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Abstract

We study Einstein manifolds admitting a transitive solvable Lie group of isometries (*solvmanifolds*). It is conjectured that these exhaust the class of noncompact homogeneous Einstein manifolds. J. Heber has shown that under a simple algebraic condition (he calls such a solvmanifold *standard*), Einstein solvmanifolds have many remarkable structural and uniqueness properties. In this paper, we prove that any Einstein solvmanifold is standard, by applying a stratification procedure adapted from one in geometric invariant theory due to F. Kirwan.

1. **Introduction**

The construction of Einstein metrics on manifolds is a classical problem in differential geometry and general relativity. A Riemannian manifold is called *Einstein* if its Ricci tensor is a scalar multiple of the metric. The Einstein equation $\mathrm{Ric}(g) = \lambda g$ is a nonlinear second order system of partial differential equations, and a general understanding of the solutions seems far from being attained (see [Ber03, 11.4]). General existence and nonexistence results are hard to obtain, and it is a natural simplification to impose additional symmetry assumptions, i.e.,to consider metrics admitting a large Lie group of isometries. In the homogeneous case, the Einstein equation becomes a subtle system of algebraic equations, and the following main general question is still open in both, the compact and noncompact cases:

> Which homogeneous spaces G/K admit a G-invariant Einstein Riemannian metric?

In this paper we shall consider this question in the noncompact case. We refer to [BWZ04] and the references therein for an account in the compact case.

All the known examples of noncompact homogeneous Einstein manifolds belong to the class of *solvmanifolds*, that is, simply connected solvable Lie groups S

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endowed with a left invariant metric (see the survey [Lau09]). According to a long standing conjecture attributed to D. Alekseevskii (see [Bes87, 7.57]), these might exhaust the class of noncompact homogeneous Einstein manifolds.

On the other hand, all the known examples of Einstein solvmanifolds satisfy the following additional condition: if $\mathfrak{s} = \mathfrak{a} \oplus \mathfrak{n}$ is the orthogonal decomposition of the Lie algebra $\mathfrak{s}$ of S with $\mathfrak{n} = [\mathfrak{s}, \mathfrak{s}]$, then $[\mathfrak{a}, \mathfrak{a}] = 0$. A solvmanifold with such a property is called *standard*. For instance, any solvmanifold of nonpositive sectional curvature is standard (see [AW76]).

Standard Einstein solvmanifolds constitute a distinguished class that has been deeply investigated by J. Heber, who has derived many remarkable structural and uniqueness results, by assuming only the standard condition (see [Heb98]). We shall review some of them. In contrast to the compact case, a standard Einstein metric is unique up to isometry and scaling among invariant metrics ([Heb98, Th. E]). Any standard Einstein solvmanifold is isometric to a solvmanifold whose underlying metric Lie algebra resembles an Iwasawa subalgebra of a semisimple Lie algebra in the sense that $\operatorname{ad} A$ is symmetric and nonzero for any $A \in \mathfrak{a}$, $A \neq 0$. Moreover, if H denotes the *mean curvature vector* of S (i.e., $\operatorname{tr} \operatorname{ad} A = \langle H, A \rangle$ for all $A \in \mathfrak{a}$), then the eigenvalues of $\operatorname{ad} H|_{\mathfrak{n}}$ form (up to scaling) a set of natural numbers, called the *eigenvalue type* of S. There are finitely many such types in each dimension. Let $\mathcal{M}$ be the moduli space of all the isometry classes of Einstein solvmanifolds of a given dimension with scalar curvature equal to -1, and let $\mathcal{M}_{\mathrm{st}}$ be the subspace of those which are standard. Then each eigenvalue type determines a compact pathwise connected component of $\mathcal{M}_{\mathrm{st}}$, which is homeomorphic to a real semialgebraic set. A main result in [Heb98] shows that $\mathcal{M}_{\mathrm{st}}$ is open in $\mathcal{M}$ in the C^{∞}-topology ([Heb98, Th. G]).

The goal of this paper is to apply an adaptation of a stratification method given in [Kir84] to prove that actually $\mathcal{M}_{\mathrm{st}} = \mathcal{M}$. In particular, all the nice structural and uniqueness results in [Heb98] are valid for any Einstein solvmanifold, and possibly for any noncompact homogeneous Einstein manifold (if the Alekseevskii's conjecture turns out to be true).

THEOREM. *Any Einstein solvmanifold is standard.*

The proof of the theorem involves a somewhat extensive study of the natural $\mathrm{GL}_n(\mathbb{R})$-action on the vector space $V_n := \Lambda^2(\mathbb{R}^n)^* \otimes \mathbb{R}^n$, from a geometric invariant theory point of view (a method already used in [Heb98, §§6.3 and 6.4] in the standard case). We recall that V_n can be viewed as a vector space containing the space of all n-dimensional Lie algebras as an algebraic subset.

We define in Section 2 a $\mathrm{GL}_n(\mathbb{R})$-invariant stratification of V_n satisfying certain boundary properties (see Theorem 2.10), by adapting a construction for reductive groups actions on projective algebraic varieties given by F. Kirwan in [Kir84, §12] in the algebraically closed case (see also [Nes84]). We note that any $\mu \in V_n$ is *unstable* (i.e., $0 \in \overline{\mathrm{GL}_n(\mathbb{R}).\mu}$). The strata are parametrized by a finite set $\mathcal{B}$ of diagonal $n \times n$ matrices, and each $\beta \in \mathcal{B}$ is (up to conjugation) the 'most responsible' direction for the instability of each μ in the stratum $\mathcal{S}_\beta$, in the sense that $e^{-t\beta}.\mu \to 0$, as $t \to \infty$ faster than any other one-parameter subgroup having a tangent vector of the same norm.

We also prove that $\mathcal{S}_\beta$ can be described in terms of semistable (i.e.,non unstable) vectors for a suitable action. This and the fact that the automorphism group of any semistable $\mu \in \mathcal{S}_\beta$ must be contained in the parabolic subgroup of $\mathrm{GL}_n(\mathbb{R})$ defined by β are crucial in the proof of several lemmas needed to prove the theorem.

The first step in the proof of the main theorem, given in Section 3, uses the following fact proved in [Lau06]: S is an Einstein solvmanifold with $\mathrm{Ric} = cI$ if and only if

$$\text{(1)} \qquad \operatorname{tr}\left(cI + \tfrac{1}{2}B + S(\operatorname{ad} H)\right) E = \tfrac{1}{4}\langle \pi(E)[\cdot,\cdot],[\cdot,\cdot]\rangle, \qquad \forall E \in \operatorname{End}(\mathfrak{s}),$$

where B is the Killing form and $S(\cdot)$ denotes the symmetric part of an operator. Here $\mathfrak{s}$ is identified with $\mathbb{R}^m$, the bracket $[\cdot,\cdot]$ of $\mathfrak{s}$ becomes a vector in V_m and π is the representation of $\mathfrak{gl}_m(\mathbb{R})$ on V_m corresponding to the $\mathrm{GL}_m(\mathbb{R})$-action. The next step is to use (1) for the right choice of E, namely, $E|_\mathfrak{a} = 0$, $E|_\mathfrak{n} = \beta + ||\beta||^2 I$, where $\mathcal{S}_\beta$ is the stratum the Lie bracket $\mu := [\cdot,\cdot]|_{\mathfrak{n}\times\mathfrak{n}}$ belongs to, as a vector of V_n. Notice that we are identifying $\mathfrak{n} = [\mathfrak{s},\mathfrak{s}]$ with $\mathbb{R}^n$. We obtain in this way many expressions which are proved to be nonnegative by using the lemmas in Section 2. Finally, by a positivity argument, all these inequalities turn into equalities, and one of them shows that $[\mathfrak{a},\mathfrak{a}] = 0$. A key fact is that $\beta + ||\beta||^2 I$ is positive definite for any stratum $\mathcal{S}_\beta$ which meets the closed subset $\mathcal{N} \subset V_n$ of all nilpotent Lie brackets. This is a special feature of this action and is the only point where we actually use that the vector $\mu = [\cdot,\cdot]|_{\mathfrak{n}\times\mathfrak{n}} \in V_n$, to which we are applying the geometric invariant theory machinery, is a nilpotent Lie algebra.

Partial results on the question if $\mathcal{M}_{\mathrm{st}} = \mathcal{M}$ were obtained by J. Heber [Heb98], D. Schueth [Sch04], E. Nikitenko and Y. Nikonorov [NN06] and Y. Nikolayevsky [Nik06]. It is proved for instance in [Nik06] that many classes of nilpotent Lie algebras cannot be the nilradical of a nonstandard Einstein solvmanifold.

More recent results on the structure of standard Einstein solvmanifolds include interplays with critical points of the square norm of a moment map and Ricci soliton metrics (see for instance [Lau09], [Nik] and the references therein).

We finally mention that the stratification in this paper has also proved to be very useful in the study of standard Einstein solvmanifolds (see [LW]). The subset $\mathcal{N} \subset V_n$ parametrizes a set of $(n+1)$-dimensional rank-one (i.e. $\dim \mathfrak{a} = 1$) solvmanifolds $\{S_\mu : \mu \in \mathcal{N}\}$, containing the set of all those which are Einstein in that dimension. The stratum of μ determines the eigenvalue type of a potential Einstein solvmanifold $S_{g.\mu}$, $g \in \mathrm{GL}_n(\mathbb{R})$ (if any), and so the stratification provides a convenient tool to produce existence results as well as obstructions for nilpotent Lie algebras to be the nilradical of an Einstein solvmanifold.

