ANNALS OF MATHEMATICS

The u-invariant of the function fields of p-adic curves

By Raman Parimala and V. Suresh

SECOND SERIES, VOL. 172, NO. 2

September, 2010

ANMAAH

Annals of Mathematics, **172** (2010), 1391–1405

The u-invariant of the function fields of p-adic curves

By RAMAN PARIMALA and V. SURESH

Abstract

The u-invariant of a field is the maximum dimension of ansiotropic quadratic forms over the field. It is an open question whether the u-invariant of function fields of p-aidc curves is 8. In this paper, we answer this question in the affirmative for function fields of nondyadic p-adic curves.

Introduction

It is an open question ([Lam05, Q. 6.7, Chap XIII]) whether every quadratic form in at least nine variables over the function fields of p-adic curves has a nontrivial zero. Equivalently, one may ask whether the u-invariant of such a field is 8. The u-invariant of a field F is defined as the maximal dimension of anisotropic quadratic forms over F. In this paper we answer this question in the affirmative if the p-adic field is nondyadic.

In [PS98, 4.5], we showed that every quadratic form in eleven variables over the function field of a p-adic curve, $p \neq 2$, has a nontrivial zero. The main ingredients in the proof were the following: Let K be the function field of a p-adic curve X and $p \neq 2$.

(1) (Saltman [Sal97, 3.4]). Every element in the Galois cohomology group
$$H^2(K, \mathbf{Z}/2\mathbf{Z})$$
is a sum of at most two symbols.

(2) (Kato [Kat86, 5.2]). The unramified cohomology group $H^3_{\text{nr}}(K/\mathscr{X}, \mathbf{Z}/2\mathbf{Z}(2))$ is zero for a regular projective model $\mathscr{X}$ of K.

If K is as above, we proved ([PS98, 3.9]) that every element in $H^3(K, \mathbf{Z}/2\mathbf{Z})$ is a symbol of the form $(f)\cdot(g)\cdot(h)$ for some $f, g, h \in K^*$ and f may be chosen to be a value of a given binary form $\langle a, b\rangle$ over K. If, further, given $\zeta = (f)\cdot(g)\cdot(h) \in H^3(K, \mathbf{Z}/2\mathbf{Z})$ and a ternary form $\langle c, d, e\rangle$, one can choose $g', h' \in K^*$ such that

The authors acknowledge partial financial support from NSF DMS-0653382 and UGC/SAP respectively .

$\zeta = (f)\cdot(g')\cdot(h')$ with g' a value of $\langle c, d, e\rangle$, then, one is led to the conclusion that $u(K) = 8$ (cf. Proposition 4.3). We in fact prove that such a choice of $g', h' \in K^*$ is possible by proving the following local global principle:

Let k be a p-adic field and $K = k(X)$ the function field of a curve X over k. For any discrete valuation v of K, let K_v denote the completion of K at v. Let l be a prime not equal to p. Assume that k contains a primitive l^{th} root of unity.

THEOREM. *Let k, K and l be as above. Let $\zeta \in H^3(K, \mu_l^{\otimes 2})$ and $\alpha \in H^2(K, \mu_l)$. Suppose that α corresponds to a degree l central division algebra over K. If $\zeta = \alpha \cup (h_v)$ for some $h_v \in K_v^*$, for all discrete valuations v of K, then there exists $h \in K^*$ such that $\zeta = \alpha \cup (h)$. In fact, one can restrict the hypothesis to discrete valuations of K centered on codimension-1 points of a regular model $\mathscr{X}$, projective over the ring of integers $\mathcal{O}_k$ of k.*

A key ingredient toward the proof of the theorem is a recent result of Saltman [Sal07] where the ramification pattern of prime degree central simple algebras over function fields of p-adic curves is completely described.

We thank J.-L. Colliot-Thélène for helpful discussions during the preparation of this paper and for his critical comments on the text.

1. Some preliminaries

In this section we recall a few basic facts from the algebraic theory of quadratic forms and Galois cohomology. We refer the reader to [CT95] and [Sch85].

Let F be a field and l a prime not equal to the characteristic of F. Let μ_l be the group of l^{th} roots of unity. For $i \geq 1$, let $\mu_l^{\otimes i}$ be the Galois module given by the tensor product of i copies of μ_l. For $n \geq 0$, let $H^n(F, \mu_l^{\otimes i})$ be the n^{th} Galois cohomology group with coefficients in $\mu_l^{\otimes i}$.

We have the Kummer isomorphism $F^*/F^{*^l} \simeq H^1(F, \mu_l)$. For $a \in F^*$, its class in $H^1(F, \mu_l)$ is denoted by (a). If $a_1, \dots, a_n \in F^*$, the cup product $(a_1)\cdots(a_n) \in H^n(F, \mu_l^{\otimes n})$ is called a *symbol*. We have an isomorphism $H^2(F, \mu_l)$ with the l-torsion subgroup ${}_lBr(F)$ of the Brauer group of F. We define the *index* of an element $\alpha \in H^2(F, \mu_l)$ to be the index of the corresponding central simple algebra in ${}_lBr(F)$.

Suppose F contains all the l^{th} roots of unity. We fix a generator ρ for the cyclic group μ_l and identify the Galois modules $\mu_l^{\otimes i}$ with $\mathbf{Z}/l\mathbf{Z}$. This leads to an identification of $H^n(F, \mu_l^{\otimes m})$ with $H^n(F, \mathbf{Z}/l\mathbf{Z})$. The element in $H^n(F, \mathbf{Z}/l\mathbf{Z})$ corresponding to the symbol $(a_1)\cdots(a_n) \in H^n(F, \mu_l^{\otimes n})$ through this identification is again denoted by $(a_1)\cdots(a_n)$. In particular, for $a, b \in F^*$, $(a)\cdot(b) \in H^2(K, \mathbf{Z}/l\mathbf{Z})$ represents the cyclic algebra (a, b) defined by the relations $x^l = a$, $y^l = b$ and $xy = \rho yx$.

Let v be a discrete valuation of F. The residue field of v is denoted by $\kappa(v)$. Suppose char$(\kappa(v)) \neq l$. Then there is a *residue* homomorphism

$$\partial_v \colon H^n(F, \mu_l^{\otimes m}) \to H^{n-1}(\kappa(v), \mu_l^{\otimes(m-1)}).$$

Let $\alpha \in H^n(F, \mu_l^{\otimes m})$. We say that α is *unramified* at v if $\partial_v(\alpha) = 0$; otherwise it is said to be *ramified* at v. If F is complete with respect to v, then we denote the kernel of ∂_v by $H^n_{\mathrm{nr}}(F, \mu_l^{\otimes m})$. Suppose α is unramified at v. Let $\pi \in K^*$ be a parameter at v and $\zeta = \alpha \cup (\pi) \in H^{n+1}(F, \mu_l^{\otimes(m+1)})$. Let $\overline{\alpha} = \partial_v(\zeta) \in H^n(\kappa(v), \mu_l^{\otimes m})$. The element $\overline{\alpha}$ is independent of the choice of the parameter π and is called the *specialization* of α at v. We say that α *specializes to* $\overline{\alpha}$ at v. The following result is well known.

LEMMA 1.1. *Let k be a field and l a prime not equal to the characteristic of k. Let K be a complete discrete valuated field with residue field k. If $H^3(k, \mu_l^{\otimes 3}) = 0$, then $H^3_{\mathrm{nr}}(K, \mu_l^{\otimes 3}) = 0$. Suppose further that every element in $H^2(k, \mu_l^{\otimes 2})$ is a symbol. Then every element in $H^3(K, \mu_l^{\otimes 3})$ is a symbol.*

Proof. Let R be the ring of integers in K. The Gysin exact sequence in étale cohomology yields an exact sequence (cf. [C, p. 21, §3.3])

$$H^3_{\text{ét}}(R, \mu_l^{\otimes 3}) \to H^3(K, \mu_l^{\otimes 3}) \xrightarrow{\partial} H^2(k, \mu_l^{\otimes 2}) \to H^4_{\text{ét}}(R, \mu_l^{\otimes 3}).$$

Since R is complete, $H^3_{\text{ét}}(R, \mu_l^{\otimes 3}) \simeq H^3(k, \mu_l^{\otimes 3})$ ([Mil80, p. 224, Cor. 2.7]). Hence $H^3_{\text{ét}}(R, \mu_l^{\otimes 3}) = 0$ by the hypothesis. In particular, $\partial\colon H^3(K, \mu_l^{\otimes 3}) \to H^2(k, \mu_l^{\otimes 2})$ is injective and $H^3_{\mathrm{nr}}(K, \mu_l^{\otimes 3}) = 0$. Let $u, v \in R$ be units and $\pi \in R$ a parameter. Then we have $\partial((u) \cdot (v) \cdot (\pi)) = (\overline{u}) \cdot (\overline{v})$. Let $\zeta \in H^3(K, \mu_l^{\otimes 3})$. Since every element in $H^2(k, \mu_l^{\otimes 2})$ is a symbol, we have $\partial(\zeta) = (\overline{u}) \cdot (\overline{v})$ for some units $u, v \in R$. Since ∂ is an isomorphism, we have $\zeta = (u) \cdot (v) \cdot (\pi)$. Thus every element in $H^3(K, \mu_l^{\otimes 3})$ is a symbol. □

COROLLARY 1.2. *Let k be a p-adic field and K the function field of an integral curve over k. Let l be a prime not equal to p. Let K_v be the completion of K at a discrete valuation of K. Then $H^3_{\mathrm{nr}}(K_v, \mu_l^{\otimes 3}) = 0$. Suppose further that K contains a primitive l^{th} root of unity. Then every element in $H^3(K_v, \mu_l^{\otimes 3})$ is a symbol.*

Proof. Let v be a discrete valuation of K and K_v the completion of K at v. The residue field $\kappa(v)$ at v is either a p-adic field or a function field of a curve over a finite field of characteristic p. In either case, the cohomological dimension of $\kappa(v)$ is 2 and hence $H^n(\kappa(v), \mu_l^{\otimes 3}) = 0$ for $n \geq 3$. By Lemma 1.1, $H^3_{\mathrm{nr}}(K_v, \mu_l^{\otimes 3}) = 0$.

