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Abstract

We prove a conjecture of Rudnick and Sarnak on the mass equidistribution of Hecke eigenforms. This builds upon independent work of the authors.

1. Introduction

A central problem in the area of quantum chaos is to understand the limiting behavior of eigenfunctions. An important example which has attracted a lot of attention is the case of Maass cusp forms of large Laplace eigenvalue for the space $X = \mathrm{SL}_2(\mathbb{Z}) \backslash \mathbb{H}$. Let ϕ denote such a Maass form, let λ denote its Laplace eigenvalue, and let ϕ be normalized so that $\int_X |\phi(z)|^2 \frac{dx dy}{y^2} = 1$. Zelditch [19] has shown¹ that as $\lambda \rightarrow \infty$, for a *typical* Maass form ϕ the measure $\mu_\phi := |\phi(z)|^2 \frac{dx dy}{y^2}$ approaches the uniform distribution measure $\frac{3}{\pi} \frac{dx dy}{y^2}$. This statement is referred to as “Quantum Ergodicity.” Rudnick and Sarnak [12] have conjectured that an even stronger result holds. Namely, that as $\lambda \rightarrow \infty$, for *every* Maass form ϕ the measure μ_ϕ approaches the uniform distribution measure. This conjecture is referred to as “Quantum Unique Ergodicity.” Lindenstrauss [7] has made great progress towards this conjecture, showing that, for Maass cusp forms that are eigenfunctions of the Laplacian and all the Hecke operators,² the only possible limiting measures are of the form $\frac{3}{\pi} c \frac{dx dy}{y^2}$ with $0 \leq c \leq 1$. Recently the second author [16] showed that $c = 1$, completing the proof of the QUE conjecture for Hecke-Maass forms on $\mathrm{SL}_2(\mathbb{Z}) \backslash \mathbb{H}$. For illuminating accounts on this conjecture we refer the reader to [6], [7], [8], [9], [10], [12], [14], [13], [18].

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¹We have given Zelditch’s result in the context of $\mathrm{SL}_2(\mathbb{Z}) \backslash \mathbb{H}$. In fact he proves more, since he considers equidistribution of the micro-local lift to $\mathrm{SL}_2(\mathbb{Z}) \backslash \mathrm{SL}_2(\mathbb{R})$. Lindenstrauss’s result stated below also holds for this micro-local lift.

²The spectrum of the Laplacian is expected to be simple, so that any eigenfunction of the Laplacian would automatically be an eigenfunction of all Hecke operators. This is far from being proved.

Here we consider a holomorphic analog of the quantum unique ergodicity conjecture. This analog is very much in the spirit of the Rudnick-Sarnak conjectures, and has been spelt out explicitly in [9], [14]. Let f be a holomorphic modular cusp form of weight k (an even integer) for $\mathrm{SL}_2(\mathbb{Z})$. Associated to f we have the measure

$$\mu_f := y^k |f(z)|^2 \frac{dx dy}{y^2},$$

which is invariant under the action of $\mathrm{SL}_2(\mathbb{Z})$, and we suppose that f has been normalized so that

$$\int_{\mathcal{X}} y^k |f(z)|^2 \frac{dx dy}{y^2} = 1.$$

The space $S_k(\mathrm{SL}_2(\mathbb{Z}))$ of cusp forms of weight k for $\mathrm{SL}_2(\mathbb{Z})$ is a vector space of dimension about $k/12$, and contains elements such as $\Delta(z)^{k/12}$ (if $12|k$, and where Δ is Ramanujan's cusp form) for which the measure will not tend to uniform distribution. Therefore, one restricts attention to a particularly nice set of cusp forms, namely those that are eigenfunctions of all the Hecke operators. The Rudnick-Sarnak conjecture in this context states that as $k \rightarrow \infty$, for *every* Hecke eigencuspform f the measure μ_f tends to the uniform distribution measure. For simplicity, we have restricted ourselves to the full modular group, but the conjecture could be formulated just as well for holomorphic newforms of level N . Luo and Sarnak [9] have shown that equidistribution holds for most Hecke eigenforms, and Sarnak [14] has shown that it holds in the special case of dihedral forms. It does not seem clear how to extend Lindenstrauss's work to the holomorphic setting.³

