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Construction of harmonic diffeomorphisms and minimal graphs

By PASCAL COLLIN and HAROLD ROSENBERG

Abstract

We study complete minimal graphs in $\mathbb{H}\times\mathbb{R}$, which take asymptotic boundary values plus and minus infinity on alternating sides of an ideal inscribed polygon Γ in $\mathbb{H}$. We give necessary and sufficient conditions on the "lengths" of the sides of the polygon (and all inscribed polygons in Γ) that ensure the existence of such a graph. We then apply this to construct entire minimal graphs in $\mathbb{H}\times\mathbb{R}$ that are conformally the complex plane $\mathbb{C}$. The vertical projection of such a graph yields a harmonic diffeomorphism from $\mathbb{C}$ onto $\mathbb{H}$, disproving a conjecture of Rick Schoen and S.-T. Yau.

1. Introduction

In 1952, E. Heinz proved there is no harmonic diffeomorphism from a disk onto the complex plane $\mathbb{C}$, with the euclidean metric [Hei52]. He used this to give another proof of Bernstein's theorem: An entire minimal graph over the euclidean plane is a plane.

Later, R. Schoen and S.-T. Yau asked whether Riemannian surfaces which are related by a harmonic diffeomorphism are quasi-conformally related. In that direction, they conjectured there is no harmonic diffeomorphism from $\mathbb{C}$ onto the hyperbolic plane $\mathbb{H}$ [Sch93], [SY97] and [Mar02].

In this paper we will construct harmonic diffeomorphisms from $\mathbb{C}$ onto $\mathbb{H}$. We will use entire minimal graphs to construct these examples (E. Heinz used the nonexistence of harmonic diffeomorphisms from $\mathbb{H}$ onto $\mathbb{C}$ to prove the nonexistence of nontrivial entire minimal euclidean graphs).

Consider the Riemannian product $\mathbb{H}\times\mathbb{R}$ and entire minimal graphs Σ defined over $\mathbb{H}$. The vertical projection $\Sigma\longrightarrow\mathbb{H}$ is a surjective harmonic diffeomorphism, so we will solve the problem by constructing entire minimal graphs Σ that are conformally $\mathbb{C}$ (Theorem 3). Notice that the horizontal projection $\Sigma\longrightarrow\mathbb{R}$ is a harmonic function on Σ when Σ is a minimal surface. Hence if this height function

is bounded on Σ, Σ is necessarily hyperbolic (conformally the unit disc). So we must look for unbounded minimal graphs.

Here is the idea of the construction. Let Γ be an ideal geodesic polygon in $\mathbb{H}$ with an even number of sides (the vertices of Γ are at infinity). We give necessary and sufficient conditions on the geometry of Γ which ensure the existence of a minimal graph u over the polygonal domain D bounded by Γ, which takes the values plus and minus infinity on alternate sides of Γ (§3). This is a Jenkins-Serrin type theorem at infinity [JS66]. We call such graphs u over D ideal Scherk graphs and we show their conformal type is $\mathbb{C}$ (§5).

We attach certain ideal quadrilaterals to all of the sides of Γ (outside of D) so that the extended polygonal domain D_1 admits an ideal Scherk graph. We do this so that D_1 depends on a small parameter τ, and a minimal Scherk function $u_1(\tau)$ defined over D_1 satisfies the following. Given a fixed compact disk K_0 in the domain D, $u_1(\tau)$ is as close as we wish to u over K_0, for τ sufficiently small. Also ideal Scherk graphs are conformally $\mathbb{C}$ so there is a compact disk K_1 (containing K_0) in D_1, so that the conformal type of the annulus in the graph of $u_1(\tau)$, over $K_1 - K_0$, is greater than one. Now fix τ and use the same process to enlarge D_1, attaching certain ideal quadrilaterals to all of the sides to obtain a domain, admitting an ideal Scherk graph $u_2(\tau')$, which is as close as we want to $u_1(\tau)$ on K_1 for τ' sufficiently small; then the conformal moduli of the annulus in the graph of $u_2(\tau')$ over $K_1 - K_0$ remains greater than one. As before take K_2 with the same condition on modulus for $u_2(\tau')$ over $K_2 - K_1$. The entire minimal graph Σ is obtained by continuing this process and choosing a convergent subsequence. The conformal type of Σ is $\mathbb{C}$ because we write $\mathbb{H}$ as the union of an increasing sequence of compact disks K_n, and each annulus in Σ over each $K_{n+1} - K_n$ has conformal modulus at least one.

2. **Preliminaries**

First we will state some properties of solutions established in [JS66] and [NR02]. By solution in D we mean a solution of the minimal surface equation in a domain D of the hyperbolic plane. We make no assumptions here on ∂D.

COMPACTNESS THEOREM. *Let $\{u_n\}$ be a uniformly bounded sequence of solutions in D. Then a subsequence converges uniformly on compact subsets of D, to a solution in D.*

MONOTONE CONVERGENCE THEOREM. *Let $\{u_n\}$ be a monotone sequence of solutions in D. If the sequence $\{|u_n|\}$ is bounded at one point of D, then there is a nonempty open set $U \subset D$ (the convergence set) such that $\{u_n\}$ converges to a solution in U. The convergence is uniform on compact subsets of U and the divergence is uniform on compact subsets of $D - U = V$ where V is called the divergence set.*

Now assume ∂D is an ideal polygon with a finite number of vertices at $\partial_\infty \mathbb{H}$, composed of geodesic arcs $A_1, \ldots, A_k$, $B_1, \ldots, B_{k'}$ joining the vertices, together with convex arcs (convex towards D), $C_1, \ldots, C_{k''}$. We assume D is simply connected, ∂D, together with the vertices, homeomorphic to S^1, and no two A's (or B's) have a vertex in common.

DIVERGENCE STRUCTURE THEOREM. *Let $\{u_n\}$ be a monotone sequence of solutions in D, each u_n continuous on $\bar{D}$. If the divergence set $V \neq \varnothing$, then* $\operatorname{int} \mathrm{V} \neq \varnothing$, *and ∂V is composed of ideal geodesics among the A_i and B_j convex arcs among the $C(i)$, and interior ideal geodesics $C \subset D$, joining two vertices of ∂D. No two interior geodesics C_1, C_2 of ∂V go to the same vertex at infinity.*

Remark 1. With the exception of the last sentence in the above theorem the proofs are in the papers cited at the beginning of this section. We will prove the last statement after stating the flux relations.

Now let $\{u_n\}$ be a sequence defined in D satisfying the hypothesis of the Divergence Structure Theorem. For each n, let $X_n = \frac{\nabla u_n}{W_n}$ be the vector field on D, $W_n^2 = 1 + |\nabla u_n|^2$. For $\mathcal{W} \subset D$, and α a boundary arc of $\mathcal{W}$, we define the flux of u_n across α to be $F_n(\alpha) = \int_\alpha \langle X_n, \nu \rangle \, ds$; here α is oriented as the boundary of $\mathcal{W}$ and ν is the outer conormal to $\mathcal{W}$ along α. More generally, for any solution u in D and an oriented arc α, we write $F_u(\alpha)$ the flux of the associated field $X = \frac{\nabla u}{W}$, $W = (1 + |\nabla u|^2)^{1/2}$.

FLUX THEOREM. *Let $\mathcal{W}$ be a domain in D. Then*

i) *If $\partial \mathcal{W}$ is a compact cycle, then $F_n(\partial \mathcal{W}) = 0$.*

ii) *If $\mathcal{W} \subset U$ (the convergence set) and α is a compact arc of $\partial \mathcal{W}$ on which the u_n diverge to $+\infty$, then α is a geodesic and*

$$\lim_{n \to \infty} F_n(\alpha) = |\alpha|.$$

If the u_n diverge to $-\infty$ on α, then α is a geodesic and

$$\lim_{n \to \infty} F_n(\alpha) = -|\alpha|.$$

iii) *If $\mathcal{W} \subset V$, and the u_n remain uniformly bounded on α, then*

$$\lim_{n \to \infty} F_n(\alpha) = -|\alpha|, \text{ if } u_n \to +\infty \text{ on } V$$

and

$$\lim_{n \to \infty} F_n(\alpha) = |\alpha|, \text{ if } u_n \to -\infty \text{ on } V.$$

Remark 2. In ii) of the Flux Theorem, the fact that α (contained in $\partial \mathcal{W}$) is a geodesic when the u_n diverge on α is called the Straight Line Lemma.

Now we can prove the last statement of the Divergence Structure Theorem.

Suppose on the contrary, that ∂V has two interior arcs C_1, C_2 going to the same vertex at infinity of D. Let $\mathcal{W} \subset D$ be a bounded domain, with $\partial \mathcal{W}$ a simple

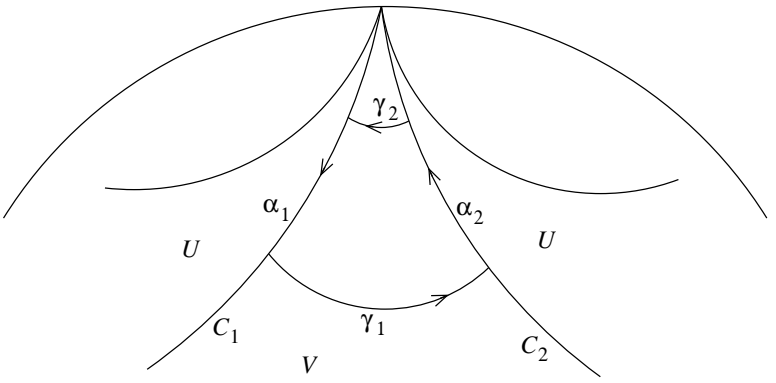

Figure 1

closed curve composed of two horocycle arcs γ_1, γ_2 and two arcs α_1, α_2 contained in C_1, C_2 respectively.

By part i) of the Flux Theorem, $F_n(\partial\mathcal{W}) = 0$, i.e.,

$$F_n(\alpha_1) + F_n(\gamma_1) + F_n(\alpha_2) + F_n(\gamma_2) = 0.$$

First assume the u_n diverge to $+\infty$ on $\mathcal{W}$; i.e. $\mathcal{W} \subset V$; cf. Figure 1. Since U is on the other side from $\mathcal{W}$ along α_1 and α_2, we have

$$\lim_{n\to\infty} F_n(\alpha_1) = -|\alpha_1| = \lim_{n\to\infty} F_n(\alpha_2),$$

by part iii) of the Flux Theorem. But $F_n(\gamma_i) \leq |\gamma_i|$ for $i = 1, 2$, and the length of γ_2 can be chosen arbitrarily small, the length of the α_i arbitrarily large, which contradicts the flux equality along $\partial\mathcal{W}$.

If the u_n diverge to $-\infty$ in $\mathcal{W}$, one obtains a similar contradiction.

When $\mathcal{W}$ is not contained in V, then $\mathcal{W} \subset U$ and the same flux equation gives a contradiction; the only change is the sign of $\lim_{n\to\infty} F_n(\alpha_1) = \lim_{n\to\infty} F_n(\alpha_2)$.

When we establish existence theorems for unbounded boundary data, we need to know solutions take on the boundary values prescribed. This is guaranteed in our situation by the following result.

