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Sieving for mass equidistribution
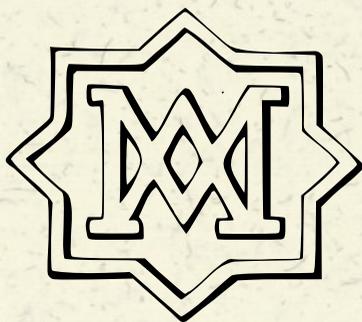
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Sieving for mass equidistribution

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Abstract

We approach the holomorphic analogue to the Quantum Unique Ergodicity conjecture through an application of the Large Sieve. We deal with shifted convolution sums as in [Hol09], with various simplifications in our analysis due to the knowledge of the Ramanujan-Petersson conjecture in this holomorphic case.

1. Introduction and statement of results

We study the *shifted convolution sums*

$$\sum_{n \leq x} \lambda_1(n)\overline{\lambda_2(n + \ell)}$$

where ℓ is a fixed nonzero integer and the λ_i are multiplicative functions. Ignoring cancellations among the summation terms, we look to obtain nontrivial upper bounds for the sums

$$\sum_{n \leq x} |\lambda_1(n)\lambda_2(n + \ell)|$$

when the multiplicative functions satisfy $|\lambda_i(n)| \leq \tau_m(n)$ for some m. Here $\tau_m(n)$ is the number of ways to represent n as the product of m natural numbers. This is a continuation of the work in [Hol09] where we studied such sums with Hecke eigenvalues of Hecke-Maass cusp forms. The results obtained here will be specifically applied to the holomorphic cusp form analogue since the eigenvalues in this case satisfy the Ramanujan-Petersson bound $|\lambda_f(n)| \leq \tau(n) := \tau_2(n)$.

Our work here complements recent work of Soundararajan [Sou10] and in [HS10] we combine our results to resolve the mass equidistribution conjecture, the motivation for this work which we now describe.

1.1. *Mass equidistribution.* Let $\mathbb{H}$ be the upper half-plane with hyperbolic measure $d\mu z := y^{-2}dxdy$. Set $\Gamma = \mathrm{SL}_2(\mathbb{Z})$ and let $X = \Gamma\backslash\mathbb{H}$. Denote by $\mathscr{L}(X)$ the Hilbert space of square integrable automorphic functions with inner product

$$\langle f, g \rangle := \int_X f(z)\overline{g(z)}d\mu z.$$

Choosing any sequence of holomorphic Hecke eigen cusp forms $\{f_k\}$ normalized so that

$$\int_X y^k |f_k(z)|^2 d\mu z = 1$$

and any fixed $\phi \in \mathscr{L}(X)$, we are interested in the behavior of $\langle \phi F_k, F_k \rangle$ as $k \longrightarrow \infty$.

Here we have simply taken $F_k(z) = y^{k/2} f_k(z)$ where $z = x + iy$. In particular, we are motivated by the following conjecture.

MASS EQUIDISTRIBUTION. *For any fixed ϕ smooth and bounded on X we have*

(1) $$\langle \phi F_k, F_k \rangle = \frac{1}{\mathrm{Vol}(X)}\langle \phi, 1 \rangle + o(1) \text{ as } k \longrightarrow \infty.$$

Here, $\mathrm{Vol}(X) := \int_X d\mu z = \pi/3$. Note that this statement is trivial for ϕ a constant function.

This equidistribution statement is a holomorphic analogue to a more natural conjecture known as Quantum Unique Ergodicity(QUE) which was originally stated for compact arithmetic surfaces by Rudnick and Sarnak [RS94] and proved in the compact case by Lindenstrauss [Lin06]. Techniques applied by Lindenstrauss from ergodic theory for QUE, however, do not seem to carry over to the analogous holomorphic statement of mass equidistribution.

The statement of QUE, for our choice of noncompact X, replaces the varying normalized holomorphic forms in (1) with an orthonormal basis of Hecke-Maass cusp forms $\{u_j\}$ and affirms that

$$\langle \phi u_j, u_j \rangle = \frac{3}{\pi}\langle \phi, 1 \rangle + o(1) \text{ as } t_j \longrightarrow \infty.$$

The u_j above are indexed with respect to their Laplace eigenvalues

$$\left(\Delta + \left(\frac{1}{4} + t_j^2\right)\right)u_j = 0,$$

where

$$\Delta := y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)$$

is the hyperbolic Laplacian. The study of the inner products $\langle \phi u_j, u_j \rangle$ as $t_j \longrightarrow \infty$ provides connections between a classically chaotic dynamical system, Hamiltonian flow on a surface of negative curvature, and its quantum model (see [Sar03] and [Zel87]).

1.2. *Main results.* We will be analyzing the inner products seen in the statement of mass equidistribution with test functions ϕ which have a Fourier series expansion of the form

$$\phi(z) = \sum_{\ell} a_{\ell}(y)e(\ell x).$$

For each Fourier coefficient $a_{\ell}(y)$, we will define the integral

$$S_{\ell}(Y) := \int_0^{\infty} g(Yy)y^{-2}\left(\int_{-1/2}^{1/2} a_{\ell}(y)e(\ell x)|F_k(z)|^2 dx\right)dy$$

from which the shifted convolution sums will arise upon integration in the variable x. Here we have chosen $g \in C_c^{\infty}(\mathbb{R}^+)$ to be some smooth, compactly supported and positive valued test function and $Y \geq 1$ to be some parameter. The role of g and Y will be to control the size of the shifts ℓ which must be considered. These parameters will be introduced in conjunction with an incomplete Eisenstein series

$$E_Y(z|g) := \sum_{\gamma \in \Gamma_{\infty} \backslash \Gamma} g(Y \operatorname{Im}(\gamma z))$$

which will act as the main "analytic tool" in manipulating the inner products $\langle \phi F_k, F_k \rangle$. In fact, our analysis starts with the inner product $\langle E_Y(z|g)\phi F_k, F_k \rangle$, in Section 3. The extra incomplete Eisenstein series will allow for an "unfolding" method (see §3.1) which will bring us to objects of the form $S_{\ell}(Y)$ and allows us to work in a Siegel domain with $y \gg 1/Y$ by the support of g. Note that in the work of [LS03], the test functions ϕ are chosen to be incomplete Eisenstein and Poincaré series. With such test functions ϕ, the unfolding of $\langle \phi F_k, F_k \rangle$ is automatic, but there is less control over the size of the shifts ℓ. In Section 3 we prove the following.