If $\kappa(v)$ is a local field, by local class field theory, every finite-dimensional central division algebra over $\kappa(v)$ is split by an unramified (cyclic) extension. If $\kappa(v)$ is a function field of a curve over a finite field, then by a classical theorem of Hasse-Brauer-Noether-Albert, every finite-dimensional central division algebra over $\kappa(v)$ is split by a cyclic extension. Since $\kappa(v)$ contains a primitive l^{th} root of unity, every element in $H^2(\kappa(v), \mathbf{Z}/l\mathbf{Z})$ is a symbol. By Lemma 1.1, every element in $H^3(K_v, \mathbf{Z}/l\mathbf{Z})$ is a symbol. □

Let $\mathscr{X}$ be a regular integral scheme of dimension d, with field of fractions F. Let $\mathscr{X}^1$ be the set of points of $\mathscr{X}$ of codimension-1. A point $x \in \mathscr{X}^1$ gives rise to a discrete valuation v_x on F. The residue field of this discrete valuation ring is denoted by $\kappa(x)$ or $\kappa(v_x)$. The corresponding residue homomorphism is denoted by ∂_x. We say that an element $\zeta \in H^n(F, \mu_l^{\otimes m})$ is *unramified* at x if $\partial_x(\zeta) = 0$; otherwise it is said to be *ramified* at x. We define the ramification divisor $\mathrm{ram}_{\mathscr{X}}(\zeta) = \sum x$ as x runs over $\mathscr{X}^1$ where ζ is ramified. The unramified cohomology on $\mathscr{X}$, denoted by $H^n_{\mathrm{nr}}(F/\mathscr{X}, \mu_l^{\otimes m})$, is defined as the intersection of kernels of the residue homomorphisms

$$\partial_x \colon H^n(F, \mu_l^{\otimes m}) \to H^{n-1}(\kappa(x), \mu_l^{\otimes (m-1)}),$$

with x running over $\mathscr{X}^1$. We say that $\zeta \in H^n(F, \mu_l^{\otimes m})$ is *unramified on* $\mathscr{X}$ if $\zeta \in H^n_{\mathrm{nr}}(F/\mathscr{X}, \mu_l^{\otimes m})$. If $\mathscr{X} = \mathrm{Spec}(R)$, then we also say that ζ is unramified on R if it is unramified on $\mathscr{X}$. Suppose C is an irreducible subscheme of $\mathscr{X}$ of codimension-1. Then the generic point x of C belongs to $\mathscr{X}^1$ and we set $\partial_x = \partial_C$. If $\alpha \in H^n(F, \mu_l^{\otimes m})$ is unramified at x, then we say that α is *unramified* at C.

Let k be a p-adic field and K the function field of a smooth, projective, geometrically integral curve X over k. By the resolution of singularities for surfaces (cf. [Lip75] and [Lip78]), there exists a regular, projective model $\mathscr{X}$ of X over the ring of integers $\mathcal{O}_k$ of k. We call such an $\mathscr{X}$ a *regular projective model of* K. Since the generic fibre X of $\mathscr{X}$ is geometrically integral, it follows that the special fibre $\overline{\mathscr{X}}$ is connected. Further if D is a divisor on $\mathscr{X}$, there exists a proper birational morphism $\mathscr{X}' \to \mathscr{X}$ such that the total transform of D on $\mathscr{X}'$ is a divisor with normal crossings (cf. [Sha66, Thm., p. 38 and Rem. 2, p. 43]). We use this result throughout this paper without further reference.

Let k be a p-adic field and K the function field of a smooth, projective, geometrically integral curve over k. Let l be a prime not equal to p. Assume that k contains a primitive l^{th} root of unity. Let $\alpha \in H^2(K, \mu_l)$. Let $\mathscr{X}$ be a regular projective model of K such that the ramification locus $\mathrm{ram}_{\mathscr{X}}(\alpha)$ is a union of regular curves with normal crossings. Let P be a closed point in the intersection of two regular curves C and E in $\mathrm{ram}_{\mathscr{X}}(\alpha)$. Suppose that $\partial_C(\alpha) \in H^1(\kappa(C), \mathbf{Z}/l\mathbf{Z})$ and $\partial_E(\alpha) \in H^1(\kappa(E), \mathbf{Z}/l\mathbf{Z})$ are unramified at P. Let $u(P), v(P) \in H^1(\kappa(P), \mathbf{Z}/l\mathbf{Z})$ be the specializations at P of $\partial_C(\alpha)$ and $\partial_E(\alpha)$ respectively. Following Saltman ([Sal07, §2]), we say that P is a *cool point* if $u(P)$ and $v(P)$ are trivial, a *chilli point* if $u(P)$ and $v(P)$ both are nontrivial, and a *hot point* if one of them is trivial and the other one nontrivial. Note that if $u(P)$ is nontrivial, then $u(P)$ generates $H^1(\kappa(P), \mathbf{Z}/l\mathbf{Z})$. Let $\mathcal{O}_{\mathscr{X},P}$ be the regular local ring at P and π, δ prime elements in $\mathcal{O}_{\mathscr{X},P}$ which define C and E respectively at P. The condition that $\partial_C(\alpha) \in H^1(\kappa(C), \mathbf{Z}/l\mathbf{Z})$ and $\partial_E(\alpha) \in H^1(\kappa(E), \mathbf{Z}/l\mathbf{Z})$ are unramified at P is equivalent to the condition $\alpha = \alpha' + (u, \pi) + (v, \delta)$ for some units $u, v \in \mathcal{O}_{\mathscr{X},P}$ and α' unramified on $\mathcal{O}_{\mathscr{X},P}$ ([Sal98, §2]). The specializations of $\partial_C(\alpha)$ and $\partial_E(\alpha)$ in $H^1(\kappa(P), \mathbf{Z}/l\mathbf{Z}) \simeq \kappa(P)^*/\kappa(P)^{*^l}$ are given by the images of u and v in $\kappa(P)$.

Let P be a closed point of a regular curve C in $\mathrm{ram}_{\mathscr{X}}(\alpha)$ which is not on any other regular curve in $\mathrm{ram}_{\mathscr{X}}(\alpha)$. We have $\alpha = \alpha' + (u, \pi)$, where α' is unramified on $\mathcal{O}_{\mathscr{X},P}$, $u \in \mathcal{O}_{\mathscr{X},P}$ is a unit and $\pi \in \mathcal{O}_{\mathscr{X},P}$ is a prime defining the curve C at P; see [Sal97, 1.2]. Therefore $\partial_C(\alpha) = (\overline{u}) \in H^1(\kappa(C), \mathbf{Z}/l\mathbf{Z})$ is unramified at P.

PROPOSITION 1.3 ([Sal07, 2.5]). *If the index of α is l, then there are no hot points for α.*

Suppose P is a chilli point. Then $v(P) = u(P)^s$ for some s with $1 \le s \le l-1$ and s is called the *coefficient* of P ([Sal97, p. 830]) with respect to π. To get some compatibility for these coefficients, Saltman associates to α and $\mathscr{X}$ the following graph: The set of vertices is the set of irreducible curves in $\mathrm{ram}_{\mathscr{X}}(\alpha)$ and there is an edge between two vertices if there is a chilli point in the intersection of the two irreducible curves corresponding to the vertices. A loop in this graph is called a *chilli loop*.