In this paper we shall establish the Rudnick-Sarnak conjecture for holomorphic Hecke eigencuspforms. The proof combines two different approaches to the mass equidistribution conjecture, developed independently by the authors [3], [17]. Either of these approaches is capable of showing that there are very few possible exceptions to the conjecture, and under reasonable hypotheses either approach would show that there are no exceptions. However, it seems difficult to show unconditionally that there are no exceptions using just one of these approaches. Fortunately, as we shall explain below, the two approaches are complementary, and the few rare cases that are untreated by one method fall easily to the other method. Both approaches use in an essential way that the Hecke eigenvalues of a holomorphic eigencuspform satisfy the Ramanujan conjecture (Deligne's theorem). The Ramanujan conjecture remains open for Maass forms, and this is the (only) barrier to using our methods in the nonholomorphic setting.⁴ At present, it is not clear how to use our methods in the case of compact quotients.

³The difficulty from the ergodic point of view concerns the invariance under the geodesic flow of the quantum limits of the micro-local lifts associated to holomorphic forms.

⁴Assuming the Ramanujan conjecture for Maass forms, our methods would obtain the stronger micro-local version of QUE. Moreover, our methods would then be able to quantify the rate at which equi-distribution is attained.

Recall that for two smooth bounded functions g_1 and g_2 on X we may define the Petersson inner product

$$\langle g_1, g_2 \rangle = \int_X g_1(z) \overline{g_2(z)} \frac{dx dy}{y^2}.$$

In this definition we could allow for one of the g_1 or g_2 to be unbounded, so long as the other function decays appropriately for the integral to converge. If f is a modular form of weight k , below we shall let $F_k(z)$ denote $y^{k/2} f(z)$ where $z = x + iy$. Let h denote a smooth bounded function on X . Considering h as fixed, and letting $k \rightarrow \infty$, the Rudnick-Sarnak conjecture asserts that for every Hecke eigencuspform f of weight k we have⁵

$$(1.1) \quad \langle hF_k, F_k \rangle \rightarrow \frac{3}{\pi} \langle h, 1 \rangle,$$

with the rate of convergence above depending on the function h .

To attack the conjecture (1.1), it is convenient to decompose the function h in terms of a basis of smooth functions on X . There are two natural ways of doing this. First we could use the spectral decomposition of a smooth function on X in terms of eigenfunctions of the Laplacian. The spectral expansion will involve (i) the constant function $\sqrt{3/\pi}$; (ii) Maass cusp forms ϕ that are also eigenfunctions of all the Hecke operators; and (iii) Eisenstein series on the $\frac{1}{2}$ line. Recall that the Eisenstein series is defined for $\text{Re}(s) > 1$ by

$$E(z, s) = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} \text{Im}(\gamma z)^s,$$

where $\Gamma = \text{SL}_2(\mathbb{Z})$ and Γ_∞ denotes the stabilizer group of the cusp at infinity (namely the set of all translations by integers). The Eisenstein series $E(z, s)$ admits a meromorphic continuation, with a simple pole at $s = 1$, and is analytic for s on the line $\text{Re}(s) = \frac{1}{2}$. For more on the spectral expansion; see Iwaniec [4]. Note that (1.1) is trivial when h is the constant eigenfunction. To establish (1.1) using the spectral decomposition, we would need to show that for a fixed Maass eigencuspform ϕ , and for a fixed real number t that

$$\langle \phi F_k, F_k \rangle, \quad \text{and} \quad \langle E(\cdot, \frac{1}{2} + it) F_k, F_k \rangle \rightarrow 0,$$

as $k \rightarrow \infty$. The inner products above may be related to values of L -functions. In the case of Eisenstein series, this is the classical work of Rankin and Selberg. In the more difficult Maass form case, this relation (to a triple product L -function) is given by a beautiful formula of Watson [18]. The connection to L -functions, and estimating such values, forms the basis for Soundararajan’s approach to (1.1).

⁵We are slightly abusing notation here, because F_k is not $\text{SL}_2(\mathbb{Z})$ -invariant. However $|F_k(z)|^2$ is $\text{SL}_2(\mathbb{Z})$ -invariant, and so the inner product in (1.1) does not depend on a choice of the fundamental domain.