2. A stratification of V_n

In this section, we define a $\mathrm{GL}_n(\mathbb{R})$-invariant stratification of a certain real representation V_n of $\mathrm{GL}_n(\mathbb{R})$ by adapting to our context the construction given by F. Kirwan in [Kir84, §12] for reductive group representations over an algebraically closed field. This construction, in turn, is based on some instability results due to G. Kempf [Kem78] and W. Hesselink [Hes78] (see also [Nes84]). We have decided to give a self-contained proof of all these results following [Kir84], bearing in mind that they are crucial in the proof of Theorem 3.1 and that a direct application of them does not seem feasible. What is really needed in the proof is the existence of a diagonal matrix β satisfying conditions (10) and those stated in Lemmas 2.15, 2.16 and 2.17.

We consider the vector space

$$V_n = \Lambda^2(\mathbb{R}^n)^* \otimes \mathbb{R}^n = \{\mu : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}^n : \mu \text{ bilinear and skew-symmetric}\},$$

on which there is a natural linear action of $\mathrm{GL}_n(\mathbb{R})$ on the left given by

$$g.\mu(X,Y) = g\mu(g^{-1}X, g^{-1}Y), \qquad X, Y \in \mathbb{R}^n, \quad g \in \mathrm{GL}_n(\mathbb{R}), \quad \mu \in V_n. \tag{2}$$

The canonical inner product $\langle \cdot, \cdot \rangle$ on $\mathbb{R}^n$ defines an $\mathrm{O}(n)$-invariant inner product on V_n by

$$\langle \mu, \lambda \rangle = \sum_{ijk} \langle \mu(e_i, e_j), e_k \rangle \langle \lambda(e_i, e_j), e_k \rangle, \tag{3}$$

where $\{e_1, \dots, e_n\}$ is the canonical basis of $\mathbb{R}^n$. A Cartan decomposition for the Lie algebra of $\mathrm{GL}_n(\mathbb{R})$ is given by $\mathfrak{gl}_n(\mathbb{R}) = \mathfrak{so}(n) \oplus \mathrm{sym}(n)$, that is, in skew-symmetric and symmetric matrices respectively. We consider the following $\mathrm{Ad}(\mathrm{O}(n))$-invariant inner product on $\mathfrak{gl}_n(\mathbb{R})$,

$$\langle \alpha, \beta \rangle = \operatorname{tr} \alpha \beta^{\mathrm{t}}, \qquad \alpha, \beta \in \mathfrak{gl}_n(\mathbb{R}). \tag{4}$$

Remark 2.1. There have been several abuses of notation concerning inner products. Recall that $\langle \cdot, \cdot \rangle$ has been used to denote an inner product on $\mathbb{R}^n$, V_n and $\mathfrak{gl}_n(\mathbb{R})$.

The action of $\mathfrak{gl}_n(\mathbb{R})$ on V_n obtained by differentiation of (2) is given by

$$\pi(\alpha)\mu = \alpha\mu(\cdot,\cdot) - \mu(\alpha\cdot,\cdot) - \mu(\cdot,\alpha\cdot), \qquad \alpha \in \mathfrak{gl}_n(\mathbb{R}), \quad \mu \in V_n. \tag{5}$$

We note that $\pi(\alpha)^t = \pi(\alpha^t)$ for any $\alpha \in \mathfrak{gl}_n(\mathbb{R})$. Let $\mathfrak{t}$ denote the set of all diagonal $n \times n$ matrices. If $\{e_1', \dots, e_n'\}$ is the basis of $(\mathbb{R}^n)^*$ dual to the canonical basis then

$$\{v_{ijk} = (e_i' \wedge e_j') \otimes e_k : 1 \le i < j \le n,\ 1 \le k \le n\}$$

is a basis of weight vectors of V_n for the action (2), where v_{ijk} is actually the bilinear form on $\mathbb{R}^n$ defined by $v_{ijk}(e_i, e_j) = -v_{ijk}(e_j, e_i) = e_k$ and zero otherwise.

The corresponding weights $\alpha_{ij}^k \in \mathfrak{t}$, $i < j$, are given by

$$\pi(\alpha)v_{ijk} = (a_k - a_i - a_j)v_{ijk} = \langle \alpha, \alpha_{ij}^k \rangle v_{ijk}, \quad \forall \alpha = \begin{bmatrix} a_1 & & \\ & \ddots & \\ & & a_n \end{bmatrix} \in \mathfrak{t},$$

where $\alpha_{ij}^k = E_{kk} - E_{ii} - E_{jj}$ and $\langle \cdot, \cdot \rangle$ is the inner product defined in (4). As usual, E_{rs} denotes the matrix whose only nonzero coefficient is 1 at the entry rs. From now on, we will always denote by μ_{ij}^k the coefficients of a vector $\mu \in V_n$ with respect to this basis, that is,

$$\mu = \sum \mu_{ij}^k v_{ijk}, \qquad \mu_{ij}^k \in \mathbb{R}.$$

Let $\mathfrak{D}$ denote the set of all $n \times n$ matrices which are diagonalizable; that is,

$$\mathfrak{D} = \bigcup_{g \in \mathrm{GL}_n(\mathbb{R})} g\mathfrak{t}g^{-1}.$$

Consider $m : V_n \times \mathfrak{D} \longrightarrow \mathbb{R}$ the function defined by letting $m(\mu, \alpha)$ be the smallest eigenvalue of $\pi(\alpha)$ such that the projection of μ onto the corresponding eigenspace is nonzero. Since μ_1 is an eigenvector of $\pi(\alpha)$ with eigenvalue a if and only if $g.\mu_1$ is an eigenvector of $\pi(g\alpha g^{-1})$ with eigenvalue a, we have that

$$m(g.\mu, g\alpha g^{-1}) = m(\mu, \alpha), \qquad \forall g \in \mathrm{GL}_n(\mathbb{R}). \tag{6}$$

It follows from the definition of m that $m(\mu, a\alpha) = am(\mu, \alpha)$ for any $a > 0$ and

$$m(\mu, \alpha) = \inf \left\{ \langle \alpha, \alpha_{ij}^k \rangle : \mu_{ij}^k \neq 0 \right\}, \quad \forall \alpha \in \mathfrak{t}.$$

For each nonzero $\mu \in V_n$ define

$$Q(\mu) = \inf_{\alpha \in \mathfrak{D}} \{ q(\alpha) : m(\mu, \alpha) \geq 1 \}$$

and

$$\Lambda(\mu) = \{ \beta \in \mathfrak{D} : q(\beta) = Q(\mu), \quad m(\mu, \beta) \geq 1 \},$$

where $q : \mathfrak{gl}_n(\mathbb{R}) \longrightarrow \mathbb{R}$ is defined by $q(\alpha) = \operatorname{tr} \alpha^2$. Note that q is invariant by $\mathrm{GL}_n(\mathbb{R})$-conjugation, $q(\alpha) > 0$ for any nonzero $\alpha \in \mathfrak{D}$ and $q(\alpha) = ||\alpha||^2$ for any $\alpha \in \mathrm{sym}(n)$.

Remark 2.2. It would actually be enough to require that $m(\mu, \alpha) = 1$ in the definition of $Q(\mu)$, and also it turns out that $m(\mu, \beta) = 1$ for any $\beta \in \Lambda(\mu)$, but the above choice is technically more convenient.

Remark 2.3. Every $\mu \in V_n$ is unstable for this $\mathrm{GL}_n(\mathbb{R})$-action, i.e.,

$$0 \in \overline{\mathrm{GL}_n(\mathbb{R}).\mu},$$

since scalar matrices act as homotheties. Recall that if $m(\mu, \alpha) > 0$ for $\alpha \in \mathfrak{D}$ then $\lim_{t \to \infty} e^{-t\alpha}.\mu = 0$, and thus the number $Q(\mu)^{-1}$ measures in some sense the degree of instability of μ as

$$Q(\mu)^{-\frac{1}{2}} = \sup_{\alpha \in \mathfrak{D}} \{m(\mu, \alpha) : q(\alpha) = 1\}.$$

We note that the existence of such a one-parameter subgroup is also necessary for a vector to be unstable, and is called the numerical criterion of stability. This notion was established by D. Hilbert to classify homogeneous polynomials, by D. Mumford [MFK94] in the general algebraically closed case and by R. Richardson and P. Slodowy [RS90] in the real case. One therefore may give the following description of the set $\Lambda(\mu)$: the elements $\beta \in \Lambda(\mu)$ are the 'most responsible' for the instability of μ, in the sense that $e^{-t\beta}.\mu$ converges to zero when $t \to \infty$ more quickly than any other α of the same norm (recall the definition of m). To show that $\Lambda(\mu)$ lies in a single conjugacy class (i.e.,that such a one-parameter subgroup is essentially unique) will be actually the main goal of this section. This was proved by G. Kempf [Kem78] in the complex reductive case.