BOUNDARY VALUES LEMMA. *Let D be a domain and let C be a compact convex arc in ∂D. Suppose $\{u_n\}$ is a sequence of solutions in D that converges uniformly on compact subsets of D to a solution u in D. Assume each u_n is continuous in $D \cup C$ and that the boundary values of u_n on C converge uniformly to a function f on C. Then u is continuous in $D \cup C$ and u equals f on C.*

Proof. One needs to show that for each $p \in C$, the sequence $\{u_n\}$ is uniformly bounded in a neighborhood of p in $D \cup C$; then standard local barriers show U takes on the correct boundary values.

When C is strictly convex in a neighborhood of p then one places a Scherk surface defined over a geodesic triangle over the graph of u; cf. Figure 2.

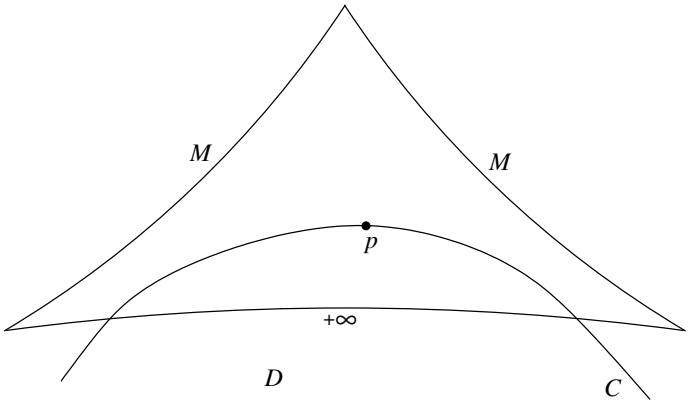

Figure 2

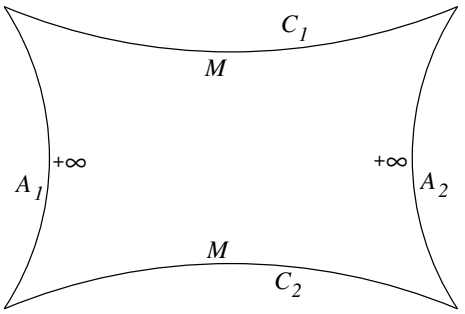

Figure 3

Here M is the maximum of f on C. Since the boundary values of this Scherk surface are above those of u_n, the Scherk surface is above the graph of u_n on the region in the triangle over $\bar{D}$. Thus the Scherk surface is above u, and this uniformly bounds u in a smaller compact neighborhood of p inside the triangle.

When the arc C contains a geodesic segment in a neighborhood of p, one uses an analogous Scherk barrier.

Consider a quadrilateral $\mathcal{P}$ composed of two ("vertical") geodesics A_1, A_2, and two geodesic curves C_1, C_2 ("horizontal"); cf. Figure 3.

If $|A_1| + |A_2| < |C_1| + |C_2|$, then there is a minimal solution v defined inside $\mathcal{P}$ taking the boundary values $+\infty$ on $A_1 \cup A_2$ and any prescribed continuous data on $C_1 \cup C_2$.

Now place the quadrilateral in $\bar{D}$, putting C_1 into C, and the rest of $\mathcal{P}$ inside D.

Let M be the maximum of u on C_2 and f on C_1. Let v be the solution in $\mathcal{P}$ that equals M on $C_1 \cup C_2$ and $+\infty$ on $A_1 \cup A_2$. Then $v \geq u_n$ on $\partial\mathcal{P}$ so that $v \geq u_n$ in $\mathcal{P}$. Thus v bounds u in a smaller compact neighborhood of p inside $\mathcal{P}$.

We complete this section by stating the existence theorem for compact domains.

EXISTENCE THEOREM. *Let $\mathcal{W}$ be a bounded domain with $\partial\mathcal{W}$ a Jordan curve. Assume there is a finite set $E \subset \partial\mathcal{W}$ and $\partial\mathcal{W} - E$ is composed of convex (towards $\mathcal{W}$) arcs. Then there is a solution to the Dirichlet problem in $\mathcal{W}$ taking on arbitrarily prescribed continuous data on $\partial\mathcal{W} - E$. The arcs need not be strictly convex.*

3. **The Dirichlet problem on unbounded domains**

Let Γ be an ideal polygon of $\mathbb{H}$; i.e., Γ is a geodesic polygon all of whose vertices are at infinity $\partial_\infty(\mathbb{H})$. We assume Γ has an even number of sides $A_1, B_1, A_2, B_2, \ldots, A_k, B_k$, ordered by traversing Γ clockwise. Let D be the interior of the convex hull of the vertices of Γ; so that $\partial D = \Gamma$ and D is a topological disk.

We consider the Dirichlet problem for the minimal surface equation in D where one prescribes the data $+\infty$ on each A_i, and $-\infty$ on each B_j.

When D is a relatively compact domain in $\mathbb{R}^2$, with boundary composed of line segments $\{A_i\}$ and $\{B_j\}$, and convex arcs $\{C_l\}$, this type of Dirichlet problem was solved completely by H. Jenkins and J. Serrin [JS66] (allowing continuous data on the $\{C_l\}$). They found necessary and sufficient conditions — in terms of the geometry of ∂D – that guaranty a solution to the Dirichlet problem, for the minimal surface equation in $\mathbb{R}^2$.

The Jenkins-Serrin theorem was extended to $\mathbb{H} \times \mathbb{R}$, for compact domains $\bar{D} \subset \mathbb{H}$, and minimal graphs over some noncompact domains were considered [NR02].

When Γ is an ideal polygon, we will find necessary and sufficient conditions, in terms of the "lengths" of the edges, which enable us to solve the Dirichlet problem. Since the lengths of the edges are infinite, we proceed as follows.

At each vertex a_i of Γ, place a horocycle H_i; do this so that $H_i \cap H_j = \varnothing$ if $i \neq j$. Let F_i be the convex horodisk with boundary H_i.

Each A_i meets exactly two horodisks. Denote by $\widetilde{A}_i$ the compact arc of A_i which is the part of A_i outside the two horodisks, $\partial\widetilde{A}_i$ is two points, each on a horocycle; $|A_i|$ is the distance between these horocycles, i.e. the length of $\widetilde{A}_i$. Define $\widetilde{B}_i$ and $|B_i|$ in the same way.

Define

$$a(\Gamma) = \sum_{i=1}^{k} |A_i| \quad \text{and}$$

$$b(\Gamma) = \sum_{i=1}^{k} |B_i|.$$

Observe that $a(\Gamma) - b(\Gamma)$ does not depend on the choice of horocycles (assuming they are pairwise disjoint).

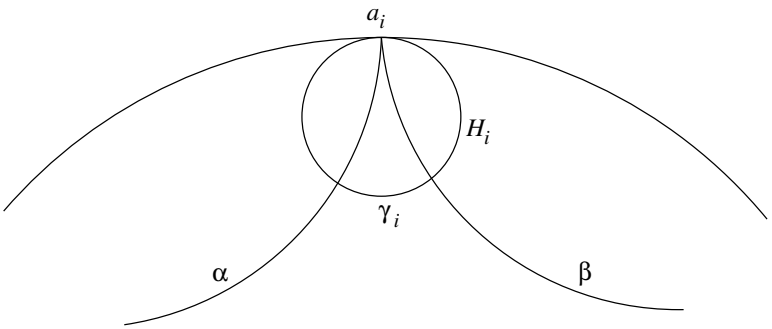

Figure 4

Definition 1. An ideal geodesic polygon $\mathcal{P}$ is said to be inscribed in D if the vertices of $\mathcal{P}$ are among the vertices of Γ; $\mathcal{P}$ is a simple closed polygon whose edges are either interior in D or equal to an A_i or B_j.

Notice that the definition of $a(\Gamma)$ and $b(\Gamma)$ extends to inscribed polygons. Also, to each inscribed polygon $\mathcal{P}$ in Γ and a choice of horocycles at the vertices, we associate a Jordan curve $\widetilde{\mathcal{P}}$ in $\mathbb{H}$ as follows.

Let α and β be geodesic edges of $\mathcal{P}$ with the same vertex a_i. Let γ_i denote the compact arc of H_i joining $\alpha \cap H_i$ to $\beta \cap H_i$. Then $\widetilde{\mathcal{P}}$ is obtained from $\mathcal{P}$ by removing the noncompact arcs of α and β in H_i and replacing them by γ_i; cf. Figure 4. Denote by $|\mathcal{P}|$ the length of the boundary arcs of $\mathcal{P}$ exterior to the horocycles at the vertices of $\mathcal{P}$; we call this the truncated length of $\mathcal{P}$.

We can now state the result.

THEOREM 1. *There is a solution to the minimal surface equation in the polygonal domain D, equal to $+\infty$ on each A_i and $-\infty$ on each B_j, if and only if the following two conditions are satisfied*

1. *When $a(\gamma)$ and $b(\gamma)$ are defined as above, one has*

$$a(\Gamma) = b(\Gamma). \tag{1}$$

2. *For each inscribed polygon $\mathcal{P}$ in Γ, $\mathcal{P} \neq \Gamma$, and for some choice of the horocycles at the vertices, one has*

$$2a(\mathcal{P}) < |\mathcal{P}| \text{ and } 2b(\mathcal{P}) < |\mathcal{P}|. \tag{2}$$

The solution is unique up to an additive constant.

Remark 3. It is easy to check that if the inequalities of condition 2 hold for some choice of horocycles, then they continue to hold for "smaller" horocycles (the inequalities get better); cf. Remark 8. Thus for a given Γ, a finite number of choices of horocycles suffice to check that the inequalities are satisfied.

Definition 2. An inscribed polygon $\mathcal{P}$ that satisfies the conditions of Theorem 1 is said to be admissible.

First we fix some notation. At each vertex a_i of Γ, let $H_i(n)$ be a sequence of nested horocycles at a_i, converging to a_i as $n \to \infty$; let $F_i(n)$ be the convex horodisk bounded by $H_i(n)$.

$H_i(n) \cap \Gamma$ consists of two points and $\gamma_i(n)$ is the compact arc of $H_i(n)$ joining these two points. We now need to work in convex domains in order to form the Jordan curves $\widehat{\Gamma}(n)$; we will not attach $\gamma_i(n)$ but the geodesic arc $\widehat{\gamma}_i(n)$, having the same end-points as $\gamma_i(n)$.

Then $\widehat{\Gamma}(n)$ is the convex Jordan curve:

$$\widehat{\Gamma}(n) = \Big[\Gamma - \bigcup_i (\Gamma \cap F_i(n))\Big] \cup \Big[\bigcup_i \widehat{\gamma}_i(n)\Big].$$

Let $D(n)$ denote the disk of $\mathbb{H}$ bounded by $\widehat{\Gamma}(n)$, so that $\bigcup_n D(n) = D$.

We establish several lemmas for the proof of the Theorem 1.

LEMMA 1. *For each $i = 1, \ldots, k$, there is a solution to the Dirichlet problem on D with boundary data $+\infty$ on A_i, zero on the rest of Γ.*

Proof. For each n, define u_n to be the minimal graph on $D(n)$, which is equal to n on $\partial D(n) \cap A_i$, and zero on the rest of $\partial D(n)$. By the maximum principle, u_n is a monotone increasing sequence on $D(\ell)$ for $n \geq \ell$. Thus it suffices to show the u_n are uniformly bounded on compact subsets of $D(\ell) - A_i$; then a diagonal process yields the solution to Lemma 1.

