Curves and symmetric spaces, II

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Abstract

Let $\text{Sym}_3 C \longrightarrow \mathbf{P}_*(k \oplus \text{Sym}_3 k \oplus \text{Sym}_3 k \oplus k) = \mathbf{P}^{13}$, $A \mapsto (1 : A : A' : \det A)$ be the Veronese embedding of the space of symmetric matrices of degree 3, where $A'$ is the cofactor matrix of $A$. The closure $\text{SpG}(3, 6)$ of this image is a 6-dimensional homogeneous variety of the symplectic group $\text{Sp}(3)$. A canonical curve $C_{16} \subset \mathbf{P}^8$ of genus 9 over a perfect field $k$ is isomorphic to a complete linear section of this projective variety $\text{SpG}(3, 6) \subset \mathbf{P}^{13}$ unless $C \otimes_k \bar{k}$, $\bar{k}$ being the algebraic closure, is a covering of degree at most 5 of the projective line. We prove this by means of linear systems of higher rank.

Introduction

Let $\text{SpG}(n, 2n)$ be the symplectic Grassmannian, that is, the Grassmannian of Lagrangian subspaces of a $2n$-dimensional symplectic vector space, over a field $k$. In the case $n = 3$, $\text{SpG}(3, 6)$ is of dimension 6 and embedded into the projective space $\mathbf{P}^{13}$ with homogeneous coordinate $(y : X : Y : x)$, where $x, y \in k$ are scalars and $X, Y \in \text{Sym}_3 k$ are symmetric matrices. Then $\text{SpG}(3, 6) \subset \mathbf{P}^{13}$ is the common zero locus of the following 21 (=6+6+9) quadratic equations

\begin{equation}
X' = yY, \quad Y' = xX \in \text{Sym}_3 k \quad \text{and} \quad XY = xyI_3 \in \text{Mat}_3 k,
\end{equation}

which will be derived in Section 2 after Proposition 2.3.

In our study of Fano 3-folds, we observed that this (symmetric) projective variety has a canonical curve section of genus 9, that is, a transversal intersection

$$[C \subset \mathbf{P}^8] = [\text{SpG}(3, 6) \subset \mathbf{P}^{13}] \cap H_1 \cap \cdots \cap H_5$$

is a curve of genus 9 embedded in $\mathbf{P}^8$ by the ratio of the differentials of the first kind. We showed that every general curve of genus 9 is obtained in this way when $k = \mathbb{C}$ ([10, Cor. 6.3]). The purpose of this article is to show the following refinement, which was partly announced in [11].

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THEOREM A. Let $C$ be a curve of genus 9 over an algebraically closed field $k$. Then $C$ is isomorphic to a transversal linear section of the 6-dimensional symplectic Grassmannian $\text{SpG}(3, 6) \subset \mathbb{P}^{13}$ if and only if $C$ is not pentagonal, i.e., $C$ has no $g_5^1$.

By Bertini’s theorem we have

COROLLARY. If $C$ satisfies the condition given in Theorem A and if $k$ is of characteristic zero, then $C$ is contained in a smooth $K3$ surface as an ample divisor.

This theorem, together with similar results [14] and [13] in genus 7 and 8, will be applied to our classification of Gorenstein-Fano 3-folds with only canonical singularities (cf. [15]). We prove the theorem using a certain simple vector bundle of rank 3. By its uniqueness (see below) and by a standard descent argument (§7), we have the following also:

THEOREM B. Let $C$ be a curve of genus 9 defined over a perfect field $k$ and assume that $C$ has no $g_5^1$ over the algebraic closure $\bar{k}$. Then we have

1. $C$ has an embedding into the 6-dimensional symplectic Grassmannian $\text{SpG}(3, 6) \subset \mathbb{P}^{13}$ over $k$ whose image is a transversal intersection with a $k$-linear subspace $P \subset \mathbb{P}^{13}$ of dimension 8, and

2. such subspaces $P$ cutting out $C$ are unique up to the action of $\text{PGSp}(3)$. More precisely, for every isomorphism $g : C = \text{SpG}(3, 6) \cap P \to C' = \text{SpG}(3, 6) \cap P'$ there exists $\gamma \in \text{PGSp}(3, k)$ such that $\gamma(P) = P'$.

Here $\text{PGSp}(3)$ is the subgroup of $\text{PGL}(6)$ stabilizing the 1-dimensional space generated by a symplectic form. Let $G(8, \mathbb{P}^{13})$ be the Grassmannian of 8-dimensional linear subspaces $P$ of $\mathbb{P}^{13}$ and $G(8, \mathbb{P}^{13})^\circ$ the open subset consisting of $P$’s such that the intersection $P \cap \text{SpG}(3, 6)$ is transversal.

COROLLARY. The weighted cardinality, or mass, of the nonpentagonal curves $C$ of genus 9 over the finite field $\mathbb{F}_q$ is equal to $\frac{\#G(8, \mathbb{P}^{13})^\circ(\mathbb{F}_q)}{\#\text{PGSp}(3, \mathbb{F}_q)}$:

$$\sum_{\text{nonpentagonal}} \frac{1}{\#\text{Aut}_FC} = \frac{\#G(8, \mathbb{P}^{13})^\circ(\mathbb{F}_q)}{q^9(q^6-1)(q^4-1)(q^2-1)}.$$
on the choice of $\alpha$ and is characterized by the following property (Proposition 5.6):

\[
\begin{align*}
\text{(i)} & \quad \bigwedge^3 E \cong K_C, \\
\text{(ii)} & \quad h^0(E) = 6, \text{ and} \\
\text{(iii)} & \quad |E| \text{ is free and semi-irreducible.}
\end{align*}
\]

Such a bundle $E$ gives rise to a morphism $\Phi_E : C \to G(H^0(E_{\text{max}}), 3)$ to the Grassmannian of 3-dimensional quotient spaces of $H^0(E_{\text{max}})$ ($\S 3$). The following is the essence of Theorems A and B:

**Theorem C.** Let $C$ be a nonhyperelliptic curve of genus 9 over an algebraically closed field and assume that a rank 3 vector bundle $E = E_{\text{max}}$ on it satisfies the condition (0.2). Then the natural linear maps

\[
\lambda_2 : \bigwedge^2 H^0(E) \to H^0\left(\bigwedge^2 E\right) \quad \text{and} \quad \lambda_3 : \bigwedge^3 H^0(E) \to H^0\left(\bigwedge^3 E\right) \cong H^0(K_C)
\]

are surjective and $\text{Ker} \lambda_2$ is generated by a nondegenerate bivector $\sigma$. The image of $\Phi_E$ is contained in the symplectic Grassmannian $G(H^0(E), \sigma)$ (see $\S 2$) and the commutative diagram

\[
\begin{array}{ccc}
C & \to & G(H^0(E), \sigma) \\
\downarrow & & \downarrow \text{Plücker} \\
\mathbb{P}^8 & \to & \mathbb{P}^* \bigwedge^3 (H^0(E), \sigma) \\
\end{array}
\]

is cartesian, where $\lambda_3$ is the linear map

\[
\bigwedge^3 (H^0(E), \sigma) := \bigwedge^3 H^0(E)/(\sigma \wedge H^0(E)) \to H^0\left(\bigwedge^3 E\right) \cong H^0(K_C)
\]

induced by $\lambda_3$.

**Notation and conventions.** For a vector space $V$, the second exterior product $\bigwedge^2 V$ is the quotient of $V \otimes V$ by the subspace generated by $v \otimes v$, $v \in V$. Similarly $S^2 V$ is the quotient generated by $u \otimes v - v \otimes u$, $u, v \in V$. An element of $\bigwedge^2 V$ is called a bivector of $V$. We denote by $G(r, V)$ and $G(V, r)$ the Grassmannians of $r$-dimensional subspaces and quotient spaces of $V$, respectively. Two projective spaces $G(1, V)$ and $G(V, 1)$ associated to $V$ are denoted by $\mathbb{P}_*(V)$ and $\mathbb{P}^*(V)$, respectively. $\mathbb{P}_*$ is a covariant functor and $\mathbb{P}^*$ is contravariant. For a vector space or vector bundle $V$, its dual is denoted by $V^\vee$. The tensor product symbol $\otimes$ between a vector bundle and a line bundle is often omitted when there seems no fear of confusion.

All (algebraic) varieties are considered over a fixed base field $k$. A smooth complete geometrically irreducible curve is simply called a curve. By a $g^r_d$, we mean a line bundle $L$ on a curve with $\deg L = d$ and $\dim H^0(L) \geq r + 1$. A saturation of a subsheaf $F \subset E$ is the largest subsheaf $\tilde{F}$ between $F$ and $E$ such that $\tilde{F}/F$ is torsion.
1. Preliminaries

We prove two lemmas on the number of global sections. Let \( \xi \) be a line bundle on a curve \( C \) and \( \eta \) the Serre adjoint \( K_C \xi^{-1} \). We denote the evaluation homomorphism \( H^0(\eta) \otimes \mathcal{O}_C \to \eta \) by \( \text{ev}_\eta \) and the dual of its kernel by \( Q_\eta \). We have an exact sequence

\[
0 \to Q_\eta^\vee \to H^0(\eta) \otimes \mathcal{O}_C \to \eta.
\]

Its dual

\[
0 \to \eta^{-1} \to H^0(\eta)^\vee \otimes \mathcal{O}_C \to Q_\eta \to 0
\]

is also exact if \( \eta \) is free. The rank of \( Q_\eta \) is equal to \( \dim |\eta| = r - 1 \), where we put \( r = h^0(\eta) \). The following is a variant of the so-called base point free pencil trick.

