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Weak subconvexity for central values of
L-functions

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Abstract

We describe a general method to obtain weak subconvexity bounds for many classes of L-functions. We give several examples of our bound, and our work has applications to a conjecture of Rudnick and Sarnak for the mass equidistribution of Hecke eigenforms

1. Introduction and statement of results

A fundamental problem in number theory is to estimate the values of L-functions at the center of the critical strip. The Langlands program predicts that all L-functions arise from automorphic representations of $\mathrm{GL}(N)$ over a number field, and moreover that such L-functions can be decomposed as a product of primitive L-functions arising from irreducible cuspidal representations of $\mathrm{GL}(n)$ over $\mathbb{Q}$. The L-functions that we consider will either arise in this manner, or will be the Rankin-Selberg L-function associated to two irreducible cuspidal representations. Note that such Rankin-Selberg L-functions are themselves expected to arise from automorphic representations, but this is not known in general.

Given an irreducible cuspidal automorphic representation π (normalized to have unitary central character), we denote the associated standard L-function by $L(s, \pi)$, and its analytic conductor (whose definition we shall recall shortly) by $C(\pi)$. There holds generally a convexity bound of the form $L(\frac{1}{2}, \pi) \ll_{\varepsilon} C(\pi)^{\frac{1}{4}+\varepsilon}$ (see Molteni [28]).[1] The Riemann hypothesis for $L(s, \pi)$ implies the Lindelöf hypothesis: $L(\frac{1}{2}, \pi) \ll C(\pi)^{\varepsilon}$. In several applications it has emerged that the convexity bound barely fails to be of use, and that any improvement over the convexity bound would have significant consequences. Obtaining such subconvexity bounds has been an active area of research, and estimates of the type $L(\frac{1}{2}, \pi) \ll C(\pi)^{\frac{1}{4}-\delta}$

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[1]Recently, Roger Heath-Brown [14] has pointed out an elegant application of Jensen's formula for strips that leads generally to the stronger convexity bound $L(\frac{1}{2}, \pi) \ll C(\pi)^{\frac{1}{4}}$.

for some $\delta > 0$ have been obtained for several important classes of L-functions. However in general the subconvexity problem remains largely open. For comprehensive accounts on L-functions and the subconvexity problem we refer to Iwaniec and Sarnak [21], and Michel [27].

In this paper we describe a method that leads in many cases to an improvement over the convexity bound for values of L-functions. The improvement is not a saving of a power of the analytic conductor, as desired in formulations of the subconvexity problem. Instead we obtain an estimate of the form $L(\frac{1}{2}, \pi) \ll C(\pi)^{\frac{1}{4}}/(\log C(\pi))^{1-\varepsilon}$, which we term *weak* subconvexity. In some applications, it appears that a suitable weak subconvexity bound would suffice in place of genuine subconvexity. In particular, by combining sieve estimates for the shifted convolution problem developed by Holowinsky together with the weak subconvexity estimates developed here, Holowinsky and I [18] have been able to resolve a conjecture of Rudnick and Sarnak on the mass equidistribution of Hecke eigenforms. We begin by giving three illustrative examples of our work before describing the general result.

Example 1. Let f be a holomorphic Hecke eigenform of large weight k for the full modular group $\mathrm{SL}_2(\mathbb{Z})$. Let t be a fixed real number (for example, $t = 0$), and consider the symmetric square L-function $L(\frac{1}{2} + it, \mathrm{sym}^2 f)$. The convexity bound for this L-function gives $|L(\frac{1}{2} + it, \mathrm{sym}^2 f)| \ll_t k^{\frac{1}{2}+\varepsilon}$, and this can be refined to $\ll_t k^{\frac{1}{2}}$ by Heath-Brown's result mentioned in footnote 1. Our method gives the weak subconvexity bound, for any $\varepsilon > 0$,

$$(1.1) \qquad |L(\tfrac{1}{2} + it, \mathrm{sym}^2 f)| \ll_\varepsilon \frac{k^{\frac{1}{2}}(1 + |t|)^{\frac{3}{4}}}{(\log k)^{1-\varepsilon}}.$$

Obtaining subconvexity bounds in this situation (with a power saving in k) remains an important open problem. In the case when k is fixed, and t gets large such a subconvexity bound has been achieved recently by Li [22]. We have assumed that the level is 1 for simplicity, and the result holds for higher level also. However, the assumption that f is holomorphic is essential, since our method makes use of the Ramanujan conjectures known here due to Deligne. Similar results would hold for Maass forms if we assume the Ramanujan conjectures, but unfortunately the partial results known towards the Ramanujan bounds are insufficient for our purpose.

Example 2. Let f be a holomorphic Hecke eigenform of large weight k for the full modular group. Let ϕ be a fixed Hecke-Maass eigencuspform for $\mathrm{SL}_2(\mathbb{Z})$. Consider the triple product L-function $L(\frac{1}{2}, f \times f \times \phi)$. The convexity bound gives $L(\frac{1}{2}, f \times f \times \phi) \ll_\phi k^{1+\varepsilon}$. Our weak subconvexity bound gives for any $\varepsilon > 0$

$$(1.2) \qquad L(\tfrac{1}{2}, f \times f \times \phi) \ll_{\phi,\varepsilon} \frac{k}{(\log k)^{1-\varepsilon}}.$$

Again we could consider higher level, but the assumption that f is holomorphic is necessary for our method.

Example 3. Let π_0 be an irreducible cuspidal automorphic representation on $\mathrm{GL}(m_0)$ over $\mathbb{Q}$ with unitary central character. We treat π_0 as fixed, and consider $L(\frac{1}{2} + it, \pi_0)$ in the t-aspect. The convexity bound here is $L(\frac{1}{2} + it, \pi_0) \ll_{\pi_0} (1 + |t|)^{\frac{m_0}{4} + \varepsilon}$ and we obtain

$$(1.3) \qquad |L(\tfrac{1}{2} + it, \pi_0)| \ll_{\pi_0, \varepsilon} \frac{(1 + |t|)^{\frac{m_0}{4}}}{(\log(1 + |t|))^{1-\varepsilon}}.$$

Similarly, if $\chi \pmod q$ is a primitive Dirichlet character with q large, then

$$(1.4) \qquad L(\tfrac{1}{2}, \pi_0 \times \chi) \ll_{\pi_0, \varepsilon} \frac{q^{\frac{m_0}{4}}}{(\log q)^{1-\varepsilon}}.$$

The most general example along these lines is the following: Let π_0 be as above, and let π be an irreducible cuspidal automorphic representation on $\mathrm{GL}(m)$ with unitary central character and *such that π satisfies the Ramanujan conjectures.* Then we would obtain a weak subconvexity bound for $L(\frac{1}{2}, \pi_0 \times \pi)$.

We now describe an axiomatic framework (akin to the Selberg class) for the class of L-functions that we consider. The properties of L-functions that we assume are mostly standard, and we have adopted this framework in order to clarify the crucial properties needed for our method. In addition to the usual assumptions of a Dirichlet series with a Euler product and a functional equation, we will need an assumption on the size of the Dirichlet series coefficients. We call this a *weak Ramanujan hypothesis,* as the condition is implied by the Ramanujan conjectures. The reader may prefer to ignore our conditions below and restrict his attention to automorphic L-functions satisfying the Ramanujan conjectures, but our framework allows us to deduce results even in cases where the Ramanujan conjectures are not known.

Let $m \geq 1$ be a fixed natural number. Let[2] $L(s, \pi)$ be given by the Dirichlet series and Euler product

$$(1.5a) \qquad L(s, \pi) = \sum_{n=1}^{\infty} \frac{a_\pi(n)}{n^s} = \prod_p \prod_{j=1}^{m} \left(1 - \frac{\alpha_{j,\pi}(p)}{p^s}\right)^{-1},$$

and we suppose that both the series and product are absolutely convergent in $\mathrm{Re}(s) > 1$. We write

$$(1.5b) \qquad L(s, \pi_\infty) = N^{\frac{s}{2}} \prod_{j=1}^{m} \Gamma_{\mathbb{R}}(s + \mu_j),$$

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[2]Here the notation is meant to suggest that π corresponds to an automorphic representation, but this is not assumed.

where $\Gamma_{\mathbb{R}}(s) = \pi^{-s/2}\Gamma(s/2)$, N denotes the conductor, and the μ_j are complex numbers. The completed L-function $L(s, \pi)L(s, \pi_\infty)$ has an analytic continuation[3] to the entire complex plane, and has finite order. Moreover, it satisfies a functional equation

$$(1.5c) \qquad L(s, \pi_\infty)L(s, \pi) = \kappa L(1-s, \tilde{\pi}_\infty)L(1-s, \tilde{\pi}),$$

where κ is the root number (a complex number of magnitude 1), and

$$(1.5d) \quad L(s, \tilde{\pi}) = \sum_{n=1}^\infty \frac{\overline{a_\pi(n)}}{n^s}, \qquad \text{and} \qquad L(s, \tilde{\pi}_\infty) = N^{\frac{s}{2}} \prod_{j=1}^m \Gamma_{\mathbb{R}}(s + \overline{\mu_j}).$$

We define the analytic conductor $C = C(\pi)$ (see [21]) by

$$(1.5e) \qquad\qquad C(\pi) = N \prod_{j=1}^m (1 + |\mu_j|).$$

Our goal is to obtain an estimate for $L(\frac{1}{2}, \pi)$ in terms of the analytic conductor $C(\pi)$.

Properties (1.5a)–(1.5d) are standard features of all interesting L-functions. We now need an assumption on the size of the numbers $\alpha_{j,\pi}(p)$. The Ramanujan conjectures, which are expected to hold for all L-functions, predict that $|\alpha_{j,\pi}(p)| \leq 1$ for all p. Further, it is expected that the numbers μ_j appearing in (1.5b) all satisfy $\text{Re}(\mu_j) \geq 0$. Towards the Ramanujan conjectures it is known (see [25]) that if π is an irreducible cuspidal representation of $\text{GL}(m)$ then $|\alpha_{j,\pi}(p)| \leq p^{\frac{1}{2}-\delta_m}$ for all p, and that $\text{Re}(\mu_j) \geq -\frac{1}{2} + \delta_m$ where $\delta_m = 1/(m^2 + 1)$. We will make the following weak Ramanujan hypothesis.

Write

$$(1.6a) \qquad\qquad -\frac{L'}{L}(s, \pi) = \sum_{n=1}^\infty \frac{\lambda_\pi(n)\Lambda(n)}{n^s},$$

where $\lambda_\pi(n) = 0$ unless $n = p^k$ is a prime power when it equals $\sum_{j=1}^m \alpha_{j,\pi}(p)^k$. We assume that for some constants $A_0, A \geq 1$, and all $x \geq 1$ there holds

$$(1.6b) \qquad\qquad \sum_{x < n \leq ex} \frac{|\lambda_\pi(n)|^2}{n}\Lambda(n) \leq A^2 + \frac{A_0}{\log ex}.$$

Note that the Ramanujan conjecture would give (1.6b) with $A = m$, and $A_0 \ll m^2$. Analogously for the parameters μ_j we assume that[4]

$$(1.6c) \qquad \text{Re}(\mu_j) \geq -1 + \delta_m, \qquad \text{for some } \delta_m > 0, \text{ and all } 1 \leq j \leq m.$$

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[3]Thus we are not allowing $L(s, \pi)$ to have any poles. It would not be difficult to modify our results to allow the completed L-function to have poles at 0 and 1.

