\alpha(t)}^t$  lands on  $O$  in  $X^t$  for each  $t$ .*

We shall refer to the function  $\alpha : [0, 1] \rightarrow S^1$  in Theorem 4.2 as the *continuous angle function*.

*Proof.* By [Mil99, Cor. 17.10],  $\alpha$  as defined in the statement of the lemma is a function. It remains to be shown that  $\alpha$  is continuous. We will first present an outline of the proof. Fix  $\varepsilon > 0$ . Let  $a(t_0)$  and  $b(t_0)$  be endpoints of a crosscut  $C(t_0)$  of  $X^{t_0}$  in  $B(O, 1/2)$  such that  $L^{t_0}$  lands in the shadow of the crosscut  $C(t_0)$  and  $\text{diam}((\varphi^{t_0})^{-1}(C(t_0))) < 2\varepsilon/3$ . Choose

$$\beta < 1/5 \min\{d(a(t_0), b(t_0)), d(L^{t_0}, \{a(t_0), b(t_0)\}), d(O, C(t_0))\}.$$

We shall choose  $K$ , a large compact subarc of  $C(t_0)$ , such that  $B(L^{t_0}, \beta) \cap C(t_0) \subset K$  and such that  $L^t \subset B(L^{t_0}, \beta)$  and  $K \cap X^t = \emptyset$  whenever  $t$  is close to  $t_0$ . We shall define  $K^t$ , a crosscut of  $X^t$ , which contains a large sub-arc of  $K$  together with two small arcs  $J(a, t)$  and  $J(b, t)$  which join points close to the endpoints of  $K$  to  $X^t$  such that  $(\varphi^t)^{-1}(K^t)$  is a small crosscut of  $\mathbb{D}$  whose shadow contains  $\alpha(t_0)$  and  $\alpha(t)$  for  $t$  sufficiently close to  $t_0$ . This completes the outline of the proof.

Let  $C(t_0)$ ,  $a(t_0)$ ,  $b(t_0)$  and  $\beta$  be defined as above and let  $\hat{a}(0)$  and  $\hat{b}(0)$  be the endpoints of  $(\varphi^{t_0})^{-1}(C(t_0))$ . Then  $\alpha(t_0)$  belongs to the interval  $(\hat{a}(0), \hat{b}(0)) \subset S^1$  which is contained in the closure of  $(\varphi^{t_0})^{-1}(\text{Int}(\text{Sh}(C(t_0))))$ . Then  $|\hat{b}(0) - \hat{a}(0)| < 2\varepsilon/3$ . Choose  $\delta_1 > 0$  such that  $\delta_1 < (1/5) \min\{|\hat{b}(0) - \alpha(t_0)|, |\alpha(t_0) - \hat{a}(0)|, \varepsilon/4, \beta\}$ . Let  $\rho < \sqrt{\rho} < \min\{\delta_1, 1\}$  such that  $\frac{2\pi}{\sqrt{\ln(1/\rho)}} < \delta_1$  and

$$(4.1) \quad (\varphi^{t_0})^{-1}(B(a(t_0), 2\sqrt{\rho}) \cap C(t_0)) \subset B(\hat{a}(0), \delta_1),$$

$$(4.2) \quad (\varphi^{t_0})^{-1}(B(b(t_0), 2\sqrt{\rho}) \cap C(t_0)) \subset B(\hat{b}(0), \delta_1),$$

and there is just one component  $K$  of  $C(t_0) \setminus [B(a(t_0), \rho/2) \cup B(b(t_0), \rho/2)]$  which meets both  $S(a(t_0), \rho/2)$  and  $S(b(t_0), \rho/2)$ .

Next, using Lemma 4.1, the continuity of  $h$  and Theorem 1.1, choose  $\delta_2 > 0$  such that for all  $|t - t_0| < \delta_2$ :

- (i)  $L^t \subset B(L^{t_0}, \rho/2)$ ,
- (ii)  $d(h^t, h^{t_0}) < \rho/2$ ,
- (iii)  $X^t \cap K = \emptyset$  and
- (iv)  $d((\varphi^t)^{-1}|_K, (\varphi^{t_0})^{-1}|_K) < \delta_1$ .

By [Pom92, Prop. 2.2] there exist  $\rho \leq r, s \leq \sqrt{\rho}$  such that if  $J(a, t)$  is a component of  $S(a(t_0), r) \setminus X^t$  which meets  $K$  and  $J(b, t)$  is a component of  $S(b(t_0), s) \setminus X^t$  which meets  $K$ , then

$$(4.3) \quad \text{diam}(\varphi^t)^{-1}(J(z, t)) \leq \frac{2\pi}{\sqrt{\ln(1/\rho)}} < \delta_1 \text{ for } z \in \{a, b\}.$$

Then  $K \cup J(a, t) \cup J(b, t)$  contains a crosscut  $C(t)$  of  $X^t$  and  $L^t$  lands in the shadow of  $C(t)$ . Since the endpoints of  $C(t)$  are joined by a subarc of  $J(z, t) \subset B(z, \sqrt{\rho})$  to  $K$  it follows from (4.1), (4.2), (4.3) and (iv) that the endpoints of  $(\varphi^t)^{-1}(C(t))$  are within  $3\delta_1 < (3/20)\varepsilon$  of the endpoints  $\hat{a}(0)$  and  $\hat{b}(0)$  of  $(\varphi^{t_0})^{-1}(C(t_0))$ . Then for  $|t - t_0| < \delta_2$ ,  $\alpha(t_0)$  is in the shadow of  $(\varphi^t)^{-1}(C(t))$ ,  $\varphi^t(L^t)$  lands in this shadow and the distance between the endpoints of  $(\varphi^t)^{-1}(C(t))$  is less than  $6\delta_1 + 2\varepsilon/3 < (6/20 + 2/3)\varepsilon < \varepsilon$  as desired.  $\square$

**THEOREM 4.3.** *Suppose  $h^t$  is an isotopy of the boundary of a nonseparating continuum  $X \subset \mathbb{C}$  such that  $h^0 = \text{id}_{\text{Bd}(X)}$ . Let  $U^t$  be the component of  $\mathbb{C}^* \setminus h^t(\text{Bd}(X))$  containing  $\infty$  and let  $\varphi^t : \mathbb{D} \rightarrow U^t$  denote the normalized Riemann map. Then there exists an isotopy  $\alpha : S^1 \times [0, 1] \rightarrow S^1$  such that  $\alpha^0 = \text{id}_{S^1}$  and if  $R_\theta^0$  lands on  $x^0 \in \text{Bd}(X^0)$ , then  $R_{\alpha(\theta, t)}^t$  lands on  $x^t \in \text{Bd}(X^t)$  for each  $t$ .*

*Proof.* Suppose that  $R_\theta^0$  lands on  $x^0 \in \text{Bd}(X^0)$ . By Theorem 4.2, there exists a continuous function  $\alpha_\theta : [0, 1] \rightarrow S^1$  such that  $\alpha_\theta(0) = \theta$  and  $R_{\alpha_\theta(t)}^t$  lands on  $x^t \in \text{Bd}(X^t)$  for each  $t$ . Let  $\mathcal{A}$  be the set of angles in  $S^1$  such that for each  $\theta \in \mathcal{A}$ ,  $R_\theta^0$  lands on a point  $x(\theta) \in \text{Bd}(X^0)$ . Define  $\alpha : \mathcal{A} \times [0, 1] \rightarrow S^1$  by  $\alpha(\theta, t) = \alpha_\theta(t)$ . Then  $\alpha$  is a circular order preserving isotopy of  $\mathcal{A}$  such that  $\alpha^0 = \text{id}_{\mathcal{A}}$ . Since  $\mathcal{A}$  is dense in  $S^1$ ,  $\alpha$  can be extended to an isotopy of all of  $S^1$ .  $\square$

We will refer to the isotopy  $\alpha$  as the *continuous angle isotopy*.