**Lemma 1.1.** For a vector bundle \( E \) of rank \( r \) on \( C \),

\[
\dim \text{Hom}(E, \xi) + \dim \text{Hom}(Q_\eta, E) \geq r(h^0(E) - \deg \eta) - \chi(E).
\]

**Proof.** Take the global section of the exact sequence (1.1) tensored with \( E \).

Then we have

\[
\dim \text{Hom}(Q_\eta, E) + h^0(E\eta) \geq rh^0(E).
\]

By the Riemann-Roch theorem and Serre duality,

\[
h^0(E\eta) - h^0(E^\vee \xi) = \chi(E\eta) = \chi(E) + r \deg \eta.
\]

Our assertion follows immediately from these.

If \( E \) is of canonical determinant, i.e., \( \bigwedge E \simeq K_C \), then

\[
\dim \text{Hom}(E, \xi) + \dim \text{Hom}(Q_\eta, E) \geq r(h^0(E) - r - s) - 2\rho + 2,
\]

since \( \chi(E) = (r-2)(1-g) \), where \( s = h^0(\xi) = h^1(\eta) \) and \( \rho \) is the Brill-Noether number of \( \eta \), or equivalently, of \( \xi \).

The number of global sections behaves specially if a vector bundle has a nondegenerate quadratic form with values in \( K_C \). The following is one of such phenomena clarified in Mumford [16].

**Proposition 1.2.** Let \( E \) and \( F \) be rank two vector bundles on a curve \( C \) such that \( (\det E) \otimes (\det F) \simeq K_C \). Then \( h^0(E \otimes F) \) is congruent to \( \deg E \mod 2 \).

**Proof.** Choose a line subbundle and express \( F \) as an extension

\[
0 \to L \to F \to M \to 0
\]

of line bundles. The alternating bihomomorphism \( E \times E \to \det E, (s,t) \mapsto s \wedge t \), induces a bilinear map

\[
\varphi : H^0(E \otimes M) \times H^1(E \otimes L) \to H^0((\det E) \otimes (\det F)) = H^0(K_C) \simeq k,
\]

which is nondegenerate by Serre duality. Let \( e \in H^1(M^{-1} \otimes L) \) be the extension class of (1.4) and \( \delta : H^0(E \otimes M) \to H^1(E \otimes L) \) be the coboundary map coming
from $E \otimes (1.4)$. Then $\varphi(s, \delta(s)) = s \cup (s \cup e) = (s \wedge s) \cup e = 0$ for $s \in H^0(E \otimes M)$. Therefore, the linear map $\delta$ is alternating with respect to the Serre pairing $\varphi$. Hence $h^0(E \otimes F)$ is congruent to

$$h^0(E \otimes L) + h^0(E \otimes M) = h^0(E \otimes L) + h^1(E \otimes L)$$

modulo 2. Since $h^0(E \otimes L) - h^1(E \otimes L)$ is congruent to $\deg(E \otimes L)$, we have our assertion.

\[\square\]

2. Symplectic Grassmannian

Let $A$ be a $k$-vector space. For a subspace $B \subset A$ the linear map $\wedge^2 B \to \wedge^2 A$ is injective.

**Definition 2.1.** A bivector $\sigma \in \wedge^2 A$ is degenerate if $\sigma$ is contained in $\wedge^2 B$ for a proper subspace $B \subset A$.

A bivector $\sigma$ is always degenerate if $\dim A$ is odd. In the case $\dim A$ is even, $\sigma$ is degenerate if and only if the value of the Pfaffian is zero. There exists a minimal subspace $B \subset A$ such that $\sigma \in \wedge^2 B$. This subspace $B$ is called the co-radical of $\sigma$.

**Definition 2.2.** A symplectic vector space is a pair $(V, \sigma)$ of a vector space $V$ and a nondegenerate bivector $\sigma \in \wedge^2 V^\vee$ of the dual vector space.

Note that $\wedge^2 V^\vee$ is the quotient of $V^\vee \otimes V^\vee$ by the subspace $SB(V)$ of symmetric bilinear forms on $V$. When the characteristic of $k$ is not 2, the equivalence class $\sigma + SB(V)$ has the unique anti-symmetric representative, say $\sigma^{AS}$, in $V^\vee \otimes V^\vee$. A subspace $U \subset V$ is Lagrangian if $2 \dim U = \dim V$ and the restriction $\sigma|_U : U \times U \to k$ of $\sigma$ to $U$ is symmetric. If $\text{char}(k) \neq 2$, then the second condition is equivalent to the usual one; that is, $\sigma^{AS}|_U = 0$. We denote the set of Lagrangian subspaces of $(V, \sigma)$ by $G(\sigma, V)$.

Two vectors $u$ and $v \in V$ are perpendicular with respect to $\sigma$ if the restriction of $\sigma$ to the subspace spanned by $u$ and $v$ is symmetric. For a nonzero vector $v \in V$, the set of vectors $u \in V$ perpendicular to $v$ is a subspace of codimension one. We denote this subspace by $v^\perp$. $\sigma$ induces a bilinear form $\bar{\sigma}$ on the quotient space $\bar{V} := v^\perp/kv$ and $(\bar{V}, \bar{\sigma})$ becomes a symplectic vector space of dimension two less. If a Lagrangian subspace $U$ of $(V, \sigma)$ contains $v$, then the quotient $U/kv$ is a Lagrangian of $(\bar{V}, \bar{\sigma})$. Conversely, if $\bar{U}$ is a Lagrangian of $(\bar{V}, \bar{\sigma})$, then its inverse image by $v^\perp \to \bar{V}$ is a Lagrangian of $(V, \sigma)$ which contains $v$. By this correspondence we identify $G(\bar{\sigma}, \bar{V})$ with the subset of $G(\sigma, V)$ consisting of $[U]$ with $v \in U$.

For our purpose, the Grassmannian of quotient spaces is more convenient than that of subspaces. A quotient space $A \xrightarrow{f} Q$ of $A$ is Lagrangian with respect to a nondegenerate bivector $\sigma$ if $2 \dim W = \dim A$ and if $(\wedge^2 f)(\sigma) = 0$. We denote the set of Lagrangian quotient spaces of the pair $(A, \sigma)$ by $G(A, \sigma)$, which coincides
with $G(\sigma, A^\vee)$. Let $\mathcal{U}$ be the universal quotient bundle on $G(A, n)$, $\dim A = 2n$. Then $\sigma \in \bigwedge^2 A$ determines a global section of $\bigwedge^2 \mathcal{U}$, which we denote by $s$. Then $G(A, \sigma)$ coincides with the zero set of $s \in H^0(G(A, n), \bigwedge^2 \mathcal{U})$. We endow $G(A, \sigma)$ with a scheme structure by considering it as the zero locus of $s$. An element of this isomorphism class is denoted by $\text{SpG}(n, 2n)$.

**Proposition 2.3.** The symplectic Grassmannian $G(A, \sigma)$ is a smooth variety of dimension $n(n + 1)/2$ and the anti-canonical class is $n + 1$ times the hyperplane section $H$ of the Plücker embedding.