[4]This assumption is very weak: from [25] we know that it holds for all automorphic L-functions, and also for the Rankin-Selberg L-function associated to two automorphic representations.

THEOREM 1. *Let $L(s, \pi)$ be an L-function satisfying the properties* (1.5a)–(1.5e) *and* (1.6a)–(1.6c)*. Then for any $\varepsilon > 0$ we have*

$$L(\tfrac{1}{2}, \pi) \ll \frac{C(\pi)^{\frac{1}{4}}}{(\log C(\pi))^{1-\varepsilon}}.$$

Here the implied constant depends on m, A, A_0, δ_m, and ε.

We now show how the examples given above fit into the framework of Theorem 1.

Example 1 (proof). Write the Euler product for $L(s, \text{sym}^2 f)$ as

$$L(s, \text{sym}^2 f) = \prod_p \left(1 - \frac{\alpha_p^2}{p^s}\right)^{-1} \left(1 - \frac{\alpha_p \beta_p}{p^s}\right)^{-1} \left(1 - \frac{\beta_p^2}{p^s}\right)^{-1},$$

where $\alpha_p = \overline{\beta_p}$ are complex numbers of magnitude 1 (by Deligne), and $\alpha_p + \beta_p$ equals the p-th Hecke eigenvalue of f. From the work of Shimura we know that the completed L-function

$$\Lambda(s, \text{sym}^2 f) = \Gamma_{\mathbb{R}}(s+1)\Gamma_{\mathbb{R}}(s+k-1)\Gamma_{\mathbb{R}}(s+k)L(s, \text{sym}^2 f)$$

is entire and satisfies the functional equation $\Lambda(s, \text{sym}^2 f) = \Lambda(1-s, \text{sym}^2 f)$. Thus criteria (1.5a)–(1.5d) hold, and the analytic conductor of (1.5e) is $\asymp k^2$. If we write $-L'/L(s, \text{sym}^2 f)$ in the notation of (1.6a), then the analogous $\lambda_{\text{sym}^2 f}(p^k)$ equals $\alpha_p^{2k} + 1 + \beta_p^{2k}$ which is ≤ 3 in magnitude. Thus criterion (1.6b) holds with $A = 3$, and A_0 being some absolute constant. Visibly, criterion (1.6c) also holds. Therefore Theorem 1 applies and yields $L(\tfrac{1}{2}, \text{sym}^2 f) \ll k^{\frac{1}{2}}/(\log k)^{1-\varepsilon}$.

We have shown (1.1) when $t = 0$. To obtain the general case, we apply the framework of Theorem 1 to the shifted function $L_t(s, \text{sym}^2 f) := L(s+it, \text{sym}^2 f)$. The criteria we require hold. The only difference is that we must make corresponding shifts to the Γ-functions appearing in the functional equation. These shifts imply that the analytic conductor is now $\asymp (1 + |t|)(k + |t|)^2 \ll k^2(1 + |t|)^3$. Applying Theorem 1, we complete the proof of (1.1).

Example 2 (proof). Write the p-th Hecke eigenvalue of f as $\alpha_f(p) + \beta_f(p)$, where $\alpha_f(p)\beta_f(p) = 1$ and $|\alpha_f(p)| = |\beta_f(p)| = 1$. Write the p-th Hecke eigenvalue of ϕ as $\alpha_\phi(p) + \beta_\phi(p)$ where $\alpha_\phi(p)\beta_\phi(p) = 1$, but we do not know here the Ramanujan conjecture that these are both of size 1. Write also the Laplace eigenvalue of ϕ as $\lambda_\phi = \tfrac{1}{4} + t_\phi^2$, where[5] $t_\phi \in \mathbb{R}$.

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[5]This is true since we are working on the full modular group. For a congruence subgroup, we could use Selberg's bound that the least eigenvalue is $\geq \frac{3}{16}$ which gives that $|\text{Im}(t_\phi)| \leq \frac{1}{4}$; see [25].

The triple product L-function $L(s, f \times f \times \phi)$ is then defined by means of the Euler product of degree 8 (absolutely convergent in $\text{Re}(s) > 1$)

$$\prod_p \left(1 - \frac{\alpha_f(p)^2 \alpha_\phi(p)}{p^s}\right)^{-1} \left(1 - \frac{\alpha_\phi(p)}{p^s}\right)^{-2} \left(1 - \frac{\beta_f(p)^2 \alpha_\phi(p)}{p^s}\right)^{-1}$$

$$\times \left(1 - \frac{\alpha_f(p)^2 \beta_\phi(p)}{p^s}\right)^{-1} \left(1 - \frac{\beta_\phi(p)}{p^s}\right)^{-2} \left(1 - \frac{\beta_f(p)^2 \beta_\phi(p)}{p^s}\right)^{-1}.$$

This L-function is not primitive and factors as $L(s, \phi) L(s, \text{sym}^2 f \times \phi)$. Consider the product of eight Γ-factors

$$L_\infty(s, f \times f \times \phi)$$
$$:= \prod_\pm \Gamma_\mathbb{R}(s + k - 1 \pm it_\phi) \Gamma_\mathbb{R}(s + k \pm it_\phi) \Gamma_\mathbb{R}(s \pm it_\phi) \Gamma_\mathbb{R}(s + 1 \pm it_\phi).$$

From the work of Garrett [6], it is known that $L(s, f \times f \times \phi) L_\infty(s, f \times f \times \phi)$ is an entire function in $\mathbb{C}$, and its value at s equals its value at $1 - s$. Thus criteria (1.5a)–(1.5d) are met, and the analytic conductor in (1.5e) is of size $\ll k^4(1 + |t_\phi|)^8$.

If we write

$$-\frac{L'}{L}(s, f) = \sum_n \lambda_f(n)\Lambda(n)n^{-s} \quad \text{and} \quad -\frac{L'}{L}(s, \phi) = \sum_n \lambda_\phi(n)\Lambda(n)n^{-s},$$

then

$$-\frac{L'}{L}(s, f \times f \times \phi) = \sum_n \frac{\lambda_{f \times f \times \phi}(n)\Lambda(n)}{n^s} = \sum_n \frac{\lambda_f(n)^2 \lambda_\phi(n)\Lambda(n)}{n^s}.$$

If $n = p^k$ then $|\lambda_f(n)| = |\alpha_f(p)^k + \beta_f(p)^k| \le 2$, and so to check (1.6b) for $f \times f \times \phi$, we need only show that

$$\sum_{x < n \le ex} \frac{|\lambda_\phi(n)|^2 \Lambda(n)}{n} \le A^2 + \frac{A_0}{\log(ex)},$$

for all $x \ge 1$, where A and A_0 are constants which *are allowed to depend on* ϕ. This condition follows from an appeal to the Rankin-Selberg theory for $L(s, \phi \times \phi)$ which is known to extend analytically to $\mathbb{C}$ except for a simple pole at $s = 1$. Since $-\frac{L'}{L}(s, \phi \times \phi) = \sum_n |\lambda_\phi(n)|^2 \Lambda(n)n^{-s}$, and $L(s, \phi \times \phi)$ has a classical zero-free region $\text{Re}(s) > 1 - c_\phi / \log(1 + |t|)$ (see Theorem 5.44 of [20]), we may deduce, arguing as in the proof of the prime number theorem, that

$$\sum_{x < n \le ex} \frac{|\lambda_\phi(n)|^2 \Lambda(n)}{n} = 1 + O_\phi\left(\frac{1}{\log(ex)}\right),$$

from which our desired weak Ramanujan estimate follows. Criterion (1.6c) is immediate from our formula for the Γ factors above.

Thus Theorem 1 applies and we obtain the desired estimate (1.2). We could also obtain weak subconvexity bounds for triple products $\phi \times f_1 \times f_2$, fixing ϕ

(a holomorphic or Maass eigencuspform) and allowing f_1 and f_2 to vary over holomorphic eigencuspforms. Or we could consider $f_1 \times f_2 \times f_3$ with all three varying over holomorphic eigencuspforms. Subconvexity bounds (with a power saving) have been obtained for triple products $\phi_1 \times \phi_2 \times \phi_3$ where ϕ_1 and ϕ_2 are considered fixed, and ϕ_3 varies over holomorphic or Maass forms, see [1] and [32].

Example 3 (proof). This example follows upon using the ideas in the proof of Examples 1 and 2. The weak Ramanujan hypothesis (1.6b) is verified by an appeal to the Rankin-Selberg theory for $L(s, \pi_0 \times \tilde{\pi}_0)$ as in Example 2. The general Rankin-Selberg theory is the culmination of work by many authors, notably Jacquet, Piatetskii-Shapiro, and Shalika, Shahidi, and Moeglin and Waldspurger; a convenient synopsis of the analytic features of this theory may be found in [30]. A narrow zero-free region (which is sufficient to check (1.6b)) for general Rankin-Selberg L-functions has been established by Brumley [3].

The value $L(\frac{1}{2} + it, \pi_0)$ is bounded by shifting L-functions as in Example 1. We omit further details.

Application to the mass equidistribution of Hecke eigenforms. Perhaps the most interesting application of our results pertains to a conjecture of Rudnick and Sarnak on the mass equidistribution of Hecke eigenforms. For simplicity, consider a holomorphic Hecke eigencuspform f of weight k for the full modular group $\Gamma = \mathrm{SL}_2(\mathbb{Z})$. Consider the measure

$$\mu_f = y^k |f(z)|^2 \frac{dx\,dy}{y^2},$$

where we suppose that f has been normalized to satisfy

$$\int_{\Gamma \backslash \mathbb{H}} y^k |f(z)|^2 \frac{dx\,dy}{y^2} = 1.$$

Rudnick and Sarnak ([29]; see also [26], [30]) have conjectured that as $k \to \infty$, the measure μ_f approaches the uniform distribution measure $\frac{3}{\pi} \frac{dx\,dy}{y^2}$ on the fundamental domain $X = \Gamma \backslash \mathbb{H}$. This is the holomorphic analog of their quantum unique ergodicity conjecture for Maass forms. Quantum unique ergodicity for Hecke-Maass forms is now known thanks to the breakthrough work of Lindenstrauss [23] together with a final ingredient (ruling out escape of mass) supplied by Soundararajan [31]. However, the ergodic theoretic methods introduced by Lindenstrauss do not seem to apply to the holomorphic case.

Set $F_k(z) = y^{k/2} f(z)$, and recall the Petersson inner product of two nice functions g_1 and g_2 on X

$$\langle g_1, g_2 \rangle = \int_X g_1(z) \overline{g_2(z)} \frac{dx\,dy}{y^2}.$$

A smooth bounded function on X has a spectral expansion in terms of the constant function, the space of Maass cusp forms which are eigenfunctions of the Laplacian

and all Hecke operators, and the Eisenstein series $E(z, \frac{1}{2} + it)$ with $t \in \mathbb{R}$; see Iwaniec [19].

Thus by an analog of Weyl's equidistribution criterion, the Rudnick-Sarnak conjecture amounts to showing that

$$\langle \phi F_k, F_k \rangle, \qquad \langle E(\cdot, \tfrac{1}{2} + it) F_k, F_k \rangle \to 0$$

as $k \to \infty$, where ϕ is a fixed Maass cusp form which is an eigenfunction of the Laplacian and all Hecke operators, and $E(z, \frac{1}{2} + it)$ denotes the Eisenstein series (with t fixed).