## 5. Extension over hyperbolic crosscuts

Suppose  $U$  is an arbitrary component of  $\mathbb{C} \setminus Z$  and  $h : Z \times [0, 1] \rightarrow \mathbb{C}$  is an isotopy such that  $h^0 = \text{id}_Z$ . Then there exists a path  $\gamma : [0, 1] \rightarrow \mathbb{C}$  such that  $\gamma(0) \in U$  and  $\gamma(t) \in \mathbb{C} \setminus Z^t$  for all  $t$ . Then we denote by  $U^t$  the component of  $\mathbb{C} \setminus Z^t$  which contains the point  $\gamma(t)$ . Note that  $U^t$  is independent of the choice of the path  $\gamma$ . Hence, by applying an inversion and rotations of the sphere if necessary, we may assume that  $U^t$  is the component of  $\mathbb{C}^* \setminus Z^t$  which contains the point  $\infty$  for all  $t \in [0, 1]$ . We have shown in the previous section that if there exists a crosscut in  $U^0$  joining the points  $a^0$  and  $b^0$ , then for each  $t$  there exists a crosscut in  $U^t$  joining the points  $a^t$  and  $b^t$ . We will show next that we can choose for each  $t$  a natural crosscut  $C^t$  joining these points such that the isotopy  $h$  can be extended over  $X^0 \cup C^0$ . For this purpose we will use hyperbolic geodesics defined by the Poincaré metric on  $\mathbb{D}$ .

Suppose that  $a^0$  and  $b^0$  are the landing points of the external rays  $R_{\theta(a)}^0$  and  $R_{\theta(b)}^0$  in  $\mathbb{C} \setminus X^0$ . By Theorem 4.2, there exist continuous angle functions  $\alpha : [0, 1] \rightarrow S^1$  and  $\beta : [0, 1] \rightarrow S^1$  such that for each  $t$ ,  $R_{\alpha(t)}^t$  and  $R_{\beta(t)}^t$  land on  $a^t$  and  $b^t$  in  $Z^t$ , respectively. Let  $G^t$  be the hyperbolic geodesic joining the points  $\alpha(t)$  and  $\beta(t)$  in  $\mathbb{D}$  (i.e.,  $G^t$  is the intersection of the round circle through the points  $\alpha(t)$  and  $\beta(t)$  with  $\mathbb{D}$  which crosses  $S^1$  perpendicularly at both of these points). Let  $C^t = \varphi^t(G^t)$ . We will call  $C^t$  the *hyperbolic crosscut of  $X^t$  joining the points  $a^t$  and  $b^t$* . In the final part of this section we will consider  $Z$  as a subset of the sphere and show that the isotopy  $h : \text{Bd}(X^0) \times [0, 1] \rightarrow \mathbb{C}^*$  can be extended to an isotopy  $H : \{\text{Bd}(X^0) \cup C^0\} \times [0, 1] \rightarrow \mathbb{C}^*$  such that  $H^t(C^0) = C^t$ , where  $C^t$  is the hyperbolic crosscut of  $X^t$  in  $U^t$  joining  $a^t$  to  $b^t$ . We will make use of the following well-known theorem [Pom92, Th. 4.20]<sup>2</sup>.

**THEOREM 5.1** (Gehring-Hayman theorem). *There exists a universal constant  $K$  such that for any conformal map  $\varphi : \mathbb{D} \rightarrow \mathbb{C}$ , if  $z_1, z_2 \in \overline{\mathbb{D}}$ ,  $\gamma$  is an arc in  $\mathbb{D}$  from  $z_1$  to  $z_2$ , and  $S$  is the hyperbolic geodesic from  $z_1$  to  $z_2$ , then  $\text{diam}(\varphi(S)) \leq K \text{diam}(\varphi(\gamma))$ .*

We may assume that  $U$  is the unbounded component of  $Z$ , that  $O \in Z$  is an arbitrary accessible point in  $\text{Bd}(U)$ , that  $Z^t \subset \mathbb{C}^* \setminus \{\infty\}$  for all  $t$  and that the isotopy  $h$  fixes the point  $O$ . Recall that  $X = \mathbb{C} \setminus U$  is a nonseparating plane continuum. Hence  $\text{Bd}(X) \subset Z$  and  $h$  is defined on  $\text{Bd}(X)$ . Each angle  $\theta \in S^1$  corresponds to a *prime end* of  $\mathbb{C}^* \setminus X$ . By a fundamental chain  $\{C_j\}$  of crosscuts we mean a sequence of crosscuts of  $X$  such that  $\lim \text{diam}(C_j) = 0$ ,  $C_i \subset \text{Sh}(C_j)$  for  $i > j$  and the arcs  $\overline{C_i}$  are all pairwise disjoint. A naturally defined equivalence class of fundamental chains is called a *prime end* of  $\mathbb{C}^* \setminus X$  (see [Mil99, p. 164] for further details).

**LEMMA 5.2.** *Let  $h$  be an isotopy of  $\text{Bd}(X)$ ,  $O \in \text{Bd}(X)$  and  $h^t(O) = O$  for all  $t$ . Suppose that  $R_\theta^0$  is a conformal external ray of  $X^0$  landing on  $O$ . Then the isotopy  $h$  can be extended to an isotopy  $H : [\text{Bd}(X) \cup R_\theta^0] \times [0, 1] \rightarrow \mathbb{C}$  such that  $H^t(R_\theta^0)$  is an external ray of  $X^t$  landing on  $O$ .*

*Proof.* By Theorem 4.2, there exists a continuous angle function  $\alpha : [0, 1] \rightarrow S^1$  such that  $\alpha(0) = \theta$  and the (conformal) external ray  $R_{\alpha(t)}^t$  lands on  $O$  for each  $t$ . Extend the isotopy  $h$  over  $R_{\alpha(0)}^0$  by

$$(5.1) \quad H(z, t) = \varphi^t \circ \rho^t \circ (\varphi^0)^{-1}(z)$$

for  $z \in R_{\alpha(0)}^0$ , where  $\rho^t$  is the rotation of  $\mathbb{D}$  by the angle  $\alpha(t) - \alpha(0)$ . By Carathéodory kernel convergence,  $H$  is an isotopy of every compact subset of  $R_{\alpha(0)}^0$ . Hence it suffices to show that if  $z_i \rightarrow O$  in  $R_{\alpha(0)}^0$  and  $t_i \rightarrow \tau$ , then  $H(z_i, t_i) \rightarrow O = H(O, \tau)$ .

<sup>2</sup>We are indebted to Paul Fabel for this reference.

To see this fix  $\varepsilon > 0$ . Recall that  $R_{\alpha(t)}^t$  is a ray which runs from  $\infty = \varphi^t(O)$  to its landing point  $O \in \text{Bd}(X^t)$ . It suffices to show that there exists an open disk  $B$  containing  $O$  with simple closed curve boundary  $S$  (constructed below) and  $\delta > 0$  such that for all  $t$  with  $|t - \tau| < \delta$ , if  $z^t$  is the first point of  $R_{\alpha(t)}^t$  (tracing  $R_{\alpha(t)}^t$  from  $\infty$ ) on  $S$  and if  $CR_{z^t}^t$  is the component of  $R_{\alpha(t)}^t \setminus z^t$  from  $z^t$  to  $O$ , then  $CR_{z^t}^t \subset B(O, \varepsilon)$ .

