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Local rigidity of partially hyperbolic actions I. KAM method and $\mathbb{Z}^k$ actions on the torus

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Abstract

We show C^∞ local rigidity for $\mathbb{Z}^k$ $(k \geq 2)$ higher rank partially hyperbolic actions by toral automorphisms, using a generalization of the KAM (Kolmogorov-Arnold-Moser) iterative scheme. We also prove the existence of irreducible genuinely partially hyperbolic higher rank actions on any torus $\mathbb{T}^N$ for any even $N \geq 6$.

1. **Introduction**

1.1. *Algebraic actions of higher rank abelian groups.* Differentiable ($C^r, r \geq 1$) rigidity of the orbit structure and its variations sets the higher rank abelian group actions apart from the classical cases of diffeomorphisms and flows (actions of $\mathbb{Z}$ and $\mathbb{R}$) where only C^0 orbit structure may be stable under small perturbations.

In this paper we begin the study of differentiable rigidity for algebraic (homogeneous of affine) *partially hyperbolic* actions of $\mathbb{Z}^k \times \mathbb{R}^l$, $k + l \geq 2$. For definitions and general background on partially hyperbolic dynamical systems, see [32]. For a survey of local rigidity of actions of other groups, see [13]. In the latter area the key work dealing with the partially hyperbolic case is [14].

The most general condition in the setting of $\mathbb{Z}^k \times \mathbb{R}^l$, $k + l \geq 2$ actions, which leads to various rigidity phenomena (cocycle rigidity, local differentiable rigidity, measure rigidity, etc.), is the following:

($\mathfrak{R}$) *The group $\mathbb{Z}^k \times \mathbb{R}^l$ contains a subgroup L isomorphic to $\mathbb{Z}^2$ such that for the suspension of the restriction of the action to L every element other than identity acts ergodically with respect to the standard invariant measure obtained from Haar measure.*

This condition should be viewed as a paradigm. We chose not to formulate specific conjectures due to various subtleties of algebraic nature which up to now have prevented even the results in the otherwise well-understood hyperbolic case from being clean and definitive; see [25]. Instead let us point out to two specific

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representative classes of algebraic actions where the above condition is satisfied and various (although not always the strongest expected) rigidity properties have been established in many cases.

(1) Actions of $\mathbb{Z}^k$, $k \geq 2$ by automorphisms or affine maps of tori or, more generally, (infra)nilmanifolds without nontrivial rank one factors for any finite cover, and suspensions of such actions.

(2) Left actions of a higher rank abelian group $A \subset G$ on the double coset space $C \setminus G/\Gamma$ where G is a connected Lie group, $\Gamma \subset G$ a cocompact lattice, and C a compact subgroup of the centralizer of A which intersects with A trivially. Furthermore, the Lie algebras of A, C and contracting subgroups of all elements of A generate the full Lie algebra $\mathfrak{g}$ of G.

1.2. *Rigidity of actions and related notions.* Let A be a finitely generated discrete group with generators $g_1, \ldots, g_k$. Let $\mathrm{Act}^r(A, M)$ be the space of A actions by diffeomorphisms of class C^r of a compact manifold M. Every $\alpha \in \mathrm{Act}^r(A, M)$ is defined by a homomorphism $\rho_\alpha \colon A \to \mathrm{Diff}^r M$. In particular, diffeomorphisms $\rho_\alpha(g_1), \ldots, \rho_\alpha(g_k)$ completely determine the action α. Thus the usual C^r topology of $\mathrm{Diff}^r(M)^k$ induces via projection a topology in the space $\mathrm{Act}^r(A, M)$ which we refer to as a C^r topology in the space of A actions on M. Similarly, for an action of a connected Lie group A on M, the C^r topology in $\mathrm{Act}^r(A, M)$ is induced by the C^r topology on vector fields which generate the action of the Lie algebra of A.

For actions of discrete groups the notions of rigidity which we consider are summarized as follows.

An action α of a finitely generated discrete group A on a manifold M is $C^{k,r,l}$ *locally rigid* if any sufficiently C^r small C^k perturbation $\tilde{\alpha}$ is C^l conjugate to α; i.e., there exists a C^l close to identity diffeomorphism $\mathcal{H}$ of M which conjugates $\tilde{\alpha}$ to α:

$$\mathcal{H} \circ \alpha(g) = \tilde{\alpha}(g) \circ \mathcal{H} \tag{1.1}$$

for all $g \in A$. $C^{\infty,1,\infty}$ local rigidity is often referred to as C^∞ *local rigidity*. The case of $C^{1,1,0}$ is known as C^1 *structural stability*.

For actions of continuous Lie groups such as $\mathbb{R}^k$, the above definition of local rigidity is too strict since one has to allow for small time changes induced by automorphisms of the acting group. An action α of a continuous Lie group A on a manifold M is $C^{k,r,l}$ *locally rigid* if any sufficiently C^r small C^k perturbation $\tilde{\alpha}$ is C^l conjugate to a small time change of α, i.e., there exists a C^l close to identity diffeomorphism $\mathcal{H}$ and $\rho \in \mathrm{Aut}(G)$ close to id such that instead of (1.1) one has

$$\mathcal{H} \circ \alpha(\rho(g)) = \tilde{\alpha}(g) \circ \mathcal{H}. \tag{1.2}$$

A weaker notion in the case of actions of continuous groups is *foliation rigidity*: one requires the diffeomorphism $\mathcal{H}$ to map an invariant foliation $\mathcal{F}$ (e.g., the orbit foliation $\mathbb{O}$ of the action α) to the perturbed foliation $\mathcal{F}'$ (e.g., the orbit foliation

$\widetilde{\mathcal{O}}$ of the perturbed action $\widetilde{\alpha}$). The foliation rigidity in the case of the orbit foliation is usually called *orbit rigidity*. $C^{1,1,0}$ orbit rigidity is called *structural stability*.

For a broad class of hyperbolic algebraic actions of $\mathbb{Z}^k$ and $\mathbb{R}^k$, $k \geq 2$, C^∞ local rigidity has been established in [25]; see Section 2.1 for a discussion. In this paper we prove for the first time local rigidity for a class of *partially hyperbolic* actions of $\mathbb{Z}^k$, namely for type **(1)** actions on tori.

The orbits of partially hyperbolic algebraic actions are parts of homogeneous isometric foliations. For such actions, e.g., those of type **(2)** differentiable rigidity should be understood in a modified sense allowing not only a linear time change in A but also an algebraic perturbation of A inside the isometric foliation; see [8], [5] for a detailed discussion.

1.3. *Statement of results.*

1.3.1. *Rigidity.* Let $\mathrm{GL}(N, \mathbb{Z})$ be the group of integer $N \times N$ matrices with determinant ± 1. Any matrix $A \in \mathrm{GL}(N, \mathbb{Z})$ defines an automorphism of the torus $\mathbb{T}^N = \mathbb{R}^N / \mathbb{Z}^N$ which we also denote by A.

An action $\alpha\colon \mathbb{Z}^k \times \mathbb{T}^N \to \mathbb{T}^N$ by automorphisms is given by an embedding $\rho_\alpha\colon \mathbb{Z}^k \to \mathrm{GL}(N, \mathbb{Z})$ so that

$$\alpha(g, x) = \rho_\alpha(g)x$$

for any $g \in \mathbb{Z}^k$ and any $x \in \mathbb{T}^N$. We will write simply $\alpha(g)$ for $\rho_\alpha(g)$.

An action $\alpha'\colon \mathbb{Z}^k \times \mathbb{T}^{N'} \to \mathbb{T}^{N'}$ is an *algebraic factor* of α if there exists an epimorphism $h\colon \mathbb{T}^N \to \mathbb{T}^{N'}$ such that $h \circ \alpha = \alpha' \circ h$.

An action α by automorphisms of $\mathbb{T}^N$ is called *irreducible* if any nontrivial algebraic factor of α has finite fibers.

An action α' is a *rank one factor* if it is an algebraic factor and if $\rho_{\alpha'}(\mathbb{Z}^k)$ contains a cyclic subgroup of finite index.

An ergodic action α by toral automorphisms has no nontrivial rank one factors if and only if $\rho_\alpha(\mathbb{Z}^k)$ contains a subgroup isomorphic to $\mathbb{Z}^2$ such that all nontrivial elements in this subgroup are ergodic toral automorphisms (see for example [34]). We call ergodic $\mathbb{Z}^k$ actions by toral automorphisms *higher rank* if they have no nontrivial rank one factors.

THEOREM 1. *Let* $\alpha\colon \mathbb{Z}^k \times \mathbb{T}^N \to \mathbb{T}^N$ *be an ergodic higher rank action of* $\mathbb{Z}^k$ ($k \geq 2$) *by automorphisms of the* N*-dimensional torus. Then there exists a constant* $l = l(\alpha, N) \in \mathbb{N}$ *such that* α *is* $C^{\infty,l,\infty}$ *locally rigid.*

The constant l in the above theorem depends on the dimension of the torus and the given linear action, and is precisely defined in the proof of Theorem 1 (Section 5).

We note that the result above extends to affine actions whose linear parts are actions described in Theorem 1. While any affine map of the torus without eigenvalue one in its linear part has a fixed point and hence is conjugate to its linear part, the case of commuting affine maps is more general (for examples of affine actions without fixed points see [17]). However the case of commuting affine maps reduces to the case of commuting linear maps by passing to a subgroup of finite rank which has a fixed point and observing that the centralizer of an ergodic partially hyperbolic action by toral automorphisms is discrete.

We also obtain rigidity for suspensions of actions considered in Theorem 1. (For the detailed description of the suspension construction see for example [23, §2.2] or [3, §5.4]).

THEOREM 2. *Suspension of a $\mathbb{Z}^k$ action by toral automorphisms with no rank one factors is $C^{\infty,l,\infty}$ locally rigid* (*up to an automorphism of the acting group $\mathbb{R}^k$*) *where l is the same as in Theorem* 1.

Theorem 2 is deduced from Theorem 1 in Section 6. It involves a general statement (Lemma 6.1), which we present for the sake of completeness, but also relies on cocycle rigidity results which are specific.

1.3.2. *Existence.* Let us call an action of $\mathbb{Z}^k$ by automorphisms of a torus *genuinely partially hyperbolic* if it is ergodic with respect to Lebesgue measure but no element of the action is hyperbolic (Anosov). It is easy to see that this is equivalent to simultaneous existence of

(1) an element of the action none of whose eigenvalues is a root of unity and
(2) an invariant linear foliation on which there is no exponential expansion/contraction for any element of the action.

As before, such an action is higher rank if and only if it contains a $\mathbb{Z}^2$ action all of whose elements are ergodic.

The method of Theorem 1 is essential in proving rigidity of higher rank genuinely partially hyperbolic actions since for hyperbolic actions earlier methods are available [25]. The following result discusses the existence of genuinely partially hyperbolic higher rank actions.

THEOREM 3. *Genuinely partially hyperbolic higher rank $\mathbb{Z}^2$ actions exist*: *on any torus of even dimension $N \geq 6$ there are irreducible examples while on any torus of odd dimension $N \geq 9$ there are only reducible examples. There are no examples on tori of dimension $N \leq 5$ and $N = 7$.*

Remark 1. In Section 9 in addition to the general construction for examples in even dimensions, we give an explicit example in dimension 6.

Remark 2. Using our constructions in Section 9, a similar statement as Theorem 3 about existence of genuinely partially hyperbolic higher rank actions of $\mathbb{Z}^k$ for $k > 2$ can be obtained. For example, an irreducible action of such kind exists in any even dimension starting from $2k + 2$.

2. Old and new approaches to rigidity of abelian group actions

Present paper, although it deals with a specific problem, is a part of a broader program of studying local differentiable rigidity for hyperbolic and partially hyperbolic actions of higher rank abelian groups. Moreover, it is the first in a series of papers by the same authors which address the partially hyperbolic situation, which is considerably more difficult that the hyperbolic one. It is natural in this context to briefly outline the developments which led to the present work, explain both successes and deficiencies of the previous methods, present the current state-of-the-art, and discuss both current difficulties and prospects of the further developments.

2.1. *The a priori regularity method.* This method first appeared in a rudimentary fashion in [21] in the context of proving rigidity of some standard lattice actions on tori, and was introduced in its general form in [25]. It can be very briefly described as follows. One considers an action with sufficiently strong hyperbolic properties. The C^0 *orbit structure* (or an essential part of it) of such an action is rigid (structural stability, Hirsch-Pugh-Shub theory [16]). Moreover the C^0 orbit equivalence is unique transversally to the orbits. The method then consists of establishing *a priori* regularity of the orbit equivalence. The central idea is that *locally* in the phase space there are invariant geometric structures on invariant foliations for both the original and perturbed action. The simplest example of such a structure is a flat affine connection (local linearization); in general the structure is more complicated, but it still has a finite-dimensional Lie group of automorphisms. Such a structure appears due to the fact that some elements of the action contract the foliations in question and this is true both in the rank one and higher rank hyperbolic situations. Higher rank is crucial in showing that the conjugacy, which is *a priori* only continuous, preserves the invariant structure and hence is smooth in the direction of each foliation. This in turn relies on the fact that in certain *critical* directions the unperturbed action acts by isometries on each leaf of the foliation. This in particular excludes the case of actions by automorphisms of the torus and of a nil-manifold in the presence on nontrivial Jordan blocks.[1]

For the hyperbolic $\mathbb{R}^k$ actions there is another step from smooth orbit equivalence to smooth conjugacy of the actions up to an automorphism of $\mathbb{R}^k$. This follows from the cocycle rigidity which allows to straighten out the time change [23]. Notice that cocycle rigidity also holds for some partially hyperbolic actions of higher rank abelian groups [24]. In both of those papers cocycle rigidity has been established using harmonic analysis.

This method of establishing regularity of a conjugacy (whose existence is obtained from a different kind of reasoning) also appears as an ingredient in the proofs of local rigidity for actions of certain groups other than abelian ([21], [22], [29], and [14]).

2.2. *Difficulties in the partially hyperbolic case.* If one tries to follow this approach for a *partially hyperbolic* algebraic action of a higher rank abelian group with semisimple linear part one starts with structural stability of the *neutral foliation* for the unperturbed action. Then two difficulties arise. First, this foliation for perturbed action is not necessarily smooth and hence smoothness of the foliation

[1] *Added in Proof*: Partial results for that case were obtained later in [11].

conjugacy along the stable and unstable directions of various elements of the action does not guarantee the global regularity. Second, even if one assumes smoothness of the neutral foliation, or, equivalently, only considers perturbations along this foliation, cocycle rigidity of the unperturbed algebraic action is not sufficient.

To overcome these difficulties we introduce two new methods: one, which is used in the present paper and has further applications, and the other, briefly outlined in Section 2.5, which is developed in [5] and further in [4].[2] The second method still involves the *a priori* regularity arguments as an essential component. Results of [5] and [4] are announced in [8].

Remark 3. Notice that for case of general lattice actions considered in [14] the partially hyperbolic case is also central. In this situation totally different methods, specific to groups with Property (T), are used.

2.3. *The* KAM/*Harmonic analysis approach.* In our new approach we do not start from a conjugacy of low regularity. Instead we construct one of high regularity by an iterative process as a fixed point of a certain nonlinear operator. We use an iterative procedure, namely an adapted version of the KAM procedure motivated by the procedure used by Moser in [31] to study small perturbations of *isometric* $\mathbb{Z}^k$ actions generated by circle rotations. Moser first noticed that commutativity along with simultaneous Diophantine condition was enough to provide a smooth solution to certain over-determined system of equations.

Notice however, a principal difference between Moser's setting and ours: The only obstruction to solving linearized conjugacy equation in the case of commuting circle rotations is the same as for a single one: it comes from the rotation number. In our case there are infinitely many obstructions for solving the linearized equation for a single element of the action but they magically vanish for the whole action due to the "higher rank trick". Thus while Moser used his setting to show that Diophantine conditions for individual elements of the action are not necessary to contain the effects of "small denominators", we obtain differentiable conjugacy for an action in the absence of even a formal conjugacy for perturbations of its individual elements.

The solution is the limit of successive approximations obtained by approximating the nonlinear problem by its linear part, and solving (approximately) the corresponding linearized equation. This scheme arises from the classical KAM, which also set the basis for generalization of classical Implicit Function Theorem to so-called hard implicit function theorems [38], or Nash-Moser inverse function theorems [15]. (For more references see for example [10].) Those general results have been applied more recently to *isometric* "large" lattice actions. We discuss relevance of the Implicit Function Theorem approach in our situation and quote other relevant work in Section 8. We note that these general results do not apply to $\mathbb{Z}^k$ actions generated by circle rotations studied by Moser in [31].

[2] *Added in Proof*: See also two recent papers by Z. J. Wang [36], [37].

In our case the first key observation is that the linearized equation in this case is “almost” a twisted cocycle equation over the unperturbed linear action. It is possible to solve such equations due to the “higher rank trick” which lies at the root of cocycle rigidity [23], [24] and which is adapted in the current paper to the twisted case.

The second key observation is that it is possible to obtain an approximate solution to the linearized equation if the linearized equation can be approximated by a twisted cohomology equation over the unperturbed action. This reduces to approximating a certain twisted cocycle over the perturbed action by a twisted cocycle over the unperturbed action. We construct this approximation cocycle explicitly by adapting and developing a method from [18].

As a by-product of our approach we remove the restriction of semi-simplicity for the unperturbed action. Our method works when the unperturbed action, whether hyperbolic or partially hyperbolic, has nontrivial Jordan blocks. A disadvantage of our method is the requirement that the perturbation is small in C^l topology for a certain large l while the method of [25], when it is applicable, can deal with small perturbations in C^1 topology.[3]

2.4. *Extensions of the* KAM*/harmonic analysis approach to other actions.* The method presented in this paper provides a blueprint for studying local rigidity for other algebraic abelian actions.

(I) A natural generalization of actions by automorphisms of a torus appears on infranilmanifolds when one considers projections of partially hyperbolic actions by $\mathbb{Z}^k$ on a simply connected nilpotent group N preserving a lattice Λ on the compact factor-space N/Λ. The work on the nilpotent case is in progress.

In [25, §4.4] it is explained (although calculations are omitted) how cocycle rigidity for the nilpotent case is deduced from the cocycle rigidity in the toral case. What is needed in addition is first, tame estimates and second, the splitting. In order to do that, Fourier analysis of Section 4 is substituted by arguments using infinite-dimensional unitary representations of nilpotent groups.

(II) Some ideas from specific constructions carried out in Section 4 have been applied recently to certain totally nonhyperbolic, namely, *parabolic* abelian actions. The simplest example is the action of the upper-triangular unipotent subgroup of $\mathrm{SL}(2,\mathbb{R})\times\mathrm{SL}(2,\mathbb{R})$ on the factor by a cocompact irreducible lattice. Cocycle rigidity with tame estimates as well as tame splitting are established by D. Mieczkowski in his 2006 Ph. D. thesis and published as [30]. Such actions are not locally rigid already among homogeneous actions. In that they are similar to the rotations on the circle. However, the iteration scheme with an iteration step relying on the results

[3] *Added in Proof*: M.Einsiedler and T. Fisher in [11] developed a version of the *a priori* regularity method to prove local differentiable rigidity for hyperbolic actions by automorphisms of a torus in the presence of Jordan blocks. They require an additional “narrow spectrum” assumption for proportional Lyapunov exponents.

of [30] can be carried out producing certain modified form of local rigidity restricted to parametric families of perturbations; see [9]. The number of parameters needed is the same as the codimension of unipotent subgroups within the ambient Lie group; it is equal to two in the basic example above. Due to the obstacles, just like in the case of commuting circle diffeomorphisms [31], explicit iteration is needed resembling the one carried out in Sections 5.3 and 5.4. This is in a contrast with other situations where one may use generalized implicit function theorems; see Section 8 for more discussion.

(III) *Added in Proof*: Very recently decisive advances have been made in extending applications of this approach to the actions of type **(2)** as well other algebraic partially hyperbolic actions. This confirms the power and fruitfulness of the KAM-based method. A key difference with the setting of the present paper is that one can apply a priori regularity method to obtain smooth rigidity of the neutral foliation of the action and hence reduce the problem to a perturbation along this foliation. This leads to untwisted linearized conjugacy equations and allows to use available estimates for decay of matrix coefficients. This work also contains a number of other new ingredients. (The paper by the second author and Z. J.Wang is in preparation)

2.5. *The geometric approach.* In [5], [4] we use another approach to the semisimple homogeneous case **(2)** which bypasses the difficulties of the KAM/ harmonic analysis method. For the sake of completeness of our account of the state of rigidity program, we present a very brief narrative outline here, referring for precise formulations and details to the papers and the published detailed announcement [8].