In the upper half-plane model of $\mathbb{H}$ consider the function [Abr], [SE08]:

$$h(x, y) = \ln\left(\frac{\sqrt{x^2 + y^2} + y}{x}\right), \ x > 0, \ y > 0.$$

Consider h defined in the domain of $\mathbb{H}$ defined by $y > 0$, $x > 0$; it satisfies the minimal surface equation, has asymptotic values zero on the arc at $\partial_\infty \mathbb{H}$, given by $x > 0$, $y = 0$. Also it converges to $+\infty$ on $x = 0$, $y > 0$.

Thus for any geodesic of $\mathbb{H}$, one has such a minimal graph. Apply this to A_i: let h be defined on the part of $\mathbb{H}$, with boundary A_i, that contains D. Clearly h is greater than each u_n by the maximum principle. Hence on any compact set $K \subset D(\ell) - A_i$, the u_n, $n \geq \ell$, are uniformly bounded.

Remark 4. This argument greatly simplifies the proofs of Theorems 2 and 3, Step 1, of [NR02]; the barrier h shows the u_n of Theorem 2, and of Step 1 of Theorem 3, are uniformly bounded on compact sets.

LEMMA 2. *For each $i = 1, \ldots, k$, there is a solution to the Dirichlet problem on D with boundary data $+\infty$ on each A_j, $j \neq i$, and zero on the rest of Γ.*

Proof. We note by $A = A_1 \cup A_2 \cup \cdots \cup A_k$, $B = B_1 \cup B_2 \cup \cdots \cup B_k$, so that $\Gamma = A \cup B$. $\widetilde{A}(n)$ and $\widetilde{B}(n)$ denote the truncated A's and B's in $\widehat{\Gamma}(n) = \partial D(n)$; $\widehat{\Gamma}(n) = \widetilde{A}(n) \cup \widetilde{B}(n) \cup \widehat{\gamma}(n)$, where $\widehat{\gamma}(n) = \widehat{\gamma}_1(n) \cup \widehat{\gamma}_2(n) \cup \cdots \cup \widehat{\gamma}_k(n)$ is the union of the remaining geodesic arcs. For convenience, suppose $i = 1$.

For each n, let u_n be the solution to the Dirichlet problem on D with boundary data equal to n on $A_2 \cup \cdots \cup A_k$, and zero on the rest of Γ. This u_n exists: first construct such a solution on each compact $D(\ell)$, then let $\ell \to \infty$ and choose a convergent subsequence by a diagonal process.

By the generalized maximum principle (whose proof we give in Theorem 2), $\{u_n\}$ is a monotone sequence so the Divergence Structure Theorem applies. If the divergence set is not empty, a connected component is an inscribed polygon in D whose geodesic edges join two vertices of Γ. We show $V = \varnothing$ to prove Lemma 2.

First suppose $V = D$ so that $\partial V = \Gamma$.

We fix ℓ and consider $n \geq 0$ and the function u_n on $D(\ell)$. Let $X_n = \frac{\nabla u_n}{W_n}$ where ∇u_n is the gradient of u_n in $\mathbb{H}$ and $W_n^2 = 1 + |\nabla u_n|^2$. Then $\operatorname{div}(X_n) = 0$, so the flux of X_n along $\partial D(\ell) = \widehat{\Gamma}(\ell)$ is zero. For each arc $\alpha \subset \widehat{\Gamma}(\ell)$, let $F_n(\alpha) = \int_\alpha \langle X_n, \nu \rangle \, ds$, where ν is the outer conormal to $D(\ell)$ along $\partial D(\ell)$.

Then the flux of X_n along $\partial D(\ell)$ yields:

$$0 = F_n(\widetilde{A}_1) + F_n(\widetilde{A}_2) + \cdots + F_n(\widetilde{A}_k) + F_n(\widetilde{B}_1) + \cdots + F_n(\widetilde{B}_k) + \sum_{j=1}^{k} F_n(\widehat{\gamma}_j(\ell)),$$

where $\widehat{\gamma}_j(\ell)$ are the small geodesic arcs in $\partial D(\ell)$. Now the u_n diverge uniformly to infinity on compact subsets of D and are bounded on $A_1 \cup B_1 \cup \cdots \cup B_k$, so that

$$\lim_{n\to\infty} F_n(\widetilde{A}_1) = -|\widetilde{A}_1|, \quad \lim_{n\to\infty} F_n(\widetilde{B}_j) = -|\widetilde{B}_j|$$

for $j = 1, \ldots, k$. Also $F_n(C) \leq |C|$ for any arc $C \subset \partial D(\ell)$. Hence letting $n \to \infty$, we obtain

$$|\widehat{\gamma}(\ell)| + |\widetilde{A}_2| + \cdots + |\widetilde{A}_k| \geq |\widetilde{A}_1| + |\widetilde{B}_1| + \cdots + |\widetilde{B}_k|,$$

where $|\widehat{\gamma}(\ell)|$ is the length of all the arcs $\widehat{\gamma}_j(\ell)$ in $\partial D(\ell)$. But for any choice of ℓ,

$$a(\Gamma) = |\widetilde{A}_1| + \cdots + |\widetilde{A}_k| = |\widetilde{B}_1| + \cdots + |\widetilde{B}_k| = b(\Gamma)$$

by hypothesis and $|\widehat{\gamma}(\ell)| \to 0$, as $\ell \to \infty$. Thus this last inequality is impossible.

It remains to show $V \neq D$, $V \neq \varnothing$ is impossible. Suppose this were the case. Fix ℓ and consider $V \cap D(\ell) = V(\ell)$. $V(\ell)$ is bounded by interior geodesic arcs $C_1, C_2, \ldots$, some arcs $\widetilde{A}_{i_1}, \widetilde{A}_{i_2}, \ldots, \widetilde{B}_{j_1}, \widetilde{B}_{j_2}, \ldots$ of $\partial D(\ell)$, and some small geodesic arcs $\widetilde{\gamma}_1, \widetilde{\gamma}_2, \ldots$ (all of these arcs depend on ℓ).

The flux of X_n along $\partial V(\ell)$ equals zero; hence:

$$\sum F_n(\widetilde{A}_i) + \sum F_n(\widetilde{B}_j) + \sum F_n(C_l) + \sum F_n(\widetilde{\gamma}_m) = 0,$$

where the sums are taken over all the arcs in $\partial V(\ell)$. The arcs $C_1, C_2, \ldots$, are interior arcs of D so that on each C_l,

$$\lim_{n\to\infty} F_n(C_l) = -|C_l|.$$

Similarly, for the $\widetilde{B}_j$ in $\partial V(\ell)$,

$$\lim_{n\to\infty} F_n(\widetilde{B}_j) = -|\widetilde{B}_j|.$$

Let $|\widetilde{\gamma}|$ be the total length of the small geodesic arcs in $\partial V(\ell) = \mathcal{P}$, so that $|\widetilde{\gamma}| \geq \sum F_n(\widetilde{\gamma}_m)$. Similarly for the $\widetilde{A}_i$ arcs we have $a(\mathcal{P}) \geq \sum F_n(\widetilde{A}_i)$.

The flux equality yields

$$a(\mathcal{P}) + |\widetilde{\gamma}| \geq b(\mathcal{P}) + |C|,$$

where $|C|$ is the sum of the lengths of the arcs C_m in $\mathcal{P}$. This last inequality can be rewritten:

$$2a(P) - a(\mathcal{P}) - b(\mathcal{P}) - |C| \geq -|\widetilde{\gamma}|.$$

Combining the previous inequality and the definition of the truncated length, one obtains

$$2a(\mathcal{P}) - |\mathcal{P}| \geq -|\widetilde{\gamma}|.$$

The inscribed polygon $\mathcal{P}$ satisfies the condition 2 for some choice of horocycles, so that $2a(\mathcal{P}) - |\mathcal{P}| < 0$ for this choice of horocycles. For smaller horocycles, this quantity decreases, so the above inequality $2a(\mathcal{P}) - |\mathcal{P}| \geq -|\widetilde{\gamma}|$ is impossible for ℓ large enough.

Remark 5. Clearly Lemma 2 also proves there is a solution on D which is $-\infty$ on each B_j, $j \neq i$, and zero on the rest of the boundary of D.

Proof of Theorem 1. Let v_n be the solution on D which is n on each A_i and zero on each B_j. This v_n is easily constructed by solving this Dirichlet problem on the disks $D(\ell)$ and taking the limit as $\ell \to \infty$.

For $0 < c < n$, define

$$E_c = \{v_n > c\} \text{ and } F_c = \{v_n < c\}.$$

Let E_c^i be the component of E_c whose closure contains A_i and let F_c^j be the component of F_c whose closure contains B_j.

We can separate any two of the A_i by a curve joining two of the B_j's, on which v_n is bounded away from n. Hence for c sufficiently close to n, the sets E_c^i are pairwise disjoint. Let $\mu(n)$ be the infimum of the constants c such that the sets E_c^i are pairwise disjoint.

We claim that each component of E_c is equal to some E_c^i. Also, there is a pair E_μ^i and E_μ^j, $i \neq j$, $\mu = \mu(n)$, such that $E_\mu^i \cap E_\mu^j = \varnothing$ and $\overline{E_\mu^i} \cap \overline{E_\mu^j} \neq \varnothing$. Hence, given any F_μ^i, there is some F_μ^j disjoint from it.

To see this claim, consider the level curves $\{v_n = c\}$, for some $0 < c < n$. There are no compact level curves by the maximum principle. Each connected level curve is proper; hence each end of the level curve is asymptotic to exactly one vertex of Γ.

Suppose $\mathcal{W}$ is a component of E_c and $\mathcal{W} \neq E_c^i$ for each i. Then $v_n|_{\partial\mathcal{W}} = c$ and $\partial\mathcal{W}$ is composed of level curves of v_n contained in D. We know $\partial\mathcal{W}$ is not compact, so it enters the ends of D and is asymptotic to the vertices at these ends. Form a compact cycle β from $\partial\mathcal{W}$ by attaching short arcs in $\mathcal{W}$ joining two branches of the curve of $\partial\mathcal{W}$ asymptotic to the same vertex; cf. Figure 5.

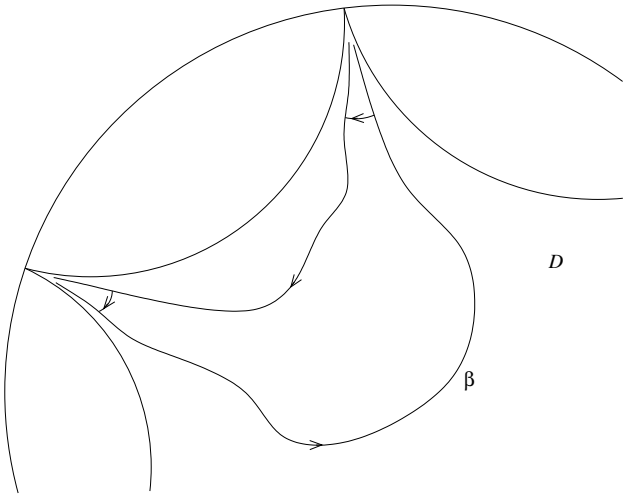

Figure 5

Let $X_n = \frac{\nabla v_n}{W_n}$. The flux of X_n along β is zero. On the arcs of β in $\partial^\infty \mathcal{W}$, the flux is bounded away from zero. Since the flux on the "short" arcs of β "inside" D is bounded by their lengths, and the lengths can be chosen arbitrarily small, this is impossible.