Using the unfolding method, it is easy to show that

$$|\langle E(\cdot, \tfrac{1}{2} + it) F_k, F_k \rangle| = \left| \pi^{\frac{3}{2}} \frac{\zeta(\frac{1}{2} + it)L(\frac{1}{2} + it, \mathrm{sym}^2 f)}{\zeta(1 + 2it)L(1, \mathrm{sym}^2 f)} \frac{\Gamma(k - \frac{1}{2} + it)}{\Gamma(k)} \right|.$$

Since $|\Gamma(k - \frac{1}{2} + it)| \le \Gamma(k - \frac{1}{2})$, $|\zeta(\frac{1}{2} + it)| \ll (1 + |t|)^{\frac{1}{4}}$, and $|\zeta(1 + 2it)| \gg 1/\log(1 + |t|)$, using Stirling's formula and our bound (1.1) it follows that

$$|\langle E(\cdot, \tfrac{1}{2} + it) F_k, F_k \rangle| \ll_\varepsilon \frac{(1 + |t|)^2}{(\log k)^{1-\varepsilon} L(1, \mathrm{sym}^2 f)}.$$

For the case of a Maass cusp form, a beautiful formula of Watson (see Theorem 3 of [33]) shows that (here ϕ has been normalized so that $\langle \phi, \phi \rangle = 1$)

$$|\langle \phi F_k, F_k \rangle|^2 = \frac{1}{8} \frac{L_\infty(\frac{1}{2}, f \times f \times \phi)L(\frac{1}{2}, f \times f \times \phi)}{\Lambda(1, \mathrm{sym}^2 f)^2 \Lambda(1, \mathrm{sym}^2 \phi)},$$

where $L(s, f \times f \times \phi)$ is the triple product L-function of Example 2, and L_∞ denotes its Gamma factors (see Example 2 (proof)), and

$$\Lambda(s, \mathrm{sym}^2 f) = \Gamma_\mathbb{R}(s + 1)\Gamma_\mathbb{R}(s + k - 1)\Gamma_\mathbb{R}(s + k)L(s, \mathrm{sym}^2 f),$$

and

$$\Lambda(s, \mathrm{sym}^2 \phi) = \Gamma_\mathbb{R}(s)\Gamma_\mathbb{R}(s + 2it_\phi)\Gamma_\mathbb{R}(s - 2it_\phi)L(s, \mathrm{sym}^2 \phi).$$

Using Stirling's formula and the bound (1.2) of Example 2, we conclude that

$$|\langle \phi F_k, F_k \rangle| \ll_\phi \frac{1}{(\log k)^{\frac{1}{2} - \varepsilon} L(1, \mathrm{sym}^2 f)}.$$

COROLLARY 1. *With notation as above, we have*

$$|\langle \phi F_k, F_k \rangle| \ll_{\phi, \varepsilon} \frac{1}{(\log k)^{\frac{1}{2} - \varepsilon} L(1, \mathrm{sym}^2 f)}.$$

Moreover,

$$|\langle E(\cdot, \tfrac{1}{2} + it) F_k, F_k \rangle| \ll_\varepsilon \frac{(1 + |t|)^2}{(\log k)^{1-\varepsilon} L(1, \mathrm{sym}^2 f)}.$$

Given $\delta > 0$, Corollary 1 shows that if f ranges over those Hecke eigencusp-forms with $L(1, \mathrm{sym}^2 f) \ge (\log k)^{-\frac{1}{2} + \delta}$ then as $k \to \infty$, the measure μ_f converges

to $\frac{3}{\pi}\frac{dx\,dy}{y^2}$. This criterion on $L(1, \text{sym}^2 f)$ is expected to hold for all eigenforms f; for example, it is implied by the Riemann hypothesis for $L(s, \text{sym}^2 f)$. Using large sieve estimates[6] one can show that the number of exceptional eigenforms f with weight $k \leq K$ for which the criterion fails is $\ll K^\varepsilon$. Our criterion complements the work of Holowinsky [17], who attacks the mass equidistribution conjecture by an entirely different method. Combining his results with ours gives a complete resolution of the Rudnick-Sarnak conjecture on mass equidistribution for eigenforms. A detailed account of this result will appear in a joint work with Holowinsky [18].

As noted earlier, the barrier to using our methods for Maass forms is the Ramanujan conjecture which remains open here. The weak Ramanujan hypothesis that we need could be verified (using the large sieve) for all but T^ε Maass forms with Laplace eigenvalue below T. Thus with at most T^ε exceptions, one could establish the equidistribution of Maass forms on $\text{SL}_2(\mathbb{Z}) \backslash \mathbb{H}$.

For simplicity we have confined ourselves to $\Gamma = \text{SL}_2(\mathbb{Z})$ above. Similar results apply to congruence subgroups of level N. One could also consider the cocompact case of quaternion division algebras. If the quaternion algebra is un-ramified at infinity, Lindenstrauss's work shows the equidistribution of Maass cusp forms of large eigenvalue. Our results would show the corresponding equidistribution for holomorphic eigenforms, allowing for a small number of exceptional cases (at most K^ε exceptions with weight below K). In the case of a ramified quaternion algebra (acting on the unit sphere S^2), the problem concerns the equidistribution of eigenfunctions on the sphere (see [2]) , and again our results establish such equidistribution except for a small number of cases (omitting at most N^ε spherical harmonics of degree below N that are also Hecke eigenforms). In these compact cases, there is no analog of Holowinsky's work, and so we are unable to obtain a definitive result as in $\text{SL}_2(\mathbb{Z}) \backslash \mathbb{H}$.

We now return to the setting of Theorem 1, and describe an auxiliary result which will be used to prove Theorem 1. For an L-function satisfying (1.5a)–(1.5e) and (1.6a)–(1.6c), we may use the convexity bound to establish that (see Lemma 4.2 below)[7]

$$(1.7) \qquad \sum_{n \leq x} a_\pi(n) \ll \frac{x}{\log x},$$

provided $x \geq C^{\frac{1}{2}}(\log C)^B$ for some positive constant B. Our main idea is to show that similar cancellation holds even when $x = C^{\frac{1}{2}}(\log C)^{-B}$ for any constant B.

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[6]Precisely, with very few exceptions one can approximate $L(1, \text{sym}^2 f)$ by a short Euler product. Such results in the context of Dirichlet L-functions are classical, and a detailed account may be found in [7]. In the context of symmetric square L-functions, see Luo [24], and Cogdell and Michel [4].

[7]We recall here that $L(s, \pi)$ was assumed not to have any poles. If we alter our framework to allow a pole at $s = 1$, say, then (1.7) would be modified to an asymptotic formula with a main term of size x. Then Theorem 2 would extrapolate that asymptotic formula to a wider region.

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THEOREM 2. *Let $L(s, \pi)$ be as in Theorem 1. For any $\varepsilon > 0$, any positive constant B, and all $x \geq C^{\frac{1}{2}}(\log C)^{-B}$ we have*

$$\sum_{n \leq x} a_\pi(n) \ll \frac{x}{(\log x)^{1-\varepsilon}}.$$

The implied constant may depend on A, A_0, m, δ_m, B and ε.

Once Theorem 2 is established, Theorem 1 will follow from a standard partial summation argument using an approximate functional equation. In Theorems 1 and 2, by keeping track of the various parameters involved, it would be possible to quantify ε. However, the limit of our method would be to obtain a bound $C^{\frac{1}{4}}/\log C$ in Theorem 1, and $x/\log x$ in Theorem 2.

We have termed the saving of $(\log C)^{1-\varepsilon}$ as weak subconvexity, and as noted above this is close to the limit of our method. One may legitimately call a saving of any fixed power of $\log C$ as weak subconvexity. For example, in the application to the Rudnick-Sarnak conjecture any log power saving together with Holowinsky's work would suffice to show that $|\langle \phi F_k, F_k \rangle| \to 0$, for ϕ a fixed Maass cusp form. However to deal with the Eisenstein series contributions in that application, a saving of a substantial power of $\log C$ is needed[8], and a small saving would not suffice. Finally, it would be very desirable to establish a version of weak subconvexity saving a large power of $\log C$. For example, if one could save $(\log C)^{2+\delta}$ for any fixed $\delta > 0$, one would obtain immediately the mass equidistribution consequences mentioned above. Similarly it would be desirable to improve upon the weak Ramanujan criterion that we have imposed.

2. Slow oscillation of mean values of multiplicative functions

We now discuss the ideas underlying Theorem 2, and state our main technical result from which the results stated in Section 1 follow. At the heart of Theorem 2 is the fact that mean values of multiplicative functions vary slowly. Knowing (1.7) in the range $x \geq C^{\frac{1}{2}}(\log C)^B$, this fact will enable us to extrapolate (1.7) to the range $x \geq C^{\frac{1}{2}}(\log C)^{-B}$.

The possibility of obtaining such extrapolations was first considered by Hildebrand [15], [16]. If f is a multiplicative function, we shall denote by $S(x) = S(x; f)$ the partial sum $\sum_{n \leq x} f(n)$. Hildebrand [16] showed that if $-1 \leq f(n) \leq 1$ is a real valued multiplicative function then for $1 \leq w \leq \sqrt{x}$

$$(2.1) \qquad \frac{1}{x} \sum_{n \leq x} f(n) = \frac{w}{x} \sum_{n \leq x/w} f(n) + O\left(\left(\log \frac{\log x}{\log 2w}\right)^{-\frac{1}{2}}\right).$$

In other words, the mean value of f at x does not change very much from the mean-value at x/w. Hildebrand [15] used this idea to show that from knowing

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[8]A saving of $(\log C)^{0.7}$ would probably be sufficient.

Burgess's character sum estimates[9] for $x \geq q^{\frac{1}{4}+\varepsilon}$ one may obtain some nontrivial cancellation even in the range $x \geq q^{\frac{1}{4}-\varepsilon}$.

Elliott [5] generalized Hildebrand's work to cover complex valued multiplicative functions with $|f(n)| \leq 1$, and also strengthened the error term in (2.1). Notice that a direct extension of (2.1) for complex valued functions is false. Consider $f(n) = n^{i\tau}$ for some real number $\tau \neq 0$. Then $S(x; f) = x^{1+i\tau}/(1+i\tau) + O(1)$, and $S(x/w; f) = (x/w)^{1+i\tau}/(1+i\tau) + O(1)$. Therefore (2.1) is false, and instead we have that $S(x)/x$ is close to $w^{i\tau}S(x/w)/(x/w)$. Building on the pioneering work of Halasz [11], [12] on mean-values of multiplicative functions, Elliott showed that for a multiplicative function f with $|f(n)| \leq 1$, there exists a real number $\tau = \tau(x)$ with $|\tau| \leq \log x$ such that for $1 \leq w \leq \sqrt{x}$

$$(2.2) \qquad S(x) = w^{1+i\tau} S(x/w) + O\Big(x\Big(\frac{\log 2w}{\log x}\Big)^{\frac{1}{19}}\Big).$$

In [8], Granville and Soundararajan give variants and stronger versions of (2.2), with $\frac{1}{19}$ replaced by $1 - 2/\pi - \varepsilon$.

In order to establish Theorem 2, we require similar results when the multiplicative function is no longer constrained to the unit disc. The situation here is considerably more complicated, and instead of showing that a suitable linear combination of $S(x)/x$ and $S(x/w)/(x/w)$ is small, we will need to consider linear combinations involving several terms $S(x/w^j)/(x/w^j)$ with $j = 0, \ldots, J$. In order to motivate our main result, it is helpful to consider two illustrative examples.