Let  $K$  be the universal constant from Theorem 5.1. By Lemma 4.1 there exists a continuously varying arc  $L^t \subset \mathbb{C} \setminus X^t$  landing on  $O$  in  $X^t$  for each  $t$  such that  $(\varphi^0)^{-1}(L^0) \cap S^1 = \{\theta\}$ . Choose a fundamental chain of crosscuts  $C_n^\tau$  of  $X^\tau$  for the prime-end  $\alpha(\tau)$ . Then both  $L^\tau$  and  $R_{\alpha(\tau)}^\tau$  cross  $C_n^\tau$  essentially (that is  $X \cup C_n^\tau$  separates the endpoints of  $L^\tau$  and also the ends of the ray  $R_{\alpha(\tau)}^\tau$ ). Hence we can choose  $n$  sufficiently large and a simple closed curve  $S$  containing  $O$  in its bounded complementary domain  $B$  such that  $C_n^\tau \subset S$ ,  $[L^\tau \cup R_{\alpha(\tau)}^\tau] \cap [S \setminus C_n^\tau] = \emptyset$  and  $\text{diam}(S) < \varepsilon/K$ . From now on fix this  $n$  and let  $a$  and  $b$  be the endpoints of  $C_n^\tau$ .

For  $t$  close to  $\tau$ , let  $w^t$  be the first point (tracing  $L^t$  from  $O$ ) of  $L^t$  on  $S$ . Let  $C^t$  be the component of  $S \setminus X^t$  containing the point  $w^\tau$ . Choose

$$\rho < (1/3)d(\{a, b\}, [L^\tau \cup R_{\alpha(\tau)}^\tau])$$

and let  $C_-^t$  be the component of  $C^t \setminus [B(a, \rho) \cup B(b, \rho)]$  which contains  $w^\tau$ . Choose  $\delta > 0$  such that if  $|t - \tau| < \delta$ , then

- (1)  $w^\tau \in \mathbb{C} \setminus [X^t \cup B(a, \rho) \cup B(b, \rho)]$ ,
- (2)  $C_-^t = C_-^\tau$ ,
- (3)  $L^t \subset B(L^\tau, \rho)$ ,
- (4) if  $z^t$  is the first point (tracing  $R_{\alpha(t)}^t$  from  $\infty$ ) of  $R_{\alpha(t)}^t$  on  $S$ , then  $z^t \in C_-^t$ .

The first and second conditions follow from the continuity of  $h$  and the third from the continuity of  $L^t$ . The last condition follows from Carathéodory kernel convergence: recall that  $d(R_{\alpha(\tau)}^\tau, S \setminus C^\tau) = \eta > 0$ . Let  $v \in R_{\alpha(\tau)}^\tau \cap B$  such that the component of  $R_{\alpha(t(\theta)}^t \setminus \{v\}$  from  $v$  to  $O$  is contained in  $B$  and let  $(\varphi^\tau)^{-1}(v) = r_0 \exp(\alpha(\tau))$ . By Carathéodory kernel convergence,  $I^t = \varphi^t(\{r \exp(\alpha(t)) \mid 0 \leq r \leq r_0\})$  converges to the segment from  $v$  to  $\infty$  in  $R_{\alpha(\tau)}^\tau$ . Hence  $d(I^t, S \setminus C^\tau) > \eta/2$  and  $d(I^t, v) < (1/2)d(v, S)$  for  $t$  close to  $\tau$ , and (4) hold for  $\delta$  sufficiently small.

By (2), (3) and (4), the sub-arc  $A^t$  of  $C^t$  joining the points  $w^t$  and  $z^t$ , is contained in  $\mathbb{C} \setminus X^t$ . Hence the union of the arcs  $A^t$  and  $[w^t, O] \subset L^t$  is an arc in  $[\mathbb{C} \setminus X^t] \cup \{O\}$ , joining  $z^t$  to  $O$ , of diameter less than  $\varepsilon/K$ . By Theorem 5.1, the terminal segment  $CR_{z^t}^t \subset B(O, \varepsilon)$  as required.  $\square$

In the remaining part of the paper we will consider  $Z$  as a subset of the unit sphere  $\mathbb{C}^* \subset \mathbb{R}^3$  with *spherical metric*  $\rho$ . Hence the distance between two points  $z, w \in \mathbb{C}^*$  is the length of the shortest arc in the great circle which is the intersection of  $\mathbb{C}^*$  and the plane through  $z, w$  and the origin in  $\mathbb{R}^3$ .

Since every hyperbolic crosscut is conformally equivalent to a diameter of  $\mathbb{D}$  it follows that we can extend the isotopy  $h^t$  over any hyperbolic crosscut  $C^0 \subset U^0$  joining two points  $a^0$  and  $b^0$  in  $Z^0$  to an isotopy  $H : Z^0 \cup C^0 \rightarrow \mathbb{C}^*$  (since in this case the point at infinity is not fixed, the range of the isotopy must be the sphere). Note that if  $C_i$  is a convergent sequence of hyperbolic crosscuts whose limit contains a nondegenerate subcontinuum  $Y \subset Z$ , then this extension of the isotopy over  $\cup C_i$  is not necessarily continuous at  $Y$ . However, we can extend over a suitable compact set of hyperbolic crosscuts in  $U$  as follows.

Suppose that  $\mathcal{H}$  is a collection of disjoint hyperbolic crosscuts in  $U$  such that the set  $\bigcup_{C \in \mathcal{H}} \overline{C}$  is compact and there exists  $\varepsilon > 0$  such that for each  $C \in \mathcal{H}$ ,  $\text{diam}(C) \geq \varepsilon$ , then we call  $\mathcal{H}$  a *compact set of disjoint hyperbolic crosscuts in  $U$* . Let  $a_C$  and  $b_C$  denote the endpoints of  $C \in \mathcal{H}$ , let  $\alpha_C$  and  $\beta_C$  be the corresponding endpoints of  $(\varphi^0)^{-1}(C)$  and let  $\mathcal{A}^0$  denote the union of all pairs of the angles  $\{\alpha_C, \beta_C\}$  for  $C \in \mathcal{H}$ . Let  $\alpha^t$  be the continuous angle isotopy and let  $\mathcal{A}^t = \alpha^t(\mathcal{A}^0)$ . Then for each  $t \in [0, 1]$ , the collection of hyperbolic chords  $\alpha^t(\alpha_C)\alpha^t(\beta_C)$ ,  $C \in \mathcal{H}$ , is a compact lamination in the unit disk in the sense of Thurston [Thu09]. We will denote the family of all such chords by  $\mathcal{L}^t$ . We will say that  $\mathcal{L}^0 = \mathcal{L}$  is the *pullback of the lamination  $\mathcal{H}$  to the unit disk*. We will call each element of  $\mathcal{L}^t$  ( $\mathcal{H}^t$ ) a *hyperbolic geodesic in  $\mathcal{L}^t$  ( $\mathcal{H}^t$ )* and we denote the union of all hyperbolic geodesics in  $\mathcal{L}^t$  ( $\mathcal{H}^t$ ) by  $\mathcal{L}^{t*}$  ( $\mathcal{H}^{t*}$ , respectively). Note that any two distinct hyperbolic geodesics in  $\mathcal{L}^t$  meet at most in a common endpoint and there exists  $\delta > 0$  such that for each  $t$  and each chord in  $\mathcal{L}^t$ ,  $\text{diam}(\alpha^t(\alpha_C)\alpha^t(\beta_C)) > \delta$ . Let  $\Lambda^t : \mathcal{L}^0 \rightarrow \mathcal{L}^t$  be the *linear isotopy on  $\mathcal{L}$*  which extends  $\alpha^t$  such that  $\Lambda^t$  maps each chord  $\alpha^0(\alpha_C)\alpha^0(\beta_C)$  in  $\mathcal{L}$  linearly onto the chord  $\alpha^t(\alpha_C)\alpha^t(\beta_C)$  in  $\mathcal{L}^t$ . Then the following theorem follows.