THEOREM 1.1. *Let $Y \geq 1$ and $\varepsilon > 0$. Fix a positive-valued test function $g \in C_c^{\infty}(\mathbb{R}^+)$ and set $c_Y := \frac{3}{\pi}\langle E(z|g), 1\rangle Y$. Fix an automorphic form ϕ with Fourier expansion*

$$\phi(z) = \sum_{\ell} a_{\ell}(y)e(\ell x).$$

If ϕ is a Hecke-Maass cusp form, then

$$\langle \phi F_k, F_k \rangle = c_Y^{-1} \sum_{0 < |\ell| < Y^{1+\varepsilon}} S_{\ell}(Y) + O\left(\frac{1}{Y^{1/2}}\right).$$

If ϕ is an incomplete Eisenstein series, then

$$\langle \phi F_k, F_k \rangle = \frac{3}{\pi}\langle \phi, 1\rangle + c_Y^{-1} \sum_{0 < |\ell| < Y^{1+\varepsilon}} S_{\ell}(Y) + O\left(\frac{1 + R_k(f)}{Y^{1/2}}\right)$$

with

$$R_k(f) = \frac{1}{k^{1/2} L(1, \text{sym}^2 f_k)} \int_{-\infty}^{+\infty} \frac{|L(\frac{1}{2} + it, \text{sym}^2 f_k)|}{(|t| + 1)^{10}} |dt|.$$

Furthermore, we have the bound

$$c_Y^{-1} S_\ell(Y)$$

$$\ll \frac{|a_\ell(Y^{-1})|}{L(1, \text{sym}^2 f_k)} \left\{ \frac{1}{Yk} \sum_n |\lambda_f(n)\lambda_f(n+\ell)| \, g\left(\frac{Y(k-1)}{4\pi(n+\frac{\ell}{2})} \right) + \frac{(Yk)^\varepsilon}{k} \right\}.$$

We therefore see interest in the study of the shifted convolution sums which appear in our bound for $c_Y^{-1} S_\ell(Y)$ above. In Section 4, we prove the following.

THEOREM 1.2. *Let* $\lambda_1(n)$ *and* $\lambda_2(n)$ *be multiplicative functions which satisfy* $|\lambda_i(n)| \leq \tau_m(n)$ *for some m. For any* $0 < \varepsilon < 1$, *any x sufficiently large with respect to* ε *and any fixed integer* $0 < |\ell| \leq x$ *we have*

$$\sum_{n \leq x} |\lambda_1(n)\lambda_2(n+\ell)| \ll x(\log x)^\varepsilon M(x)\tau(|\ell|),$$

where

$$M(x) := \frac{1}{(\log x)^2} \prod_{p \leq z} \left(1 + \frac{|\lambda_1(p)|}{p} \right) \left(1 + \frac{|\lambda_2(p)|}{p} \right)$$

with $z = \exp(\frac{\log x}{\varepsilon \log \log x})$.

Remark. For the sake of completeness, we provide a direct proof of Theorem 1.2 using the Large Sieve. However, an alternate proof may be derived from the works of Nair [Nai92] and Nair-Tenenbaum [NT98].

Using Theorem 1.2 with $\lambda_1 = \lambda_2 = \lambda_f$ for each Hecke eigen cusp form f_k and bounding $|a_\ell(Y^{-1})|$ in Theorem 1.1 by an application of Lemma 2.1 below (with $A = 0$), leaves us with evaluating a simple sum over the shifts $0 < |\ell| < Y^{1+\varepsilon}$. In the cusp form case we are left with

$$\frac{(Y \log k)^\varepsilon}{Y^{1/2}(\log k)^2 L(1, \text{sym}^2 f_k)} \prod_{p \leq k} \left(1 + \frac{2|\lambda_f(p)|}{p} \right) \sum_{0 < |\ell| < Y^{1+\varepsilon}} |\rho(\ell)|\tau(|\ell|).$$

In the incomplete Eisenstein series case, $|\rho(\ell)|$ is replaced by $\tau(|\ell|)$. Cauchy's inequality and the bound (3) below (used only in the cusp form case) gives the following corollary upon choosing Y optimally.

COROLLARY 1.3. *Fix an automorphic form* ϕ. *Define*

$$M_k(f) := \frac{1}{(\log k)^2 L(1, \text{sym}^2 f_k)} \prod_{p \leq k} \left(1 + \frac{2|\lambda_f(p)|}{p} \right).$$

If ϕ *is a Hecke-Maass cusp form, then*

$$\langle \phi F_k, F_k \rangle \ll (\log k)^\varepsilon M_k(f)^{1/2}$$

for any $\varepsilon > 0$. If ϕ is an incomplete Eisenstein series, then

$$\langle \phi F_k, F_k \rangle = \frac{3}{\pi} \langle \phi, 1 \rangle + O\left((\log k)^{\varepsilon} M_k(f)^{1/2}(1 + R_k(f))\right)$$

for any $\varepsilon > 0$ with

$$R_k(f) \ll \frac{1}{k^{1/2} L(1, \mathrm{sym}^2 f_k)} \int_{-\infty}^{+\infty} \frac{|L\left(\frac{1}{2} + it, \mathrm{sym}^2 f_k\right)|}{(|t| + 1)^{10}} |dt|.$$

Note. In Theorem 1.1 we ask that $Y \geqslant 1$. The bounds in Corollary 1.3 come from choosing $Y = M_k(f)^{-1}$. We trivially have that $M_k(f)^{\varepsilon} \ll (\log k)^{\varepsilon}$. If $M_k(f) > 1$ for some form f_k, then simply choosing $Y = 1$ in Theorem 1.1 (i.e., ignoring the shifted sums) produces the better result. In either case, the bounds in Corollary 1.3 still hold true.