Example 2.1. Let k be a natural number, and take $f(n) = d_k(n)$, the k-th divisor function. Then, it is easy to show that $S(x) = xP_k(\log x) + O(x^{1-1/k+\varepsilon})$ where P_k is a polynomial of degree $k - 1$. If $k \geq 2$, it follows that $S(x)/x - S(x/w)/(x/w)$ is of size $(\log w)(\log x)^{k-2}$, which is not $o(1)$. However, if $1 \leq w \leq x^{1/2k}$, the linear combination

$$\sum_{j=0}^{k}(-1)^j \binom{k}{j} \frac{S(x/w^j)}{x/w^j} = \sum_{j=0}^{k}(-1)^j \binom{k}{j} P_k(\log x/w^j) + O(x^{-\frac{1}{2k}}) = O(x^{-\frac{1}{2k}})$$

is very small.

Example 2.2. Let $\tau_1, \ldots, \tau_R$ be distinct real numbers, and let $k_1, \ldots, k_R$ be natural numbers. Let f be the multiplicative function defined by $F(s) = \sum_{n=1}^{\infty} f(n)n^{-s} = \prod_{j=1}^{R} \zeta(s - i\tau_j)^{k_j}$. Consider here the linear combination (for $1 \leq w \leq x^{1/(2(k_1+\cdots+k_R))}$)

$$\frac{1}{x} \sum_{j_1=0}^{k_1} \cdots \sum_{j_R=0}^{k_R} (-1)^{j_1+\cdots+j_R} \binom{k_1}{j_1}$$

$$\cdots \binom{k_R}{j_R} w^{j_1(1+i\tau_1)+\cdots+j_R(1+i\tau_R)} S\Big(\frac{x}{w^{j_1+\cdots+j_R}}\Big).$$

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[9]For simplicity, suppose that q is cube-free.

By Perron's formula we may express this as, for $c > 1$,

$$\frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \prod_{j=1}^{R} \zeta(s - i\tau_j)^{k_j} (1 - w^{1+i\tau_j-s})^{k_j} x^{s-1} \frac{ds}{s}.$$

Notice that the poles of the zeta-functions at $1 + i\tau_j$ have been canceled by the factors $(1 - w^{1+i\tau_j-s})^{k_j}$. Thus the integrand has a pole only at $s = 0$, and a standard contour shift argument shows that this integral is $\ll x^{-\delta}$ for some $\delta > 0$.

Fortunately, it turns out that Example 2.2 captures the behavior of mean-values of the multiplicative functions of interest to us. In order to state our result, we require some notation. Let f denote a multiplicative function and recall that

$$(2.3) \qquad\qquad S(x) = S(x; f) = \sum_{n \le x} f(n).$$

We shall write

$$(2.4) \qquad\qquad F(s) = \sum_{n=1}^{\infty} \frac{f(n)}{n^s},$$

and we shall assume that this series converges absolutely in $\mathrm{Re}(s) > 1$. Moreover, we write

$$(2.5) \qquad -\frac{F'}{F}(s) = \sum_{n=1}^{\infty} \frac{\lambda_f(n)\Lambda(n)}{n^s} = \sum_{n=1}^{\infty} \frac{\Lambda_f(n)}{n^s},$$

where $\lambda_f(n) = \Lambda_f(n) = 0$ unless n is the power of a prime p. We next assume the analog of the weak Ramanujan hypothesis (1.6b). Namely, we suppose that there exist constants $A, A_0 \ge 1$ such that for all $x \ge 1$ we have

$$(2.6) \qquad \sum_{x < n \le ex} \frac{|\lambda_f(n)|^2 \Lambda(n)}{n} \le A^2 + \frac{A_0}{\log(ex)}.$$

Let R be a natural number, and let $\tau_1, \ldots, \tau_R$ denote R real numbers. Let $\underline{\ell} = (\ell_1, \ldots, \ell_R)$ and $\underline{j} = (j_1, \ldots, j_R)$ denote vectors of nonnegative integers, with the notation $\underline{j} \le \underline{\ell}$ indicating that $0 \le j_1 \le \ell_1, \ldots, 0 \le j_R \le \ell_R$. Define

$$(2.7) \qquad\qquad \binom{\underline{\ell}}{\underline{j}} = \binom{\ell_1}{j_1} \cdots \binom{\ell_R}{j_R}.$$

Finally, we define a measure of the oscillation of the mean-values of f by setting

$$(2.8)$$

$$\mathbb{O}_{\underline{\ell}}(x, w) = \mathbb{O}_{\underline{\ell}}(x, w; \tau_1, \ldots, \tau_R)$$

$$= \sum_{\underline{j} \le \underline{\ell}} (-1)^{j_1 + \cdots + j_R} \binom{\underline{\ell}}{\underline{j}} w^{j_1(1+i\tau_1) + \cdots + j_R(1+i\tau_R)} S\left(\frac{x}{w^{j_1 + \cdots + j_R}}\right).$$

THEOREM 2.1. *Keep in mind the conditions and notation* (2.3) *through* (2.8). *Let $X \geq 10$ and $1 \geq \varepsilon > 0$ be given. Let $R = [10A^2/\varepsilon^2] + 1$ and put $L = [10AR]$, and $\underline{L} = (L, \ldots, L)$. Let w be such that $0 \leq \log w \leq (\log X)^{\frac{1}{3R}}$. There exist real numbers $\tau_1, \ldots, \tau_R$ with $|\tau_j| \leq \exp((\log \log X)^2)$ such that for all $2 \leq x \leq X$ we have*

$$|\mathbb{O}_{\underline{L}}(x, w; \tau_1, \ldots, \tau_R)| \ll \frac{x}{\log x}(\log X)^\varepsilon.$$

The implied constant above depends on A, A_0 and ε.

For a general multiplicative function, we cannot hope for any better bound for the oscillation than $x/\log x$. To see this, suppose $w \geq 2$, and consider the multiplicative function f with $f(n) = 0$ for $n \leq x/2$ and $f(p) = 1$ for primes $x/2 < p \leq x$. Then $S(x) \gg x/\log x$ whereas $S(x/w^j) = 1$ for all $j \geq 1$, and therefore for any choice of the numbers $\tau_1, \ldots, \tau_R$ we would have $\mathbb{O}_{\underline{L}}(x, w) \gg x/\log x$.

Our proof of Theorem 2.1 builds both on the techniques of Halasz (as developed in [5] and [8]), and also the idea of *pretentious* multiplicative functions developed by Granville and Soundararajan (see [10] and [9]). In Section 5 we will describe the choice of the numbers $\tau_1, \ldots, \tau_R$ appearing in Theorem 2.1, and develop relevant estimates for the Dirichlet series $F(s)$. Then the proof of Theorem 2.1 will be completed in Section 6.

3. Some preliminary lemmas

We collect together here some lemmas that will be useful below. We begin with a combinatorial lemma.

LEMMA 3.1. *Let $b(1), b(2), \ldots$ be a sequence of complex numbers. Define the sequence $a(0) = 1, a(1), a(2), \ldots$ by means of the formal identity*

$$(3.1) \qquad \exp\Big(\sum_{k=1}^{\infty} \frac{b(k)}{k} x^k\Big) = \sum_{n=0}^{\infty} a(n)x^n.$$

For $j = 1$ or 2, define the sequences $A_j(0) = 1, A_j(1), A_j(2), \ldots$ by means of the formal identity

$$\exp\Big(\sum_{k=1}^{\infty} \frac{|b(k)|^j}{k} x^k\Big) = \sum_{n=0}^{\infty} A_j(n)x^n.$$

Then $A_j(n) \geq |a(n)|^j$ for all n.

Proof. If we expand out the left-hand side of (3.1) and equate coefficients we obtain that

$$(3.2) \qquad a(n) = \sum_{\substack{\underline{\lambda} \\ \lambda_1 + \cdots + \lambda_r = n \\ \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r \geq 1}} b(\lambda_1) \cdots b(\lambda_r)W(\underline{\lambda}),$$

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where the sum is over all partitions $\underline{\lambda}$ of n, and $W(\underline{\lambda}) > 0$ is a weight attached to each partition which is independent of the sequence $b(k)$. Although we can write down explicitly what the weight $W(\underline{\lambda})$ is, we do not require this. All we need is that

$$\sum_{\underline{\lambda}} W(\underline{\lambda}) = 1,$$

which follows from (3.2) by taking $b(k) = 1$ for all k so that $\exp(\sum_{k=1}^{\infty} b(k)x^k/k)$ $= 1/(1-x) = \sum_{n=0}^{\infty} x^n$ whence $a(n) = 1$. When $j = 1$ the lemma follows by the triangle inequality. In the case $j = 2$, Cauchy-Schwarz gives

$$|a(n)|^2 \le \Big(\sum_{\underline{\lambda}} |b(\lambda_1)\cdots b(\lambda_r)|^2 W(\underline{\lambda}) \Big)\Big(\sum_{\underline{\lambda}} W(\underline{\lambda}) \Big)$$

$$= \Big(\sum_{\underline{\lambda}} |b(\lambda_1)\cdots b(\lambda_r)|^2 W(\underline{\lambda}) \Big) = A_2(n). \qquad \square$$

The significance of Lemma 3.1 for us is the following. Let f be a multiplicative function with the Euler factor at p being (compare (1.6a) and (2.5))

$$\sum_{n=0}^{\infty} \frac{f(p^n)}{p^{ns}} = \exp\Big(\sum_{k=1}^{\infty} \frac{\lambda_f(p^k)}{kp^{ks}} \Big);$$

then the lemma guarantees that if we write

$$\exp\Big(\sum_{k=1}^{\infty} \frac{|\lambda_f(p^k)|^2}{kp^{ks}} \Big) = \sum_{n=0}^{\infty} \frac{f^{(2)}(p^n)}{p^{ns}},$$

and use this to define a multiplicative function $f^{(2)}(n)$, then $|f(n)|^2 \le f^{(2)}(n)$. In the case that $f(n) = a_\pi(n)$ corresponds to the coefficients of an automorphic L-function π, this means that the coefficient[10] $a_{\pi \times \tilde{\pi}}(n)$ of the Rankin-Selberg L-function exceeds $|a_\pi(n)|^2$. The reader may compare this lemma with a similar result of Molteni, [28, Prop. 6]. Although we are only interested in Lemma 3.1 in the cases $j = 1$ and 2, an application of Hölder's inequality shows that a similar result holds for all $j \ge 1$.

LEMMA 3.2. *Let f be a multiplicative function, and keep the notation (2.4) and (2.5), and suppose that the criterion (2.6) holds. For all $x \ge 2$ we have*

$$\sum_{n \le x} \frac{|f(n)|}{n} \ll (\log x)^A, \qquad and \qquad \sum_{n \le x} \frac{|f(n)|^2}{n} \ll (\log x)^{A^2}.$$

Moreover, for all $2 \ge \sigma > 1$ we have

$$|F(\sigma + it)| \ll \Big(\frac{1}{\sigma - 1} \Big)^A \qquad and \qquad |F'(\sigma + it)| \ll \Big(\frac{1}{\sigma - 1} \Big)^{A+1}.$$

The implied constants may depend on A and A_0.

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[10] We assume that n is the power of an unramified prime.

