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Abstract

We prove the Conley conjecture for a closed symplectically aspherical symplectic manifold: a Hamiltonian diffeomorphism of such a manifold has infinitely many periodic points. More precisely, we show that a Hamiltonian diffeomorphism with finitely many fixed points has simple periodic points of arbitrarily large period. This theorem generalizes, for instance, a recent result of Hingston establishing the Conley conjecture for tori.

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1. **Introduction**

We show that a Hamiltonian diffeomorphism of a closed symplectically aspherical manifold has infinitely many periodic points. More precisely, we prove that such a diffeomorphism with finitely many fixed points has simple periodic points of arbitrarily large period. For tori, this fact, recently established by Hingston, [Hin09], was conjectured by Conley, [Con84], [SZ92] and is frequently referred to as the *Conley conjecture*. (See also [FH03], [LC06] and references therein for similar results for Hamiltonian diffeomorphisms and homeomorphisms of surfaces.) The proof given here uses some crucial ideas from [Hin09], but is completely self-contained.

1.1. *Principal results.* The main result of the paper is

THEOREM 1.1. *Let $\varphi: W \to W$ be a Hamiltonian diffeomorphism of a closed symplectically aspherical manifold W. Assume that the fixed points of φ are isolated. Then φ has simple periodic points of arbitrarily large period.*

We refer the reader to Section 2.1.1 for the definitions. Here we only point out that a Hamiltonian diffeomorphism is the time-one map of a time-dependent Hamiltonian flow and that the manifolds W with $\pi_2(W) = 0$ (e.g., tori and surfaces of genus greater than zero) are among symplectically aspherical manifolds. Thus, Theorem 1.1 implies in particular the Conley conjecture for tori, [Hin09], and the results of [FH03] on Hamiltonian diffeomorphisms of such surfaces.

COROLLARY 1.2. *A Hamiltonian diffeomorphism φ of a closed symplectically aspherical manifold has infinitely many simple periodic points.*

Remark 1.3. The example of an irrational rotation of S^2 shows that in general the requirement that W is symplectically aspherical cannot be completely eliminated; see, however, [FH03]. Let H be a periodic in time Hamiltonian giving rise to φ. Since periodic points of φ are in one-to-one correspondence with periodic orbits of the time-dependent Hamiltonian flow φ_H^t, Theorem 1.1 and Corollary 1.2 can be viewed as results about periodic orbits of H. Then, in both of the statements, the periodic orbits can be assumed to be contractible. (It is not hard to see that contractibility is a property of a fixed point rather than of an orbit, independent of the choice of H.) Finally note that, as simple examples show, the assumption of

Theorem 1.1 that the fixed points of φ are isolated cannot be dropped as long as the periodic orbits are required to be contractible.

There are numerous parallels between the Hamiltonian Conley conjecture considered here and its Lagrangian counterpart; see, e.g., [Lon00], [Lu09], [Maz08] and references therein. The similarity between the two problems goes beyond the obvious analogy of the statements and can also easily be seen on the level of the proofs, although the methods utilized in [Lon00], [Lu09], [Maz08] are quite different from the Floer homological techniques used in the present paper. Thus, for instance, our Proposition 4.7 plays the same role as Bangert's homological vanishing method originating from [Ban80], [BK83] in, e.g., [Lon00], [Maz08].

1.2. *Methods.* In the framework of symplectic topology, there are two essentially different approaches to proving results along the lines of the Conley conjecture. The first approach, due to Conley and Salamon and Zehnder, [CZ84], [SZ92], is based on an iteration formula for the Conley-Zehnder index, asserting that the index of an isolated weakly nondegenerate orbit either grows linearly under iterations or its absolute value does not exceed $n-1$, where $2n = \dim W$. This, in particular, implies that the local Floer homology of such an orbit eventually becomes zero in degree n as the order of iteration grows, provided that the orbit remains isolated. (We refer the reader to Sections 2 and 3 for the definitions. The argument of Salamon and Zehnder, [SZ92], does not rely on the notion of local Floer homology, but this notion becomes indispensable in the proof of Theorem 1.1.) Since the Floer homology of W in degree n is nonzero, it follows that when all one-periodic orbits are weakly nondegenerate, new simple orbits must be created by large prime iterations to generate the Floer homology in degree n; see [SZ92] for details.

The second approach comprises a broad class of methods and is based on the idea that a Hamiltonian H with sufficiently large variation must have one-periodic orbits with nonvanishing action. Since iterating a Hamiltonian diffeomorphism φ has the same effect as, roughly speaking, increasing the variation of H, one can expect φ to have infinitely many periodic points. When a sufficiently accurate upper or lower bound on the action is available, the orbits can be shown to be simple. The results obtained along these lines are numerous and use a variety of symplectic topological techniques and assumptions on W and H.

For instance, if the support of H is displaceable and the variation of H is greater than the displacement energy e of the support, one-periodic orbits with action in the range $(0, e]$ have been shown to exist for many classes of symplectic manifolds and Hamiltonians; see, e.g., [CGK04], [FH94], [FHW94], [FS07], [Gür08], [HZ94], [Sch00], [Vit92]. Then, the *a priori* bound on action implies the existence of simple periodic orbits with nonzero action and arbitrarily large period. These methods do not rely on particular requirements on the fixed points of φ, but the assumption that the support is displaceable appears at this moment to be crucial. Within this broad class is also a group of methods applicable to Hamiltonians H

with sufficiently degenerate large or "flat" maximum and detecting orbits with action slightly greater than the maximum of H; see, e.g., [Gin07], [GG04], [Hin09], [HZ90], [HZ94], [Ker05], [KL03], [LM95], [MS01], [Oh02]. Iterating φ can be viewed as stretching H near its maximum, and thus increasing its variation. Hence, methods from this group can also be used in some instances to prove the existence of simple periodic orbits of large period. Here the condition that the maximum is in a certain sense flat is crucial, but the assumption that the support of the Hamiltonian is displaceable is less important and, in some cases, not required at all. In fact, what appears to matter is that the set where the maximum is attained is relatively small (e.g., symplectic as in [GG04] or displaceable as in [LM95] or just isolated as in [Gin07], [KL03]). It is one of these methods, combined with the Conley-Salamon-Zehnder approach, that we use in the proof of the Conley conjecture.

A work of Hingston [Hin09] clearly suggests the idea, which is central to our proof, that the two approaches outlined above can be extended to cover the case of an arbitrary Hamiltonian. Namely, the method of [SZ92] detects infinitely many simple periodic points of arbitrarily large periods, unless there exists a strongly degenerate τ-periodic point p such that the local Floer homology groups of the τ-th iteration φ^τ and of a large iteration of φ^τ at p are nonzero in degree n; see Section 4. Then we show (Proposition 4.5) that the φ^τ is the time-τ flow of a τ-periodic Hamiltonian, say H_t, such that p is a (constant) local maximum of H_t for all t and this maximum is in a certain sense very degenerate; cf. [Hin09]. (However, the Hessian $d^2(H_t)_p$ need not be identically zero.) Finally, we prove (Proposition 4.7) that large iterations of H have periodic orbits with actions arbitrarily close to the action of the iterated Hamiltonian at p. These orbits are necessarily simple due to the lower and upper bounds on the action. Proposition 4.7 is established by using a simple squeezing argument akin to the ones from [BPS03], [GG04]. This concludes the proof of the theorem.

This argument is extremely flexible and readily extends to manifolds convex at infinity or geometrically bounded and wide; see [FS07] and [Gür08] for the definitions. We will give a detailed proof of the Conley conjecture for such manifolds elsewhere.

1.3. *Organization of the paper.* In Section 2, we set notation and conventions, briefly review elements of Floer theory, and also discuss the properties of loops of Hamiltonian diffeomorphisms relevant to the proof. Local Floer homology is the subject of Section 3. In Section 4, we state Propositions 4.5 and 4.7 mentioned above and derive Theorem 1.1 from these propositions. Proposition 4.5 reduces the problem to the case of a Hamiltonian with strict, but "flat", local maximum. This proposition is proved in Sections 5 and 6 by adapting an argument from [Hin09]. Proposition 4.7 asserting the existence of simple periodic orbits of large period for such a Hamiltonian and completing the proof of Theorem 1.1 is established in Section 7.

2. **Preliminaries**

2.1. *Notation and conventions.*

2.1.1. *General.* Throughout the paper, a smooth m-dimensional manifold is denoted by M and W stands for a symplectic manifold of dimension $2n$, equipped with a symplectic form ω. The manifold (W, ω) is always assumed to be closed and *symplectically aspherical*, i.e., $\omega|_{\pi_2(W)} = 0 = c_1|_{\pi_2(W)}$, where c_1 is the first Chern class of W; see, e.g., [MS04]. An almost complex structure J on W is said to be *compatible with* ω if $\omega(\cdot, J\cdot)$ is a Riemannian metric on W. When $J = J_t$ depends on an extra parameter t (time), this condition is required to hold for every t. A (time-dependent) metric of the form $\omega(\cdot, J\cdot)$ is said to be compatible with ω.