Alternatively, one could expand the function h in terms of incomplete Poincaré and Eisenstein series. Let ψ denote a smooth function, compactly supported in $(0, \infty)$. For an integer m the incomplete Poincaré series is defined by

$$P_m(z \mid \psi) = \sum_{\gamma \in \Gamma_\infty \setminus \Gamma} e(m\gamma z)\psi(\text{Im}(\gamma z)).$$

In the special case $m = 0$ we obtain incomplete Eisenstein series $E(z \mid \psi) = P_0(z \mid \psi)$. For an account on approximating a smooth function h using incomplete Poincaré series see [8]. Luo and Sarnak [9] noted that in this approach to (1.1) one faces the problem of estimating the shifted convolution sums (for m fixed, and as $k \rightarrow \infty$)

$$\sum_{n \succ k} \lambda_f(n)\lambda_f(n + m),$$

where the sum is over n of size k , and $\lambda_f(n)$ denotes the Hecke eigenvalue of f normalized so that Deligne’s bound reads $|\lambda_f(n)| \leq d(n)$. The study of these shifted sums using sieve methods forms the basis for Holowinsky’s approach to (1.1).

We are now in a position to state our main result, after which we will describe the main theorems of Holowinsky and Soundararajan, and how those combine.

THEOREM 1. *Let f be a Hecke eigencuspform of weight k for $\text{SL}_2(\mathbb{Z})$, and write $F_k(z) = y^{k/2} f(z)$.*

- (i) *Let ϕ be a Maass cusp form which is also an eigenfunction of all Hecke operators. Then*

$$|\langle \phi F_k, F_k \rangle| \ll_{\phi, \varepsilon} (\log k)^{-\frac{1}{30} + \varepsilon}.$$

- (ii) *Let ψ be a fixed smooth function compactly supported in $(0, \infty)$. Then*

$$\left| \langle E(\cdot \mid \psi) F_k, F_k \rangle - \frac{3}{\pi} \langle E(\cdot \mid \psi), 1 \rangle \right| \ll_{\psi, \varepsilon} (\log k)^{-\frac{2}{15} + \varepsilon}.$$

Remark 1. We have made no attempt to optimize the rate of decay given above. Our methods would not appear to lead to any decay better than $(\log k)^{-1}$. The generalized Riemann hypothesis is known to imply a rate of decay $k^{-\frac{1}{2} + \varepsilon}$ which would be optimal; see [6], [10].

Remark 2. A striking consequence of Theorem 1 is that the zeros of the modular form f lying in X (there are about $k/12$ such zeros) become equidistributed with respect to the measure $\frac{3}{\pi} \frac{dx \, dy}{y^2}$. This follows from the work of Rudnick [11], who derived this consequence from the mass equidistribution conjecture.

Remark 3. With more effort, we could keep track of the dependence on ϕ and ψ in Theorem 1. This would require some careful book-keeping in the works [3] and [17]. Keeping track of these dependencies would allow one to give a rate of decay for the discrepancy (for example, the spherical cap discrepancy defined in [8]) between the measure μ_f and the uniform distribution measure.

We now describe the results of Holowinsky and Soundararajan, and how they pertain to Theorem 1. In [3] Holowinsky attacks the inner products in Theorem 1 by an unfolding method, which leads him to the estimation of shifted convolution sums of the Hecke eigenvalues. He then develops a sieve method to estimate these shifted convolution sums, obtaining the following result.

THEOREM 2 (Holowinsky). *Keep the notations of Theorem 1. Let $\lambda_f(n)$ denote the Hecke eigenvalue of f for the n -th Hecke operator normalized so that $|\lambda_f(n)| \leq d(n)$. Let $L(s, \text{sym}^2 f)$ denote the symmetric square L -function attached to f , and define*

$$M_k(f) := \frac{1}{(\log k)^2 L(1, \text{sym}^2 f)} \prod_{p \leq k} \left(1 + \frac{2|\lambda_f(p)|}{p}\right).$$

(i) *For a Maass cusp form ϕ we have*

$$|\langle \phi F_k, F_k \rangle| \ll_{\phi, \varepsilon} (\log k)^\varepsilon M_k(f)^{\frac{1}{2}}.$$