**Proof.** Since $\bigwedge^2 A$ generates $\bigwedge^2 \mathcal{U}$, $G(A, \tilde{\sigma})$ is locally a smooth complete intersection for general $\tilde{\sigma}$ by the Bertini theorem for vector bundles ([12, Th. 1.10]). Since the $GL(2n)$-orbit of nondegenerate bivectors is dense in $\bigwedge^2 A$, $G(A, \sigma)$ is isomorphic to $G(A, \tilde{\sigma})$. It is of dimension $n^2 - \text{rank} \bigwedge^2 \mathcal{U} = n(n + 1)/2$. It is irreducible since the symplectic group $\text{Sp}(n)$ acts transitively. The conormal bundle $\mathcal{J}/\mathcal{J}^2$ of $G(A, \sigma)$ is the restriction of $(\bigwedge^2 \mathcal{U})^\vee$, where $\mathcal{J}$ is the ideal sheaf. ($\mathcal{J}$ is the image of $(\bigwedge^2 \mathcal{U})^\vee \to \mathcal{O}_{G(A, n)}$ and $[(\bigwedge^2 \mathcal{U})^\vee / \mathcal{J}] \otimes \mathcal{O}_{G(A, \sigma)}$ is an isomorphism.) Since $c_1(G(A, n)) = 2nH$ and $c_1(\bigwedge^2 \mathcal{U}) = (n - 1)H$, the anti-canonical class of $G(A, \sigma)$ is equal to the restriction of $c_1(G(A, n)) - c_1(\bigwedge^2 \mathcal{U}) = (n + 1)H$. \hfill $\square$

Choose a pair of Lagrangian subspaces $U_0$ and $U_\infty$ of a symplectic vector space $(V, \sigma)$ with $U_0 \cap U_\infty = 0$. For a linear map $f : U_0 \to U_\infty$ the graph $\Gamma_f \subset U_0 \times U_\infty = V$ is Lagrangian if and only if $f \in \text{Hom}(U_0, U_\infty) \simeq U_\infty \otimes U_\infty$ is a symmetric tensor. The Plücker coordinate of $\Gamma_f$ is equal to

$$1 + f + (f \wedge f) + (f \wedge f \wedge f) + \cdots$$

(cf. [14, §1]). Hence, for example, the 9-dimensional Grassmannian $G(3, 6)$ is the closure of the *Veronese embedding* of the space of square matrices of degree 3,

$$\text{Mat}_3 \mathbb{C} \longrightarrow \mathbb{P}_*(k \oplus \text{Mat}_3 k \oplus \text{Mat}_3 k \oplus k), \quad A \mapsto (1 : A : A' : \det A),$$

where $A'$ is the cofactor matrix of $A$. It is the common zero locus of the Plücker equations

$$X' = yY, \quad Y' = xX \in \text{Mat}_3 k \quad \text{and} \quad XY = YX = xyI_3 \in \text{Mat}_3 k,$$

in the projective space $\mathbb{P}^{19}$ with homogeneous coordinate $(y : X : Y : x)$, where $x, y \in k$ are scalars and $X, Y \in \text{Mat}_3 k$ are square matrices. Restricting ourselves to symmetric matrices, we have the equations $(0.1)$ of $\text{SpG}(3, 6) \subset \mathbb{P}^{13}$.

The divisor class group of the Grassmannian $G(n, 2n)$ is generated by the hyperplane section class $H$. Its Chow group of codimension 2 cycles is generated by two Schubert subvarieties:

$$Y = \{[U] \mid U \cap W \neq 0\} \quad \text{and} \quad Y' = \{[U] \mid U + W' \neq V\}$$

for a subspace $W$ of dimension $n - 1$ and $W'$ of codimension $n - 1$. It is well known that the self intersection $H \cdot H$ is (rationally) equivalent to their sum. On
the symplectic Grassmannian, obviously $Y$ and $Y'$ are equivalent and hence we have

$$H \cdot H \sim Y + Y' \sim 2Y.$$  

Let $a$ be a nonzero vector of $A$. The image $\overset{\circ}{\sigma}$ of $\sigma$ in $\wedge^2(A/ka)$ is degenerate since $\dim(A/ka)$ is odd. In fact, the co-radical $A$ of $\overset{\circ}{\sigma}$ is of codimension one. Similar to the inclusion $G(\overset{\circ}{\sigma}, V) \hookrightarrow G(\sigma, V)$, we have a natural inclusion $G(\overset{\circ}{A}, \overset{\circ}{\sigma}) \hookrightarrow G(A, \sigma)$. Moreover, $G(\overset{\circ}{A}, \overset{\circ}{\sigma})$ is the scheme of zeros of the global section of $\mathscr{E} = \mathcal{U}|_{G(A, \sigma)}$ corresponding to $a \in A$.

Let $G(A, n) \subset \mathbf{P}^* (\wedge^n A)$ be the Plücker embedding of the Grassmannian $G(A, n)$. The tautological line bundle $\mathcal{O}_G(1)$ is isomorphic to $\wedge^n \mathcal{U}$. Since $\sigma$ vanishes on $G(A, \sigma)$, so do all the linear forms $\sigma \wedge (\wedge^{n-2} A) \subset \wedge^n A$. Let $\wedge^n (A, \sigma)$ be the quotient space of $\wedge^n A$ by the subspace $\sigma \wedge (\wedge^{n-2} A)$. Then $G(A, \sigma)$ is contained in the subspace $\mathbf{P}^* (\wedge^n (A, \sigma))$ and we have a commutative diagram

$$\begin{align*}
G(A, \sigma) & \longrightarrow \mathbf{P}^* (\wedge^n (A, \sigma)) \\
G(A, n) & \longrightarrow \mathbf{P}^* (\wedge^n A).
\end{align*}$$

Plücker

$G(A, \sigma)$ coincides with $G(A, 1) = \mathbf{P}^1$ for $n = 1$ and is a smooth hyperplane section of the smooth 4-dimensional quadric $G(A, 2) \subset \mathbf{P}^5$ for $n = 2$.

Now we set $n = 3$ and investigate the conormal space of $G(A, \sigma) \subset \mathbf{P}^3 (\wedge^3 A, \sigma)$ and an important cubic cone in it. Let $A \rightarrow Q$ be a 3-dimensional quotient space and put $W = \text{Ker } [A \rightarrow Q]$. Then we have a filtration of subspaces

$$F_0 = \wedge^3 W \subset F_1 = \left( \wedge^2 W \right) \wedge A \subset F_2 = W \wedge \wedge^2 A \subset F_3 = \wedge^3 A.$$  

Then $\wedge^3 A \rightarrow F_3 / F_2 \simeq \wedge^3 Q$ is the Plücker coordinate of $Q$. $F_2 / F_1$ is isomorphic to $W \otimes (\wedge^2 Q)$. $F_2 = F_1 \otimes \det Q^{-1} \simeq \text{Hom } (Q, W)$ is canonically isomorphic to the cotangent space of $G(A, 3)$ at $[Q]$. $F_1 \otimes \det Q^{-1}$ is canonically isomorphic to the conormal space of $G(A, 3) \subset \mathbf{P}^* (\wedge^3 A)$. Hence we have an exact sequence

$$0 \longrightarrow k \longrightarrow F_1 \otimes \det W^{-1} \longrightarrow \text{Hom } (W, Q) \longrightarrow 0.$$  

$$\downarrow$$  

$$N^\vee_{G(A,3)/\mathbf{P}} \otimes \det \mathcal{Q} \otimes \det W^{-1}$$

Assume that $[A \rightarrow Q] \in G(A, \sigma)$ is Lagrangian. Then $\sigma$ belongs to $W \wedge A \subset \wedge^2 A$. Let

$$\overline{F}_0 \subset \overline{F}_1 \subset \overline{F}_2 \subset \overline{F}_3,$$  

$$\overline{F}_i = F_i / (F_i \cap \sigma \wedge A),$$

be the quotient filtration of (2.4) by $\sigma \wedge A \subset F_2$. Then $\overline{F}_3 / \overline{F}_2 \simeq \wedge^3 Q$ is the Plücker coordinate of $Q$. The cotangent space of $G(3, \sigma)$ at $[Q]$ is $\overline{F}_2 / \overline{F}_1 \otimes \det Q^{-1} \simeq S^2 W$. The conormal space is isomorphic to $\overline{F}_1 \otimes \det Q$ and we have an exact
More precisely, we put \( a \) for \( U \) be a basis of \( \text{Ker} \) is a basis of \( \text{Ker} \). By the exact sequence (2.5),

\[
\begin{array}{c}
0 \to k \to \bar{F}_1 \otimes \det Q \to S^2 Q \to 0.
\end{array}
\]

(2.5)

Let

\[
\alpha : P_*\left( 3 \bigwedge A \right) \cdots \to P_*\left( 3 \bigwedge (A, \sigma) \right)
\]

be the projection with center \( P_*(\sigma \wedge A) \). Since \( \sigma \) is nondegenerate, \( G(3, A) \) is disjoint from the center. We consider the image of the Schubert subvariety \( S_Q = \{ [U] \mid \rk [U \to A \to Q] \leq 1 \} \subset G(3, A) \) by \( \alpha \) for a Lagrangian quotient space \( A \to Q \) (cf. (3.3) and (4.1)). \( S_Q \) is a 5-dimensional subvariety of

\[
P_*\left( \left( 2 \bigwedge W \right) \wedge A \right) = P_*\left( N_{G(A, 3)}/P, Q \right)
\]

and \( \alpha(S_Q) \) is a subvariety of

\[
P_*\left( \bar{F}_1 \right) = P_*\left( N_{G(A, \sigma)}/P, Q \right) = P^6.
\]

By the exact sequence (2.5), \( P^*(N_{G(A, \sigma)}/P, Q) \) has the distinguished point corresponding to \( \text{Ker} \{ A \to Q \} \), which we denote by \( \kappa_Q \), and the special projection onto \( P^*(S^2 Q) \). \( \alpha(S_Q) \) contains the point \( \kappa_Q \).

**Proposition 2.4.** The image \( \alpha(S_Q) \) is a cubic hypersurface of

\[
P^*(N_{G(A, \sigma)}/P, Q).
\]

More precisely, it is the cone over the discriminant hypersurface of \( P^*(S^2 Q) \) with vertex \( \kappa_Q \).