Applying the Ramanujan-Petersson bound trivially in Corollary 1.3, one fails to properly control the growth rate of $M_k(f)$. The usefulness of Theorem 1.2 might therefore initially be unapparent. However, we still expect the main term $M_k(f)$ to be relatively small due to the conjectured Sato-Tate distribution. More specifically, we expect that

$$\exp\left(2 \sum_{p \leqslant k} \frac{|\lambda_f(p)| - 1}{p}\right) = o(L(1, \mathrm{sym}^2 f_k))$$

as $k \longrightarrow \infty$. To help illustrate this point, we turn to an idea seen in [EMS84]. Simply by the Ramanujan-Petersson bound we have

$$2|\lambda_f(p)| - 2 \leqslant (\lambda_f^2(p) - 1) - \frac{1}{9}(\lambda_f^2(p) - 1)^2$$

for each prime p. The Hecke relations for the coefficients of the symmetric square then give that

$$\sum_{p \leqslant k} \frac{2|\lambda_f(p)| - 2}{p} \leqslant \sum_{p \leqslant k} \frac{\lambda_f(p^2)}{p} - \frac{1}{9} \sum_{p \leqslant k} \frac{\lambda_f^2(p^2)}{p}.$$

We expect the contribution from the sum over terms of the form $\lambda(p^m)p^{-1}$ to be roughly of size $\log L(1, \mathrm{sym}^m f_k)$ and with $\lambda_f^2(p^2) = 1 + \lambda_f(p^2) + \lambda_f(p^4)$ one therefore hopes to obtain a bound of the form

$$M_k(f) \ll \{(\log k) L(1, \mathrm{sym}^2 f_k) L(1, \mathrm{sym}^4 f_k)\}^{-1/9} \ll_f (\log k)^{-\delta}$$

for some $\delta > 0$. Recall that the holomorphicity and nonvanishing of the symmetric m-th power L-functions in the half-plane $\mathrm{Re}(s) \geqslant 1$ is known for $m \leqslant 8$ by the works of [KS00] and [Kim03]. Although it seems difficult, by means of current technology, to show that the above bound for $M_k(f)$ is uniform for all f, one may at least show that the above bound holds uniformly for some $\delta > 0$ for all but at most $O(k^{\varepsilon})$ forms of weight k by a zero-free region argument.

In his recent work [Sou10], Soundararajan attacks the problem of mass equidistribution through the analysis of L-functions at their central values. Soundararajan's method also produces mass equidistribution with a small number of possible exceptions. In [HS10], we combine our two different arguments to remove the possibility of any exceptions, thus proving mass equidistribution with an effective rate of convergence.

2. Fourier coefficients of the forms ϕ and f_k

For ϕ a fixed Maass cusp form or incomplete Eisenstein series, we can express ϕ as a Fourier series expansion of the type

$$\phi(z) = a_0(y) + \sum_{\ell \neq 0} a_\ell(y) e(\ell x), \tag{2}$$

with $a_0(y) = \langle \phi, 1 \rangle = 0$ in the cusp form case. If $\phi(z)$ is a fixed Maass cusp form with Δ eigenvalue $1/4 + r^2$, then we have the expansion

$$\phi(z) = \sqrt{y} \sum_{\ell \neq 0} \rho(\ell) K_{ir}(2\pi |\ell| y) e(\ell x)$$

where the $\rho(\ell)$ are complex numbers. We know by [Iwa95, Th. 3.2] that the Fourier coefficients of a cusp form satisfy

$$\sum_{|\ell| \leqslant x} |\rho(\ell)|^2 \ll (r + x) e^{\pi r}. \tag{3}$$

For $\phi(z) = E(z|\psi)$ the incomplete Eisenstein series

$$E(z|\psi) := \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} \psi(\operatorname{Im} \gamma z),$$

where $\psi(y)$ is a smooth function, compactly supported on $\mathbb{R}^+$, the coefficients $a_\ell(y)$ can be determined in terms of the coefficients of the Eisenstein series

$$E(z, s) := \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} (\operatorname{Im} \gamma z)^s. \tag{4}$$

The latter has the Fourier expansion

$$E(z, s) = y^s + \varphi(s) y^{1-s} + \sqrt{y} \sum_{\ell \neq 0} \varphi_{|\ell|}(s) K_{s-\frac{1}{2}}(2\pi |\ell| y) e(\ell x), \tag{5}$$

where

$$\varphi(s) = \sqrt{\pi} \frac{\Gamma(s - \frac{1}{2}) \zeta(2s - 1)}{\Gamma(s) \zeta(2s)} = \frac{\theta(1 - s)}{\theta(s)},$$

$$\theta(s) = \pi^{-s} \Gamma(s) \zeta(2s),$$

$$\varphi_\ell(s) = \frac{2}{\theta(s)} \sum_{ab = \ell} \left(\frac{a}{b}\right)^{s - \frac{1}{2}}, \text{ if } \ell \geqslant 1.$$

Note that

$$\text{(6)} \qquad \text{Res}_{s=1} E(z, s) = \text{Res}_{s=1} \varphi(s) = \frac{3}{\pi}.$$

We have

$$E(z|\psi) = \frac{1}{2\pi i} \int_{(2)} \Psi(-s) E(z, s) ds$$

where $\Psi(s)$ is the Mellin transform of $\psi(y)$. This is an entire function with rapid decay in vertical strips; specifically

$$\Psi(s) \ll (|s| + 1)^{-A}$$

for any $A \geqslant 0$, $-2 \leqslant \text{Re}(s) \leqslant 2$, with the implied constant depending only on ψ and A.