LEMMA 2.4. *Q is $\mathrm{GL}_n(\mathbb{R})$-invariant, $Q(\mu) > 0$ for any $\mu \neq 0$ and*

$$\Lambda(g.\mu) = g\Lambda(\mu)g^{-1}, \qquad \forall \mu \in V_n,\ g \in \mathrm{GL}_n(\mathbb{R}).$$

Proof. $Q(\mu)$ is always positive for a nonzero μ since the eigenvalues of $\pi(\alpha)$ converge to zero when α converges to zero. It follows from (6) that:

$$\begin{aligned} Q(g.\mu) &= \inf_{\alpha \in \mathfrak{D}} \{q(\alpha) : m(g.\mu, \alpha) \geq 1\} \\ &= \inf_{\alpha \in \mathfrak{D}} \{q(g^{-1}\alpha g) : m(\mu, g^{-1}\alpha g) \geq 1\} = Q(\mu). \end{aligned}$$

By definition, $\beta \in \Lambda(\mu)$ if and only if $m(\mu, \beta) \geq 1$ and $q(\beta) \leq q(\alpha)$ for any $\alpha \in \mathfrak{D}$ such that $m(\mu, \alpha) \geq 1$, which is equivalent to saying that $m(g.\mu, g\beta g^{-1}) \geq 1$ and $q(g\beta g^{-1}) \leq q(g\alpha g^{-1})$ for any $g\alpha g^{-1} \in \mathfrak{D}$ such that $m(g.\mu, g\alpha g^{-1}) \geq 1$; that is, $g\beta g^{-1} \in \Lambda(g.\mu)$. □

Let T be the subgroup of $\mathrm{GL}_n(\mathbb{R})$ consisting of the diagonal invertible matrices. In an analogous way, we consider for the action of T on V_n the number

$$Q_T(\mu) = \inf_{\alpha \in \mathfrak{t}} \{||\alpha||^2 : m(\mu, \alpha) \geq 1\}$$

and

$$\Lambda_T(\mu) = \{\beta \in \mathfrak{t} : ||\beta||^2 = Q_T(\mu), \quad m(\mu, \beta) \geq 1\}.$$

Given a finite subset X of $\mathfrak{t}$, denote by $\mathrm{CH}(X)$ the convex hull of X and by $\mathrm{mcc}(X)$ the *minimal convex combination of* X, that is, the (unique) vector of minimal norm in $\mathrm{CH}(X)$. Each nonzero $\mu \in V_n$ uniquely determines an element $\beta_\mu \in \mathfrak{t}$ given by

$$\beta_\mu = \mathrm{mcc}\left\{\alpha_{ij}^k : \mu_{ij}^k \neq 0\right\}, \qquad \mu = \sum \mu_{ij}^k v_{ijk}.$$

We note that β_μ is always nonzero since $\operatorname{tr}\alpha_{ij}^k = -1$ for all $i < j$ and consequently $\operatorname{tr}\beta_\mu = -1$.

LEMMA 2.5. $\Lambda_T(\mu) = \left\{\frac{\beta_\mu}{||\beta_\mu||^2}\right\}$ *and* $Q_T(\mu) = \frac{1}{||\beta_\mu||^2}$.

Proof. Since $\langle\beta_\mu, \alpha_{ij}^k\rangle \geq ||\beta_\mu||^2$ for all $\mu_{ij}^k \neq 0$, we obtain that $m\left(\mu, \frac{\beta_\mu}{||\beta_\mu||^2}\right) \geq 1$. On the other hand, for $\alpha \in \mathfrak{t}$, if $m(\mu, \alpha) \geq 1$ then $1 \leq \langle\alpha, \alpha_{ij}^k\rangle$ for all $\mu_{ij}^k \neq 0$ and so $1 \leq \langle\alpha, \beta_\mu\rangle \leq ||\alpha||\, ||\beta_\mu||$. Thus $||\alpha|| \geq \frac{1}{||\beta_\mu||} = \left|\left|\frac{\beta_\mu}{||\beta_\mu||^2}\right|\right|$, and the equality holds if and only if $\alpha = \frac{\beta_\mu}{||\beta_\mu||^2}$. □

Remark 2.6. Let T_1 be any maximal torus of $\mathrm{GL}_n(\mathbb{R})$ and define $\Lambda_{T_1}(\mu)$ as above but use the Lie algebra of T_1 instead of $\mathfrak{t}$. By considering the weights of the T_1-action on V_n one can prove exactly as above that $\Lambda_{T_1}(\mu)$ consists of a single element: the minimal convex combination of those weights having nonzero projection of μ onto their weight spaces.

If μ runs through V_n, there are only finitely many possible vectors β_μ, so that we can define for each $\beta \in \mathfrak{t}$ the set

$$\mathcal{S}_\beta = \left\{\mu \in V_n \setminus \{0\} : \beta \text{ is an element of maximal norm in } \{\beta_{g.\mu} : g \in \mathrm{GL}_n(\mathbb{R})\}\right\}.$$

It is clear that $\mathcal{S}_\beta$ is $\mathrm{GL}_n(\mathbb{R})$-invariant for any $\beta \in \mathfrak{t}$,

$$V_n \setminus \{0\} = \bigcup_{\beta \in \mathfrak{t}} \mathcal{S}_\beta,$$

and the set $\{\beta \in \mathfrak{t} : \mathcal{S}_\beta \neq \varnothing\}$ is finite.

LEMMA 2.7. $\frac{\beta}{||\beta||^2} \in \Lambda(\mu)$ *for all* $\mu \in \mathcal{S}_\beta$ *such that* $\beta_\mu = \beta$. *In particular*, $Q(\mu) = \frac{1}{||\beta||^2}$ *for any* $\mu \in \mathcal{S}_\beta$ *and*

$$\mathcal{S}_\beta = \mathrm{GL}_n(\mathbb{R}).\left\{\mu \in \mathcal{S}_\beta : \frac{\beta}{||\beta||^2} \in \Lambda(\mu)\right\}.$$

Proof. If $\mu \in V_n$ and $g \in \mathrm{GL}_n(\mathbb{R})$ then from Lemmas 2.4 and 2.5 we obtain that

$$\Lambda(\mu) \cap g^{-1}\mathfrak{t}g = g^{-1}\left(\Lambda(g.\mu) \cap \mathfrak{t}\right) g \subset g^{-1}\Lambda_T(g.\mu)g = \left\{g^{-1}\frac{\beta_{g.\mu}}{||\beta_{g.\mu}||^2}g\right\},$$

and $m\left(\mu, g^{-1}\frac{\beta_{g.\mu}}{||\beta_{g.\mu}||^2}g\right) = m\left(g.\mu, \frac{\beta_{g.\mu}}{||\beta_{g.\mu}||^2}\right) \geq 1$. Since $\Lambda(\mu) \subset \mathfrak{D}$ we have that

$$Q(\mu) = \inf_{g \in \mathrm{GL}_n(\mathbb{R})} \left\{q\left(g^{-1}\frac{\beta_{g.\mu}}{||\beta_{g.\mu}||^2}g\right)\right\} = \inf_{g \in \mathrm{GL}_n(\mathbb{R})} \left\{\frac{1}{||\beta_{g.\mu}||^2}\right\}.$$

So if $\mu \in \mathcal{S}_\beta$ then $Q(\mu) = \frac{1}{||\beta||^2}$ and if in addition $\beta_\mu = \beta$, then $\frac{\beta}{||\beta||^2} \in \Lambda(\mu)$ since in this case $m\left(\mu, \frac{\beta}{||\beta||^2}\right) \geq 1$. The last assertion thus also follows. □

Let us consider the Weyl chamber of $\mathfrak{gl}_n(\mathbb{R})$ and its closure, respectively given by

$$\mathfrak{t}^+ = \left\{ \begin{bmatrix} a_1 & & \\ & \ddots & \\ & & a_n \end{bmatrix} \in \mathfrak{t} : a_1 < \cdots < a_n \right\}, \quad \overline{\mathfrak{t}^+} = \left\{ \begin{bmatrix} a_1 & & \\ & \ddots & \\ & & a_n \end{bmatrix} \in \mathfrak{t} : a_1 \leq \cdots \leq a_n \right\}.$$

For $\alpha \in \overline{\mathfrak{t}^+}$ we define the parabolic subgroup $P_\alpha := B\mathrm{GL}_n(\mathbb{R})_\alpha$, where B is the subgroup of $\mathrm{GL}_n(\mathbb{R})$ of all lower triangular invertible matrices and $\mathrm{GL}_n(\mathbb{R})_\alpha = \{g \in \mathrm{GL}_n(\mathbb{R}) : g\alpha g^{-1} = \alpha\}$. In general, for $\alpha' \in \mathfrak{D}$, we let $P_{\alpha'} := gP_\alpha g^{-1}$ if $\alpha' = g\alpha g^{-1}$, $\alpha \in \overline{\mathfrak{t}^+}$. This is well defined since $h\alpha h^{-1} = g\alpha g^{-1}$ implies that $h^{-1}g \in \mathrm{GL}_n(\mathbb{R})_\alpha \subset P_\alpha$ and so $h^{-1}gP_\alpha g^{-1}h = P_\alpha$.