**Proof.** Choose a basis \( \{ v_1, v_2, v_3, v_{-1}, v_{-2}, v_{-3} \} \) of \( A \) such that \( \{ v_1, v_2, v_3 \} \) is a basis of \( \text{Ker} \{ A \to Q \} \) and \( \sigma = v_1 \wedge v_{-1} + v_2 \wedge v_{-2} + v_3 \wedge v_{-3} \). Let \( \{ u_1, u_2, u_3 \} \) be a basis of \( U \in S_Q \) such that \( u_1, u_2 \in \text{Ker} \{ U \to Q \} \). The exterior product \( u_1 \wedge u_2 \) is equal to

\[
a_1 v_2 \wedge v_3 + a_2 v_3 \wedge v_1 + a_3 v_1 \wedge v_2 \in \bigwedge^2 \text{Ker} \{ A \to Q \}
\]

for \( a_1, a_2 \) and \( a_3 \in k \). Put \( u_3 = a_4 v_1 + a_5 v_2 + a_6 v_3 + b_1 v_{-1} + b_2 v_{-2} + b_3 v_{-3} \). Then the Plücker coordinate \( u_1 \wedge u_2 \wedge u_3 \) of \( U \) is

\[
a_0 v_1 \wedge v_2 \wedge v_3 + (a_1 v_2 \wedge v_3 + a_2 v_3 \wedge v_1 + a_3 v_1 \wedge v_2) \wedge (b_1 v_{-1} + b_2 v_{-2} + b_3 v_{-3})
\]

\[
= a_0 v_1 \wedge v_2 \wedge v_3 + (a_2 b_1 v_{12} - a_1 b_2 v_{21}) + (a_1 b_3 v_{31} - a_3 b_1 v_{13})
\]

\[
+ (a_3 b_2 v_{23} - a_2 b_3 v_{32}) + \sum_{i=1}^{3} a_i b_i v_{ii},
\]

where we put \( a_0 = a_1 a_4 + a_2 a_5 + a_3 a_6 \).

\[
v_{11} = v_{-1} \wedge v_2 \wedge v_3, \quad v_{22} = v_1 \wedge v_{-2} \wedge v_3, \quad v_{33} = v_1 \wedge v_2 \wedge v_{-3}
\]
and $v_{jk} = v_i \wedge v_j \wedge v_{-j}$ for every $\{i, j, k\} = \{1, 2, 3\}$. Since $v_{jk} + v_{kj} \in A \wedge \sigma$ for every $j \neq k$, $u_1 \wedge u_2 \wedge u_3$ is congruent to

$$a_0 v_1 \wedge v_2 \wedge v_3 - (a_1 b_2 + a_2 b_1) v_{12} - (a_1 b_3 + a_3 b_1) v_{13} + (a_2 b_3 + a_3 b_2) v_{23} + \sum_{i=1}^{3} a_i b_i v_{ii}$$

modulo $A \wedge \sigma$. Hence $\omega(S_Q)$ consists of those $\gamma_0 v_1 \wedge v_2 \wedge v_3 + \sum_{1 \leq i \leq j \leq 3} \gamma_{ij} v_{ij}$ such that the quadratic form $\sum_{1 \leq i \leq j \leq 3} \gamma_{ij} X_i X_j$ is of rank $\leq 2$ (or equivalently, $4\gamma_{11}\gamma_{22}\gamma_{33} - \gamma_{11}\gamma_{23}^2 - \gamma_{22}\gamma_{13}^2 - \gamma_{33}\gamma_{12}^2 + \gamma_{12}\gamma_{13}\gamma_{23} = 0$).

\section{Linear systems of higher rank}

A linear system of rank $r$ is a pair $(E, A)$ of a vector bundle $E$ of rank $r$ and a space of global sections $A \subset H^0(E)$. The special one with $A = H^0(E)$ is called a complete linear system and denoted by $|E|$. A linear system $(E, A)$ on an algebraic variety $C$ is free if the evaluation homomorphism $ev_{E,A} : A \otimes k(C) \to E$ is surjective. If this holds, we obtain a morphism $\Phi_{E,A}$ of $C$ to the Grassmannian $G(A, r)$ of $r$-dimensional quotient spaces. It is characterized by the property that $\Phi_{E,A}^*(\mathcal{U}, A) = (E, A)$, where $\mathcal{U}$ is the universal quotient bundle on $G(A, r)$ and $\Phi_{|E|}$ is abbreviated to $\Phi_E$.

Let

$$\bigwedge^m ev_{E,A} : \bigwedge^m A \otimes k(C) \to \bigwedge^m E$$

be the exterior product of the evaluation homomorphism $ev_{E,A}$. It induces the linear map

$$\bigwedge^m A \to H^0\left(\bigwedge^m E\right),$$

which we denote by $\lambda_m$. The image $\lambda_m(s_1 \wedge \cdots \wedge s_m)$ of a simple $m$-vector $s_1 \wedge \cdots \wedge s_m$ is zero if and only if $m$ global sections $s_1, \ldots, s_m \in A \subset H^0(E)$ are linearly dependent at the generic point of $C$, that is, they generate a subsheaf of rank less than $m$. This linear map is most important when $m = r$. Assume that $\lambda_r : \bigwedge^r A \to H^0(\det E)$ is surjective. Then the map

$$\Psi : \mathbb{P}^*(H^0(\det E)) \to \mathbb{P}^*\left(\bigwedge^r A\right)$$

induced by $\lambda_r$ is a linear embedding and the following diagram is commutative:

$$\begin{array}{ccc}
C & \xrightarrow{\Phi_E} & G(A, r) \\
\cap & \xrightarrow{} & \cap \\
\bigwedge^r A & \xrightarrow{\Psi} & \mathbb{P}^*\left(\bigwedge^r A\right).
\end{array}$$

Even when $\lambda_r$ is not surjective, the above is still commutative though $\Psi = \mathbb{P}^*\lambda_r$ is only a rational map. The linear map $\lambda_r$ is important in analyzing $E$ itself also.
Now we assume that the base field $k$ is algebraically closed (until the end of §6). The dual Grassmannian $G(r, A) \subset \mathbb{P}_*(\wedge^r A)$ is also important for understanding $(E, A)$.

**Definition 3.1.** A linear system $(E, A)$ of rank $r$ is **irreducible** if it satisfies the following equivalent conditions:

i) For every $r$-dimensional linear subspace $U$ of $A$ the image of $U \otimes_k \mathbb{C} \rightarrow E$ is of rank $r$, and

ii) The kernel of the natural linear map $\lambda_r : \wedge^r A \rightarrow H^0(C, \det E)$ contains no nonzero simple $r$-vectors; that is, $G(r, A) \cap \mathbb{P}_*(\Ker \lambda_r) = \emptyset$.

The following is known as Castelnuovo’s trick (cf. [2, Chap. 10]):

**Proposition 3.2.** If $r (\dim A - r) \geq h^0(\det E)$, then $(E, A)$ is reducible.

**Proof.** The left-hand side of the inequality is the dimension of $G(r, A)$. The codimension of $\mathbb{P}_*(\Ker \lambda_r) \subset \mathbb{P}_*(\wedge^r H^0(E))$ is at most $h^0(\det E)$. Hence, if the inequality holds, then the intersection $G(r, A) \cap \mathbb{P}_*(\Ker \lambda_r)$ is not empty. \hfill $\square$

A line bundle is irreducible. But irreducibility seems a strong condition in general. Irreducible bundles of rank $\geq 2$ will not appear in the sequel. Instead the following concept plays a crucial role in our proof.

**Definition 3.3.** A linear system $(E, A)$ of rank $r$ on a (smooth complete) curve $C$ is **semi-irreducible** if the evaluation homomorphism $\text{ev}_U : U \otimes_k \mathbb{C} \rightarrow E$ is either injective or everywhere of rank $r - 1$ for every $r$-dimensional subspace $U$ of $A$.

For an $r$-dimensional quotient space $A \rightarrow Q$, we denote by $S_Q$ the Schubert subvariety

$$[[U] | \ \rk [U \rightarrow A \rightarrow Q] \leq r - 2] \subset G(r, A)$$

associated to $Q$. Also, $S_Q$ is contained in the projective space $\mathbb{P}_*((\wedge^2 W) \wedge (\wedge^{r-2} A))$, which is the projectivisation $\mathbb{P}_*(N^\vee_{G(A, r)/\mathbb{P}_*[Q]})$ of the conormal space of $G(A, r) \subset \mathbb{P}_*(\wedge^r A)$ at $[Q]$. The following is obvious:

**Lemma 3.4.** $(E, A)$ is semi-irreducible if and only if $S_{E_p} \cap \mathbb{P}_*(\Ker \lambda_r) = \emptyset$ for every fiber $E_p$ of $E$, $p \in C$.

Now we restrict ourselves to complete linear systems for simplicity.