Proof. These are simple consequences of our weak Ramanujan assumption (2.6). By splitting the sum into intervals $e^k < n \le e^{k+1}$ and using (2.6), we see that for any $2 \ge \sigma > 1$

$$(3.3) \qquad \sum_{n=2}^{\infty} \frac{|\lambda_f(n)|^2 \Lambda(n)}{n^\sigma \log n} \le A^2 \log\left(\frac{1}{\sigma-1}\right) + O(1),$$

where the error term above depends on A and A_0. By Cauchy-Schwarz it follows also that

$$(3.4) \qquad \sum_{n=2}^{\infty} \frac{|\lambda_f(n)| \Lambda(n)}{n^\sigma \log n} \le A \log\left(\frac{1}{\sigma-1}\right) + O(1).$$

Using Lemma 3.1 and (3.4) we see that

$$\sum_{n \le x} \frac{|f(n)|}{n} \ll \sum_{n=1}^{\infty} \frac{|f(n)|}{n^{1+1/\log(ex)}} \le \exp\left(\sum_{n \ge 2} \frac{|\lambda_f(n)|\Lambda(n)}{n^{1+1/\log(ex)}\log n}\right) \ll (\log x)^A.$$

This proves our first inequality. The second inequality follows in the same way, using (3.3) instead of (3.4). The third inequality follows easily from (3.4). Finally, for $2 \ge \sigma > 1$,

$$\left|\frac{F'}{F}(\sigma + it)\right| \le \sum_{n \ge 2} \frac{|\lambda_f(n)|\Lambda(n)}{n^\sigma} \ll \frac{1}{\sigma-1}$$

using Cauchy-Schwarz and (2.6) as in the proof of (3.4). The lemma follows. $\square$

LEMMA 3.3. *Let f be a multiplicative function as in Lemma 3.2. Then for all $x \ge 2$*

$$\sum_{n \le x} |f(n)| \ll x(\log x)^A.$$

Moreover, for $1 \le y \le x$ we have

$$\sum_{x < n \le x+y} |f(n)| \ll (yx)^{\frac{1}{2}}(\log x)^{A^2/2}.$$

Proof. Since $\sum_{n \le x} |f(n)| \le x \sum_{n \le x} |f(n)|/n$, the first assertion follows from Lemma 3.2. Cauchy-Schwarz gives

$$\left|\sum_{x < n \le x+y} |f(n)|\right|^2 \le y \sum_{x < n \le x+y} |f(n)|^2 \ll yx \sum_{n \le 2x} \frac{|f(n)|^2}{n},$$

and our second assertion also follows from Lemma 3.2. $\square$

4. **Deduction of the main results from Theorem** 2.1

In this section we shall show how Theorems 1 and 2 follow from Theorem 2.1. We begin with Theorem 2, whose proof will require the following simple convexity bound for our L-functions.

LEMMA 4.1. *Let $L(s, \pi)$ be an L-function satisfying the properties* (1.5a)–
(1.5e) *and* (1.6a)–(1.6c). *Then for all $t \in \mathbb{R}$ we have*

$$|L(\tfrac{1}{2} + it, \pi)| \ll C(\pi)^{\frac{1}{4}} (1 + |t|)^{\frac{m}{4}+1} (\log C(\pi))^A.$$

Proof. Define $\Lambda(s) = L(s, \pi) L(s, \pi_\infty) e^{(s-\frac{1}{2}-it)^2}$. Using the Phragmen-
Lindelöf principle we may bound $|\Lambda(\frac{1}{2} + it)|$ by the maximum value taken by
$|\Lambda(s)|$ on the lines $\text{Re}(s) = 1 + 1/\log C$, and $\text{Re}(s) = -1/\log C$. The functional
equation shows that the maximum on the line $\text{Re}(s) = -1/\log C$ is the same as
the maximum on the line $\text{Re}(s) = 1 + 1/\log C$. Therefore, using Lemma 3.2,

$$|L(\tfrac{1}{2} + it, \pi)| \le \max_{y \in \mathbb{R}} \frac{|\Lambda(1 + 1/\log C + it + iy)|}{|L(\frac{1}{2} + it, \pi_\infty)|}$$

$$\ll (\log C)^A \max_{y \in \mathbb{R}} e^{-y^2} \frac{|L(1 + 1/\log C + it + iy, \pi_\infty)|}{|L(\frac{1}{2} + it, \pi_\infty)|}.$$

Using Stirling's formula, we may show that

$$\left| \frac{\Gamma_{\mathbb{R}}(1 + 1/\log C + it + iy + \mu_j)}{\Gamma_{\mathbb{R}}(\frac{1}{2} + it + \mu_j)} \right| \ll e^{2|y|} (1 + |t| + |\mu_j|)^{\frac{1}{4} + \frac{1}{2\log C}},$$

where we used that $\text{Re}(\mu_j) \ge -1 + \delta_m$ to ensure that the numerator stays away
from poles of the Γ-function. The lemma follows immediately. $\square$

Our next lemma establishes the result (1.7) stated in the introduction.

LEMMA 4.2. *Let $L(s, \pi)$ be as above. In the range*

$$x \ge x_0 := C(\pi)^{\frac{1}{2}} (\log C(\pi))^{50mA^2}$$

we have

$$\sum_{n \le x} a_\pi(n) \ll \frac{x}{\log x}.$$

Proof. Observe that for any $c > 0$, $y > 0$, any $\lambda > 0$, and any natural number K,

$$\frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{y^s}{s} \left(\frac{e^{\lambda s} - 1}{\lambda s} \right)^K ds$$

$$= \frac{1}{\lambda^K} \int_0^\lambda \cdots \int_0^\lambda \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} (y e^{x_1 + \cdots + x_K})^s \frac{ds}{s} dx_1 \ldots dx_K$$

$$= \begin{cases} 1 & \text{if } y \ge 1 \\ \in [0, 1] & \text{if } 1 > y \ge e^{-\lambda K} \\ 0 & \text{if } y < e^{-\lambda K}. \end{cases}$$

Therefore, for any $c > 1$,

(4.1)
$$\frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} L(s, \pi) \frac{x^s}{s} \left(\frac{e^{\lambda s} - 1}{\lambda s} \right)^K ds = \sum_{n \le x} a_\pi(n) + O\left(\sum_{x < n \le e^{K\lambda} x} |a_\pi(n)| \right).$$

We shall take here $K = [m/4] + 3$ and $\lambda = (\log x)^{-2-A^2}$. Then for large x we have $e^{K\lambda} \leq 2$, and so applying Lemma 3.3 to the multiplicative function $a_\pi(n)$ we see that the error term above is

$$(4.2) \qquad \ll (e^{K\lambda} - 1)^{\frac{1}{2}} x (\log x)^{A^2/2} \ll \frac{x}{\log x}.$$

Now we move the line of integration in the left-hand side of (4.1) to the line $c = 1/2$. Using Lemma 4.1 we see that the integral on the $1/2$ line is

$$\ll C(\pi)^{\frac{1}{4}} x^{\frac{1}{2}} \lambda^{-K} (\log C)^A \int_{-\infty}^{\infty} (1 + |t|)^{m/4+1} \frac{dt}{(1 + |t|)^{K+1}}$$
$$\ll C(\pi)^{\frac{1}{4}} x^{\frac{1}{2}} (\log(Cx))^{12mA^2}.$$

Combining this with (4.1) and (4.2) we conclude that for

$$x \geq x_0 := C(\pi)^{\frac{1}{2}} (\log C(\pi))^{50mA^2}$$

we have

$$\sum_{n \leq x} a_\pi(n) \ll \frac{x}{\log x},$$

proving our lemma. $\qquad\qquad\square$

Proof of Theorem 2. To prove Theorem 2, we invoke Theorem 2.1. Let $R = [10A^2/\varepsilon^2] + 1$ and $L = [10AR]$ be as in Theorem 2.1. Let x_0 be as in Lemma 4.2, and let $x_0 \geq x \geq C^{\frac{1}{2}}/(\log C)^B$. Take $w = x_0/x$ and $X = xw^{LR}$. Applying Theorem 2.1 to the multiplicative function a_π (note that (1.6b) gives the assumption (2.6)) we find that for an appropriate choice of $\tau_1, \ldots, \tau_R$ that

$$(4.3) \qquad |\mathbb{O}_{\underline{L}}(X, w)| \ll \frac{X}{(\log X)^{1-\varepsilon}}.$$

But, by definition, the left-hand side above is

$$(4.4) \qquad w^{LR} \Big| \sum_{n \leq X/w^{LR}} a_\pi(n) \Big| + O\Big(\sum_{j=0}^{LR-1} w^j \Big| \sum_{n \leq X/w^j} a_\pi(n) \Big| \Big).$$

Now $X/w^{LR} = x$, and for $0 \leq j \leq LR - 1$ we have $X/w^j \geq xw = x_0$ so that the bound of Lemma 4.2 applies. Therefore (4.4) equals

$$w^{LR} \Big| \sum_{n \leq x} a_\pi(n) \Big| + O\Big(\frac{X}{\log X} \Big),$$

From (4.3) we conclude that

$$\Big| \sum_{n \leq x} a_\pi(n) \Big| \ll w^{-LR} \frac{X}{(\log X)^{1-\varepsilon}} \ll \frac{x}{(\log x)^{1-\varepsilon}},$$

which proves Theorem 2. $\qquad\qquad\square$

Deduction of Theorem 1 *from Theorem* 2. Theorem 1 follows from Theorem 2 by a standard argument using an "approximate functional equation" for $L(\frac{1}{2}, \pi)$ (see for example Harcos [13, Th. 2.1]) and partial summation. For the sake of completeness we provide a brief argument. We start with, for $c > \frac{1}{2}$,

$$\frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} L(s + \tfrac{1}{2}, \pi) \frac{L(s + \tfrac{1}{2}, \pi_\infty)}{L(\tfrac{1}{2}, \pi_\infty)} e^{s^2} \frac{ds}{s},$$

and move the line of integration to $\mathrm{Re}(s) = -c$. We encounter a pole at $s = 0$, and so the above equals

$$L(\tfrac{1}{2}, \pi) + \frac{1}{2\pi i} \int_{-c-i\infty}^{-c+i\infty} L(s + \tfrac{1}{2}, \pi) \frac{L(s + \tfrac{1}{2}, \pi_\infty)}{L(\tfrac{1}{2}, \pi_\infty)} e^{s^2} \frac{ds}{s}.$$

Now we use the functional equation above, and make a change of variables $s \to -s$. In this way we obtain that

$$L(\tfrac{1}{2}, \pi) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} L(s + \tfrac{1}{2}, \pi) \frac{L(s + \tfrac{1}{2}, \pi_\infty)}{L(\tfrac{1}{2}, \pi_\infty)} e^{s^2} \frac{ds}{s}$$

$$+ \frac{\kappa}{2\pi i} \int_{c-i\infty}^{c+i\infty} L(s + \tfrac{1}{2}, \tilde{\pi}) \frac{L(s + \tfrac{1}{2}, \tilde{\pi}_\infty)}{L(\tfrac{1}{2}, \pi_\infty)} e^{s^2} \frac{ds}{s}.$$

Consider the first integral above; the second is estimated similarly. Using

$$L(s + \tfrac{1}{2}, \pi) = (s + \tfrac{1}{2}) \int_1^\infty \sum_{n \leq x} a_\pi(n) \frac{dx}{x^{s + \frac{3}{2}}},$$

we see that the first integral above equals

$$(4.5) \qquad \int_1^\infty \sum_{n \leq x} a_\pi(n) \Big(\frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} (s + \tfrac{1}{2}) \frac{L(s + \tfrac{1}{2}, \pi_\infty)}{L(\tfrac{1}{2}, \pi_\infty)} e^{s^2} x^{-s} \frac{ds}{s} \Big) \frac{dx}{x^{\frac{3}{2}}}.$$

To estimate the inner integral over s, we move the line of integration either to $\mathrm{Re}(s) = \frac{1}{2} - \frac{\delta_m}{2}$, or to $\mathrm{Re}(s) = 2$. Using Stirling's formula, we see that this inner integral is $\ll \min((\sqrt{C}/x)^{\frac{1}{2} - \frac{\delta_m}{2}}, (\sqrt{C}/x)^2)$. Thus (4.5) is

$$\ll C^{\frac{1}{4} - \frac{\delta_m}{4}} \int_1^{\sqrt{C}} \Big| \sum_{n \leq x} a_\pi(n) \Big| \frac{dx}{x^{2 - \frac{\delta_m}{2}}} + C \int_{\sqrt{C}}^\infty \Big| \sum_{n \leq x} a_\pi(n) \Big| \frac{dx}{x^{\frac{7}{2}}}.$$

We now split into the ranges $x \leq \sqrt{C}/(\log C)^{4A/\delta_m}$ and $x > \sqrt{C}/(\log C)^{4A/\delta_m}$. In the first range we use Lemma 3.3 to bound $|\sum_{n \leq x} a_\pi(n)|$ by $\ll x(\log x)^A$, and in the second range we use $\sum_{n \leq x} a_\pi(n) \ll x/(\log x)^{1-\varepsilon}$ by Theorem 2. Inserting these bounds above, we conclude that the quantity in (4.5) is $\ll C^{\frac{1}{4}}/(\log C)^{1-\varepsilon}$, and Theorem 1 follows.