It is easy to see that there is an ordered basis of V_n with respect to which the action of g on V_n is lower triangular for any $g \in B$, and furthermore the eigenvalues of $\pi(\alpha)$ are increasing for any $\alpha \in \overline{\mathfrak{t}^+}$. It is then easy to check that

$$m(\mu, g\alpha g^{-1}) = m(g.\mu, \alpha) \geq 1 \qquad \forall \alpha \in \overline{\mathfrak{t}^+},\ m(\mu, \alpha) \geq 1,\ g \in P_\alpha,$$

from which it follows that

$$g\alpha g^{-1} \in \Lambda(\mu) \qquad \forall \alpha \in \Lambda(\mu) \cap \overline{\mathfrak{t}^+},\ g \in P_\alpha. \tag{7}$$

More generally, if $\alpha' \in \Lambda(\mu)$ then there exists $h \in \mathrm{GL}_n(\mathbb{R})$ such that $\alpha' = h\alpha h^{-1}$, $\alpha \in \overline{\mathfrak{t}^+}$, and so any $g \in P_{\alpha'}$ is of the form $g = hg_1h^{-1}$ with $g_1 \in P_\alpha$. It follows from Lemma 2.4 that $\alpha \in \Lambda(h^{-1}.\mu)$ and then again from Lemma 2.4 and (7) we get that $g\alpha' g^{-1} = hg_1\alpha g_1^{-1}h^{-1} \in \Lambda(\mu)$. Thus

$$g\alpha' g^{-1} \in \Lambda(\mu) \qquad \forall \alpha' \in \Lambda(\mu),\ g \in P_{\alpha'}. \tag{8}$$

PROPOSITION 2.8. *For all $\alpha, \beta \in \Lambda(\mu)$ we have that $P_\alpha = P_\beta$, and any such P_α acts transitively on $\Lambda(\mu)$ by conjugation.*

Proof. P_α and P_β are parabolic subgroups of $\mathrm{GL}_n(\mathbb{R})$ and so there is a maximal torus T_1 of $\mathrm{GL}_n(\mathbb{R})$ contained in $P_\alpha \cap P_\beta$ (see for instance [FdV69, Th. 74.2]). This implies the existence of elements $g \in P_\alpha$, $h \in P_\beta$ such that $g\alpha g^{-1}$ and $h\beta h^{-1}$ both lie in $\mathrm{Lie}(T_1)$, the Lie algebra of T_1. It follows from (8) that $g\alpha g^{-1}$ and $h\beta h^{-1}$ both belong to $\Lambda(\mu) \cap \mathrm{Lie}(T_1) \subset \Lambda_{T_1}(\mu)$, and so $g\alpha g^{-1} = h\beta h^{-1}$ since $\Lambda_{T_1}(\mu)$ consists of a single element (Remark 2.6). Thus

$$P_\alpha = gP_\alpha g^{-1} = P_{g\alpha g^{-1}} = P_{h\beta h^{-1}} = hP_\beta h^{-1} = P_\beta$$

and P_α acts transitively on $\Lambda(\mu)$. □

Definition 2.9. [Kir84, 2.11] A finite collection $\{S_i : i \in I\}$ of subsets of a topological space X form a *stratification* of X if X is the disjoint union of the S_i, $i \in I$, and there is a partial order $>$ on the indexing set I such that

$$\overline{S_i} \subset S_i \cup \bigcup_{j>i} S_j \qquad \forall i \in I.$$

The S_i's are called the *strata* of X.

For each $\beta \in \mathfrak{t}$ we define

$$W_\beta = \{\mu \in V_n : \langle \beta, \alpha_{ij}^k \rangle \geq ||\beta||^2, \quad \forall \mu_{ij}^k \neq 0\},$$

that is, the direct sum of all the eigenspaces of $\pi(\beta)$ with eigenvalues $\geq ||\beta||^2$. We also consider $\mathcal{B} = \{\beta \in \overline{\mathfrak{t}^+} : \mathcal{S}_\beta \neq \varnothing\}$, and we can now state the main result of this section.

THEOREM 2.10. *The collection* $\{\mathcal{S}_\beta : \beta \in \mathcal{B}\}$ *is a* $\mathrm{GL}_n(\mathbb{R})$*-invariant stratification of* $V_n \setminus \{0\}$:

(i) $V_n \setminus \{0\} = \bigcup_{\beta \in \mathcal{B}} \mathcal{S}_\beta \qquad$ *(disjoint union).*

(ii) $\overline{\mathcal{S}_\beta} \setminus \mathcal{S}_\beta \subset \bigcup_{||\beta'|| > ||\beta||} S_{\beta'}$, *where* $\overline{\mathcal{S}_\beta}$ *is the closure of* $\mathcal{S}_\beta$ *relative to the usual topology of* V_n. *In particular, each stratum* $\mathcal{S}_\beta$ *is a locally closed subset of* $V_n \setminus \{0\}$.

Furthermore, for any $\beta \in \mathcal{B}$ *we have that*

(iii) $W_\beta \setminus \{0\} \subset \mathcal{S}_\beta \cup \bigcup_{||\beta'|| > ||\beta||} \mathcal{S}_{\beta'}$.

(iv) $\mathcal{S}_\beta \cap W_\beta = \{\mu \in \mathcal{S}_\beta : \beta_\mu = \beta\}$.

(v) $\mathcal{S}_\beta = \mathrm{O}(n).\big(\mathcal{S}_\beta \cap W_\beta\big)$.

Proof. We first prove (i). Let σ be a permutation of $\{1, \dots, n\}$ and define $g_\sigma \in \mathrm{GL}_n(\mathbb{R})$ by $g_\sigma e_i = e_{\sigma(i)}$, $i = 1, \dots, n$. It is easy to see that if $\mu = \sum \mu_{ij}^k v_{ijk} \in V_n$ then

$$g_\sigma.\mu = \sum \mu_{\sigma^{-1}(i)\sigma^{-1}(j)}^{\sigma^{-1}(k)} v_{ijk},$$

and since $g_\sigma E_{ii} g_{\sigma^{-1}} = E_{\sigma(i)\sigma(i)}$ for any i we have that

$$\begin{aligned}\{\alpha_{ij}^k : (g_\sigma.\mu)_{ij}^k \neq 0\} &= \{\alpha_{ij}^k : \mu_{\sigma^{-1}(i)\sigma^{-1}(j)}^{\sigma^{-1}(k)} \neq 0\} = \{\alpha_{\sigma(i)\sigma(j)}^{\sigma(k)} : \mu_{ij}^k \neq 0\} \\ &= \{g_\sigma \alpha_{ij}^k g_\sigma^{-1} : \mu_{ij}^k \neq 0\} = g_\sigma \left\{\alpha_{ij}^k : \mu_{ij}^k \neq 0\right\} g_\sigma^{-1}.\end{aligned}$$

This implies that

$$\mathrm{CH}\{\alpha_{ij}^k : (g_\sigma.\mu)_{ij}^k \neq 0\} = g_\sigma\, \mathrm{CH}\left\{\alpha_{ij}^k : \mu_{ij}^k \neq 0\right\} g_\sigma^{-1}$$

and thus $\beta_{g_\sigma.\mu} = g_\sigma \beta_\mu g_\sigma^{-1}$ for any $\mu \in V_n$ and permutation σ. We can therefore guarantee the existence of an element of maximal norm in $\{\beta_{g.\mu} : g \in \mathrm{GL}_n(\mathbb{R})\}$ which lies in $\overline{\mathfrak{t}^+}$ (recall that $g_\sigma \in \mathrm{O}(n)$ and so $||g_\sigma \beta g_\sigma^{-1}|| = ||\beta||$ for any $\beta \in \mathfrak{t}$), and so

$$V_n \setminus \{0\} = \bigcup_{\beta \in \mathcal{B}} \mathcal{S}_\beta.$$

Let us now prove the disjointness of this union. Assume that $\varnothing \neq \mathcal{S}_\beta \cap \mathcal{S}_{\beta'}$, $\beta, \beta' \in \mathcal{B}$. Thus there exists $\mu \in \mathcal{S}_\beta \cap \mathcal{S}_{\beta'}$ such that $\beta_\mu = \beta$, $\beta_{g.\mu} = \beta'$ for some $g \in \mathrm{GL}_n(\mathbb{R})$ and $||\beta|| = ||\beta'||$. It then follows from Lemma 2.7 that $\frac{\beta}{||\beta||^2} \in \Lambda(\mu)$ and $\frac{\beta'}{||\beta'||^2} \in \Lambda(g.\mu)$, or equivalently, $g^{-1}\frac{\beta'}{||\beta'||^2}g \in \Lambda(\mu)$ (Lemma 2.4). By Proposition 2.8 we get that β and β' are conjugate and therefore $\beta = \beta'$ since both are in $\overline{\mathfrak{t}^+}$.

We now prove part (iii). For any $\mu \in W_\beta$ we have that $\langle \beta, \beta_\mu \rangle \geq ||\beta||^2$, and hence either $\beta = \beta_\mu$ or $||\beta|| < ||\beta_\mu||$. Thus if $\mu \in \mathcal{S}_{\beta_\mu}$ we are done. Otherwise, $\mu \in \mathcal{S}_{\beta'}$ for some $\beta' \in \overline{\mathfrak{t}^+}$ such that $||\beta'|| \geq ||\beta_\mu||$, and so $||\beta'|| > ||\beta||$ and (iii) follows.

To prove (ii) recall first that for each $\mu \in \mathcal{S}_\beta$ there exists $g \in G$ such that $\beta_{g.\mu} = \beta$ and thus $g.\mu \in W_\beta$. This implies that $\mathcal{S}_\beta \subset \mathrm{GL}_n(\mathbb{R})W_\beta$, a closed subset since W_β is a subspace and $\mathrm{GL}_n(\mathbb{R})/P_\beta$ is compact (recall that W_β is P_β-invariant), and hence $\overline{\mathcal{S}}_\beta \subset \mathrm{GL}_n(\mathbb{R})W_\beta$. It now follows from (iii) that

$$\overline{\mathcal{S}}_\beta \smallsetminus \mathcal{S}_\beta \subset \mathrm{GL}_n(\mathbb{R})W_\beta \smallsetminus \mathcal{S}_\beta \subset \bigcup_{||\beta'||>||\beta||} \mathcal{S}_{\beta'},$$

as was to be shown. The last assertion in (ii) follows from:

$$\mathcal{S}_\beta = \overline{\mathcal{S}}_\beta \smallsetminus \bigcup_{||\beta'||>||\beta||} S_{\beta'}$$

and $\bigcup_{||\beta'||>||\beta||} S_{\beta'}$ is also a closed subset of $V_n \smallsetminus \{0\}$. Thus $\mathcal{S}_\beta$ is the intersection of an open subset and a closed subset, that is, $\mathcal{S}_\beta$ is locally closed.