**THEOREM 5.3.** *Suppose that  $\mathcal{H} = \mathcal{H}^0$  is a compact set of disjoint hyperbolic crosscuts in  $U^0$ . Then the isotopy  $h : Z^0 \times [0, 1] \rightarrow \mathbb{C}^*$  can be extended to an isotopy  $H : [Z \cup \mathcal{H}^*] \times [0, 1] \rightarrow \mathbb{C}^*$  such that  $H^t(\mathcal{H}^*) = \mathcal{H}^{t*} = \varphi^t(\mathcal{L}^{t*})$  and  $H$  is defined by:*

$$H^t(z) = \begin{cases} h^t(z), & \text{if } z \in Z^0; \\ \varphi^t \circ \Lambda^t \circ (\varphi^0)^{-1}(z) & \text{if } z \in \mathcal{H}^*, \end{cases}$$

where  $\mathcal{L}$  is the pullback of  $\mathcal{H}$  and  $\Lambda^t$  is the linear extension of the angle isotopy  $\alpha^t$  over  $\mathcal{L}$ .

We will say that the extended isotopy  $H$  defined in Theorem 5.3 is the *linear extended isotopy which preserves hyperbolic crosscuts in  $\mathcal{H}$* .

## 6. Existence of short crosscuts

It follows from the results of the previous section that if  $C$  is the hyperbolic crosscut of  $Z$  which joins the points  $a$  and  $b$  in a complementary domain  $U$  of  $Z$  in  $\mathbb{C}^*$ , then we can extend the isotopy to an isotopy  $H$  of  $Z \cup C$  such that

$H^t(C) = C^t$  is the hyperbolic crosscut joining the points  $a^t$  and  $b^t$ . We need to show that if the crosscut  $C$  has small diameter, then the crosscut  $C^t$  also has small diameter. If  $C$  is contained in the component  $U$  of  $\mathbb{C}^* \setminus Z$ , then we denote by  $U^t$  the component of  $\mathbb{C}^* \setminus Z^t$  which contains  $C^t$ .

Given a hyperbolic crosscut  $C$  of a continuum  $Z \subset \mathbb{C}^*$ , we say that  $C$  is a  $\delta$ -hyperbolic crosscut if the diameter of  $C$  is less than  $\delta$ . Note that we see  $Z$  as a subset of the sphere  $\mathbb{C}^*$  with the spherical metric  $\rho$ .

**THEOREM 6.1.** *For each  $\varepsilon > 0$  there exists  $\delta > 0$  such that if  $x, y \in Z$  can be joined by a  $\delta$ -hyperbolic crosscut  $C \subset U$ , where  $U$  is a component of  $\mathbb{C}^* \setminus Z$ , then if  $H_C : Z \cup C \rightarrow \mathbb{C}^*$  is the linear extended isotopy which preserves the hyperbolic crosscut  $C$  and  $H_C^t(C) = C^t$  is the hyperbolic crosscut joining  $x^t$  to  $y^t$  in  $U^t$ , then  $C^t$  is an  $\varepsilon$ -hyperbolic crosscut.*

*Proof.* The main idea of the proof is a variant of the following notion. Let  $S \subset \mathbb{C}$  be a simple closed curve. Call a complementary domain  $U$  of  $S$  *odd* if there exists an arc  $J \subset \mathbb{C}^*$  such that  $J$  is transverse to  $S$ , one endpoint of  $J$  is  $\infty$ , the other endpoint is in  $U$  and  $|J \cap S|$  is odd. Given an isotopy  $p : S^1 \times [0, 1] \rightarrow \mathbb{C}$  the odd components of  $\mathbb{C}^* \setminus p^t(S^1)$  change continuously. (See [OT82, Lemmas 2.2 and 2.3] where this idea was used for a continuous family of paths  $p^t : S^1 \rightarrow \mathbb{C}$ .)

We will first provide an overview of the proof. By Theorem 5.3 for every component  $U$  of  $\mathbb{C}^* \setminus Z$  and every hyperbolic crosscut  $C \subset U$  joining points  $x$  and  $y$  in  $\text{Bd}(U)$ , we can extend the isotopy  $h$  to an isotopy  $H_C : Z \cup C \rightarrow \mathbb{C}^*$  such that  $H_C^t(C) = C^t$  is a hyperbolic crosscut joining the points  $x^t$  and  $y^t$  in  $\text{Bd}(U^t)$ . Suppose that the theorem fails for some  $\varepsilon > 0$ . We first note that we may assume that  $O \in Z$ ,  $h^t(O) = O$  for all  $t \in [0, 1]$  and there exists  $\delta > 0$  such that if  $h^t(z) = z^t \in Z^t \cap B(O, \delta)$  for some  $t$ , then  $z^s \in B(O, \varepsilon/3K)$  for all  $s \in [0, 1]$ , where  $K$  is the Gehring-Hayman constant from Theorem 5.1. We will show that there exist a component  $U = U^0$  of  $\mathbb{C} \setminus Z$  and three accessible points  $x, y, w$  in  $\text{Bd}(U)$  such that the hyperbolic geodesic  $C = C^0$ , which joins  $x^0$  to  $y^0$  in  $U^0$ , is contained in  $B(O, \delta)$  and such that for all  $t$ ,  $\{x^t, y^t\} \subset B(O, \delta)$  while  $w^t \in \mathbb{C} \setminus \overline{B(O, \delta)}$ , and for some  $v$  the hyperbolic geodesic  $C^v = H_C^v(C)$ , joining  $x^v$  to  $y^v$ ,  $C^v \setminus B(O, \varepsilon/2) \neq \emptyset$ . By Theorem 5.1 if there exists a crosscut  $A$  in  $U^v$ , homotopic to  $C^v$  in  $U^v$  with endpoints fixed, joining  $x^v$  to  $y^v$  such that  $A \subset B(\varepsilon/2K)$  then  $C^v \subset B(O, \varepsilon/2)$ , a contradiction.

To see that such an arc  $A$  exists we proceed as follows. Put  $D = B(O, \delta)$  and  $P^t = D \cup C^t$ , then  $P^t$  depends continuously on  $t$ . Components of  $\mathbb{C}^* \setminus P^t$  consist of two kinds: those that can be connected by an arc  $J \subset \mathbb{C}^* \setminus D$  to the point  $w^t$  such that every intersection of  $J$  and  $C^t$  is transverse and  $|J \cap C^t|$  is even, and those for which  $|J \cap C^t|$  is odd. Call the latter ones *odd domains* of  $\mathbb{C}^* \setminus P^t$ . Note that both even and odd domains of  $\mathbb{C}^* \setminus P^t$  may contain points of  $Z^t$ . Let  $Q^t$  be the union of  $P^t$  and all odd domains of  $\mathbb{C}^* \setminus P^t$ . We will show that  $Q^t$  is continuous in  $t$  and, even though odd domains may contain points of  $Z^t$ , if

$z^t \in Z^t \cap Q^t$ , then

$$(6.1) \quad \rho(z^t, O) < \varepsilon/2K.$$

This means that if we can join two points of  $C^t$  by an arc  $J$  contained in  $Y^t \setminus B(O, \varepsilon/2K)$ , where  $Y^t$  is an odd domain of  $\mathbb{C}^* \setminus P^t$ , then  $J \subset U^t$ . We will use this fact to obtain the required arc  $A$ .