Now is a good time to make a slight digression.

THEOREM 2 (Generalized Maximum Principle). *Let D be a domain with ∂D an ideal geodesic polygon. Let $\mathcal{U} \subset D$ be a domain and $u, v \in C^0(\overline{\mathcal{U}})$, two solutions of the minimal surface equation in $\mathcal{U}$ with $u \leq v$ on $\partial^\infty \mathcal{U}$. Then $u \leq v$ in $\mathcal{U}$.*

Proof. If this is not true, then we can suppose (after a possible small vertical translation of the graph of v) that $\mathcal{W} = \{p \in \mathcal{U} | u(p) > v(p)\}$ is a nonempty domain with smooth boundary. $\overline{\mathcal{W}}$ is not compact by the maximum principle, so $\mathcal{W}$ has branches going into the ends of D. As in the previous argument, we form a compact cycle β composed of (long) arcs on $\partial^\infty \mathcal{W}$ and (short) arcs in the ends of $D \cap \mathcal{W}$; cf. Figure 5.

Let $X = \frac{\nabla u}{W_u}$ and $Y = \frac{\nabla v}{W_v}$. The flux of $X - Y$ across β is 0. On the short arcs of β, this flux is bounded by twice the length of the short arcs, hence can be made arbitrarily small. It remains to prove that the flux of $X - Y$ on the long arcs of β is bounded away from zero. This is a well-known argument which we repeat here.

Since $u - v > 0$ in $\mathcal{W}$, $u = v$ on $\partial^\infty \mathcal{W}$, we have $\nabla(u - v) = \lambda \eta$ on $\partial^\infty \mathcal{W}$, where η, the inner pointing conormal to ∂W in W, orients the level curve; λ is a strictly positive function on $\partial^\infty \mathcal{W}$, by the boundary maximum principle.

An algebraic identity ([NR02]) shows $\langle \nabla(u - v), X - Y \rangle \geq 0$ with equality if and only if $\nabla(u - v)$ is zero. Thus $\langle X - Y, \eta \rangle$ is bounded away from zero on the long arcs of β, which proves the generalized maximum principle.

Remark 6. Theorem 2 extends to possible infinite boundary values for the solutions along geodesic arcs α of $\partial^\infty \mathcal{U}$. It is immediate if the data are distinct by

restricting $\mathcal{U}$. If u and v take the same infinite value along α, a careful analysis of X and Y in a neighborhood of α gives $X = Y$ along α and the arc α cannot generate flux, which allows us to conclude as above. In particular, this proves the unicity part of Theorem 1.

Now the existence proof of Theorem 1 proceeds as in Jenkins-Serrin [JS66].

Let u_i^+, $i = 1, \dots, k$ be the solutions which take the value $+\infty$ on A_i and zero on the rest of Γ. Similarly, let u_i^- be the solution in D which takes the value $-\infty$ on $B_1, \dots B_{i-1}, B_{i+1}, \dots B_k$, and zero on the rest of Γ; they exist by Lemmas 1 and 2. Define $u_n = v_n - \mu(n)$ and, for each $p \in D$,

$$u^+(p) = \max_i \{u_i^+(p)\},\ u^-(p) = \min_i \{u_i^-(p)\}.$$

Observe that

$$u^- \le u_n \le u^+ \text{ in } D.$$

To see this, notice that if, for some $p \in D$, $u_n(p) > 0$ then $p \in E_\mu^i$ for some $i \in \{1, \dots, k\}$ and by the generalized maximum principle we have $u_n \le u_i^+$ in E_μ^i. If $u_n(p) < 0$, then $p \in F_\mu^i$ for some i. There is some $j \ne i$ such that $F_\mu^i \cap F_\mu^j = \varnothing$. Then the generalized maximum principle in F_μ^i yields $u_n(p) \ge u_j^-(p)$.

Thus the sequence u_n is uniformly bounded on compact subsets of D, so a subsequence of the u_n converges to a solution u in D. It remains to show $u = +\infty$ on each A_i, and $u = -\infty$ on each B_j.

Observe that the sequence of constants $\mu(n)$ diverges to infinity. Otherwise we would have a subsequence (again called u_n) such that u_n converges to u, u is $+\infty$ on the A_i, and $u = -\mu_0$ on the B_j.

Fix ℓ and calculate the flux of u along $\partial D(\ell)$; this gives

$$\sum_i F_u(\widetilde{A}_i) + \sum_i F_u(\widetilde{B}_i) + \sum_i F_u(\widehat{\gamma}_i(\ell)) = 0.$$

Here $\widetilde{A}_i$, $\widetilde{B}_i$, $\widehat{\gamma}_i(\ell)$ are the arcs on $\partial D(\ell) = \widehat{\Gamma}(\ell)$; the $\widehat{\gamma}_i(\ell)$ are short geodesic arcs. We have, using the Flux Theorem:

$$\sum_i F_u(\widetilde{A}_i) = a(\Gamma) = |\widetilde{A}_1| + \cdots + |\widetilde{A}_k|,$$

$$\sum_i F_u(\widehat{\gamma}_i(\ell)) \le |\widehat{\gamma}(\ell)| = |\widehat{\gamma}_1(\ell)| + \cdots + |\widehat{\gamma}_k(\ell)|,$$

and

$$\sum_i F_u(\widetilde{B}_i) \le b(\Gamma) - \delta \text{ for some } \delta > 0.$$

Since $|\widehat{\gamma}(\ell)| \to 0$, as $\ell \to \infty$, and δ can be chosen independent of ℓ, this contradicts $a(\Gamma) = b(\Gamma)$.

In the same way, one concludes $n - \mu(n)$ diverges to infinity. This completes the proof of Theorem 1.

4. **Ideal polygons with convex arcs**

Now let Γ be an ideal polygon with a finite number of vertices at infinity and geodesic sides $(A_i)_{i=1,\dots,k}$, $(B_j)_{j=1,\dots,k'}$ as before, but we also allow convex arcs $(C_l)_{l=1,\dots,k''}$ in Γ, joining vertices of Γ at infinity. We assume the C_l are convex with respect to the domain D bounded by Γ and we do not require the C_l to be strictly convex.

We make an important assumption: when a convex arc C in Γ, has a point $b \in \partial_\infty \mathbb{H}$ as a vertex, then the other arc α of Γ having b as vertex is asymptotic to C at b. This means that for a sequence $x_n \in \alpha$, converging to b, one has $\operatorname{dist}_{\mathbb{H}}(x_n, C) \to 0$ as $n \to \infty$.

This assumption is what we need to assure the Generalized Maximum Principle (Theorem 2) holds in D. Its proof is similar to the proof we gave when there are no convex arcs in Γ.

We now state the expected result.

THEOREM 3. *Let Γ be as described above and let f be continuous on the convex arcs C_l of Γ (we assume there are convex arcs C_l). Then there is a unique solution in D which is $+\infty$ on each A_i, $-\infty$ on each B_j and f on the C_l, if and only if*

$$2a(\mathcal{P}) < |\mathcal{P}|$$

and

$$2b(\mathcal{P}) < |\mathcal{P}|,$$

for all inscribed polygons $\mathcal{P}$ in Γ.

Recall that we constructed an exhaustion of D by compact convex disks $D(\ell)$ using geodesic arcs at the vertices of Γ, when there were no convex arcs in Γ. Now we construct $D(\ell)$ as follows. At each vertex b of Γ with an A_i and C_j asymptotic to b, choose points $x_\ell \in A_i$ converging to b and let $\gamma(\ell)$ be the minimizing geodesic joining x_ℓ to C_j. Clearly the $\gamma(\ell)$ are pairwise disjoint. An analogous sequence can be constructed when there are B's and C's at the same vertex b or two arcs C_i, C_j at b. Then we define $D(\ell)$ to be the disk bounded by the $\gamma(\ell)$ together with the compact arcs on the A's, B's and C's they bound.

First we solve the Dirichlet problem of Theorem 3 when there are no B's, and f is bounded below.

In each $D(\ell)$, let $u_n(\ell)$ be the solution that equals n on the A_i in $\partial D(\ell)$, zero on the $\gamma(\ell)$'s and $\min(n, f)$ on the C_l in $\partial D(\ell)$. By the maximum principle, $u_n(\ell) \le u_n(\ell')$ on $D(\ell)$ for $\ell \le \ell'$. Then locally, $u_n(\ell)$ is an increasing bounded sequence as ℓ tends to infinity and therefore, by the Monotone Convergence Theorem, it converges to a solution u_n in D that equals n on the A_i and $\min(f, n)$ on the C_l.

By the generalized maximum principle, $\{u_n\}$ is a monotone increasing sequence which is uniformly bounded in a neighborhood of strictly convex arcs C_l

by the Boundary Values Lemma. Hence if the sequence $\{u_n\}$ has a nonempty divergence set V, then a connected component of V is bounded by an inscribed polygon $\mathcal{P}$ whose edges are geodesics (some of the C's may be in $\mathcal{P}$) joining vertices of Γ. Then the same flux calculation we did in Lemma 2 contradicts our hypothesis $2a(\mathcal{P}) < |\mathcal{P}|$.

Thus the u_n converge to a solution u in D that equals $+\infty$ on the A_i and f on the C_l from the Boundary Values Lemma. The same argument shows that if there are no A's in Γ and f is bounded above, then there is a solution in D that equals $-\infty$ on the B_j and f on C_l.

Let us now prove the theorem when there are A's, B's and C's.

Let u^+ be the solution in D equal to $+\infty$ on the A_i, zero on the B_j, and $\max(f, 0)$ on C_l; and similarly, let u^- be the solution in D equal to zero on the A_i, $-\infty$ on the B_j, and $\min(f, 0)$ on C_l. These solutions exist by our previous discussion.

Let v_n be the solution in D equal to n on the A_i, $-n$ on the B_j and f_n on C_l, where f_n is f truncated above by n and below by $-n$.

By the generalized maximum principle,

$$u^- \leq v_n \leq u^+ \text{ in } D.$$

Therefore the sequence $\{v_n\}$ is uniformly bounded on compact subsets of D so that a subsequence converges uniformly on compact sets to a solution v in D. By the Boundary Values Lemma v takes on the desired boundary values on Γ.

5. **Conformal type**

We present here the main result of this paper concerning harmonic diffeomorphisms. Clearly, any conformal representation of the graphs given by the following theorem gives rise to a harmonic diffeomorphism of $\mathbb{C}$ onto $\mathbb{H}$, by projection.