We now prove (iv). If $\mu \in W_\beta$ then $(\beta, \beta_\mu) \geq ||\beta||^2$, and if in addition $\mu \in \mathcal{S}_\beta$, then $||\beta_\mu|| \leq ||\beta||$. Thus $\beta = \beta_\mu$ for any $\mu \in W_\beta \cap \mathcal{S}_\beta$. Conversely, if $\mu \in \mathcal{S}_\beta$ and $\beta_\mu = \beta$, then $\mu \in W_\beta$ since $\beta_\mu = \mathrm{mcc}\{\alpha_{ij}^k : \mu_{ij}^k \neq 0\}$.

Finally, (v) follows from the fact that W_β is B-invariant and hence

$$\mathcal{S}_\beta = \mathrm{GL}_n(\mathbb{R}).(\mathcal{S}_\beta \cap W_\beta) = (\mathrm{O}(n)B).(\mathcal{S}_\beta \cap W_\beta) = \mathrm{O}(n)(\mathcal{S}_\beta \cap W_\beta).$$

This concludes the proof of the theorem. $\square$

It follows from Proposition 2.8 that for each nonzero $\mu \in V_n$ there exists a parabolic subgroup $P_\mu \subset \mathrm{GL}_n(\mathbb{R})$ acting transitively on $\Lambda(\mu)$, which satisfies $P_\mu = P_\alpha$ for any $\alpha \in \Lambda(\mu)$. If $\alpha \in \Lambda(\mu)$ and $g \in \mathrm{GL}_n(\mathbb{R})$ satisfies $g\alpha g^{-1} \in \Lambda(\mu)$, then there exists $h \in P_\mu$ such that $hg\alpha g^{-1}h^{-1} = \alpha$. Thus $hg \in \mathrm{GL}_n(\mathbb{R})_\alpha \subset P_\mu$ and so $g \in P_\mu$. This implies that

$$P_\mu = \{g \in \mathrm{GL}_n(\mathbb{R}) : \mathrm{Ad}(g)\alpha \in \Lambda(\mu)\}, \qquad \forall \alpha \in \Lambda(\mu),$$

which in turn gives $\mathrm{Aut}(\mu) \subset P_\mu$, where $\mathrm{Aut}(\mu)$ is the automorphism group of the algebra μ. Indeed, $\Lambda(\mu)$ is $\mathrm{Aut}(\mu)$-invariant since $m(\mu, \alpha) = m(g.\mu, g\alpha g^{-1}) = m(\mu, g\alpha g^{-1})$ for all $g \in \mathrm{Aut}(\mu)$. We therefore obtain that

$$\mathrm{Der}(\mu) \subset \mathfrak{p}_\mu, \tag{9}$$

where $\mathrm{Der}(\mu) = \{\alpha \in \mathfrak{gl}_n(\mathbb{R}) : \pi(\alpha)\mu = 0\}$ is the Lie algebra of derivations of μ and $\mathfrak{p}_\mu$ is the Lie algebra of P_μ. We note that if $\mu \in \mathcal{S}_\beta \cap W_\beta$, $\beta \in \mathcal{B}$, then $\frac{\beta}{||\beta||^2} \in \Lambda(\mu)$ (see Lemma 2.7) and so $P_\mu = P_\beta = B\mathrm{GL}_n(\mathbb{R})_\beta$. It is then easy to check by use of (9) that

$$\langle[\beta, D], D\rangle \geq 0, \qquad \forall D \in \mathrm{Der}(\mu), \quad \mu \in \mathcal{S}_\beta \cap W_\beta. \tag{10}$$

We will now give a description of the strata in terms of semistable vectors. For each $\beta \in \mathfrak{t}$ consider the sets

$$Z_\beta = \{\mu \in V_n : \langle\beta, \alpha_{ij}^k\rangle = ||\beta||^2, \quad \forall \mu_{ij}^k \neq 0\},$$

$$Y_\beta = \{\mu \in W_\beta : \langle\beta, \alpha_{ij}^k\rangle = ||\beta||^2 \text{ for at least one } \mu_{ij}^k \neq 0\}.$$

Thus $Z_\beta \subset Y_\beta \subset W_\beta$, and Z_β is actually the eigenspace of $\pi(\beta)$ with eigenvalue $||\beta||^2$. Z_β is therefore $\mathrm{GL}_n(\mathbb{R})_\beta$-invariant and since W_β is so, Y_β turns out to be $\mathrm{GL}_n(\mathbb{R})_\beta$-invariant as well. Let $\mathfrak{g}_\beta$ denote the Lie algebra of $\mathrm{GL}_n(\mathbb{R})_\beta$; that is, $\mathfrak{g}_\beta = \{\alpha' \in \mathfrak{gl}_n(\mathbb{R}) : [\alpha', \alpha] = 0\}$.

LEMMA 2.11. *For any* $\mu \in Z_\beta$, $\Lambda(\mu) \cap \mathfrak{g}_\beta \neq \emptyset$. *In particular, there exists* $g \in \mathrm{GL}_n(\mathbb{R})_\beta$ *such that* $Q(\mu) = \frac{1}{||\beta_{g.\mu}||^2}$.

Proof. If $\mu \in Z_\beta$ then $\beta + ||\beta||^2 I \in \mathrm{Der}(\mu)$ and so $\beta + ||\beta||^2 I \in \mathfrak{p}_\mu$ (see (9)), where $\Lambda(\mu)$ is contained. Thus for any $\alpha \in \Lambda(\mu)$ there exists $h \in P_\mu$ such that $[h\alpha h^{-1}, \beta] = 0$, and so $h\alpha h^{-1} \in \Lambda(\mu) \cap \mathfrak{g}_\beta$ and the first assertion follows. For the second one, we first note that if $\gamma \in \Lambda(\mu) \cap \mathfrak{g}_\beta$ then there exists $g \in \mathrm{GL}_n(\mathbb{R})_\beta$ such that $g\gamma g^{-1} \in \Lambda(\mu) \cap \mathfrak{t} \subset \Lambda_T(\mu)$, which by Lemma 2.5 gives $g\gamma g^{-1} = \frac{\beta_{g.w}}{||\beta_{g.w}||^2}$. Thus

$$Q(\mu) = q(\gamma) = q(g\gamma g^{-1}) = \frac{1}{||\beta_{g.\mu}||^2},$$

as asserted. □

PROPOSITION 2.12. *For any* $\beta \in \mathcal{B}$ *the* $\mathrm{GL}_n(\mathbb{R})_\beta$*-invariant subsets* $Z_\beta^{\mathrm{ss}} := Z_\beta \cap \mathcal{S}_\beta$ *and* $Y_\beta^{\mathrm{ss}} := Y_\beta \cap \mathcal{S}_\beta$ *satisfy*:

(i) $Y_\beta^{\mathrm{ss}} = \mathcal{S}_\beta \cap W_\beta$; *in particular* $\mathcal{S}_\beta = \mathrm{O}(n).Y_\beta^{\mathrm{ss}}$.

(ii) $Y_\beta^{\mathrm{ss}} = \{\mu \in Y_\beta : p_\beta(\mu) \in Z_\beta^{\mathrm{ss}}\}$, *where* $p_\beta : W_\beta \longrightarrow Z_\beta$ *is the orthogonal projection on* Z_β.

Proof. It is clear that $Y_\beta^{\mathrm{ss}} \subset \mathcal{S}_\beta \cap W_\beta$ as $Y_\beta \subset W_\beta$. Conversely, if $\mu \in \mathcal{S}_\beta \cap W_\beta$, then $\beta_\mu = \beta$ and it is not possible to have $\langle\beta, \alpha_{ij}^k\rangle > ||\beta||^2$ for any $\mu_{ij}^k \neq 0$ since this would contradict the fact that $\beta = \mathrm{mcc}\{\alpha_{ij}^k : \mu_{ij}^k \neq 0\}$. This implies that $\mu \in Y_\beta$ and so is in Y_β^{ss}, and hence the first assertion in (i) follows. The second one follows from Theorem 2.10, (v).