Choose an arc  $M \subset U^t$ , joining  $x^t$  to  $y^t$ , satisfying similar conditions as  $C^t$  (in particular (6.1) but now with regard to odd domains of  $D \cup M$ ; see below) such that all intersections of  $M$  with the boundary of  $B(O, \varepsilon/2K)$  are transverse and  $n = |M \cap \text{Bd}(B(O, \varepsilon/2K))|$  is minimal. If  $n = 0$  we are done, hence we may assume that  $n > 0$  and then  $n \geq 2$ . In this case we change the set  $P^t = D \cup C^t$  to  $P_M^t = D \cup M$  and define odd regions of  $\mathbb{C}^* \setminus P_M^t$  as above but now counting intersections with  $M$  rather than  $C^t$ . Similarly, we define the set  $Q_M^t$  as the union of  $P_M^t$  and all odd domains of  $\mathbb{C}^* \setminus P_M^t$ . Finally we construct a shortcut  $J \subset Y_M \setminus B(O, \varepsilon/2K)$ , where  $Y_M$  is an odd domain of  $\mathbb{C}^* \setminus P_M^t$ , joining two points  $j_1, j_2 \in M$  such that if we replace the subarc of  $M$  between  $j_1$  and  $j_2$  by  $J$ , then we obtain a new crosscut  $M' \subset U^t$ , satisfying all required conditions, joining  $x^t$  to  $y^t$  with  $|M' \cap \text{Bd}(B(O, \varepsilon/2K))| < n$ . This contradiction with the minimality of  $n$  will complete the proof. This completes the outline of the proof.

We proceed with the details; suppose the theorem fails for  $\varepsilon > 0$ . Then there exist  $x_n, y_n \in Z^0 = Z$  and a sequence of  $1/n$ -hyperbolic crosscuts  $C_n$  in complementary domains  $U_n$  joining them and  $t_n \in [0, 1]$  such that the points  $x_n^{t_n}$  and  $y_n^{t_n}$  are not joined by an  $\varepsilon$ -hyperbolic crosscut in  $U_n^{t_n}$ . Without loss of generality, the origin  $O \in Z$ ,  $\lim C_n = \{O\}$  and  $h^t(O) = O$  for all  $t$ . Then there exists  $0 < \varepsilon' < \varepsilon$  and  $w_n \in \text{Bd}(U_n)$ , accessible from  $U_n$ , such that  $\rho(w_n^t, O) > \varepsilon'$  for all  $n$  and all  $t \in [0, 1]$ .

Let  $K$  be the universal constant from Theorem 5.1. Choose  $0 < \delta < \varepsilon'/3$  such that if  $h^t(z) \in \overline{B(O, \delta)}$  for some  $t \in [0, 1]$ , then  $h^s(z) \in B(O, \varepsilon/3K)$  for all  $s \in [0, 1]$ . Choose  $n_0$  such that  $C_{n_0} \subset B(O, \delta)$  and  $\{x_{n_0}^s, y_{n_0}^s\} \subset B(O, \delta)$  for all  $s \in [0, 1]$ . From now on we fix this  $n = n_0$  and, hence, we can omit  $n$  from the notation. In particular we have a fixed component  $U$  of  $\mathbb{C}^* \setminus Z$ , three points  $x, y, w \in \text{Bd}(U)$  with  $x$  and  $y$  joined by the hyperbolic crosscut  $C \subset U \cap B(O, \delta)$ , with  $x^s, y^s \in B(O, \delta)$  and  $w^s \in \mathbb{C} \setminus \overline{B(O, \delta)}$  for all  $s \in [0, 1]$ . By Theorem 5.3 we can extend the isotopy  $h$  to an isotopy  $H$  of  $Z \cup C$  such that  $H^t(C) = C^t$  is the hyperbolic crosscut joining  $x^t$  to  $y^t$  in  $U^t \subset \mathbb{C}^* \setminus Z^t$  for each  $t$ .

Let  $D$  be the closed  $\delta$ -ball centered at  $O$ . For each  $t \in [0, 1]$ , let  $P^t = D \cup C^t$ . Since  $\text{Bd}(P^t)$  is contained in  $S(O, \delta) \cup C^t$ , which is a finite union of arcs,  $\text{Bd}(P^t)$  contains no continuum of convergence and each sub-continuum of  $\text{Bd}(P^t)$  is locally connected and arcwise connected [Why42, V.2.1].

Since  $C^t$  is an arc, the components  $\{T_i\}$  of  $C^t \setminus D$  form a null sequence. For each  $i$ ,  $\overline{T_i}$  is an arc and  $\overline{T_i} \cap D$  consists of the endpoints of  $T_i$ . Each point of  $\mathbb{C}^* \setminus P^t$  can be joined to  $w^t$  by an arc in  $\mathbb{C}^* \setminus D$  which meets  $P^t$  in a finite set.

Suppose that  $V$  is a component of  $\mathbb{C}^* \setminus P^t$ . We say that  $V$  is an *odd domain* (respectively *even domain*) of  $\mathbb{C}^* \setminus P^t$  if there is a closed arc  $A \subset \mathbb{C}^* \setminus D$  from  $w^t$  to a point in  $V$  such that  $|A \cap C^t|$  is odd (respectively, even) and  $A$  is transverse to  $C^t$  at each point of  $A \cap C^t$ . This definition is independent of the choice of the arc and the point in  $V$ .

Let  $Q^t = P^t \cup \bigcup \{V \mid V \text{ is an odd domain of } \mathbb{C}^* \setminus P^t\}$ . The boundary of each odd domain  $V$  of  $\mathbb{C}^* \setminus P^t$  is a simple closed curve which meets  $D$  and there exists a  $T_i$  such that  $T_i$  is contained in  $\text{Bd}(V)$  and  $T_i \cup D$  separates  $V$  from  $w^t$  in  $\mathbb{C} \setminus D$ . Also each  $T_i$  is contained in the boundary of exactly one odd domain of  $\mathbb{C}^* \setminus P^t$ . Since the odd domains form a null family,  $Q^t$  is a locally connected continuum.

Let  $t_i$  converge to  $t \in [0, 1]$ . We prove that  $\lim Q^{t_i} = Q^t$ . Note that  $\lim P^{t_i} = P^t$ . Let  $z \in \mathbb{C}^* \setminus P^t$ . It suffices to prove that  $z \in Q^t$  if and only if  $z \in Q^{t_i}$  for all sufficiently large  $i$ . Let  $A \subset \mathbb{C}^* \setminus B(O, \rho(O, z))$  be a piecewise linear arc from  $z$  to  $w^t$  which witnesses whether or not  $z \in Q^t$ . Then  $A$  meets only finitely many, without loss of generality  $T_1, \dots, T_n$ , of the open arcs  $T_j$ . Let  $H : Z \cup C \rightarrow \mathbb{C}^*$  be the extended linear isotopy of Theorem 5.3 such that  $H^t(C) = C^t$ . Let  $\delta < \delta' < \min(\rho(z, O), 2\varepsilon'/3)$  then for all  $i$  sufficiently large  $\overline{B(O, \delta')} \cup T_j$  separates  $z$  from  $w^t$  if and only if  $\overline{B(O, \delta')} \cup H^{t_i}((H^t)^{-1}(T_j))$  does for each  $j = 1, \dots, n$  and  $T_j \cap A \neq \emptyset$  if and only if  $H^{t_i}((H^t)^{-1}(T_j)) \cap A \neq \emptyset$  for all  $j$ .