From these formulas, we find the coefficients of (2) in the incomplete Eisenstein series case

$$a_0(y) = \frac{1}{2\pi i} \int_{(2)} \Psi(-s)(y^s + \varphi(s) y^{1-s}) ds = \psi(y) + O(y^{-1}),$$

and for $\ell \neq 0$ we move the integration to the line $\text{Re}(s) = 1/2$ to get

$$\text{(7)} \quad a_\ell(y) = \left(\frac{y}{\pi}\right)^{\frac{1}{2}} \int_{-\infty}^{+\infty} \frac{\pi^{it} \Psi(-\frac{1}{2} - it)}{\Gamma(\frac{1}{2} + it) \zeta(1 + 2it)} \left(\sum_{ab=|\ell|} \left(\frac{a}{b}\right)^{it}\right) K_{it}(2\pi|\ell|y) dt.$$

Doing the same for $a_0(y)$ we get

$$a_0(y) = \frac{3}{\pi} \Psi(-1) + O(\sqrt{y}).$$

On the other hand, by unfolding the incomplete Eisenstein series ϕ, we derive that

$$\text{(8)} \qquad \langle \phi, 1 \rangle = \int_{-1/2}^{1/2} \int_0^\infty \psi(y) d\mu z = \Psi(-1).$$

Therefore, we have

$$a_0(y) = \frac{3}{\pi} \langle \phi, 1 \rangle + O(\sqrt{y})$$

which conveniently contains the expected main term $3/\pi \langle \phi, 1 \rangle$ in the statement of mass equidistribution. For $\ell \neq 0$ we apply to (7) the bound

$$K_{it}(w) \ll |\Gamma(\frac{1}{2} + it)| \left(\frac{1 + |t|}{w}\right)^A \left(1 + \frac{1 + |t|}{w}\right)^\varepsilon$$

known for real w and imaginary order it with any $\varepsilon > 0$ and any integer $A \geqslant 0$ by repeated integration by parts of the integral representation of the K-Bessel function

$$K_{it}(w) = \pi^{-1/2} \Gamma\left(\frac{1}{2} + it\right) \left(\frac{w}{2}\right)^{-it} \int_0^{+\infty} (v^2 + 1)^{-it-1/2} \cos(vw) dv,$$

in order to obtain

$$a_\ell(y) \ll \tau(|\ell|) \sqrt{y} \left(\frac{1}{|\ell|y}\right)^A \left(1 + \frac{1}{|\ell|y}\right)^\varepsilon.$$

Here $\tau(\ell)$ is the divisor function. In the case of the cusp form, when $a_\ell(y) = \rho(\ell)K_{ir}(2\pi|\ell|y)$, we get a similar bound. We state the results in the following lemma.

LEMMA 2.1. *Let $\phi \in \mathcal{A}(X)$ be an automorphic function on $X = \mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}$ with Fourier series expansion*

$$\phi(z) = a_0(y) + \sum_{\ell \neq 0} a_\ell(y)e(\ell x).$$

If ϕ is a Maass cusp form with Δ eigenvalue $1/4 + r^2$, then $a_0(y) = 0$ and for $\ell \neq 0$ we have

$$a_\ell(y) \ll |\rho(\ell)|\sqrt{y}\Big(\frac{1+|r|}{|\ell|y}\Big)^A\Big(1 + \frac{1+|r|}{|\ell|y}\Big)^\varepsilon$$

for any $\varepsilon > 0$ and any integer $A \geqslant 0$. If ϕ is an incomplete Eisenstein series, then

$$a_0(y) = \frac{3}{\pi}\langle\phi, 1\rangle + O(\sqrt{y})$$

and for $\ell \neq 0$ we have

$$a_\ell(y) \ll \tau(|\ell|)\sqrt{y}\Big(\frac{1}{|\ell|y}\Big)^A\Big(1 + \frac{1}{|\ell|y}\Big)^\varepsilon$$

for any $\varepsilon > 0$ and any integer $A \geqslant 0$.

For f_k a holomorphic Hecke eigen cusp form of weight k with Petersson norm

$$\int_X y^k|f_k(z)|^2 d\mu z = 1,$$

we can express f_k as a Fourier series expansion of the form

$$f_k(z) = \sum_{n \geqslant 1} a_f(n)e(nz).$$

The coefficients $a_f(n)$ are proportional to the Hecke eigenvalues $\lambda_f(n)$

$$(9) \qquad\qquad a_f(n) = \lambda_f(n)a_f(1)n^{(k-1)/2}$$

with the first Fourier coefficient satisfying

$$(10) \qquad\qquad |a_f(1)|^2 = \frac{(4\pi)^{k-1}}{\Gamma(k-1)}\frac{2\pi^2}{(k-1)L(1, \mathrm{sym}^2 f_k)}.$$

The Ramanujan-Petersson conjecture, $|\lambda_f(p)| \leqslant 2$ for prime p, is known for holomorphic forms f_k by the work of Deligne and it is also known ([HL94], [GHL94]) that

$$(\log k)^{-1} \ll L(1, \mathrm{sym}^2 f_k) \ll (\log k)^3.$$

3. **Proof of Theorem** 1.1

We demonstrate how one may relate the inner products

$$(11) \qquad \langle \phi F_k, F_k \rangle = \int_X \phi(z) |F_k(z)|^2 d\mu z$$

in the question of mass equidistribution to the study of a controlled number of shifted convolution sums. As mentioned in the introduction, we will introduce an additional incomplete Eisenstein series along with a parameter Y which will act as our main analytic tools.

3.1. *Construction of our main object $I_\phi(Y)$.* Let f_k be a normalized holomorphic Hecke eigen cusp form of weight k and ϕ an automorphic function which is smooth and bounded on $\mathbb{H}$. Let $F_k(z) := y^{k/2} f_k(z)$. Fix a positive valued function $g(y) \in C_c^\infty(\mathbb{R}^+)$, smooth and compactly supported on $\mathbb{R}^+$, and let

$$G(s) := \int_0^{+\infty} g(y) y^{s-1} dy$$

be its Mellin transform. Therefore, $G(s)$ is entire and

$$G(s) \ll (|s| + 1)^{-A}$$

for any $A \geq 0$, uniformly in vertical strips, where the implied constant depends on g and A. Let $Y \geq 1$ and consider the integral

$$(12) \qquad I_\phi(Y) := \frac{1}{2\pi i} \int_{(\sigma)} G(-s) Y^s \int_X E(z, s) \phi(z) |F_k(z)|^2 d\mu z \, ds$$

with $\sigma > 1$. The integral $I_\phi(Y)$ will provide the connection between our inner products (11) and shifted convolution sums. We compute $I_\phi(Y)$ asymptotically in two ways.