By the structural stability result of Hirsch-Pugh-Shub [16], the problem of obtaining a Hölder conjugacy reduces to problem of reducing a Hölder cocycle over the *perturbed action* to a constant cocycle. Obstructions to this can be described via *periodic cycle functionals* defined on closed paths along coarse Lyapunov foliations [20], [7]. For the unperturbed action such closed paths have a structure coming from the algebraic structure of the covering group, and it is proved in [7] that this structure implies vanishing of obstructions to cocycle trivialization. The main part of the geometric approach is to show that the fine structure of Lyapunov foliations for the unperturbed action is robust under small perturbations, so that it can be used to imply vanishing of obstructions to cocycle trivialization for small perturbations and produce a Hölder conjugacy with the unperturbed action. Using the *a priori* regularity method (see Section 2.1) one shows that conjugacy is smooth along the unipotent foliations for the unperturbed action. Then the usual regularity along stable directions and Hörmander theorems guarantee that the conjugacy is C^∞.[4]

[4] *Added in Proof*: This method has been successfully applied in [36], [37] to a variety of new cases of type **(2)** including those with nonsplit groups G. This work is based on advances in calculating appropriate families of generating relations for those groups.

2.6. *Conclusion.* Notice that for partially hyperbolic actions the geometric approach requires sufficient noncommutativity for various stable foliations for their brackets to generate the neutral directions and hence it cannot be applied in principle to partially hyperbolic actions on the torus where all invariant distributions commute. Thus the KAM and the geometric approach nicely complement each other in advancing significantly the solution of the local differentiable rigidity problem for algebraic partially hyperbolic actions.

Results of the current paper are announced in [6]. We would like to thank Elon Lindenstrauss who pointed out an inaccuracy in an early version of this paper. As a result Section 4.5 has been reworked. We also thank David Fisher by bringing to our attention the relevant work of Benveniste.

3. Setting of the problem and overview of the KAM scheme

3.1. *Ergodic toral automorphisms.* An automorphism of the torus $\mathbb{T}^N$ is induced by an invertible integer matrix A. We will use the same notation A both for an integer matrix and for the toral automorphism induced by it. A induces the map given by the transpose matrix A^t on the dual group $\mathbb{Z}^N$. If $e_n(x) = e^{2\pi i n\cdot x}$ are the characters, then A^t acts as

$$e_{A^t n}(x) = e^{2\pi i A^t n\cdot x} = e^{2\pi i n\cdot Ax} = e_n(Ax),$$

which implies that

$$\widehat{(f\circ A)}_n = \hat{f}_{(A^t)^{-1}n}$$

for Fourier coefficients of any C^∞ function f on $\mathbb{T}^N$. We call $(A^t)^{-1}$ the dual map on $\mathbb{Z}^N$. To simplify the notation in the rest of the paper, whenever there is no confusion as to which map we refer to we will denote the dual map by the same symbol A. When we wish to clearly distinguish the two maps, we use for the dual map the notation A^*. The dual orbits of A are $\mathbb{O}(n) = \{A^i n \mid i \in \mathbb{Z}\}$.

The following is a simple characterization of ergodicity of toral automorphisms in terms of the corresponding dual action. Even though this characterization is a well-known fact we include also a quick proof here, for completeness.

LEMMA 3.1. (i) *An automorphism of* $\mathbb{T}^N$ *induced by* $N\times N$ *integer matrix* A *is ergodic if and only if all nontrivial orbits of the dual map* A^* *on* $\mathbb{Z}^N$ *are infinite.*

(ii) *An automorphism of* $\mathbb{T}^N$ *induced by* $N\times N$ *integer matrix* A *is ergodic if and only if* A *has no roots of unity in the spectrum.*

Proof. (i) Suppose there exists a nonzero $n\in\mathbb{Z}^N$ with a finite orbit, i.e., $A^m n = n$ for some m. Then the function $f = \sum_{i=0}^{m-1} e_{A^i n}$ is an A invariant nonconstant function, which contradicts ergodicity.

Conversely, if A is not ergodic and if $f = \sum_n \hat{f}_n e_n$ is an invariant function, then $\hat{f}_{An} = \hat{f}_n$ for all $n\in\mathbb{Z}^N$, therefore if the orbit of n is infinite then $\hat{f}_n = 0$. This implies that f is constant. Thus A is ergodic.

(ii) follows from (i) and from the fact that the dual map has a root of unity in the spectrum if and only if there exists a nontrivial $n \in \mathbb{Z}^N$ with finite orbit. □

3.2. *Higher rank actions by toral automorphisms.* Any action α of $\mathbb{Z}^k$ by toral automorphisms, induces a *dual action* α^* on $\mathbb{Z}^N$ given by

$$\rho_{\alpha^*}(i)n = (A_1^*)^{i_1}(A_2^*)^{i_2}\cdots(A_k^*)^{i_k}n$$

for $i = (i_1, \dots, i_k) \in \mathbb{Z}^k$ and $n \in \mathbb{Z}^N$. Then for every $f = \sum_{n\in\mathbb{Z}^N} \hat{f}_n e_n$ we have

$$f(\rho_\alpha(i)x) = \sum_{n\in\mathbb{Z}^N} \hat{f}_{\alpha^i n} e_n$$

where, to simplify the notation (whenever there is no confusion), we use the notation $\alpha^i n$ for the dual action $\rho_{\alpha^*}(i)n$ on $\mathbb{Z}^N$.

Since we are interested in actions with no rank one factors we will assume further on that the action α satisfies the following:

($\mathfrak{G}$) *There exist $g_1, g_2 \in \mathbb{Z}^k$ with $A \stackrel{\text{def}}{=} \rho_\alpha(g_1)$ and $B \stackrel{\text{def}}{=} \rho_\alpha(g_2)$ such that any $A^l B^k$, for a nonzero $(l,k) \in \mathbb{Z}^2$, is ergodic. A and B will be referred to as ergodic generators.*

The following is a simple lemma which shows that obtaining a C^∞ conjugacy for one ergodic generator suffices for the proof of Theorem 1.

LEMMA 3.2. *Let α be a $\mathbb{Z}^k$ action by automorphisms of $\mathbb{T}^N$ such that for some $g \in \mathbb{Z}^k$ the automorphism $\alpha(g)$ is ergodic. Let $\tilde{\alpha}$ be a C^1 small perturbation of α such that there exists a C^∞ map $H : \mathbb{T}^N \to \mathbb{T}^N$ which is C^1 close to identity and satisfies*

$$\tilde{\alpha}(g) \circ H = H \circ \alpha(g). \tag{3.1}$$

Then H conjugates the corresponding maps for all the other elements of the action; i.e., for all $h \in \mathbb{Z}^k$ we have

$$\tilde{\alpha}(h) \circ H = H \circ \alpha(h).$$

Proof. Let h be any element in $\mathbb{Z}^k$, other than g. It follows from (3.1) and commutativity that

$$\tilde{\alpha}(g) \circ \tilde{H} = \tilde{H} \circ \alpha(g),$$

where

$$\tilde{H} = \tilde{\alpha}(h)^{-1} \circ H \circ \alpha(h).$$

Therefore,

$$\alpha(g) \circ \tilde{H}^{-1} \circ H = \tilde{H}^{-1} \circ H \circ \alpha(g).$$

Denote the lift of $\tilde{H}^{-1} \circ H$ to $\mathbb{R}^N$ by $\mathrm{id} + \Omega$ where Ω is a C^1 small $\mathbb{Z}^N$-periodic C^∞ map, and use the same notation $\alpha(g)$ for the lift of $\alpha(g)$. Then

$$\alpha(g) \circ (\mathrm{id} + \Omega) = (\mathrm{id} + \Omega) \circ \alpha(g). \tag{3.2}$$

If A is diagonalizable, equation (3.2) reduces to several equations of the following type:

$$\lambda\omega = \omega \circ A,$$

where $A = \alpha(g)$, ω is C^∞ function, and λ is an eigenvalue of A. Then, passing to the dual action (while, as mentioned before, we use the same notation for the dual and for the original action), we have

$$\lambda\widehat{\omega}_n = \widehat{\omega}_{An},$$

which implies

$$\lambda^i\widehat{\omega}_n = \widehat{\omega}_{A^i n}$$

for all $i \in \mathbb{Z}$. Therefore

$$\sum_{i\in\mathbb{Z}}^{\infty} \lambda^i\widehat{\omega}_n = \sum_{i\in\mathbb{Z}} \widehat{\omega}_{A^i n}.$$

Since A is ergodic and ω is C^∞ the right-hand side converges absolutely. The left-hand side can converge only if $\widehat{\omega}_n = 0$ for all $n \neq 0$. Since $\lambda \neq 1$ we also have $\widehat{\omega}_0 = 0$, i.e., $\omega \equiv 0$. This implies $\Omega \equiv 0$ and $H \equiv \widetilde{H}$.

If there are Jordan blocks for A, the argument above is still sufficient to deduce that Ω has to be 0. Namely, if there is, say, a 3-Jordan block, then equation (3.2) reduces to

$$\begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{pmatrix} \begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix} \begin{pmatrix} \omega_1 \circ A \\ \omega_2 \circ A \\ \omega_3 \circ A \end{pmatrix}.$$

This implies

$$\lambda\omega_1 + \omega_2 - \omega_1 \circ A = 0,$$

$$\lambda\omega_2 + \omega_3 - \omega_2 \circ A = 0,$$

$$\lambda\omega_3 - \omega_3 \circ A = 0.$$

From the third equation above, using ergodicity of A as in the case of a simple eigenvalue, we deduce that $\omega_3 = 0$. Substituting into the second equation, we obtain $\omega_2 = 0$ and finally, using this fact in the first equation above, we obtain $\omega_1 = 0$. One can obviously by induction obtain $\Omega = 0$ for Jordan blocks of any dimension. Therefore $\widetilde{H} = H$ and

$$H = \alpha(h)^{-1} \circ H \circ \widetilde{\alpha}(h)$$

for an arbitrary $h \in \mathbb{Z}^k$. □

3.3. *Overview of the* KAM *scheme.* Let α be a linear action as described in Theorem 1. Let $\widetilde{\alpha}$ be its small perturbation (the topology in which the perturbation is made will become apparent from the proof). The goal is to prove the existence of a C^∞ map $H\colon \mathbb{T}^N \to \mathbb{T}^N$ such that $\widetilde{\alpha} \circ H = H \circ \alpha$.

One can consider the problem of finding a conjugacy as a problem of solving the following nonlinear functional equation

$$\mathcal{N}(\widetilde{\alpha}, H) \stackrel{\text{def}}{=} \widetilde{\alpha} \circ H - H \circ \alpha = 0.$$

Following the ideas of the elementary Newton method and assuming the existence of a linear structure in the neighborhood of the identity, the identity may be viewed as an approximate solution (usually called the initial guess) of the nonlinear problem. The linearization of the operator $\mathcal{N}$ at (α, id) is

$$\begin{aligned}\mathcal{N}(\widetilde{\alpha}, H) &= \mathcal{N}(\alpha, \mathrm{id}) + D_1\mathcal{N}(\alpha, \mathrm{id})(\widetilde{\alpha} - \alpha)\\ &\quad + D_2\mathcal{N}(\alpha, \mathrm{id})(\Omega) + \mathrm{Res}(\widetilde{\alpha} - \alpha, \Omega)\\ &= \widetilde{\alpha} - \alpha + \alpha \circ \Omega - \Omega \circ \alpha + \mathrm{Res}(\widetilde{\alpha} - \alpha, \Omega),\end{aligned}$$

where $\Omega = H - \mathrm{id}$, and $\mathrm{Res}(\widetilde{\alpha} - \alpha, \Omega)$ is quadratically small with respect to $\widetilde{\alpha} - \alpha$ and Ω. Here $D_1\mathcal{N}(\alpha, \mathrm{id})$ and $D_2\mathcal{N}(\alpha, \mathrm{id})$ denote Frechét derivatives of the map $\mathcal{N}$ in the first and second variable, respectively, at the point (α, id). Also, since α is a linear action, $\alpha \circ \Omega$ for each generator is given simply by a matrix applied to a vector valued map. To stress this fact we will use the notation $\alpha\Omega$ for $\alpha \circ \Omega$.

If one finds H so that the linear part of the equation above is zero, i.e.,

$$\alpha(H - \mathrm{id}) - (H - \mathrm{id}) \circ \alpha = -(\widetilde{\alpha} - \alpha),$$

then such H is a better approximate solution of the equation $\mathcal{N}(\widetilde{\alpha}, H) = 0$ than the identity is. After obtaining a better solution, the linearization procedure and solving the linearized equation may be repeated for the new perturbation leading to an even better approximation. The difficulties which arise, in particular applications of this iterative scheme, are of two kinds: one is to solve (or solve approximately) the linearized equation, and the other has to do with obtaining good estimates for the solution so that the sequence of approximate solutions produced by this scheme converges in some reasonable function space.

We now adapt this general scheme to our specific problem concerning toral automorphisms. Any map of the torus $\mathbb{T}^N$ into itself can be lifted to the universal cover $\mathbb{R}^N$. For every $g \in \mathbb{Z}^k$, the lift of $\alpha(g)$ is a linear map of $\mathbb{R}^N$, i.e., a matrix with integer entries and with determinant ± 1, which is also denoted by $\alpha(g)$. The lift of $\widetilde{\alpha}(g)$ is $\alpha(g) + R(g)$ where $R(g)$ is an $\mathbb{Z}^N$-periodic map for every g, i.e., $R(g)(x+m) = R(g)(x)$ for $m \in \mathbb{Z}^N$. The lift of H is $\mathrm{id} + \Omega$ with a $\mathbb{Z}^N$-periodic Ω.

In terms of Ω the nonlinear conjugacy problem becomes

$$\alpha\Omega - \Omega \circ \alpha = -R \circ (\mathrm{id} + \Omega) \tag{3.3}$$

and the corresponding linearized equation is

$$\alpha\Omega - \Omega \circ \alpha = -R. \tag{3.4}$$

If Ω is a solution for the linearized equation (3.4) (or at least an approximate solution, i.e., it solves (3.4) with an error which is small with respect to R), then one may expect that the new perturbation defined by

$$\widetilde{\alpha}^{(1)} \stackrel{\text{def}}{=} H^{-1} \circ \widetilde{\alpha} \circ H,$$

where $H = \mathrm{Id} + \Omega$, is much closer to α than $\widetilde{\alpha}$; i.e., the new error

$$R^{(1)} \stackrel{\text{def}}{=} \widetilde{\alpha}^{(1)} - \alpha$$

is expected to be small with respect to the old error R.

The comparison between the two errors as usual cannot be realized in the same function space. The norm of the old error in some function space is only small compared to the norm of the new error in some larger space, thus one has some loss of regularity. The loss however is fixed; it depends only on the initial linear action and the dimension of the torus. In the proof that we present here all the maps involved will be C^∞. We have loss of regularity at an individual iterative step, in the sense described above. Thus we work with the family of C^r norms ($r \in \mathbb{N}$) on the space of C^∞ maps. In Section 7.1 we treat perturbations of finite regularity: we consider the space of C^{r_0} maps with fixed $r_0 > 0$ along with a family of C^r norms for $r \le r_0$. In general, if the loss of regularity is fixed and if the family of spaces used in the iterative scheme admits smoothing operators (see Section 5.1), then the interpolation inequalities hold and it is often the case (as it is here) that the iterative procedure then can be set to converge to a smooth solution of the nonlinear equation (see for example [38]).

At an individual iterative step, the new error can be expressed as

$$\begin{aligned} R^{(1)} = \widetilde{\alpha}^{(1)} - \alpha = \Big[\Omega \circ \widetilde{\alpha}^{(1)} - \Omega \circ \alpha + R \circ (\mathrm{id} + \Omega) - R\Big] \\ + \Big[R - (\alpha\Omega - \Omega \circ \alpha)\Big]. \end{aligned}$$

The error term in the first bracket comes from the linearization of the problem and is easy to estimate providing Ω is of the same order as R. The difficulty lies in estimating the part of the error in the second bracket, namely solving the linearized equation (3.4) approximately, with an error quadratically small with respect to R.

In order to understand better where commutativity of the action and its perturbation come into the picture, we look more closely into the linearized equation. Equation (3.4) actually consists of infinitely many equations corresponding to different elements g_i of the action

$$\alpha(g_i)\Omega - \Omega \circ \alpha(g_i) = -R(g_i) \tag{3.5}$$

and we need a common approximate solution Ω to all the equations above.

If such Ω exists, then it is easy to check that due to commutativity of each pair $\alpha(g_i)$ and $\alpha(g_j)$, the map $R\colon G \times \mathbb{T}^N \to \mathbb{R}^N$ satisfies the following condition:

$$\alpha(g_i)R(g_j) - R(g_j) \circ \alpha(g_i) = \alpha(g_j)R(g_i) - R(g_i) \circ \alpha(g_j). \tag{3.6}$$

At this point we need to introduce some terminology.

Definition 1. Let $\alpha\colon A\times M\to M$ be an action of an abelian group A on a manifold M, let $(H,*)$ be a topological group and let β_0 be a homomorphism $\beta_0\colon A\to \operatorname{Aut} H$. Then a map $\beta\colon A\times M\to H$ is called a β_0 twisted cocycle over α if

$$\beta(a+b,x)=\beta(a,\alpha(b,x))*\beta_0(a)(\beta(b,x)). \tag{3.7}$$

A β_0 twisted cocycle β is a β_0 twisted coboundary over α if there exists $P\colon M\to H$ such that for all $a\in A$

$$\beta(a,x)=\beta_0(a)(P(x))*(P(\alpha(a,x)))^{-1}. \tag{3.8}$$

If every C^∞ β_0 twisted cocycle β over α is a β_0 twisted coboundary over α via a C^∞ map $P\colon M\to H$, then α is said to be C^∞ β_0 cocycle rigid.

If β_0 is the identity automorphism, then β_0 is dropped from the notations above.

If $R\colon \mathbb{Z}^k\times M\to\mathbb{R}^N$ satisfies (3.6), then R is an α twisted cocycle over the action α (here we use the same notation α for the linear action on $\mathbb{T}^N$ and for its lift to $\mathbb{R}^N$). Then the system of equations (3.5) can be viewed as an α twisted coboundary equation over the *linear* action α. However, this is not quite true: R is an α twisted cocycle over the *perturbed* action $\tilde\alpha$, not over the linear action α.

LEMMA 3.3. *If $\tilde\alpha=\alpha+R$ is a small perturbation of α then the map R is an α twisted cocycle over $\tilde\alpha$.*

Proof. As before, we use the same notation here for the linear action α and for its lift to $\mathbb{R}^N$:

$$\begin{aligned}\tilde\alpha(a+b,x)&=\tilde\alpha(a,\tilde\alpha(b,x))=\alpha(a,\tilde\alpha(b,x))+R(a,\tilde\alpha(b,x))\\&=\alpha(a)\tilde\alpha(b,x)+R(a,\tilde\alpha(b,x))\\&=\alpha(a)\alpha(b)x+\alpha(a)R(b,x)+R(a,\tilde\alpha(b,x)).\end{aligned} \tag{3.9}$$

On the other hand,

$$\tilde\alpha(a+b,x)=\alpha(a+b,x)+R(a+b,x)=\alpha(a)\alpha(b)x+R(a+b,x). \tag{3.10}$$

This implies

$$R(a+b,x)=\alpha(a)R(b,x)+R(a,\tilde\alpha(b,x)). \tag{3.11}$$

Thus R is an α twisted cocycle over $\tilde\alpha$. □

There are two difficulties in solving the linearized equation (3.4). First is that R is a twisted cocycle *not* over α but over $\tilde\alpha$ thus (3.4) is not a twisted coboundary equation over the linear action. Second is that even if (3.4) is a twisted coboundary equation, solving it requires more care than the corresponding untwisted equation. In fact, the twist produces greater loss of regularity for the solution of the twisted coboundary problem, compared to the untwisted case, as we will see in Lemma 4.2.