The group of linear symplectic transformations of a finite-dimensional linear symplectic space (V, ω) is denoted by $\mathrm{Sp}(V)$. We will also need the fact that $\pi_1(\mathrm{Sp}(V)) \cong \mathbb{Z}$ (see, e.g., [MS95]), and hence $\mathrm{H}_1(\mathrm{Sp}(V); \mathbb{Z}) \cong \mathbb{Z}$. To be more specific, fixing a linear complex structure J on V, compatible with ω, gives rise to an inclusion $\mathrm{U}(V) \hookrightarrow \mathrm{Sp}(V)$ of the unitary group into the symplectic group. This inclusion is a homotopy equivalence. The isomorphism $\pi_1(\mathrm{Sp}(V)) \cong \mathbb{Z}$ is the composition of the isomorphism $\pi_1(\mathrm{Sp}(V)) = \pi_1(\mathrm{U}(V))$, the isomorphism of the fundamental groups induced by $\det\colon U(V) \to S^1$ (the unit circle in $\mathbb{C}$), and the identification $\pi_1(S^1) \cong \mathbb{Z}$ arising from fixing the counter clock-wise orientation of S^1. Note that the resulting isomorphism is independent of the choice of J. The *Maslov index* of a loop in $\mathrm{Sp}(V)$ is the class of this loop in $H_1(\mathrm{Sp}(V); \mathbb{Z}) \cong \mathbb{Z}$.

2.1.2. *Hamiltonians and periodic orbits.* We use the notation S^1 for the circle $\mathbb{R}/\mathbb{Z}$ and the circle $\mathbb{R}/T\mathbb{Z}$ of circumference $T > 0$ is denoted by S^1_T. All Hamiltonians H on W considered in this paper are assumed to be T-periodic (in time), i.e., $H\colon S^1_T \times W \to \mathbb{R}$. We set $H_t = H(t, \cdot)$ for $t \in S^1_T$. The Hamiltonian vector field X_H of H is defined by $i_{X_H}\omega = -dH$.

Let $\gamma\colon S^1_T \to W$ be a contractible loop. The action of H on γ is defined by

$$A_H(\gamma) = A(\gamma) + \int_{S^1_T} H_t(\gamma(t))\, dt.$$

Here $A(\gamma)$ is the negative symplectic area bounded by γ, i.e.,

$$A(\gamma) = -\int_z \omega,$$

where $z\colon D^2 \to W$ is such that $z|_{S^1_T} = \gamma$.

The least action principle asserts that the critical points of A_H on the space of all contractible maps $\gamma\colon S^1_T \to W$ are exactly the contractible T-periodic orbits of the time-dependent Hamiltonian flow φ^t_H of H. When the period T is clear from the context and, in particular, if $T = 1$, we denote the time-T map φ^T_H by φ_H. The action spectrum $\mathcal{S}(H)$ of H is the set of critical values of A_H. This is a zero measure, closed set; see, e.g., [HZ94], [Sch00]. In this paper we are only concerned with contractible periodic orbits. *A periodic orbit is always assumed to be contractible, even if this is not explicitly stated.*

Definition 2.1. A T-periodic orbit γ of H is *nondegenerate* if the linearized return map $d\varphi_H^T\colon T_{\gamma(0)}W \to T_{\gamma(0)}W$ has no eigenvalues equal to one. Following [SZ92], we call γ *weakly nondegenerate* if at least one of the eigenvalues is different from one. When all eigenvalues are equal to one, the orbit is said to be *strongly degenerate*.

When γ is nondegenerate or even weakly nondegenerate, the so-called *Conley-Zehnder index* $\mu_{\mathrm{CZ}}(\gamma) \in \mathbb{Z}$ is defined, up to a sign, as in [Sal99], [SZ92]. More specifically, in this paper, the Conley-Zehnder index is the negative of that of [Sal99]. In other words, we normalize μ_{CZ} so that $\mu_{\mathrm{CZ}}(\gamma) = n$ when γ is a nondegenerate maximum of an autonomous Hamiltonian with small Hessian. More generally, when H is autonomous and γ is a nondegenerate critical point of H such that the eigenvalues of the Hessian (with respect to a metric compatible with ω) are less than $2\pi/T$, the Conley-Zehnder index of γ is equal to one half of the negative signature of the Hessian.

Sometimes, the same Hamiltonian can be treated as T-periodic for different values of $T > 0$. For instance, an autonomous Hamiltonian is T-periodic for every $T > 0$ and a T-periodic Hamiltonian can also be viewed as Tk-periodic for any integer k. In this paper, it will be essential to keep track of the period. Unless specified otherwise, every Hamiltonian H considered here is originally one-periodic and T is always an integer. When we wish to view H as a T-periodic Hamiltonian, we denote it by $H^{(T)}$ and refer to it as the *T-th iteration* of H. (The parentheses here are used to distinguish iterated Hamiltonians from families of Hamiltonians, say H^s, parametrized by s.) Since $H^{(T)}$ is regarded as a T-periodic Hamiltonian, it makes sense to speak only about T-periodic (or Tk-periodic) orbits of $H^{(T)}$. Clearly, T-periodic orbits of $H^{(T)}$ are simply T-periodic orbits of H. When $\gamma\colon S^1 \to W$ is a one-periodic orbit of H, its *T-th iteration* is the obviously defined map $\gamma^{(T)}\colon S^1_T \to W$ obtained by composing the T-fold covering map $S^1_T \to S^1$ with γ. Thus, $\gamma^{(T)}$ is a T-periodic orbit of H and $H^{(T)}$. We call a T-periodic orbit *simple* if it is not an iteration of an orbit of a smaller period.

As is well-known, the fixed points of $\varphi_H := \varphi_H^1$ are in one-to-one correspondence with (not-necessarily contractible) one-periodic orbits of H. Likewise, the T-periodic points of φ_H, i.e., the fixed points of φ_H^T, are in one-to-one correspondence with (not-necessarily contractible) T-periodic orbits. In the proof of Theorem 1.1, we will work with (contractible!) periodic orbits of a Hamiltonian H whose time-one map is φ. In fact, as is easy to see from Section 2.3, the free homotopy type of the one-periodic orbit of H through a fixed point p of φ is completely determined by φ and p and is independent of the choice of H. The same holds for T-periodic points and orbits. Hence, "contractible fixed points or periodic points" of φ, i.e., those with contractible orbits, are well-defined and we will establish Theorem 1.1 for points in this class.

When K and H are two (say, one-periodic) Hamiltonians, the composition $K\#H$ is defined by the formula $(K\#H)_t = K_t + H_t \circ (\varphi_K^t)^{-1}$. The flow of $K\#H$

is $\varphi_K^t \circ \varphi_H^t$. In general, $K\#H$ is not one-periodic. However, this will be the case if, for example, $H_0 \equiv 0 \equiv H_1$. Another instance when the composition $K\#H$ of two one-periodic Hamiltonians is automatically one-periodic is when the flow φ_K^t is a loop of Hamiltonian diffeomorphisms, i.e., $\varphi_K^1 = \mathrm{id}$.

2.1.3. *Norms with respect to a coordinate system.* In what follows, it will be convenient to use a somewhat unconventional terminology and work with C^k-norms of functions, vector fields, etc. taken with respect to a coordinate system.

Let ξ be a coordinate system on a neighborhood U of a point $p \in M^m$, i.e., ξ is a diffeomorphism $U \to \xi(U) \subset \mathbb{R}^m$ sending p to the origin. (Thus, U is a part of the data ξ.) Let f be a function on U or on the entire manifold M. The C^k-norm $\|f\|_{C^k(\xi)}$ of f with respect to ξ is, by definition, the C^k-norm of f on U with respect to the flat metric associated with ξ, i.e., the pull-back by ξ of the standard metric on $\mathbb{R}^m$. The C^k-norm with respect to ξ of a vector field or a field of operators on U is defined in a similar fashion.

Likewise, the norm $\|v\|_\Xi$ of a vector v in a finite-dimensional vector space V with respect to a basis Ξ is the norm of v with respect to the Euclidean inner product for which Ξ is an orthonormal basis. The norm of an operator $V \to V$ with respect to Ξ is defined in a similar way. When ξ is a coordinate system near p, we denote by ξ_p the natural coordinate basis in T_pM arising from ξ.

Example 2.2. Let $A\colon V \to V$ be a linear map with all eigenvalues equal to zero. Then $\|A\|_\Xi$ can be made arbitrarily small by choosing a suitable basis Ξ. In other words, for any $\sigma > 0$ there exists Ξ such that $\|A\|_\Xi < \sigma$. Indeed, in some basis A is given by an upper triangular matrix with zeros on the diagonal; Ξ is then obtained by appropriately scaling the elements of this basis.