LEMMA 3.1a (contour shift). *For ϕ a fixed Hecke-Maass cusp form or incomplete Eisenstein series we have*

$$I_\phi(Y) = c_Y \langle \phi F_k, F_k \rangle + O(Y^{1/2}),$$

where

$$(13) \qquad c_Y := \frac{3}{\pi} \langle E(z|g), 1 \rangle Y.$$

Proof. Starting with equation (12) and moving the contour of integration to the line $\text{Re}(s) = 1/2$, we write

$$I_\phi(Y) = c_Y \langle \phi F_k, F_k \rangle + R_\phi(Y)$$

with c_Y as in (13) coming from the pole of the Eisenstein series at $s = 1$ (see (8)) and $R_\phi(Y)$ the remaining term

$$R_\phi(Y) := \int_X p(z) \phi(z) |F_k(z)|^2 d\mu z,$$

where

$$p(z) := \frac{1}{2\pi i} \int_{(1/2)} G(-s)Y^s E(z,s)ds.$$

From the Fourier series expansion (5) for the Eisenstein series $E(z,s)$ and Lemma 2.1 (with $A = 2$) we have $E(z,s) \ll \sqrt{y} + |s|^2 y^{-3/2}(1 + |s|/y)^\varepsilon$ on the line $\mathrm{Re}(s) = 1/2$. Hence $p(z) \ll \sqrt{yY}$ if $y \geqslant 1/2$. Assuming that $\sqrt{y}|\phi(z)|$ is bounded on $\mathbb{H}$, we conclude that $R_\phi(Y) \ll_{\phi,g} \sqrt{Y}$. Such is the case for ϕ a cusp form or incomplete Eisenstein series. $\qquad\square$

LEMMA 3.1b (unfolding). *For ϕ a Hecke-Maass cusp form or incomplete Eisenstein series we have*

$$(14) \qquad I_\phi(Y) = \int_0^\infty g(Yy)y^{-2}\left(\int_{-1/2}^{1/2} \phi(z)|F_k(z)|^2 dx\right) dy.$$

Proof. Unfolding the inner integral in (12) by the definition of the Eisenstein series (4) and then integrating in s we get the result. $\qquad\square$

Remark. Since $g(Yy)$ is supported on $y \asymp 1/Y$, $|x| \leqslant 1/2$ and $\phi(z)$ is bounded, we have by Lemma 2.10 in [Iwa95], which states that there are roughly Y copies of the standard fundamental domain in that region, that $I_\phi(Y) \ll_{\phi,g} Y$. Note that the same bound holds for $\phi \equiv 1$ the constant function.

Combining Lemma 3.1a with Lemma 3.1b we get the following.

PROPOSITION 3.2. *For ϕ a Hecke-Maass cusp form or incomplete Eisenstein series we have*

$$c_Y \langle \phi F_k, F_k \rangle = \int_0^\infty g(Yy)y^{-2}\left(\int_{-1/2}^{1/2} \phi(z)|F_k(z)|^2 dx\right) dy + O(Y^{1/2})$$

where $c_Y = \dfrac{3}{\pi}\langle E(z|g), 1\rangle Y$.

The remainder of Section 3 will be devoted to the analysis of the integral in (14) to provide a useful relation between $\langle \phi F_k, F_k \rangle$ and shifted convolution sums via Proposition 3.2.

3.2. *Truncating the test function $\phi(z)$.* We are left to extract shifted convolution sums from the integral

$$I_\phi(Y) = \int_0^\infty g(Yy)y^{-2}\left(\int_{-1/2}^{1/2} \phi(z)|F_k(z)|^2 dx\right) dy.$$

We first choose to truncate our fixed form ϕ in order to restrict the size of the shifts ℓ which must be considered. Recall that the fixed form $\phi(z)$ has Fourier expansion

$$(15) \qquad \phi(z) = \sum_\ell a_\ell(y)e(\ell x)$$

with coefficient bounds as seen in Lemma 2.1. We ignore the dependence on the spectral parameter for bounds involving ϕ since it is fixed in this case.

If ϕ is an incomplete Eisenstein series, then we find that the contribution to $I_\phi(Y)$ from the tail of (15) with $|\ell| \geq Y^{1+\varepsilon}$ for any $\varepsilon > 0$ is bounded by

$$\left(\int_0^\infty \int_{-1/2}^{1/2} g(Yy)|F_k(z)|^2 \frac{dx\,dy}{y^2} \right) \frac{Y^{A+\varepsilon}}{\sqrt{Y}} \sum_{|\ell| \geq Y^{1+\varepsilon}} \frac{\tau(\ell)}{|\ell|^A} \ll Y^{3/2+\varepsilon(1-A)}$$

by the support of g and Lemma 2.1. See the remark after Lemma 3.1b. Hence the contribution of these terms to $I_\phi(Y)$ is $\ll Y^{1/2}$ after A is chosen sufficiently large with respect to ε. A similar argument works when ϕ is a fixed cusp form.

PROPOSITION 3.3. *For ϕ a Hecke-Maass cusp form or incomplete Eisenstein series we have*

$$c_Y \langle \phi F_k, F_k \rangle = \int_0^\infty g(Yy)y^{-2} \left(\int_{-1/2}^{1/2} \phi^*(z)|F_k(z)|^2 dx \right) dy + O(Y^{1/2}),$$

where $c_Y = \dfrac{3}{\pi} \langle E(z|g), 1 \rangle Y$ and

$$\phi^*(z) := \sum_{|\ell| < Y^{1+\varepsilon}} a_\ell(y)e(\ell x).$$

3.3. *Extracting shifted convolution sums.* We now consider each Fourier coefficient $a_\ell(y)$ of $\phi^*(z)$ separately and write

$$(16) \qquad c_Y \langle \phi F_k, F_k \rangle = S_0(Y) + \sum_{0 < |\ell| < Y^{1+\varepsilon}} S_\ell(Y) + O(Y^{1/2})$$

where for any integer ℓ we define

$$(17) \qquad S_\ell(Y) := \int_0^\infty g(Yy)y^{-2} \left(\int_{-1/2}^{1/2} a_\ell(y)e(\ell x)|F_k(z)|^2 dx \right) dy.$$

The aim of this section is to analyze the objects $S_\ell(Y)$ so that when we divide through equation (16) by c_Y we will have the equations and bounds seen in Theorem 1.1. We start by noting that $S_0(Y) \equiv 0$ for ϕ a cusp form. By Lemma 2.1 we have

$$(18) \quad S_0(Y) = \left\{ \frac{3}{\pi}\langle \phi, 1 \rangle + O(Y^{-1/2}) \right\} \int_0^\infty g(Yy)y^{-2} \left(\int_{-1/2}^{1/2} |F_k(z)|^2 dx \right) dy$$

for ϕ an incomplete Eisenstein series. We treat $S_0(Y)$ in (18) and $S_\ell(Y)$ in (17) for $\ell \neq 0$ simultaneously.