Lemma 3.2 shows that it is enough to produce a conjugacy for one ergodic generator. It is clear however that, in general, it is not possible to produce a C^∞ conjugacy for a single element of the action, since a single genuinely partially hyperbolic toral automorphism is not even structurally stable. Indeed, Lemma 4.2 in Section 4 shows that there are infinitely many obstructions to solving the linearized equation for one generator. Therefore, we consider two ergodic generators, and reduce the problem of solving the linearized equation (3.4) to solving simultaneously the following system:

$$\begin{aligned} A\Omega - \Omega \circ A &= -R_A, \\ B\Omega - \Omega \circ B &= -R_B, \end{aligned} \tag{3.12}$$

where A and B are ergodic generators: $A := \alpha(g_1)$, $B := \alpha(g_2)$ and

$$R_A := R(g_1), \qquad R_B := R(g_2). \tag{3.13}$$

The system (3.12) splits further into several simpler systems using an appropriate basis and the fact that A and B commute. The linear problem (3.12) is reduced to several equations of the kind

$$\begin{aligned} J_A\Omega - \Omega \circ A &= \Theta \\ J_B\Omega - \Omega \circ B &= \Psi, \end{aligned} \tag{3.14}$$

where J_A is a matrix consisting of Jordan blocks of A corresponding to an eigenvalue of A, J_B is the corresponding block of B, and Θ and Ψ are small vector valued $\mathbb{Z}^N$-periodic maps given by the perturbation maps R_A and R_B. In particular, if λ, μ are simple eigenvalues of A, B respectively, then we have

$$\begin{aligned} \lambda\omega - \omega \circ A &= \theta \\ \mu\omega - \omega \circ B &= \psi, \end{aligned} \tag{3.15}$$

where θ and ψ are small $\mathbb{Z}^N$-periodic functions. In Lemma 4.4 we show that it is possible to solve the linearized equation (3.12) and to estimate the solution with the fixed loss of regularity if the following condition is satisfied by R:

$$L(R_A, R_B) \stackrel{\text{def}}{=} (R_A \circ B - BR_B) - (R_B \circ A - AR_B) = 0. \tag{3.16}$$

As mentioned above, R does not satisfy this condition. However the fact that it satisfies the twisted cocycle condition over the *perturbed* action $\widetilde{\alpha}$ implies that it almost satisfies the twisted cocycle condition (3.16); i.e., (3.16) is satisfied up to an error which is small with respect to R. More precisely, Lemma 4.7 shows that if $\widetilde{\alpha} = \alpha + R$ is a commutative action, then $L(R_A, R_B)$ is small with respect to R.

The key step is to show that if R is *almost* a twisted cocycle over α then R is *close* to an actual twisted cocycle over α. In Section 4.5 we construct a projection $\mathcal{P}R$ of R to the space of twisted cocycles over α so that the difference $\mathcal{E}R = \mathcal{P}R - R$ is small with respect to R. More precisely, in Lemma 4.6 we show that even if R_A and R_B do not satisfy the solvability condition (3.16), it is possible to approximate both by maps $\mathcal{P}R_A$ and $\mathcal{P}R_B$ which satisfy the condition (3.16) with an error bounded by the size of $L(R_A, R_B)$.

Lemmas 4.6, 4.4 and 4.7 combined give an approximate solution to the linearized equation (3.12).

In Section 5.1 the smoothing operators are introduced to overcome the fixed loss of regularity at each iterative step and further in Section 5 the iteration process is set and is carried out, producing a C^∞ conjugacy which works for the $\mathbb{Z}^2$ action generated by the two ergodic generators, thus, according to Lemma 3.2, it works for all the other elements of the $\mathbb{Z}^k$ action α.

4. **Approximate solution of the linearized equation**

In this section we prove the existence of an approximate solution to the linearized equation (3.12). We first introduce in Section 4.1 some notation which will be used in the rest of the paper. In Section 4.2 we describe the obstructions for solving a single linear equation in (3.15), which is a twisted coboundary equation over an ergodic toral automorphism. We show, providing the obstructions vanish, that there exists a C^∞ solution with tame estimates. This is a version of the arguments in [35] adapted to the case of a twisted coboundary equation. An essential ingredient in the proof of Lemma 4.2, just like in [35], is an exponential growth estimate on the dual orbit of an ergodic toral automorphism which reflects arithmetic properties of ergodic toral automorphisms and was proved by Katznelson in [26].

Section 4.3 contains corresponding growth estimate for the dual orbits of a $\mathbb{Z}^k$ higher rank ergodic action by toral automorphisms, and applications to estimating decay of Fourier coefficients for specifically defined functions which are used in Section 4.5.

In Section 4.4 we show that the obstructions to solving each equation in (3.12) vanish providing that the maps R_A and R_B satisfy (3.16). Then in Lemma 4.7 we show that the criterion (3.16) is almost satisfied, up to an error which is small with respect to R. Finally in Section 4.5 we construct a projection $(\mathcal{P}R_A, \mathcal{P}R_B)$ such that $(\mathcal{P}R_A, \mathcal{P}R_B)$ satisfies (3.16) and therefore one can solve tamely the system (3.12) with $(\mathcal{P}R_A, \mathcal{P}R_B)$ instead of (R_A, R_B). Moreover, the projection in Section 4.5 is such that the error

$$(\mathcal{E}R_A := R_A - \mathcal{P}R_A,\ \mathcal{E}R_B := R_B - \mathcal{P}R_B)$$

is small with respect to (R_A, R_B). This produces an approximate solution to the linearized equation (3.12).

4.1. *Some notation.* Let A and B be the two ergodic generators for α.

(1) The dual map A^* on $\mathbb{Z}^N$ induces a decomposition of $\mathbb{R}^N$ into expanding, neutral and contracting subspaces. We will denote the expanding subspace by $V_1(A)$, the contracting subspace by $V_3(A)$ and the neutral subspace by $V_2(A)$.

$$\mathbb{R}^N = V_1(A) \oplus V_2(A) \oplus V_3(A). \tag{4.1}$$

All three subspaces $V_i(A)$, $i = 1, 2, 3$ are A invariant and

$$\|A^i v\| \geq C\rho^i \|v\|, \quad \rho > 1, \quad i \geq 0, \quad v \in V_1(A), \tag{4.2}$$
$$\|A^i v\| \geq C\rho^{-i} \|v\|, \quad \rho > 1, \quad i \leq 0, \quad v \in V_3(A),$$
$$\|A^i v\| \geq C(|i|+1)^{-N} \|v\|, \quad i \in \mathbb{Z}, \quad v \in V_2(A).$$

(2) For $n \in \mathbb{Z}^N$, $|n| \stackrel{\text{def}}{=} \max\{\|\pi_1(n)\|, \|\pi_2(n)\|, \|\pi_3(n)\|\}$, where $\|\cdot\|$ is Euclidean norm and $\pi_i(n)$ are projections of n to subspaces $V_i (i = 1, 2, 3)$ from (4.2), that is, to the expanding, neutral, and contracting subspaces of $\mathbb{R}^N$ for A (or B; we will use the norm which is more convenient in a particular situation; those are equivalent norms, the choice does not affect any results).

(3) For $n \in \mathbb{Z}^N$ we say *n is mostly in $i(A)$* for $i = 1, 2, 3$ and will write $n \hookrightarrow i(A)$, if the projection $\pi_i(n)$ of n to the subspace V_i corresponding to A is sufficiently large:

$$|n| = \|\pi_i(n)\|.$$

The notation $n \hookrightarrow 1, 2(A)$ will be used for n which is mostly in $1(A)$ or mostly in $2(A)$.

(4) Given a complex number λ and a function φ on the torus, define the twisted coboundary operators:

$$\Delta_A^\lambda \varphi \stackrel{\text{def}}{=} \lambda\varphi - \varphi \circ A, \qquad \Delta_A^\lambda \widehat{\varphi}_n \stackrel{\text{def}}{=} \lambda\widehat{\varphi}_n - \widehat{\varphi}_{An}.$$

In what follows λ will usually be an eigenvalue of A, and μ will usually denote an eigenvalue of B, so we will often use the following simpler notation:

$$\Delta^\lambda \stackrel{\text{def}}{=} \Delta_A^\lambda, \qquad \Delta^\mu \stackrel{\text{def}}{=} \Delta_B^\mu,$$
$$\Delta^\lambda \widehat{\varphi}_n \stackrel{\text{def}}{=} \Delta_A^\lambda \widehat{\varphi}_n, \quad \Delta^\mu \widehat{\varphi}_n \stackrel{\text{def}}{=} \Delta_B^\mu \widehat{\varphi}_n.$$

(5) Similarly, define the following operator

$$\Delta^A \mathcal{F} \stackrel{\text{def}}{=} A\mathcal{F} - \mathcal{F} \circ A.$$

(6) For the functions θ, ψ and the maps $\mathcal{F}, \mathcal{G}$ define the following operators:

$$L(\theta, \psi) \stackrel{\text{def}}{=} \Delta_B^\mu \theta - \Delta_A^\lambda \psi, \qquad L(\mathcal{F}, \mathcal{G}) \stackrel{\text{def}}{=} \Delta^B \mathcal{F} - \Delta^A \mathcal{G}.$$

(7) We introduce the notation for the following sums:

$$\sum\nolimits_+^A \widehat{\varphi}_n \stackrel{\text{def}}{=} \sum_{i=0}^{\infty} \lambda^{-(i+1)} \widehat{\varphi}_{A^i n},$$
$$\sum\nolimits_-^A \widehat{\varphi}_n \stackrel{\text{def}}{=} \sum_{i=-\infty}^{-1} \lambda^{-(i+1)} \widehat{\varphi}_{A^i n},$$
$$\sum\nolimits^A \widehat{\varphi}_n \stackrel{\text{def}}{=} \sum_{i=-\infty}^{+\infty} \lambda^{-(i+1)} \widehat{\varphi}_{A^i n}.$$

Here A is used to denote the dual action A^* induced on $\mathbb{Z}^N$ by a toral automorphism given by an integer matrix A (Section 3.1). $\widehat{\varphi}_n$ are Fourier coefficients of a C^∞ function φ. The corresponding notation will be used for B instead of A and μ instead of λ, where μ and λ are corresponding eigenvalues of A and B. We will also sometimes abbreviate the following notation as follows:

$$\sum_l{}_- \stackrel{\text{def}}{=} \sum_{l=-\infty}^{-1}, \qquad \sum_l{}_+ \stackrel{\text{def}}{=} \sum_{l=0}^{\infty}, \quad \text{and} \quad \sum_l \stackrel{\text{def}}{=} \sum_{l=-\infty}^{l=+\infty}.$$

(8) In what follows, C will denote any constant that depends only on the given linear action α with chosen ergodic generators and on the dimension of the torus. $C_{x,y,z,\dots}$ will denote any constant that in addition to the above depends also on parameters $x, y, z, \dots$.

(9) Let f be a C^∞ function $f = \sum_n \hat{f}_n e_n$. Then

(i) $\|f\|_a \stackrel{\text{def}}{=} \sup_n |\hat{f}_n||n|^a, \qquad a > 0.$

(ii) $D_j \stackrel{\text{def}}{=} \frac{\partial}{\partial x_j}$, $D^k \stackrel{\text{def}}{=} D_1^{k_1} D_2^{k_2} \cdots D_k^{k_N}$, $|k| = \sum_{i=1}^N k_i$, $k = (k_1, \dots, k_N)$,

$$\|f\|_{C^r} \stackrel{\text{def}}{=} \max_{0 \le l \le r} \sup_{\{x \in \mathbb{T}^N, |k| = l\}} |D^k f x|, \quad r \in \mathbb{N}_0.$$

(iii) The following relations hold (see, for example, Section 3.1 of [10]):

$$\|f\|_r \le C\|f\|_{C^r} \qquad \text{and} \qquad \|f\|_{C^r} \le C_r \|f\|_{r+\sigma},$$

where $\sigma > N + 1$, and $r \in \mathbb{N}_0$. In particular, one may take $\sigma = N + 1 + \delta$ with a small $\delta > 0$.

(iv) For a map $\mathcal{F}$ with coordinate functions f_i $(i = 1, \dots, k)$ define $\|\mathcal{F}\|_a \stackrel{\text{def}}{=} \max_{1 \le i \le k} \|f_i\|_a$. For two maps $\mathcal{F}$ and $\mathcal{G}$ define $\|\mathcal{F}, \mathcal{G}\|_a \stackrel{\text{def}}{=} \max\{\|\mathcal{F}\|_a, \|\mathcal{G}\|_a\}$. $\|\mathcal{F}\|_{C^r}$ and $\|\mathcal{F}, \mathcal{G}\|_{C^r}$ are defined similarly.

(v) *Note.* Instead of norms $\|\cdot\|_a$ and $\|\cdot\|_{C^r}$ one can use Sobolev norms $\|\cdot\|_{H^s}$ for which the relation to Fourier coefficients is immediate:

$$\|f\|_{H^s} = \Big(\sum_{n \in \mathbb{Z}} (1 + |n|^2)^s |\hat{f}_n|^2\Big)^{\frac{1}{2}}.$$

Then Sobolev embedding theorem would imply comparison with C^r norms with loss of $\sigma = N/2$. However, the explicit calculations are simpler with norms $\|\cdot\|_a$, and, at the expense of somewhat better estimates with Sobolev norms, we choose to use $\|\cdot\|_a$ norms in order to keep the calculations more transparent.

(10) If β is a cocycle (or a twisted cocycle) over a higher rank $\mathbb{Z}^k$, $k \ge 2$ action α with generators $\alpha(g_1, \cdot)$ and $\alpha(g_2, \cdot)$, then we define the C^r norm of β by

$$\|\beta\|_{C^r} := \|\beta(g_1, \cdot), \beta(g_2, \cdot)\|_{C^r}.$$

4.2. *Twisted coboundary equation over an ergodic toral automorphism.* In what follows we make frequent use of the following lemma in which arithmetic properties of ergodic toral automorphisms play the main role. In the subsequent section we prove its generalization to higher rank actions by toral automorphisms.

LEMMA 4.1 (Katznelson, [26]). *Let A be an $N \times N$ matrix with integer coefficients. Assume that $\mathbb{R}^N$ splits as $\mathbb{R}^N = V \oplus V'$ with V and V' invariant under A and such that $A|_V$ and $A|_{V'}$ have no common eigenvalues. If $V \cap \mathbb{Z}^N = \{0\}$, then there exists a constant γ such that $d(n, V) \geq \gamma \|n\|^{-N}$ for all $n \in \mathbb{Z}^N$ where $\|\cdot\|$ is Euclidean norm and d is Euclidean distance.*

Remark 4. This can be viewed as a version of the Liouville's theorem about rational approximation of algebraic irrationals; i.e., $|\alpha - \frac{m}{n}| \geq C n^{-N}$ for any nonzero integers m and n, where α is an irrational first-order root of an integer polynomial of degree N. The proof of this classical result inspires the proof of Lemma 4.1 in [26] which we summarize here since it gives some insight on arithmetic vs. dynamical properties of toral automorphisms.

Proof. Any polynomial p sufficiently close to the minimal polynomial f of A on V satisfies the condition $p(A)n \neq 0$ for all $n \in \mathbb{Z}^N, n \neq 0$, because its null space is contained in V and $V \cap \mathbb{Z}^N = \{0\}$ by assumption. Then one can construct a polynomial f_Q with rational coefficients of that kind. The choice is made as $|a_j - r_j/q| \leq \frac{1}{qQ}$, where a_j are coefficients of f, r_j/q coefficients of f_Q and $q \leq Q^k$. Since A is an integer matrix we have $\|f_Q(A)n\| > \frac{1}{q}$ for any nonzero n. Now if n_V is the projection of n to V, then

$$f_Q(A)n = f_Q(A)(n - n_V) + (f_Q(A) - f(A))n_V.$$

This implies $\frac{1}{q} \leq C(d(n, V) + \frac{\|n\|}{qQ})$. Then by choosing $Q = C\|n\|$, where C is a constant depending on A, the estimate follows:

$$d(n, V) > \frac{1}{Cq} \geq \frac{1}{CQ^k} > C_1\|n\|^{-k} > C_1\|n\|^{-N}$$

with C_1 being a positive constant depending only on A. □

Remark 5. In particular, if A is ergodic and $V = V_3 \oplus V_2$ from (4.1), then $V \cap \mathbb{Z}^N = \{0\}$. Then the above lemma implies for $n \in \mathbb{Z}^N$:

$$\|\pi_1(n)\| \geq \gamma \|n\|^{-N},$$

where $\pi_1(n)$ is the projection of n to V_1, the expanding subspace for A.

Obstructions to solving a one-cohomology equation over a hyperbolic toral automorphism in C^∞ category are sums of Fourier coefficients of the given function along a dual orbit of the automorphism. Equivalently, obstructions are sums along the periodic orbits of the automorphism. This is the content of the Livšic theorem for hyperbolic toral automorphisms. The same characterization holds however for ergodic toral automorphisms as well, due to the estimate in Lemma 4.1. This is due to Veech [35]. Even though the Livšic theorem generalizes to Anosov diffeomorphisms, for partially hyperbolic non-Anosov diffeomorphisms, it is a rare occurrence that such description of obstructions is possible.

We first obtain a solution to a twisted one-cohomology equation with estimates for the norm of the solution. The proof below follows closely the proof of Veech for the untwisted case [35].

LEMMA 4.2. *Let θ be a C^∞ function on the torus and $\lambda \in \mathbb{C}$, $\lambda \neq 1$. Let A be an integer matrix in* $\mathrm{GL}(N, \mathbb{Z})$ *defining an ergodic automorphism of $\mathbb{T}^N$ such that for all nonzero $n \in \mathbb{Z}^N$, the following sums along the dual orbits are zero, i.e.,*

$$\widehat{\theta}_n = \sum_{i=-\infty}^{+\infty} \lambda^{-(i+1)} \widehat{\theta}_{A^i n} = 0. \tag{4.3}$$

Then the equation

$$\lambda\omega - \omega \circ A = \theta \tag{4.4}$$

has a C^∞ solution ω, and the following estimate:

$$\|\omega\|_{a-\delta} \leq \frac{C_a}{\delta^\nu} \|\theta\|_a \tag{4.5}$$

holds for $\delta > 0$, $\nu = aN + 1$, and $a > \frac{|\log|\lambda||}{\log \rho}$. Here $\rho > 1$ is the expansion rate for A from (4.2). *Thus for $r \geq 0$*

$$\|\omega\|_{C^r} \leq C_r \|\theta\|_{C^{r+\sigma}}, \tag{4.6}$$

where σ is an integer greater than $\max\{N+1, \frac{|\lg|\lambda||}{\lg \rho}\}$.

Proof. Suppose ω is a C^∞ solution to (4.4) and let $\widehat{\omega}_n$ and $\widehat{\theta}_n$ denote Fourier coefficients of ω and θ. Then the equation $\lambda\omega - \omega \circ A = \theta$ in the dual space has the form

$$\lambda\widehat{\omega}_n - \widehat{\omega}_{An} = \widehat{\theta}_n, \quad \forall n \in \mathbb{Z}^N.$$

For $n = 0$, since $\lambda \neq 1$, we can immediately calculate $\widehat{\omega}_0 = \frac{\widehat{\theta}_0}{\lambda - 1}$. For $n \neq 0$ the dual equation has two solutions

$$\widehat{\omega}_n^{\overset{+}{(-)}} = \overset{+}{(-)} \sum_{\substack{i \geq 0 \\ (i \leq -1)}} \lambda^{-(i+1)} \widehat{\theta}_{A^i n}, \qquad n \neq 0.$$

Each sum converges absolutely since θ is C^∞ and all nonzero integer vectors have nontrivial projections to expanding and contracting subspaces for A due to the ergodicity assumption on A. By assumption $\sum^A \widehat{\theta}_n = 0$, $\forall n \neq 0$, i.e., $\widehat{\omega}_n^+ = \widehat{\omega}_n^- \overset{\text{def}}{=} \widehat{\omega}_n$. This gives a formal solution $\omega = \sum \widehat{\omega}_n^+ e_n = \sum \widehat{\omega}_n^- e_n$. We estimate each $\widehat{\omega}_n$ using both of its forms in order to show that ω is C^∞.

If n is mostly contracting, i.e., if $n \hookrightarrow 3(A)$, then we may use the $\widehat{\omega}_n^-$ form for the solution to obtain the following bound on the n-th Fourier coefficient:

$$
\begin{aligned}
|\widehat{\omega}_n| &= \left| \sum_{k\leq 0} \lambda^{-(k+1)} \widehat{\theta}_{A^k n} \right| \leq \sum_{k\leq 0} |\lambda|^{-(k+1)} |\widehat{\theta}_{A^k n}| \\
&\leq \|\theta\|_a \sum_{k\leq 0} |\lambda|^{-(k+1)} |A^k n|^{-a} \leq \|\theta\|_a \sum_{k\leq 0} |\lambda|^{-(k+1)} \|A^k \pi_3(n)\|^{-a} \\
&\leq \|\theta\|_a C^{-a} \sum_{k\leq 0} |\lambda|^{-(k+1)} \rho^{ak} \|\pi_3(n)\|^{-a} \leq C_a \|\theta\|_a |n|^{-a}
\end{aligned}
$$

where $a > \frac{\log|\lambda|}{\log \rho}$.

Similarly, if $n \hookrightarrow 1(A)$, using the form $\widehat{\omega}_n = \widehat{\omega}_n^+$, then the estimate $|\widehat{\omega}_n| \leq C_a \|\theta\|_a |n|^{-a}$ holds, if $a > \frac{\log|\lambda|^{-1}}{\log \rho}$.