Example 2.3. Restricting ξ to a smaller neighborhood of p reduces the norm of f. However, one cannot make, say, $\|f\|_{C^1(\xi)}$ arbitrarily small by shrinking U unless $f(p) = 0$ and $df_p = 0$. Indeed, $\|f\|_{C^1(\xi)} \geq \max\{|f(p)|, \|df_p\|_{\xi_p}\}$. It is clear that for a fixed basis Ξ in T_pM and a function f near p there exists a coordinate system ξ with $\xi_p = \Xi$, such that $\|f\|_{C^1(\xi)}$ is arbitrarily close to $\max\{|f(p)|, \|df_p\|_\Xi\}$.

2.2. *Floer homology.* In this section, we briefly recall the notion of filtered Floer homology for closed symplectically aspherical manifolds. All definitions and results mentioned here are quite standard and well-known and we refer the reader to Floer's papers [Flo88a], [Flo88b], [Flo89a], [Flo89c], to [BPS03], [FHW94], [FH94], [FHS95], [SZ92], [Sch00], or to [HZ94], [MS04], [Sal90], [Sal99] for introductory accounts of the construction of Floer homology in this (or more general) setting.

2.2.1. *Definitions.* Let us first focus on one-periodic Hamiltonians. Consider a Hamiltonian H such that all one-periodic orbits of H are nondegenerate. This is a generic condition and we will call such Hamiltonians *nondegenerate*. Let

$J = J_t$ be a (time-dependent: $t \in S^1$) almost complex structure on W compatible with ω. For two one-periodic orbits $x^\pm$ of H denote by $\mathcal{M}_H(x^-, x^+, J)$ the space of solutions $u\colon S^1 \times \mathbb{R} \to W$ of the Floer equation

$$\frac{\partial u}{\partial s} + J_t(u)\frac{\partial u}{\partial t} = -\nabla H_t(u) \tag{2.1}$$

which are asymptotic to $x^\pm$ at $\pm\infty$, i.e., $u(s,t) \to x^\pm(t)$ point-wise as $s \to \pm\infty$.

The energy $E(u)$ of a solution u of the Floer equation, (2.1), is

$$E(u) = \int_{-\infty}^{\infty} \left\| \frac{\partial u}{\partial s} \right\|^2_{L^2(S^1)} ds = \int_{-\infty}^{\infty} \int_{S^1} \left\| \frac{\partial u}{\partial t} - J\nabla H(u) \right\|^2 dt\, ds,$$

where we set $u(s) := u(s, \cdot)\colon S^1 \to W$. Every finite energy solution of (2.1) is asymptotic to some $x^\pm$ and

$$A_H(x^-) - A_H(x^+) = E(u).$$

When J meets certain standard regularity requirements that hold generically (see, e.g., [FHS95], [SZ92]), the space $\mathcal{M}_H(x^-, x^+, J)$ is a smooth manifold of dimension $\mu_{CZ}(x^+) - \mu_{CZ}(x^-)$. This space carries a natural $\mathbb{R}$-action $(\tau \cdot u)(t,s) = u(t, s+\tau)$ and we denote by $\widehat{\mathcal{M}}_H(x^-, x^+, J)$ the quotient $\mathcal{M}_H(x^-, x^+, J)/\mathbb{R}$. When $\mu_{CZ}(x^+) - \mu_{CZ}(x^-) = 1$, the set $\widehat{\mathcal{M}}_H(x^-, x^+, J)$ is finite and we denote the number, mod 2, of points in this set by $\#\big(\widehat{\mathcal{M}}_H(x, y, J)\big)$.

Let $a < b$ be two points outside $\mathcal{S}(H)$. Denote by $\mathrm{CF}_k^{(a,b)}(H)$ the vector space over $\mathbb{Z}_2$ generated by one-periodic orbits of H with $\mu_{CZ}(x) = k$ and $a < A_H(x) < b$. The Floer differential

$$\partial\colon \mathrm{CF}_k^{(a,b)}(H) \to \mathrm{CF}_{k-1}^{(a,b)}(H)$$

is defined by

$$\partial x = \sum_y \#\big(\widehat{\mathcal{M}}_H(x, y, J)\big) \cdot y,$$

where the summation is over all y such that $\mu_{CZ}(y) = k-1$ and $a < A_H(y) < b$. As is well-known, $\partial^2 = 0$. The homology $\mathrm{HF}_*^{(a,b)}(H)$ of the resulting complex is called the *filtered Floer homology* of H for the interval (a, b). Thus, $\mathrm{HF}_*(H) := \mathrm{HF}_*^{(-\infty,\infty)}(H)$ is the ordinary Floer homology. It is a standard fact that $\mathrm{HF}_*(H) = \mathrm{H}_{*+n}(W; \mathbb{Z}_2)$. In general, $\mathrm{HF}_*^{(a,b)}(H)$ depends on the Hamiltonian H, but not on J.

The subcomplexes $\mathrm{CF}_*^{(-\infty,b)}(H)$, where $b \in \mathbb{R} \smallsetminus \mathcal{S}(H)$, form a filtration of the total Floer complex $\mathrm{CF}_*(H) := \mathrm{CF}_*^{(-\infty,\infty)}(H)$, called the *action filtration*, and $\mathrm{CF}_*^{(a,b)}(H)$ can be identified with the complex $\mathrm{CF}_*^{(-\infty,b)}(H)/\mathrm{CF}_*^{(-\infty,a)}(H)$; see, e.g., [FH94], [Sch00]. Let now $a < b < c$ be three points outside $\mathcal{S}(H)$. Then, similarly, $\mathrm{CF}_*^{(a,b)}(H)$ is a subcomplex of $\mathrm{CF}_*^{(a,c)}(H)$, and $\mathrm{CF}_*^{(b,c)}(H)$ is naturally isomorphic to $\mathrm{CF}_*^{(a,c)}(H)/\mathrm{CF}_*^{(a,b)}(H)$. As a result, we have the long exact sequence

$$(2.2)\quad \cdots \to \mathrm{HF}_*^{(a,b)}(H) \to \mathrm{HF}_*^{(a,c)}(H) \to \mathrm{HF}_*^{(b,c)}(H) \to \mathrm{HF}_{*-1}^{(a,b)}(H) \to \dots .$$

The filtered Floer homology of H is defined even when the periodic orbits of H are not necessarily nondegenerate, provided that $a < b$ are outside $\mathcal{S}(H)$. Namely, let $\tilde{H}$ be a C^2-small perturbation[1] of H with nondegenerate one-periodic orbits. The filtered Floer homology $\mathrm{HF}_*^{(a,b)}(H)$ of H is by definition $\mathrm{HF}_*^{(a,b)}(\tilde{H})$. (Clearly, $a < b$ are still outside $\mathcal{S}(\tilde{H})$.) These groups are canonically isomorphic for different choices of $\tilde{H}$ (close to H) and the results discussed here hold for $\mathrm{HF}_*^{(a,b)}(H)$; see, e.g., [BPS03], [FH94], [FHW94], [Sch00], [Vit99]. In fact, when an assertion concerns individual Hamiltonians (as opposed to families of Hamiltonians), it is usually sufficient to prove the assertion in the nondegenerate case, for then it extends "by continuity" to all Hamiltonians.

These constructions carry over to T-periodic Hamiltonians word-for-word by replacing one-periodic orbits with T-periodic ones. When H is one-periodic, but we treat it as T-periodic for some integer $T > 0$, we denote the resulting Floer homology groups $\mathrm{HF}_*^{(a,b)}\big(H^{(T)}\big)$.

2.2.2. *Homotopy maps.* Consider two nondegenerate Hamiltonians H^0 and H^1. Let H^s be a homotopy from H^0 to H^1. By definition, this is a family of Hamiltonians parametrized by $s \in \mathbb{R}$ such that $H^s \equiv H^0$ when s is large negative and $H^s \equiv H^1$ when s is large positive. (Strictly speaking, the notion of homotopy includes also a family of almost complex structures J_s; see, e.g., [BPS03], [FH94], [FHW94], [Sal99]. We suppress this part of the homotopy structure in the notation.) Assume, in addition, that the homotopy is *monotone decreasing*, i.e., $H_t^s(p)$ is a decreasing function of s for all $p \in W$ and $t \in S^1$. (Thus, in particular, $H^0 \geq H^1$.) Then, whenever $a < b$ are outside $\mathcal{S}(H^0)$ and $\mathcal{S}(H^1)$, the homotopy H^s induces a homomorphism of complexes $\Psi_{H^0,H^1}\colon \mathrm{CF}_*^{(a,b)}(H^0) \to \mathrm{CF}_*^{(a,b)}(H^1)$ by the standard continuation construction; see, e.g., [BPS03], [FHW94], [FH94], [Sch00]. Namely, for a one-periodic orbit x of H^0 and a one-periodic orbit y of H^1, let $\mathcal{M}_H(x, y, J)$ be the space of solutions of (2.1) with H^s on the right hand side, asymptotic to x and, respectively, y at $\pm\infty$. Under the well-known regularity requirements on J and H^s, the space $\mathcal{M}_{H^s}(x, y, J)$ is a smooth manifold of dimension $\mu_{\mathrm{CZ}}(y) - \mu_{\mathrm{CZ}}(x)$; see, e.g., [FH94], [Sal99], [SZ92], [Sch00], [Sch93]. Moreover, $\mathcal{M}_{H^s}(x, y, J)$ is a finite collection of points when $\mu_{\mathrm{CZ}}(y) = \mu_{\mathrm{CZ}}(x)$. The map Ψ_{H^0,H^1} is defined by

$$(2.3)\qquad \Psi_{H^0,H^1}(x) = \sum_y \#\big(\mathcal{M}_H(x, y, J)\big) \cdot y,$$

where the summation is over all y such that $\mu_{\mathrm{CZ}}(y) = \mu_{\mathrm{CZ}}(x)$ and $a < A_H(y) < b$.