THEOREM 4. *In $\mathbb{H} \times \mathbb{R}$, there exist entire minimal graphs over $\mathbb{H}$ which are conformally the complex plane $\mathbb{C}$.*

Proof. We will recursively use Propositions 1 and 2 (that follow) to construct an exhaustion of $\mathbb{H}$ by compact disks K_n with $K_n \subset \mathring{K}_{n+1}$ and a sequence of minimal graphs u_n over K_n (the restriction of ideal Scherk graphs defined over ideal polygonal domains D_n containing K_n) satisfying the following:

i) $\| u_{n+1} - u_n \|_{C^2(K_n)} < \varepsilon_n$, for some sequence $\varepsilon_n > 0$, with $\sum_{n=0}^{\infty} \varepsilon_n < +\infty$.

ii) For each j, $0 \leq j < n$, the conformal modulus of the annulus in the graph of u_n over the domain $K_{j+1} - \mathring{K}_j$ is greater than one.

For that, let ε_n be a sequence of positive real numbers such that $\sum_{n=0}^{\infty} \varepsilon_n < +\infty$. We assume (D_j, u_j, K_j) are constructed for $0 \leq j \leq n$ and satisfy the properties i) and ii) we require above.

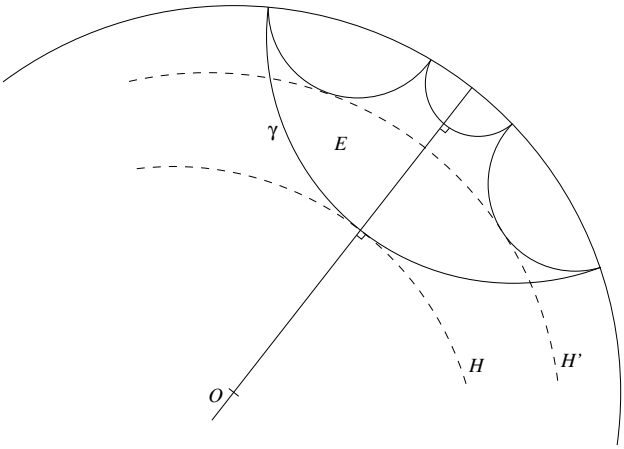

Figure 6

Using Proposition 2, attach (perturbed) elementary quadrilaterals E_τ, E'_τ, to *all* of the pairs of sides of ∂D_n, to obtain an ideal Scherk graph u_{n+1} over an enlarged polygonal domain D_{n+1}. The E_τ, E'_τ are attached successively to the pairs of sides of ∂D_n; the parameter τ of each pair attached depends on the previous expanded polygons.

Also, to be sure we will be able to construct an exhaustion, we need to choose carefully the initial nonperturbed regular quadrilateral of Proposition 2. For that, consider at each side γ of ∂D_n, the reflexion across the geodesic orthogonal to γ and passing through a fixed point $O \in K_0$, and the special regular quadrilateral along this side γ which is invariant by the corresponding reflexion; cf. Figure 6. Then use these quadrilaterals to construct the E_τ and E'_τ.

Moreover, in Proposition 2, we can choose ε small enough to get u_{n+1} as close as we want to u_n in the C^2-topology on the compact K_n, so that properties i) and ii) are satisfied. For we can easily ensure that first $\| u_{n+1} - u_n \|_{C^2(K_n)} < \varepsilon_n$, and secondly, as for u_n, the graph of u_{n+1} over each annulus $K_{j+1} - \overset{\circ}{K}_j$, $0 \leq j < n$, has conformal modulus greater than one, since the closer the graphs are, the closer are the conformal moduli.

Now the graph of u_{n+1} is conformally $\mathbb{C}$ by Proposition 1, so there is a compact E in this graph satisfying:

- E is a disk that contains F in its interior, and
- the conformal modulus of the annulus $E - F$ is greater than one.

Here F is the graph of u_{n+1} over $\overset{\circ}{K}_n$.

Then define K_{n+1} to be the vertical projection of E; eventually enlarge E in order that K_{n+1} has its boundary in a tubular neighborhood of radius one of ∂D_{n+1}. By the above construction, this K_{n+1} satisfies property ii). Then the sequence is constructed, and the argument will be complete if we prove the K_n exhaust $\mathbb{H}$.

For each n, using the particular geometry of the perturbed quadrilateral we attach to all of the sides of ∂D_n, we get that the boundary of D_{n+1} is a fixed constant farther from the fixed point O we have chosen in K_0 (if the quadrilateral we add were regular, it would be at least $\ln(1+\sqrt{2})$ by an elementary computation; cf. Figure 6, where the equidistant curves H and H' are two parallel horocycles). Hence the ∂D_n diverge to infinity with n. But we constructed K_n so that ∂K_n is uniformly close to ∂D_n.

Next we let n tend to infinity. We obtain an entire graph u, since at any $x \in \mathbb{H}$, the $u_n(x)$ form a Cauchy sequence. Since on each $K_{j+1} - \mathring{K}_j$, the u_n converge uniformly to u in the C^2-topology, the modulus of the graph of u over $K_{j+1} - K_j$ is at least one. Hence, by the Grötzsch Lemma [Vas02], the conformal type of the graph of u is $\mathbb{C}$.

PROPOSITION 1. *The conformal type of an ideal Scherk graph is* $\mathbb{C}$.

Proof. Let Γ be an ideal Scherk polygon, D the convex hull of Γ and let u be a minimal solution defined in D taking the values $+\infty$ on the A_i in Γ and $-\infty$ on the B_j. Let Σ be the graph of u.

Σ is stable and so has uniformly bounded curvature by Schoens' curvature estimates (Σ is complete).

Then there is a $\delta > 0$ such that for each $x \in \Sigma$, Σ is a graph (in a neighborhood of x) over $D_\delta(x) \subset T_x\Sigma$, where $D_\delta(x)$ is the disk of radius δ in the tangent space $T_x\Sigma$ of Σ at x, centered at the origin of $T_x\Sigma$. Moreover this local graph has bounded geometry, independent of $x \in \Sigma$. Let $G_\delta(x)$ denote this local graph, $G_\delta(x) \subset \Sigma$.

For $p \in D$, we denote by $\Sigma_\delta(p)$ the local graph $G_\delta(x)$ translated vertically so that x goes to height zero; here $x = (p, u(p))$.

Let γ be one of the geodesic components of Γ (an A_i or B_j) and let $p_n \in D$, $q \in \gamma$ and $\lim_{n\to\infty} p_n = q$.

We claim the local surfaces $\Sigma_\delta(p_n)$ converge uniformly to $\gamma \times \mathbb{R}$ as $p_n \to q$.

More precisely, they converge uniformly to $\gamma_\delta(q) \times [-\delta, \delta]$, where $\gamma_\delta(q)$ is the δ-interval of γ centered at q.

Suppose this claim were not the case. First observe the tangent planes to $\Sigma_\delta(p_n)$, at p_n, must converge to the vertical plane $\gamma \times \mathbb{R}$ at q. If not, let q_n denote a subsequence of p_n such that the $T_{q_n}\Sigma_\delta(q_n)$ converge to a plane P at q, distinct from $Q = \gamma \times \mathbb{R}$. Since the graphs $\Sigma_\delta(q_n)$ have bounded geometry, for n large, $\Sigma_\delta(q_n)$ is a graph over the $\delta/2$ disk in P, centered at q, noted $P_{\delta/2}(q)$. A subsequence of these graphs converges uniformly to a minimal graph F over $P_{\delta/2}(q)$. Since $P \neq Q$, there are points of F near q whose horizontal projection to $\mathbb{H}$ is outside of $\overline{D}$. But the $\Sigma_\delta(q_n)$ converge uniformly to F and so for n large, $\Sigma_\delta(q_n)$ would not be a vertical graph over a domain in D, a contradiction.

Thus the tangent planes at p_n to $\Sigma_\delta(p_n)$ converge to the tangent plane to $\gamma \times \mathbb{R}$ at q. Reasoning similar to the previous arguments shows the $\Sigma_\delta(p_n)$ converge to

$\gamma_\delta(q)\times[-\delta,\delta]$. A subsequence of the $\Sigma_\delta(p_n)$ converges to a minimal graph F over $\gamma\times\mathbb{R}$. Were F different from $\gamma\times\mathbb{R}$, F would have points near q whose horizontal projection is outside of $\bar{D}$. But then $\Sigma_\delta(p_n)$ would also have points near q whose horizontal projection is outside $\bar{D}$, for n large. This is impossible.

Now let $\ell>0$ and suppose $\gamma(\ell)$ is a segment of γ of length ℓ. For $\varepsilon>0$, there exists a height $h=h(\ell,\varepsilon)$ and a tubular neighborhood T of $\gamma(\ell)$ in $\bar{D}$, such that the graph of u over T is ε-close in the C^2-topology, to $\gamma(\ell)\times[h,+\infty)$ when γ is an A_i, and Σ is ε-close over T to $\gamma(\ell)\times(-\infty,h]$ when γ is a B_j. This follows from our previous discussion by analytic continuation of the disks of radius δ on Σ that converge to δ-disks on $\gamma\times\mathbb{R}$ as one converges to γ.

We denote by $\Sigma(\gamma(\ell))$ this part of Σ, above (or below) height h, that is ε-close to $\gamma(\ell)\times[h,\infty)$ (or to $\gamma(\ell)\times(-\infty,h]$). As one goes higher (or lower), the $\Sigma(\gamma(\ell))$ converge to $\gamma(\ell)\times\mathbb{R}$. In particular, the horizontal projection of $\Sigma(\gamma(\ell))$ to $\gamma(\ell)\times\mathbb{R}$ is a quasi-isometry.

Now consider a vertex of Γ and let A_i and B_i be the edges of Γ at this vertex. Let F_i be a horocycle at the vertex and $E_i\subset D$ the inside of the horocycle H_i.

By choosing H_i ("small") we can guarantee that each point of E_i is as close to $A_i\cup B_i$ as we wish. Then for $\varepsilon>0$, there exists an H_i such that the part of Σ over E_i (we call $\Sigma(E_i)$) is ε-close to $A_i\times\mathbb{R}$ (and to $B_i\times\mathbb{R}$). We choose ε small so that the horizontal projection of $\Sigma(E_i)$ to $A_i\times\mathbb{R}$ is a quasi-isometry onto its image.

Let $H_1,\dots,H_{2k}$ be small horocycles at each of the vertices of Γ (we assume Γ has $2k$ vertices) so that each $\Sigma(E_i)$ is quasi-isometric to $A_i\times\mathbb{R}$.

Let $\widetilde{A}_i$ and $\widetilde{B}_i$ denote the compact arcs on each A_i and B_i outside of each F_i. For $|h|$ sufficiently large and T a small tubular neighborhood of $\bigcup_{i=1}^{k}(\widetilde{A}_i\cup\widetilde{B}_i)$, each component of the part of Σ over T projects horizontally to $\widetilde{A}_i\times[h,\infty)$ or to $\widetilde{B}_i\times(-\infty,h]$, quasi-isometrically.

To prove Σ is conformally the complex plane $\mathbb{C}$, we write $\Sigma=\bigcup_{j=0}^{\infty}K_j$ where each K_j is a disk, $K_j\subset\mathring{K}_{j+1}$ for each j, and the conformal modulus of each annulus $K_{j+1}-\mathring{K}_j$ is at least one.