Note that  $|T_j \cap A|$  is odd if and only if  $\overline{B(O, \delta')} \cup T_j$  separates  $z$  from  $w^t$ . Fix any large  $i$  and choose an arc  $M$  very close to  $A$  which witnesses whether  $z$  is in  $Q^{t_i}$ . Then  $|M \cap H^{t_i}((H^t)^{-1}(T_j))| = |A \cap T_j| \pmod 2$  for  $j = 1, \dots, n$  and  $M \cap H^{t_i}((H^t)^{-1}(T_j)) = \emptyset$  for all  $j > n$ . Hence  $z \in Q^{t_i}$  if and only if  $z \in Q^t$  as desired.

Let  $z^t \in Q^t \cap Z^t$ . We prove that  $\rho(z^t, O) < \varepsilon/3K$ . We may assume that  $z^t \notin \{x^t, y^t\} \cup D$ . Let  $s_0 = \inf\{s \in [0, 1] \mid z^s \in Q^s \setminus D\}$ . Since  $Q^0 = D$  and  $Z^s \cap C^s = \emptyset$  for all  $s$ ,  $z^{s_0} \in D$ . Hence, by the choice of  $\delta$ ,  $\rho(O, z^t) < \varepsilon/3K$ .

It remains to prove the following:

CLAIM.  $U^t \cap B(O, \varepsilon/2K)$  contains an arc  $A$  such that  $A \cap Z^t = \{x^t, y^t\}$ .

The difficulty in showing that such an arc exists is a consequence of the fact that both even and odd domains of  $\mathbb{C}^* \setminus P^t$  may contain points of  $Z^t$ . Hence the required arc  $A \subset U^t$  must “weave around  $Z^t$ ” while at the same time staying close to  $O$ . The proof below shows that this is possible.

*Proof of the claim.* Fix  $t \in [0, 1]$ . Then  $C^t \subset U^t$  is an arc such that  $C^t \cap Z^t = \{x^t, y^t\}$ . After a small perturbation of  $C^t$  we may assume that  $C^t \cap S(O, \varepsilon/2K)$  is finite and all intersections are transverse. Then  $C^t$  satisfies conditions (1)-(3) below. Note that the definition of an odd domain of  $\mathbb{C}^* \setminus P^t$  was with respect to  $P^t = D \cup C^t$ . In what follows we will use the same definition but now with respect to  $P_M^t = D \cup M$ , where  $M \subset U^t \cup \{x^t, y^t\}$  is an arc homotopic to  $C^t$  in  $U^t \cup \{x^t, y^t\}$  with endpoints fixed such that:

- (1)  $M \setminus \{x^t, y^t\} \subset U^t$  and  $x^t$  and  $y^t$  are endpoints of  $M$ ,

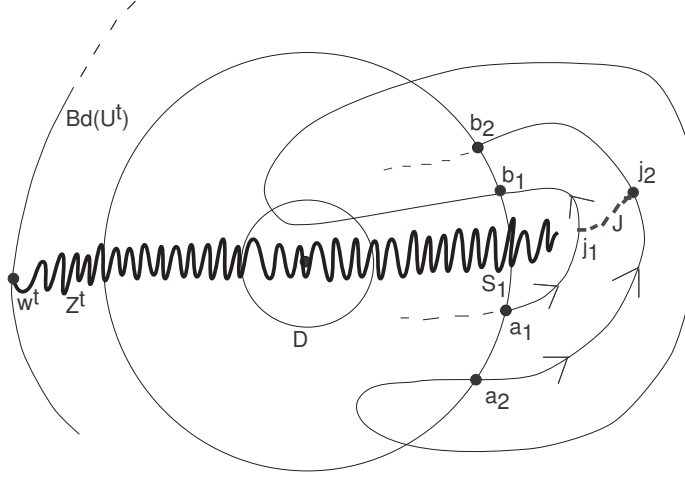
- (2)  $M \cap S(O, \varepsilon/2K)$  is finite and all intersections are transverse,
- (3) for each odd domain  $V$  of  $\mathbb{C}^* \setminus P_M^t$  and each  $z^t \in Z^t \cap V$ ,  $\rho(z^t, O) < \varepsilon/3K$ ,
- (4)  $n = |M \cap S(O, \varepsilon/2K)|$  is minimal.

If  $n = 0$  we are done. Note that  $n = 1$  is impossible since all intersections of  $S(O, \varepsilon/2K)$  and  $M$  are transverse and both endpoints of  $M$  are in  $B(O, \varepsilon/2K)$ . Hence, assume  $n > 1$ . Let  $Q_M^t = P_M^t \cup \bigcup \{V \mid V \text{ is an odd domain of } \mathbb{C}^* \setminus P_M^t\}$ . Let  $S_i$  be all components of  $S(O, \varepsilon/2K) \setminus M$ . Since each component of  $M \setminus D$  is an arc which locally separates the plane, points on one side of such a component are in an even domain and points on the other side are in an odd domain. Hence, each arc  $S_i$  is contained in a complementary domain  $V_i$  of  $\mathbb{C}^* \setminus P_M^t$  and these domains are alternately even and odd moving around the circle  $S(O, \varepsilon/2K)$ . In particular,  $n$  is even. We may order  $M$  so that  $x^t < y^t$  and we write intervals in  $M$  as in  $\mathbb{R}$ .

Let  $\mathcal{M} = \{M_i\}$  be the collection of all components of  $M \setminus \overline{B(O, \varepsilon/2K)}$ . We can define a partial order  $<$  on  $\mathcal{M}$  by  $M_1 < M_2$  if  $M_2$  separates  $M_1$  from  $w^t$  in  $\mathbb{C} \setminus \overline{B(O, \varepsilon/2K)}$ . Assume that  $M_1 = (a_1, b_1)$  (with  $a_1 < b_1$ ) is a minimal element of  $\mathcal{M}$ . Then  $M_1 \cup \overline{B(O, \varepsilon/2K)}$  bounds a disk  $V_1$  whose closure meets  $\overline{B(O, \varepsilon/2K)}$  in an arc  $S_1 \subset S(O, \varepsilon/2K)$  and  $S_1 \cap M = \{a_1, b_1\}$ . Then  $S_1$  is either contained in an even or an odd domain of  $\mathbb{C}^* \setminus P_M^t$ .

*Subcase 0.* Suppose that  $Z^t \cap S_1 = \emptyset$  (by (3) this must be the case if  $S_1$  is contained in an odd domain). In this case choose  $a'_1 < a_1 < b_1 < b'_1$ , with  $a'_1$  in  $B(O, \varepsilon/2K) \cap M$  very close to  $a_1$  and  $b'_1$  in  $B(O, \varepsilon/2K) \cap M$  very close to  $b_1$ , and an arc  $S'_1 \subset B(O, \varepsilon/2K)$  very close to  $S_1$  from  $a'_1$  to  $b'_1$  such that  $S'_1 \cap Z^t = \emptyset$ . Then  $Z^t$  is disjoint from the bounded complementary domain  $B$  of the simple closed curve  $F = S'_1 \cup (a'_1, b'_1)$ . Hence there exists a homotopy of the plane which is the identity on  $Z^t \cup S'_1 \cup [x^t, a'_1] \cup [b'_1, y^t]$  and shrinks  $B$  to  $S'_1$ . Let  $M' = S'_1 \cup [M \setminus (a_1, b_1)]$ . Then  $z \in Z^t$  lies in an odd domain of  $\mathbb{C}^* \setminus P_M^t$  if and only if  $z$  lies in an odd domain of  $\mathbb{C}^* \setminus [D \cup M']$ . Thus  $M'$  satisfies (3). Clearly  $M'$  satisfies (1-2) and since  $|M' \cap S(O, \varepsilon/2K)| < n$  we have a contradiction with the minimality of  $n$ .