(ii) *For an incomplete Eisenstein series $E(z | \psi)$ we have*

$$\left| \langle E(\cdot | \psi) F_k, F_k \rangle - \frac{3}{\pi} \langle E(\cdot | \psi), 1 \rangle \right| \ll_{\psi, \varepsilon} (\log k)^\varepsilon M_k(f)^{\frac{1}{2}} (1 + R_k(f)),$$

where

$$R_k(f) = \frac{1}{k^{\frac{1}{2}} L(1, \text{sym}^2 f)} \int_{-\infty}^{+\infty} \frac{|L(\frac{1}{2} + it, \text{sym}^2 f)|}{(1 + |t|)^{10}} dt.$$

Although this is not immediately apparent, the quantity $M_k(f)$ appearing above is expected to be small in size. One can show that there are at most K^ε eigenforms f with weight below K for which $M_k(f) \geq (\log k)^{-\delta}$ for some fixed $\delta > 0$. A weak form of the generalized Riemann hypothesis could be used to show that $M_k(f) \leq (\log k)^{-\delta}$ for some $\delta > 0$. Moreover, one can show that

$$(1.2) \quad M_k(f) \ll (\log k)^\varepsilon \exp\left(-\sum_{p \leq k} \frac{(|\lambda_f(p)| - 1)^2}{p}\right),$$

so that one would expect $M_k(f)$ to be small unless $|\lambda_f(p)| \approx 1$ for most $p \leq k$. This last possibility is not expected to hold,⁶ and can be shown to be rare,⁷ but is difficult to rule out completely. In the Eisenstein series case ((ii) above), one also needs to bound $R_k(f)$; again this can be shown to be small in all but very rare cases.

As mentioned earlier, if we approach (1.1) through the spectral expansion of h we are led to estimating central values of L -functions. Here it turns out that an easy convexity bound for L -values barely fails to be of use, and improved subconvexity estimates (saving a power of the analytic conductor) would solve

⁶Since we would expect $\lambda_f(p)$ to be distributed in $[-2, 2]$ according to the Sato-Tate measure.

⁷By bounding the quantity in (1.2) in terms of $L(1, \text{sym}^2 f)$ and $L(1, \text{sym}^4 f)$; see [3] for details.

the problem completely (see, for example, [6]). In [17] Soundararajan developed a general method which gives weak subconvexity bounds for central values of L -functions. Instead of obtaining a power saving of the analytic conductor, one obtains a saving of powers of the logarithm of the analytic conductor.

THEOREM 3 (Soundararajan). *Keep the notations of Theorems 1 and 2.*

(i) *For a Maass eigencuspform ϕ we have*

$$|\langle \phi F_k, F_k \rangle| \ll_{\phi, \varepsilon} \frac{(\log k)^{-\frac{1}{2} + \varepsilon}}{L(1, \text{sym}^2 f)}.$$

(ii) *For the Eisenstein series $E(z, \frac{1}{2} + it)$ we have*

$$|\langle E(\cdot, \frac{1}{2} + it) F_k, F_k \rangle| \ll_{\varepsilon} (1 + |t|)^2 \frac{(\log k)^{-1 + \varepsilon}}{L(1, \text{sym}^2 f)}.$$

If $L(1, \text{sym}^2 f) \geq (\log k)^{-\frac{1}{2} + \delta}$ for some $\delta > 0$ then Theorem 3 would establish the Rudnick-Sarnak conjecture. This bound on $L(1, \text{sym}^2 f)$ is certainly expected to hold; for example it follows from a weak form of the generalized Riemann hypothesis. Moreover, one can show that this bound fails to hold for at most K^ε eigencuspforms f with weight below K . However, we know only that $L(1, \text{sym}^2 f) \gg (\log k)^{-1}$ (see §2 below), and it seems difficult to rule out the possibility of small values of $L(1, \text{sym}^2 f)$ completely.