If $n \hookrightarrow 2(A)$ and $|\lambda| \geq 1$, using the form $\widehat{\omega}_n^+$ of the solution, then it follows that

$$
\begin{aligned}
|\widehat{\omega}_n| &\leq \|\theta\|_a \sum_{k\geq 0} |\lambda|^{-(k+1)} |A^k n|^{-a} \\
&\leq \|\theta\|_a C^{-a} \sum_{k\geq 0} |\lambda|^{-(k+1)} (1+k)^{N\alpha} \|\pi_2(n)\|^{-a}.
\end{aligned} \tag{4.7}
$$

However, the above sum need not converge. This is where we use again the fact that A is ergodic. Namely, according to the remark following Lemma 4.1 no integer vector can stay mostly in the neutral direction for too long. After the time which is approximately $\ln|n|$ the expanding direction takes over. More precisely, from Lemma 4.1 it follows that $\|\pi_1(n)\| \geq \gamma |n|^{-N}$ for some γ and all n. Therefore

$$
|A^k n| \geq \|A^k \pi_1(n)\| \geq C\rho^k \|\pi_1(n)\| \geq \gamma C \rho^k |n|^{-N} \geq \gamma C \rho^{k-k_0} |n|
$$

for $k \geq k_0$ and $k_o = [\frac{(1+N)\log|n|}{\log \rho}] + 1$. This fact can be used to estimate all but finitely many terms of the series in (4.7). For the rest the polynomial estimate in (4.2) for vectors in V_2 holds. Hence

$$
\begin{aligned}
|\widehat{\omega}_n| \leq \|\theta\|_a \sum_{k=0}^{k_0-1} |\lambda|^{-(k+1)} |k|^{Na} \|\pi_2(n)\|^{-a} \\
+ C\|\theta\|_a \sum_{k=k_0}^{\infty} |\lambda|^{-(k+1)} \rho^{-a(k-k_0)} |n|^{-a}.
\end{aligned}
$$

Thus using that $n \hookrightarrow 2(A)$ and $|\lambda| > 1$ we have

$$
|\widehat{\omega}_n| \leq C\|\theta\|_a |k_0|^{Na+1} |n|^{-a} + C\|\theta\|_a |n|^{-a}.
$$

Now by choice of k_0, $k_0 \sim \log|n|$. This implies the following estimate:

$$|\widehat{\omega}_n| \leq C_a (\log|n|)^{Na+1} |n|^{-a} \|\theta\|_a.$$

For $|\lambda| < 1$ the same estimate follows using the second form for $\widehat{\omega}_n$, i.e., the negative sum and the fact that A^{-1} is also an ergodic toral automorphism thus going backwards in time, the contracting direction takes over; that is, we use the Lemma 4.1 for A^{-1}. Therefore for all $n \in \mathbb{Z}^N$ we have

$$|\widehat{\omega}_n|\, |n|^{a-\delta} \leq \frac{C_a}{\delta^{\nu}} \|\theta\|_a$$

for $a > \frac{|\log|\lambda||}{\log\rho}$ and $\delta > 0$. This implies the estimate (4.5) for $\|\omega\|_{a-\delta}$. The estimate for C^r norms follows immediately using the norm comparison from Section 4.1. In particular if θ is C^∞, then ω is also C^∞. □

4.3. *Orbit growth for the dual action.* In this section the crucial estimates for the exponential growth along individual orbits of the dual action are obtained. They may be viewed as a generalization of Lemma 4.1 to higher rank actions by toral automorphisms. The implications of the growth estimates to certain estimates of the C^r norms of specifically defined functions are immediate and are formulated and proved in Lemma 4.3.

The existence of such estimates in case of $\mathbb{Z}^d$ actions with $d \geq 2$ relies fundamentally on the higher rank assumption.

LEMMA 4.3. *Let α be a $\mathbb{Z}^d$ action by ergodic automorphisms of $\mathbb{T}^N$. Then there exist constants $\tau > 0$ and $C > 0$ depending on the action only, such that*

a) *For every integer vector $n \in \mathbb{Z}^N$ and for all $k \in \mathbb{Z}^d$,*

$$|\alpha^k n| \geq C \exp\{\tau\|k\|\} |n|^{-N}.$$

b) *For any C^∞ function φ on the torus, any nonzero $n \in \mathbb{Z}^N$ and any vector $y \in \mathbb{R}^d$ the following sums:*

$$S_K(\varphi, n) = \sum_{k \in K} y^k \widehat{\varphi}_{\alpha^k n},$$

where $y^k \stackrel{\text{def}}{=} \prod_{i=1}^d y_i^{k_i}$, converge absolutely for any $K \subset \mathbb{Z}^d$.

c) *Assume in addition to the assumptions in* b) *that for an $n \in \mathbb{Z}^N$ and for every $k \in K = K(n) \subset \mathbb{Z}^d$ we have $P(\|k\|)|\alpha^k n| \geq |n|$ where P is a polynomial of degree N. Then*

$$|S_K(\varphi, n)| \leq C_{a,y} \|\varphi\|_a |n|^{-a+\kappa_{y,\alpha}} \tag{4.8}$$

for any $a > \kappa_{y,\alpha} \stackrel{\text{def}}{=} \frac{N+1}{\tau} \sum_{i=1}^d |\log|y_i||$.

d) *If the assumptions of* c) *are satisfied for every $n \in \mathbb{Z}^N$, then the function*

$$S(\varphi) \stackrel{\text{def}}{=} \sum_{n \in \mathbb{Z}^N} S_{K(n)}(\varphi, n) e_n$$

is a C^∞ function if φ is. Moreover, the following norm comparison holds:

$$\|S(\varphi)\|_{C^r} \leq C_{r,y}\|\varphi\|_{C^{r+\sigma}}, \tag{4.9}$$

for $r > 0$ and any $\sigma > N + 2 + [\kappa_{y,\alpha}]$.

Proof of a). When $d = 1$, i.e., the action is given by a single ergodic toral automorphism, then the inequality a) is an immediate consequence of Lemma 4.1.

When $d \geq 2$, we first notice that it is enough to obtain the constant τ and to show the exponential estimate in a) in the semisimple case, i.e., when the action is generated by matrices $A_1, \dots, A_d$ which are simultaneously diagonalizable over $\mathbb{C}$. If the action is not semisimple, only polynomial growth may occur in addition, thus the same estimate holds with slightly smaller τ and with possibly larger C (for more details see [18]).

Here we give the proof in the case when the action is irreducible, which shows the main idea, but is technically simpler and we refer to [18] for the proof in the general case.

In the case when the action is irreducible, we may project a nontrivial $n \in \mathbb{Z}^N$ to the Lyapunov directions corresponding to nonzero Lyapunov exponents of the action. Lyapunov exponents are defined as

$$\chi_i(k) = \sum_{j=1}^{d} k_j \ln|\lambda_{ij}|,$$

where $k = (k_1, \dots, k_d) \in \mathbb{Z}^d$, $i = 1, \dots, r$, and $\lambda_{1j}, \dots, \lambda_{rj}$ are the eigenvalues of A_j for $j = 1, \dots, d$. Individual Lyapunov directions are irrational, and, due to the irreducibility assumption, each of the projections of the vector n to the Lyapunov directions is nontrivial. Thus one may apply Lemma 4.1 to each of these projections and choose τ as the minimum of the function $f(t) := \max_i \chi_i(t)$ for t on the unit sphere in $\mathbb{R}^d$. Let us assume that the minimal value τ of the function f on the unit sphere is achieved at some point t^0.

Claim. $\tau > 0$.

Proof of the claim. If $\tau \leq 0$ then for all $i = 1, \dots, r$ we have $\chi_i(t^0) \leq 0$. Since $\sum_{i=1}^{r} \chi_i(t) = 0$ for all t, it follows that $\chi_i(t^0) = 0$ for all $i = 1, \dots, r$ and consequently $\tau = 0$. Since t^0 is a point on the unit circle in particular it is not zero, this implies existence of a line l in $\mathbb{R}^d$ such that for all points on l all Lyapunov functionals take value zero. It is a result of Kronecker [27] which states that an integer matrix with all eigenvalues on the unit circle has to have all eigenvalues roots of unity. The line l then cannot contain any integer vectors $k \in \mathbb{Z}^d$ because of the ergodicity of all the nontrivial elements α^k of the action. Thus there are lattice points arbitrary close to l, in other words, there is a sequence $k^j = (k_1^j, \dots, k_d^j) \in \mathbb{Z}^d$ such that α^{k_j} is a sequence of integer matrices whose eigenvalues all tend to 1 in absolute value as $j \to \infty$. At this point we may use any of the improvements of the Kronecker's result, for example [2], which implies that for a fixed dimension N of the torus, there exists a number $b(N) > 1$ such that any integer matrix on $\mathbb{T}^N$ with all eigenvalues in absolute value less than $b(N)$, has all eigenvalues roots of unity. Thus we conclude that for some $j > 0$ the eigenvalues of α^{k_j} are roots of unity, which again contradicts the assumption on ergodicity of all elements of the action. Therefore, τ is strictly positive.

Note. The constant τ as defined above depends on the action. However by [2] it is possible to choose τ independently of the action as $\tau = \log(1 + \frac{1}{52N \log 6N})$ where N is the dimension of the torus.

Now that we have $\tau > 0$, we proceed by using the following norm:

$$\|\alpha^k n\|_\chi = \sum_{i=1}^{r} \|n_i\| \exp \chi_i(k),$$

where n_i are projections of n to the corresponding Lyapunov directions. By applying Lemma 4.1, we obtain the needed estimate

$$|\alpha^k n| \geq C \|\alpha^k n\|_\chi \geq C \exp\{\tau \|k\|\} \min_i \|n_i\| \geq C \exp\{\tau \|k\|\} \|n\|^{-N}.$$

Proof of b). The claim in b) follows from the estimate in a) and the fast decay of Fourier coefficients:

$$|S_K| \leq \|\varphi\|_a \sum_{k \in K} |y|^k |\alpha^k n|^{-a} \leq C_a \|\varphi\|_a |n|^{Na} \sum_{k \in K} (|y|^{\operatorname{sgn} k} e^{-a\tau})^{\|k\|}.$$

The last sum clearly converges providing $a > \max_{i=1,\dots,d} \frac{|\log |y_i||}{\tau}$, and for a C^∞ function φ we can choose a as large as needed.

Proof of c). Here we use the norm where $\|k\| = \max_{1 \leq i \leq d} |k_i|$ for $k = (k_1, \dots, k_d) \in \mathbb{Z}^d$. Let us choose an appropriate $k_0 \in \mathbb{N}$ in order to split the sum $S_K(\varphi, n)$ into two sums: the finite one with $\|k\| < k_0$ where the polynomial estimate for $|\alpha^k n|$ can be used, and the infinite one for $\|k\| > k_0$ where the exponential estimate obtained in a) prevails. The estimate in a) allows us to locate k_0 so that it is not very large. Namely, as in Lemma 4.2 we have

$$\begin{aligned} |\alpha^k n| &\geq C\gamma \exp\{\tau \|k\|\} |n|^{-N} \\ &\geq C\gamma \exp\{\tau(\|k\| - k_0)\} \exp\{\tau k_0\} |n|^{-N} \\ &\geq C \exp\{\tau(\|k\| - k_0)\} |n| \end{aligned}$$

providing $\|k\| \geq k_0 > \frac{N+1}{\tau} \log |n|$.

After choosing $k_0 = [\frac{N+1}{\tau} \log |n|] + 1$, we split the sum S_K as follows:

$$S_K(\varphi, n) = S_{\|k\| < k_0}(\varphi, n) + S_{\|k\| \geq k_0}(\varphi, n).$$

To estimate the first sum we use the polynomial estimate for $k \in K(n)$:

$$\begin{aligned}|S_{\|k\|<k_0}(\varphi,n)| \le& C\|\varphi\|_a \sum_{\{k\in K:\|k\|<k_0\}} |y|^k P(\|k\|)^a |n|^{-a}\\ \le& C\|\varphi\|_a k_0^d \prod_{i=1}^{d} \max\{|y_i|,|y_i|^{-1}\}^{k_0} k_0^{daN}|n|^{-a}\\ \le& C_a\|\varphi\|_a k_0^{d(1+aN)}|n|^{-a+\kappa_1}\\ \le& C_a\|\varphi\|_a(\log|n|)^{d(1+aN)}|n|^{-a+\kappa_1}\\ \le& \frac{C_a}{\delta^\nu}\|\varphi\|_a|n|^{-a+\kappa_1+\delta}\end{aligned}$$

for any $\delta > 0$ and any $a > \kappa_1 = \frac{N+1}{\tau}\sum_{i=1}^{d} |\log|y_i||$. The constant $\nu = \nu(N,d,a)$ depends only on the dimension of the torus, rank of the action, and a. For the second sum we use the exponential estimate obtained in a):

$$\begin{aligned}|S_{\|k\|\ge k_0}(\varphi,n)| &\le C\|\varphi\|_a \sum_{\{k\in K:\|k\|\ge k_0\}} |y|^k|\alpha^k n|^{-a}\\ &\le C\|\varphi\|_a|n|^{-a}\sum_{i=1}^{d}\sum_{\{k\in K:|k_i|\ge k_0\}} |y_i|^{k_i}e^{-a\tau(|k_j|-k_0)}\prod_{i\ne j=1}^{j=d}|y_j|^{k_j}e^{-a\tau|k_j|}\\ &\le C\|\varphi\|_a|n|^{-a}\sum_{i=1}^{d} C_{a,y}\max\{|y_i|,|y_i|^{-1}\}^{k_0}\\ &\le C_{a,y}\|\varphi\|_a|n|^{-a+\kappa_2}\end{aligned}$$

for any $a > \kappa_2 = \frac{N+1}{\tau}\max_{1\le i\le d}|\log|y_i||$.

By combining the estimates obtained above we have

$$|S_K(\varphi,n)| \le C_{a,y}\|\varphi\|_a|n|^{-a+\kappa_{y,\alpha}}$$

for any $a > \kappa_{y,\alpha} \stackrel{\text{def}}{=} \frac{N+1}{\tau}\sum_{i=1}^{d}|\log|y_i||$.

Proof of d). If for every n the set $K(n)$ is chosen to satisfy the assumptions of c) then the estimate (4.8) clearly implies

$$\|S(\varphi)\|_{a-\kappa_{y,\alpha}} \le C_{a,y}\|\varphi\|_a,$$

which has as a consequence the C^r estimate (4.9) with the loss of $N+2+[\kappa_{y,\alpha}]$ derivatives. □

4.4. *Higher rank trick*: *solution to a twisted coboundary equation over a higher rank action by toral automorphisms.* If A and B commute and there exists a solution to (3.12), then it is immediate that $L(R_A,R_B) := \Delta^B R_A - \Delta^A R_B = 0$. In this section we show that if every nontrivial $A^l B^k$ is ergodic then the condition

$L(R_A, R_B) = 0$ is not only necessary but is also sufficient for the existence of a solution to (3.12). This argument is essentially the argument used in [23] adapted to the twisted and not necessarily semisimple situation. After showing the vanishing of obstructions, the tame estimates for the common solution of (3.12) follow from Lemma 4.2.

LEMMA 4.4. *If* $L(R_A, R_B) = 0$ (*where* R_A, R_B *are* C^∞ *maps described in Section* 3.3, *see* (3.5) *and* (3.13)), *then the equations* (3.12)

$$\Delta^A \Omega = R_A, \qquad \Delta^B \Omega = R_B$$

have a common C^∞ *solution satisfying*

$$\|\Omega\|_{C^r} \le C_r \|R_A, R_B\|_{C^{r+\sigma}}, \tag{4.10}$$

for any $r > 0$ *and* $\sigma > M_0 = \max\{N+2, [m_0(A, B)]\}$, *where* $m_0(A, B)$ *is a positive constant defined explicitly in* (4.27) *below*, *depending on the eigenvalues of* A *and* B.

Proof. (i) *The semisimple case.* Assume that A and B are simultaneously diagonalizable. Then the equations $L(R_A, R_B) \equiv 0$ and (3.12) split into finitely many equations of the form

$$L(\theta, \psi) = \Delta^\mu \theta - \Delta^\lambda \psi \equiv 0 \tag{4.11}$$

and

$$\Delta^\lambda \omega = \theta, \qquad \Delta^\mu \omega = \psi, \tag{4.12}$$

where θ and ψ are C^∞ functions and λ and μ are corresponding eigenvalues of A and B, respectively.

The assumption $L(\theta, \psi) \equiv 0$ implies $\Delta^\mu \theta \equiv \Delta^\lambda \psi$, which after passing to the dual action implies

$$\sum\nolimits^B (\Delta^\mu \widehat{\theta}_n) = \sum\nolimits^B (\Delta^\lambda \widehat{\psi}_n)$$

$$\sum\nolimits^A (\Delta^\lambda \widehat{\psi}_n) = \sum\nolimits^A (\Delta^\mu \widehat{\theta}_n).$$

Consider now the first equation above. Since all the sums involved converge absolutely we have

$$\Delta^\mu \sum\nolimits^B \widehat{\theta}_n = 0$$

which implies

$$\sum\nolimits^B (\Delta^\lambda \widehat{\psi}_n) = 0.$$

Using Lemma 4.3 b), this implies that the obstruction for ψ is not only multiplied by μ under the action of B, but is also multiplied by λ under the action of A, i.e., $\lambda \sum^B \widehat{\psi}_n = \sum^B \widehat{\psi}_{An}$. By iterating this equation we obtain

$$\lambda^k \sum\nolimits^B \widehat{\psi}_n = \sum\nolimits^B \widehat{\psi}_{A^k n},$$

for every $k \in \mathbb{Z}$. Therefore

$$\sum_{k\in\mathbb{Z}} \lambda^k \sum\nolimits^B \widehat{\psi}_n = \sum_{k\in\mathbb{Z}} \sum\nolimits^B \widehat{\psi}_{A^k n}. \tag{4.13}$$

The series in the left-hand side of (4.13) does not converge unless $\sum^B \widehat{\psi}_n = 0$ while the right-hand side of (4.13) converges absolutely by Lemma 4.3 b). Therefore $\sum^B \widehat{\psi}_n = 0$, for all $n \neq 0$. Similarly $\sum^A \widehat{\theta}_n = 0$, for all $n \neq 0$.

By Lemma 4.2, the formal solutions of each equation in (4.12) are C^∞ functions. Moreover, they coincide. Indeed, if ω solves the second equation, i.e., $\Delta^\mu \omega = \psi$, then

$$\Delta^\lambda \Delta^\mu \omega = \Delta^\lambda \psi = \Delta^\mu \theta.$$

Since operators Δ^λ and Δ^μ commute this implies

$$\Delta^\mu (\Delta^\lambda \omega - \theta) = 0.$$

As in Lemma 3.2 the ergodicity of A and B implies that Δ^μ and Δ^λ are injective operators on C^∞. Therefore $\Delta^\lambda \omega = \theta$, i.e., ω solves the first equation as well.

(ii) *The general case.* If A and B are not simultaneously diagonalizable then choose a basis in which A has its Jordan normal form. Since A and B commute any root space for A is B invariant. Therefore, in this basis B has a block diagonal form. Let $J_A = (a_{ij})$ be an $m \times m$ matrix which consists of blocks of A corresponding to the eigenvalue λ; i.e., let $a_{ii} = \lambda$ for all $i = 1, \dots, m$ and $a_{i,i+1} = *_i \in \{0, 1\}$ for all $i = 1, \dots, m-1$. Let $J_B = (b_{ij})$ be the corresponding block of B where $b_{ii} = \mu$ for all $i = 1, \dots, m$ (μ is an eigenvalue of B) and $b_{ij} = 0$ for all $m \geq i > j \geq 1$. Then because of the fact that A and B commute, by simply comparing coefficients, it is easy to obtain the following relation which the coefficients of A and B must satisfy

$$*_i b_{ki} = *_k b_{k+1,i+1} \tag{4.14}$$

for any fixed k between 1 and $m-1$ and for all $i = k+1, \dots, m-1$.

For any such pair of blocks J_A and J_B the equations (3.12) split into equations of the form

$$\begin{aligned} J_A \Omega - \Omega \circ A &= \Theta \\ J_B \Omega - \Omega \circ B &= \Psi \end{aligned} \tag{4.15}$$

and the condition $L(R_A, R_B) = 0$ splits as

$$J_B \Theta - \Theta \circ B = J_A \Psi - \Psi \circ A. \tag{4.16}$$

Let the coordinate functions of Θ and Ψ be θ_i and ψ_i ($i = 1, \dots, m$), respectively. Then we look for functions ω_i ($i = 1, \dots, m$) which solve the equations above and whose norm can be compared to the norm of Θ and Ψ.