[1] For the sake of brevity, we refer to the function $\tilde{H}$, rather than the difference $\tilde{H} - H$, as a perturbation of H. However, it is the difference $\tilde{H} - H$ that is required to be C^2-small.

The induced *homotopy map* in the filtered Floer homology, also denoted by Ψ_{H^0,H^1}, is independent of the decreasing homotopy H^s and commutes with the maps from the long exact sequence (2.2); see, e.g., [BPS03], [FH94], [Sal99], [SZ92], [Sch93], [Sch00]. By "continuity" in the Hamiltonians and the homotopy, this construction extends to all (not necessarily nondegenerate) Hamiltonians and all decreasing (but not necessarily regular) homotopies as long as $a < b$ are not in $\mathcal{S}(H^0)$ and $\mathcal{S}(H^1)$.

A (nonmonotone) homotopy H^s from H^0 to H^1 with a and b outside $\mathcal{S}(H^s)$ for all s gives rise to an isomorphism between the groups $\mathrm{HF}_*^{(a,b)}(H^s)$, and hence, in particular,

$$\mathrm{HF}_*^{(a,b)}(H^0) \cong \mathrm{HF}_*^{(a,b)}(H^1); \tag{2.4}$$

see [BPS03], [Vit99]. This isomorphism is defined by breaking the homotopy H^s into a composition of homotopies $H^{i,s}$ close to the identity. Each of the homotopies $H^{i,s}$ and its inverse homotopy increase action by no more than some small $\varepsilon > 0$. Then, it is shown that the map in $\mathrm{HF}_*^{(a,b)}$ induced by $H^{i,s}$ is an isomorphism. Although this construction requires additional choices, it is not hard to see that the isomorphism (2.4) is uniquely determined by the homotopy H^s. Furthermore, (2.4) commutes with the maps from the long exact sequence (2.2), provided that all three points $a < b < c$ are outside $\mathcal{S}(H^s)$ for all s; see [Gin07]. Note also that when H^s is a decreasing homotopy, the isomorphism (2.4) coincides with Ψ_{H^0,H^1}.

Example 2.4. A homotopy H^s is said to be *isospectral* if $\mathcal{S}(H^s)$ is independent of s. In this case, the isomorphism (2.4) is defined for any $a < b$ outside $\mathcal{S}(H^s)$.

For instance, let ψ_s^t, where $t \in S^1$ and $s \in [0, 1]$, be a family of loops of Hamiltonian diffeomorphisms based at id, i.e., $\psi_s^0 = \mathrm{id}$ for all s. In other words, ψ_s^t is a based homotopy from the loop ψ_0^t to the loop ψ_1^t. Let G_t^s be a family of one-periodic Hamiltonians generating these loops and let H be a fixed one-periodic Hamiltonian. Then $H^s := G^s \# H$ is an isospectral homotopy, provided that G^s are suitably normalized. (Namely, $A(G^s) = 0$ for all s; see §2.3.)

It is easy to see that if $K \geq H^s$ for all s, the isomorphism (2.4) intertwines monotone homotopy homomorphisms from K to H^0 and to H^1, i.e., the diagram

$$\begin{tikzcd} \mathrm{HF}_*^{(a,b)}(K) \arrow[d, "\Psi_{K,H^0}"'] \arrow[dr, "\Psi_{K,H^1}"] & \\ \mathrm{HF}_*^{(a,b)}(H^0) \arrow[r, "\cong"] & \mathrm{HF}_*^{(a,b)}(H^1) \end{tikzcd} \tag{2.5}$$

is commutative. Note that it is not at all clear whether the same is true if we only require that $K \geq H^0$ and $K \geq H^1$.

2.2.3. *Nontriviality criterion for a homotopy map.* We conclude this section by establishing a technical result, used later in the proof, giving a criterion for a monotone homotopy map to be nonzero.

LEMMA 2.5. *Let H^s be a monotone decreasing homotopy such that a point p is a nondegenerate constant one-periodic orbit of H^s and $H_t^s(p) = c$ for all s and t. Then the monotone homotopy map*

$$\Psi_{H^0,H^1}\colon \mathrm{HF}_*^{(a,\,b)}(H^0) \to \mathrm{HF}_*^{(a,\,b)}(H^1)$$

is nontrivial, provided that $\mathcal{S}(H^0) \cap (a,\,b) = \{c\} = \mathcal{S}(H^1) \cap (a,\,b)$.

Remark 2.6. In fact, we will prove a stronger result. Let us perturb H^0 and H^1 away from p, making these Hamiltonians nondegenerate. Then p is a cycle in $\mathrm{CF}_*^{(a,\,b)}(H^0)$ and $\mathrm{CF}_*^{(a,\,b)}(H^1)$ and, moreover, this cycle is not homologous to any cycle that does not include p. (This is easy to see from the energy estimates; see, e.g., [Sal90], [Sal99].) In particular, $[p] \neq 0$ in $\mathrm{HF}_*^{(a,\,b)}(H^0)$ and $\mathrm{HF}_*^{(a,\,b)}(H^1)$ and we will show that Ψ_{H^0,H^1} sends $[p] \in \mathrm{HF}_*^{(a,\,b)}(H^0)$ to $[p] \in \mathrm{HF}_*^{(a,\,b)}(H^1)$.

Moreover, a simple modification of our argument proves the following: There exist C^2-small nondegenerate perturbations $\widehat{H}^0$ of H^0 and $\widehat{H}^1$ of H^1 for which p is still a nondegenerate constant one-periodic orbit, and a regular monotone decreasing homotopy $\widehat{H}^s$ from $\widehat{H}^0$ to $\widehat{H}^1$ such that the cycle p for $\widehat{H}^0$ is connected to p for $\widehat{H}^1$ by an odd number of homotopy trajectories and all such trajectories are contained in a small neighborhood of p. (Note that we do not assume that p is a nondegenerate constant one-periodic orbit of $\widehat{H}^s$ for all s or that $\widehat{H}^0(p) = \widehat{H}^1(p)$.) Of course, the lemma can be further generalized. For instance, the constant orbit p can be replaced by a fixed one-periodic orbit.

Proof of Lemma 2.5. Let us perturb the homotopy on the complement of a neighborhood U of p, keeping the homotopy monotone decreasing, to ensure that all but a finite number of the Hamiltonians H^s are nondegenerate. In particular, we will assume that H^0 and H^1 are such. This can be achieved by an arbitrarily small perturbation of H^s. We keep the notation H^s for the perturbed homotopy.

If the homotopy H^s were regular, we would simply argue that the constant connecting trajectory $u \equiv p$ is the only connecting trajectory from p for H^0 to p for H^1. Indeed, $0 \leq E(v) = A_{H^0}(p) - A_{H^1}(p) = 0$ for any such connecting trajectory v, and thus v must be constant. However, while it is easy to guarantee that H^s is regular away from p by reasoning as in, e.g., [FH94], [FHS95], it is not *a priori* obvious that the transversality requirements can be satisfied for $u \equiv p$ because of the constraint $H_t^s(p) = c$. Rather than checking regularity of u by a direct calculation, we chose to circumvent this difficulty.