We have seen how either of the approaches in Theorems 2 and 3 should work always, and that both fail in rare circumstances which are difficult to rule out. Now we shall see that in the rare circumstances that one of these results fails, the other result succeeds. To gain an intuitive understanding of this phenomenon, note that Theorem 3 fails only when $L(1, \text{sym}^2 f) \leq (\log k)^{-\frac{1}{2} + \delta}$ is small. But this L -value is small only if for most primes $p \leq k$ we have $\lambda_f(p^2) \approx -1$ (a Siegel zero type phenomenon). But this means $\lambda_f(p)^2 - 1 \approx -1$ so that $\lambda_f(p) \approx 0$. Recall now that the quantity $M_k(f)$ appearing in Holowinsky’s work is small unless $|\lambda_f(p)| \approx 1$ for most $p \leq k$. Evidently, both situations cannot happen simultaneously.

More precisely, in Lemma 3 below we shall show that

$$M_k(f) \ll (\log k)^{\frac{1}{6}} (\log \log k)^{\frac{9}{2}} L(1, \text{sym}^2 f)^{\frac{1}{2}}.$$

Therefore, if $L(1, \text{sym}^2 f) < (\log k)^{-\frac{1}{3} - \delta}$ for some small $\delta > 0$ it follows from Theorem 2(i) that $\langle \phi F_k, F_k \rangle$ is small. However, if $L(1, \text{sym}^2 f) > (\log k)^{-\frac{1}{3} - \delta} > (\log k)^{-\frac{1}{2} + \delta}$ then as noted above Theorem 3(i) shows that $\langle \phi F_k, F_k \rangle$ is small. This shows how Theorems 2 and 3 complement each other in the cusp form case. In the case of Eisenstein series, we first show how the weak subconvexity results in [17] lead to a satisfactory bound for the term $R_k(f)$ appearing in Theorem 2(ii) (see Lemma 1 below). Then the argument follows as in case (i).

2. Preliminaries on the symmetric square L -function

We collect together here some properties of $L(s, \text{sym}^2 f)$ that we shall need for the proof of Theorem 1. We shall require several important results on $L(s, \text{sym}^2 f)$ that are known thanks to the works of Shimura [15], Gelbart and Jacquet [1], Hoffstein and Lockhart [2], and Goldfeld, Hoffstein and Lieman [2]. If we write

$$L(s, f) = \sum_{n=1}^{\infty} \frac{\lambda_f(n)}{n^s} = \prod_p \left(1 - \frac{\alpha_p}{p^s}\right)^{-1} \left(1 - \frac{\beta_p}{p^s}\right)^{-1},$$

where α_p and $\beta_p = \overline{\alpha_p}$ are complex numbers of magnitude 1, then the symmetric square L -function is

$$L(s, \text{sym}^2 f) = \sum_{n=1}^{\infty} \frac{\lambda_f^{(2)}(n)}{n^s} = \prod_p \left(1 - \frac{\alpha_p^2}{p^s}\right)^{-1} \left(1 - \frac{1}{p^s}\right)^{-1} \left(1 - \frac{\beta_p^2}{p^s}\right)^{-1}.$$

The series and product above converge absolutely in $\text{Re}(s) > 1$, and by the work of Shimura [15], we know that $L(s, \text{sym}^2 f)$ extends analytically to the entire complex plane, and satisfies the functional equation

(2.1)

$$\Lambda(s, \text{sym}^2 f) = \Gamma_{\mathbb{R}}(s+1)\Gamma_{\mathbb{R}}(s+k-1)\Gamma_{\mathbb{R}}(s+k)L(s, \text{sym}^2 f) = \Lambda(1-s, \text{sym}^2 f),$$

where $\Gamma_{\mathbb{R}}(s) = \pi^{-s/2}\Gamma(s/2)$.

Gelbart and Jacquet [1] have shown that $L(s, \text{sym}^2 f)$ arises as the L -function of a cuspidal automorphic representation of $\text{GL}(3)$. Therefore, invoking the Rankin-Selberg convolution for $\text{sym}^2 f$, one can establish a classical zero-free region for $L(s, \text{sym}^2 f)$. For example, from Theorem 5.42 (or Theorem 5.44) of Iwaniec and Kowalski [5] one obtains that for some constant $c > 0$ the region

$$\mathcal{R} = \left\{s = \sigma + it : \sigma \geq 1 - \frac{c}{\log k(1 + |t|)}\right\}$$

does not contain any zeros of $L(s, \text{sym}^2 f)$ except possibly for a simple real zero. The work of Hoffstein and Lockhart (see the appendix by Goldfeld, Hoffstein and Lieman [2]) shows that $c > 0$ may be chosen so that there is no real zero in our region \mathcal{R} . Thus $L(s, \text{sym}^2 f)$ has no zeros in \mathcal{R} . Moreover, the work of Goldfeld, Hoffstein, and Lieman [2] shows that

(2.2)
$$L(1, \text{sym}^2 f) \gg \frac{1}{\log k}.$$

To be precise, the work of Goldfeld, Hoffstein, and Lieman considers symmetric square L -functions of Maass forms in the eigenvalue aspect, but our case is entirely analogous, and follows upon making minor modifications to their argument.