PROPOSITION 1.4 ([Sal07, 2.6, 2.9]). *There exists a projective model $\mathscr{X}$ of K such that there are no chilli loops and no cool points on $\mathscr{X}$ for α.*

Let F be a field of characteristic not equal to 2. The u-invariant of F, denoted by $u(F)$, is defined as follows:

$$u(F) = \sup\{\mathrm{rk}(q) \mid q \text{ an anisotropic quadratic form over } F\}.$$

For $a_1, \dots, a_n \in F^*$, we denote the diagonal quadratic form $a_1X_1^2 + \cdots + a_nX_n^2$ by $\langle a_1, \dots, a_n\rangle$. Let $W(F)$ be the Witt ring of quadratic forms over F and $I(F)$ be the ideal of $W(F)$ consisting of even dimension forms. Let $I^n(F)$ be the n^{th} power of the ideal $I(F)$. For $a_1, \dots, a_n \in F^*$, let $\langle\langle a_1, \dots, a_n\rangle\rangle$ denote the n-fold Pfister form $\langle 1, a_1\rangle \otimes \cdots \otimes \langle 1, a_n\rangle$. The abelian group $I^n(F)$ is generated by n-fold Pfister forms. The dimension modulo 2 gives an isomorphism $e_0\colon W(F)/I(F) \to H^0(F, \mathbf{Z}/2\mathbf{Z})$. The discriminant gives an isomorphism

$$e_1\colon I(F)/I^2(F) \to H^1(F, \mathbf{Z}/2\mathbf{Z}).$$

The classical result of Merkurjev [Mer81], asserts that the Clifford invariant gives an isomorphism $e_2\colon I^2(F)/I^3(F) \to H^2(F, \mathbf{Z}/2\mathbf{Z})$.

Let $P_n(F)$ be the set of isometry classes of n-fold Pfister forms over F. There is a well-defined map ([Ara75])

$$e_n\colon P_n(F) \to H^n(F, \mathbf{Z}/2\mathbf{Z})$$

given by $e_n(\langle 1, a_1\rangle \otimes \cdots \otimes \langle 1, a_n\rangle) = (-a_1)\cdots(-a_n) \in H^n(F, \mathbf{Z}/2\mathbf{Z})$.

A quadratic form version of the Milnor conjecture asserts that e_n induces a surjective homomorphism $I^n(F) \to H^n(F, \mathbf{Z}/2\mathbf{Z})$ with kernel $I^{n+1}(F)$. This conjecture was proved by Voevodsky, Orlov and Vishik. In this paper we are interested in fields of 2-cohomological dimension at most 3. For such fields, Milnor's conjecture above has already been proved by Arason, Elman and Jacob [AEJ86, Cor. 4 and Th. 2], using the theorem of Merkurjev [Mer81].

Let q_1 and q_2 be two quadratic forms over F. We write $q_1 = q_2$ if they represent the same class in the Witt group $W(F)$. We write $q_1 \simeq q_2$, if q_1 and

q_2 are isometric quadratic forms. We note that if the dimensions of q_1 and q_2 are equal and $q_1 = q_2$, then $q_1 \simeq q_2$.

2. Divisors on arithmetic surfaces

In this section we recall a few results from a paper of Saltman [Sal07] on divisors on arithmetic surfaces.

Let $\mathcal{Z}$ be a connected, reduced scheme of finite type over a Noetherian ring. Let $\mathcal{O}^*_{\mathcal{Z}}$ be the sheaf of units in the structure sheaf $\mathcal{O}_{\mathcal{Z}}$. Let $\mathcal{P}$ be a finite set of closed points of $\mathcal{Z}$. For each $P \in \mathcal{P}$, let $\kappa(P)$ be the residue field at P and $\iota_P\colon \mathrm{Spec}(\kappa(P)) \to \mathcal{Z}$ be the natural morphism. Consider the sheaf

$$\mathcal{P}^* = \oplus_{P\in\mathcal{P}} \iota^*_P \kappa(P)^*,$$

where $\kappa(P)^*$ denotes the group of units in $\kappa(P)$. Then there is a surjective morphism of sheaves $\mathcal{O}^*_{\mathcal{Z}} \to \mathcal{P}^*$ given by the evaluation at each $P \in \mathcal{P}$. Let $\mathcal{O}^{*(1)}_{\mathcal{Z},\mathcal{P}}$ be its kernel. When there is no ambiguity we denote $\mathcal{O}^{*(1)}_{\mathcal{Z},\mathcal{P}}$ by $\mathcal{O}^{*(1)}_{\mathcal{P}}$. Let $\mathcal{K}$ be the sheaf of total quotient rings on $\mathcal{Z}$ and $\mathcal{K}^*$ be the sheaf of groups given by units in $\mathcal{K}$. Every element $\gamma \in H^0(\mathcal{Z}, \mathcal{K}^*/\mathcal{O}^*)$ can be represented by a family $\{U_i, f_i\}$, where U_i are open sets in $\mathcal{Z}$, $f_i \in \mathcal{K}^*(U_i)$ and $f_i f_j^{-1} \in \mathcal{O}^*(U_i \cap U_j)$. We say that an element $\gamma = \{U_i, f_i\}$ of $H^0(\mathcal{Z}, \mathcal{K}^*/\mathcal{O}^*)$ *avoids* $\mathcal{P}$ if each f_i is a unit at P for all $P \in U_i \cap \mathcal{P}$. Let $H^0_{\mathcal{P}}(\mathcal{Z}, \mathcal{K}^*/\mathcal{O}^*)$ be the subgroup of $H^0(\mathcal{Z}, \mathcal{K}^*/\mathcal{O}^*)$ consisting of those γ which avoid $\mathcal{P}$. Let $K^* = H^0(\mathcal{Z}, \mathcal{K}^*)$ and $K^*_{\mathcal{P}}$ be the subgroup of K^* consisting of those functions which are units at all $P \in \mathcal{P}$. We have a natural inclusion $K^*_{\mathcal{P}} \to H^0_{\mathcal{P}}(\mathcal{Z}, \mathcal{K}^*/\mathcal{O}^*) \oplus (\oplus_{P\in\mathcal{P}} \kappa(P)^*)$.

Now, we have

PROPOSITION 2.1 ([Sal07, 1.6]). *Let $\mathcal{Z}$ be a connected, reduced scheme of finite type over a Noetherian ring. Then*

$$H^1(\mathcal{Z}, \mathcal{O}^{*(1)}_{\mathcal{P}}) \simeq \frac{H^0_{\mathcal{P}}(\mathcal{Z}, \mathcal{K}^*/\mathcal{O}^*) \oplus (\oplus_{P\in\mathcal{P}} \kappa(P)^*)}{K^*_{\mathcal{P}}}.$$

Let k be a p-adic field and $\mathcal{O}_k$ the ring of integers of k. Let $\mathcal{X}$ be a connected regular surface with a projective morphism $\eta\colon \mathcal{X} \to \mathrm{Spec}(\mathcal{O}_k)$. Let $\overline{\mathcal{X}}$ be the reduced special fibre of η. Assume that $\overline{\mathcal{X}}$ is connected. Note that $\overline{\mathcal{X}}$ is connected if the generic fibre is geometrically integral. Let $\mathcal{P}$ be a finite set of closed points in $\mathcal{X}$. Since every closed point of $\mathcal{X}$ is in $\overline{\mathcal{X}}$, $\mathcal{P}$ is also a subset of closed points of $\overline{\mathcal{X}}$. Let m be an integer coprime with p.

PROPOSITION 2.2 ([Sal07, 1.7]). *The canonical map*

$$H^1(\mathcal{X}, \mathcal{O}^{*(1)}_{\mathcal{X},\mathcal{P}}) \to H^1(\overline{\mathcal{X}}, \mathcal{O}^{*(1)}_{\overline{\mathcal{X}},\mathcal{P}})$$

induces an isomorphism

$$\frac{H^1(\mathcal{X}, \mathcal{O}^{*(1)}_{\mathcal{X},\mathcal{P}})}{mH^1(\mathcal{X}, \mathcal{O}^{*(1)}_{\mathcal{X},\mathcal{P}})} \simeq \frac{H^1(\overline{\mathcal{X}}, \mathcal{O}^{*(1)}_{\overline{\mathcal{X}},\mathcal{P}})}{mH^1(\overline{\mathcal{X}}, \mathcal{O}^{*(1)}_{\overline{\mathcal{X}},\mathcal{P}})}.$$

Let $\mathscr{X}$ be as above. Suppose that $\overline{\mathscr{X}}$ is a union of regular curves $F_1, \dots, F_m$ on $\mathscr{X}$ with only normal crossings. Let $\mathscr{P}$ be a finite set of closed points of $\mathscr{X}$ including all the points of $F_i \cap F_j$, $i \neq j$ and at least one point from each F_i. Let E be a divisor on $\mathscr{X}$ whose support does not pass through any point of $\mathscr{P}$. In particular, no F_i is in the support of E. Hence there are only finitely many closed points $Q_1, \dots, Q_n$ on the support of E. For each closed point Q_i on the support of E, let D_i be a regular curve on $\mathscr{X}$ not contained in the special fiber of $\mathscr{X}$ such that Q_i is the multiplicity one intersection of D_i and $\overline{\mathscr{X}}$. Such a curve exists by ([Sal07, 1.1]). We note that any closed point on $\mathscr{X}$ is a point of codimension-2 and there is a unique closed point on any geometric curve on $\mathscr{X}$ (cf. §1).

The following is extracted from [Sal07, §5].