As in the proof of (2.4) (see, e.g., [BPS03], [Gin07], [Vit99]), we can break the homotopy H^s by reparametrization of s into a composition of homotopies $K^{i,s}$ from $K^i := H^{s_i}$ to $K^{i+1} := H^{s_{i+1}}$ with $K^0 = H^0$ and $K^k = H^1$. These homotopies are monotone, since H^s is monotone, and close to the identity homotopy. For every $\varepsilon > 0$, this can be done so that the inverse homotopy Ψ_{K^{i+1},K^i} from K^{i+1} to K^i increases the actions by no more than $\varepsilon > 0$. Without loss of generality, we may assume that the Hamiltonians K^i are nondegenerate. Since all "direct" homotopies are monotone decreasing, we have

$$\Psi_{H^0,H^1} = \Psi_{K^{k-1},K^k} \circ \cdots \circ \Psi_{K^1,K^2} \circ \Psi_{K^0,K^1}.$$

Observe that it suffices to establish the lemma for a and b arbitrarily close to c. Let U be so small that p is the only one-periodic trajectory of H^s entering U for all s. (Since p is isolated for all H^s, such a neighborhood U exists.) There exists a constant $\varepsilon_U > 0$ such that every Floer anti-gradient trajectory v connecting p with any other one-periodic orbit with action in the range (a, b) has energy $E(v) > \varepsilon_U$ for any regular Hamiltonian in the family H^s; cf. [Sal90], [Sal99]. In particular, this holds for K^i and K^{i+1} and, moreover, for every regular Hamiltonian in the homotopy $K^{i,s}$. We will pick a and b so that $c - \varepsilon_U < a$ and $b < c + \varepsilon_U$. Then, for every K^i the point p is a cycle in $\mathrm{CF}_*^{(a,b)}(K^i)$ and p is not homologous to any cycle that does not include p.

Now it is sufficient to prove that $\Psi_{K^i,K^{i+1}}$ sends $[p] \in \mathrm{HF}_*^{(a,b)}(K^i)$ to $[p] \in \mathrm{HF}_*^{(a,b)}(K^{i+1})$. To this end, let us first prove that $\Psi_{K^i,K^{i+1}}([p]) \neq 0$. We may assume that none of the points a, $a+\varepsilon$, b, $b+\varepsilon$ is in $\mathcal{S}(K^i)$ or in $\mathcal{S}(K^{i+1})$.

It is easy to see (see, e.g., [Gin07]) that

$$\Psi_{K^{i+1},K^i} \circ \Psi_{K^i,K^{i+1}} \colon \mathrm{HF}_*^{(a,b)}(K^i) \to \mathrm{HF}_*^{(a+\varepsilon,b+\varepsilon)}(K^i) \tag{2.6}$$

is the natural "quotient-inclusion" map, i.e., the composition of the "quotient" and "inclusion" maps $\mathrm{HF}_*^{(a,b)}(K^i) \to \mathrm{HF}_*^{(a+\varepsilon,b)}(K^i) \to \mathrm{HF}_*^{(a+\varepsilon,b+\varepsilon)}(K^i)$. Note that ε_U is completely determined by H^s and U and is independent of how H^s is broken into the homotopies $K^{i,s}$, and thus of $\varepsilon > 0$. Pick $\varepsilon > 0$ so that $a + \varepsilon < c$ and $b + \varepsilon < c + \varepsilon_U$. Then p is a cycle in $\mathrm{CF}_*^{(a+\varepsilon,b+\varepsilon)}(K^i)$, which is not homologous to any cycle that does not include p. As a consequence, $[p] \neq 0$ in both of the Floer homology groups in (2.6) and $\Psi_{K^{i+1},K^i} \circ \Psi_{K^i,K^{i+1}}([p]) = [p]$. Therefore, $\Psi_{K^i,K^{i+1}}([p]) \neq 0$ in $\mathrm{HF}_*^{(a,b)}(K^{i+1})$.

To show that $\Psi_{K^i,K^{i+1}}([p]) = [p]$, we need to refine our choice of ε_U. Note that there exists $\varepsilon_U > 0$ such that, in addition to the above requirements, every $K^{i,s}$-homotopy trajectory starting at p and leaving U has energy greater than ε_U. Then, clearly, the same is true for every sufficiently C^2-small perturbation $\widehat{K}^{i,s}$ of $K^{i,s}$. Again, ε_U depends only on H^s and U, but not on breaking H^s into the homotopies $K^{i,s}$. (The existence of $\varepsilon_U > 0$ with these properties readily follows from energy estimates for connecting trajectories; cf. [Sal90], [Sal99].) Pick a C^2-small regular perturbation $\widehat{K}^{i,s}$ of $K^{i,s}$. We may still assume that $\widehat{K}^{i,s}$ is monotone decreasing and p is a nondegenerate constant orbit of $\widehat{K}^i$ and $\widehat{K}^{i+1}$. However, p is not required to be a constant one-periodic orbit of $\widehat{K}^{i,s}$ for all s; nor is $\widehat{K}_t^{i,s}(p)$

constant as a function of s and t. Clearly, the inverse homotopy to $\widehat{K}^{i,s}$ does not increase action by more than $\varepsilon > 0$. Hence, in the homological analysis of the previous paragraph we can replace K^i and K^{i+1} by $\widehat{K}^i$ and $\widehat{K}^{i+1}$. In fact, the original and perturbed Hamiltonians have equal filtered Floer homology for relevant action intervals and the maps $\Psi_{K^i,K^{i+1}}$ and Ψ_{K^{i+1},K^i} are induced by the maps for $\widehat{K}^{i,s}$ acting on the level of complexes. Therefore, $\Psi_{\widehat{K}^i,\widehat{K}^{i+1}}([p]) = \Psi_{K^i,K^{i+1}}([p]) \neq 0$ in $\mathrm{HF}_*^{(a,b)}(\widehat{K}^{i+1}) = \mathrm{HF}_*^{(a,b)}(K^{i+1})$, and thus $\Psi_{\widehat{K}^i,\widehat{K}^{i+1}}(p) \neq 0$ in $\mathrm{CF}_*^{(a,b)}(\widehat{K}^{i+1})$. Since $c-\varepsilon_U < a$ and $c+\varepsilon_U < b$ and every connecting orbit leaving U must have energy greater than ε_U, we conclude that $\Psi_{\widehat{K}^i,\widehat{K}^{i+1}}(p) = p$ in the Floer complexes, and hence in the Floer homology. □

2.3. *Loops of Hamiltonian diffeomorphisms.* In this section, we recall a few well-known facts about loops of Hamiltonian diffeomorphisms of W. We will focus on loops parametrized by S^1, but obviously all results discussed here hold for loops of any period. Furthermore, throughout the paper all loops $\psi = \psi^t$ are assumed to be based at id, i.e., $\psi^1 = \psi^0 = \mathrm{id}$; contractible loops are thus required to be contractible in this class.

Recall that, as is proved in [Sch00], the filtered Floer homology of the Hamiltonian H is determined, up to a shift of filtration, entirely by the time-one map φ_H and is independent of the Hamiltonian H. This fact translates to geometric properties of loops of Hamiltonian diffeomorphisms, which are briefly reviewed below, and is actually proved by first establishing these properties.

2.3.1. *Global loops.* Let $\psi^t = \varphi_G^t$, $t \in S^1$, be a loop generated by a periodic Hamiltonian G. Then all orbits $\gamma(t) = \psi^t(p)$ of ψ^t with $t \in S^1$ and $p \in W$ are one-periodic and lie in the same homotopy class. Hence, every orbit of G is contractible by the Arnold conjecture. The action $A(G) := A_G(\gamma)$ is independent of $p \in W$ (see, e.g., [HZ94], [Sch00]) and $A(G) = \mathrm{vol}(W)^{-1}\int_0^1\int_W G\,\omega^n\,dt$, where $\mathrm{vol}(W)$ is the symplectic volume of W. The latter identity is easy to prove when the loop ψ^t is contractible (see, e.g., [Gin07], [Sch00]); in the general case, this is a nontrivial result, [Sch00].

For γ as above pick a trivialization of $TW|_\gamma$ that extends to a trivialization of TW along a disk bounded by γ. Using this trivialization, we can view the maps $d\psi^t\colon T_{\gamma(0)}W \to T_{\gamma(t)}W$ as a loop in $\mathrm{Sp}(T_{\gamma(0)}W)$. Hence, the linearization $d\psi^t$ along γ gives rise to an element in $\pi_1(\mathrm{Sp}(T_{\gamma(0)}W)) = \mathbb{Z}$, which could be called the Maslov index, $\mu(\psi)$, of the loop ψ^t if it were nontrivial; cf. [SZ92], [Sal99]. The Maslov index is well-defined: it is independent of γ, the trivialization, and the disc. (The latter follows from the fact that $c_1(W)|_{\pi_2(W)} = 0$.) However, as is well-known and as we will soon reprove, $\mu(\psi) = 0$; see also, e.g., [Sch00].