**Proposition 3.5.** Assume that a complete linear system $|E|$ of rank $r$ is free and semi-irreducible.

1. If $F$ is a proper nonzero subbundle, then $h^0(F) \leq r(F) + 1$, where $r(F)$ is the rank of $F$.

2. If $h^0(E) \geq r + 2$ and if $F$ is a subbundle of rank $\leq r - 2$, then $h^0(F) \leq r(F)$.

3. If $h^0(E) \geq r + 3$, then $E$ is simple, i.e., $\text{End } E = k$. 
Proof. (1) Assume that \( F \) is of rank \( r - 1 \) and \( h^0(F) \geq r \). Then the evaluation homomorphism \( B \otimes_k \mathcal{O}_C \to F \) is surjective for every \( r \)-dimensional subspace \( B \subset H^0(F) \) by semi-irreducibility. Hence we have \( h^0(F) \leq r \). The general case follows from this since, for every proper subbundle \( F \), there exists a subsheaf \( F' \subset E \) of rank \( r - 1 \) which contains \( F \) and \( h^0(F') \geq h^0(F) + r(F') - r(F) \).

(2) By the same reason as above, we may assume that \( F \) is of rank \( r + 1 \). We prove \( h^0(F) \leq r + 1 \) by contradiction. Assume that \( h^0(F) > r + 1 \) and put \( G = E/F \). We regard the quotient space \( H^0(E)/H^0(F) \) as a subspace of \( H^0(G) \). Since \( \dim H^0(E)/H^0(F) = h^0(F) - (r - 1) \geq 3 \) and since \( G \) is of rank 2, there exists a global section \( s \in H^0(E) \setminus H^0(F) \) such that \( \tilde{s} \in H^0(G) \) vanishes at a point on \( C \). Then \( H^0(F) \) and \( s \) do not generate a subsheaf of rank \( r \) or a subbundle of rank \( r - 1 \), which contradicts the semi-irreducibility of \( |E| \). Therefore, we have \( h^0(F) \leq r + 1 \) by (1).

(3) It suffices to show that every endomorphism \( \phi : E \to E \) is either zero or an isomorphism. Assume that \( \phi \) is neither. Then both the kernel and the image are proper subsheaves and we have

\[
h^0(E) \leq h^0(\ker \phi) + h^0(\text{im} \phi) \leq r(\ker \phi) + 1 + r(\text{im} \phi) + 1 = r + 2
\]

by (1), which is a contradiction. \( \square \)

The following is proved similarly.

Lemma 3.6. Assume that two complete linear systems \(|E|\) and \(|E'|\) are free, semi-irreducible and of the same rank \( r \) and assume further that \( h^0(E) \geq r + 3 \). Then every nonzero homomorphism \( E \to E' \) is injective.

4. Linear sections of the symplectic Grassmannian

Throughout this section \( C \subset \mathbb{P}^8 \) is a transversal linear section \( \text{SpG}(3, 6) \cap H_1 \cap \cdots \cap H_5 \) of the 6-dimensional symplectic Grassmannian.

Lemma 4.1. \( C \) is of genus 9 and the restriction of tautological line bundle \( \mathcal{O}(1) \) is isomorphic to the canonical bundle \( K_C \) of \( C \).

Proof. By Proposition 2.3 and by adjunction, we have \( K_C \simeq \mathcal{O}_C(K_{\text{SpG}} + H_1 + \cdots + H_5) \simeq \mathcal{O}_C(1) \). The Chern class of the universal quotient bundle \( \mathcal{U} \) on \( G(3, 6) \) is the sum \( 1 + \sigma_1 + \sigma_2 + \sigma_3 \) of the special Schubert cycles ([8, Chap. 1]). By Pieri’s formula, we have

\[
2g(C) - 2 = \deg[\text{SpG}(3, 6) \subset \mathbb{P}^{13}] = \left( c_3 \left( \bigwedge^2 \mathcal{U} \right) . c_1(\mathcal{U})^6 \right) = (\sigma_1 \sigma_2 - \sigma_3 . \sigma_1^6) = 21 - 5 = 16,
\]

since \( \text{SpG}(3, 6) \) is the zero locus of a global section of \( \bigwedge^2 \mathcal{U} \). Hence \( C \) is of genus 9. \( \square \)

Let \( G(A, \sigma) \), \( \dim A = 6 \), be a representative of \( \text{SpG}(3, 6) \).
LEMMA 4.2. The linear map \( \bigwedge^3(A, \sigma) \rightarrow H^0(K_C) \) is surjective and its kernel is generated by the linear forms \( f_1, \ldots, f_5 \in \bigwedge^3(A, \sigma) \) defining the five hyperplanes \( H_1, \ldots, H_5 \).

Proof. Let \( X_i \) be the common zero locus of the first \( i \) linear forms \( f_1, \ldots, f_i \) for \( 1 \leq i \leq 5 \). Then we obtain a ladder
\[
C = X_5 \subset X_4 \subset X_3 \subset X_2 \subset X_1 \subset X_0 := G(A, \sigma).
\]
Since \( C \) is irreducible, so is each \( X_i \). Hence the kernel of the restriction map \( H^0(X_i, \mathcal{O}_X(1)) \rightarrow H^0(X_{i+1}, \mathcal{O}_X(1)) \) is generated by \( f_{i+1} \), for every \( 1 \leq i \leq 4 \). Hence \( \bigwedge^3(A, \sigma)/(f_1, \ldots, f_5) \rightarrow H^0(K_C) \) is injective. This map is also surjective because the source and the target have the same dimension. \( \Box \)

Let \( \mathfrak{C} \) be the restriction of \( \mathfrak{U} \) to \( G(A, \sigma) \) and \( E \) the restriction to \( C \).

LEMMA 4.3. The restriction map \( A \rightarrow H^0(E) \) is injective.

Proof. Assume the contrary. Then for each of the Lagrangian quotient spaces \( A \rightarrow Q \) parametrized by \( C \), \( \text{Ker}[A \rightarrow Q] \) contains a nonzero common vector \( a \). Hence \( C \) is contained in the symplectic Grassmannian \( G(\widetilde{A}, \widetilde{\sigma}) \), where \( \widetilde{A} \) is the co-radical of \( A/ka \). This contradicts the preceding lemma since \( G(\widetilde{A}, \widetilde{\sigma}) \) lies in a 4-dimensional linear subspace.

By this lemma we identify \( A \) with its image in \( H^0(E) \).

LEMMA 4.4. (1) A nonzero global section \( s \in A \) of \( E \) has at most two zeros (counted with multiplicity); that is, \( A \cap H^0(E(\widetilde{-D})) = 0 \) for every effective divisor \( D \) of degree 3 on \( C \).

(2) If \( A' \subset A \) is a subspace of codimension one, then the cokernel of the evaluation homomorphism \( A' \otimes_k \mathcal{O}_C \rightarrow E \) is of length \( \leq 2 \).

Proof. Assume that \( s \) has at least three zeros. Then we have an exact sequence \( E^\vee \rightarrow \mathcal{O}_C \rightarrow \mathcal{O}_D \rightarrow 0 \) for an effective divisor \( D \) of degree \( \geq 3 \). Let \( G(\widetilde{A}, \widetilde{\sigma}) \subset G(A, \sigma) \) be the 3-dimensional symplectic Grassmannian determined by \( s \in A \). Then the intersection \( G(\widetilde{A}, \widetilde{\sigma}) \cap C \) contains \( D \). Since \( G(\widetilde{A}, \widetilde{\sigma}) \) is a quadric, its intersection with the linear span \( \langle D \rangle \) is of positive dimension, which is a contradiction. This shows (1). The proof of (2) is similar. \( \Box \)

Let \( U \subset A \) be a 3-dimensional subspace and \( H_U \subset \mathbb{P}^* \bigwedge^3 A \) the hyperplane corresponding to it. Then the intersection \( H_U \cap G(A, r) \) consists of the \( r \)-dimensional quotient spaces \( A \rightarrow Q \) such that the composite \( U \leftarrow A \rightarrow Q \) is not an isomorphism. It is singular along the Schubert subvariety
\[
\{[A \rightarrow Q] \mid \text{rank} [U \leftarrow A \rightarrow Q] \leq 1\}.
\]

If \( H_U \not\supset C \), then the evaluation homomorphism \( \text{ev}_U : U \otimes \mathcal{O}_C \rightarrow E \) is of rank 3 at the generic point. Hence it is injective. If \( H_U \supset C \), then \( H_U \) belongs to \( \langle [H_1], \ldots, [H_3] \rangle \). Since the intersection \( C = H_1 \cap \cdots \cap H_5 \cap G(A, \sigma) \) is transversal, \( H_U \cap G(A, \sigma) \) must be smooth along \( C \). Hence \( \text{ev}_U \) is of rank 2 everywhere. So
we have proved the following, which indicates that the semi-irreducibility is a key concept for canonical curves of genus 9.

**Proposition 4.5.** The induced rank three linear system \((A, E)\) on \(C = G(A, \sigma) \cap H_1 \cap \cdots \cap H_5\) is semi-irreducible.