Since J_A and J_B are upper diagonal, it is easy to obtain ω_m. Namely, from (4.16) we have that θ_m and ψ_m satisfy the condition $L(\theta_m, \psi_m) = \Delta^\mu \theta_m - \Delta^\lambda \psi_m = 0$. Therefore, using part (i), there exist ω_m which solves simultaneously the last of m pairs of equations in (4.15), namely the equations $\Delta^\lambda \omega_m = \theta_m$ and $\Delta^\mu \omega_m = \psi_m$. Moreover, the estimate

$$\|\omega_m\|_{a-\delta} \leq \frac{C_a}{\delta^\nu} \|\theta_m, \psi_m\|_a \leq \frac{C_a}{\delta^\nu} \|\Theta, \Psi\|_a \tag{4.17}$$

follows from Lemma 4.2 for $a \geq \max\{\frac{|\log|\lambda||}{\log \rho}, \frac{|\log|\mu||}{\log \eta}\}$ where ρ and η are growth rates in the hyperbolic direction corresponding to A and B, respectively. Now the $(m-1)$-st pair of equations in (4.15) is

$$\begin{aligned} \Delta^\lambda \omega_{m-1} + *_{m-1}\omega_m &= \theta_{m-1} \\ \Delta^\mu \omega_{m-1} + b_{m-1,m}\omega_m &= \psi_{m-1}, \end{aligned} \tag{4.18}$$

while the cocycle condition for θ_{m-1} and ψ_{m-1} from (4.16) is

$$\Delta^\mu \theta_{m-1} + b_{m-1,m} = \Delta^\lambda \psi_{m-1} + *_{m-1}\psi_{m-1}. \tag{4.19}$$

By substituting $\theta_m = \Delta^\lambda \omega_m$ and $\psi_m = \Delta^\mu \omega_m$ into (4.19), we obtain that

$$L(\theta_{m-1} - *_{m-1}\omega_m, \psi_{m-1} - b_{m-1,m}\omega_m) = 0,$$

where the norm of both functions, on which the operator L acts, due to the estimate (4.17), can be bounded by the norm of Θ and Ψ with a small loss. Now we may use the part (i) again to conclude that there exists some ω_{m-1} solving the system (4.18) and such that the following estimate holds:

$$\|\omega_{m-1}\|_{a-2\delta} \leq \frac{C_a}{\delta^{2\nu}} \|\Theta, \Psi\|_a. \tag{4.20}$$

Now we proceed by induction. Fix k between 1 and $m-2$ and assume that for all $i \geq k$, we have obtained a solution ω_i with the appropriate estimate, i.e., for every $i = k+1, \dots, m$ we have a C^∞ function ω_i which solves the i-th pair of equations of (4.15):

$$\begin{aligned} \Delta^\lambda \omega_i + *_i\omega_{i+1} &= \theta_i \\ \Delta^\mu \omega_i + \sum_{l=i+1}^{m} b_{il}\omega_l &= \psi_i \end{aligned} \tag{4.21}$$

and that the following estimates hold:

$$\|\omega_i\|_{a-(m-i+1)\delta} \leq \frac{C_a}{\delta^{(m-i+1)\nu}} \|\Theta, \Psi\|_a. \tag{4.22}$$

We wish to find ω_k that solves the k-th pair of equations in (4.15):

$$\Delta^\lambda \omega_k + *_k \omega_{k+1} = \theta_k \tag{4.23}$$
$$\Delta^\mu \omega_k + \sum_{i=k+1}^{m} b_{ki}\omega_i = \psi_k,$$

providing that the k-th equation in (4.16) is satisfied by θ_k and ψ_k; i.e.,

$$\Delta^\mu \theta_k + \sum_{i=k+1}^{m} b_{ki}\theta_i = \Delta^\lambda \psi_k + *_k \psi_{k+1}. \tag{4.24}$$

Now we use the fact that all the subsequent pairs of equations are solved; i.e., we substitute all θ_i for $i = k+1, \dots, m$ and the ψ_{k+1} into (4.24) using their expression as in (4.21). This implies

$$\Delta^\mu \theta_k + \sum_{i=k+1}^{m} (b_{ki}\Delta^\lambda \omega_i + *_i b_{ki}\omega_{i+1})$$
$$= \Delta^\lambda \psi_k + *_k \Delta^\mu \omega_{k+1} + \sum_{i=k+1}^{m} b_{k+1,i+1}\omega_{i+1}.$$

Since A and B commute, we can use the equations (4.14) for the coefficients and the linearity of operators Δ^λ and Δ^μ to simplify the above expression to

$$\Delta^\mu(\theta_k - *_k\omega_{k+1}) = \Delta^\lambda\Big(\psi_k - \sum_{i=k+1}^{m} b_{ki}\omega_i\Big).$$

Thus the functions $\theta_k - *_k\omega_{k+1}$ and $\psi_k - \sum_{i=k+1}^{m} b_{ki}\omega_i$ satisfy the solvability condition $L(\theta_k - *_k\omega_{k+1}, \psi_k - \sum_{i=k+1}^{m} b_{ki}\omega_i) = 0$, they are C^∞ and therefore we may use the part (i) again to conclude that the pair of equations (4.23) has a common C^∞ solution ω_k. As a consequence of assumptions (4.22) this solution satisfies the estimate

$$\|\omega_k\|_{a-(m-k+1)\delta} \le \frac{C_a}{\delta^{(m-k+1)\nu}} \|\Theta, \Psi\|_a. \tag{4.25}$$

Since k is an arbitrary integer between 1 and $m-1$ it follows that there exists a solution Ω to (4.15) providing that the condition (4.16) is satisfied. This can be repeated for all corresponding blocks of A and B. Since the maximal size of a Jordan block is bounded by N, we obtain the following estimate for the norm of the C^∞ solution Ω of the system $\Delta^A\Omega = R_A$, $\Delta^B\Omega = R_B$:

$$\|\Omega\|_{a-N\delta} \le \frac{C_a}{\delta^{N\nu}} \|R_A, R_B\|_a. \tag{4.26}$$

Here, a is any number satisfying $a > m_0 = m_0(A, B)$, and

$$m_0(A, B) \overset{\text{def}}{=} \max_i \max\left\{\frac{|\log|\lambda_i||}{\log\rho}, \frac{|\log|\mu_i||}{\log\eta,}\right\} \tag{4.27}$$

where the first maximum is taken over all pairs of eigenvalues λ_i, μ_i of A and B, respectively. Of course, the constant C_a has been changing throughout the procedure. It only depends on the matrices A and B and the dimension N of the torus, besides a.

As before, by fixing δ and using the norm comparison, this implies the estimate (4.10) for C^r norm as well, with the loss of $\max\{N+2, [m_0(A, B)]\}$ derivatives. □

4.5. *Construction of projections.* As before, let α denote the linear action generated by A and B. Given smooth functions θ, ψ and φ, which satisfy the equation $L(\theta, \psi) = \varphi$, we construct a *tame* solution for the same equation, namely we construct C^∞ functions $\mathscr{E}\theta$ and $\mathscr{E}\psi$ such that $L(\mathscr{E}\theta, \mathscr{E}\psi) = \varphi$ and such that their norms are bounded by norms of φ with fixed loss of regularity. The tame estimates are obtained by using the growth estimates for the orbits of the dual action from Section 4.3.

Applying Lemma 4.5 inductively, Lemma 4.6 gives a tame solution for the equation $L(\mathscr{E}\Theta, \mathscr{E}\Psi) = \Phi$ given that $L(\Theta, \Psi) = \Phi$ for some C^∞ maps Θ, Ψ, Φ on the torus.

The operator L is linear thus in the set-up of Section 3.3, the maps $\mathscr{P}R_A := R_A - \mathscr{E}R_A$ and $\mathscr{P}R_B := R_B - \mathscr{E}R_B$ are C^∞ maps of the same order as R, so that the pair $(\mathscr{P}R_A, \mathscr{P}R_B)$ is a projection of (R_A, R_B) to the space of pairs of maps which induce α twisted cocycles over α. In other words, the projections satisfy $L(\mathscr{P}R_A, \mathscr{P}R_B) = 0$ and are of the same order as R_A and R_B. The appropriate estimates are contained in Lemmas 4.5 and 4.6.

Remark. Even though we use the same letter $\mathscr{P}$ to denote projections of both maps R_A and R_B we note that the construction for $\mathscr{P}R_A$ and $\mathscr{P}R_B$ is not the same for both maps. So in this context $\mathscr{P}$ should be viewed as a projection of the *pair* (R_A, R_B) into the space of pairs which generate α twisted cocycle over the linear action α.

The construction of projections gives an approximate solution to the linearized equation (3.4). Namely, since $L(\mathscr{P}R_A, \mathscr{P}R_B) = 0$, Lemma 4.4 implies that there exists a common smooth solution to the equations $\Delta^A\Omega = \mathscr{P}R_A$ and $\Delta^B\Omega = \mathscr{P}R_B$ which is of the same order as (R_A, R_B). Then Ω is an approximate solution of (3.12) because the error is of the order of $(\mathscr{E}R_A, \mathscr{E}R_B)$ which is of the order of $L(R_A, R_B)$, and this is small with respect to (R_A, R_B) as demonstrated in Lemma 4.7. This estimate for the error is then used in Section 5 as a base for KAM iteration.

To give some intuition behind the construction of projections we recall first that from Lemma 4.4 the kernel of the operator L coincides with the space of twisted coboundaries, so the obstructions in the dual space described in Lemma 4.2 all vanish for any pair of maps in the kernel of L. Thus to construct projection to the kernel of L we simply subtract the obstructions (the part we subtract is

$(\mathcal{E}\theta, \mathcal{E}\psi)$ and the rest $(\mathcal{P}\theta, \mathcal{P}\psi)$ is in the kernel of L). The obstructions are defined on dual orbits of B (or A) and are invariant under B (or A), so we have some freedom in choosing the projection in such a way that the estimates are optimal. To construct $(\mathcal{E}\theta, \mathcal{E}\psi)$ we concentrate the obstruction on each dual orbit for one of the generators, say B, on a single point of the dual orbit and we choose this point so that it has large expanding and large contracting components with respect to the action of B. Then the needed comparison that $\mathcal{E}\psi$ projection is of the same order as $L(\theta, \psi)$ comes from the fact that obstructions of ψ along the dual orbit of B are comparable to the sum of Fourier coefficients of $L(\theta, \psi)$ along the two-dimensional dual orbit of the action generated by A and B. This is where the estimates for the growth along the dual orbits of the full $\mathbb{Z}^2$ action generated by A and B from Lemma 4.3, as well as the assumption that all nontrivial $A^k B^l$ are ergodic, play the crucial role.

LEMMA 4.5. *Let θ, ψ, and φ be C^∞ functions such that $L(\theta, \psi) = \Delta^\mu \theta - \Delta^\lambda \psi = \varphi$, then it is possible to split θ and ψ as*

$$\theta = \mathcal{P}\theta - \mathcal{E}\theta$$
$$\psi = \mathcal{P}\psi + \mathcal{E}\psi$$

so that $L(\mathcal{P}\theta, \mathcal{P}\psi) = 0$, $L(\mathcal{E}\theta, \mathcal{E}\psi) = \varphi$ and the following estimates hold:

$$\|\mathcal{E}\theta, \mathcal{E}\psi\|_{C^r} \le C \|\varphi\|_{C^{r+\sigma}} \tag{4.28}$$

for any $r > 0$ and any $\sigma > \widetilde{M}_{\lambda,\mu}$ and

$$\|\mathcal{P}\theta, \mathcal{P}\psi\|_{C^r} \le C \|\theta, \psi\|_{C^{r+\sigma}} \tag{4.29}$$

for any $r > 0$ and any $\sigma > \dot{M}_{\lambda,\mu}$. As λ and μ are eigenvalues of A and B, constants $\widetilde{M}_{\lambda,\mu}$ and $\dot{M}_{\lambda,\mu}$ depend only on A, B and the dimension of the torus and are precisely defined below (*see* (4.37) *and* (4.44)).

Proof. (i) *Construction of $\mathcal{P}\theta, \mathcal{P}\psi, \mathcal{E}\theta$ and $\mathcal{E}\psi$.* Let us call $n \in \mathbb{Z}^2$ minimal and denote it by $n_{\min}$ if n is the lowest point on its B orbit in the sense that $n \hookrightarrow 3(B)$ and $Bn \hookrightarrow 1, 2(B)$ (for the definition of "$\hookrightarrow$", see Section 4.1). There is one such minimal point on each nontrivial dual B orbit, we choose one on each dual B orbit and denote it by $n_{\min}$.

Now let $\mathcal{E}\psi \stackrel{\text{def}}{=} \sum \widehat{\mathcal{E}\psi_n} e_n$ where

$$\widehat{\mathcal{E}\psi_n} \stackrel{\text{def}}{=} \begin{cases} \mu \sum^B \widehat{\psi}_n, & n = n_{\min}, \\ 0, & \text{otherwise} \end{cases} \tag{4.30}$$

for $n \neq 0$ and $\widehat{\mathcal{E}\psi_0} \stackrel{\text{def}}{=} 0$. Let $\omega \stackrel{\text{def}}{=} \sum \widehat{\omega}_n e_n$ where

$$\widehat{\omega}_n \stackrel{\text{def}}{=} \begin{cases} \sum_+^B \widehat{\psi}_n, & n \hookrightarrow 1, 2(B) \\ -\sum_-^B \widehat{\psi}_n, & n \hookrightarrow 3(B) \end{cases}$$

for $n \in \mathbb{Z}^N \setminus \{0\}$ and $\widehat{\omega}_0 = (\mu - 1)^{-1}\widehat{\psi}_0$. Let

$$\mathcal{P}\psi \stackrel{\text{def}}{=} \Delta^{\mu}\omega = \mu\omega - \omega \circ B. \tag{4.31}$$

Then it is easy to check that

$$\psi = \mathcal{P}\psi + \mathcal{E}\psi.$$

In (ii) and (iii) below we will show that both $\mathcal{P}\psi$ and $\mathcal{E}\psi$ are smooth functions such that $\mathcal{P}\psi$ is of the order of ψ, and $\mathcal{E}\psi$ is the order of φ.

Let us define $\mathcal{P}\theta$ as

$$\mathcal{P}\theta \stackrel{\text{def}}{=} \Delta^{\lambda}\omega. \tag{4.32}$$

Then $L(\mathcal{P}\theta, \mathcal{P}\psi) = 0$ since operators Δ^{λ} and Δ^{μ} commute due to the commutativity of the generators A and B. Therefore, by defining $\mathcal{E}\theta$ as

$$\mathcal{E}\theta \stackrel{\text{def}}{=} \theta - \mathcal{P}\theta, \tag{4.33}$$

we obtain $L(\mathcal{E}\theta, \mathcal{E}\psi) = \varphi$; i.e.,

$$\Delta^{\mu}\mathcal{E}\theta = \Delta^{\lambda}\mathcal{E}\psi + \varphi. \tag{4.34}$$

Since operators Δ^{μ} and Δ^{λ} are bounded, if $\mathcal{E}\psi$ is proved to be smooth with norm comparable to some norm of φ, then by Lemma 4.2 the same holds true for $\mathcal{E}\theta$ as a solution of equation (4.34). (The operator Δ^{μ} is injective on C^{∞} whenever $\mu \neq 1$. This fact is contained in the proof of the Lemma 3.2 and is a consequence of the ergodicity of B.)

(ii) *Estimates for $\mathcal{P}\theta$ and $\mathcal{P}\psi$.* The following estimate follows from the definition of ω in (i) and is obtained as in the proof of Lemma 4.2:

$$\|\omega\|_{a-\delta-\kappa_{\mu,B}} \leq \frac{C_a}{\delta^{\nu}} \|\psi\|_a \tag{4.35}$$

for any $a > \kappa_{\mu,B}$ and any $\delta > 0$, where

$$\kappa_{\mu,B} \stackrel{\text{def}}{=} \frac{N+1}{\log \eta} |\log |\mu||. \tag{4.36}$$

The extra loss of $\kappa_{\mu,B}$ appears here because the obstructions $\sum^{B} \widehat{\psi}$ do not vanish and $|\mu|$ may be different than 1, just as in the proof of Lemma 4.3, parts b) and c). Here, $\eta > 1$ is the constant coming from the exponential growth in the hyperbolic direction for B (as ρ is for A in (4.2)). Since operators Δ^{μ} and Δ^{λ} are bounded in any $\|\cdot\|_a$ norm, this also implies the following estimate for $\mathcal{P}\psi$ and $\mathcal{P}\theta$:

$$\|\mathcal{P}\psi, \mathcal{P}\theta\|_{a-\delta-\kappa_{\mu,B}} \leq \frac{C_a}{\delta^{\nu}} \|\psi\|_a,$$

which, in particular, implies the corresponding estimate (4.29) for C^r norms for $\mathcal{P}\psi$ and $\mathcal{P}\theta$ with the loss of

$$\dot{M}_{\lambda,\mu} \stackrel{\text{def}}{=} N + 2 + [\kappa_{\mu,B}] \tag{4.37}$$

derivatives.

(iii) *Estimates for* $\mathscr{E}\psi$ *and* $\mathscr{E}\theta$. To estimate $\mathscr{E}\psi$ we need to bound $\sum^B \widehat{\psi}_n$ in case $n \hookrightarrow 3(B)$ and $Bn \hookrightarrow 1, 2(B)$ with respect to φ. Since $\Delta^\mu \theta = \Delta^\lambda \psi + \varphi$, the obstructions for $\Delta^\lambda \psi + \varphi$ with respect to B vanish; therefore

$$\Delta^\lambda \sum^B \widehat{\psi}_n = -\sum^B \widehat{\varphi}_n.$$

Iterating this equation with respect to A we obtain

$$\sum^B \widehat{\psi}_n + \lambda^{-l} \lim_{l \to \infty} \sum^B \lambda^{-l} \widehat{\psi}_{A^l n} = -\sum_{i=0}^{l} \sum^B \widehat{\varphi}_{A^i n}.$$

From Lemma 4.3 b) the limit above is 0. By iterating backwards and applying the same reasoning, we obtain

$$\sum^B \widehat{\psi}_n = \sum_{-}^{A} \sum^B \widehat{\varphi}_n = -\sum_{+}^{A} \sum^B \widehat{\varphi}_n. \tag{4.38}$$

In the notation of Lemma 4.3 c), (4.38) implies that for $n \in \mathbb{Z}^N$ which is minimal on its B orbit, we have

$$\widehat{\mathscr{E}\psi_n} = S_{H^+}(\varphi, n) = -S_{H^-}(\varphi, n),$$

where H^+ is the set of lattice points (l, k) in $\mathbb{Z}^2$ with positive l and H^- the set of points with negative l. Then according to Lemma 4.3 d), the needed estimate for $\mathscr{E}\psi$ with respect to φ follows if in at least one of the half-spaces H^- and H^+ the dual action satisfies some polynomial lower bound for every $n = n_{\min}$.

In case $Bn \hookrightarrow 2(B)$ for all l and all k we obviously have

$$|A^l B^k n| \geq C |l|^{-N} |k|^{-N} |n|; \tag{4.39}$$

thus the polynomial estimate needed for the application of part c) of Lemma 4.3 is satisfied both in H^+ and H^- for such n.

However in the other case, i.e., when $Bn \hookrightarrow 1(B)$, the same estimate holds either in H^+ or in H^-. This follows from the fact that in this case ($n \hookrightarrow 3(B)$ and $Bn \hookrightarrow 1(B)$), n is substantially large both in the expanding and in the contracting direction for B.

To see this we let n_{i_1} and n_{i_3} be (large) projections of n to some expanding and contracting Lyapunov subspaces V_{i_1} and V_{i_2} for B with Lyapunov exponents χ_{i_1} and χ_{i_2}, respectively; i.e., let

$$\|n_{i_1}\| \geq C |n| \qquad \text{and} \qquad \|n_{i_3}\| \geq C |n|,$$

where C is some fixed positive number. Then (assuming for the moment that α is semisimple) this implies

$$\begin{aligned} |A^l B^k n| &\geq C \sum_{i=1}^{r} \exp \chi_i(l, k) \|n_i\| \\ &\geq C(\exp \chi_{i_1}(l, k) + \exp \chi_{i_3}(l, k)) |n|. \end{aligned} \tag{4.40}$$

Now we notice that the union H of half-spaces $\{(l,k) : \chi_{i_1}(l,k) \geq 0\}$ and $\{(l,k) : \chi_{i_3}(l,k) \geq 0\}$ covers either H^+ or H^-. Namely, for any $k \in \mathbb{Z}$, (l,k) is in H if $l\left(\frac{\log|\lambda_{i_1}|}{\log|\mu_{i_1}|} - \frac{\log|\lambda_{i_3}|}{\log|\mu_{i_3}|}\right) \geq 0$ and this is true for $l \geq 0$ or for $l \leq 0$ depending on the $\operatorname{sgn}\left(\frac{\log|\lambda_{i_1}|}{\log|\mu_{i_1}|} - \frac{\log|\lambda_{i_3}|}{\log|\mu_{i_3}|}\right)$. Here $\lambda_{i_3}, \lambda_{i_1}$ and μ_{i_3}, μ_{i_1} are corresponding eigenvalues of A and B, respectively. Therefore, from (4.40) we obtain

$$|A^l B^k n| \geq C|n|$$

in H^+ or in H^- if α is semisimple. If α is not semisimple, then it decomposes a product of its semisimple and its unipotent part. For the semisimple part we use the estimate above and in the unipotent part only a polynomial growth may occur. This implies that (4.39) holds in H^+ or in H^- for a general (not necessarily semisimple) α.