Let H be a periodic Hamiltonian on W. Recall that $G\#H$ is the Hamiltonian generating the flow $\varphi_G^t\varphi_H^t$. This Hamiltonian is automatically one-periodic and its time-one map is φ_H. We claim that there exists an isomorphism of filtered Floer homology

$$\mathrm{HF}_*^{(a,b)}(H) \cong \mathrm{HF}_*^{(a+A(G),\, b+A(G))}(G\#H). \tag{2.7}$$

Indeed, composition with $\psi^t = \varphi_G^t$ sends one-periodic orbits of H to one-periodic orbits of $G\#H$ with shift of action by $A(G)$ and shift of Conley-Zehnder indices by $-2\mu(\psi)$. (See, e.g., [Sal99], [Sch00]; the negative sign is a result of the difference in conventions.) Furthermore, let u be a Floer anti-gradient trajectory for H and a time-dependent almost complex structure J. Then, as a straightforward calculation shows, $\psi(u)(s,t) := \psi^t(u(s,t))$ is a Floer anti-gradient trajectory for $G\#H$ and the almost complex structure $\tilde{J}_t := d\psi^t \circ J_t \circ (d\psi^t)^{-1}$. Furthermore, it is clear that the transversality requirements are satisfied for (H, J) if and only if they are satisfied for $(G\#H, \tilde{J})$. Therefore, the composition with ψ commutes with the Floer differential and thus induces an isomorphism of Floer complexes (and hence homology groups) shifting action by $A(G)$ and grading by $-2\mu(\psi)$. Applying this construction to the full Floer homology $\mathrm{HF}_*(H) \cong H_{*+n}(W) \cong \mathrm{HF}_*(G\#H)$, we see that the grading shift must be zero, i.e., $\mu(\psi) = 0$.

Remark 2.7. When the loop ψ is contractible, the existence of an isomorphism (2.7) readily follows from (2.4); see Example 2.4. However, it is not clear whether this is the same isomorphism as constructed above.

2.3.2. *Local loops.* Let now ψ^t be a loop of (the germs of) Hamiltonian diffeomorphisms at $p \in W$ generated by G. In other words, the maps ψ^t and the Hamiltonian G are defined on a small neighborhood of p and $\psi^t(p) = p$ for all $t \in S^1$. Then the action $A(G)$ and the Maslov index $\mu(\psi)$ are introduced exactly as above with the orbit γ taken sufficiently close to p. In fact, we can set $\gamma \equiv p$ and hence $A(G) = \int_0^1 G_t(p)\, dt$ and $\mu(\psi)$ is just the Maslov index of the loop $d\psi_p^t$ in $\mathrm{Sp}(T_pW)$. Note that in this case $\mu(\psi)$ need not be zero.

We conclude this section by giving a necessary and sufficient condition, to be used later, for ψ to extend to a loop of global Hamiltonian diffeomorphisms of W.

LEMMA 2.8. *Let ψ^t, $t \in S^1$, be a loop of germs of Hamiltonian diffeomorphisms at $p \in W$. The following conditions are equivalent*:

(i) *the loop ψ extends to a loop of global Hamiltonian diffeomorphisms of W*,

(ii) *the loop ψ extends to a loop of global Hamiltonian diffeomorphisms of W, contractible in the class of loops fixing p*,

(iii) *the loop ψ is contractible in the group of germs of Hamiltonian diffeomorphisms at p*,

(iv) $\mu(\psi) = 0$.

Proof. The implications (ii)⇒(i) and (iii)⇒(iv) are clear and (i)⇒(iv) is established above.

To prove that (iv)⇒(iii), we identify a neighborhood of p in W with a neighborhood of the origin in $\mathbb{R}^{2n}$. Then, as is easy to see, the loop ψ^t is homotopy equivalent to its linearization $d\psi^t$, a loop of (germs of) linear maps. By the definition of the Maslov index, $d\psi^t$ is contractible in $\mathrm{Sp}(\mathbb{R}^{2n})$ if and only if $\mu(\psi) := \mu(d\psi) = 0$.

To complete the proof of the lemma, it remains to show that (iii)⇒(ii).

To this end, let us first analyze the case where ψ^t is C^1-close to the identity. Fixing a small neighborhood U of p, we identify a neighborhood of the diagonal in $U \times U$ with a neighborhood of the zero section in T^*U. Then the graphs of ψ^t in $U \times U$ turn into Lagrangian sections of T^*U. These sections are the graphs of exact forms df_t on U, where all f_t are C^2-small and $f_0 \equiv 0 \equiv f_1$. Then we extend (the germs of) the functions f_t to C^2-small functions $\tilde{f}_t$ on W such that $\tilde{f}_0 \equiv 0 \equiv \tilde{f}_1$. The graphs of $d\tilde{f}_t$ in T^*W form a loop of exact Lagrangian submanifolds which are C^1-close to the zero section. Thus, this loop can be viewed as a loop of Hamiltonian diffeomorphisms of W. It is clear that the resulting loop is contractible in the class of loops fixing p.

To deal with the general case, consider a family ψ_s, $s \in [0, 1]$, of local loops with $\psi_0 \equiv \mathrm{id}$ and $\psi_1^t = \psi^t$. Let $0 = s_0 < s_1 < \cdots < s_k = 1$ be a partition of the interval $[0, 1]$ such that the loops ψ_{s_i} and $\psi_{s_{i+1}}$ are C^1-close for all $i = 0, \dots, k-1$. In particular, the loop ψ_{s_1} is C^1-close to $\psi_0 = \mathrm{id}$, and thus extends to a contractible loop $\widetilde{\psi}_{s_1}$ on W. Arguing inductively, assume that a contractible extension $\widetilde{\psi}_{s_i}$ of ψ_{s_i} has been constructed. Consider the loop $\eta^t = \psi^t_{s_{i+1}}(\psi^t_{s_i})^{-1}$ defined near p. This loop is C^1-close to the identity, for $\psi_{s_{i+1}}$ and ψ_{s_i} are C^1-close. Hence, η extends to a contractible loop $\tilde{\eta}$ on W. Then $\widetilde{\psi}^t_{s_{i+1}} := \tilde{\eta}^t \widetilde{\psi}^t_{s_i}$ is the required extension of $\psi_{s_{i+1}}$, contractible in the class of loops fixing p. □

Remark 2.9. It is clear from the proof of Lemma 2.8 that the extension of the germ of a loop near p to a global loop fixing p can be carried out with some degree of control of the C^k-norm and the support of the loop. We will need the following simple fact, which can be easily verified by adapting the proof of the implication (iii)⇒(ii).

Assume that ψ is the germ of a loop near p and the linearization of ψ at p is equal to the identity: $d\psi^t_p = I$ for all t. Then ψ extends to a loop $\widetilde{\psi}$ of global Hamiltonian diffeomorphisms of W such that $\widetilde{\psi}$ is contractible in the class of loops fixing p and having identity linearization at p.

3. **Local Floer homology**

3.1. *Local Morse homology.* Let $f\colon M^m \to \mathbb{R}$ be a smooth function on a manifold M and let $p \in M$ be an isolated critical point of f. Fix a small neighborhood U of p containing no other critical points of f and consider a small generic perturbation $\tilde{f}$ of f in U. To be more precise, $\tilde{f}$ is Morse inside U and C^1-close to f. Then, as is easy to see, for any two critical points of $\tilde{f}$ in U, all anti-gradient trajectories connecting these two points are contained in U. Moreover, the same is true for broken trajectories connecting these two points. As a consequence, the vector space (over $\mathbb{Z}_2$) generated by the critical points of $\tilde{f}$ in U is a complex with (Morse) differential defined in the standard way. (See, e.g., [Sch93].) Furthermore, the continuation argument shows that the homology of this complex, denoted here by $\mathrm{HM}_*^{\mathrm{loc}}(f, p)$ and referred to as the *local Morse homology* of f at p, is independent of the choice of $\tilde{f}$. This construction is a particular case of the one from [Flo89c].

Example 3.1. Assume that p is a nondegenerate critical point of f of index k. Then $\mathrm{HM}_*^{\mathrm{loc}}(f, p) = \mathbb{Z}_2$ when $* = k$ and $\mathrm{HM}_*^{\mathrm{loc}}(f, p) = 0$ otherwise.

Example 3.2. When p is a strict local maximum of f, we have $\mathrm{HM}_m^{\mathrm{loc}}(f, p) = \mathbb{Z}_2$. Indeed, in this case, as is easy to see from standard Morse theory,

$$\mathrm{HM}_m^{\mathrm{loc}}(f, p) = \mathrm{H}_m(\{f \geq f(p) - \varepsilon\}, \{f = f(p) - \varepsilon\}) = \mathbb{Z}_2,$$

where $\varepsilon > 0$ is assumed to be small and such that $f(p) - \varepsilon$ is a regular value of f.

We will need the following two properties of local Morse homology:

(LM1) Let f_s, $s \in [0, 1]$, be a family of smooth functions with *uniformly isolated* critical point p, i.e., p is the only critical point of f_s, for all s, in some neighborhood of p independent of s. Then $\mathrm{HM}_*^{\mathrm{loc}}(f_s, p)$ is constant throughout the family, and hence $\mathrm{HM}_*^{\mathrm{loc}}(f_0, p) = \mathrm{HM}_*^{\mathrm{loc}}(f_1, p)$; cf. [GM69, Lemma 4].

(LM2) The function f has a (strict) local maximum at p if and only if $\mathrm{HM}_m^{\mathrm{loc}}(f, p) \neq 0$, where $m = \dim M$.