To prove (ii), we first note that if $\mu \in Y_\beta$ and $\mu \notin \mathcal{S}_\beta$ then μ must lie in one of the strata $\mathcal{S}_{\beta'}$ with $||\beta'|| > ||\beta||$ as $\mu \in W_\beta$ (see Theorem 2.10, (iii)). We have that

$$\lambda := p_\beta(\mu) = \lim_{t\to\infty} e^{-t(\beta+||\beta||^2 I)}.\mu \in \overline{\mathrm{GL}_n(\mathbb{R}).\mu}, \tag{11}$$

but then $\lambda \in \overline{\mathcal{S}}_{\beta'}$ and so $\lambda \notin \mathcal{S}_\beta$ by Theorem 2.10, (ii). The set on the right-hand side is then contained in Y_β^{ss}. Conversely, if $\mu \in Y_\beta^{ss}$ and $\lambda = p_\beta(\mu)$ then it follows from Theorem 2.10, (iv) that $\beta = \beta_\mu = \beta_\lambda$. Let us assume that $\lambda \in \mathcal{S}_{\beta'}$ with $||\beta'|| > ||\beta||$ and $\beta' = \beta_{h.\lambda}$ for some $h \in \mathrm{GL}_n(\mathbb{R})$. From Lemma 2.11 we obtain that $Q(\lambda)^{-1} = ||\beta_{g.\lambda}||^2$ for some $g \in \mathrm{GL}_n(\mathbb{R})_\beta$, and so $g.\lambda \in Z_\beta$ and $\beta_{g.\lambda} = \beta_{g.\mu}$, which gives

$$||\beta||^2 \geq ||\beta_{g.\mu}||^2 = ||\beta_{g.\lambda}||^2 = \tfrac{1}{Q(\lambda)} \geq ||\beta_{h.\lambda}||^2 = ||\beta'||^2,$$

a contradiction. Thus, since $\lambda \in \overline{\mathcal{S}_\beta}$, it follows from Theorem 2.10, (ii) that $\lambda \in \mathcal{S}_\beta$, as claimed. □

Let H_β be the connected Lie subgroup of $\mathrm{GL}_n(\mathbb{R})$ with Lie algebra $\mathfrak{h}_\beta$, the orthogonal complement of β in $\mathfrak{g}_\beta$. If $\beta = \beta_\mu$ for some nonzero $\mu \in V_n$, then all its entries are in $\mathbb{Q}$, as β is the minimal convex combination of a subset of $\{\alpha_{ij}^k\}$ and all these matrices have rational entries. In particular, β is rational and H_β is therefore a real reductive algebraic group for any $\beta \in \mathfrak{t}$ such that $\mathcal{S}_\beta \neq \varnothing$. In that case, $\mathfrak{h}_\beta = (\mathfrak{so}(n) \cap \mathfrak{g}_\beta) \oplus (\mathrm{sym}(n) \cap \mathfrak{h}_\beta)$ is a Cartan decomposition and $\mathfrak{t} \cap \mathfrak{h}_\beta = \{\alpha \in \mathfrak{t} : \langle \alpha, \beta\rangle = 0\}$ is a maximal abelian subalgebra of $\mathrm{sym}(n) \cap \mathfrak{h}_\beta$.

Definition 2.13. A vector $\mu \in V_n$ is called *H_β-semistable* if $0 \notin \overline{H_\beta.\mu}$.

PROPOSITION 2.14. *For any $\beta \in \mathfrak{t}$ such that $\mathcal{S}_\beta \neq \varnothing$, the following conditions hold*:

(i) *Z_β^{ss} is the set of H_β-semistable vectors in Z_β.*

(ii) *Y_β^{ss} is the set of H_β-semistable vectors in W_β.*

Proof. We first prove (i). Assume that a $\mu \in Z_\beta^{ss}$ is not H_β-semistable. Thus $\{0\}$ is the only closed orbit in $\overline{H_\beta.\mu}$ (see [RS90, 9.3]) and so there exists $\alpha \in \mathrm{sym}(n) \cap \mathfrak{h}_\beta$ such that $\lim_{t\to\infty} e^{-t\alpha}.\mu = 0$ (see [RS90, Lemma 3.3]). We take $g \in \mathrm{O}(n)_\beta$ such that $g\alpha g^{-1} \in \mathfrak{t}$, but then $g.\mu \in Z_\beta^{ss}$ and $\lim_{t\to\infty} e^{-tg\alpha g^{-1}}.(g.\mu) = 0$ as well, from which it follows that we can assume $\alpha \in \mathfrak{t}$. This implies that $\lim_{t\to\infty} \sum \mu_{ij}^k e^{-t\langle\alpha,\alpha_{ij}^k\rangle} v_{ijk} = 0$ and consequently $\langle\alpha, \alpha_{ij}^k\rangle > 0$ for all $\mu_{ij}^k \neq 0$. Thus $\langle\alpha, \beta_\mu\rangle > 0$, a contradiction, since $\beta = \beta_\mu$ by Theorem 2.10, (iv) and hence $\alpha \perp \beta$.

Conversely, if $\mu \in Z_\beta$ is H_β-semistable and $\mu \notin \mathcal{S}_\beta$ then $\mu \in \mathcal{S}_{\beta'}$ with $||\beta'|| > ||\beta||$ (recall that $\mu \in W_\beta$ and see Theorem 2.10, (iii)). If $g \in \mathrm{GL}_n(\mathbb{R})_\beta$ and $\beta_{g.\mu} = \alpha + a\beta$ for a nonzero $\alpha \in \mathfrak{t}$, $\alpha \perp \beta$, then

$$0 = \lim_{t\to\infty} \sum (g.\mu)_{ij}^k e^{-t\langle\beta_{g.\mu}-a\beta,\alpha_{ij}^k\rangle} v_{ijk} = \lim_{t\to\infty} e^{-t\alpha}.(g.\mu) \in \overline{H_\beta.\mu},$$

since $g.\mu \in Z_\beta \cap \mathcal{S}_{\beta'}$ and hence

$$\langle \beta_{g.\mu}, \alpha_{ij}^k \rangle \geq ||\beta_{g.\mu}||^2 > ||a\beta||^2 = \langle a\beta, \alpha_{ij}^k \rangle, \qquad \forall (g.\mu)_{ij}^k \neq 0.$$

This contradicts the fact that μ is H_β-semistable and so $\beta_{g.\mu}$ is a scalar multiple of β. This gives $\beta_{g.\mu} = \beta$ for any $g \in \mathrm{GL}_n(\mathbb{R})_\beta$ as $\operatorname{tr} \beta_{g.\mu} = \operatorname{tr} \beta = -1$. It then follows from Lemma 2.11 that $Q(\mu)^{-1} = ||\beta||^2$, a contradiction, since $Q(\mu)^{-1} = ||\beta'||^2$ by Lemma 2.7. This concludes the proof of (i).

Let $\mu \in W_\beta$. In order to prove (ii), we must show that $\mu \in \mathcal{S}_\beta$ if and only if μ is H_β-semistable, and since both conditions imply as in (i) that $\beta_\mu = \beta$, we may assume that $\mu \in Y_\beta$. If $\lambda = p_\beta(\mu) \in Z_\beta$ then $\beta_\lambda = \beta_\mu$, and since $p_\beta(g.\mu) = g.p_\beta(\mu)$ for any $g \in \mathrm{GL}_n(\mathbb{R})_\beta$ we obtain that μ is H_β-semistable if and only if λ is so. Thus (ii) follows from (i) and the fact that $\mu \in \mathcal{S}_\beta$ if and only if $\lambda \in \mathcal{S}_\beta$ (see Proposition 2.12, (ii)). □

In what follows, we prove a series of lemmas which will be needed in the proof of Theorem 3.1.

LEMMA 2.15. *If $\mu \in Y_\beta^{\mathrm{ss}}$ then $\operatorname{tr} \beta D = 0$ for any $D \in \mathrm{Der}(\mu)$.*

Proof. It follows from (9) that $D \in \mathfrak{p}_\mu = \mathfrak{b} + \mathfrak{gl}_n(\mathbb{R})_\beta$, where $\mathfrak{b}$ is the Lie subalgebra of $\mathfrak{gl}_n(\mathbb{R})$ of all lower triangular matrices. Thus $\lim\limits_{t\to\infty} e^{-t\beta} D e^{t\beta}$ exists and it is an element $A \in \mathfrak{gl}_n(\mathbb{R})_\beta$. If $\lambda = p_\beta(\mu)$ then we can use (11) to show that

$$\pi(A)\lambda = \lim_{t\to\infty} \pi(e^{-t\beta} D e^{t\beta}) e^{-t(\beta + ||\beta||^2 I)}.\mu = \lim_{t\to\infty} e^{t||\beta||^2} e^{-t\beta}.\pi(D)\mu = 0.$$

That is, $A \in \mathrm{Der}(\lambda)$. We decompose A as $A = a\beta + A'$ with $A' \perp \beta$. By using that $e^{tD}.\mu = \mu$ for all t we obtain that $e^{tA'}.\lambda = e^{-ta\beta}.\lambda = e^{-ta||\beta||^2}\lambda$ (recall that $\lambda \in Z_\beta^{\mathrm{ss}}$ by Proposition 2.12, (ii)) and hence $a = 0$ since otherwise $0 \in \overline{H_\beta.\lambda}$, contradicting the fact that λ is H_β-semistable (see Proposition 2.14, (ii)). This implies that $\operatorname{tr} \beta D = \operatorname{tr} \beta A = \operatorname{tr} \beta A' = 0$, as was to be shown. □

LEMMA 2.16. *$\langle \pi(\beta + ||\beta||^2 I)\mu, \mu \rangle \geq 0$ for any $\mu \in W_\beta$.*

Proof. If $\mu = \sum \mu_{ij}^k v_{ij}^k \in W_\beta$ then $\langle \beta, \alpha_{ij}^k \rangle \geq ||\beta||^2$ for all $\mu_{ij}^k \neq 0$ and henceforth

$$\begin{aligned} \langle \pi(\beta + ||\beta||^2 I)\mu, \mu \rangle &= \langle \pi(\beta)\mu, \mu \rangle - ||\beta||^2 ||\mu||^2 \\ &= \sum (\mu_{ij}^k)^2 \langle \beta, \alpha_{ij}^k \rangle - ||\beta||^2 ||\mu||^2 \geq 0, \end{aligned}$$

as claimed. □

The space of all n-dimensional nilpotent Lie algebras can be parametrized by the set

$$\mathcal{N} = \{\mu \in V_n : \mu \text{ satisfies the Jacobi identity and is nilpotent}\},$$

which is an algebraic subset of V_n as the Jacobi identity and the nilpotency condition can both be expressed as zeroes of polynomial functions. Note that $\mathcal{N}$

is $\mathrm{GL}_n(\mathbb{R})$-invariant and Lie algebra isomorphism classes are precisely $\mathrm{GL}_n(\mathbb{R})$-orbits.