Hence we may assume that  $S_1 \cup V_1$  is contained in an even domain and  $Z^t \cap S_1 \neq \emptyset$ . Then there exists  $M_2 = (a_2, b_2) \in \mathcal{M}$  (with  $a_2 < b_2$ ) such that  $M_2$  is the immediate successor of  $M_1$  in  $\mathcal{M}$  (i.e., no element of  $\mathcal{M}$  between  $M_1$  and  $M_2$  separates  $M_1$  from  $\infty$ ). Let  $V_2$  be the component of  $\mathbb{C}^* \setminus [\overline{V_1 \cup B(O, \varepsilon/2K)} \cup M_2]$  whose closure contains the arc  $(a_1, b_1)$ . Since  $M_2$  is the immediate successor of  $M_1$  in  $\mathcal{M}$ , there exists an arc  $J \subset [V_2 \setminus M] \cup \{j_1, j_2\}$  with one endpoint of  $J$ ,  $j_1 \in (a_1, b_1)$  and the other endpoint of  $J$ ,  $j_2 \in (a_2, b_2)$ . Moreover, since  $V_1$  was even,  $J$  is contained in an odd domain and  $J \cap Z^t = \emptyset$ . We will examine the circular (counterclockwise) order  $<_C$  of the four points  $a_1, b_1, a_2, b_2$  around the circle  $S(O, \varepsilon/2K)$ .



**Figure 1.** Subcase 1 in the proof of Theorem 6.1

*Subcase 1.*  $a_2 <_C a_1 <_C b_1 <_C b_2$ . We have either  $a_1 < b_1 < a_2 < b_2$  (see Figure 1) or  $a_2 < b_2 < a_1 < b_1$ . Since  $w^t \in Z^t \cap \text{Bd}(U^t)$ ,  $Z^t \cap S_1 \neq \emptyset$  and  $M \cap Z^t = \{x^t, y^t\}$ , in either case (see the gate theorems in [Bec74, p. 36]), the simple closed curve  $F = J \cup [j_1, j_2]$  (where  $[j_1, j_2]$  is the arc in  $M$  with endpoints  $j_1$  and  $j_2$ ) separates  $x^t$  from  $y^t$ . (To see this note (see Figure 1) that the points  $a_1$  and  $b_2$  are separated by  $F$ . Since the subarcs  $[x, a_1]$  and  $[b_2, y]$  of  $M$  are disjoint from  $F$ ,  $F$  also separates  $x$  and  $y$ .) Since  $Z^t \cap F = \emptyset$ , this contradicts the connectedness of  $Z^t$ .

*Subcase 2.*  $b_2 <_C a_1 <_C b_1 <_C a_2$ . Then either  $a_1 < b_1 < a_2 < b_2$  or  $a_2 < b_2 < a_1 < b_1$ . Since  $M \cap Z^t = \{x^t, y^t\}$ ,  $w^t \in Z^t \cap \text{Bd}(U^t)$  and  $Z^t \cap S_1 \neq \emptyset$ ,  $x^t$  and  $y^t$  are contained in the unbounded component of  $F = J \cup [j_1, j_2]$  and the proof proceeds as in Subcase 0, where  $F$  is now  $J \cup [j_1, j_2]$ . To see this note that since  $M \cap Z^t = \{x^t, y^t\}$ ,  $w^t \in Z^t \cap \text{Bd}(U^t)$  and  $Z^t \cap S_1 \neq \emptyset$ ,  $a_1$  and  $b_2$  are now contained in the same component of  $\mathbb{C} \setminus F$  as  $Z^t$ . Then  $M' = (M \setminus [j_1, j_2]) \cup J$  is an arc with endpoints  $x^t$  and  $y^t$ ,  $M' \cap Z^t = \{x^t, y^t\}$  and  $M'$  is homotopic to  $M$  with fixed endpoints in  $U^t \cup \{x^t, y^t\}$ ,  $M'$  is transverse to  $S(O, \varepsilon/2K)$  and  $|M' \cap S(O, \varepsilon/2K)| < |M \cap S(O, \varepsilon/2K)|$ . Moreover, if  $z \in Z^t$  is in an odd domain of  $\mathbb{C}^* \setminus P_M^t$ , then  $z$  is also in an odd domain of  $\mathbb{C}^* \setminus P_{M'}^t$  and (3) holds. This contradicts the minimality of  $M$ .

Up to interchanging clockwise with counterclockwise, these are all the cases. This completes the proof of the claim.

Hence for each  $t$  there exists a crosscut  $M(t)$ , homotopic to  $C^t$ , joining  $x^t$  to  $y^t$  in  $U^t$  such that  $\text{diam}(M(t)) < \varepsilon/K$ . By Theorem 5.1, the diameter of the hyperbolic crosscut  $C^t$  is less than  $\varepsilon$  for all  $t$ . This contradiction completes the proof.  $\square$

## 7. Extending the isotopy over $\mathbb{C}$

Now that we know how to extend the isotopy over hyperbolic crosscuts, it remains to define the extension over all complementary domains  $U$  of  $Z$ . Easy examples show that if we choose the hyperbolic crosscuts without care the extension may not be continuous. Fortunately a suitable set of hyperbolic crosscuts exists (see [FMOT08, §§3 and 4] for more details). Fix a component  $U$  of  $\mathbb{C}^* \setminus Z$  and let  $\mathcal{B}$  be the collection of all maximal open balls  $B(z, r) \subset U$  (that is open balls in the spherical metric and such that  $|S(z, r) \cap Z| \geq 2$ ). Let  $\mathcal{C}$  be the collection of all centers of such balls and for  $c \in \mathcal{C}$  let  $r(c)$  be the corresponding radius. Note that for each  $c \in \mathcal{C}$ ,  $B(c, r(c))$  is conformally equivalent with the unit disk  $\mathbb{D}$  and, hence, can be endowed with the hyperbolic metric. Let  $F(c)$  be the convex hull of the set  $S(c, r(c)) \cap X$  in  $B(c, r(c))$  using the hyperbolic metric on the ball  $B(c, r(c))$ . The following theorem is due to Kulkarni and Pinkall:

**THEOREM 7.1** ([KP94]). *For each  $z \in U$  there exists a unique  $c \in \mathcal{C}$  such that  $z \in F(c)$ .*

Note that the collection of chords in the boundaries of all  $F(c)$  form a “lamination” of  $U$ . As in [Thu09] we will call the chords in this lamination *leaves*.