The first assertion, (LM1), is again established by the continuation argument; cf. [Sch93]. We emphasize that here the assumption that p is uniformly isolated is essential and cannot be replaced by the weaker condition that p is just an isolated critical point of f_s for all s. (Example: $f_s(x) = sx^2 + (1-s)x^3$ on $\mathbb{R}$ with $p = 0$. The author is grateful to Doris Hein for this remark.)

Regarding (LM2) first note that, by Example 3.2, $\mathrm{HM}_m^{\mathrm{loc}}(f, p) \neq 0$ when f has a strict local maximum at p. The converse requires a proof although the argument is quite standard.

Proof of the implication ($\Leftarrow$) *in* (LM2). Denote by ψ^t the anti-gradient flow of f. Let B be a closed connected neighborhood of p with piecewise smooth boundary ∂B such that whenever $x \in B$ and $\psi^t(x) \in B$ the entire trajectory segment $\psi^\tau(x)$ with $\tau \in [0, t]$ is also in B, and p is the only critical point of f contained in B. We call B a Gromoll-Meyer neighborhood of p. It is not hard to show that p has an (arbitrarily small) Gromoll-Meyer neighborhood; see [Cha93, pp. 49–50] or [GM69]. (Strictly speaking, the above definition is slightly different from the one used in [Cha93]. However, the existence proof given in [Cha93, pp. 49–50] goes through with no modifications.) When $\tilde{f}$ is a C^2-small generic perturbation of f supported in B, the Morse complex of $\tilde{f}|_B$ is defined and its homology is equal to $\mathrm{HM}_*^{\mathrm{loc}}(f, p)$. (The fact that $\partial^2 = 0$ follows from the requirements on B.) Assume that $\mathrm{HM}_m^{\mathrm{loc}}(f, p) \neq 0$. Then there exists a nonzero cycle C of degree m in the Morse complex of $\tilde{f}|_B$. Let V be the closure of the union of the unstable manifolds of $\tilde{f}|_B$ for all local maxima entering C. The set V is the closure of a domain with piecewise smooth boundary. The condition that C is a cycle implies that for every critical point x of $\tilde{f}$ in V, the intersection of the unstable manifold of x with B is contained entirely in the interior of V. Hence, $\partial V \subset \partial B$, and thus $B = V$. It follows that at every smooth point $z \in \partial B$, the gradient $\nabla \tilde{f}(z) = \nabla f(z)$ either points inward or is tangent to ∂B.

Consider a Gromoll-Meyer neighborhood N of p. Note that for a small generic $\varepsilon > 0$ the connected component B of $N \cap \{f(p) - \varepsilon \leq f \leq f(p) + \varepsilon\}$ containing p is also a Gromoll-Meyer neighborhood. Clearly, when p is not a local maximum of f, there are smooth points on ∂B where ∇f points inward, provided that $\varepsilon > 0$ is small. As a consequence of the above analysis, $\mathrm{HM}_m^{\mathrm{loc}}(f, p) = 0$ if p is not a local maximum. This completes the proof of the implication ($\Leftarrow$). $\square$

Remark 3.3. Generalizing Example 3.2 and the proof of (LM2), it is not hard to relate local Morse homology to local homology of a function, introduced in [Mor96]; see also [Cha93], [GM69]. However, we do not touch upon this question, for such a generalization is not necessary for the proof of Theorem 1.1. In the setting of local homology, the analogues of (LM1) and (LM2) are established in [Cha93], [GM69] and, respectively, in [Hin93], [Hin09].

3.2. *Local Floer homology*: *the definition and basic properties.* Let γ be an isolated one-periodic orbit of a Hamiltonian $H\colon S^1 \times W \to \mathbb{R}$. Pick a sufficiently small tubular neighborhood U of γ and consider a nondegenerate C^2-small perturbation $\widetilde{H}$ of H supported in U. More specifically, let U be a neighborhood of $\gamma(S^1)$, where γ is viewed as a curve in the extended phase space $S^1 \times W$, and let $\widetilde{H}$ be a Hamiltonian C^2-close to H, equal to H outside of U, and such that all one-periodic orbits of $\widetilde{H}$ that enter U are nondegenerate. (Such perturbations $\widetilde{H}$ do exist; see [SZ92, Theorem 9.1].) Abusing notation, we will treat U simultaneously as an open set in W and in $S^1 \times W$.

Consider one-periodic orbits of $\widetilde{H}$ contained in U. Every anti-gradient trajectory u connecting two such orbits is also contained in U, provided that $\|\widetilde{H} - H\|_{C^2}$ and $\operatorname{supp}(\widetilde{H} - H)$ are small enough. Indeed, the energy $E(u)$ is equal to the difference of action values on the periodic orbits, and thus is bounded from above by $O(\|\widetilde{H} - H\|_{C^2})$. The C^2-norm of $\widetilde{H}$ is bounded from above by a constant independent of $\widetilde{H}$, say $2\|H\|_{C^2}$. Therefore, $|\partial_s u|$ is pointwise uniformly bounded by $O(\|\widetilde{H} - H\|_{C^2})$, and it follows that u takes values in U; see [Sal90], [Sal99]. Note also that for a suitable small perturbation of a fixed almost complex structure on W the transversality requirements are satisfied for moduli spaces of Floer anti-gradient trajectories connecting one-periodic orbits $\widetilde{H}$ contained in U; see [FHS95], [SZ92].

By the compactness theorem, every broken anti-gradient trajectory u connecting two one-periodic orbits in U lies entirely in U. Hence, the vector space (over $\mathbb{Z}_2$) generated by one-periodic orbits of $\widetilde{H}$ in U is a complex with (Floer) differential defined in the standard way. The continuation argument (see, e.g., [SZ92]) shows that the homology of this complex is independent of the choice of $\widetilde{H}$ and of the almost complex structure. We refer to the resulting homology group $\mathrm{HF}^{\mathrm{loc}}_*(H,\gamma)$ as the *local Floer homology* of H at γ. Homology groups of this type were first considered (in a more general setting) by Floer in [Flo89c], [Flo89b]; see also [Poź99, §3.3.4]. In fact, an orbit γ can be replaced by a connected isolated set Γ of one-periodic orbits of H; see [Flo89c], [Flo89b], [Poź99]. (Note that $A_H|_\Gamma$ is constant, for A_H is continuous and $\mathcal{S}(H)$ is nowhere dense.)

Example 3.4. Assume γ is nondegenerate and $\mu_{\mathrm{CZ}}(\gamma)=k$. Then $\mathrm{HF}^{\mathrm{loc}}_*(H,\gamma) = \mathbb{Z}_2$ when $*=k$ and $\mathrm{HF}^{\mathrm{loc}}_*(H,\gamma)=0$ otherwise.

We will need the following properties of local Floer homology:

(LF1) Let H^s, $s\in[0,\,1]$, be a family of Hamiltonians such that γ is a *uniformly isolated* one-periodic orbit for H^s, i.e., γ is the only periodic orbit of H^s, for all s, in some open set independent of s. Then $\mathrm{HF}^{\mathrm{loc}}_*(H^s,\gamma)$ is constant throughout the family, and hence $\mathrm{HF}^{\mathrm{loc}}_*(H^0,\gamma)=\mathrm{HF}^{\mathrm{loc}}_*(H^1,\gamma)$.

This is again an immediate consequence of the continuation argument. However, it is worth pointing out that unless H^s is monotone decreasing, the isomorphism $\mathrm{HF}^{\mathrm{loc}}_*(H^0,\gamma)=\mathrm{HF}^{\mathrm{loc}}_*(H^1,\gamma)$ is not induced by the homotopy H^s in the same sense as the homomorphism Ψ_{H^0,H^1} is induced by a monotone homotopy; see (2.3). The isomorphism in question is constructed similarly to (2.4) by breaking H^s into a composition of homotopies close to the identity.

Local Floer homology spaces are building blocks for filtered Floer homology. Namely, essentially by definition, we have the following

(LF2) Let $c\in\mathbb{R}$ be such that all one-periodic orbits γ_i of H with action c are isolated. (As a consequence, there are only finitely many such orbits.) Then, if $\varepsilon>0$ is small enough,

$$\mathrm{HF}^{(c-\varepsilon,\,c+\varepsilon)}_*(H)=\bigoplus_i \mathrm{HF}^{\mathrm{loc}}_*(H,\gamma_i).$$

In particular, if all one-periodic orbits γ of H are isolated and $\mathrm{HF}^{\mathrm{loc}}_k(H,\gamma)=0$ for some k and all γ, we have $\mathrm{HF}_k(H)=0$ by the long exact sequence (2.2) of filtered Floer homology.

The effect on local Floer homology of the composition of H with a loop of Hamiltonian diffeomorphisms is the same as in the global setting and is established in a similar fashion; see Section 2.3.