Let K_0 be the part of Σ over $D-\big(T\cup\big(\bigcup_{i=1}^{2k}E_i\big)\big)$. Choose h_1 large so that $\Sigma\cap(\mathbb{H}\times[-h_1,h_1])$ contains an annulus K_1-K_0, K_1 compact, of conformal modulus at least one. Similarly, choose $h_2>h_1$ so that $\Sigma\cap(\mathbb{H}\times[-h_2,h_2])$ contains an annulus K_2-K_1 of conformal modulus at least one. The part of Σ outside these K_j converges to geodesics $\times\mathbb{R}$, so the K_j exist for all j. Thus each ideal Scherk surface is conformally $\mathbb{C}$. Notice that our proof describes the geometry of the end of an ideal Scherk graph.

Remark 7. We now give another proof that an ideal Scherk graph is conformally $\mathbb{C}$. A complete Riemannian surface of finite total curvature is conformally equivalent to a compact Riemann surface punctured in a finite number of points [Hub57]. Since a Scherk graph is complete and simply connected, it suffices to show the graph has finite total curvature.

Recall the notation in the proof of Lemma 2: $\hat{\Gamma}(\ell) = \partial D(\ell) = \widetilde{A}(\ell) \cup \widetilde{B}(\ell) \cup \hat{\gamma}(\ell)$. For n fixed, we denote by $v_n(\ell)$ the minimal solution on $D(\ell)$, which equals n on $\widetilde{A}(\ell)$ and zero on the rest of the boundary of $D(\ell)$. As ℓ goes to infinity, the $v_n(\ell)$ converge to the function v_n on D which equals n on the A_i, $i = 1, \ldots, k$, and equals zero on the B_j, $j = 1, \ldots .k$. By the Gauss-Bonnet formula, the graphs of the $v_n(\ell)$ have total curvature equal to $2\pi - 3k\frac{\pi}{2}$. Since the $v_n(\ell)$ converge uniformly on compact sets to v_n, the absolute value of total curvature of the graph of v_n is at most $|(2-2k)\pi|$.

After renormalization (let $u_n = v_n - \mu_n$), the u_n converge uniformly on compact sets to a Scherk graph u defined on D. Thus the graph of u has finite total curvature. Our analysis of the geometry of the Scherk end shows the total curvature equals $(2-2k)\pi$.

6. **Extending an ideal Scherk surface**

We now identify $\mathbb{H}$ with the unit disk in the complex plane $\{z \in \mathbb{C} \,/\, |z| < 1\}$, and $\partial_\infty \mathbb{H}$ with the unit circle S^1. Let $(d_1, d_2, \ldots, d_n)$ be n distinct points of S^1, ordered clockwise, and denote by $P(d_1, d_2, \ldots, d_n)$ the convex hull of the n points in $\mathbb{H}$ (for the hyperbolic metric). We say $P(d_1, d_2, d_3, d_4)$ is a regular quadrilateral when the cross ratio $\frac{(d_1-d_3)(d_2-d_4)}{(d_2-d_3)(d_1-d_4)}$ equals 2. When n is even, we make the convention that ideal Scherk graphs on $P(d_1, d_2, \ldots, d_n)$ take the value $+\infty$ on the geodesic sides $[d_i, d_{i+1}]$, i even, and $-\infty$ on the other sides when i is odd. We denote by (A_i) the geodesics $[d_{2i}, d_{2i+1}]$ and by B_i the geodesics $[d_{2i+1}, d_{2i+2}]$.

Let $\mathcal{P} = \partial P(d_1, d_2, \ldots, d_n)$ be an ideal polygon. As in the previous section, place a horocycle H_i at each vertex d_i, and let $|\mathcal{P}| = \sum_{i=0}^{i=n-1} |d_i d_{i+1}|$ denote the truncated perimeter, where $d_0 = d_n$ and $|d_i d_j|$ is the distance between the horocycles H_i and H_j. $|\mathcal{P}|$ represents the total length of arcs of $\mathcal{P}$ exterior to all the horocycles. The quantity $|\mathcal{P}|$, as the distances $|d_i d_j|$, extends naturally to geodesic polygons with vertices in $\mathbb{H}$, place horocycles only at vertices which are at infinity. The same extension can be done for the quantities $a(\mathcal{P})$ and $b(\mathcal{P})$ if the polygon $\mathcal{P}$ comes from a Dirichlet problem.

Remark 8. The utilization of the truncated perimeter $|\mathcal{P}|$ gives rise practically to quantities associated to $\mathcal{P}$ which are independent of the choice of the horocyle at each vertex d_i. This allows us to check the conditions of Theorem 1 for a choice of horocycles *a priori*, and a restricted class of inscribed polygons. For, under the hypothesis of Theorem 1, let $\mathcal{P}$ be an inscribed polygon, $\mathcal{P} \neq \Gamma$. Notice that, on the one hand at a vertex of $\mathcal{P}$ with a side A_i (necessarily unique), the quantity $|\mathcal{P}| - 2a(\mathcal{P})$ does not depend on the choice of the horocycle at that point; on the other hand, for the remaining vertices, this quantity increases arbitrarily for a choice of horocycles «small» enough. Consequently, if the sides A_i alternate on $\mathcal{P}$, the quantity $|\mathcal{P}| - 2a(\mathcal{P})$ depends only on $\mathcal{P}$. Thus the polygon $\mathcal{P}$ will

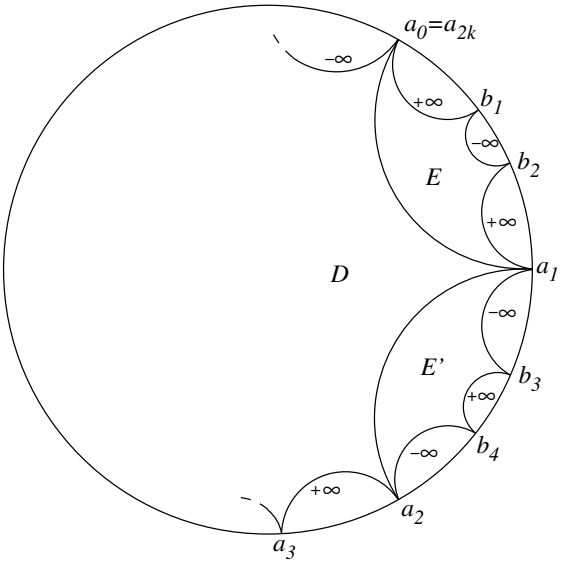

Figure 7

satisfy the first inequality (2) if and only if $|\mathcal{P}| - 2a(\mathcal{P}) > 0$. In all the other cases, for a choice of horocycles «small» enough at vertices where no A_i arrives, the quantity $|\mathcal{P}| - 2a(\mathcal{P})$ becomes arbitrarily large. The first condition of (2) is automatically satisfied. Similarly, the second inequality is equivalent to checking that $|\mathcal{P}| - 2b(\mathcal{P}) > 0$, when the B_j alternate on $\mathcal{P}$ (then this quantity depends only on $\mathcal{P}$).

PROPOSITION 2. *Let u be an ideal Scherk graph on a polygonal domain $D = P(a_1, \ldots, a_{2k})$ and K a compact of D. Let $D_0 = P(b_1, b_2, a_1, b_3, b_4, a_2 \ldots, a_{2k})$ be the polygonal domain D to which we attach two regular polygons*

$$E = P(b_1, b_2, a_1, a_0) \quad \text{and} \quad E' = P(a_1, b_3, b_4, a_2);$$

E is attached to the side $[a_0, a_1] = [a_{2k}, a_1]$ of D and E' to the side $[a_1, a_2]$; cf. Figure 7.

Then for all $\varepsilon > 0$, there exists $(b_i')_{i=2,3}$ and v an ideal Scherk graph on $P(b_1, b_2', a_1, b_3', b_4, a_2 \ldots, a_{2k})$ such that:

$$|b_i' - b_i| \leq \varepsilon \text{ and } \| v - u \|_{\mathscr{C}^2(K)} \leq \varepsilon .$$

We will show that D_0 satisfies the conditions of Theorem 1 except for some particular inscribed polygons. This will allow us, by a small variation of $(b_i)_{i=2,3}$, to ensure them completely. Also, it is easy to check that the regularity of E and E' permits the choice, step-by-step, of the horocycles such that:

$$|a_0 b_1| = |b_1 b_2| = |b_2 a_1| = |a_1 a_0| = a,$$
$$|a_1 b_3| = |b_3 b_4| = |b_4 a_2| = |a_2 a_1| = a.$$

First we establish a Triangle inequality at infinity, using horocycles as usual to define lengths.

LEMMA 3 (Triangle inequality at infinity). *For any triangle with vertices p, q, and r (ideal or not) and small enough pairwise disjoint horocycles placed at the vertices at infinity, $|pq| \leq |pr| + |rq|$. If p and q are in $\partial_\infty \mathbb{H}$, and $r \in \mathbb{H}$, the inequality is true independently of disjoint horocycles placed at p and q.*

Proof. If $r \in \partial_\infty \mathbb{H}$, then the inequality is true for small enough horocycles placed at r. If r is in $\mathbb{H}$, and if p (or q) is at infinity, then the geodesic $[r, p]$ (or $[r, q]$) is asymptotic to $[p, q]$ at p (or q). Then $\forall \alpha > 0$, there exist horocycles small enough so that $|pq| \leq |pr| + |rq| + \alpha$. However, the quantity $|pr| + |rq| - |pq|$ does not depend on the horocycles placed at p and q if any. Then, passing to the limit, we get a triangle inequality at infinity $0 \leq |pr| + |rq| - |pq|$.

In the particular case where $r \in \mathbb{H}$, p and q at infinity, denote by H_p and H_q, the horocycles placed at p and q respectively. If r is in the convex side of one of these horocycles, H_p say, so that $|pr| = 0$, change H_q until touching H_p (on the geodesic $[p, q]$). The Triangle inequality becomes $0 \leq |rq|$ which is true. If r is outside the horocycles, change one of the horocycles, say H_p, until $r \in H_p$, and use the previous computation.

LEMMA 4. *All the inscribed polygons of D_0 are admissible except the boundaries of E, E' and their complements $D_0 \setminus E$, $D_0 \setminus E'$.*

Proof. For the entire polygon D_0, a simple computation gives that, as for the initial polygon D, we have $a(\Gamma_0) = b(\Gamma_0)$, where $\Gamma_0 = \partial D_0$. We only prove the inequalities (2) for values $+\infty$. Then by symmetry of the problem, we will get the inequalities for the $-\infty$ data on the boundary.

We assume now that $\mathcal{P} = \partial P$ is an inscribed polygon of D_0, $P = \partial P(d_1, \ldots, d_n)$ where the $(d_i)_{i=1,\ldots,n}$ are vertices of D_0, and moreover that $P \neq D_0$, $P \neq E$, $P \neq E'$, $P \neq D_0 \setminus E$, $P \neq D_0 \setminus E'$. By Remark 8, it is enough to prove that $|\mathcal{P}| - 2a(\mathcal{P}) > 0$ for an a priori choice of disjoint horocycles H_i at d_i $(i = 1, \ldots, n)$. When the sides A_i where $v = +\infty$ alternate on $\mathcal{P}$, we assume this hypothesis from now on.