Squaring out $|F_k(z)|^2$ and integrating in x gives

$$S_0(Y) = \left\{ \frac{3}{\pi}\langle \phi, 1 \rangle + O(Y^{-1/2}) \right\} \sum_{n \geq 1} |a_f(n)|^2 \left(\int_0^\infty g(Yy)y^{k-2}e^{-4\pi ny}dy \right)$$

and

$$S_\ell(Y) = \sum_{n \geqslant 1} \overline{a_f(n)} a_f(n+\ell) \left(\int_0^\infty g(Yy) a_\ell(y) y^{k-2} e^{-2\pi(2n+\ell)y} \, dy \right)$$

for $\ell \neq 0$ which satisfies

$$S_\ell(Y) \ll |a_\ell(Y^{-1})| \sum_{n \geqslant 1} |a_f(n) a_f(n+\ell)| \left(\int_0^\infty g(Yy) y^{k-2} e^{-2\pi(2n+\ell)y} \, dy \right)$$

by the nonnegativity of our test function g. Appealing to the Mellin transform of $g(Yy)$ and applying the normalizations (9) and (10) for $a_f(n)$ and $|a_f(1)|^2$, we integrate in y and define for all integers n and ℓ with $n \geqslant 1$ and $n + \ell \geqslant 1$

$$W_{n,\ell}(Y) := \left(\frac{\sqrt{n(n+\ell)}}{n + \frac{\ell}{2}} \right)^{k-1} \frac{1}{2\pi i} \int_{(\sigma)} G(-s) \left(\frac{Y}{4\pi(n+\frac{\ell}{2})} \right)^s \frac{\Gamma(s+k-1)}{\Gamma(k-1)} \, ds$$

for any $\sigma > 1$. Note that $\sqrt{n(n+\ell)} \leqslant n + \ell/2$ with equality when $\ell = 0$.

LEMMA 3.3a. *Let $Y \geqslant 1$. When $\ell = 0$ we have $S_0(Y) \equiv 0$ for ϕ a cusp form and*

$$S_0(Y) = \left\{ \frac{3}{\pi} \langle \phi, 1 \rangle + O(Y^{-1/2}) \right\} \frac{2\pi^2}{(k-1)L(1, \mathrm{sym}^2 f_k)} \sum_n |\lambda_f(n)|^2 W_{n,0}(Y)$$

for ϕ an incomplete Eisenstein series. When $\ell \neq 0$,

$$S_\ell(Y) \ll \frac{|a_\ell(Y^{-1})|}{kL(1, \mathrm{sym}^2 f_k)} \sum_n |\lambda_f(n)\lambda_f(n+\ell)| W_{n,\ell}(Y)$$

for both such ϕ.

In the case of the $\ell = 0$ shift, we make the extra observation that

$$\sum_n |\lambda_f(n)|^2 W_{n,0}(Y) = \frac{1}{2\pi i} \int_{(\sigma)} G(-s) L(s, f_k \otimes f_k) \left(\frac{Y}{4\pi} \right)^s \frac{\Gamma(s+k-1)}{\Gamma(k-1)} \, ds,$$

where

$$L(s, f_k \otimes f_k) := \sum_{n \geqslant 1} \lambda_f^2(n) n^{-s}$$

is the Rankin-Selberg convolution. Moving the contour of integration to the $\sigma = 1/2$ line we pick up the pole at $s = 1$ giving

$$\sum_n |\lambda_f(n)|^2 W_{n,0}(Y) = \frac{G(-1) Y(k-1) L(1, \mathrm{sym}^2 f_k)}{4\pi \zeta(2)} + E_{1/2}(Y),$$

where

$$E_{1/2}(Y) := \frac{1}{2\pi i} \int_{(1/2)} G(-s) L(s, f_k \otimes f_k) \left(\frac{Y}{4\pi} \right)^s \frac{\Gamma(s+k-1)}{\Gamma(k-1)} \, ds.$$

We now make use of a lemma seen in the work of Luo and Sarnak. By [LS03, (2.3)] we have

$$(19) \qquad \frac{\Gamma(s+k-1)}{\Gamma(k-1)} = (k-1)^s \left(1 + O_{a,b}((|s|+1)^2 k^{-1})\right),$$

which holds by Stirling's formula for any vertical strip $0 < a \leqslant \mathrm{Re}(s) \leqslant b$. Applying this in the case of the zero shift for $E_{1/2}(Y)$ tells us that

$$E_{1/2}(Y) \ll (Yk)^{1/2} \int_{-\infty}^{+\infty} \frac{|L(\frac{1}{2}+it, \mathrm{sym}^2 f_k)|}{(|t|+1)^A} |dt|.$$

for any $A > 0$. For the nonzero shifts ℓ, (19) and integration in s with $\sigma = 1 + \varepsilon$ for some $\varepsilon > 0$ gives

$$(20) \quad W_{n,\ell}(Y) = \left(\frac{\sqrt{n(n+\ell)}}{n+\frac{\ell}{2}}\right)^{k-1} g\left(\frac{Y(k-1)}{4\pi(n+\frac{\ell}{2})}\right) + O\left(k^\varepsilon \left(\frac{Y}{n+\frac{\ell}{2}}\right)^{1+\varepsilon}\right).$$

Combining the above, we get the following by applying Lemma 3.3a along with a trivial application of Cauchy's inequality and Deligne's bound to control the contribution from the error term in (20) above.