Now choose the half-space in which the estimate (4.39) holds, that is choose one of the sums $S_{H^+}(\varphi, n)$ or $S_{H^-}(\varphi, n)$. Then the assumptions of d) in Lemma 4.3 are satisfied for one of the sums above S_{H^+} or S_{H^-} and therefore the estimate for $\mathcal{E}\psi$ follows:

$$\|\mathcal{E}\psi\|_{a-\delta-\kappa_{(\lambda,\mu),\alpha}} \leq \frac{C_a}{\delta^\nu}\|\varphi\|_a \tag{4.41}$$

for any $a > \kappa_{(\lambda,\mu),\alpha}$ and any $\delta > 0$, where

$$\kappa_{(\lambda,\mu),\alpha} \stackrel{\text{def}}{=} \frac{N+1}{\tau}(|\log|\mu|| + |\log|\lambda||). \tag{4.42}$$

Here, $\tau = \tau(A, B) > 0$ is the constant chosen as in the Lemma 4.3 a).

As we mentioned in part (i), by construction we have $\Delta^\mu \mathcal{E}\theta = \Delta^\lambda \mathcal{E}\psi + \varphi$. This by using Lemma 4.2 implies the following estimate for $\mathcal{E}\theta$ with respect to φ:

$$\|\mathcal{E}\theta\|_{a-\delta-\kappa_{(\lambda,\mu),\alpha}} \leq \frac{C_a}{\delta^\nu}\|\varphi\|_a \tag{4.43}$$

for any $a > \kappa_{(\lambda,\mu),\alpha}$ and any $\delta > 0$. This implies the C^r estimate (4.28) for $\mathcal{E}\psi$ and $\mathcal{E}\theta$ with the loss $\sigma > \widetilde{M}_{\lambda,\mu}$, where

$$\widetilde{M}_{\lambda,\mu} \stackrel{\text{def}}{=} N + 2 + [\kappa_{(\lambda,\mu),\alpha}] \tag{4.44}$$

where $\kappa_{(\lambda,\mu),\alpha}$ is defined in (4.42). □

LEMMA 4.6. *For two C^∞ maps R_A and R_B with $L(R_A, R_B) = \Phi$, there exists a splitting*

$$R_A = \mathcal{P}R_A + \mathcal{E}R_A$$
$$R_B = \mathcal{P}R_B + \mathcal{E}R_B$$

such that

$$L(\mathcal{P}R_A, \mathcal{P}R_B) = 0, \qquad L(\mathcal{E}R_A, \mathcal{E}R_B) = \Phi$$

$$\|\mathcal{P}R_A, \mathcal{P}R_B\|_{C^r} \leq C_r \|R_A, R_B\|_{C^{r+\sigma}} \tag{4.45}$$

$$\|\mathcal{E}R_A, \mathcal{E}R_B\|_{C^r} \leq C_r \|\Phi\|_{C^{r+\sigma}} \tag{4.46}$$

for any $r > 0$ *and* $\sigma > M = M(A, B, N)$, *where constant* M *depends only on the dimension of the torus and the linear action and is defined below* (*see* (4.60)).

Proof. If A and B are semisimple, then the statement follows directly from Lemma 4.5 as the condition $L(R_A, R_B) = \Phi$ splits into finitely many equations of the type

$$\Delta^\mu \theta - \Delta^\lambda \psi = \varphi,$$

where θ, ψ, and φ are C^∞ functions.

Now assume that A and B are not simultaneously diagonalizable and choose a basis in which A is in its Jordan normal form with some nontrivial Jordan blocks. Then in the same basis B has block diagonal form and as in Lemma 4.4 we can take $m \times m$ blocks J_A and J_B corresponding to eigenvalues λ and μ of A and B, respectively, and split $L(R_A, R_B) = \Phi$ into equations

$$J_B \Theta - \Theta \circ B - J_A \Psi + \Psi \circ A = \Phi, \tag{4.47}$$

where, as in Lemma 4.4, we take $J_A = (a_{ij})$ to be an $m \times m$ matrix which consists of blocks of A corresponding to the eigenvalue λ, i.e., $a_{ii} = \lambda$ for all $i = 1, \ldots, m$ and $a_{i,i+1} = *_i \in \{0, 1\}$ for all $i = 1, \ldots, m-1$ and $J_B = (b_{ij})$ to be the corresponding block of B where $b_{ii} = \mu$ for all $i = 1, \ldots, m$ (μ is an eigenvalue of B) and $b_{ij} = 0$ for all $m \geq i > j \geq 1$. Equation (4.47) splits into m equations. For every $k = 1, \ldots, m$ we have the following equation which we call $(EQ)_k$:

$$\left(\Delta^\mu \theta_k + \sum_{i=k+1}^{m} b_{ki} \theta_i\right) - (\Delta^\lambda \psi_k + *_k \psi_{k+1}) = \varphi_k \tag{4.48}$$

where θ_i, ψ_i, and φ_i are coordinate functions of Θ, Ψ, and Φ, respectively, in the basis in which A is in its Jordan normal form. In the special case when $\varphi_k = 0$, then we denote equation (4.48) by $(EQ)_k^0$. Clearly, for $k = m$ the equation $(EQ)_m$ is simply

$$\Delta^\mu \theta_m - \Delta^\lambda \psi_m = \varphi_m,$$

which by Lemma 4.5 implies the existence of the splitting

$$\begin{aligned} \theta_m &= \mathcal{P}\theta_m + \mathcal{E}\theta_m = \Delta^\lambda \omega_m + \mathcal{E}\theta_m \\ \psi_m &= \mathcal{P}\psi_m + \mathcal{E}\psi_m = \Delta^\mu \omega_m + \mathcal{E}\psi_m, \end{aligned} \tag{4.49}$$

where ω_m, $\mathcal{E}\theta_m$, $\mathcal{E}\psi_m$, $\mathcal{P}\theta_m = \Delta^\lambda \omega_m$, and $\mathcal{P}\psi_m = \Delta^\mu \omega_m$ are C^∞ functions satisfying the estimates:

(4.50)
$$\begin{aligned}\|\mathcal{E}\theta_m, \mathcal{E}\psi_m\|_{a-(\delta+\kappa)} &\le \frac{C_a}{\delta^\nu}\,\|\Phi\|_a,\\ \|\omega_m\|_{a-(\delta+\kappa)} &\le \frac{C_a}{\delta^\nu}\,\|\Theta,\Psi\|_a,\\ \|\mathcal{P}\theta_m, \mathcal{P}\psi_m\|_{a-(\delta+\kappa)} &\le \frac{C_a}{\delta^\nu}\,\|\Theta,\Psi\|_a,\end{aligned}$$

where we let $\kappa \overset{\text{def}}{=} \max\{\kappa_{(\lambda,\mu),\alpha}, \kappa_{\mu,B}\}$ (see the proof of Lemma 4.5, (4.42), (4.36)).

Now we proceed by induction. Fix k between 1 and $m-1$ and assume that for all $i = k+1,\dots,m$ we already have the splitting

(4.51)
$$\begin{aligned}\theta_i &= \mathcal{P}\theta_i + \mathcal{E}\theta_i = \Delta^\lambda\omega_i + \mathcal{E}\theta_i,\\ \psi_i &= \mathcal{P}\psi_i + \mathcal{E}\psi_i = \Delta^\mu\omega_i + \mathcal{E}\psi_i,\end{aligned}$$

where ω_i, $\mathcal{P}\theta_i = \Delta^\lambda\omega_i$, $\mathcal{P}\psi_i = \Delta^\mu\omega_i$, $\mathcal{E}\theta_i$ and $\mathcal{E}\psi_i$ are C^∞ functions satisfying the following estimates:

(4.52)
$$\begin{aligned}\|\mathcal{E}\theta_i, \mathcal{E}\psi_i\|_{a-(m-i+1)(\delta+\kappa)} &\le \frac{C_a}{\delta^{(m-i+1)\nu}}\,\|\Phi\|_a,\\ \|\omega_i\|_{a-(m-i+1)(\delta+\kappa)} &\le \frac{C_a}{\delta^{(m-i+1)\nu}}\,\|\Theta,\Psi\|_a,\\ \|\mathcal{P}\theta_i, \mathcal{P}\psi_i\|_{a-(m-i+1)(\delta+\kappa)} &\le \frac{C_a}{\delta^{(m-i+1)\nu}}\,\|\Theta,\Psi\|_a,\end{aligned}$$

and such that $\mathcal{P}\theta_i$ and $\mathcal{P}\psi_i$ satisfy the equation $(EQ)_i^0$ (4.48).

By substituting from (4.51) the expressions for θ_i and ψ_i for all $i = k+1,\dots,m$ into (4.48), we obtain

(4.53)
$$\Delta^\mu(\theta_k - *_k\omega_{k+1}) - \Delta^\lambda\Big(\psi_k - \sum_{i=k+1}^{m} b_{ki}\omega_i\Big) = \varphi_k - *_k\mathcal{E}\psi_{k+1} - \sum_{i=k+1}^{m} b_{ki}\mathcal{E}\theta_i.$$

Then using Lemma 4.5 again we can obtain ω_k, $\mathcal{E}\theta_k$ and $\mathcal{E}\psi_k$ such that

(4.54)
$$\begin{aligned}\theta_k - *_k\omega_{k+1} &= \Delta^\lambda\omega_k + \mathcal{E}\theta_k\\ \psi_k - \sum_{i=k+1}^{m} b_{ki}\omega_i &= \Delta^\mu\omega_k + \mathcal{E}\psi_k\end{aligned}$$

with estimates for $\mathcal{E}\theta_k$ and $\mathcal{E}\psi_k$ following from (4.52) and (4.53):

(4.55)
$$\begin{aligned}\|\mathcal{E}\theta_k, \mathcal{E}\psi_k\|_{a-(m-k+1)(\delta+\kappa)} &\le \frac{C_a}{\delta^{(m-k+1)\nu}}\,\|\Phi\|_a,\\ \|\omega_k\|_{a-(m-k+1)(\delta+\kappa)} &\le \frac{C_a}{\delta^{(m-k+1)\nu}}\,\|\Theta,\Psi\|_a,\\ \|\mathcal{P}\theta_k, \mathcal{P}\psi_k\|_{a-(m-k+1)(\delta+\kappa)} &\le \frac{C_a}{\delta^{(m-k+1)\nu}}\,\|\Theta,\Psi\|_a,\end{aligned}$$

where we define $\mathcal{P}\theta_k$ and $\mathcal{P}\psi_k$ as

$$\mathcal{P}\theta_k = \Delta^\lambda \omega_k + *_k\omega_{k+1}, \tag{4.56}$$

$$\mathcal{P}\theta_k = \Delta^\lambda \omega_k + \sum_{i=k+1}^{m} b_{ki}\omega_i .$$

Now checking that $\mathcal{P}\theta_k$ and $\mathcal{P}\psi_k$ are "good", i.e., that they satisfy the equation $(EQ)_k^0$, is easily done just by substitution. At this point however one has to use the coefficients relations (4.14) which were derived in Lemma 4.4 from the fact that A and B commute.

Since the maximal size of a Jordan block of A is less than N, we can estimate $\mathcal{E}\Theta$ and $\mathcal{E}\Psi$ in the $\|\cdot\|_{a-N(\delta+\kappa)}$ norm

$$\|\mathcal{E}\Theta, \mathcal{E}\Psi\|_{a-N(\delta+\kappa)} \le \frac{C_a}{\delta^{N\nu}}\|\Phi\|_a \tag{4.57}$$

and similarly for the maps $\mathcal{P}\theta$ and $\mathcal{P}\psi$:

$$\|\mathcal{P}\Theta, \mathcal{P}\Psi\|_{a-N(\delta+\kappa)} \le \frac{C_a}{\delta^{N\nu}}\|\Theta, \Psi\|_a . \tag{4.58}$$

If repeated for all Jordan blocks, this produces the required splitting

$$R_A = \mathcal{P}R_A + \mathcal{E}R_A \qquad R_B = \mathcal{P}R_B + \mathcal{E}R_B, \tag{4.59}$$

which satisfies the conditions and the estimates in the statement. Since we use repeatedly Lemma 4.5 the C^r estimates (4.45) and (4.46) hold for $\sigma > M = M(A, B, N)$, where

$$M \stackrel{\text{def}}{=} N + 2 + N[\max\{\max_i \kappa_{\mu_i,B}, \max_i \kappa_{(\lambda_i,\mu_i),\alpha}\}], \tag{4.60}$$

where the first maximum in i is taken over all eigenvalues μ_i of B (here $\kappa_{\mu_i,B}$ are numbers defined in (4.36)) and the second maximum in i is taken over all pairs of corresponding eigenvalues λ_i and μ_i of A and B (here $\kappa_{(\lambda_i,\mu_i),\alpha}$ are the numbers defined in (4.42)). □

The following lemma shows that $\Phi = L(R_A, R_B)$ cannot be large if $\alpha + R$ is a commutative action. It is in fact quadratically small with respect to R.

LEMMA 4.7. *If $\tilde{\alpha} = \alpha + R$ is a commutative C^∞ action of an abelian group $\mathcal{A}$ by toral automorphisms, with α linear, then for $r \ge 0$*

$$\|L(R_A, R_B)\|_{C^r} \le C_r \|R_A, R_B\|_{C^r} \|R_A, R_B\|_{C^{r+1}} \tag{4.61}$$

where $R_A = R(g_1)$, $R_B = R(g_2)$ and $g_1, g_2 \in \mathcal{A}$.

Proof.

$$\tilde{\alpha}_A \circ \tilde{\alpha}_B = \tilde{\alpha}_B \circ \tilde{\alpha}_A$$

$$(A + R_A) \circ (B + R_B) = (B + R_B) \circ (A + R_A)$$

$$R_A \circ (B + R_B) - BR_A = R_B \circ (A + R_A) - AR_B .$$

Therefore,

$$L(R_A, R_B) = -R_A \circ B + BR_A + R_B \circ A - AR_B$$
$$= -(R_B \circ (A + R_A) - R_B \circ A) + (R_A(B + R_B) - R_A \circ B).$$

Then from the Taylor's formula with integral remainder

$$R_B \circ (A + R_A) - R_B \circ A = \int_0^1 DR_B(A + tR_A)R_A dt$$

just as in [31], it follows that

$$\|R_B \circ (A + R_A) - R_B \circ A\|_{C^0} \le C_r \|R_B\|_{C^1} \|R_A\|_{C^0}.$$

A similar estimate holds for $R_A(B + R_B) - R_A \circ B$.

The estimate (4.61) for C^r norms, even if a bit less obvious, follows similarly (see for example [28, Appendix II]):

$$\|L(R_A, R_B)\|_{C^r} \le C_r(\|R_B\|_{C^{r+1}} \|R_A\|_{C^r} + \|R_A\|_{C^{r+1}} \|R_B\|_{C^r})$$
$$\le C_r \|R_A, R_B\|_{C^r} \|R_A, R_B\|_{C^{r+1}}. \qquad \square$$

5. **Proof of Theorem** 1

Assuming $\tilde{\alpha}$ is a C^∞ perturbation of α, and that the difference $R = \tilde{\alpha} - \alpha$ is small in some C^l norm (where l is fixed and will be determined in the proof; see (5.12)) we show that $\tilde{\alpha}$ is smoothly conjugate to α. The conjugacy is produced for the two ergodic generators and by Lemma 3.2 it works for all elements of the action. This proof is similar to the iterative proof in [31]. We refer to Section 8 for a short discussion on possible variations of the proof below and possible applications of generalized implicit function theorems in this set-up.

5.1. *Smoothing.* Following the scheme described in Section 3.3 at each step of the iterative procedure we solve the linearized equation (3.12)

$$A\Omega - \Omega \circ A = -R_A, \qquad B\Omega - \Omega \circ A = -R_B$$

approximately. By results of the Section 4.5 we have that the linearized equation (3.12) has an approximate solution Ω which is C^∞ although we can only compare its norm in C^r to the norm of R in $C^{r+\sigma}$ where σ is large but fixed. The error is $\mathscr{E}R$ and is by construction in Section 4.5, comparable to $L(R_A, R_B)$. Thus it is small with respect to R by Lemma 4.7, but comparison again comes with fixed loss of derivatives. The loss of derivatives might be a large number (depending of the hyperbolicity properties of the linear action), but it only depends on the dimension of the torus and the *unperturbed* linear action. To overcome this fixed loss of derivatives at each step of the iteration process, it is standard (see for example [38]) to introduce a family of smoothing operators $\{S_J, J \in \mathbb{N}\}$. Then instead of solving approximately (3.12) we solve approximately the following system:

$$\begin{aligned} A\Omega - \Omega \circ A &= -S_J R_A \\ B\Omega - \Omega \circ B &= -S_J R_B. \end{aligned} \tag{5.1}$$

For a C^∞ function $f = \sum_n \hat{f}_n \chi_n$ we define $S_J f$ as

$$S_J f \stackrel{\text{def}}{=} \sum_{|n|<J} \hat{f}_n \chi_n.$$

Then the smoothing operators satisfy

$$\begin{aligned} \|S_J f\|_{a+b} &\le J^b \|f\|_a \\ \|S_J f\|_{C^{a+b}} &\le J^{b+\sigma} \|f\|_{C^a}, \end{aligned} \tag{5.2}$$

where $a > b > 0$ and $\sigma > N + 1$. Also,

$$\begin{aligned} \|(I - S_J) f\|_{a-b} &\le J^{-b} \|f\|_a \\ \|(I - S_J) f\|_{C^{a-b}} &\le C J^{-b+\sigma} \|f\|_{C^a} \end{aligned} \tag{5.3}$$

for $a > b > \sigma > N + 1$. These simple smoothing operators are convenient in our setting since they are well behaved with respect to the operator L. Namely, we have

$$L(S_J f, S_J g) = S_{\frac{J}{\xi}}(L(f,g)) + \mathcal{F}_{>\frac{J}{\xi}},$$

where ξ is a constant depending on A and B and the last term in the expression above consists of pieces of Fourier series for f and g involving only terms with $|n| > \frac{J}{\xi}$. This implies the following estimates:

$$\begin{aligned} \|L(S_J f, S_J g)\|_a &\le \|S_{\frac{J}{\xi}}(L(f,g))\|_a + C J^{-b} \|f, g\|_{a+b} \\ \|L(S_J f, S_J g)\|_{C^r} &\le \|S_{\frac{J}{\xi}}(L(f,g))\|_{C^{r+\sigma}} + C J^{-b+\sigma} \|f, g\|_{C^{r+b}} \\ &\le J^{2\sigma} \|L(f,g)\|_{C^r} + C J^{-b+\sigma} \|f, g\|_{C^{r+b}} \end{aligned} \tag{5.4}$$

for any $a > 0$, $b > \sigma > N + 1$, $r \ge 0$.

5.2. *Iterative step and the error estimate.* At each step of the iterative scheme we first choose an appropriate smoothing operator S_J. In order to solve approximately (5.1) we use Lemma 4.6 to obtain the splitting

$$\begin{aligned} S_J R_A &= \mathcal{P}(S_J R_A) + \mathcal{E}(S_J R_A) \\ S_J R_B &= \mathcal{P}(S_J R_B) + \mathcal{E}(S_J R_B) \end{aligned}$$

so that $L(\mathcal{P}(S_J R_A), \mathcal{P}(S_J R_B)) = 0$. Now from Lemma 4.4 the system

$$\begin{aligned} A\Omega - \Omega \circ A &= -\mathcal{P}(S_J R_A) \\ B\Omega - \Omega \circ B &= -\mathcal{P}(S_J R_B) \end{aligned}$$

has an approximate solution Ω such that

$$\begin{aligned}\|\Omega\|_{C^r} &\le C_r\|\mathcal{P}(S_J R_A), \mathcal{P}(S_J R_B)\|_{C^{r+\sigma}} \\ &\le C_r\|S_J R_A, S_J R_B\|_{C^{r+2\sigma}} \\ &\le C_r J^{3\sigma}\|R_A, R_B\|_{C^r} \le C_r J^{3\sigma}\|R\|_{C^r}.\end{aligned} \tag{5.5}$$

Here we used the estimates from Lemma 4.6 and the properties of smoothing (5.2). As mentioned in the Section 4.1, $\|R\|_{C^r}$ stands for $\max\{\|R_A\|_{C^r}, \|R_A\|_{C^r}\}$.