Moreover, some inequalities used in the proof of Lemma 4 implicitly assume a further choice of horocycles at d_i $(i = 1, \ldots, n)$. This hypothesis allows us to make that change; cf. Remark 8.

Consider $\mathcal{P}' = \partial P'$, $P' = P \setminus E'$.

CLAIM 1. *If $|\mathcal{P}'| - 2a(\mathcal{P}') > 0$, then $|\mathcal{P}| - 2a(\mathcal{P}) > 0$.*

Proof of the claim. Consider $\mathcal{P}'$; if $\mathcal{P}' = \mathcal{P}$ there is nothing to prove; otherwise we can suppose the geodesic $[b_3, b_4]$ where $v = +\infty$ is in $\mathcal{P}$ (the following argument simplifies if it is not in $\mathcal{P}$). For convenience, change the notation so that $\mathcal{P} = \partial P(d_1, b_3, b_4, d_2 \ldots, d_n)$. Let $q_1 = [d_1, b_3] \cap [a_1, a_2]$ and $q_2 = [d_2, b_4] \cap [a_1, a_2]$; cf. Figure 8. Notice that if $a_1 \in \mathcal{P}$ (resp. $a_2 \in \mathcal{P}$), then

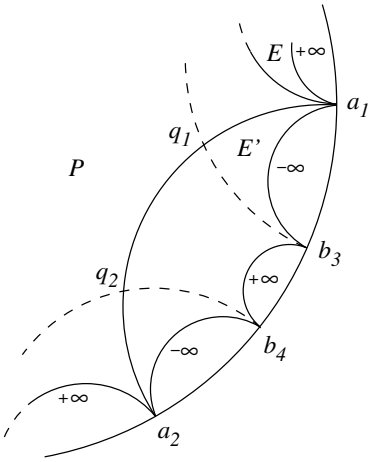

Figure 8

$q_1 = a_1$ and $|a_1q_1| = 0$ by convention (resp. $q_2 = a_2$ and $|a_2q_2| = 0$). We have the relations:

$$a(\mathcal{P}) = a(\mathcal{P}') + a \quad \text{and} \quad |\mathcal{P}| = |\mathcal{P}'| - |q_1q_2| + (|q_1b_3| + a + |q_2b_4|).$$

Now, $|\mathcal{P}'| - 2a(\mathcal{P}') > 0$; hence by substitution:

$$\begin{aligned} |\mathcal{P}| - 2a(\mathcal{P}) &> (|q_1b_3| + |q_2b_4|) - (a + |q_1q_2|) \\ &= (|q_1b_3| + |q_2b_4|) - (2a - |a_1q_1| - |a_2q_2|) \\ &= (|a_1q_1| + |q_1b_3| - a) + (|a_2q_2| + |q_2b_4| - a). \end{aligned}$$

By the triangle inequality at infinity (directly if $q_1 = a_1$ or $q_2 = a_2$):

$$|a_1q_1| + |q_1b_3| - a \geq 0 \quad \text{and} \quad |a_2q_2| + |q_2b_4| - a \geq 0.$$

Hence, $|\mathcal{P}| - 2a(\mathcal{P}) > 0$ and the claim is proved.

So it remains to prove $|\mathcal{P}'| - 2a(\mathcal{P}') > 0$. For that, define $\mathcal{P}'' = \partial P''$, $P'' = P' \setminus E$.

The key point is a flux inequality for $\mathcal{P}''$ coming from the initial solution u defined on D. We have $P'' \subset D$ and there exists the divergence free field $X = \frac{\nabla u}{W}$ ($W = (1 + |\nabla u|^2)^{1/2}$) on P''. Moreover, on the arcs of $\mathcal{P}''$, $X = \nu$ if $u = +\infty$, $X = -\nu$ if $u = -\infty$, where ν is the outward normal of P''. Let us write $\mathcal{P}''$ as $I_0 \cup I_1 \cup J$: I_0 is the union of all geodesics A_i (where $u = +\infty$) contained in $\mathcal{P}''$ and disjoint from $[a_0, a_1]$, $I_1 = \mathcal{P}'' \cap [a_0, a_1]$ and J the union of the remaining arcs.

The flux of X along $\mathcal{P}'' = \partial P''$ is zero, which yields:

$$0 = a(\mathcal{P}'') + |I_1| + F_u(J) + \rho.$$

Here the flux $F_u(J)$ is taken on the compact part of J outside the horocycles and ρ is a residual term corresponding to the flux of X along some parts of horocycles.

Next we have the truncated perimeter of $\mathcal{P}''$:

$$|\mathcal{P}''| = a(\mathcal{P}'') + |I_1| + |J|.$$

Adding these last two equalities, we have

$$|\mathcal{P}''| - 2(a(\mathcal{P}'') + |I_1|) = |J| + F_u(J) + \rho.$$

Note that the condition on the A_i sides of $\mathcal{P}$ (alternate) yields: on the one hand the quantity we have to estimate $|\mathcal{P}'| - 2a(\mathcal{P}')$ is independent of the choice of horocycles, so this allows us to change this choice as necessary; on the other hand, we have that $P'' \neq D$ and $P'' \neq \varnothing$. For, if $P'' = D$, a careful analysis of the possibilities of inscribed polygons $\mathcal{P}$ with alternate A_i sides (using a_0 and a_1 are vertices of $\mathcal{P}$, cf. Figure 10 below) leads to $P = D_0$ or $P = D_0 \setminus E$ which are excluded by hypothesis. Similarly, if $P'' = \varnothing$, then $P \subset E$ or $P \subset E'$ and, in this case, the only possibility of an inscribed polygon with alternate A_i sides is E which is excluded too. Therefore J contains interior arcs and as the horocycles at vertices of $\mathcal{P}''$ diverge:

$$\exists c_0 > 0 \text{ so that } |J| + F_u(J) \geq c_0.$$

We can ensure $|\rho| < c_0$ for a suitable choice of horocycles. Hence we get the following flux inequality:

$$|\mathcal{P}''| - 2(a(\mathcal{P}'') + |I_1|) > 0. \tag{3}$$

We have three cases to consider.

Case 1. Suppose $[a_0, b_1] \cup [b_2, a_1] \subset \mathcal{P}$. In this case, $E \subset P'$ and

$$\begin{aligned} a(\mathcal{P}') &= a(\mathcal{P}'') + 2a, \\ |\mathcal{P}'| &= |\mathcal{P}''| + 2a, \\ |I_1| &= a. \end{aligned}$$

The flux inequality (3) directly gives:

$$0 < (|\mathcal{P}'| - 2a) - 2(a(\mathcal{P}') - a) = |\mathcal{P}'| - 2a(\mathcal{P}').$$

Case 2. Suppose only one of the $[a_0, b_1]$ or $[b_2, a_1]$ is contained in $\mathcal{P}$, $[a_0, b_1]$ say. If we denote $I_1 = [a_0, q]$ (cf. Figure 9), then we have:

$$\begin{aligned} a(\mathcal{P}') &= a(\mathcal{P}'') + a, \\ |\mathcal{P}'| &= |\mathcal{P}''| - |I_1| + a + |b_1 q|. \end{aligned}$$

The flux inequality (3) yields:

$$\begin{aligned} &0 < (|\mathcal{P}'| + |I_1| - a - |b_1 q|) - 2(a(\mathcal{P}') - a + |I_1|), \\ &0 < |\mathcal{P}'| - 2a(\mathcal{P}') - |b_1 q| - |I_1| + a. \end{aligned}$$

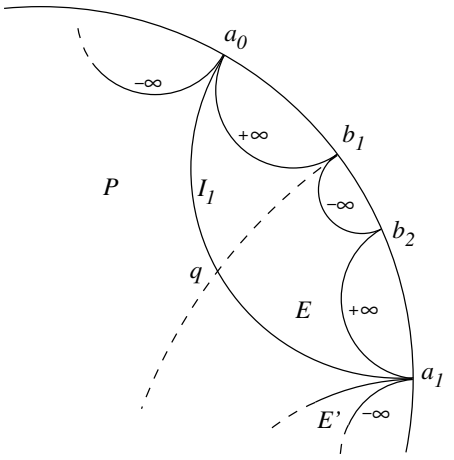

Figure 9

Hence, using the triangle inequality at infinity (also valid if $q = a_1$, which does not occur), we obtain:

$$|\mathcal{P}'| - 2a(\mathcal{P}') > |b_1 q| + |q a_0| - a \geq 0.$$

Case 3. The remaining case is for $P' \subset D$. Then the flux inequality (3) gives directly the result for $\mathcal{P}' = \partial P'$. This completes the proof of Lemma 4.

The only obstructions to the existence of an ideal Scherk graph on D_0 come from the polygons E, E' and their complements where we have some cases of equality in (2). In the next lemma, we will ensure them by a perturbation of D_0.

LEMMA 5. *There exists* $\tau_0 > 0$ *such that for all* $\tau \in (0, \tau_0]$, *there exist* v_τ, *an ideal Scherk graph on* $D_\tau = P(b_1, b_2(\tau), a_1, b_3(\tau), b_4, a_2, \ldots, a_{2k})$ *with*: $|b_i(\tau) - b_i| \leq \tau$ *for* $i = 2, 3$.

Proof. Let $\tau > 0$ and consider moving b_2 and b_3 in the direction of a_1, such that $|b_i(\tau) - b_i| \leq \tau$ $(i = 2, 3)$; cf. Figure 10. By such variations, the quantities $(|a_0 a_1| - |a_1 b_2(\tau)| + |b_2(\tau) b_1| - |b_1 a_0|)$ and $(|a_2 a_1| - |a_1 b_3(\tau)| + |b_3(\tau) b_4| - |b_4 a_2|)$, which are independent of the choice of horocycles, increase. They are zero for the initial polygons E andE'. Then there exist such variations $b_2(\tau)$ and $b_3(\tau)$ so that, for $\tau > 0$:

$$\varphi(\tau) = |a_0 a_1| - |a_1 b_2(\tau)| + |b_2(\tau) b_1| - |b_1 a_0| > 0,$$
$$\varphi(\tau) = |a_2 a_1| - |a_1 b_3(\tau)| + |b_3(\tau) b_4| - |b_4 a_2| > 0.$$

Hence, for the polygon $\Gamma_\tau = \partial D_\tau$, we have the condition (1) of Theorem 1, $a(\Gamma_\tau) = b(\Gamma_\tau)$ for all τ small. On the other hand, for this choice of variation, the $\varphi(\tau)$-perturbed polygons $E_\tau = P(b_1, b_2(\tau), a_1, a_0)$ and $E'_\tau = P(a_1, b_3(\tau), b_4, a_2)$ satisfy

$$|\partial E_\tau| - 2a(\partial E_\tau) = |\partial E'_\tau| - 2b(\partial E'_\tau) = \varphi(\tau) > 0,$$

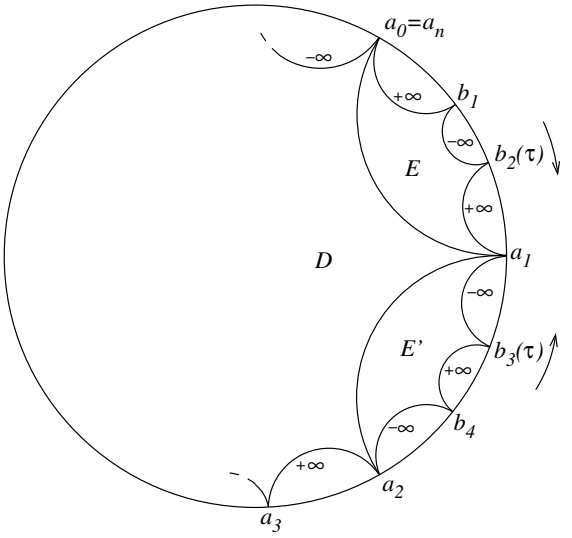

Figure 10

and the inequalities (2) are true for these polygons (use Remark 8). A similar computation gives the inequalities (2) for $D_\tau \setminus E_\tau$ and $D_\tau \setminus E'_\tau$.