PROPOSITION 2.3. *Let $\mathscr{X}, \mathscr{P}, E, Q_i, D_i$ be as above. For each closed point Q_i, let m_i be the intersection multiplicity of the support of E and the special fibre $\overline{\mathscr{X}}$ at Q_i. Let l be a prime not equal to p. Then there exist $\nu \in K^*$ and a divisor E' on $\mathscr{X}$ such that*

$$(\nu) = -E + \sum_{i=1}^{n} m_i D_i + lE'$$

and $\nu(P) \in \kappa(P)^{*^l}$ for each $P \in \mathscr{P}$.

Proof. Let F be the divisor on $\mathscr{X}$ given by $\sum F_i$. Let $\gamma \in \mathrm{Pic}(\mathscr{X})$ be the line bundle equivalent to the class of the divisor $-E$ and $\overline{\gamma} \in \mathrm{Pic}(\overline{\mathscr{X}})$ its image. Since the support of E does not pass through the points of $\mathscr{P}$ and $\mathscr{P}$ contains all the points of intersection of distinct F_i, E and F intersect only at smooth points of $\overline{\mathscr{X}}$. In particular, $\overline{\gamma} = -\sum m_i Q_i$. Let $\gamma' \in H^1(\mathscr{X}, \mathbb{O}^*_{\mathscr{P}})$ be the element which, under the isomorphism of Proposition 2.1, corresponds to the class of the element $(-E + \sum m_i D_i, 1)$ in $H^0_{\mathscr{P}}(\mathscr{X}, \mathscr{K}^*/\mathbb{O}^*) \oplus (\oplus_{P \in \mathscr{P}} \kappa(P)^*)$. Since the m_i's are intersection multiplicities of E and $\overline{\mathscr{X}}$ at Q_i and the image of $\sum m_i D_i$ in $H^0_{\mathscr{P}}(\overline{\mathscr{X}}, \mathscr{K}^*/\mathbb{O}^*)$ is $\sum m_i Q_i$, the image $\overline{\gamma'}$ of γ' in $H^1(\overline{\mathscr{X}}, \mathbb{O}^*_{\mathscr{P}})$ is zero. By Proposition 2.2, we have $\gamma' \in lH^1(\mathscr{X}, \mathbb{O}^*_{\mathscr{P}})$. Using Proposition 2.1, there exists $(E', (\lambda_P)) \in H^0_{\mathscr{P}}(\mathscr{L}, \mathscr{K}^*/\mathbb{O}^*) \oplus (\oplus_{P \in \mathscr{P}} \kappa(P)^*)$ such that $(-E + \sum m_i D_i, 1) = l(E', (\lambda_P)) = (lE', (\lambda^l_P))$ modulo $K^*_{\mathscr{P}}$. Thus there exists $\nu \in K^*_{\mathscr{P}} \subset K^*$ such that $(\nu) = (-E + \sum m_i D_i, 1) - (lE', (\lambda^l_P))$.That is, $(\nu) = -E + \sum m_i D_i - lE'$ and $\nu(P) = \lambda^l_P$ for each $P \in \mathscr{P}$. □

3. A local-global principle

Let k be a p-adic field, $\mathbb{O}_k$ be its ring of integers and K the function field of a smooth, projective, geometrically integral curve over k. Let l be a prime not equal to p. Throughout this section, except in Remark 3.6, we assume that k contains a primitive l^{th} root of unity. We fix a generator ρ for μ_l and identify μ_l with $\mathbf{Z}/l\mathbf{Z}$.

LEMMA 3.1. *Let $\alpha \in H^2(K, \mu_l)$. Let $\mathscr{X}$ be a regular projective model of K. Assume that the ramification locus* $\mathrm{ram}_{\mathscr{X}}(\alpha)$ *is a union of regular curves $\{C_1, \dots, C_r\}$ with only normal crossings. Let T be a finite set of closed points of $\mathscr{X}$ including the* *points of $C_i \cap C_j$, for all $i \neq j$. Let D be an irreducible curve on $\mathscr{X}$ which is not in the ramification locus of α and does not pass through any point in T. Then D intersects C_i at points P where $\partial_{C_i}(\alpha)$ is unramified. Suppose further that at such points P, $\partial_{C_i}(\alpha)$ specializes to* 0 *in $H^1(\kappa(P), \mathbf{Z}/l\mathbf{Z})$. Then α, which is unramified at D, specializes to* 0 *in $H^2(\kappa(D), \mu_l)$.*

Proof. Since k contains a primitive l^{th} root of unity, we fix a generator ρ for μ_l and identify the Galois modules $\mu_l^{\otimes j}$ with $\mathbf{Z}/l\mathbf{Z}$.

Let P be a point in the intersection of D and the support of $\text{ram}_{\mathscr{X}}(\alpha)$. Since D does not pass through the points of T and T contains all the points of intersection of distinct C_j, the point P belongs to a unique curve C_i in the support of $\text{ram}_{\mathscr{X}}(\alpha)$. Thus $\partial_{C_i}(\alpha) = (\overline{u}) \in H^1(\kappa(C_i), \mathbf{Z}/l\mathbf{Z})$ is unramified at P (cf. §1).

Suppose that $\partial_{C_i}(\alpha)$ specializes to zero in $H^1(\kappa(P), \mathbf{Z}/l\mathbf{Z})$. Since D is not in the ramification locus of α, α is unramified at D. Let $\overline{\alpha}$ be the specialization of α in $H^2(\kappa(D), \mathbf{Z}/l\mathbf{Z})$. Since $\kappa(D)$ is either a p-adic field or a function field of a curve over a finite field, to show that $\overline{\alpha}$ is zero, by class field theory it is enough to show that $\overline{\alpha}$ is unramified at every discrete valuation of $\kappa(D)$.

Let v be a discrete valuation of $\kappa(D)$ and R the corresponding discrete valuation ring. Then there exists a closed point P of D such that R is a localization of the integral closure of the one-dimensional local ring $\mathbb{O}_{D,P}$ of P on D. The local ring $\mathbb{O}_{D,P}$ is a quotient of the local ring $\mathbb{O}_{\mathscr{X},P}$.

Suppose P is not on the ramification locus of α. Then α is unramified on $\mathbb{O}_{\mathscr{X},P}$ and hence $\overline{\alpha}$ on $\overline{\mathbb{O}_{D,P}}$. In particular, $\overline{\alpha}$ is unramified at R.

Suppose P is on the ramification locus of α. As before, we have $\alpha = \alpha' + (u, \pi)$, where α' is unramified on $\mathbb{O}_{\mathscr{X},P}$, $u \in \mathbb{O}_{\mathscr{X},P}$ is a unit and $\pi \in \mathbb{O}_{\mathscr{X},P}$ is a prime defining the curve C_i at P. Therefore $\partial_{C_i}(\alpha) = \overline{u}$ in $\kappa(C_i)^*/\kappa(C_i)^{*^l}$. Since, by the assumption, $\partial_{C_i}(\alpha)$ specializes to 0 at P, $u(P) \in \kappa(P)^{*^l}$. We have $\overline{\alpha} = \overline{\alpha'} + (\overline{u}, \overline{\pi}) \in H^2(\kappa(D), \mathbf{Z}/l\mathbf{Z})$. Since α' is unramified at P, the residue of $\overline{\alpha}$ at R is $(u(P))^{\nu(\overline{\pi})}$. Since $\kappa(P)$ is contained in the residue field of the discrete valuation ring R and $u(P)$ is an l^{th} power in $\kappa(P)$, it follows that $\overline{\alpha}$ is unramified at R. □

PROPOSITION 3.2. *Let K and l be as above. Let $\alpha \in H^2(K, \mu_l)$ with index l. Let $\mathscr{X}$ be a regular projective model of K such that the ramification locus* $\text{ram}_{\mathscr{X}}(\alpha)$ *and the special fibre of $\mathscr{X}$ are a union of regular curves with only normal crossings and α has no cool points and no chilli loops on $\mathscr{X}$ (cf. Proposition* 1.4*). Let s_i be the corresponding coefficients (cf. §1). Let $F_1, \ldots, F_r$ be irreducible regular curves on $\mathscr{X}$ which are not in* $\text{ram}_{\mathscr{X}}(\alpha) = \{C_1, \ldots, C_n\}$ *and such that* $\{F_1, \ldots, F_r\} \cup \text{ram}_{\mathscr{X}}(\alpha)$ *have only normal crossings. Let $m_1, \ldots, m_r$ be integers. Then there exists $f \in K^*$ such that*

$$\text{div}_{\mathscr{X}}(f) = \sum s_i C_i + \sum m_s F_s + \sum n_j D_j + lE',$$

where $D_1, \ldots, D_t$ are irreducible curves which are not equal to C_i and F_s for all i and s and α specializes to zero at D_j for all j and $(n_j, l) = 1$.

Proof. Let T be a finite set of closed points of $\mathscr{X}$ containing all the points of intersection of distinct C_i and F_s, and at least one point from each C_i and F_s. By a semilocal argument, we choose $g \in K^*$ such that $\mathrm{div}_{\mathscr{X}}(g) = \sum s_i C_i + \sum m_s F_s + G$ where G is a divisor on $\mathscr{X}$ whose support does not contain any of C_i or F_s and does not intersect T.