By Proposition 3.2, there exists a 3-dimensional subspace \(U\) of \(A\) such that \(H_U \supset C\). Let \(F\) and \(\alpha\) be the image and the cokernel of \(ev_U\). Then \(\alpha\) is a line bundle, \(\det F\) is isomorphic to \(\mathcal{O}_C^3\) and we have exact sequences

\[
0 \to \beta^{-1} \to \mathcal{O}_C^{\oplus 3} \to F \to 0 \quad \text{and} \quad 0 \to F \to E \to \alpha \to 0.
\]

By (2.2), the line bundles \(\alpha\) and \(\beta\) are both of degree 8.

**Proposition 4.6.** \(C\) is nonpentagonal.

*Proof.* It is obvious that \(C\) is nonhyperelliptic. Since \(\SpG(3, 6) \subset \mathbb{P}^{13}\) is an intersection of quadrics (see (0.1)), so is \(C \subset \mathbb{P}^8\). In particular, \(C \subset \mathbb{P}^8\) has no tri-secant lines. By the geometric version of the Riemann-Roch theorem ([1, Chap I, §2]), \(C\) has no \(g_1^1\). Also \(C\) has no \(g_2^2\) either, since the (geometric) genus of a plane quintic is at most 6. Let \(\xi\) be a \(g_3^3\) on \(C\). Then we have \(h^0(\xi) = 2\). Let \(U\) and \(F\) be as above. Taking the global section of the exact sequence

\[
0 \to F^\vee \to \mathcal{O}_C^{\oplus 3} \to \beta \to 0 \otimes \xi,
\]

we have

\[
6 \leq 3h^0(\xi) \leq \dim \text{Hom} (F, \xi) + h^0(\xi\beta) = \dim \text{Hom} (F, \xi) + 5 + h^1(\xi\beta).
\]

Hence we have

\[
(4.3) \quad \dim \text{Hom} (F, \xi) + \dim \text{Hom} (\xi, \alpha) \geq 1.
\]

Assume that there exists a nonzero homomorphism \(F \to \xi\) and let \(s\) be a nonzero global section in the kernel of \(U \to H^0(F) \to H^0(\xi)\). Then \(s\) has at least three zeros since \(\deg F - \deg \xi = 3\). If \(\text{Hom}(F, \xi)\) is zero, then \(\text{Hom}(\xi, \alpha)\) is not by (4.3). Hence \(\alpha\) contains a subsheaf isomorphic to \(\xi\). Let \(A'\) be the inverse image of \(H^0(\xi)\) by \(A \to H^0(\alpha)\). Then the cokernel of the evaluation homomorphism \(A' \otimes_k \mathcal{O}_C \to E\) is of length 3. Both contradict Lemma 4.4.

**Remark 4.7.** (1) For a curve of genus 9, the nonexistence of \(g_3^3\) is equivalent to its Clifford index which equals 4 (Martens [9, Beispiel 9]).

(2) Green’s property \((\mathcal{N}_p)\) ([6]) gives another proof of the proposition: First a general curve of genus 9 satisfies \((\mathcal{N}_3)\) by Ein [3]. Hence \(\SpG(3, 6) \subset \mathbb{P}^{13}\) and its complete linear section do so. By the converse of Green’s conjecture (Green-Lazarsfeld [7]), \(C\) is nonpentagonal.

By the proposition and (1) of the remark, \(C\) has no \(g_3^3\). Hence we have \(h^0(\alpha) = h^0(\beta) = 3\). By Lemma 5.1 below, we have \(h^0(E) \leq h^0(\alpha) + H^0(\mathcal{Q}_\beta) \leq 6\). Combining this with Lemma 4.3, we have
PROPOSITION 4.8. The restriction map $A \rightarrow H^0(E)$ is an isomorphism.

In the following sections we aim at a kind of converse of Proposition 4.5.

5. Rank 3 linear systems on a nonpentagonal curve

Throughout this section we assume that $C$ is a nonpentagonal curve of genus 9. In particular, $C$ has no $g_7^2$. Let $\alpha$ be a $g_8^5$, $\beta$ its Serre adjoint and $Q_\beta$ the cokernel of $\text{ev}_\beta$ as in the introduction and in (1.1). The image of $\Phi_\beta : C \rightarrow \mathbb{P}^2$ is a singular plane curve of degree 8. Hence there exists a pair $(p, q)$ of points (not necessarily distinct) such that $h^0(\beta(-p - q)) = 2$. By assumption $\xi := \beta(-p - q)$ is a free $g_6$. Hence we have a commutative diagram

$$0 \rightarrow \beta^{-1} \rightarrow \mathcal{O}_C^\oplus 3 \rightarrow Q_\beta \rightarrow 0$$

(5.1)

and an exact sequence

$$0 \rightarrow \mathcal{O}_C(p + q) \rightarrow Q_\beta \rightarrow \xi \rightarrow 0.$$  

(5.2)

**Lemma 5.1.** (1) $h^0(Q_\beta) = 3$;

(2) $H^0(\alpha^{-1} Q_\beta) = 0$.

*Proof.* (1) $h^0(Q_\beta) = 3$ is obvious from the defining exact sequence of $Q_\beta$. The opposite inequality $h^0(Q_\beta) \leq 3$ follows from (5.2).

(2) $Q_\beta$ is isomorphic to $\beta Q_\beta'$ since it is of rank 2. Hence $Q_\beta$ is a subbundle of $\beta^\oplus 3$. If $\alpha \not\sim \beta$, then $H^0(\alpha^{-1} \beta) = 0$ and hence $H^0(\alpha^{-1} Q_\beta) = 0$. If $\alpha \sim \beta$, then $H^0(\alpha^{-1} Q_\beta) \simeq H^0(Q_\beta') = 0$ by the exact sequence (1.1). $\square$

We consider extensions

$$0 \rightarrow Q_\beta \rightarrow E \rightarrow \alpha \rightarrow 0$$

(5.3)

which are $\Gamma$-split; that is, $H^0(E) \rightarrow H^0(\alpha)$ is surjective.

**Lemma 5.2.** There exists a nontrivial extension $E$ of $\alpha$ by $Q_\beta$ with $h^0(E) = 6$.

*Proof.* The extensions with $h^0(E) = 6$ are parametrized by the kernel of the natural linear map $\varphi : \text{Ext}^1(\alpha, Q_\beta) \rightarrow H^0(\alpha)^\vee \otimes H^1(Q_\beta)$, which is equal to the first cohomology $H^1$ of the homomorphism

$$[\alpha^{-1} \rightarrow H^0(\alpha)^\vee \otimes \mathcal{O}_C] \otimes Q_\beta.$$  

Since its cokernel is $Q_\alpha \otimes Q_\beta$, we have an exact sequence

$$H^0(\alpha)^\vee \otimes H^0(Q_\beta) \xrightarrow{\psi} H^0(Q_\alpha \otimes Q_\beta) \rightarrow H^1(\alpha^{-1} Q_\beta) \xrightarrow{\varphi} H^0(\alpha)^\vee \otimes H^1(Q_\beta).$$

(5.4)

The first map $\psi$ is injective by (2) of Lemma 5.1 and $h^0(Q_\alpha \otimes Q_\beta)$ is even by Proposition 1.2. Since $h^0(\alpha) h^0(Q_\beta) = 9$, $\varphi$ is not injective. $\square$
PROPOSITION 5.3. Let $E$ be as in the preceding lemma. Then the complete linear system $|E|$ is free and semi-irreducible.

Proof. $|E|$ is free since both $|Q_\beta|$ and $|\alpha|$ are. Let $U \subset H^0(E)$ be a 3-dimensional subspace and $F \subset E$ the saturation of the subsheaf $F'$ generated by $U$. Obviously $h^0(F) \geq 3$. If $F$ is of rank one, then deg $F \geq 8$ by our assumption. Since $F \not\subset Q_\beta$, the extension (5.3) splits, which is a contradiction. Hence $F$ is of rank two. Let $\xi$ be the quotient line bundle $E/F$. Since $|E|$ is free, so is $\xi$. Since $\text{Hom}(E, \mathcal{O}_C) = 0$, we have $h^0(\xi) \geq 2$. By duality and (1) of Remark 4.7, $h^0(\text{det } F) = h^1(\xi) \leq g + 1 - 4 - h^0(\xi) \leq 4$. Assume that $h^0(F) \geq 4$. Then $F$ contains a line subbundle $\xi$ with $h^0(\xi) \geq 2$ by Proposition 3.2. Since $\xi \not\subset Q_\beta$, $\xi$ is isomorphic to a proper subsheaf of $\alpha$. Hence we have $h^0(\xi) = 2$. Let $\eta$ be the quotient line bundle $F/\xi$. Then we have $h^0(\eta) \geq h^0(F) - h^0(\xi) = 2$. Since $\text{deg } \xi + \text{deg } \eta + \text{deg } \xi = 16$, one of the three line bundles is of degree $\leq 5$, which is a contradiction. Hence we have $h^0(F) = 3$ and $h^0(\xi) \geq h^0(E) - h^0(F) = 3$. Since $h^1(\xi) = h^0(\text{det } F) \geq 3$, $\xi$ is a $g^2_5$ and $F'$ is isomorphic to $Q_\xi$. In particular, $F' = F$ and $F'$ is a subbundle. $\square$

Now conversely we study a uniqueness.