In order to prove the inequalities (2) for all other inscribed polygons of D_τ, we use Lemma 4. For the inscribed polygons $\mathcal{P}$ of D_0 (except ∂D_0 and those excluded by Lemma 4), the inequalities (2) are strict and so are stable by small enough perturbations of vertices (and attached horocycles). There are a finite number of such admissible polygons; thus there exists $\tau_0 > 0$ such that for all $0 < \tau \leq \tau_0$, and variations $b_2(\tau)$ and $b_3(\tau)$ as above, the conditions of Theorem 1 are satisfied. This ensures the existence of v_τ on D_τ.

Proof of Proposition 2. The main step of the proof consists in establishing that $\lim_{\tau\to 0} \nabla v_\tau|_D = \nabla u$. For that, consider the divergence-free fields associated to v_τ and u: $X_\tau = \frac{\nabla v_\tau}{W_\tau}$ and $X = \frac{\nabla u}{W}$ (with $W_\tau = (1+|\nabla v_\tau|^2)^{1/2}$, $W = (1+|\nabla u|^2)^{1/2}$); we will prove that $\lim_{\tau\to 0} X_\tau|_D = X$.

On the boundary, let ν be the outer pointing normal to ∂D. On $\partial D \setminus ([a_0, a_1] \cup [a_1, a_2])$, v_τ and u take the same infinite values. Hence $X_\tau = X = \pm\nu$. On $[a_0, a_1]$, $u = +\infty$ so that $X = \nu$. On the other side, consider the boundary of the domain E_τ truncated by horocycles. Denote the four horocycle arcs by $\widetilde{\gamma}$. An estimate of the flux of X_τ yields:

$$0 = |a_0 b_1| - |b_1 b_2(\tau)| + |b_2(\tau) a_1| + \int_{[a'_0, a'_1]} \langle X_\tau, (-\nu)\rangle \, ds + F_{v_\tau}(\widetilde{\gamma});$$

the integral is on $[a'_0, a'_1]$, the compact part of $[a_0, a_1]$ joining the horocycles. Then

$$0 = -\varphi(\tau) + \int_{[a'_0, a'_1]} (1 - \langle X_\tau, \nu\rangle) \, ds + F_{v_\tau}(\widetilde{\gamma}).$$

For a diverging sequence of nested horocycles, we get the convergence of the integral on the whole geodesic and the equality:

$$\int_{[a_0,a_1]} (1 - \langle X_\tau, \nu \rangle)\, ds = \varphi(\tau).$$

In the same way, on $[a_1, a_2]$ we get a convergent integral

$$\int_{[a_1,a_2]} (1 + \langle X_\tau, \nu \rangle)\, ds = \varphi(\tau).$$

Because of the value of X on $[a_0, a_1] \cup [a_1, a_2]$, for any family α of disjoint arcs of ∂D

$$\left| \int_\alpha \langle X - X_\tau, \nu \rangle\, ds \right| \leq \int_{[a_0,a_1]\cup[a_1,a_2]} |\langle X - X_\tau, \nu \rangle|\, ds = 2\varphi(\tau). \tag{4}$$

For the study of the field $X_\tau - X$ on the interior of D, we consider the flux of $X_\tau - X$ along a level curve through an interior point p. This level curve goes to the boundary of D where we create a closed cycle by attaching short curves and a curve on the boundary of D to the level curve. Then the flux is zero along the closed cycle, and small along the curve we attach to the level curve. Thus the flux is small along the level curve, which implies the tangent planes are close; then bounded curvature of the graphs gives the fact that the solutions are close. We now make this precise.

As in the previous section, Σ the graph of u and Σ_τ the graph of v_τ are stable, complete and satisfy uniform curvature estimates. Then

$$\forall \mu > 0,\ \exists \rho > 0 \text{ (independent of } \tau) \text{ such that } \forall p \in D :$$
$$q \in \Sigma_\tau \cap B((p, v_\tau(p)), \rho) \Longrightarrow \| n_\tau(q) - n_\tau(p) \| \leq \mu.$$

Here, n_τ denotes the normal to Σ_τ pointing down and $B((p, v_\tau(p)), \rho)$ denotes the ball of radius ρ, centered at $(p, v_\tau(p)) \in \mathbb{H} \times \mathbb{R}$. We have the same estimates for Σ.

Fix any $\mu > 0$ and $p \in D$, this gives a $\rho_1 \leq \rho/2$ (independent of τ) such that $\forall q \in D(p, \rho_1)$, the disk of $\mathbb{H}$ with center p and radius ρ_1, we have $|u(q) - u(p)| \leq \rho/2$.

Assume now that $\| n(p) - n_\tau(p) \| \geq 3\mu$. Let $\Omega_\tau(p)$ be the connected component of $\{u - v_\tau > u(p) - v_\tau(p)\}$ with p in its boundary and Λ_τ the component of $\partial\Omega_\tau$ containing p. Now Λ_τ, as the level curve of $u - v_\tau$, is piecewise smooth. Above $\Lambda_\tau \cap D(p, \rho_1)$, there are two parallel curves: $\sigma \subset \Sigma$ and $\sigma_\tau \subset \Sigma_\tau$. Moreover on σ:

$$\forall q \in \Lambda_\tau \cap D(p, \rho_1),\ |(q, u(q)) - (p, u(p))| \leq \rho_1 + \rho/2 \leq \rho.$$

Hence $\| n(q) - n(p) \| \leq \mu$.

By a vertical translation of height $(v_\tau(p) - u(p))$:

$$(q, v_\tau(q)) \in \sigma_\tau \text{ and } v_\tau(q) - v_\tau(p) = u(q) - u(p).$$

Then

$$\| (q, v_\tau(q)) - (p, v_\tau(p)) \| \leq \rho \text{ and } \| n_\tau(q) - n_\tau(p) \| \leq \mu.$$

Combining the two last estimates with the assumption on the normals at p, we obtain:

$$\forall q \in \Lambda_\tau \cap D(p, \rho_1),\ \| n(q) - n_\tau(q) \| \geq \| n(p) - n_\tau(p) \| - 2\mu \geq \mu.$$

Apply Lemma A.1 below to conclude

$$\int_{\Lambda_\tau \cap D(p,\rho_1)} \langle X - X_\tau, \eta \rangle \, ds \geq \frac{\rho_1 \mu^2}{2}.$$

As in the proof of the maximum principle, $\langle X - X_\tau, \eta \rangle$ is nonnegative outside the isolated points where $\nabla(u - v_\tau) = 0$; then for all compact arcs $\beta \subset \Lambda_\tau$, containing $\Lambda_\tau \cap D(p, \rho_1)$ we have:

$$\int_\beta \langle X - X_\tau, \eta \rangle \, ds \geq \frac{\rho_1 \mu^2}{2}. \tag{5}$$

By the maximum principle, Λ_τ is noncompact in D; so its two infinite branches are close to ∂D. Then there exists a connected compact part β of Λ_τ, containing $\Lambda_\tau \cap D(p, \rho_1)$, and two arcs $\widetilde{\gamma}$ in D small enough and joining the extremities of β to ∂D. Eventually truncating by a family of horocycles, the flux formula for $X - X_\tau$ yields:

$$0 = \int_\beta \langle X - X_\tau, (-\eta) \rangle \, ds + \int_\alpha \langle X - X_\tau, \nu \rangle \, ds + F_{u - v_\tau}(\widetilde{\gamma} \cup \widetilde{\gamma}'),$$

where α is contained in ∂D and $\widetilde{\gamma}'$ is contained in the horocycles and correctly oriented. Using (4) and (5) we obtain

$$\frac{\rho_1 \mu^2}{2} \leq 2\varphi(\tau) + F_{u - v_\tau}(\widetilde{\gamma} \cup \widetilde{\gamma}').$$

When the length of $\widetilde{\gamma} \cup \widetilde{\gamma}'$ goes to zero, we conclude

$$\frac{\rho_1 \mu^2}{2} \leq 2\varphi(\tau).$$

Hence,

$$\varphi(\tau) \leq \frac{\rho_1 \mu^2}{4} \Longrightarrow \| X(p) - X_\tau(p) \| \leq \| n(p) - n_\tau(p) \| \leq 3\mu.$$

This gives precisely the behavior of X_τ and v_τ for τ close to zero. After the renormalisation $v_\tau(p_0) = u(p_0)$ for a fixed $p_0 \in D$, we have $\lim_{\tau \to 0} v_{\tau|_D} = u$. The convergence being uniform and $\mathcal{C}^\infty$ on compact sets, for τ small enough we can ensure $\| v_\tau - u \|_{\mathcal{C}^2(K)} \leq \varepsilon$. By Lemma 5, the existence of the ideal Scherk graph of Proposition 2 is established.

We present here the geometric estimate of flux along level curves needed at the end of the proof of Proposition 2.

LEMMA A.1. *Let w and w' be two minimal graphs of $\mathbb{H}\times\mathbb{R}$, above a domain $\Omega\subset\mathbb{H}$ where n and n' are their respective normals. Then at any regular point of $w'-w$:*

$$\langle X'-X,\eta\rangle_{\mathbb{H}}\geq\frac{\|\,n'-n\,\|^2}{4}\geq\frac{|X'-X|^2}{4},$$

where X (resp. X') is the projection of n (resp. n') on $\mathbb{H}$ and $\eta=\frac{\nabla(w'-w)}{|\nabla(w'-w)|}$ orients the level curve at this regular point.

Proof. We write $X=\frac{\nabla w}{W}$, $X'=\frac{\nabla w'}{W'}$ (the normals point down). Classically [CK91],

$$\begin{aligned}\langle X'-X,\nabla w'-\nabla w\rangle_{\mathbb{H}}&=\langle n'-n,W'n'-Wn\rangle_{\mathbb{H}\times\mathbb{R}}\\&=(W+W')\big(1-\langle n,n'\rangle_{\mathbb{H}\times\mathbb{R}}\big)\\&=(W+W')\frac{\|\,n'-n\,\|^2}{2}.\end{aligned}$$

Also $\frac{|\nabla w'-\nabla w|}{W'+W}\leq\frac{|\nabla w'|}{W'}+\frac{|\nabla w|}{W}\leq 2$. Hence

$$\langle X'-X,\eta\rangle=\frac{W+W'}{|\nabla(w'-w)|}\,\frac{\|\,n'-n\,\|^2}{2}\geq\frac{\|\,n'-n\,\|^2}{4}.$$

The last inequality of Lemma A.1 simply arises by projection.

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