PROPOSITION 3.4. *Let $Y \geqslant 1$. When $\ell = 0$ we have $S_0(Y) \equiv 0$ for ϕ a cusp form and*

$$c_Y^{-1} S_0(Y) = \left\{\frac{3}{\pi} \langle \phi, 1 \rangle + O(Y^{-1/2})\right\}$$

$$\times \left\{1 + O\left((Yk)^{-1/2} \int_{-\infty}^{+\infty} \frac{|L(\frac{1}{2}+it, \mathrm{sym}^2 f_k)|}{(|t|+1)^{10} L(1, \mathrm{sym}^2 f_k)} |dt|\right)\right\}$$

for ϕ an incomplete Eisenstein series. For $\ell \neq 0$ we have for both such ϕ,

$$c_Y^{-1} S_\ell(Y) \ll \frac{|a_\ell(Y^{-1})|}{L(1, \mathrm{sym}^2 f_k)} \left\{\frac{1}{Yk} \sum_n |\lambda_f(n)\lambda_f(n+\ell)| \, g\left(\frac{Y(k-1)}{4\pi(n+\frac{\ell}{2})}\right) + \frac{(Yk)^\varepsilon}{k}\right\}$$

for any $\varepsilon > 0$.

With this and Proposition 3.3, we conclude the proof of Theorem 1.1.

4. **Proof of Theorem** 1.2

We first rearrange and partition our shifted sums

$$(21) \qquad S_\ell(x) := \sum_{n \leqslant x} |\lambda_1(n)\lambda_2(n+\ell)|,$$

into objects which may either be treated by elementary methods or by a Large Sieve. Recall that we are working with general multiplicative functions satisfying $|\lambda_i(n)| \leqslant \tau_m(n)$ for some m. For future reference, we assume that $0 < |\ell| \leqslant x$ and given $0 < \varepsilon < 1$ we will be working throughout Section 4 with the choice of

variables $2 \leqslant z \leqslant y \leqslant x$ satisfying

$$(22) \qquad\qquad z := x^{1/s},$$

$$s := \varepsilon \log \log x,$$

$$y := x^{\varepsilon},$$

$$x \geqslant \exp(\exp(\exp\{(4 + m^4)(2\varepsilon)^{-1}\})).$$

4.1. *Factorization of n and $n + \ell$ and partitioning of $S_\ell(x)$.* Factoring n and $n + \ell$ uniquely as

$$(23) \qquad\qquad n = ab \text{ and } n + \ell = a_\ell b_\ell,$$

such that for every prime p dividing $n(n + \ell)$,

$$(24) \qquad\qquad p|aa_\ell \Rightarrow p \leqslant z \text{ and } p|bb_\ell \Rightarrow p > z,$$

we partition the sum $S_\ell(x)$ into parts depending on the sizes of a and a_ℓ. We shall refer to the factorizations in (23)–(24) as $\mathscr{F}_z$. The partitioning of the sum $S_\ell(x)$ is both convenient and necessary for the Large Sieve application. We denote by $\mathscr{S}^y(x)$ the parts where a or a_ℓ are greater than y,

$$(25) \quad \mathscr{S}^y(x) := \sum_{\substack{n \leqslant x \\ \mathscr{F}_z \text{ and } a > y}} |\lambda_1(n)\lambda_2(n + \ell)| + \sum_{\substack{n \leqslant x \\ \mathscr{F}_z \text{ and } a_\ell > y}} |\lambda_1(n)\lambda_2(n + \ell)|,$$

and the part where both a and a_ℓ are less than or equal to y we denote by $\mathscr{S}_y(x)$

$$(26) \qquad\qquad \mathscr{S}_y(x) := \sum_{\substack{n \leqslant x \\ \mathscr{F}_z \text{ and } a, a_\ell \leqslant y}} |\lambda_1(n)\lambda_2(n + \ell)|$$

so that

$$(27) \qquad\qquad S_\ell(x) \leqslant \mathscr{S}^y(x) + \mathscr{S}_y(x).$$

In equations (25)–(26), $\mathscr{F}_z$ denotes that we are considering n and $n + \ell$ in terms of their factorizations given by (23)–(24).

4.2. *Treating $\mathscr{S}^y(x)$ by elementary methods.* We first handle the part with a or a_ℓ large. The technique need only be elementary, because the number of such a and a_ℓ with small prime factors should be relatively small.

By Hölder's inequality and $|\lambda_i(n)| \leqslant \tau_m(n)$ we get

$$\mathscr{S}^y(x) \ll \left(\sum_{\substack{y < a \leqslant x \\ p|a \Rightarrow p \leqslant z}} \sum_{\substack{b \leqslant x/a \\ p|b \Rightarrow p > z}} 1 \right)^{1/2} \left(\sum_{n \leqslant x} \tau_m^4(n) \right)^{1/2}$$

for any $|\ell| \leqslant x$. We know (see for example [IK04, (1.80)]) that

$$\sum_{n \leqslant x} \tau_m^4(n) \ll x(\log x)^{m^4 - 1}.$$

By a classical Rankin's method argument ([MV07, Th. 7.6]), the number of integers up to x, containing only small prime factors, can be bounded by $\ll x(\log x)^{-A}$ for any $A > 0$. Partial summation and our choice of $2 \leqslant z \leqslant y \leqslant x$ in (22) then gives that

$$(28) \qquad \mathscr{S}^y(x) \ll \frac{x}{(\log x)^2}.$$

Note. The result holds even for $\ell = 0$ and in this case there is only one factorization $n = ab$. The fact that $\ell \neq 0$ is crucial, however, for what follows in Section 4.3.

4.3. *Treating $\mathscr{S}_y(x)$ by the Large Sieve.* From our definition (26) of $\mathscr{S}_y(x)$, we write n and $n + \ell$ in terms of their factorizations $\mathscr{F}_z$ and are left with evaluating

$$(29) \qquad \mathscr{S}_y(x) = \sum_{\substack{a, a_\ell \leqslant y \\ p|aa_\ell \Rightarrow p \leqslant z}} |\lambda_1(a)\lambda_2(a_\ell)| \sum_{\substack{n \leqslant x \\ n \equiv 0 \,(\mathrm{mod}\, a) \\ n \equiv -\ell \,(\mathrm{mod}\, a_\ell) \\ p|bb_\ell \Rightarrow p > z}} |\lambda_1(b)\lambda_2(b_\ell)|.$$