5. **Successive maxima**

Recall the conditions and notation (2.3) through (2.8). As in Theorem 2.1, $X \geq 10$ and $1 \geq \varepsilon > 0$ are given, and $R = [10A^2/\varepsilon^2] + 1$. In this section we define the points $\tau_1, \ldots, \tau_R$ appearing in Theorem 2.1, and collect together some estimates for the Dirichlet series $F(s)$.

From now on, we shall write $T = \exp((\log \log X)^2)$. We define τ_1 as that point t in the compact set $\mathscr{C}_1 = [-T, T]$ where the maximum of $|F(1 + 1/\log X + it)|$ is attained. Now remove the interval $(\tau_1 - (\log X)^{-\frac{1}{R}}, \tau_1 + (\log X)^{-\frac{1}{R}})$ from $\mathscr{C}_1 = [-T, T]$, and let $\mathscr{C}_2$ denote the remaining compact set. We define τ_2 to be that point t in $\mathscr{C}_2$ where the maximum of $|F(1 + 1/\log X + it)|$ is attained. Next remove the interval $(\tau_2 - (\log X)^{-\frac{1}{R}}, \tau_2 + (\log X)^{-\frac{1}{R}})$ from $\mathscr{C}_2$ leaving behind the compact set $\mathscr{C}_3$. Define τ_3 to be the point where the maximum of $|F(1 + 1/\log X + it)|$ for $t \in \mathscr{C}_3$ is attained. We proceed in this manner, defining the successive maxima $\tau_1, \ldots, \tau_R$, and the nested compact sets $\mathscr{C}_1 \supset \mathscr{C}_2 \supset \cdots \supset \mathscr{C}_R$. Notice that all the points $\tau_1, \ldots, \tau_R$ lie in $[-T, T]$, and moreover are well-spaced: $|\tau_j - \tau_k| \geq (\log X)^{-\frac{1}{R}}$ for $j \neq k$.

Lemma 3.2 bounds $|F(1 + 1/\log X + it)|$ by $\ll (\log X)^4$. For $t \in [-T, T]$ we will show that a much better bound holds, unless t happens to be near one of the points $\tau_1, \ldots, \tau_R$. The next lemma is inspired by the ideas in [10] and [9].

LEMMA 5.1. *Let $1 \leq j \leq R$ and let t be a point in $\mathscr{C}_j$. Then*

$$|F(1 + 1/\log X + it)| \ll (\log X)^{A\sqrt{1/j + (j-1)/(jR)}}.$$

In particular, if $t \in \mathscr{C}_R$ we have $|F(1 + 1/\log X + it)| \ll (\log X)^{\varepsilon/2}$.

Proof. If $t \in \mathscr{C}_j$, then for all $1 \leq r \leq j$

$$|F(1 + 1/\log X + it)| \leq |F(1 + 1/\log X + i\tau_j)| \leq |F(1 + 1/\log X + i\tau_r)|.$$

Therefore,

$$
\begin{aligned}
|F(1 + 1/\log X + i\tau_j)| &\leq \Big(\prod_{r=1}^{j} |F(1 + 1/\log X + i\tau_r)| \Big)^{\frac{1}{j}} \\
&\leq \exp\Big(\operatorname{Re}\frac{1}{j} \sum_{n \geq 2} \frac{\lambda_f(n)\Lambda(n)}{n^{1+1/\log X}(\log n)}(n^{-i\tau_1} + \cdots + n^{-i\tau_j})\Big).
\end{aligned}
$$

By Cauchy-Schwarz

$$
\begin{aligned}
\sum_{n \geq 2} &\frac{|\lambda_f(n)|\Lambda(n)}{n^{1+1/\log X}\log n}\Big| \sum_{r=1}^{j} n^{-i\tau_r}\Big| \\
&\leq \Big(\sum_{n \geq 2} \frac{|\lambda_f(n)|^2\Lambda(n)}{n^{1+1/\log X}\log n}\Big)^{\frac{1}{2}} \Big(\sum_{n \geq 2} \frac{\Lambda(n)}{n^{1+1/\log X}\log n}\Big| \sum_{r=1}^{j} n^{-i\tau_r}\Big|^2\Big)^{\frac{1}{2}}.
\end{aligned}
$$

By (2.6) the first factor above is $\leq (A^2 \log \log X + O(1))^{\frac{1}{2}}$. To handle the second factor, we expand out the square and obtain

$$
\sum_{n \geq 2} \frac{\Lambda(n)}{n^{1+1/\log X} \log n} \Big| \sum_{r=1}^{j} n^{-i\tau_r} \Big|^2
$$

$$
= j \sum_{n \geq 2} \frac{\Lambda(n)}{n^{1+1/\log X} \log n} + 2\mathrm{Re} \sum_{\leq r < s \leq j} \sum_{n \geq 2} \frac{\Lambda(n)}{n^{1+1/\log X + i(\tau_r - \tau_s)} \log n}
$$

$$
= j(\log \log X + O(1)) + 2 \sum_{1 \leq r < s \leq j} \log |\zeta(1 + 1/\log X + i(\tau_r - \tau_s))|.
$$

Now note that $(\log X)^{-\frac{1}{R}} \leq |\tau_r - \tau_s| \leq 2T$ and hence $|\zeta(1 + 1/\log X + i(\tau_r - \tau_s))| \leq (\log X)^{\frac{1}{R}} + O(1)$. Using this above, the lemma follows. $\square$

Let $\underline{\ell} = (\ell_1, \ldots, \ell_R)$ be a vector of nonnegative integers. In our proof of Theorem 2.1 we will encounter (recall Example 2.2 from §2 where a similar quantity arises)

$$
\mathscr{F}_{\underline{\ell}}(s) = F(s) \prod_{j=1}^{R} (1 - w^{1+i\tau_j - s})^{\ell_j}.
$$

We will need good bounds for this quantity, and we record such estimates in the next two lemmas.

LEMMA 5.2. *Let $\sigma \geq 1 + 1/\log X$. Then*

$$
\max_{|t| \leq T/2} |\mathscr{F}_{\underline{\ell}}(\sigma + it)| \leq \max_{|t| \leq T} |\mathscr{F}_{\underline{\ell}}(1 + 1/\log X + it)| + O((\log X)^{-1}).
$$

Proof. This follows from the argument of Lemma 2.2 in Granville and Soundararajan [8].

For completeness we give a proof. Put $\sigma = 1 + 1/\log X + \alpha$, and assume that $\alpha > 0$. The Fourier transform of $k(z) = e^{-\alpha|z|}$ is $\hat{k}(\xi) = \int_{-\infty}^{\infty} e^{-\alpha|z| - i\xi z} dz = \frac{2\alpha}{\alpha^2 + \xi^2}$. By Fourier inversion, we have for any $z \geq 1$

$$
z^{-\alpha} = k(\log z) = k(-\log z) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{k}(\xi) z^{-i\xi} d\xi
$$

$$
= \frac{1}{\pi} \int_{-T/2}^{T/2} \frac{\alpha}{\alpha^2 + \xi^2} z^{-i\xi} d\xi + O\left(\frac{\alpha}{T}\right).
$$

Using this relation appropriately we obtain that

$$
\mathscr{F}_{\underline{\ell}}(\sigma + it)l = \frac{1}{\pi} \int_{-T/2}^{T/2} \frac{\alpha}{\alpha^2 + \xi^2} \mathscr{F}_{\underline{\ell}}(1 + 1/\log X + it + i\xi) d\xi
$$

$$
+ O\left(\frac{\alpha}{T} \sum_{n=1}^{\infty} \frac{|f(n)|}{n^{1+1/\log X}}\right).
$$

Using Lemma 3.2 and partial summation, the error term above is $O((\log X)^A/T) = O((\log X)^{-1})$. If $|t| \le T/2$ then $|t + \xi| \le T$, and so

$$\max_{|t| \le T/2} |\mathscr{F}_{\underline{\ell}}(\sigma + it)|$$

$$\le \max_{|y| \le T} |\mathscr{F}_{\underline{\ell}}(1 + 1/\log X + iy)| \frac{1}{\pi} \int_{-T/2}^{T/2} \frac{\alpha}{\alpha^2 + \xi^2} d\xi + O((\log X)^{-1}),$$

and the lemma follows. $\qquad\square$

LEMMA 5.3. *Suppose $\ell_j \ge L - 1$ for all $1 \le j \le R$ where we recall that $L = [10AR]$. Then provided $0 \le \log w \le (\log X)^{1/(3R)}$ we have*

$$\max_{|t| \le T} |\mathscr{F}_{\underline{\ell}}(1 + 1/\log X + it)| \ll (\log X)^{\varepsilon/2}.$$

Proof. Suppose first that $|t| \le T$ but $|t - \tau_j| > (\log X)^{-1/R}$ for all $1 \le j \le R$. Then Lemma 5.1 gives that $|F(1 + 1/\log X + it)| \ll (\log X)^{\varepsilon/2}$ and so $|\mathscr{F}_{\underline{\ell}}(1 + 1/\log X + it)| \ll (\log X)^{\varepsilon/2}$ as well.

Now suppose that $|t - \tau_j| \le (\log X)^{-1/R}$ for some $1 \le j \le R$. By Lemma 3.2 we have that $|F(1 + 1/\log X + it)| \ll (\log X)^A$. Moreover,

$$|1 - w^{-1/\log X - it + i\tau_j}|^{\ell_j} \ll \left(\frac{\log w}{(\log X)^{\frac{1}{R}}}\right)^{\ell_j} \le (\log X)^{-\frac{2(L-1)}{3R}} \ll (\log X)^{-A},$$

and hence $|\mathscr{F}_{\underline{\ell}}(1 + 1/\log X + it)| \ll 1$. The lemma follows. $\qquad\square$

Using our work so far, we can record a preliminary estimate for the oscillation which we shall refine in the next section to obtain Theorem 2.1.