Since α has no cool points and no chilli loops on $\mathscr{X}$, by [Sal07, Prop. 4.6], there exists $u \in K^*$ such that $\mathrm{div}_{\mathscr{X}}(ug) = \sum s_i C_i + \sum m_s F_s + E$, where E is a divisor of $\mathscr{X}$ whose support does not contain any C_i or F_s, does not pass through the points in T and either E intersects C_i at a point P where the specialization of $\partial_{C_i}(\alpha)$ is 0 or the intersection multiplicity $(E \cdot C_i)_P$ is a multiple of l.

Suppose C_i for some i is a geometric curve on $\mathscr{X}$. Then the closed point of C_i is in T. Since the support of E avoids all the points in T, the support of E does not intersect C_i. Thus the support of E intersects only those C_i which are in the special fibre $\overline{\mathscr{X}}$. Let $Q_1, \ldots, Q_t$ be the points of intersection of the support of the divisor E and the special fibre with intersection multiplicity n_j at Q_j coprime with l. For each Q_j, let D_j be a regular geometric curve on $\mathscr{X}$ such that Q_j is the multiplicity one intersection of D_j and $\overline{\mathscr{X}}$ (cf. paragraph after Proposition 2.2). Then by Proposition 2.3 there exists $\nu \in K^*$ such that $\mathrm{div}_{\mathscr{X}}(\nu) = -E + \sum n_j D_j + lE'$ and $\nu(P) \in \kappa(P)^{*^l}$ for all $P \in T$. Let $f = ug\nu \in K^*$. Then

$$\mathrm{div}_{\mathscr{X}}(f) = \sum s_i C_i + \sum m_s F_s + \sum n_j D_j + lE'.$$

Since each Q_j is the only closed point on D_j and $\partial_{C_i}(\alpha)$ specializes to zero at Q_j, by Lemma 3.1, the α specializes to 0 at D_j. Thus f has all the required properties. □

LEMMA 3.3. *Let $\alpha \in H^2(K, \mu_l)$ and let v be a discrete valuation of K. Let $u \in K^*$ be a unit at v such that $\overline{u} \in \kappa(v)^* \setminus \kappa(v)^{*^l}$. Suppose further that if α is ramified at v, $\partial_v(\alpha) = [L] \in H^1(\kappa(v), \mathbf{Z}/l\mathbf{Z})$, where $L = K(u^{\frac{1}{l}})$. Then, for any $g \in L^*$, the image of $\alpha \cup (N_{L/K}(g)) \in H^3(K_v, \mu_l^{\otimes 2})$ is zero.*

Proof. We identify the Galois modules $\mu_l^{\otimes j}$ with $\mathbf{Z}/l\mathbf{Z}$ as before. Since u is a unit at v and $\overline{u} \notin \kappa(v)^{*^l}$, there is a unique discrete valuation $\tilde{v}$ of L extending the valuation v of K, which is unramified with residual degree l. In particular, $v(N_{L/K}(g))$ is a multiple of l. Thus if $\alpha' \in H^2(K_v, \mathbf{Z}/l\mathbf{Z})$ is unramified at v, then $\alpha' \cup (N_{L/K}(g)) \in H^3(K_v, \mathbf{Z}/l\mathbf{Z})$ is unramified. Since $H^3_{\mathrm{nr}}(K_v, \mathbf{Z}/l\mathbf{Z}) = 0$ (cf. Corollary 1.2), we have $\alpha' \cup (N_{L/K}(g)) = 0$ for any $\alpha' \in H^2(K_v, \mathbf{Z}/l\mathbf{Z})$ which is unramified at v. In particular, if α is unramified at v, then $\alpha \cup (N_{L/K}(g)) = 0$.

Suppose that α is ramified at v. Then by the choice of u, we have $\alpha = \alpha' + (u) \cdot (\pi_v)$, where π_v is a parameter at v and $\alpha' \in H^2(K_v, \mathbf{Z}/l\mathbf{Z})$ is unramified at v. Thus we have

$$\begin{aligned}\alpha \cup (N_{L/K}(g)) &= \alpha' \cup (N_{L/K}(g)) + (N_{L/K}(g)) \cdot (u) \cdot (\pi_v)\\ &= (N_{L/K}(g)) \cdot (u) \cdot (\pi_v) \in H^3(K_v, \mathbf{Z}/l\mathbf{Z}).\end{aligned}$$

Since $L_v = K_v(u^{\frac{1}{l}})$, we have $((N_{L/K}(g)) \cdot (u) = 0 \in H^2(K_v, \mathbf{Z}/l\mathbf{Z})$ and $\alpha \cup (N_{L/K}(g)) = 0$ in $H^3(K_v, \mathbf{Z}/l\mathbf{Z})$. □

THEOREM 3.4. *Let K and l be as above. Let $\alpha \in H^2(K, \mu_l)$ and $\zeta \in H^3(K, \mu_l^{\otimes 2})$. Assume that the index of α is l. Let $\mathscr{X}$ be a regular projective model of K. Suppose that for each $x \in \mathscr{X}^1$, there exists $f_x \in K_x^*$ such that $\zeta = \alpha \cup (f_x) \in H^3(K_x, \mu_l^{\otimes 2})$, where K_x is the completion of K at the discrete valuation given by x. Then there exists $f \in K^*$ such that $\zeta = \alpha \cup (f) \in H^3(K, \mu_l^{\otimes 2})$.*

Proof. We identify the Galois modules $\mu_l^{\otimes j}$ with $\mathbf{Z}/l\mathbf{Z}$ as before. By weak approximation, we may find $f \in K^*$ such that $(f) = (f_v) \in H^1(K_v, \mathbf{Z}/l\mathbf{Z})$ for all the discrete valuations corresponding to the irreducible curves in $\mathrm{ram}_{\mathscr{X}}(\alpha) \cup \mathrm{ram}_{\mathscr{X}}(\zeta)$. Let

$$\mathrm{div}_{\mathscr{X}}(f) = C' + \sum m_i F_i + lE,$$

where C' is a divisor with support contained in $\mathrm{ram}_{\mathscr{X}}(\alpha) \cup \mathrm{ram}_{\mathscr{X}}(\zeta)$, F_i's are distinct irreducible curves which are not in $\mathrm{ram}_{\mathscr{X}}(\alpha) \cup \mathrm{ram}_{\mathscr{X}}(\zeta)$, m_i is coprime with l and E is some divisor on $\mathscr{X}$.

For any $C_j \in \mathrm{ram}_{\mathscr{X}}(\zeta) \setminus \mathrm{ram}_{\mathscr{X}}(\alpha)$, let $\lambda_j \in \kappa(C_j)^* \setminus \kappa(C_j)^{*^l}$. By weak approximation, we choose $u \in K^*$ with $\overline{u} = \partial_{C_i}(\alpha) \in H^1(\kappa(C_i), \mathbf{Z}/l\mathbf{Z})$ for all $C_i \in \mathrm{ram}_{\mathscr{X}}(\alpha)$, $\nu_{F_i}(u) = m_i$, where ν_{F_i} is the discrete valuation at F_i and $\overline{u} = \lambda_j$ for any $C_j \in \mathrm{ram}_{\mathscr{X}}(\zeta) \setminus \mathrm{ram}_{\mathscr{X}}(\alpha)$. In particular, u is a unit at the generic point of C_j and $\overline{u} \notin \kappa(C_j)^{*^l}$ for any $C_j \in \mathrm{ram}_{\mathscr{X}}(\zeta) \setminus \mathrm{ram}_{\mathscr{X}}(\alpha)$.

Let $L = K(u^{\frac{1}{l}})$. Let $\eta\colon \mathscr{Y} \to \mathscr{X}$ be the normalization of $\mathscr{X}$ in L. Since $\nu_{F_i}(u) = m_i$ and m_i is coprime with l, $\eta\colon \mathscr{Y} \to \mathscr{X}$ ramified at F_i. In particular, there is a unique irreducible curve $\widetilde{F_i}$ in $\mathscr{Y}$ such that $\eta(\widetilde{F_i}) = F_i$ and $\kappa(F_i) = \kappa(\widetilde{F_i})$.

Let $\pi\colon \widetilde{\mathscr{Y}} \to \mathscr{Y}$ be a proper birational morphism such that the ramification locus $\mathrm{ram}_{\widetilde{\mathscr{Y}}}(\alpha_L)$ of α_L on $\widetilde{\mathscr{Y}}$ and the strict transform of the curves $\widetilde{F_i}$ on $\widetilde{\mathscr{Y}}$ is a union of regular curves with only normal crossings and there are no cool points and no chilli loops for α_L on $\widetilde{\mathscr{Y}}$ (cf. Proposition 1.4). We denote the strict transforms of $\widetilde{F_i}$ by $\widetilde{F_i}$ again. By Proposition 3.2, there exists $g \in L^*$ such that

$$\mathrm{div}_{\widetilde{\mathscr{Y}}}(g) = C + \sum -m_i \widetilde{F_i} + \sum n_j D_j + lD,$$

where the support of C is contained in $\mathrm{ram}_{\widetilde{\mathscr{Y}}}(\alpha_L)$ and D_j's are irreducible curves which are not in $\mathrm{ram}_{\widetilde{\mathscr{Y}}}(\alpha_L)$ and α_L specializes to zero at all D_j's.