LEMMA 2.17. *$\beta + ||\beta||^2$ is positive definite for every $\beta \in \mathcal{B}$ such that $\mathcal{S}_\beta \cap \mathcal{N} \neq \emptyset$.*

Proof. By Proposition 2.12, (i) there exists $\mu \in Y_\beta^{ss} \cap \mathcal{N}$, which is therefore H_β-semistable by Proposition 2.14, (ii). Thus there exists a nonzero

$$\lambda \in \overline{H_\beta.\mu} \subset Y_\beta^{ss} \cap \mathcal{N}$$

such that $||\lambda|| \leq ||\lambda'||$ for any $\lambda' \in \overline{H_\beta.\mu}$. Let us consider $\mathrm{R}_\lambda \in \mathfrak{gl}_n(\mathbb{R})$ defined implicitly by

$$\langle \mathrm{R}_\lambda, \alpha \rangle = \tfrac{1}{4}\langle \pi(\alpha)\lambda, \lambda \rangle, \qquad \forall \alpha \in \mathfrak{gl}_n(\mathbb{R}).$$

We note that $\mathrm{R}_\lambda \in \mathrm{sym}(n)$ as for any $\alpha \in \mathfrak{so}(n)$, $\langle \pi(\alpha)\lambda, \lambda \rangle = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|_{t=0}||e^{t\alpha}.\lambda||^2 = 0$. Since λ is a vector of minimal norm in $H_\beta.\lambda$ we obtain

$$\langle \mathrm{R}_\lambda, \alpha \rangle = \tfrac{1}{4}\tfrac{\mathrm{d}}{\mathrm{d}t}|_{t=0}\langle e^{t\alpha}.\lambda, \lambda \rangle = \tfrac{1}{4}\tfrac{\mathrm{d}}{\mathrm{d}t}|_{t=0}||e^{\frac{t}{2}\alpha}.\lambda||^2 = 0, \qquad \forall \alpha \in \mathfrak{h}_\beta \cap \mathrm{sym}(n),$$

and hence the orthogonal projection of R_λ on $\mathfrak{g}_\beta$ is a scalar multiple of β. Thus, such a projection equals $\frac{||\lambda||^2}{4}\beta$ as $\mathrm{tr}\,\mathrm{R}_\lambda = \langle \mathrm{R}_\lambda, I \rangle = \frac{1}{4}\langle \pi(I)\lambda, \lambda \rangle = -\frac{||\lambda||^2}{4}$ (recall that $\mathrm{tr}\,\beta = -1$). This implies that if $\lambda = \sum \lambda_{ij}^k v_{ijk}$ then

$$||\beta||^2 = \tfrac{4}{||\lambda||^2}\langle \mathrm{R}_\lambda, \beta \rangle = \tfrac{1}{||\lambda||^2}\langle \pi(\beta)\lambda, \lambda \rangle = \sum \tfrac{(\lambda_{ij}^k)^2}{||\lambda||^2}\langle \beta, \alpha_{ij}^k \rangle. \tag{12}$$

But $\lambda \in Y_\beta^{ss} \subset W_\beta$ and so

$$\langle \beta, \alpha_{ij}^k \rangle \geq ||\beta||^2, \qquad \forall \lambda_{ij}^k \neq 0,$$

which implies by (12) that these must all be equalities and hence $\lambda \in Z_\beta$.

On the other hand, for any $\alpha \in \mathfrak{gl}_n(\mathbb{R})$ we have that

$$\langle [\mathrm{R}_\lambda, \beta], \alpha \rangle = -\langle \mathrm{R}_\lambda, [\beta, \alpha] \rangle = -\tfrac{1}{4}\langle \pi([\beta, \alpha])\lambda, \lambda \rangle = 0$$

since $\pi(\beta)$ is symmetric and $\pi(\beta)\lambda = ||\beta||^2\lambda$. We therefore obtain that $\mathrm{R}_\lambda \in \mathfrak{g}_\beta$ and so $\mathrm{R}_\lambda = \frac{||\lambda||^2}{4}\beta$.

If $D := \frac{||\lambda||^2}{4}(\beta + ||\beta||^2 I)$ and $DX = dX$, $X \in \mathbb{R}^n \setminus \{0\}$, then since $D \in \mathrm{Der}(\lambda)$,

$$d\,\mathrm{ad}_\lambda X = \mathrm{ad}_\lambda(DX) = [D, \mathrm{ad}_\lambda X] = [\mathrm{R}_\lambda, \mathrm{ad}_\lambda X],$$

where ad_λ denotes the adjoint representation of the Lie algebra λ. Thus

$$\begin{aligned} d\,\mathrm{tr}\,\mathrm{ad}_\lambda X(\mathrm{ad}_\lambda X)^t &= \mathrm{tr}\,[\mathrm{R}_\lambda, \mathrm{ad}_\lambda X](\mathrm{ad}_\lambda X)^t = \langle \mathrm{R}_\lambda, [\mathrm{ad}_\lambda X, (\mathrm{ad}_\lambda X)^t] \rangle \\ &= \tfrac{1}{4}\langle \pi([\mathrm{ad}_\lambda X, \mathrm{ad}_\lambda X)^t])\lambda, \lambda \rangle = \tfrac{1}{4}||\pi((\mathrm{ad}_\lambda X)^t)\lambda||^2, \end{aligned}$$

which implies that $d \geq 0$ as long as $\mathrm{ad}_\lambda X \neq 0$. If $\mathrm{ad}_\lambda X = 0$, since $\langle \mathrm{R}_\lambda X, X\rangle = (d - \frac{||\beta||^2||\lambda||^2}{4})||X||^2$ and

$$\langle \mathrm{R}_\lambda X, X\rangle = -\tfrac{1}{2}\sum_{ij}\langle \lambda(X, e_i), e_j\rangle^2 + \tfrac{1}{4}\sum_{ij}\langle \lambda(e_i, e_j), X\rangle^2, \tag{13}$$

(see [Lau06, Props. 3.5, 4.2]) we obtain that $d > 0$. If $\mathrm{ad}_\lambda X \neq 0$ and $d = 0$ then $(\mathrm{ad}_\lambda X)^t \in \mathrm{Der}(\lambda)$, which is a contradiction since λ is a nilpotent Lie algebra (consider the orthogonal decomposition $\mathbb{R}^n = \mathfrak{n}_1 \oplus \cdots \oplus \mathfrak{n}_r$ such that $\mathfrak{n}_s \oplus \cdots \oplus \mathfrak{n}_r$, $s = 1, \dots, r$ is the central descendent series). We have therefore obtained that in any case $d > 0$, and hence $\beta + ||\beta||^2 I$ is positive definite. □

Remark 2.18. Notice that Lemma 2.17 is the only result in this section where we need μ to be a nilpotent Lie algebra, and not just any vector in V_n. It is known for instance that semisimple Lie algebras lie in the stratum $\mathscr{S}_\beta$ for $\beta = -\frac{1}{n}I$, and consequently $\beta + ||\beta||^2 I = 0$ (see [Lau03]).

3. **Einstein solvmanifolds**

We now apply the results obtained in Section 2 to prove our main theorem, namely that Einstein solvmanifolds are all standard.

Let S be a solvmanifold, that is, a simply connected solvable Lie group endowed with a left invariant Riemannian metric. Let $\mathfrak{s}$ be the Lie algebra of S and let $\langle\cdot,\cdot\rangle$ denote the inner product on $\mathfrak{s}$ determined by the metric. We consider the orthogonal decomposition $\mathfrak{s} = \mathfrak{a} \oplus \mathfrak{n}$, where $\mathfrak{n} = [\mathfrak{s}, \mathfrak{s}]$. A solvmanifold S is called *standard* if $[\mathfrak{a}, \mathfrak{a}] = 0$.

The mean curvature vector of S is the only element $H \in \mathfrak{a}$ which satisfies $\langle H, A\rangle = \mathrm{tr}\,\mathrm{ad}\, A$ for any $A \in \mathfrak{a}$. If B denotes the symmetric map defined by the Killing form of $\mathfrak{s}$ relative to $\langle\cdot,\cdot\rangle$ then $B(\mathfrak{a}) \subset \mathfrak{a}$ and $B|_{\mathfrak{n}} = 0$ as $\mathfrak{n}$ is contained in the nilradical of $\mathfrak{s}$. The Ricci operator Ric of S is given by (see for instance [Bes87, 7.38]):

$$\mathrm{Ric} = R - \tfrac{1}{2}B - S(\mathrm{ad}\, H), \tag{14}$$

where $S(\mathrm{ad}\, H) = \frac{1}{2}(\mathrm{ad}\, H + (\mathrm{ad}\, H)^t)$ is the symmetric part of $\mathrm{ad}\, H$ and R is the symmetric operator defined by

$$\langle Rx, y\rangle = -\tfrac{1}{2}\sum_{ij}\langle [x, x_i], x_j\rangle\langle [y, x_i], x_j\rangle + \tfrac{1}{4}\sum_{ij}\langle [x_i, x_j], x\rangle\langle [x_i, x_j], y\rangle, \tag{15}$$

for all $x, y \in \mathfrak{s}$, where $\{x_i\}$ is any orthonormal basis of $(\mathfrak{s}, \langle\cdot,\cdot\rangle)$. A solvmanifold S is called *Einstein* if $\mathrm{Ric} = cI$ for some $c \in \mathbb{R}$. We refer to [Bes87] for a detailed exposition on Einstein manifolds (see also the surveys in [LW99] and [Ber03, 11.4]).