LEMMA 5.4. Nontrivial extensions $E$ of $\alpha$ by $Q_\beta$ with $h^0(E) = 6$ are unique.

Proof. The assertion is equivalent to $h^0(Q_\alpha \otimes Q_\beta) \leq 10$ by the exact sequence (5.4). Take the global section of the exact sequence

\[(5.2) \otimes Q_\alpha : 0 \longrightarrow Q_\alpha (p + q) \longrightarrow Q_\alpha \otimes Q_\beta \longrightarrow Q_\alpha \xi \longrightarrow 0.\]

Now

\[h^0(Q_\alpha \otimes Q_\beta) \leq h^0(Q_\alpha (p + q)) + h^0(Q_\alpha \xi)\]

\[= h^0(Q_\alpha (p + q)) + h^1(Q_\alpha (p + q))\]

\[= 2h^0(Q_\alpha (p + q)) - \chi(Q_\alpha (p + q)).\]

Since $\chi(Q_\alpha (p + q)) = -4$, it suffices to show $h^0(Q_\alpha (p + q)) \leq 3$. Assume the contrary:

The case where $h^0(Q_\alpha (p + q)) = 4$. Let $\{s_1, s_2, s_3, s_4\}$ be a basis of the vector space $H^0(Q_\alpha (p + q))$ such that $s_1, s_2, s_3 \in H^0(Q_\alpha)$ and $F$ is the image of the evaluation homomorphism $H^0(Q_\alpha (p + q)) \otimes_k \mathcal{O}_C \longrightarrow Q_\alpha (p + q)$. Then the quotient $F/Q_\alpha$ is generated by the image of $s_4$. Hence, deg $F \leq \deg Q_\alpha + 2 = 10$. We have $h^0(\text{det } F) \leq 4$ by the nonexistence of $g^2_6$. Since $h^0(F) \geq 4$, there exists a 2-dimensional subspace of $H^0(F)$ which generates a rank one subsheaf by Proposition 3.2. This contradicts the nonexistence of $g^1_5$.

The case where $h^0(Q_\alpha (p + q)) \geq 5$. Since $\deg Q_\alpha (p + q) = 12$ and since $C$ has no $g^1_4$, we have $h^0(\text{det}(Q_\alpha (p + q))) \leq 5$. By Proposition 3.2, there exists an
exact sequence
\[ 0 \longrightarrow \xi \longrightarrow Q_\alpha (p + q) \longrightarrow \eta \longrightarrow 0 \]
such that \( h^0(\xi) \geq 2 \). Since \( \eta(-p - q) \) is a quotient of \( Q_\alpha \), we have \( h^0(\eta(-p - q)) \geq 2 \) and \( \deg \eta(-p - q) \geq 6 \), which implies \( \deg \xi \leq 4 \). This is a contradiction. \( \square \)

We strengthen this lemma.

**Lemma 5.5.** A rank 3 vector bundle \( E \) on \( C \) which satisfies

i) \( \bigwedge^3 E \cong K_C \),

ii) \( h^0(E) \geq 6 \), and

iii) \( |E| \) is semi-irreducible

is an extension of \( \alpha \) by \( Q_\beta \).

**Proof.** By Lemma 1.1, or by (1.3), we have

\[ \dim \text{Hom}(Q_\beta, E) + \dim \text{Hom}(E, \alpha) \geq 2. \]

(\( h^0(E) = r + s \) and the Brill-Noether number \( \rho \) is equal to 0.) Hence there exists a nonzero homomorphism either \( f : Q_\beta \rightarrow E \) or \( g : E \rightarrow \alpha \).

If the image of \( f \) is a line bundle \( L \), then \( h^0(L) \geq 2 \) since \( \text{Hom}(Q_\beta, \mathcal{O}_C) = 0 \). This contradicts (1) of Proposition 3.5. Hence \( f \) is injective. By semi-irreducibility, the cokernel is a line bundle and is isomorphic to \( \alpha \).

If \( g : E \rightarrow \alpha \) is not surjective, then the kernel of \( H^0(E) \rightarrow H^0(\alpha) \) is of dimension \( \geq 4 \), which contradicts semi-irreducibility. Hence \( g \) is surjective and its kernel is isomorphic to \( Q_\beta \).

By the two lemmas above, we have the following:

**Proposition 5.6.** Vector bundles \( E \) on \( C \) which satisfy the condition of the lemma are unique up to isomorphism.

This vector bundle is denoted by \( E_{\text{max}} \).

**Corollary.** If \( E \) is a rank 3 vector bundle of canonical determinant on \( C \) and if \( |E| \) is semi-irreducible, then \( h^0(E) \leq 6 \).

**Remark 5.7.** (1) By the proposition and its proof, we obtain an explicit bijection between two sets: \( W^2_8(C) \), the set of \( g^2_8 \)'s of \( C \), and the intersection

\[ G(3, H^0(E_{\text{max}})) \cap \mathbb{P}^{10}. \]

It is known that the cardinality of \( W^r_d^{-1}(C) \) of a general curve \( C \) of genus \( g \) is equal to the degree of a \( g \)-dimensional Grassmannian when the Brill-Noether number \( \rho \) is zero (cf. [1, Chap. VII, Th. (4.4)] and [4, Ex. 14.4.5]).

(2) By (1) of Proposition 3.5, it is easy to show that \( E_{\text{max}} \) is stable. It is also easy to show a converse: if \( E \) is stable, \( \bigwedge^3 E \cong K_C \) and \( h^0(E) \geq 6 \), then \( |E| \) is semi-irreducible.
6. Linear section theorems

We prove Theorem C in several steps. Assume that $E = E_{\text{max}}$ satisfies the condition (0.2). Since $E$ is a rank 3 vector bundle of canonical determinant, $K_C E^\vee$ is isomorphic to $\bigwedge^2 E$. Hence, by the Riemann-Roch theorem, we have

$$h^0(E) - h^0\left(\bigwedge^2 E\right) = \deg E + 3(1 - 9) = -8.$$ 

and $h^0(\bigwedge^2 E) = 14$. Since $\dim \bigwedge^2 H^0(E) = 15$, the linear map

$$\lambda_2 : \bigwedge^2 H^0(E) \longrightarrow H^0\left(\bigwedge^2 E\right)$$

is not injective.

**Step 1.** Every nonzero bivector $\sigma$ in $\text{Ker} \lambda_2$ is nondegenerate.

**Proof.** The rank of $\sigma$ is either 2, 4 or 6. If $\sigma$ is of rank 2, then $\sigma$ is equal to $s_1 \wedge s_2$ for a pair of global sections $s_1$ and $s_2$ which are linearly independent in $H^0(E)$ and generate a rank-one subsheaf in $E$. This contradicts (2) of Proposition 3.5. Assume that $\sigma$ is of rank 4. Then $\sigma$ is equal to $s_1 \wedge s_2 - s_3 \wedge s_4$ for $s_1, s_2, s_3$ and $s_4 \in H^0(E)$. By semi-irreducibility, $s_1$ and $s_2$ generate a rank two subsheaf in $E$. Let $F$ be its saturation. Since $\lambda_2(s_1 \wedge s_2) = \lambda_2(s_3 \wedge s_4)$, we have $\lambda_3(s_1 \wedge s_2 \wedge s_i) = \lambda_3(s_3 \wedge s_4 \wedge s_i) = 0$ for $i = 3, 4$. Hence $s_3$ and $s_4$ are contained in $H^0(F)$ and we have $h^0(F) \geq 4$. This contradicts the semi-irreducibility of $|E|$ by Proposition 3.5. \hfill \Box

The nondegeneracy of $\sigma$ is equivalent to the nonvanishing of the Pfaffian. Hence $\text{Ker} \lambda_2$ is of dimension one and $\lambda_2$ is surjective. Since $|E|$ is free, we obtain a morphism $\Phi_E : C \longrightarrow G(A, 3)$ to the Grassmannian of 3-dimensional quotient spaces of $A := H^0(E)$. Its image is contained in the symplectic Grassmannian $G(A, \sigma)$ and we obtain the commutative diagram (0.3), where $\sigma$ is a generator of $\text{Ker} \lambda_2$. Since $\bigwedge^3 (A, \sigma)$ is of dimension 14, the kernel of $\lambda_3 : \bigwedge^3 (A, \sigma) \longrightarrow H^0(K_C)$ is of dimension $\geq 14 - 9 = 5$. Let $f_1, \ldots, f_k$, $k \geq 5$, be its basis and $H_1, \ldots, H_k$ the hyperplanes corresponding to them. Since $|E|$ is semi-irreducible, the intersection $S_{E_p} \cap P_3 \text{Ker} \lambda_3$ is empty for every $p \in C$ by Lemma 3.4. Hence so is $\alpha(S_{E_p}) \cap P_3 \text{Ker} \lambda_3$ for the projection $\alpha$ in (2.6).