We now claim that $\zeta = \alpha \cup (fN_{L/K}(g))$. Since the group $H^3_{\mathrm{nr}}(K/\mathscr{X}, \mathbf{Z}/l\mathbf{Z}) = 0$ ([K, 5.2]), it is enough to show that $\zeta - \alpha \cup (fN_{L/K}(g))$ is unramified on $\mathscr{X}$. Let S be an irreducible curve on $\mathscr{X}$. Since the residue map ∂_S factors through the completion K_S, it suffices to show that $\zeta - \alpha \cup (fN_{L/K}(g)) = 0$ over K_S.

Suppose S is not in $\mathrm{ram}_{\mathscr{X}}(\alpha) \cup \mathrm{ram}_{\mathscr{X}}(\zeta) \cup \mathrm{Supp}(fN_{L/K}(g))$. Then each of ζ and $\alpha \cup (fN_{L/K}(g))$ is unramified at S.

Suppose that S is in $\mathrm{ram}_{\mathscr{X}}(\alpha) \cup \mathrm{ram}_{\mathscr{X}}(\zeta)$. Then by the choice of f we have $(f) = (f_v) \in H^1(K_v, \mathbf{Z}/l\mathbf{Z})$ where v is the discrete valuation associated to S.

Hence $\zeta = \alpha \cup (f)$ over the completion K_S of K at the discrete valuation given by S. It follows from Lemma 3.3 that $(N_{L/K}(g)) \cup \alpha = 0$ over K_S and $\zeta = \alpha \cup (f N_{L/K}(g))$ over K_S.

Suppose that S is in the support of $\mathrm{div}_{\mathscr{X}}(f N_{L/K}(g))$ and not in $\mathrm{ram}_{\mathscr{X}}(\alpha) \cup \mathrm{ram}_{\mathscr{X}}(\zeta)$. Then α and ζ are unramified at S. We show that in this case $\alpha \cup (f N_{L/K}(g)) = \zeta = 0$ over K_S. Now,

$$\begin{aligned}\mathrm{div}_{\mathscr{X}}(f N_{L/K}(g)) &= \mathrm{div}_{\mathscr{X}}(f) + \mathrm{div}_{\mathscr{X}}(N_{L/K}(g)) \\ &= C' + \sum m_i F_i + lE + \eta_* \pi_* \Big(C + \sum -m_i \widetilde{F_i} + \sum n_j D_j + lD \Big) \\ &= C' + \eta_* \pi_*(C) + \sum n_j \eta_* \pi_*(D_j) + lE'\end{aligned}$$

for some E'. We note that if D_j maps to a point, then $\eta_* \pi_*(D_j) = 0$. Since the support of C is contained in $\mathrm{ram}_{\widetilde{\mathscr{Y}}}(\alpha_L)$, the support of $\eta_* \pi_*(C)$ is contained in $\mathrm{ram}_{\mathscr{X}}(\alpha)$. Thus S is in the support of $\eta_* \pi_*(D_j)$ for some j or S is in the support of $l \eta_* \pi_*(E)$. In the later case, clearly $\alpha \cup (f N_{L/K}(g))$ is unramified at S and hence $\alpha \cup (f N_{L/K}(g)) = 0$ over K_S. Suppose S is in the support of $\eta_* \pi_*(D_j)$ for some j. In this case, if D_j lies over an inert curve, then $\eta_* \pi_*(D_j)$ is a multiple of l and we are done. Suppose that D_j lies over a split curve. Since α_L specializes to zero at D_j, it follows that α specializes to zero at $\eta_* \pi_*(D_j)$ and we are done. □

THEOREM 3.5. *Let k be a p-adic field and K a function field of a curve over k. Let l be a prime not equal to p. Suppose that all the l^{th} roots of unity are in K. Then every element in $H^3(K, \mu_l^{\otimes 3})$ is a symbol.*

Proof. We again identify the Galois modules $\mu_l^{\otimes j}$ with $\mathbf{Z}/l\mathbf{Z}$.

Let v be a discrete valuation of K and K_v the completion of K at v. By Corollary 1.2, every element in $H^3(K_v, \mathbf{Z}/l\mathbf{Z})$ is a symbol.

Let $\zeta \in H^3(K, \mathbf{Z}/l\mathbf{Z})$ and $\mathscr{X}$ be a regular projective model of K. Let v be a discrete valuation of K corresponding to an irreducible curve in $\mathrm{ram}_{\mathscr{X}}(\zeta)$. Then we have $\zeta = (f_v) \cdot (g_v) \cdot (h_v)$ for some $f_v, g_v, h_v \in K_v^*$. By weak approximation, we can find $f, g \in K^*$ such that $(f) = (f_v)$ and $(g) = (g_v)$ in $H^1(K_v, \mathbf{Z}/l\mathbf{Z})$ for all discrete valuations v corresponding to the irreducible curves in $\mathrm{ram}_{\mathscr{X}}(\zeta)$. Let v be a discrete valuation of K corresponding to an irreducible curve C in $\mathscr{X}$. If C is in $\mathrm{ram}_{\mathscr{X}}(\zeta)$, then by the choice of f and g we have $\zeta = (f) \cdot (g) \cdot (h_v) \in H^3(K_v, \mathbf{Z}/l\mathbf{Z})$. If C is not in $\mathrm{ram}_{\mathscr{X}}(\zeta)$, then $\zeta \in H^3_{\mathrm{nr}}(K_v, \mathbf{Z}/l\mathbf{Z}) \simeq H^3(\kappa(v), \mathbf{Z}/l\mathbf{Z}) = 0$. In particular, we have $\zeta = (f) \cdot (g) \cdot (1) \in H^3(K_v, \mathbf{Z}/l\mathbf{Z})$. Let $\alpha = (f) \cdot (g) \in H^2(K, \mathbf{Z}/l\mathbf{Z})$. Then we have $\zeta = \alpha \cup (h'_v) \in H^3(K_v, \mathbf{Z}/l\mathbf{Z})$ for some $h'_v \in K_v^*$ for each discrete valuation v of K associated to any point of $\mathscr{X}^1$. By Theorem 3.4, there exists $h \in K^*$ such that $\zeta = \alpha \cup (h) = (f) \cdot (g) \cdot (h) \in H^3(K, \mathbf{Z}/l\mathbf{Z})$. □

Remark 3.6. We note that all the results of this section can be extended to the situation where k does not necessarily contain a primitive l^{th} root of unity. This can be achieved by going to the extension k' of k obtained by adjoining a primitive

l^{th} root of unity to k and noting that the extension k'/k is unramified of degree $l-1$. We do not use this remark in the sequel.

4. **The u-invariant**

In Proposition 4.1 and Proposition 4.2 below, we give some necessary conditions for a field k to have the u-invariant less than or equal to 8. If K is the function field of a curve over a p-adic field and K_v is the completion of K at a discrete valuation v of K, then the residue field $\kappa(v)$ of K_v, which is either a global field of positive characteristic or a p-adic field, has u-invariant 4. By a theorem of Springer, $u(K_v) = 8$ and we use Propositions 4.1 and 4.2 for K_v.

PROPOSITION 4.1. *Let K be a field of characteristic not equal to* 2. *Suppose that $u(K) \le 8$. Then $I^4(K) = 0$ and every element in $I^3(K)$ is a* 3*-fold Pfister form. Further, if ϕ is a* 3*-fold Pfister form and q_2 a rank* 2 *quadratic form over K, then there exists $f, g, h \in K^*$ such that f is a value of q_2 and $\phi = \langle 1, f\rangle\langle 1, g\rangle\langle 1, h\rangle$.*

Proof. Suppose that $u(K) = 8$. Then every 4-fold Pfister form is isotropic and hence hyperbolic; in particular, $I^4(K) = 0$. Let ϕ be an anisotropic quadratic form representing an element in $I^3(K)$. Since $u(K) \le 8$, the rank of ϕ is 8 (cf. [Sch85, p. 156, Th. 5.6]). Then ϕ is a scalar multiple of a 3-fold Pfister form (cf. [Lam05, Ch. X, Th. 5.6]). Since $I^4(K) = 0$, ϕ is a 3-fold Pfister form.