It is proved in [Lau06, Props. 3.5, 4.2] that R is the only symmetric operator on $\mathfrak{s}$ such that

$$\operatorname{tr} RE = \tfrac{1}{4}\langle \pi(E)[\cdot,\cdot],[\cdot,\cdot]\rangle, \qquad \forall E \in \operatorname{End}(\mathfrak{s}), \tag{16}$$

where we are identifying $\mathfrak{s}$ with $\mathbb{R}^m$, $[\cdot,\cdot]$ with a vector in V_m, and so $\langle\cdot,\cdot\rangle$ is the inner product defined in (3) and π is the representation given in (5) (see the notation in §2).

We therefore obtain from (14) and (16) that S is an Einstein solvmanifold with $\operatorname{Ric} = cI$, if and only if, for any $E \in \operatorname{End}(\mathfrak{s})$,

$$\operatorname{tr}\left(cI + \tfrac{1}{2}B + S(\operatorname{ad} H)\right) E = \tfrac{1}{4}\sum_{ij}\langle E[x_i,x_j] - [Ex_i,x_j] - [x_i,Ex_j],[x_i,x_j]\rangle. \tag{17}$$

We are now in a position to prove the main result of this paper.

THEOREM 3.1. *Any Einstein solvmanifold is standard.*

Proof. Let S be an Einstein solvmanifold with $\operatorname{Ric} = cI$. We can assume that S is not unimodular by using the following fact proved by I. Dotti [DM82]: a unimodular Einstein solvmanifold must be flat and consequently standard (see [Heb98, Prop. 4.9]). Thus $H \neq 0$ and $\operatorname{tr}\operatorname{ad} H = ||H||^2 > 0$. By letting $E = \operatorname{ad} H$ in (17) and using that $\operatorname{ad} H \in \operatorname{Der}(\mathfrak{s})$ we get

$$c = -\tfrac{\operatorname{tr} S(\operatorname{ad} H)^2}{\operatorname{tr} S(\operatorname{ad} H)}. \tag{18}$$

In order to apply the results in Section 2, we identify $\mathfrak{n}$ with $\mathbb{R}^n$ via an orthonormal basis $\{e_1,\dots,e_n\}$ of $\mathfrak{n}$ and we set $\mu := [\cdot,\cdot]|_{\mathfrak{n}\times\mathfrak{n}}$. In this way, μ can be viewed as an element of $\mathcal{N} \subset V_n$. If $\mu \neq 0$ then μ lies in a unique stratum $\mathcal{S}_\beta$, $\beta \in \mathcal{B}$, by Theorem 2.10, (i). It then follows from Proposition 2.12, (i) that there exists $g \in \mathrm{O}(n)$ such that $g.\mu \in Y_\beta^{\mathrm{ss}}$. Let $\tilde{g}$ denote the orthogonal map of $(\mathfrak{s},\langle\cdot,\cdot\rangle)$ defined by $\tilde{g}|_{\mathfrak{a}} = I$, $\tilde{g}|_{\mathfrak{n}} = g$. We let $\widetilde{S}$ be the solvmanifold whose Lie algebra $\tilde{\mathfrak{s}}$ is $\mathfrak{s}$ as a vector space and has Lie bracket $\tilde{g}.[\cdot,\cdot] = \tilde{g}[\tilde{g}^{-1}\cdot,\tilde{g}^{-1}\cdot]$. The left invariant metric on $\widetilde{S}$ is determined by the same inner product $\langle\cdot,\cdot\rangle$ on $\mathfrak{s}$. We therefore have that $\widetilde{S}$ is isometric to S, as $\tilde{g}$ is an isometric isomorphism between the two metric Lie algebras. Thus $\widetilde{S}$ is also Einstein, and since S is standard if and only if $\widetilde{S}$ is so, we can assume that $\mu \in Y_\beta^{\mathrm{ss}} = \mathcal{S}_\beta \cap W_\beta$.

We now apply (17) to $E \in \operatorname{End}(\mathfrak{s})$ defined by

$$E := \begin{bmatrix} 0 & 0 \\ 0 & \beta + ||\beta||^2 I \end{bmatrix};$$

that is, $E|_{\mathfrak{a}} = 0$ and $E|_{\mathfrak{n}} = \beta + ||\beta||^2 I$. If $\{A_1, \dots, A_m\}$ is an orthonormal basis of $\mathfrak{a}$ then the right-hand side of (17) is given by

$$\begin{aligned}\tfrac{1}{4}\sum_{ij}&\langle E[e_i, e_j] - [Ee_i, e_j] - [e_i, Ee_j], [e_i, e_j]\rangle \\ &+ \tfrac{1}{4}\sum_{rs}\langle E[A_r, A_s], [A_r, A_s]\rangle \\ &+ \tfrac{1}{2}\sum_{ri}\langle E[A_r, e_i], [A_r, e_i]\rangle - \tfrac{1}{2}\sum_{ri}\langle [A_r, Ee_i], [A_r, e_i]\rangle,\end{aligned}$$

which in turn equals

$$\begin{aligned}\tfrac{1}{4}\langle \pi(\beta + ||\beta||^2 I)\mu, \mu\rangle &+ \tfrac{1}{4}\sum_{rs}\langle(\beta + ||\beta||^2 I)[A_r, A_s], [A_r, A_s]\rangle \\ &+ \tfrac{1}{2}\sum_{ri}\langle(\beta \,\mathrm{ad}\, A_r - \mathrm{ad}\, A_r \beta)(e_i), \mathrm{ad}\, A_r(e_i)\rangle.\end{aligned} \tag{19}$$

The first and second terms in (19) are ≥ 0 by Lemma 2.16 and Lemma 2.17, respectively, and the last one equals $\frac{1}{2}\sum_r \langle[\beta, \mathrm{ad}\, A_r], \mathrm{ad}\, A_r\rangle$, which is ≥ 0 by (10) since $\mathrm{ad}\, A_r|_{\mathfrak{n}} \in \mathrm{Der}(\mu)$ for all r.

We therefore obtain from (17) and (18) that

$$-\frac{\mathrm{tr}\, S(\mathrm{ad}\, H)^2}{\mathrm{tr}\, S(\mathrm{ad}\, H)}\,\mathrm{tr}\, E + \mathrm{tr}\, S(\mathrm{ad}\, H)E \geq 0. \tag{20}$$

Recall that $\mathrm{tr}\,\beta = -1$ and so

$$\begin{aligned}\mathrm{tr}\, E^2 &= \mathrm{tr}(\beta^2 + ||\beta||^4 I + 2||\beta||^2\beta) = ||\beta||^2(1 + n||\beta||^2 - 2) \\ &= ||\beta||^2(-1 + n||\beta||^2) = ||\beta||^2\,\mathrm{tr}\, E.\end{aligned} \tag{21}$$

On the other hand, we have that

$$\mathrm{tr}\, S(\mathrm{ad}\, H)E = \mathrm{tr}\,\mathrm{ad}\, H|_{\mathfrak{n}}(\beta + ||\beta||^2) = ||\beta||^2\,\mathrm{tr}\, S(\mathrm{ad}\, H) \tag{22}$$

by Lemma 2.15. We now use (20), (21) and (22) and obtain

$$\mathrm{tr}\, S(\mathrm{ad}\, H)^2\,\mathrm{tr}\, E^2 \leq (\mathrm{tr}\, S(\mathrm{ad}\, H)E)^2,$$

a 'backwards' Cauchy-Schwartz inequality. This turns all inequalities mentioned after (19) into equalities, in particular the second term:

$$\tfrac{1}{4}\sum_{rs}\langle(\beta + ||\beta||^2 I)[A_r, A_s], [A_r, A_s]\rangle = 0.$$

We therefore get that $[\mathfrak{a}, \mathfrak{a}] = 0$ since $\beta + ||\beta||^2 I$ is positive definite by Lemma 2.17.

It only remains to consider the case $\mu = 0$. Here we argue in the same way but with E chosen as $E|_{\mathfrak{a}} = 0$ and $E|_{\mathfrak{n}} = I$. It then follows from (17) that

$$\sum_{rs}||[A_r, A_s]||^2 = -\frac{\mathrm{tr}\, S(\mathrm{ad}\, H)^2}{\mathrm{tr}\, S(\mathrm{ad}\, H)}n + \mathrm{tr}\, S(\mathrm{ad}\, H) = \frac{\mathrm{tr}\, S(\mathrm{ad}\, H)^2}{\mathrm{tr}\, S(\mathrm{ad}\, H)}\left(\frac{(\mathrm{tr}\, S(\mathrm{ad}\, H))^2}{\mathrm{tr}\, S(\mathrm{ad}\, H)^2} - n\right) \leq 0,$$

and thus $[\mathfrak{a}, \mathfrak{a}] = 0$. This concludes the proof of the theorem. $\square$

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