LEMMA 1. *For any $t \in \mathbb{R}$ we have*

$$|L(\frac{1}{2} + it, \text{sym}^2 f)| \ll \frac{k^{\frac{1}{2}}(1 + |t|)^{\frac{3}{4}}}{(\log k)^{1-\varepsilon}}.$$

Therefore the quantity $R_k(f)$ appearing in Theorem 2(ii) satisfies

$$R_k(f) \ll \frac{(\log k)^\varepsilon}{(\log k)L(1, \text{sym}^2 f)} \ll (\log k)^\varepsilon.$$

Proof. The bound on $L(\frac{1}{2} + it, \text{sym}^2 f)$ follows from the results of [17] on weak subconvexity; see Example 1 there. Using this bound in the definition of $R_k(f)$ immediately gives the stated estimate. \square

LEMMA 2. *We have*

$$L(1, \text{sym}^2 f) \gg (\log \log k)^{-3} \exp\left(\sum_{p \leq k} \frac{\lambda_f(p^2)}{p}\right).$$

Proof. Let $1 \leq \sigma \leq \frac{5}{4}$, and consider for some $c > 1$, and $x \geq 1$

$$(2.3) \quad \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} -\frac{L'}{L}(s + \sigma, \text{sym}^2 f) \frac{2x^s}{s(s+2)} ds.$$

We write

$$-\frac{L'}{L}(s, \text{sym}^2 f) = \sum_{n=1}^{\infty} \frac{\Lambda_{\text{sym}^2 f}(n)}{n^s}$$

say where $\Lambda_{\text{sym}^2 f}(n) = 0$ unless $n = p^k$ is a prime power, in which case $\Lambda_{\text{sym}^2 f}(p) = \lambda_f(p^2) \log p$ and for $k \geq 2$, $|\Lambda_{\text{sym}^2 f}(p^k)| \leq 3 \log p$. Using this in (2.3) and integrating term by term, we see that

$$(2.4) \quad \begin{aligned} & \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} -\frac{L'}{L}(s + \sigma, \text{sym}^2 f) \frac{2x^s}{s(s+2)} ds \\ &= \sum_{p \leq x} \frac{\lambda_f(p^2) \log p}{p^\sigma} \left(1 - \left(\frac{p}{x}\right)^2\right) + O(1) \\ &= \sum_{p \leq x} \frac{\lambda_f(p^2) \log p}{p^\sigma} + O(1). \end{aligned}$$

We next evaluate the left-hand side of (2.4) by shifting the line of integration to the line $\text{Re}(s) = -\frac{3}{2}$. We encounter poles at $s = 0$, and when $s = \rho - \sigma$ for a nontrivial zero $\rho = \beta + i\gamma$ of $L(s, \text{sym}^2 f)$. Computing these residues, we obtain that the quantity in (2.4) equals

$$\begin{aligned} & -\frac{L'}{L}(\sigma, \text{sym}^2 f) + O\left(\sum_{\rho} \frac{x^{\beta-\sigma}}{|\rho - \sigma| |\rho - \sigma + 2|}\right) \\ & \quad + \frac{1}{2\pi i} \int_{-\frac{3}{2}-i\infty}^{-\frac{3}{2}+i\infty} -\frac{L'}{L}(s + \sigma, \text{sym}^2 f) \frac{2x^s ds}{s(s+2)}. \end{aligned}$$

To estimate the integral above, we differentiate the functional equation (2.1) logarithmically, and use Stirling’s formula. Thus if $s = -\frac{3}{2} + it$ we obtain that

$$-\frac{L'}{L}(s + \sigma, \text{sym}^2 f) \ll \log(k(1 + |t|)) + \left| \frac{L'}{L}(1 - s - \sigma, \text{sym}^2 f) \right| \ll \log(k(1 + |t|)).$$