Let $\phi = \langle 1, a\rangle\langle 1, b\rangle\langle 1, c\rangle$ be a 3-fold Pfister form and ϕ' be its pure subform. Let q_2 be a quadratic form over K of dimension 2. Since $\dim(\phi') = 7$ and $u(K) \le 8$, the quadratic form $\phi' - q_2$ is isotropic. Therefore there exists $f \in K^*$ which is a value of q_2 and $\phi' \simeq \langle f\rangle + \phi''$ for some quadratic form ϕ'' over K. Hence by [Sch85, p. 143], $\phi = \langle 1, f\rangle\langle 1, b'\rangle\langle 1, c'\rangle$ for some $b', c' \in K^*$. □

PROPOSITION 4.2. *Let K be a field of characteristic not equal to* 2. *Suppose that $u(K) \le 8$. Let $\phi = \langle 1, f\rangle\langle 1, a\rangle\langle 1, b\rangle$ be a* 3*-fold Pfister form over K and q_3 a quadratic form over K of dimension* 3. *Then there exist $g, h \in K^*$ such that g is a value of q_3 and $\phi = \langle 1, f\rangle\langle 1, g\rangle\langle 1, h\rangle$.*

Proof. Let $\psi = \langle 1, f\rangle\langle a, b, ab\rangle$. Since $u(K) \le 8$, the quadratic form $\psi - q_3$ is isotropic. Hence there exists $g \in K^*$ which is a common value of q_3 and ψ. Thus, $\psi \simeq \langle g\rangle + \psi_1$ for some quadratic form ψ_1 over K. Since ψ is hyperbolic over $K(\sqrt{-f})$, $\psi_1 \simeq \langle 1, f\rangle\langle a_1, b_1\rangle + \langle g_1\rangle$ for some $a_1, b_1, g_1 \in K^*$. By comparing the determinants, we get $g_1 = gf$ modulo squares. Hence $\psi = \langle 1, f\rangle\langle g, a_1, b_1\rangle$ and $\phi = \langle 1, f\rangle + \psi = \langle 1, f\rangle\langle 1, g, a_1, b_1\rangle$. The form ϕ is isotropic and hence hyperbolic over the function field of the conic given by $\langle f, g, fg\rangle$. Hence, as in Proposition 4.1, $\phi = \lambda\langle 1, f\rangle\langle 1, g\rangle\langle 1, h\rangle$ for some $\lambda, h \in K^*$. Since $I^4(K) = 0$, $\phi = \langle 1, f\rangle\langle 1, g\rangle\langle 1, h\rangle$ with g a value of q_3. □

PROPOSITION 4.3. *Let K be a field of characteristic not equal to* 2. *Assume the following*:

(1) *Every element in $H^2(K, \mathbf{Z}/2\mathbf{Z})$ is a sum of at most two symbols.*

(2) *Every element in* $I^3(K)$ *is equal to a* 3*-fold Pfister form.*

(3) *If* ϕ *is a* 3*-fold Pfister form and* q_2 *is a quadratic form over* K *of dimension* 2*, then* $\phi = \langle 1, f\rangle\langle 1, g\rangle\langle 1, h\rangle$ *for some* $f, g, h \in K^*$ *with* f *a value of* q_2.

(4) *If* $\phi = \langle 1, f\rangle\langle 1, a\rangle\langle 1, b\rangle$ *is a* 3*-fold Pfister form and* q_3 *a quadratic form over* K *of dimension* 3*, then* $\phi = \langle 1, f\rangle\langle 1, g\rangle\langle 1, h\rangle$ *for some* $g, h \in K^*$ *with* g *a value of* q_3.

(5) $I^4(K) = 0$.

Then $u(K) \leq 8$.

Proof. Let q be a quadratic form over K of dimension 9. Since every element in $H^2(K, \mathbf{Z}/2\mathbf{Z})$ is a sum of at most two symbols, as in [PS98, proof of 4.5], we find a quadratic form $q_5 = \lambda\langle 1, a_1, a_2, a_3, a_4\rangle$ over K such that $\phi = q + q_5 \in I^3(K)$. By assumptions (2), (3) and (4), there exist $f, g, h \in K^*$ such that $\phi = \langle 1, f\rangle\langle 1, g\rangle\langle 1, h\rangle$ and f is a value of $\langle a_1, a_2\rangle$ and g is a value of $\langle fa_1a_2, a_3, a_4\rangle$. We have $\langle a_1, a_2\rangle \simeq \langle f, fa_1a_2\rangle$ and $\langle fa_1a_2, a_2, a_3\rangle \simeq \langle g, g_1, g_2\rangle$ for some $g_1, g_2 \in K^*$. Since $I^4(K) = 0$, we have $\lambda\phi = \phi$ and

$$\begin{aligned}
\lambda q &= \lambda q + \lambda q_5 - \lambda q_5 \\
&= \lambda\phi - \lambda q_5 \\
&= \phi - \lambda q_5 \\
&= \langle 1, f\rangle\langle 1, g\rangle\langle 1, h\rangle - \langle 1, a_1, a_2, a_3, a_4\rangle \\
&= \langle 1, f\rangle\langle 1, g\rangle\langle 1, h\rangle - \langle 1, f, g, g_1, g_2\rangle \\
&= \langle gf\rangle + \langle 1, f\rangle\langle h, gh\rangle - \langle g_1, g_2\rangle.
\end{aligned}$$

The above equalities are in the Witt group of K. Since the dimension of λq is 9 and the dimension of $\langle gf\rangle \perp \langle 1, f\rangle\langle h, gh\rangle - \langle g_1, g_2\rangle$ is 7, it follows that λq, and hence q, is isotropic over K. ☐

PROPOSITION 4.4. *Let* k *be a* p*-adic field,* $p \neq 2$ *and* K *a function field of a curve over* k*. Let* ϕ *be a* 3*-fold Pfister form over* K *and* q_2 *a quadratic form over* K *of dimension* 2*. Then there exist* $f, a, b \in K^*$ *such that* f *is a value of* q_2 *and* $\phi = \langle 1, f\rangle\langle 1, a\rangle\langle 1, b\rangle$.

Proof. Let $\zeta = e_3(\phi) \in H^3(K, \mathbf{Z}/2\mathbf{Z})$. Let $\mathscr{X}$ be a projective regular model of K. Let C be an irreducible curve on $\mathscr{X}$ and v be the discrete valuation given by C. Let K_v be the completion of K at v. Since the residue field $\kappa(v) = \kappa(C)$ is either a p-adic field or a function field of a curve over a finite field, $u(\kappa(v)) = 4$ and $u(K_v) = 8$ ([Sch85, p. 209]). By Proposition 4.1, there exist $f_v, a_v, b_v \in K_v^*$ such that f_v is a value of q_2 over K_v and $\phi = \langle 1, f_v\rangle\langle 1, a_v\rangle\langle 1, b_v\rangle$ over K_v. By weak approximation, we can find $f, a \in K^*$ such that f is a value of q_2 over K and $f = f_v, a = a_v$ modulo $K_v^{*^2}$ for all discrete valuations v corresponding to the irreducible curves C in the support of $\mathrm{ram}_{\mathscr{X}}(\zeta)$. Let C be any irreducible curve on $\mathscr{X}$ and v be the discrete valuation of K given by C. If C is in the support of $\mathrm{ram}_{\mathscr{X}}(\zeta)$, then by the choice of f and a, we have $\zeta = e_3(\phi) = (-f)\cdot(-a)\cdot(-b_v)$

over K_v. If C is not in the support of $\mathrm{ram}_{\mathscr{X}}(\zeta)$, then $\zeta \in H^3_{\mathrm{nr}}(K_v, \mathbf{Z}/2\mathbf{Z}) \simeq H^3(\kappa(v), \mathbf{Z}/2\mathbf{Z}) = 0$. In particular, we have $\zeta = (-f)\cdot(-a)\cdot(1)$ over K_v. Let $\alpha = (-f)\cdot(-a) \in H^2(K, \mathbf{Z}/2\mathbf{Z})$. By Theorem 3.4, there exists $b \in K^*$ such that $\zeta = \alpha \cup (-b) \in H^3(K, \mathbf{Z}/2\mathbf{Z})$. Since $e_3\colon I^3(K) \to H^3(K, \mathbf{Z}/2\mathbf{Z})$ is an isomorphism, we have $\phi = \langle 1, f\rangle\langle 1, a\rangle\langle 1, b\rangle$ as required. □

There is a different proof of Proposition 4.4 in [PS98, 4.4]!