PROPOSITION 5.4. *Suppose $\ell_j \ge L - 1$ for all $1 \le j \le R$, and that $0 \le \log w \le (\log X)^{\frac{1}{3R}}$. For $x \le X$ we have*

$$\mathbb{O}_{\underline{\ell}}(x, w) \ll x(\log X)^{2\varepsilon/3}.$$

Proof. Since $S(x) \ll x(\log x)^A$ by Lemma 3.3, we may assume that $\log x \ge (\log X)^{\varepsilon/(2A)}$, and in particular $x \ge w^{2RL}$ is large. By Perron's formula we have that for $c > 1$

$$\frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} F(s)z^s \left(\frac{e^{s/\sqrt{T}} - 1}{s/\sqrt{T}}\right) \frac{ds}{s} = \sum_{n \le z} f(n) + O\left(\sum_{z < n \le ze^{1/\sqrt{T}}} |f(n)|\right).$$

By Lemma 3.3, the error term above is easily seen to be $O(z/\log z)$ in the range $T \le z \le X$. Using the above formula in the definition of the oscillation, we obtain that for $w^{2RL} \le x \le X$

$$\mathbb{O}_{\underline{\ell}}(x, w) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} F(s) \prod_{j=1}^{R} (1 - w^{1 + i\tau_j - s})^{\ell_j} \left(\frac{e^{s/\sqrt{T}} - 1}{s/\sqrt{T}}\right) \frac{ds}{s} + O\left(\frac{x}{\log x}\right).$$

We choose $c = 1 + 1/\log X$ and split the integral into two parts: when $|\text{Im}(s)| \leq T$ and when $|\text{Im}(s)| > T$. For the first range we use Lemma 5.3, and so this portion of the integral contributes

$$\ll x(\log X)^{\varepsilon/2} \int_{|\text{Im}(s)| \leq T} \frac{|ds|}{|s|} \ll x(\log X)^{2\varepsilon/3}.$$

For the second region we use that $|F(s)| \ll (\log X)^A$ and deduce that this integral contributes

$$\ll x(\log X)^A \int_{|\text{Im}(s)| > T} \frac{\sqrt{T}}{|s|} \frac{|ds|}{|s|} \ll x.$$

The proposition follows.   □

6. Proof of Theorem 2.1

In this section we shall prove Theorem 2.1, with the points $\tau_1, \ldots, \tau_R$ being the successive maxima defined in Section 5. Recall the conditions and notation (2.3) through (2.8), and the notation introduced in Section 5. Recall that $R = [10A^2/\varepsilon^2] + 1$, that $L = [10AR]$ and $\underline{L} = (L, \ldots, L)$. Throughout we assume that $\log w \leq (\log X)^{\frac{1}{3R}}$, that $x \leq X$, and all implicit constants will be allowed to depend on A, A_0 and ε.

LEMMA 6.1. *With the above notation, we have*

$$(\log x)\mathbb{O}_{\underline{L}}(x, w) = \sum_{d \leq x} \Lambda_f(d)\mathbb{O}_{\underline{L}}(x/d, w) + O(x(\log X)^{\varepsilon}).$$

Proof. Write $\log x = \log(x/w^{j_1 + \cdots + j_R}) + (j_1 + \cdots + j_R)\log w$. Hence, we may express $(\log x)\mathbb{O}_{\underline{L}}(x, w)$ as

$$\sum_{\leq \underline{L}} (-1)^{j_1 + \cdots + j_R} \binom{\underline{L}}{\underline{j}} \log(x/w^{j_1 + \cdots + j_R}) S(x/w^{j_1 + \cdots + j_R}) w^{j_1(1+i\tau_1) + \cdots + j_R(1+i\tau_R)}$$

$$+ \log w \sum_{\underline{j} \leq \underline{L}} (-1)^{j_1 + \cdots + j_R}(j_1 + \cdots + j_R) \binom{\underline{L}}{\underline{j}} S(x/w^{j_1 + \cdots + j_R}) w^{j_1(1+i\tau_1) + \cdots + j_R(1+i\tau_R)}.$$

The second term above is readily seen to be

$$(6.1) \qquad\qquad -\sum_{k=1}^{R} Lw^{1+i\tau_k} \log w \; \mathbb{O}_{\underline{L} - \underline{e}_k}(x/w, w),$$

where we let $\underline{e}_k$ denote the vector with 1 in the k-th place and 0 elsewhere. Note that the coordinates of $\underline{L} - \underline{e}_k$ are all at least $L - 1$, and so by Proposition 5.4, the quantity in (6.1) is $\ll x(\log X)^{\varepsilon}$.

To analyze the first term, we write

$$
\begin{aligned}
(\log x) S(x) &= \sum_{n \le x} f(n) \log n + \sum_{n \le x} f(n) \log(x/n) \\
&= \sum_{n \le x} \sum_{d \mid n} \Lambda_f(d) f(n/d) + \int_1^x S(t) \frac{dt}{t} \\
&= \sum_{d \le x} \Lambda_f(d) S(x/d) + \int_1^x S(t) \frac{dt}{t}.
\end{aligned}
$$

Therefore the first term equals

$$
\sum_{d \le x} \Lambda_f(d) \mathbb{O}_{\underline{L}}(x/d, w) + \int_1^x \mathbb{O}_{\underline{L}}(t, w) \frac{dt}{t},
$$

where we used that $\int_1^{x/w^j} S(t) dt/t = \int_1^x S(t/w^j) dt/t$. By Proposition 5.4, the integral above is $\ll x(\log X)^\varepsilon$. The lemma follows. $\qquad\square$

LEMMA 6.2. *For $1 \le z \le y$ with $y + z \le X$ we have*

$$
\left| |\mathbb{O}_{\underline{L}}(y, w)|^2 - |\mathbb{O}_{\underline{L}}(y + z, w)|^2 \right| \ll y(\log X)^\varepsilon \sum_{j=0}^{LR} w^j \sum_{y/w^j < n \le (y+z)/w^j} |f(n)|.
$$

Proof. The quantity we wish to estimate is

$$
\le \left(|\mathbb{O}_{\underline{L}}(y, w)| + |\mathbb{O}_{\underline{L}}(y + z, w)| \right) \left| \mathbb{O}_{\underline{L}}(y + z, w) - \mathbb{O}_{\underline{L}}(y, w) \right|.
$$

By Proposition 5.4, the first factor is $\ll y(\log X)^\varepsilon$. The second factor above is

$$
\ll \sum_{j=0}^{LR} w^j \left| S((y + z)/w^j) - S(y/w^j) \right| \ll \sum_{j=0}^{LR} w^j \sum_{y/w^j < n \le (y+z)/w^j} |f(n)|. \quad \square
$$

PROPOSITION 6.3. *We have*

$$
\log x \, |\mathbb{O}_{\underline{L}}(x, w)| \ll x(\log \log x)^{\frac{1}{2}} \left(\int_1^x \log(ey) |\mathbb{O}_{\underline{L}}(y, w)|^2 \frac{dy}{y^3} \right)^{\frac{1}{2}} + x(\log X)^\varepsilon.
$$

Proof. We start with Lemma 6.1, and are faced with estimating

$$
\sum_{d \le x} |\Lambda_f(d)| |\mathbb{O}_{\underline{L}}(x/d, w)|.
$$

We split this sum into the terms $d \le D := [\exp((\log \log X)^6)]$ and $d > D$. For the first category of terms we use Proposition 5.4 and obtain that this contribution is

$$
\sum_{d \le D} |\Lambda_f(d)| \frac{x}{d} (\log X)^{2\varepsilon/3} \ll x(\log X)^\varepsilon,
$$

upon using (2.6).

It remains to estimate the contribution of the terms $d > D$. Define temporarily the function $g(t) = t \log(ex/t)$ for $1 \leq t \leq x$. By Cauchy-Schwarz we have

$$(6.2) \qquad \sum_{D < d \leq x} |\Lambda_f(d)| |\mathbb{O}_{\underline{L}}(x/d, w)|$$

$$\leq \Big(\sum_{D < d \leq x} \frac{|\lambda_f(d)|^2 \Lambda(d)}{g(d)} \Big)^{\frac{1}{2}} \Big(\sum_{D < d \leq x} g(d) \Lambda(d) |\mathbb{O}_{\underline{L}}(x/d, w)|^2 \Big)^{\frac{1}{2}}$$

$$\ll (\log \log x)^{\frac{1}{2}} \Big(\sum_{D < d \leq x} g(d) \Lambda(d) |\mathbb{O}_{\underline{L}}(x/d, w)|^2 \Big)^{\frac{1}{2}},$$

where the last estimate follows from (2.6) and partial summation.

Put $\psi_0(x) = \sum_{n \leq x} (\Lambda(n) - 1) = \psi(x) - x$ so that $\psi_0(x) = O(x \exp(-c\sqrt{\log x}))$ by the prime number theorem. Then

$$(6.3) \qquad \sum_{D < d \leq x} g(d) \Lambda(d) |\mathbb{O}_{\underline{L}}(x/d, w)|^2$$

$$= \sum_{D < d \leq x} g(d) |\mathbb{O}_{\underline{L}}(x/d, w)|^2 + \sum_{D < d \leq x} (\psi_0(d) - \psi_0(d-1)) g(d) |\mathbb{O}_{\underline{L}}(x/d, w)|^2.$$

We may rewrite the second term in the right-hand side of (6.3) as

$$(6.4) \qquad \sum_{D < d \leq x} \psi_0(d) \Big(g(d) |\mathbb{O}_{\underline{L}}(x/d, w)|^2 - g(d+1) |\mathbb{O}_{\underline{L}}(x/(d+1), w)|^2 \Big)$$

$$-\psi_0(D) g(D+1) |\mathbb{O}_{\underline{L}}(x/(D+1), w)|^2.$$

Now we use that for $d > D$, $\psi_0(d) = O(d \exp(-(\log \log x)^2)) = O(d/(\log X)^{A+2})$. Hence the second term in (6.4) is $\ll D^2 (\log X)^{-1} |\mathbb{O}_{\underline{L}}(x/(D+1), w)|^2 \ll x^2$ upon using Proposition 5.4. The first term in (6.4) is

$$\ll \sum_{D < d \leq x} \frac{d}{(\log X)^{A+2}} \Big(g(d) \Big| |\mathbb{O}_{\underline{L}}(x/d, w)|^2 - |\mathbb{O}_{\underline{L}}(x/(d+1), w)|^2 \Big|$$

$$+ |g(d+1) - g(d)| |\mathbb{O}_{\underline{L}}(x/(d+1), w)|^2 \Big).$$

By Proposition 5.4 the second term above contributes $\ll x^2$, while by Lemma 6.2 we have that the first term above is

$$\ll \sum_{D < d \leq x} \frac{d^2}{(\log X)^A} \frac{x}{d} \sum_{j=0}^{LR} w^j \sum_{x/((d+1)w^j) < n \leq x/(dw^j)} |f(n)|$$

$$\ll \frac{x}{(\log X)^A} \sum_{j=0}^{LR} w^j \sum_{n \leq x/w^j} |f(n)| \frac{x}{nw^j} \ll x^2,$$

where the final estimate follows from Lemma 3.2. We conclude that the second term in the right-hand side of (6.3) is $\ll x^2$.