To help deal with certain co-primality conditions which come up during analysis, we pull out the greatest common divisor v of a and a_ℓ, writing now $n_v := n/v = ab$ and $n_v + w = a_\ell b_\ell$ with $(a, a_\ell) = (aa_\ell, w) = 1$ so that
$$(30)$$
$$\mathscr{S}_y(x) = \sum_{vw=\ell} \sum_{\substack{a, a_\ell \leqslant y/v \\ p|aa_\ell \Rightarrow p \leqslant z \\ (a,a_\ell)=(aa_\ell,w)=1}} |\lambda_1(va)\lambda_2(va_\ell)| \sum_{\substack{n_v \leqslant x/v \\ n_v \equiv r \,(\mathrm{mod}\, aa_\ell) \\ p|bb_\ell \Rightarrow p > z}} |\lambda_1(b)\lambda_2(b_\ell)|.$$

Here we applied the Chinese remainder theorem so that the residue class r in the innermost sum satisfies $r \equiv 0 \,(\mathrm{mod}\, a)$ and $r \equiv -w \,(\mathrm{mod}\, a_\ell)$. Finally, we take advantage of positivity by applying the Ramanujan-Petersson conjecture to obtain

$$(31) \qquad |\lambda_1(b)\lambda_2(b_\ell)| \ll (\log x)^{2m\varepsilon}$$

by our choice of s in (22). Indeed, we have $|\lambda_1(p^\alpha)| \leqslant \tau_m(p^\alpha) \leqslant 2^{\alpha+m-1}$ for prime p and $b = p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_t^{\alpha_t}$ with $\alpha_1 + \alpha_2 + \ldots + \alpha_t \leqslant s$.

We proceed to bound the count

$$(32) \qquad \sum_{\substack{n_v \leqslant x/v \\ n_v \equiv r \,(\mathrm{mod}\, aa_\ell) \\ p|bb_\ell \Rightarrow p > z}} 1.$$

Writing $n_v = (aa_\ell)m + r$ with $0 \leqslant r < aa_\ell$, we note the following equivalences between divisibility conditions for primes $p \leqslant z$:

$$(33) \qquad p \nmid b \iff p \nmid (a_\ell m + r/a),$$

$$(34) \qquad p \nmid b_\ell \iff p \nmid (am + (r + w)/a_\ell).$$

For fixed $(a, a_\ell) = (aa_\ell, w) = 1$ with $a, a_\ell \leqslant y/v$, we see that the count in (32) is bounded by $S = |\mathcal{S}(\mathcal{M}, \mathcal{P}, \Omega)|$ where we define the *sifted set* to be

$$\mathcal{S}(\mathcal{M}, \mathcal{P}, \Omega) = \{m \in \mathcal{M}; \ m \ (\text{mod } p) \notin \Omega_p \text{ for all } p \in \mathcal{P}\}$$

with

$$\mathcal{M} := \{m \in \mathbb{Z} \mid 0 < m(vaa_\ell) \leqslant x\},$$

$$\mathcal{P} := \{p \text{ prime} \mid 2 < p \leqslant z\},$$

and the set of residue classes to be "sieved out" $\Omega := \bigcup_{p \in \mathcal{P}} \Omega_p$ is given by

$$\Omega_p := \begin{cases} \{r_1 \ (\text{mod } p)\} & \text{for } p|a \\ \{r_2 \ (\text{mod } p)\} & \text{for } p|a_\ell \\ \{r_1 \ (\text{mod } p), r_2 \ (\text{mod } p)\} & \text{for } p \nmid aa_\ell \end{cases}$$

where $r_1 \equiv -\overline{a_\ell} r/a \ (\text{mod } p)$ and $r_2 \equiv -\overline{a}(r+w)/a_\ell \ (\text{mod } p)$. Here the overline means multiplicative inverse modulo p. Indeed, if $p|a$ (or $p|a_\ell$) we see that the condition (34) (resp. (33)) is redundant with $p \nmid w$ and therefore only one residue class is "sieved out" in these cases. Recall that $(a, a_\ell) = (aa_\ell, w) = 1$ so that $p \leqslant z$ cannot divide both a and a_ℓ. Setting $\omega(p) = |\Omega_p|$ for all $p \in \mathcal{P}$ and applying the Large Sieve as stated in [IK04, Th. 7.14], we have

$$S \leqslant \frac{N + Q^2}{H}$$

for any $Q \geqslant 1$, where $N = x/(vaa_\ell)$,

$$H = \sum_{q \leqslant Q} h(q) = \sum_{\substack{q_1 \leqslant Q \\ q_1 | aa_\ell}} h(q_1) \sum_{\substack{q_2 \leqslant Q/q_1 \\ (q_2, aa_\ell) = 1}} h(q_2)$$

and $h(q)$ is the multiplicative function supported on square-free integers with prime divisors in $\mathcal{P}$ such that

$$h(p) = \frac{\omega(p)}{p - \omega(p)}.$$

Choosing say $Q = x^{1/4}$, we get by our values of $\omega(p)$ that

$$H \gg \left(\frac{\varphi(aa_\ell)}{aa_\ell}\right)^2 \sum_{\substack{q_1 \leqslant Q \\ q_1 | aa_\ell}} h(q_1) \sum_{q_2 \leqslant Q/q_1} \frac{\tau(q_2)}{q_2} \gg \frac{\varphi(aa_\ell)}{aa_\ell}(\log z)^2$$

so that S and the count in (32) is

$$\ll \frac{x}{v\varphi(aa_\ell)(\log z)^2}.$$

Plugging the above bound for (32) along with (31) back into (30) gives

$$\mathscr{S}_y(x) \ll \frac{x}{(\log x)^{2-\varepsilon}} \sum_{vw=\ell} \frac{1}{v} \sum_{\substack{a,a_\ell \leqslant y/v \\ p|aa_\ell \Rightarrow p \leqslant z \\ (a,a_\ell)=(aa_\ell,w)=1}} \frac{|\lambda_1(va)\lambda_2(va_\ell)|}{\varphi(aa_\ell)}$$

and therefore,

$$\mathscr{S}_y(x) \ll \frac{x}{(\log x)^{2-\varepsilon}} \prod_{p \leqslant z} \left(1 + \frac{|\lambda_1(p)|}{p}\right)\left(1 + \frac{|\lambda_2(p)|}{p}\right) \tau(|\ell|).$$

By our partition of $S_\ell(x)$ in (27), combining the above bound for $\mathscr{S}_y(x)$ with the bound for $\mathscr{S}^y(x)$ in (28) concludes the proof of Theorem 1.2.

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