Thus we deduce that

$$(2.5) \quad \sum_{p \leq x} \frac{\lambda_f(p^2) \log p}{p^\sigma} = -\frac{L'}{L}(\sigma, \text{sym}^2 f) + O\left(\sum_{\rho} \frac{x^{\beta - \sigma}}{|\rho - \sigma| |\rho - \sigma + 2|}\right) + O(x^{-\frac{3}{2}} \log k).$$

To bound the sum over zeros above, we split the sum into intervals where $n \leq |\gamma| < n + 1$ for $n = 0, 1, \dots$. Each such interval contains at most $\ll \log(k(1 + n))$ zeros, and moreover they all lie outside the zero-free region \mathcal{R} . Therefore the sum over zeros in (2.5) is

$$\ll x^{-c/\log k} (\log k)^2 + \sum_{n=1}^{\infty} x^{-c/\log(k(1+n))} \frac{\log(k(1+n))}{n^2} \ll x^{-c/(2 \log k)} (\log k)^2 + 1.$$

Choose $x = k^{4(\log \log k)/c}$ so that the above becomes $\ll 1$. Combining this estimate with (2.5) we conclude that for this choice of x ,

$$-\frac{L'}{L}(\sigma, \text{sym}^2 f) = \sum_{p \leq x} \frac{\lambda_f(p^2) \log p}{p^\sigma} + O(1),$$

and integrating both sides from $\sigma = 1$ to $\frac{5}{4}$ that

$$\log L(1, \text{sym}^2 f) = \sum_{p \leq x} \frac{\lambda_f(p^2)}{p} + O(1).$$

Since $\sum_{k < p \leq x} \lambda_f(p^2)/p \leq \sum_{k < p \leq x} 3/p = 3 \log \log \log x + O(1)$, our lemma follows. □

Remark. The factor $(\log \log k)^{-3}$ above is extraneous. With more effort one should be able to show that $L(1, \text{sym}^2 f) \asymp \exp(\sum_{p \leq k} \lambda_f(p^2)/p)$.

From Lemma 2 we shall obtain an estimate for the quantity $M_k(f)$ appearing in Theorem 2.

LEMMA 3. *We have*

$$M_k(f) \ll (\log k)^{\frac{1}{6}} (\log \log k)^{\frac{9}{2}} L(1, \text{sym}^2 f)^{\frac{1}{2}}.$$

Proof. From the inequality $2|x| \leq \frac{2}{3} + \frac{3}{2}x^2$, and the Hecke relations, we obtain that

$$2 \sum_{p \leq k} \frac{|\lambda_f(p)|}{p} \leq \frac{2}{3} \sum_{p \leq k} \frac{1}{p} + \frac{3}{2} \sum_{p \leq k} \frac{\lambda_f(p)^2}{p} = \frac{13}{6} \sum_{p \leq k} \frac{1}{p} + \frac{3}{2} \sum_{p \leq k} \frac{\lambda_f(p^2)}{p}.$$

Using Lemma 2, and that $\sum_{p \leq k} 1/p = \log \log k + O(1)$, the lemma follows. \square

Remark. The *ad hoc* inequality $2|x| \leq \frac{2}{3} + \frac{3}{2}x^2$ used above was chosen for the simplicity of the statement in Lemma 3. If we write $L(1, \text{sym}^2 f) = (\log k)^\theta$ (so that $\theta \geq -1 + o(1)$ by (2.2)), then a more complicated, but more precise, bound is

$$M_k(f) \ll (\log k)^{-2-\theta+2\sqrt{1+\theta+\varepsilon}}.$$

This follows upon using $2|x| \leq \sqrt{1+\theta+\varepsilon} + x^2/\sqrt{1+\theta+\varepsilon}$ in the argument above.

3. Proof of Theorem 1

Case (i): The inner product with Maass cusp forms. Suppose that $L(1, \text{sym}^2 f) \geq (\log k)^{-\frac{7}{15}}$. Then Theorem 3(i) gives that $|\langle \phi F_k, F_k \rangle| \ll_\phi (\log k)^{-\frac{1}{30}+\varepsilon}$.