PROPOSITION 4.5. *Let k be a p-adic field, $p \neq 2$ and K be a function field of a curve over k. Let $\phi = \langle 1, f\rangle\langle 1, a\rangle\langle 1, b\rangle$ be a 3-fold Pfister form over K and q_3 a quadratic form over K of dimension 3. Then there exist $g, h \in K^*$ such that g is a value of q_3 and $\phi = \langle 1, f\rangle\langle 1, g\rangle\langle 1, h\rangle$.*

Proof. Let $\zeta = e_3(\phi) = (-f)\cdot(-a)\cdot(-b) \in H^3(K, \mathbf{Z}/2\mathbf{Z})$. Let $\mathscr{X}$ be a projective regular model of K. Let C be an irreducible curve on $\mathscr{X}$ and v be the discrete valuation of K given by C. Let K_v be the completion of K at v. Then as in the proof of Proposition 4.4, we have $u(K_v) = 8$. Thus by Proposition 4.2, there exist $g_v, h_v \in K_v^*$ such that g_v is a value of the quadratic form q_3 and $\phi = \langle 1, f\rangle\langle 1, g_v\rangle\langle 1, h_v\rangle$ over K_v. By weak approximation, we can find $g \in K^*$ such that g is a value of q_3 over K and $g = g_v$ modulo ${K_v^*}^2$ for all discrete valuations v of K given by the irreducible curves C in $\mathrm{ram}_{\mathscr{X}}(\zeta)$. Let C be an irreducible curve on $\mathscr{X}$ and v be the discrete valuation of K given by C. By the choice of g it is clear that $\zeta = e_3(\phi) = (-f)\cdot(-g)\cdot(-h_v)$ for all the discrete valuations v of K given by the irreducible curves C in the support of $\mathrm{ram}_{\mathscr{X}}(\zeta)$. If C is not in the support of $\mathrm{ram}_{\mathscr{X}}(\zeta)$, then as in the proof of Proposition 4.4, we have $\zeta = (-f)\cdot(-g)\cdot(1)$ over K_v. Let $\alpha = (-f)\cdot(-g) \in H^2(K, \mathbf{Z}/2\mathbf{Z})$. By Theorem 3.4, there exists $h \in K^*$ such that $\zeta = \alpha \cup (-h) = (-f)\cdot(-g)\cdot(-h)$. Since $e_3\colon I^3(K) \to H^3(K, \mathbf{Z}/2\mathbf{Z})$ is an isomorphism, $\phi = \langle 1, f\rangle\langle 1, g\rangle\langle 1, h\rangle$. □

THEOREM 4.6. *Let K be a function field of a curve over a p-adic field k. If $p \neq 2$, then $u(K) = 8$.*

Proof. Let K be a function field of a curve over a p-adic field k. Assume that $p \neq 2$. By a theorem of Saltman ([Sal97, 3.4]; cf. [Sal98]), every element in $H^2(K, \mathbf{Z}/2\mathbf{Z})$ is a sum of at most two symbols. Since the cohomological dimension of K is 3, we also have $I^4(K) \simeq H^4(K, \mathbf{Z}/2\mathbf{Z}) = 0$ ([AEJ86]). Now the theorem follows from Propositions 4.3, 4.4 and 4.5. □

References

[Ara75] J. K. ARASON, Cohomologische invarianten quadratischer Formen, *J. Algebra* **36** (1975), 448–491. MR 52 #10592 Zbl 0314.12104

[AEJ86] J. K. ARASON, R. ELMAN, and B. JACOB, Fields of cohomological 2-dimension three, *Math. Ann.* **274** (1986), 649–657. MR 87m:12006 Zbl 0576.12025

[CT95] J.-L. COLLIOT-THÉLÈNE, Birational invariants, purity and the Gersten conjecture, in *K-Theory and Algebraic Geometry*: *Connections with Quadratic Forms and Division Algebras* (Santa Barbara, CA, 1992), *Proc. Sympos. Pure Math.* **58**, Amer. Math. Soc., Providence, RI, 1995, pp. 1–64. MR 96c:14016 Zbl 0834.14009

[Kat86] K. KATO, A Hasse principle for two-dimensional global fields, *J. Reine Angew. Math.* **366** (1986), 142–183. MR 88b:11036 Zbl 0576.12012

[Lam05] T. Y. LAM, *Introduction to Quadratic Forms over Fields*, *Grad. Stud. Math.* **67**, Amer. Math. Soc., Providence, RI, 2005. MR 2005h:11075 Zbl 1068.11023

[Lip75] J. LIPMAN, Introduction to resolution of singularities, in *Algebraic Geometry* (Humboldt State Univ., Arcata, Calif., 1974), *Proc. Sympos. Pure Math.* **29**, Amer. Math. Soc., Providence, R.I., 1975, pp. 187–230. MR 52 #10730 Zbl 0306.14007

[Lip78] ———, Desingularization of two-dimensional schemes, *Ann. Math.* **107** (1978), 151–207. MR 58 #10924 Zbl 0349.14004

[Mer81] A. S. MERKUR′EV, On the norm residue symbol of degree 2, *Dokl. Akad. Nauk SSSR* **261** (1981), 542–547. MR 83h:12015 Zbl 0496.16020

[Mil80] J. S. MILNE, *Étale Cohomology*, *Princeton Mathematical Series* **33**, Princeton Univ. Press, Princeton, N.J., 1980. MR 81j:14002 Zbl 0433.14012

[PS98] R. PARIMALA and V. SURESH, Isotropy of quadratic forms over function fields of p-adic curves, *Inst. Hautes Études Sci. Publ. Math.* (1998), 129–150 (1999). MR 2000m:11034 Zbl 0972.11020

[Sal97] D. J. SALTMAN, Division algebras over p-adic curves, *J. Ramanujan Math. Soc.* **12** (1997), 25–47. MR 98d:16032 Zbl 0902.16021

[Sal98] ———, Correction to: "Division algebras over p-adic curves" [J. Ramanujan Math. Soc. **12** (1997), no. 1, 25–47; MR1462850 (98d:16032)], *J. Ramanujan Math. Soc.* **13** (1998), 125–129. MR 99k:16036 Zbl 0920.16008

[Sal07] ———, Cyclic algebras over p-adic curves, *J. Algebra* **314** (2007), 817–843. MR 2008i:16018 Zbl 1129.16014

[Sch85] W. SCHARLAU, *Quadratic and Hermitian Forms*, *Grundl. Math. Wissen.* **270**, Springer-Verlag, New York, 1985. MR 86k:11022 Zbl 0584.10010

[Sha66] I. R. SHAFAREVICH, *Lectures on Minimal Models and Birational Transformations of Two Dimensional Schemes*, *Tata Inst. Fund. Res. Lect. Math. Phys.* **37**, Tata Institute of Fundamental Research, Bombay, 1966. MR 36 #163 Zbl 0164.51704

(Received August 23, 2007)
(Revised October 19, 2008)

E-mail address: parimala@mathcs.emory.edu
DEPARTMENT OF MATHEMATICS AND COMPUTER SCIENCE, EMORY UNIVERSITY,
400 DOWMAN DRIVE, ATLANTA, GEORGIA 30322, UNITED STATES
http://www.mathcs.emory.edu/~parimala/

E-mail address: vssm@uohyd.ernet.in
DEPARTMENT OF MATHEMATICS AND STATISTICS, UNIVERSITY OF HYDERABAD,
GAHCIBOWLI, HYDERABAD - 500046, INDIA

ISSN 0003-486X

ANNALS OF MATHEMATICS

This periodical is published bimonthly by the Department of Mathematics at Princeton University with the cooperation of the Institute for Advanced Study. Annals is typeset in TEX by Sarah R. Warren and produced by Mathematical Sciences Publishers. The six numbers each year are divided into two volumes of three numbers each.

Editorial correspondence

Papers submitted for publication and editorial correspondence should be addressed to Maureen Schupsky, Annals of Mathematics, Fine Hall-Washington Road, Princeton University, Princeton, NJ, 08544-1000 U.S.A. The e-mail address is annals@math.princeton.edu.

Preparing and submitting papers

The Annals requests that all papers include an abstract of about 150 words which explains to the nonspecialist mathematician what the paper is about. It should not make any reference to the bibliography. Authors are encouraged to initially submit their papers electronically and in PDF format. Please send the file to: annals@math.princeton.edu or to the Mathematics e-print arXiv: front.math.ucdavis.edu/submissions. If a paper is submitted through the arXiv, then please e-mail us with the arXiv number of the paper.

Proofs

A PDF file of the galley proof will be sent to the corresponding author for correction. If requested, a paper copy will also be sent to the author.

Offprints

Authors of single-authored papers will receive 30 offprints. (Authors of papers with one co-author will receive 15 offprints, and authors of papers with two or more co-authors will receive 10 offprints.) Extra offprints may be purchased through the editorial office.

Subscriptions

The price for a print and online subscription, or an online-only subscription, is $390 per year for institutions. In addition, there is a postage surcharge of $40 for print subscriptions that are mailed to countries outside of the United States. Individuals interested in subscriptions for their own personal use should contact the publisher at the address below. Subscriptions and changes of address should be sent to Mathematical Sciences Publishers, Department of Mathematics, University of California, Berkeley, CA 94720-3840 (e-mail: contact@mathscipub.org; phone: 1-510-643-8638; fax: 1-510-295-2608). (Checks should be made payable to "Mathematical Sciences Publishers".)

Back issues and reprints

Orders for missing issues and back issues should be sent to Mathematical Sciences Publishers at the above address. Claims for missing issues must be made within 12 months of the publication date. Online versions of papers published five or more years ago are available through JSTOR (www.jstor.org).

Microfilm

Beginning with Volume 1, microfilm may be purchased from NA Publishing, Inc., 4750 Venture Drive, Suite 400, PO Box 998, Ann Arbor, MI 48106-0998; phone: 1-800-420-6272 or 734-302-6500; email: info@napubco.com, website: www.napubco.com/contact.html.

Printed in U.S.A. by Sheridan Printing Company, Inc., Alpha, NJ

TABLE OF CONTENTS