Then two such leaves do not cross each other (i.e., if  $\ell \neq \ell'$  are leaves, then  $\ell \cap \ell' \cap U = \emptyset$ ) and any convergent sequence of leaves is either a leaf, or a point in  $Z$ . In particular, the subcollection of leaves of diameter greater or equal to  $\varepsilon$  is compact for each  $\varepsilon > 0$ . Consider the “lamination” of the disk  $\mathbb{D}$  obtained by pulling back all these leaves to the unit disk under the conformal map  $\varphi : \mathbb{D} \rightarrow U$ . Note that in this case it is possible that uncountably many distinct leaves join the same pair of angles  $\alpha, \beta \in S^1$ . Nevertheless, this collection of leaves will naturally provide us with the required collection of hyperbolic crosscuts by simply replacing each leaf  $C$  in the lamination by the unique hyperbolic crosscut  $G$  joining its endpoints such that  $G$  and  $C$  are homotopic in  $U$  with endpoints fixed. The collection  $\mathcal{H}$  of such hyperbolic crosscuts will be called the *hyperbolic KP-lamination* of  $U^0$ . The union of all the hyperbolic crosscuts in  $\mathcal{H}$  will be denoted by  $\mathcal{H}^*$ . A *gap*  $G$  of  $\mathcal{H}$  is a component of  $U \setminus \mathcal{H}^*$ . By Theorems 5.3 and 6.1 we can extend the isotopy  $h$  over  $\mathcal{H}^*$ . To finish the proof we must extend the isotopy over all gaps. The following lemma follows from a result by Jørgensen [Pom92, p. 91].

**LEMMA 7.2** ([FMOT08, Lemma 4.2]). *Suppose that  $c \in \mathcal{C}$  and  $C$  is the chord in the boundary of  $F(c)$  joining the points  $a, b \in \text{Bd}(U)$ . Let  $G \in \mathcal{H}$  be the hyperbolic geodesic in  $U$ , which is homotopic to  $C$  with endpoints fixed, joining the points  $a$  and  $b$ . Then  $G \subset B(c, r(c))$ .*

**THEOREM 7.3.** *Suppose that  $h^t$  is an isotopy of a plane continuum  $Z$ , which we consider as a subset of the sphere  $\mathbb{C}^*$ , with  $h^0 = \text{id}|_Z$ . Then there exists an extension to an isotopy  $H^t : \mathbb{C}^* \rightarrow \mathbb{C}^*$  such that  $H^0 = \text{id}_{\mathbb{C}^*}$ .*

*Proof.* Let  $\{U_n\}$  be all the components of  $\mathbb{C}^* \setminus Z$ . For each  $n$  let  $\mathcal{H}_n$  be the hyperbolic KP-lamination of  $U_n$ . Since the diameter of maximal balls contained in distinct components  $U_n$  converges to 0, it follows from Lemma 7.2 that for any sequence  $C_n \in \mathcal{H}_n$ , such that  $U_n \neq U_m$  when  $n \neq m$ ,

$$(7.1) \quad \lim \text{diam}(C_n) = 0.$$

By Theorems 6.1 and 5.3 we can extend the isotopy  $h$  of  $Z$  to an isotopy  $H_n$  of  $Z \cup \mathcal{H}_n^*$  such that  $H_n$  preserves the hyperbolic crosscuts in  $\mathcal{H}_n^0$  (i.e.,  $H_n^t(\mathcal{H}_n^0) = \mathcal{H}_n^t$ ). It remains to extend the isotopy over all gaps of the laminations  $\mathcal{H}_n$ .

At this point it will be convenient to use the Cayley-Klein model<sup>3</sup> of the hyperbolic disk. The homeomorphism  $g : \overline{\mathbb{D}} \rightarrow \overline{\mathbb{D}}$  defined by  $g(r, \theta) = (\frac{2r}{1+r^2}, \theta)$  (in polar coordinates) maps a point in the Poincaré model of the disk to the corresponding point in the Cayley-Klein model. This homeomorphism is the identity on the boundary  $S^1$  of  $\mathbb{D}$  and preserves radial line segments. For any two points  $\theta_1, \theta_2 \in S^1$  the hyperbolic geodesic  $G$  joining  $\theta_1$  to  $\theta_2$  is mapped to the Euclidean straight line segment  $\theta_1\theta_2$ , which is a chord of the unit disk with endpoints  $\theta_1$  and  $\theta_2$ . Recall that  $\mathcal{L}_n = \varphi^{-1}(\mathcal{H}_n)$  is the hyperbolic pullback lamination of  $\mathcal{H}_n$  in the unit disk  $\mathbb{D}$ . Fix  $n$  and put  $\mathcal{L}_n = \mathcal{L}$ . Let  $\mathcal{E} = g(\mathcal{L})$  be the Euclidean lamination in the disk  $\mathbb{D}$  where each hyperbolic chord in  $\mathcal{L}$  is replaced by the corresponding Euclidean chord. Then it is easy to extend the angle isotopy  $\alpha : S^1 \times [0, 1] \rightarrow S^1$  to an isotopy  $\Theta_n : \overline{\mathbb{D}} \times [0, 1] \rightarrow \overline{\mathbb{D}}$  of the entire closed disk as follows. First extend  $\alpha$  linearly over all leaves  $\mathcal{E}$  (i.e.,  $\Theta_n^t$  maps the Euclidean chord  $\theta\gamma \in \mathcal{E}$  linearly onto the Euclidean chord joining the points  $\alpha^t(\theta)\alpha^t(\gamma)$ ). Denote the resulting lamination  $\Theta_n^t(\mathcal{E})$  by  $\mathcal{E}^t$ . Next extend  $\Theta_n^t$  over all gaps by mapping the barycenter  $b_G$  of each gap  $G$  of  $\mathcal{E}$  to the barycenter  $b_G^t$  of the corresponding gap  $G^t$  of  $\mathcal{E}^t$ . The map can now be extended over all of  $G$  by mapping, for each  $x \in \text{Bd}(G)$  the straight line segment  $xb_G$  linearly onto the straight line segment  $\Theta_n^t(x)b_G^t$ . Now extend  $h$  over  $U_n$  by

$$H_n(x, t) = \begin{cases} h(x, t), & \text{if } x \in \text{Bd}(U_n), \\ \varphi^t \circ g^{-1} \circ \Theta_n^t \circ g \circ \varphi^{-1}(x), & \text{if } x \in U_n. \end{cases}$$

Let  $H = \cup H_n$ . Then  $H : \mathbb{C}^* \times [0, 1] \rightarrow \mathbb{C}^*$  is continuous by (7.1) and Theorem 6.1. Hence  $H$  is the required extension of  $h$ .  $\square$

Theorem 7.3 shows that we can extend an isotopy  $h$  of a planar continuum  $Z$ , starting at the identity, to an isotopy  $H : \mathbb{C}^* \times [0, 1] \rightarrow \mathbb{C}^*$  of the sphere. Let  $U$  denote the component of  $\mathbb{C}^* \setminus Z$  containing the point at infinity. By composing the isotopy  $H$  by an isotopy  $K$  of the sphere such that  $K^0 = \text{id}_{\mathbb{C}^*}$  and  $K^t|_{\mathbb{C}^* \setminus U} =$

<sup>3</sup>We are indebted to Nandor Simanyi for suggesting the Cayley-Klein model.

$\text{id}_{\mathbb{C}^* \setminus U}$ , and  $K^t(H^t(\infty)) = \infty$  for all  $t \in [0, 1]$  we obtain an isotopy which extends  $h$  and fixes the point at infinity. Hence the following theorem follows.

**THEOREM 7.4.** *Suppose that  $h^t$  is an isotopy of a plane continuum  $Z \subset \mathbb{C}$  with  $h^0 = \text{id}|_Z$ . Then there exists an extension to an isotopy  $H^t : \mathbb{C} \rightarrow \mathbb{C}$  such that  $H^0 = \text{id}$ .*

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