The estimate (5.5) holds for any σ large enough: $\sigma > \max\{M, M_0\}$, so that the estimates in Lemma 4.4 and Lemma 4.6 hold. Then we form $H \stackrel{\text{def}}{=} \mathrm{id} + \Omega$ (since Ω is made small in C^1 throughout the iteration, H is invertible) and

$$\tilde{\alpha}^{(1)} \stackrel{\text{def}}{=} H^{-1} \circ \tilde{\alpha} \circ H.$$

The new error is

$$R^{(1)} \stackrel{\text{def}}{=} \tilde{\alpha}^{(1)} - \alpha$$

and it has two parts:

- the error coming from solving the linearized equation only approximately:

$$E_1 = \mathcal{E}(S_J R) + (I - S_J)R, \qquad \text{and}$$

- the standard error coming from the linearization

$$E_2 = \Omega \circ \tilde{\alpha}^{(1)} - \Omega \circ \alpha + R \circ (\mathrm{id} + \Omega) - R.$$

Estimate for E_1. Using Lemma 4.6 and the properties of smoothing (5.4), for every $b > \sigma$ we have

$$\begin{aligned}\|\mathcal{E}(S_J R)\|_{C^0} &\le C\|L(S_J R_A, S_J R_B)\|_{C^\sigma} \\ &\le C\|S_{\frac{J}{\xi}} L(R_A, R_B)\|_{C^{2\sigma}} + CJ^{-b+\sigma}\|R\|_{C^{\sigma+b}} \\ &\le C\left[J^{3\sigma}\|L(R_A, R_B)\|_{C^0} + J^{-b+\sigma}\|R\|_{C^{\sigma+b}}\right].\end{aligned}$$

Also, using (5.3),

$$\|(I - S_J)R\|_{C^0} \le C_l J^{-l+\sigma}\|R\|_{C^l}$$

for any $l \ge \sigma$. Let $b = l - \sigma$. Thus, using Lemma 4.7, we have

$$\|E_1\|_{C^0} \le CJ^{3\sigma}\|R\|_{C^0}\|R\|_{C^1} + CJ^{-l+2\sigma}\|R\|_{C^l} \tag{5.6}$$

for $l > 2\sigma$.

Estimate for E_2. The first part of E_2 is estimated as follows:

$$\begin{aligned}\|\Omega \circ \tilde{\alpha}^{(1)} - \Omega \circ \alpha\|_{C^0} &\le C\|\Omega\|_{C^1}\|\tilde{\alpha}^{(1)} - \alpha\|_{C^0} \\ &= C\|\Omega\|_{C^1}\|R^{(1)}\|_{C^0} \le \frac{1}{4}\|R^{(1)}\|_{C^0};\end{aligned}$$

thus this part is absorbed by $\|R^{(1)}\|_{C^0}$ providing $\|\Omega\|_{C^1}$ is bounded throughout the procedure. The second part of E_2 we estimate by using (5.5)

$$\|R(\mathrm{id} + \Omega) - R\|_{C^0} \le C\|R\|_{C^1}\|\Omega\|_{C^0} \le CJ^{3\sigma}\|R\|_{C^1}\|R\|_{C^0}. \tag{5.7}$$

Estimate of the new error $R^{(1)}$. By combining (5.6) and (5.7) we obtain an estimate for the new error:

$$\|R^{(1)}\|_{C^0} \le CJ^{3\sigma}\|R\|_{C^1}\|R\|_{C^0} + CJ^{-l+2\sigma}\|R\|_{C^l} \tag{5.8}$$

for any $l > 2\sigma$. From $\tilde{\alpha}^{(1)} = H^{-1} \circ \tilde{\alpha} \circ H$ using the fact that Ω is $\mathbb{Z}^N$-periodic and satisfies the estimate (5.5) we have

$$\|R^{(1)}\|_{C^l} \le C_l J^{3\sigma}\left(1 + \|R\|_{C^l}\right). \tag{5.9}$$

Therefore we have obtained what usually constitutes the basis for the iteration (see [38] for example), i.e., the "quadratic" estimate (5.8) for the low norm of the new error and the "linear" estimate (5.9) for some high norm of the new error with respect to the initial error R.

5.3. *Setting up the iterative process.* To set up the iterative process we first let

$$R^{(0)} = R, \qquad \tilde{\alpha}^{(0)} = \tilde{\alpha}, \qquad H^{(0)} = \mathrm{id}.$$

Now construct $R^{(n)}$ inductively for every n for $R^{(n)}$ choose an appropriate integer number J_n to obtain $S_{J_n}R^{(n)}$ which produces, after solving approximately the linearized equation, new $\Omega^{(n)}$. Then define

$$\begin{aligned} H^{(n)} &= \mathrm{id} + \Omega^{(n)} \\ \tilde{\alpha}^{(n+1)} &= \left(H^{(n)}\right)^{-1} \circ \tilde{\alpha}^{(n)} \circ H^{(n)} \\ R^{(n+1)} &= \tilde{\alpha}^{(n+1)} - \alpha. \end{aligned} \tag{5.10}$$

Consequently,

$$\begin{aligned} \tilde{\alpha}^{(n+1)} &= \left(H^{(n)}\right)^{-1} \circ \left(H^{(n-1)}\right)^{-1} \circ \cdots \circ \left(H^{(0)}\right)^{-1} \circ \tilde{\alpha} \circ H^{(0)} \circ \cdots \circ H^{(n)} \\ &= \mathcal{H}_n^{-1} \circ \tilde{\alpha} \circ \mathcal{H}_n \end{aligned}$$

where $\mathcal{H}_n \stackrel{\text{def}}{=} H^{(0)} \circ \cdots \circ H^{(n)}$. To ensure the convergence of the process, set

$$\begin{aligned} \|R^{(n)}\|_{C^0} &< \varepsilon_n = \varepsilon^{(k^n)} \\ \|R^{(n)}\|_{C^l} &< \varepsilon_n^{-1} \\ \|\Omega^{(n)}\|_{C^1} &< \varepsilon_n^{1/2} \\ J_n &= \varepsilon_n^{-\frac{1}{3(3\sigma+2)}} \end{aligned} \tag{5.11}$$

where $k = \frac{4}{3}$. At this point, fix l:

$$l = 23\sigma + 15, \tag{5.12}$$

where $\sigma = \sigma(A, B, N) = \max\{M, M_0\}$ is a constant for which the estimates (4.60) and (4.27) hold. The constant l is chosen so that the process converges and the convergence is proved in the subsequent section.

5.4. *Convergence.* By induction it is proved that all the bounds (5.11) hold for every $n \in \mathbb{N}$:

$$\begin{aligned}\|R^{(n+1)}\|_{C^l} &\le C_l J_n^{3\sigma}\left(1+\|R^{(n)}\|_{C^l}\right) \le C_l J_n^{3\sigma}\left(1+\varepsilon_n^{-1}\right)\\ &\le 2C_l J_n^{3\sigma}\varepsilon_n^{-1} \le 2C_l\varepsilon_n^{-\frac{3\sigma}{3(3\sigma+2)}}\varepsilon_n^{-1} < \varepsilon_n^{-\frac{1}{3}-1} = \varepsilon_n^{-\frac{4}{3}}\\ &= (\varepsilon_{n+1})^{-1}.\end{aligned}$$

From interpolation inequalities it follows that

$$\|R^{(n)}\|_{C^1} \le C_l\|R^{(n)}\|_{C^0}^{1-\frac{1}{l}}\|R^{(n)}\|_{C^l}^{\frac{1}{l}}.$$

Along with (5.8) this implies

$$\begin{aligned}\|R^{(n+1)}\|_{C^0} &\le C\left[\varepsilon_n^{-\frac{3\sigma}{3(3\sigma+2)}}\varepsilon_n^{1-\frac{1}{l}}\varepsilon_n^{-\frac{1}{l}}\varepsilon_n + \varepsilon_n^{\frac{l-2\sigma}{3(3\sigma+2)}}\varepsilon_n^{-1}\right]\\ &= C\left[\varepsilon_n^{-\frac{\sigma}{3\sigma+2}+2\left(1-\frac{1}{l}\right)} + \varepsilon_n^{\frac{l-2\sigma}{3(3\sigma+2)}-1}\right] = C\left[\varepsilon_n^x + \varepsilon_n^y\right]\\ &\le \varepsilon_n^{\frac{4}{3}} = \varepsilon_{n+1}\end{aligned}$$

providing $x > \frac{4}{3}$, $y > \frac{4}{3}$; i.e.,

$$\begin{aligned}x &= -\frac{\sigma}{3\sigma+2} + 2\left(1-\frac{1}{l}\right) > \frac{4}{3}\\ y &= \frac{l-2\sigma}{3(3\sigma+2)} - 1 > \frac{4}{3}.\end{aligned}$$

Both inequalities above are satisfied for $l \ge 23\sigma + 15$. We note here that with more precision (by changing the rate of convergence) the constant l can be made somewhat smaller (see Section 7.1). Using (5.5) we may check the C^1 bound for Ω:

$$\begin{aligned}\|\Omega^{(n+1)}\|_{C^1} &\le C\cdot J_n^{3\sigma+1}\|R^{(n)}\|_{C^0} \le C\cdot J_n^{3\sigma+1}\varepsilon_n = C\cdot\varepsilon_n^{-\frac{3\sigma+1}{3(3\sigma+2)}+1}\\ &< \varepsilon_n^{\frac{2}{3}} = \varepsilon_{n+1}^{\frac{1}{2}}.\end{aligned}$$

Thus for sufficiently small $\|R\|_{C^0}$ and $\|R\|_{C^l}$ the process converges to a solution $\Omega \in C^1$ with $\|\Omega\|_{\mathscr{C}^1} < \frac{1}{4}$.

Now using interpolation inequalities, the fact that the process converges in any C^r norm follows easily just as in [31] or as in [38]. We repeat this argument here for completeness. For arbitrary m from (5.9) we have

$$\|R^{(n+1)}\|_{C^m} \le C_m J_n^{3\sigma}\left(1+\|R^{(n)}\|_{C^m}\right) \le \varepsilon_n^{-\frac{1}{3}}\left(1+\|R^{(n)}\|_{C^m}\right),$$

$$\|R^{(n)}\|_{C^m} \le C_m\prod_{\nu=1}^{n-1}\varepsilon_\nu^{-\frac{1}{3}}\left(1+\|R\|_{C^m}\right) \le \varepsilon_n^{-1}C_m,$$

where in the second line above the constant C_m is $C_m := C_m\,(1 + \|R\|_{C^m})$. Let $m = 3k$. Then

$$\|R^{(n)}\|_{C^k} \le C_k \|R^{(n)}\|_{C^0}^{\frac{2}{3}} \|R^{(n)}\|_{C^{3k}}^{\frac{1}{3}} < C_k \varepsilon_n^{\frac{2}{3}} \varepsilon_n^{-\frac{1}{3}} = C_k \varepsilon_n^{\frac{1}{3}},$$

$$\|\Omega^{(n)}\|_{C^k} \le C_k J_n^{3\sigma} \|R^{(n)}\|_{C^k} \le C_k \varepsilon_n^{-\frac{3\sigma}{3(3\sigma+2)}} \varepsilon_n^{\frac{1}{3}} = C_k \varepsilon_n^{\delta},$$

with $\delta = \frac{2}{3(3\sigma+2)} > 0$ and the constant C_k changing throughout the above procedure, but it depends only on k and the C^{3k} norm of the initial perturbation R.

This implies the convergence of the sequence $\mathcal{H}_n$ in C^k norm for every $k \in \mathbb{N}$, i.e., the limit $\mathcal{H}$ is a C^∞ map.

6. **Proof of Theorem** 2

Definition 2. A C^∞ foliation $\mathcal{F}$ of a manifold M is $C^{\infty,l,\infty}$ *locally rigid* if for any sufficiently C^l close C^∞ foliation $\mathcal{F}'$ there exists a C^∞ diffeomorphism H of M taking leaves of $\mathcal{F}$ to leaves of $\mathcal{F}'$.

Definition 3. An action α of $\mathbb{R}^k$ on M is $C^{\infty,l,\infty}$ *orbit foliation rigid* if its orbit foliation is $C^{\infty,l,\infty}$ locally rigid.

The following lemma is completely general and quite possibly may be found in the literature. Since we are not aware of a specific reference we provide a proof.

LEMMA 6.1. *$C^{\infty,l,\infty}$ local rigidity for an action γ of $\mathbb{Z}^k$ by diffeomorphisms on a manifold M implies $C^{\infty,l,\infty}$ orbit foliation rigidity for the suspension action $\alpha(\gamma)$ on the suspension manifold $N(\gamma)$.*

Proof. Let $\Gamma\colon \mathbb{Z}^k \to \mathrm{Diff}^\infty(M)$ be a representation of $\mathbb{Z}^k$ defining a $\mathbb{Z}^k$ action γ by diffeomorphisms of M. Let $\alpha(\gamma)\colon \mathbb{R}^k \times N(\gamma) \to N(\gamma)$ be the corresponding suspension action with the orbit foliation $\mathcal{F}(\gamma)$ and let $\tilde{\alpha}$ be a C^l small C^∞ perturbation of $\alpha(\gamma)$ with the orbit foliation $\widetilde{\mathcal{F}(\gamma)}$ which is C^l close to $\mathcal{F}(\gamma)$.

If the perturbation is sufficiently small the orbit foliation of the perturbation is still transversal to the fibers M of the suspension manifold $N(\gamma)$. Let $\tilde{\Gamma}\colon \mathbb{Z}^k \to \mathrm{Diff}^\infty(M)$ be the holonomy of the perturbed foliation $\widetilde{\mathcal{F}(\gamma)}$, defining an action $\tilde{\gamma}$ of $\mathbb{Z}^k$ on M. Then there exists a C^∞ diffeomorphism $H_1\colon N(\tilde{\gamma}) \to N(\gamma)$ taking leaves of the foliation $\mathcal{F}(\tilde{\gamma})$ of the suspension action over $\tilde{\gamma}$ to the leaves of $\widetilde{\mathcal{F}(\gamma)}$ (Theorem 3 in [3, §5.4]).

On the other hand, since γ and $\tilde{\gamma}$ are sufficiently C^l close, C^∞ $\mathbb{Z}^k$ actions there is a C^∞ conjugacy $h\colon M \to M$ such that $h \circ \gamma = \tilde{\gamma} \circ h$ according to our assumption on the local rigidity of γ. This implies the existence of a C^∞ orbit equivalence for the corresponding suspensions, i.e., there exists $H_2\colon N(\gamma) \to N(\tilde{\gamma})$ taking leaves of $\mathcal{F}(\gamma)$ to leaves of $\mathcal{F}(\tilde{\gamma})$ (Theorem 2 in [3, §5.4]).

Thus the C^∞ diffeomorphism $H = H_1 \circ H_2$ on $N(\gamma)$ is an orbit equivalence for actions $\alpha(\gamma)$ and $\tilde{\alpha}$ which implies $C^{\infty,l,\infty}$ orbit foliation rigidity for $\alpha(\gamma)$. □

It follows from Lemma 6.1 and Theorem 1 that for a given suspension action and its small perturbation there exists a C^∞ orbit equivalence taking orbits of the original to the orbits of the perturbed action. This reduces the question of local rigidity of suspensions to considering small perturbations along the leaves of orbit foliation only. However such perturbations are given by $\mathbb{R}^k$ valued cocycles over $\alpha(\gamma)$.

It is proved in [23, §4.2] that C^∞ cocycle rigidity of a $\mathbb{Z}^k$ action γ by toral automorphisms implies C^∞ cocycle rigidity for suspension $\alpha(\gamma)$. For the proof of C^∞ cocycle rigidity of a $\mathbb{Z}^k$ action γ with $k \geq 2$ and all nontrivial elements of the ergodic action, we refer to [24] or [18]. (The proof of this fact is also contained in part (i) of the proof of Lemma 4.4 for untwisted cocycles.) Thus $\alpha(\gamma)$ is C^∞ cocycle rigid, which implies that perturbations in orbit direction are conjugate to the original action up to an automorphism of $\mathbb{R}^k$. This completes the proof of Theorem 2.

7. Comments on finitely differentiable and analytic case

7.1. *Finitely differentiable case.* It is clear that all the arguments in the proof of Theorem 1 only use a finite number of derivatives so that our rigidity results in a modified form hold when the perturbed action α is only finitely differentiable. We discuss now specific modifications which appear this way.

Let R be C^m. Then the estimate (4.10) for the approximate solution of the linearized equation obtained in Lemma 4.4 still holds if $m > \sigma$, where $\sigma > M_0$ and M_0 is defined in (4.27). Similarly, the estimates (4.45) and (4.46) obtained in Lemma 4.6 which are used later in the proof of the Theorem 1 (Section 5) hold when R is C^m and $m > M$, where M is defined in (4.60). Now the iterative proof in Section 5 applies in the following setting.

Let α be a $\mathbb{Z}^k$ action by automorphisms of $\mathbb{T}^N$ as in the statement of Theorem 1. Let $\sigma \in \mathbb{N}$ be a fixed number greater than constants M (in (4.60)) and M_0 (in (4.27)) defined in Lemmas 4.4 and 4.6.

The convergence set-up is the same as in the Section 5.3, the modification only comes in determining the speed of the convergence. Instead of (5.11) now we set

$$\begin{aligned} &\|R^{(n)}\|_{C^0} < \varepsilon_n = \varepsilon^{(k^n)}, \\ &\|R^{(n)}\|_{C^l} < \varepsilon_n^{-1}, \\ &\|\Omega^{(n)}\|_{C^1} < \varepsilon_n^{\frac{2-k}{k}}, \\ &J_n = \varepsilon_n^{-\frac{k-1}{(3\sigma+2)}}, \end{aligned} \tag{7.1}$$

where $1 < k < 2$. While making sure that these bounds hold for every n, we will obtain a lower bound on l which depends on the speed of convergence k.

We first check the bound for the C^l norm using the estimate (5.9):

$$\|R^{(n+1)}\|_{C^l} \leq C_l J_n^{3\sigma}(1 + \|R^{(n)}\|_{C_l}) \leq C_l' \varepsilon_n^{-(k-1)} \varepsilon_n^{-1} = \varepsilon_{n+1}^{-1}. \tag{7.2}$$

Using (5.5) it is easy to check the bound for Ω:

$$\|\Omega^{(n)}\|_{C^1} \le C\varepsilon_n^{-\frac{k-1}{3\sigma+2}(3\sigma+1)}\varepsilon_n \le \varepsilon_n^{2-k} \le \varepsilon_{n+1}^{\frac{2-k}{k}}. \tag{7.3}$$

Now for the C^0 bound we have, as in Section 5.4, using (5.8) and the interpolation estimates:

$$\begin{aligned}\|R^{(n+1)}\|_{C^l} &\le C_l J_n^{3\sigma}\|R^{(n)}\|_{C^1}\|R^{(n)}\|_{C^0} + C_l J_n^{-l+2\sigma}\|R^{(n)}\|_{C^l}\\ &\le C_l J_n^{3\sigma}\|R^{(n)}\|_{C^0}^{2-\frac{1}{l}}\|R^{(n)}\|_{C^l}^{\frac{1}{l}} + C_l J_n^{-l+2\sigma}\|R^{(n)}\|_{C^l}\\ &\le C_l\varepsilon_n^{-\frac{3\sigma(k_1)}{3\sigma+2}}\varepsilon_n^{2-\frac{1}{l}}\varepsilon_n^{-\frac{1}{l}} + \varepsilon_n^{-\frac{(k-1)(-l+2\sigma)}{3\sigma+2}}\varepsilon_n^{-1}\\ &\le \varepsilon_n^{-k+3-\frac{2}{l}} + \varepsilon_n^{\frac{(k-1)(l-2\sigma)}{3\sigma+2}-1}.\end{aligned}$$

Thus if $-k+3-\frac{2}{l} > k$ and $\frac{(k-1)(l-2\sigma)}{3\sigma+2} - 1 > k$ then we have $\|R^{(n+1)}\|_{C^l} \le \varepsilon_{n+1}$. The first condition gives $l > \frac{2}{3-2k}$ and the second one $l > 2\sigma + \frac{k+1}{k-1}(3\sigma+2)$. The first condition is satisfied already for $l > 2$. The second one actually gives the dependence on k. Namely, as the speed of convergence k approaches 2, the lower bound for l approaches $l_0 := 11\sigma + 6$. Thus if one chooses $l = l_0 + \delta$ for $\delta > 0$, then it is possible to choose the rate of convergence $k = k(\delta)$ so that all the bounds (7.1) hold for every n.

This implies the convergence of the procedure in C^1 norm. Therefore, the perturbation needs to be close to the initial action in C^l with l (only) strictly larger than l_0, in order to obtain a C^1 solution.

To obtain more derivatives for the solution, we need to assume that the perturbation is more regular than $C^{l_0+\delta}$. Again, at this point we use interpolation inequalities. Assume that the initial perturbation is C^m and let $1 < r < m$. As in Section 5.4 it is easy to check that one has $\|R^{(n)}\|_{C^m} \le C_m\varepsilon_n^{-1}\|R^{(0)}\|_{C^m}$. Therefore we have

$$\begin{aligned}\|R^{(n)}\|_{C^r} &\le C\|R^{(n)}\|_{C^0}^{1-\frac{r}{m}}\|R^{(n)}\|_{C^m}^{\frac{r}{m}} < C_m\varepsilon_n^{1-2\frac{r}{m}}\\ \|\Omega^{(n)}\|_{C^r} &\le CJ_n^{3\sigma}\|R^{(n)}\|_{C^r} \le C_m\varepsilon_n^{-(k-1)\frac{3\sigma}{(3\sigma+2)}}\varepsilon_n^{1-2\frac{r}{m}}.\end{aligned}$$

Therefore, in order to obtain a solution of order r, we need that $\frac{r}{m} < 1 - \frac{k}{2}$. So if the perturbation is C^∞ then for any $l > l_0$ the solution is C^∞ with k chosen close to 2 if l is close to l_0 as described above.