Now suppose that $L(1, \text{sym}^2 f) < (\log k)^{-\frac{7}{15}}$. Then Lemma 3 gives that $M_k(f) \ll (\log k)^{-\frac{1}{15}+\varepsilon}$, and using this in Theorem 2(i), we obtain again that $|\langle \phi F_k, F_k \rangle| \ll_\phi (\log k)^{-\frac{1}{30}+\varepsilon}$.

In either case, the bound stated in Theorem 1(i) holds.

Case (ii): The inner product with incomplete Eisenstein series. We begin by showing how Theorem 3(ii) applies to incomplete Eisenstein series. By Mellin inversion we may write

$$(3.1) \quad E(z | \psi) = \frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} \widehat{\psi}(-s) E(z, s) ds,$$

where $\sigma > 1$ so that we are in the range of absolute convergence of the Eisenstein series $E(z, s)$, and $\widehat{\psi}$ denotes the Mellin transform

$$\widehat{\psi}(s) = \int_0^\infty \psi(y) y^s \frac{dy}{y}.$$

Since ψ is smooth and compactly supported inside $(0, \infty)$, we have that $\widehat{\psi}$ is an analytic function, and repeated integration by parts shows that $|\widehat{\psi}(s)| \ll_{A, \psi} (1 + |s|)^{-A}$ for any positive integer A . We now move the line of integration in (3.1) to the line $\text{Re}(s) = \frac{1}{2}$. The pole of $E(z, s)$ at $s = 1$ leaves a residue $\frac{3}{\pi}$ and so

$$(3.2) \quad E(z | \psi) = \frac{3}{\pi} \widehat{\psi}(-1) + \frac{1}{2\pi i} \int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \widehat{\psi}(-s) E(z, s) ds.$$

From (3.2) it follows that

$$(3.3) \quad \langle E(\cdot | \psi) F_k, F_k \rangle = \frac{3}{\pi} \widehat{\psi}(-1) + \frac{1}{2\pi i} \int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \widehat{\psi}(-s) \langle E(\cdot, s) F_k, F_k \rangle ds.$$

By unfolding we see that

$$\langle E(\cdot, \psi), 1 \rangle = \int_0^\infty \int_{-\frac{1}{2}}^{\frac{1}{2}} \psi(y) \frac{dx dy}{y^2} = \widehat{\psi}(-1),$$

and by Theorem 3(ii) it follows that

$$\begin{aligned} & \frac{1}{2\pi i} \int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \widehat{\psi}(-s) \langle E(\cdot, s) F_k, F_k \rangle ds \\ & \ll \int_{-\infty}^\infty |\widehat{\psi}(-\frac{1}{2} - it)| \frac{(1 + |t|)^2}{(\log k)^{1-\varepsilon} L(1, \text{sym}^2 f)} dt \ll \frac{(\log k)^{-1+\varepsilon}}{L(1, \text{sym}^2 f)}. \end{aligned}$$

Using these in (3.3) we conclude that

$$\left| \langle E(\cdot | \psi) F_k, F_k \rangle - \frac{3}{\pi} \langle E(\cdot | \psi), 1 \rangle \right| \ll \frac{(\log k)^{-1+\varepsilon}}{L(1, \text{sym}^2 f)}.$$

Thus if $L(1, \text{sym}^2 f) \geq (\log k)^{-\frac{13}{15}}$, we obtain the bound stated in Theorem 1(ii).

Suppose now that $L(1, \text{sym}^2 f) \leq (\log k)^{-\frac{13}{15}}$. Then by Lemma 3 we have $M_k(f) \ll (\log k)^{-\frac{4}{15}+\varepsilon}$, and by Theorem 2(ii) (using Lemma 1 to estimate $R_k(f)$ there) we obtain

$$\left| \langle E(\cdot | \psi) F_k, F_k \rangle - \frac{3}{\pi} \langle E(\cdot | \psi), 1 \rangle \right| \ll (\log k)^\varepsilon M_k(f)^{\frac{1}{2}} \ll (\log k)^{-\frac{2}{15}+\varepsilon},$$

so that the bound stated in Theorem 1(ii) follows in this case also.

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