We now turn to the first term in (6.3). For any $D < d \le x$ and $d - 1 \le t \le d$ we have that $g(d) = g(t) + O(\log x)$, and by Lemma 6.2 that

$$|\mathbb{O}_{\underline{L}}(x/d, w)|^2 = |\mathbb{O}_{\underline{L}}(x/t, w)|^2$$
$$+ O\Big(\frac{x}{d}(\log X)^\varepsilon \sum_{j=0}^{LR} w^j \sum_{x/(dw^j)<n\le x/((d-1)w^j)} |f(n)|\Big).$$

Therefore, using also Proposition 5.4,

$$g(d)|\mathbb{O}_{\underline{L}}(x/d, w)|^2 = \int_{d-1}^{d} g(t)|\mathbb{O}_{\underline{L}}(x/t, w)|^2 dt + O\Big(\frac{x^2}{d^2}(\log X)^{1+2\varepsilon}\Big)$$
$$+ O\Big(x(\log X)^{1+\varepsilon} \sum_{j=0}^{LR} w^j \sum_{x/(dw^j)<n\le x/((d-1)w^j)} |f(n)|\Big).$$

Summing this over all $D < d \le x$ we get from the main term above the contribution

$$\int_D^x t \log(ex/t)|\mathbb{O}_{\underline{L}}(x/t, w)|^2 dt \le x^2 \int_1^x |\mathbb{O}_{\underline{L}}(y, w)|^2 \log(ey)\frac{dy}{y^3}.$$

The error terms contribute

$$\ll \frac{x^2}{D}(\log X)^{1+2\varepsilon} + x(\log X)^{1+\varepsilon} \sum_{j=0}^{LR} w^j \sum_{n\le x/(Dw^j)} |f(n)|$$

$$\ll x^2 + \frac{x^2}{D}(\log X)^{1+\varepsilon} \sum_{j=0}^{LR} \sum_{n\le x/(Dw^j)} \frac{|f(n)|}{n} \ll x^2.$$

The proof of the proposition is complete. $\qquad\square$

We must now analyze the integral appearing in Proposition 6.3. To this end, we write

$$\widetilde{S}(x) = \sum_{n\le x} f(n) \log n,$$

and define

$$\widetilde{\mathbb{O}}_{\underline{\ell}}(x, w) = \sum_{\underline{j}\le\underline{\ell}}(-1)^{j_1+\cdots+j_R}\binom{\underline{\ell}}{\underline{j}} w^{j_1(1+i\tau_1)+\cdots+j_R(1+i\tau_R)}\widetilde{S}(x/w^{j_1+\cdots+j_R}).$$

LEMMA 6.4. *We have*

$$\Big(\int_1^x |\mathbb{O}_{\underline{L}}(t, w)|^2 \log(et)\frac{dt}{t^3}\Big)^{\frac{1}{2}} \ll \Big(\int_1^x |\widetilde{\mathbb{O}}_{\underline{L}}(t, w)|^2 \frac{dt}{t^3 \log(et)}\Big)^{\frac{1}{2}} + (\log X)^{7\varepsilon/8}.$$

Proof. We start as in the proof of Lemma 6.1. Thus we may write

$$(\log t)\mathbb{O}_{\underline{L}}(t, w) = -\sum_{k=1}^{R} L w^{1+i\tau_k} (\log w) \mathbb{O}_{\underline{L}-\underline{e}_k}(t/w, w)$$

$$+ \sum_{\underline{j} \leq \underline{L}} (-1)^{j_1+\cdots+j_R} \binom{\underline{L}}{\underline{j}} \log(t/w^{j_1+\cdots+j_R}) S(t/w^{j_1+\cdots+j_R}) w^{j_1(1+i\tau_1)+\cdots+j_R(1+i\tau_R)}.$$

Since

$$(\log z)S(z) = \tilde{S}(z) + \sum_{n \leq z} f(n)\log(z/n) = \tilde{S}(z) + \int_1^z \frac{S(y)}{y} dy,$$

the second term above may be written as

$$\tilde{\mathbb{O}}_{\underline{L}}(t, w) + \int_1^t \frac{\mathbb{O}_{\underline{L}}(y, w)}{y} dy.$$

Putting these remarks together, and using Proposition 5.4 we conclude that

$$(\log t)\mathbb{O}_{\underline{L}}(t, w) = \tilde{\mathbb{O}}_{\underline{L}}(t, w) + O(t(1+\log w)(\log X)^{2\varepsilon/3})$$
$$= \tilde{\mathbb{O}}_{\underline{L}}(t, w) + O(t(\log X)^{5\varepsilon/6}).$$

The lemma follows.   □

Putting together Proposition 6.3 and Lemma 6.4, we have that

$$|\mathbb{O}_{\underline{L}}(x, w)| \ll \frac{x}{\log x} (\log\log x)^{\frac{1}{2}} \left(\int_1^x |\tilde{\mathbb{O}}_{\underline{L}}(t, w)|^2 \frac{dt}{t^3 \log(et)} \right)^{\frac{1}{2}} + \frac{x}{\log x} (\log X)^\varepsilon.$$

Theorem 2.1 will now follow from the following proposition.

PROPOSITION 6.5. *We have*

$$\int_1^x |\tilde{\mathbb{O}}_{\underline{L}}(t, w)|^2 \frac{dt}{t^3 \log(et)} \ll (\log X)^{3\varepsilon/2}.$$

Proof. We make the substitution $t = e^y$, obtaining

(6.5)
$$\int_1^x |\tilde{\mathbb{O}}_{\underline{L}}(t, w)|^2 \frac{dt}{t^3 \log(et)} = \int_0^{\log x} |\tilde{\mathbb{O}}_{\underline{L}}(e^y, w)|^2 e^{-2y} \frac{dy}{1+y}$$
$$\ll \int_{1/\log X}^\infty e^{-2\alpha} \int_0^\infty |\tilde{\mathbb{O}}_{\underline{L}}(e^y, w)|^2 e^{-2y(1+\alpha)} dy \, d\alpha.$$

The idea now is to estimate the integral over y in (6.5) using Plancherel's formula.

Note that the Fourier transform of $\widetilde{\mathbb{O}}_L(e^y, w)e^{-y(1+\alpha)}$ is

$$\int_{-\infty}^{\infty} \widetilde{\mathbb{O}}_L(e^y, w)e^{-y(1+\alpha+it)} dy$$

$$= \sum_{\underline{j} \leq \underline{L}} (-1)^{j_1+\cdots+j_R} \binom{\underline{L}}{\underline{j}} w^{j_1(1+i\tau_1)+\cdots+j_R(1+i\tau_R)}$$

$$\times \int_{-\infty}^{\infty} \sum_{n \leq e^y/w^{j_1+\cdots+j_R}} f(n) \log n \ e^{-y(1+\alpha+it)} dy$$

$$= \sum_{\underline{j} \leq \underline{L}} (-1)^{j_1+\cdots+j_R} \binom{\underline{L}}{\underline{j}} w^{j_1(i\tau_1-\alpha-it)+\cdots+j_R(i\tau_R-\alpha-it)}$$

$$\times \sum_{n=1}^{\infty} \frac{f(n) \log n}{n^{1+\alpha+it}} \int_0^{\infty} e^{-y(1+\alpha+it)} dy$$

$$= -\frac{1}{(1+\alpha+it)} \prod_{k=1}^{R} (1 - w^{-\alpha-it+i\tau_k})^L F'(1+\alpha+it).$$

Therefore, by Plancherel's formula we have that

(6.6) $$\int_0^{\infty} |\widetilde{\mathbb{O}}_L(e^y, w)|^2 e^{-y(2+2\alpha)} dy$$

$$\ll \int_{-\infty}^{\infty} |F'(1+\alpha+it)|^2 \prod_{k=1}^{R} \left| 1 - w^{-\alpha-it+i\tau_k} \right|^{2L} \frac{dt}{|1+\alpha+it|^2}.$$

We split the integral in (6.6) into the regions when $|t| \leq T/2$ and when $|t| > T/2$. For the latter region we use Lemma 3.2 which gives $|F'(1+\alpha+it)| \ll (\log X)^{A+1}$ (note that $\alpha \geq 1/\log X$ in (6.5)), so that this integral contributes

$$\ll (\log X)^{2A+2} \int_{|t|>T/2} \frac{dt}{|1+\alpha+it|^2} \ll 1.$$

For the first region we use Lemmas 5.2 and 5.3 to obtain that

$$|F'(1+\alpha+it)| \prod_{k=1}^{R} |1 - w^{-\alpha-it+i\tau_k}|^L \ll \left| \frac{F'}{F}(1+\alpha+it) \right| |\mathscr{F}_L(1+\alpha+it)|$$

$$\ll \left| \frac{F'}{F}(1+\alpha+it) \right| (\log X)^{\varepsilon/2}.$$

Therefore the integral over the first region contributes

(6.7) $$\ll (\log X)^{\varepsilon} \int_{|t| \leq T/2} \left| \frac{F'}{F}(1+\alpha+it) \right|^2 \frac{dt}{|1+\alpha+it|^2}$$

$$\ll (\log X)^{\varepsilon} \int_{-\infty}^{\infty} \left| \frac{F'}{F}(1+\alpha+it) \right|^2 \frac{dt}{|1+\alpha+it|^2}.$$

Now the Fourier transform of the function $e^{-y(1+\alpha)}\sum_{n\leq e^y}\Lambda_f(n)$ is

$$\int_{-\infty}^{\infty}\sum_{n\leq e^y}\Lambda_f(n)e^{-y(1+\alpha+it)}dy = \sum_n \frac{\Lambda_f(n)}{n^{1+\alpha+it}}\frac{1}{(1+\alpha+it)}$$

$$= -\frac{F'}{F}(1+\alpha+it)\frac{1}{(1+\alpha+it)},$$

and so using Plancherel once again we obtain that the quantity in (6.7) is

$$\ll (\log X)^\varepsilon \int_0^\infty \Big|\sum_{n\leq e^y}\Lambda_f(n)\Big|^2 e^{-(2+2\alpha)y}dy.$$

We conclude that
(6.8)
$$\int_0^\infty |\widetilde{\mathbb{O}}_{\underline{L}}(e^y,w)|^2 e^{-y(2+2\alpha)}dy \ll 1 + (\log X)^\varepsilon \int_0^\infty \Big|\sum_{n\leq e^y}\Lambda_f(n)\Big|^2 e^{-(2+2\alpha)y}dy.$$

Injecting the bound (6.8) in (6.5) we obtain that

$$\int_1^x |\widetilde{\mathbb{O}}_{\underline{L}}(t,w)|^2 \frac{dt}{t^3\log(et)}$$

$$\ll 1 + (\log X)^\varepsilon \int_{1/\log X}^\infty e^{-2\alpha}\int_0^\infty \Big|\sum_{n\leq e^y}\Lambda_f(n)\Big|^2 e^{-(2+2\alpha)y}dy\, d\alpha.$$

Expanding, we obtain that the double integrals above are

$$\ll \sum_{2\leq n_1\leq n_2}|\Lambda_f(n_1)\Lambda_f(n_2)|\int_{1/\log X}^\infty\int_{\log n_2}^\infty e^{-(2+2\alpha)y-2\alpha}dy\,d\alpha$$

$$\ll \sum_{2\leq n_1\leq n_2}|\lambda_f(n_1)\lambda_f(n_2)|\frac{\Lambda(n_1)\Lambda(n_2)}{n_2^{2+2/\log X}\log n_2}$$

$$\ll \sum_{2\leq n_1\leq n_2}(|\lambda_f(n_1)|^2+|\lambda_f(n_2)|^2)\frac{\Lambda(n_1)\Lambda(n_2)}{n_2^{2+2/\log X}\log n_2}$$

$$\ll \sum_{2\leq n}\frac{|\lambda_f(n)|^2\Lambda(n)}{n^{1+2/\log X}\log n}\ll (\log\log X),$$

upon using the prime number theorem in the penultimate step, and (2.6) for the last step. The proposition follows, and with it Theorem 2.1. □

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