Now if the perturbation is only C^m and close to the unperturbed action in C^l for some $l > l_0$, then choose k so that $l > 2\sigma + \frac{k+1}{k-1}(3\sigma+2)$, that is choose k close to but larger than $k_0(l) = \frac{l+5\sigma+2}{l-3\sigma-2} \in (1,2)$. Then the solution is C^r for every $r < (1 - \frac{k_0(l)}{2})m$.

7.2. *Analytic case.* Now suppose the perturbed action α is real analytic. A natural question is whether the unique C^∞ conjugacy with the linear action is also analytic.

Both estimates in Lemma 4.2 and in part d) in Section 4.3 can be obtained in the category of analytic spaces A_σ allowing arbitrary small loss of domain δ in all directions and assuming that there is no growth for the action in the neutral direction.

The algebraic part of the proof of Lemma 4.5 holds too, thus one can define maps $\mathcal{E}\psi$, $\mathcal{E}\theta$, $\mathcal{P}\psi$, and $\mathcal{P}\theta$. The analytic estimates for the $\mathcal{E}\psi$ and $\mathcal{P}\psi$ with small loss of domain hold (for estimating $\mathcal{P}\psi$ we need to use that θ and ψ are small with respect to $L(\theta, \psi)$ and thus quadratically small with respect to ψ which follows by using Cauchy estimates for $L(\theta, \psi)$). The problem is that the estimates for other functions, $\mathcal{E}\theta$ and $\mathcal{P}\theta$, can be obtained by this method only in a domain which is in some directions much smaller than the initial one (multiplied by a constant less than one coming from the contraction of some directions by the action of A). Such a loss at every iterative step cannot lead to a convergence. Thus, lack of analytic result may be due to specific constructions in Section 4.5 but it may also be due to the highly hyperbolic nature of the problem at hand.

8. **On the application of general implicit function theorems**

The iterative procedure carried out in Sections 5.3 and 5.4 relies essentially on the result of Lemma 4.6. Taking into account Lemmas 4.4 and 4.7, the result of Lemma 4.6 can be interpreted as *there exists an approximate right inverse for the operator*

$$T_0 \colon \Omega \mapsto \alpha\Omega - \Omega \circ \alpha$$

in the C^∞ category, with tame estimates for the C^r norms. Operator T_0 maps C^∞ maps on the torus to C^∞ α-twisted cocycles over the action α. (*Note.* As before α denotes here the unperturbed linear $\mathbb{Z}^k$ action on the torus $\mathbb{T}^N$ by ergodic automorphisms.) Now, operator T_0 is simply the linearization of the conjugacy operator

$$\mathcal{T} \colon H \to H^{-1} \circ \alpha \circ H$$

which takes the space of C^∞ diffeomorphisms of $\mathbb{T}^N$ to the space $\mathrm{Act}^\infty(\mathbb{Z}^k, \mathbb{T}^N)$ of smooth actions. Thus what we needed for the convergence in Section 5.4 is *only* an approximate right inverse of the linearization of $\mathcal{T}$ *at the identity*. The identity is in our set-up the *initial guess* for the solution. Let us denote by T_H the linearization of $\mathcal{T}$ at H when H is not the identity.

It is obvious that the problem of finding a conjugacy for an arbitrary small perturbation of an action is a problem of inverting a nonlinear operator, and it can also be viewed as an implicit function problem.

After the first results of Kolmogorov, Arnold and Moser where the method of linearization and successive iterations was applied to produce a solution to a nonlinear problem, there were several generalizations and refinements of those iterative schemes. These results are labeled in literature as "hard implicit function theorems" or "generalized implicit function theorems". Typically, these results state that under certain conditions on the linearization of the nonlinear operator *in some neighborhood of the initial guess* (most importantly, assuming existence of an approximate or an exact right inverse of the linearization) there exists a solution to the nonlinear problem.

Below we discuss two such results and we discuss why they can be applied to the problem treated in Theorem 1, or not.

One of early generalized implicit function theorems is due to Zehnder in 1975 [38, Theorem 3.1]. The main requirement for the application of Zehnder's theorem in the set-up of Theorem 1 is existence of an *approximate* right inverse of the linearization T_H for H *in a neighborhood of the initial guess*. However, his condition on how approximate should the right inverse be at the initial guess, is too strong for our purposes: it requires *an exact* right inverse at the initial guess. This is clearly something we cannot produce in our situation: existence of an exact right inverse at the identity would imply that the space of perturbations lifts to a space of α twisted cocycles, which is not true. (As explained in Section 3.3 the linearization produces only *almost* α twisted cocycles, not actual α twisted cocycles.)

On the other hand, at the end of his paper Zehnder makes a very important remark that due to the algebraic structure of conjugacy classes, the main requirement of his result concerning existence of an approximate right inverse in a neighborhood, can be relaxed to existence of an approximate right inverse of the linearization operator *at the initial guess only*. However, this is not proved explicitly in [38] and it is not clear from the remark what would be exactly the modifications in the result [38, Theorem 3.1] if one restricts it to conjugacy operator. It is possible that Zehnder's result can be adapted in this direction, so that it can be applied in the set-up of Theorem 1 to substitute for explicit proof of convergence in Section 5.4. This would only slightly improve the result in the finitely differentiable case due to the more optimal use of smoothing operator's in Zehnder's work, but it would result in considerable divergence from the main issue of the current paper.

The second result we wish to discuss applies in the set-up of Theorem 1, although it does not apply in the finitely differentiable set-up. This is the Nash-Moser theorem for exact sequences due to Hamilton [15, Theorem 3.1.1] in 1982, an extremely convenient result when one is dealing with rigidity of group actions. We refer to [12] for an excellent account on application of Nash-Moser theorem for exact sequences to rigidity of actions of finitely generated finitely presented groups, and we restrict here just to few remarks relative to the current paper.

The commutativity relation in the acting group induces an operator $\mathcal{M}$ which takes a pair of diffeomorphisms to their commutator. Let $M_{\mathcal{T}(H)}$ be the linearization of $\mathcal{M}$ at $\mathcal{T}(H)$, and let M_0 be the linearization of $\mathcal{M}$ at $\alpha = \mathcal{T}(id)$. The main requirement for the application of Hamilton's theorem is existence of a splitting of the short exact sequence of operators $M_{\mathcal{T}(H)} T_H = 0$ in the neighborhood of the initial guess. In the current paper Lemma 4.4 shows that $M_0 T_0 = 0$ indeed holds, and Lemma 4.6 shows the existence of the splitting for the short exact sequence $M_0 T_0 = 0$. (The operator M_0 restricted to two generators is the operator L which is discussed in Section 4.)

Zehnder's remark that the algebraic structure of conjugacy problems relaxes the conditions of his theorem, has been formalized and proved in the context of Hamilton's result by Benveniste [1, Lemma 4.3] in 2000. Namely, [1, Lemma 4.3] shows that if the nonlinear problem under consideration is the conjugacy problem, then existence of a tame splitting of $M_0 T_0 = 0$ implies existence of a tame splitting of $M_{\mathcal{T}(H)} T_H = 0$ for H in some small neighborhood of the identity. Benveniste applied this fact along with the Hamilton's result to prove local rigidity for isometric actions by lattices in simple Lie groups. More recently similar approach was used by Fisher [12] to prove local rigidity for isometric actions of discrete groups with property (T). Lemma 4.3 in [1], combined with Lemmas 4.6, 4.4 and 4.7, imply that Hamilton's theorem can be applied in the set-up of Theorem 1 to give a C^∞ conjugacy. However, Hamilton's result does not give explicitly the topology in which the perturbation has to be small nor how it depends on the unperturbed action and it does not apply to the finitely differentiable situation.

9. **Existence of genuinely partially hyperbolic actions**

To prove the statement of Theorem 3 we first give a proof that there are no irreducible automorphisms in odd dimensions with nontrivial neutral direction. This is proved in [33], but the proof we give here is considerably simpler. Then we eliminate the possibility of having genuinely partially hyperbolic actions on $\mathbb{T}^2$ and $\mathbb{T}^4$. This leaves open the question of existence of examples on $\mathbb{T}^N$ for $N \geq 6$. We then give an outline of an *explicit construction* of an irreducible example of a genuinely partially hyperbolic action on $\mathbb{T}^6$ with two-dimensional neutral direction. This construction can be used further as a model for constructing examples with various properties (with neutral direction of any even dimension, for example). Finally, we show that there exist irreducible examples of genuinely partially hyperbolic actions in any even dimension $N \geq 6$. By combining these results we obtain reducible examples in any odd dimension $N \geq 9$.

9.1. *There are no irreducible examples of genuinely partially hyperbolic actions on $\mathbb{T}^N$ for N odd.* As mentioned in the introduction to this section, there are no irreducible genuinely partially hyperbolic toral automorphisms in any odd dimension, thus there are no such actions in any odd dimension either. This was proved in [33]. This fact also follows from a simple number theoretic argument. Namely, if we let $p(x)$ be the irreducible characteristic polynomial of an integer matrix A of degree N, let ψ be root of p of absolute value 1, and let $\theta \stackrel{\text{def}}{=} \psi + \psi^{-1}$, then for the corresponding number fields $L = \mathbb{Q}(\psi)$ and $K = \mathbb{Q}(\theta)$ we have

$$|L : K||K : \mathbb{Q}| = N.$$

On the other hand K is real since $\theta = \psi + \psi^{-1} = \psi + \overline{\psi} \in \mathbb{R}$ so $|L : K| \geq 2$. But we also have that $\psi^2 - \psi\theta + 1 = 0$; therefore $|L : K| \leq 2$. This implies that $|L : K| = 2$; thus N has to be even.

9.2. *There are no irreducible examples of genuinely partially hyperbolic actions on $\mathbb{T}^N$ for $N = 2$ or $N = 4$.*

Case $N = 2$. An integer 2×2 matrix which induces an ergodic toral automorphism must have a nontrivial expanding and a nontrivial contraction direction. Since it is of dimension 2, there are no other directions, in particular, it cannot have any eigenvalues on the unit circle. We note that it is, of course, possible to have one hyperbolic integer matrix of determinant one in dimension 2, but no two commuting ones (that are not powers of each other) exist due to the Dirichlet Units Theorem [19].

Case $N = 4$. In dimension 4 it is possible to have one matrix with desired properties, namely a matrix that is integer and has two complex conjugate eigenvalues of absolute value one, and two real eigenvalues λ and λ^{-1}. To produce such matrix it is enough to choose a quadratic irreducible polynomial $p(x)$ with two real eigenvalues one bigger than 2 and the other less than 2 in absolute value. Then the substitution $x^2 p(x + \frac{1}{x})$ gives a fourth degree polynomial which is a characteristic polynomial of a matrix with desired properties. However, it is not possible to have two commuting matrices with the properties above and with common neutral subspace (this last one is a necessary requirement for otherwise the action generated by the two matrices would be hyperbolic). Indeed, if B commutes with A and has real eigenvalues μ and μ^{-1}, then, because of irreducibility requirement, both λ and μ are irrational, thus we can choose an integer vector (l, k) in $\mathbb{Z}^2$ such that $\lambda^l \mu^k$ is close to 1. But then the same holds true for the remaining real eigenvalues. Moreover, rotations in the neutral direction are assumed irrational; therefore, by choosing k and l large enough, they can be made close to one, also. This implies that some power of matrix $A^l B^k$ is integer matrix close to identity. This is not possible unless the matrix is identity itself, in which case the action has a nontrivial nonergodic element, thus would not be a higher rank action.

9.3. *Construction of an irreducible partially hyperbolic action with two-dimensional neutral direction in dimension* 6. The following example of irreducible genuinely partially hyperbolic $\mathbb{Z}^2$ action on the torus $\mathbb{T}^6$ was produced by S. Katok with the use of PARI program. For the background on use of algebraic number theory for producing examples of higher rank actions by toral automorphisms we refer to [19].

Starting with an irreducible cubic polynomial $x^3 - 2x^2 - 8x + 1$ with two real roots of absolute value larger than 2 and one less than 2 in absolute value, by substituting $x \to x + \frac{1}{x}$ we obtain an irreducible recurrent polynomial

$$f(x) = x^6 - 2x^5 - 5x^4 - 3x^3 - 5x^2 - 2x + 1,$$

which has four real roots and a pair of complex conjugate roots. The complex conjugate roots have to be of absolute value 1.

Let $K=\mathbb{Q}(\lambda)$ be the number field corresponding to $f(x)$. The fundamental units are $\lambda_1=\lambda, \lambda_2=\lambda^5-3\lambda^4-2\lambda^3-\lambda^2-4\lambda+1, \lambda_3=\lambda^4-2\lambda^3-6\lambda^2-\lambda+1$, $\lambda_4=2\lambda^5-6\lambda^4-3\lambda^3-6\lambda^2-6\lambda$. It turns out that λ_4 produces a matrix with same neutral subspace as A.

With respect to the basis $\{1,\lambda,\lambda^2,\lambda^3,\lambda^4,\lambda^5\}$ in $\mathbb{Z}[\lambda]$ multiplications by λ_1 and λ_4 are given by the matrices

$$A=\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0\\ 0 & 0 & 1 & 0 & 0 & 0\\ 0 & 0 & 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0 & 0 & 1\\ -1 & 2 & 5 & 3 & 5 & 2 \end{pmatrix},$$

$$B=\begin{pmatrix} 0 & -6 & -6 & -3 & -6 & 2\\ -2 & 4 & 4 & 0 & 7 & -2\\ 2 & -6 & -6 & -2 & -10 & 3\\ -3 & 8 & 9 & 3 & 13 & -4\\ 4 & -11 & -12 & -3 & -17 & 5\\ -5 & 14 & 14 & 3 & 22 & -7 \end{pmatrix}.$$

Since $B=2A^5-6A^4-3A^3-6A^2-6A$, $AB=BA$, and since the minimal polynomial of the matrix B is recurrent, $x^6+23x^5+16x^4-60x^3+16x^2+23x+1$ and also has four real roots, the two complex conjugate roots are of absolute value 1.

9.4. *Existence of irreducible examples of genuinely partially hyperbolic actions in any even dimension greater than* 6. In this section we use the Dirichlet units theorem to prove the main part of Theorem 3 stated in Section 1.3, that is, to produce examples of $\mathbb{Z}^2$ genuinely partially hyperbolic actions in any even dimension $N\geq 6$.

Let $q(x)$ be an irreducible integer polynomial of degree n with all real roots. Let $r(x)=q(x+\frac{1}{x})x^n$ be the recurrent polynomial of degree $2n$ given by q. Let $L=\mathbb{Q}(\psi)$ and $K=\mathbb{Q}(\theta)$ be the number fields corresponding to q and r respectively with $\psi+\psi^{-1}=\theta$. Let σ_i be embeddings of K into $\mathbb{C}$. Since all roots of r come in pairs, let $\sigma_i(\psi)\sigma_{n+i}(\psi)=1$ for all $i=1,\dots,n$. Let $\alpha=p(\psi)\in L$ with $\deg(p)\leq 2n$ be any other element in L. Then $\sigma_i(p(\psi^{-1}))=p(\sigma_i(\psi^{-1}))=p((\sigma_i(\psi))^{-1})=p(\sigma_{n+i}(\psi))=\sigma_{n+i}(p(\psi))$. This implies that $p(\psi)$ and $p(\psi^{-1})$ have same norms $N(p(\psi))=N(p(\psi^{-1})=\sigma_i(\psi)\cdots\sigma_{2n}(\psi)$. Therefore, $p(\psi)$ is a unit if and only if $p(\psi^{-1})$ is.

Let U_L and U_K be groups of units of L and K, respectively. Define a homomorphism $f\colon U_L\to U_K$ by

$$f(p(\psi))\stackrel{\text{def}}{=}p(\psi)p(\psi^{-1}).$$

We first show that $f(U_L)$ is indeed in U_K. Since for any integer polynomial $p(x)$, we have $p(x)p(x^{-1}) = P(x + x^{-1})$ where P is a rational polynomial, it follows that $f(p(\psi)) = P(\theta)$ thus $f(U_L) \subset K$. Also $p(\psi)$ being a unit implies that $p(\psi^{-1})$ is also a unit and thus $P(\theta)$ is a unit in U_L. Since $\psi^2 - \theta\psi + 1 = 0$ we have $|L : K| \leq 2$. With $|L : L \cap \mathbb{R}| = 2$ and the fact that K is real, this implies $K = L \cap \mathbb{R}$. Therefore, a real unit in U_L must lie in U_K. This proves that the image of f is in U_K. It is then easy to check that f is a group homeomorphism.

In order to obtain matrices that commute with matrix A whose characteristic polynomial is $r(x)$ it is enough to show that the kernel of f contains at least two independent units. We show that the kernel contains s independent units where s is the number of roots of $q(x)$ of absolute value bigger than 2 (which implies $2s$ real roots for $r(x)$). The rest $t = n - s$ roots of $q(x)$ are of absolute value less than 2 and they induce t pairs of complex conjugate roots of $r(x)$ of absolute value 1. By Dirichlet units theorem the structure of groups U_L and U_K is known. Namely, every unit in U_L can be expressed as $\rho u_1^{j_1} u_2^{j_2} \cdots u_{s+n-1}^{j_{s+n-1}}$, where u_i are independent units. The images $f(u_i)$ of fundamental units of U_L are units in U_K. By Dirichlet units theorem U_K can have at most $n-1$ independent units. Therefore, in the set $S = \{f(u_i) | i = 1, \ldots, n+s-1\}$ there can be at most $n-1$ independent units, without loss of generality assume those are $f(u_i), i = 1, \ldots, n-1$. This implies that there are at least s relations of the kind:

$$1 = f(u_1)^{j_{1,k}} f(u_2)^{j_{2,k}} \cdots f(u_{n-1})^{j_{n-1,k}} f(u_k)^{j_k},$$

where $k = n, \ldots, n+s$. Since f is a homomorphism, this implies that there are s units in the kernel of f:

$$v_k = u_1^{j_{1,k}} u_2^{j_{2,k}} \cdots u_{n-1}^{j_{n-1,k}} u_k^{j_k}$$

for $k = n, \ldots, n+s$. Moreover, all v_k are independent since u_i are.

This implies that if $s \geq 2$ then there exists a unit $v = p(\psi)$ such that $B = p(A)$ commutes with A, A and B are multiplicatively independent, and B has the same neutral eigenspace as A. Namely, since v is in the kernel of f we have $\sigma_i(f(v)) = 1$ for all i, i.e., $1 = \sigma_i(p(\psi))\sigma_i(p(\psi^{-1}))p(\sigma_i(\psi))p(\sigma_i(\psi^{-1})) = p(\psi_i)p(\psi_i^{-1})$.

In particular, if ψ_i is a root of r of absolute value 1, then $\psi_i^{-1} = \overline{\psi_i}$; therefore $1 = p(\psi_i)p(\overline{\psi_i}) = p(\psi_i)\overline{p(\psi_i)} = |p(\psi_i)|$. Thus B has neutral eigenvalues in the same direction as A and the action generated by A and B is genuinely partially hyperbolic.

9.5. *Existence of reducible examples of genuinely partially hyperbolic actions in any odd dimension greater than* 9. Reducible examples of $\mathbb{Z}^2$ actions on torus $\mathbb{T}^N$ viewed as $\mathbb{T}^{n_1} \times \mathbb{T}^{n_2}$ $(N = n_1 + n_2)$ are obtained from generator g_i of $\mathbb{Z}^2$ acting by $A_{i,j}$ on $\mathbb{T}^{n_j}$, where $i, j \in \{1, 2\}$ and $A_{i,j}$ are matrices in $\mathrm{GL}(n_j, \mathbb{Z})$. If such an action is genuinely partially hyperbolic then one of the reduced actions (on tori of dimension n_1 or n_2) would have to be genuinely partially hyperbolic. This, along with the preceding discussion on dimensions less than 5 implies that there are no reducible examples on the torus of dimension $N = 7$.

In any odd dimension $N \geq 9$, as proven in Section 9.1, there are no irreducible examples. However reducible examples exist as products of purely hyperbolic actions (which exist already in dimension 3) and the irreducible examples previously produced in Section 9.4. For example, on the nine-dimensional torus, in the construction above take $n_1 = 3$, $n_2 = 6$, let $A_{1,2}$ and $A_{2,2}$ be commuting matrices from the Section 9.3 and for $A_{1,1}$ and $A_{2,1}$ choose any two 3×3 commuting hyperbolic integer matrices which generate an irreducible $\mathbb{Z}^2$ action (for various examples see [19]).

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