**Step 2.** There exists a point $p \in C$ such that the intersection $G(A, \sigma) \cap H_1 \cap \cdots \cap H_k$ is transversal at $\Phi_E(p)$.

**Proof.** Assume the contrary. Then, for every $p \in C$, there exists a member $H_p$ of $\left(\{H_1\}, \ldots, \{H_k\}\right) = P_3 \text{Ker} \lambda_3$ such that the intersection $G(A, \sigma) \cap H_p$ is singular at $\Phi_E(p)$. The intersection $P_3(N_{G(A, \sigma)/P_3(E_p)}) \cap P_3 \text{Ker} \lambda_3$ is a point for every $p$ by Proposition 2.4. Therefore, we obtain a section of the $P^6$-bundle $P^*(\Phi_E^* N_{G(A, \sigma)/P_3})$ over $C$ which is disjoint from $\bigsqcup_{p \in C} \alpha(S_{E_p})$. By projecting from $\bigsqcup_{p \in C} k_p$, we obtain a section of $P_3(S^2 E)$ over which the discriminant form
\[\delta \in H^0(S^3(S^2E)^Y \otimes (\det E)^{\otimes 2})\] has no zeros. Let \(\xi \subset S^2E\) be the line subbundle corresponding to the section. Then \(\delta\) induces a nowhere-vanishing global section of \(\xi^{-3} \otimes (\det E)^{\otimes 2}\). This implies \(3 \deg \xi = 2 \deg E = 32\), which is absurd. \(\square\)

In particular, we have \(k = 5\) and hence the linear map \(\overline{\lambda}_3\) is surjective. Therefore, \(P^*\overline{\lambda}_3\) is a linear embedding. Since the canonical morphism \(\Phi_K\) is an embedding, so is \(\Phi_E\) by the commutative diagram (0.3). We identify \(\mathcal{C}\) with its image \(\Phi_E(\mathcal{C})\).

By Step 2, the intersection \(G(A, \sigma) \cap H_1 \cap \cdots \cap H_5\) is complete on a nonempty open subset \(C_0\) of \(C\). Hence the twisted normal bundle \(N_{C/G(A, \sigma)}(-1)\) is generated by the five global sections induced from \(f_1, \ldots, f_5\) over \(C_0\). It is generated over \(C\), since \(N_{C/G(A, \sigma)}(-1)\) is of trivial determinant. Therefore, the intersection is complete along \(C\) and contains it as a connected component. By the connectedness of linear sections (Fulton-Lazarsfeld [5, Th. 2.1]), the intersection coincides with \(C\), which completes the proof of Theorem C. (If we use the refined Bézout theorem (Fulton[4, Th. 12.3]), the proof finishes at the last paragraph.)

Theorem A is an immediate consequence of Theorem C, Proposition 5.3 and Proposition 4.6.

7. **Proof of Theorem B**

We do not assume that \(k\) is algebraically closed anymore. Let \(C \simeq G(A, \sigma') \cap P'\) be another expression of \(C = G(A, \sigma) \cap P\) as a complete linear section of a 6-dimensional symplectic Grassmannian and \(\mathcal{E}'|_C\) the restriction of the universal quotient bundle. Both \(|\mathcal{E}|_C\) and \(|\mathcal{E}'|_C\) are semi-irreducible (over \(\bar{k}\)) by Proposition 4.5. Hence they are isomorphic to each other over \(\bar{k}\) by Proposition 5.6 and there exists a nonzero homomorphism \(f : \mathcal{E}|_C \longrightarrow \mathcal{E}'|_C\) over \(k\). This is an isomorphism by Lemma 3.6. Since the diagram

\[
\begin{array}{ccc}
\bigwedge^2 A & \xrightarrow{\bigwedge^2 H^0(f)} & \bigwedge^2 H^0(\mathcal{E}|_C) \\
H^0(\bigwedge^2 \mathcal{E}|_C) & \xrightarrow{\bigwedge^2 H^0(\mathcal{E}'|_C)} & H^0(\bigwedge^2 \mathcal{E}'|_C) \\
H^0(f) & \xrightarrow{\bigwedge^2 f} & H^0(\bigwedge^2 f)
\end{array}
\]

is commutative, the isomorphism \(H^0(f)\) maps \(k\sigma\) onto \(k\sigma'\). Thus we have proved (2) of Theorem B.

Assume that \(k\) is perfect and let \(\overline{E}\) be a vector bundle on \(\overline{C} = C \otimes_k \bar{k}\). We consider a descent problem of \(\overline{E}\) under the following condition:

\((*)\) \(\overline{E}\) is simple and \(\sigma^* \overline{E} \simeq \overline{E}\) for every element \(\sigma\) of the Galois group \(\text{Gal} k\) of \(\bar{k}/k\).

As is well known, the obstruction \(\text{ob}(\overline{E})\) for \(\overline{E}\) to descend to \(C\) is an element of the second Galois cohomology group \(H^2(\text{Gal} k, \text{Aut} \overline{E})\). Choose an isomorphism
For each \( \sigma \in \text{Gal} \bar{k} \). Then \( \text{ob}(\bar{E}) \) is the cohomology class of the cocycle \( \{ c_{\sigma, \tau} \}_{\sigma, \tau \in \text{Gal} \bar{k}} \) defined by \( c_{\sigma, \tau} = f_{\sigma, \tau}^{-1} \circ \tau^* \circ f_\sigma \circ \text{Aut}_k \bar{E} \). In other words, \( \text{ob}(\bar{E}) \) is the factor set of the extension

\[
1 \longrightarrow \text{Aut}_k \bar{E} \longrightarrow \text{Aut}_k \bar{E} \longrightarrow \text{Gal} k \longrightarrow 1.
\]

**Lemma 7.1.** If \( \dim H^i(\bar{C}, \bar{E}) = n > 0 \), then the obstruction \( \text{ob}(\bar{E}) \) is \( n \)-torsion.

**Proof.** Let \( \{ s_1, \ldots, s_n \} \) be a basis of \( H^1(\bar{C}, \bar{E}) \) and \( A_{\sigma} \in M_n(\bar{k}) \) the matrix representing

\[
H^1(f_{\sigma}) : H^1(\bar{C}, \bar{E}) \longrightarrow H^1(\bar{C}, \sigma^* \bar{E})
\]

with respect to the bases \( \{ s_1, \ldots, s_n \} \) and \( \{ \sigma^* s_1, \ldots, \sigma^* s_n \} \). Then

\[
\det H^1(c_{\sigma, \tau}) = (\det A_{\sigma \tau})^{-1} \tau(\det A_\sigma) \det A_\tau
\]

in \( \bar{k}^\times \). Therefore, \( \{ \det H^1(c_{\sigma, \tau}) \}_{\sigma, \tau \in \text{Gal} \bar{k}} \) is cohomologous to zero. Since \( c_{\sigma, \tau} \) are all constant multiplications, \( \det H^1(c_{\sigma, \tau}) \) are equal to \( c^n_{\sigma, \tau} \). Hence \( \text{ob}(\bar{E}) \) is \( n \)-torsion. \( \square \)

Now we prove (1) of Theorem B. Let \( C \) be a nonpentagonal curve of genus 9 defined over \( k \). It suffices to show the following:

**Proposition 7.2.** Assume that \( C \) has no \( g_5 \) over \( \bar{k} \). Then there exists a vector bundle \( E \) on \( C \) such that \( E \otimes_k \bar{k} \) is isomorphic to the vector bundle \( E_{\text{max}} \) on \( C \otimes_k \bar{k} \).

**Proof.** By (3) of Proposition 3.5 and Proposition 5.6, \( E_{\text{max}} \) satisfies \( (*) \). Hence the obstruction \( \text{ob}(E_{\text{max}}) \) belongs to \( H^2(\text{Gal} k, \text{Aut} \bar{k} E_{\text{max}}) = H^2(\text{Gal} k, \bar{k}^\times) \). Let

\[
\text{Det} : H^2(\text{Gal} k, \text{Aut} \bar{k} E_{\text{max}}) \longrightarrow H^2(\text{Gal} k, \text{Aut} \bar{k} \det E_{\text{max}})
\]

be the determinant homomorphism. Since \( \det E_{\text{max}} \) is the canonical bundle, it descends to \( C \). Hence \( \text{ob}(E_{\text{max}}) \) belongs to the kernel and is 3-torsion. On the other hand, \( \text{ob}(E_{\text{max}}) \) is 14-torsion by the preceding lemma since \( \dim H^1(E_{\text{max}}) = 14 \). Therefore, \( \text{ob}(E_{\text{max}}) \) vanishes and \( E_{\text{max}} \) descends to \( C \). (This is a Galois group variant of an argument of Mumford-Newstead [17].) \( \square \)

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