(LF3) Let $\psi^t=\varphi^t_G$ be a loop of Hamiltonian diffeomorphisms of W. Then

$$\mathrm{HF}^{\mathrm{loc}}_*(G\#H,\psi(\gamma))=\mathrm{HF}^{\mathrm{loc}}_*(H,\gamma)$$

for every isolated one-periodic orbit γ of H, where $\psi(\gamma)$ stands for the one-periodic orbit $\psi^t(\gamma(t))$ of $G\#H$ corresponding to γ; see Section 2.3.

As is clear from the definition of local Floer homology, H need not be a function on the entire manifold W — it is sufficient to consider Hamiltonians defined only on a neighborhood of γ. For the sake of simplicity, we focus on the particular case, relevant here, where $\gamma(t) \equiv p$ is a constant orbit, and hence $dH_t(p) = 0$ for all $t \in S^1$. Then (LF1) still holds and (LF3) takes the following form:

(LF4) Let $\psi^t = \varphi_G^t$ be a loop of Hamiltonian diffeomorphisms defined on a neighborhood of p and fixing p (i.e., $\psi^t(p) = p$ for all $t \in S^1$). Then

$$\mathrm{HF}_*^{\mathrm{loc}}(G\#H, p) = \mathrm{HF}_{*-2\mu}^{\mathrm{loc}}(H, p),$$

where μ is the Maslov index of the loop $t \mapsto d\psi_p^t \in \mathrm{Sp}(T_pW)$.

Note that in (LF3), in contrast with (LF4), we *a priori* know that $\mu = 0$ as is pointed out in Section 2.3. Hence, the shift of degrees does not occur when ψ^t is a global loop. In other words, comparing (LF3) and (LF4), we can say that the group $\mathrm{HF}_*^{\mathrm{loc}}(H, \gamma)$ is completely determined by the Hamiltonian diffeomorphism $\varphi_H \colon W \to W$ and its fixed point $\gamma(0)$, while the germ of φ_H at p determines $\mathrm{HF}_*^{\mathrm{loc}}(H, p)$ only up to a shift in degree. The degree depends on the class of φ_H^t in the universal covering of the group of germs of Hamiltonian diffeomorphisms.

Finally note that in the construction of local Floer homology the Hamiltonian H need not have period one. The definitions and results above extend word-for-word to T-periodic Hamiltonians and, in particular, to the T-th iteration $H^{(T)}$ of a one-periodic Hamiltonian H as long as the T-periodic orbit in question is isolated.

3.3. *Local Floer homology via local Morse homology.* A fundamental property of Floer homology is that $\mathrm{HF}_*(H)$ is equal to the Morse homology of a smooth function on W (and thus to the homology of W). The key to establishing this fact is identifying $\mathrm{HF}_*(H)$ with $\mathrm{HM}_{*+n}(H)$, when the Hamiltonian H is autonomous and C^2-small; see [FHS95], [SZ92]. A similar identification holds for local Floer homology. We consider here the case of T-periodic Hamiltonians, for this is the (superficially more general) situation where the results will be applied in the subsequent sections.

Example 3.5. Assume that p is an isolated critical point of an autonomous Hamiltonian F and

$$T \cdot \|d^2 F_p\| < 2\pi. \tag{3.1}$$

Then $\mathrm{HF}_*^{\mathrm{loc}}\left(F^{(T)}, p\right) = \mathrm{HM}_{*+n}^{\mathrm{loc}}(F, p)$. Indeed, when the condition (3.1) is satisfied, the Hamiltonians sF, $s \in (0, 1]$, have no nontrivial T-periodic orbits (uniformly) near p. (See [HZ94, pp. 184–185] or the proof of Lemma 3.6 below.) Thus, p is a uniformly isolated T-periodic orbit of sF for $s \in [\varepsilon, 1]$ when $\varepsilon > 0$

is small, and $\mathrm{HF}_*^{\mathrm{loc}}\big(sF^{(T)}, p\big)$ is constant throughout this family by (LF1). The argument of [FHS95], [SZ92] shows that the Floer complex of $sF^{(T)} = sT \cdot F$ is equal to the local Morse complex of F when s is close to zero.

In what follows, we will need a slightly more general version of this fact, where the Hamiltonian is "close" to a function independent of time.

LEMMA 3.6. *Let F be a smooth function and let K be a T-periodic Hamiltonian, both defined on a neighborhood of a point p. Assume that p is a constant T-periodic orbit of K and an isolated critical point of F, and that the following conditions are satisfied*:

- *The inequalities $\|X_{K_t} - X_F\| \leq \varepsilon \|X_F\|$ and $\|\dot{X}_{K_t}\| \leq \varepsilon \|X_F\|$ hold pointwise near p for all $t \in S^1_T$ and some $\varepsilon > 0$. (The dot stands for the derivative with respect to time.)*
- *The Hessians $d^2(K_t)_p$ and $d^2 F_p$ and the constant $\varepsilon > 0$ are sufficiently small. Namely, $\varepsilon < 1$ and*

$$T \cdot \Big(\varepsilon(1-\varepsilon)^{-1} + \max_t \|d^2(K_t)_p\| + \|d^2 F_p\|\Big) < 2\pi. \tag{3.2}$$

Then p is an isolated T-periodic orbit of K. Furthermore,

(a) $\mathrm{HF}_*^{\mathrm{loc}}(K^{(T)}, p) = \mathrm{HM}_{*+n}^{\mathrm{loc}}(F, p)$;

(b) *if* $\mathrm{HF}_n^{\mathrm{loc}}(K^{(T)}, p) \neq 0$, *the functions K_t for all t and F have a strict local maximum at p.*

Remark 3.7. The requirement of this lemma, asserting that K is in a certain sense close to F, plays a crucial role in our proof of Theorem 1.1 (cf. Lemmas 5.2 and 6.1) and in the argument of [Hin09]. To the best of the author's knowledge, this requirement is originally introduced in [Hin09, Lemma 4] as that K is relatively autonomous. In what follows, we will sometimes call F a *reference function* and say that the pair (F, K) meets the requirements of Lemma 3.6.

Note also that the condition that p is a constant T-periodic orbit of K is superfluous: it is a consequence of other hypotheses of the lemma. Indeed, we have $\|X_{K_t}(p) - X_F(p)\| \leq \varepsilon \|X_F(p)\|$, where $X_F(p) = 0$ since p is a critical point of F. Hence, $X_{K_t}(p) = 0$ for all $t \in S^1_T$.

Proof. The statement is local and we may assume that $p = 0 \in \mathbb{R}^{2n} = W$. Consider the family of Hamiltonians $K^s = (1-s)K + sF$ starting with $K^0 = K$ and ending with $K^1 = F$. We claim that $\gamma \equiv p$ is a uniformly isolated T-periodic orbit of K^s for $s \in [0, 1]$.

We show this by adapting the proof of [HZ94, Proposition 17, p. 184]. Fix $r > 0$ and let B_r be the ball of radius r centered at p. Since p is a constant T-periodic orbit of K^s, every T-periodic orbit γ of K^s with $\gamma(0)$ sufficiently close to p is contained in B_r. Recall also that $2\pi \|z\|_{L^2} \leq T \|\dot{z}\|_{L^2}$ for any map $z\colon S^1_T \to \mathbb{R}^{2n}$

with zero mean. Applying this inequality to $z = \dot{\gamma} = X_{K^s}(\gamma)$, we obtain

$$\begin{aligned}\frac{2\pi}{T} \cdot \|\dot{\gamma}\|_{L^2} &\leq \|\ddot{\gamma}\|_{L^2} \\ &= \left\| \frac{d}{dt} X_{K^s}(\gamma) \right\|_{L^2} \\ &\leq \left\| \dot{X}_{K^s}(\gamma) \right\|_{L^2} + \left\| \nabla^2 K^s(\gamma)\dot{\gamma} \right\|_{L^2} \\ &\leq \varepsilon \|X_F(\gamma)\|_{L^2} + \Big(\max_t \|d^2(K^s_t)_p\| + O(r) \Big) \|\dot{\gamma}\|_{L^2} .\end{aligned}$$

Furthermore, from the first requirement on X_F and X_K, it is easy to see that

$$\|X_F\| \leq (1-(1-s)\varepsilon)^{-1} \|X_{K^s}\| \leq (1-\varepsilon)^{-1} \|X_{K^s}\| \tag{3.3}$$

pointwise. Hence,

$$\begin{aligned}\frac{2\pi}{T} \cdot \|\dot{\gamma}\|_{L^2} &\leq \varepsilon(1-\varepsilon)^{-1} \|X_{K^s}(\gamma)\|_{L^2} + \Big(\max_t \|d^2(K^s_t)_p\| + O(r) \Big) \|\dot{\gamma}\|_{L^2} \\ &\leq \varepsilon(1-\varepsilon)^{-1} \|\dot{\gamma}\|_{L^2} + \Big(\max_t \|d^2(K^s_t)_p\| + O(r) \Big) \|\dot{\gamma}\|_{L^2} \\ &= \Big(\varepsilon(1-\varepsilon)^{-1} + \max_t \|d^2(K^s_t)_p\| + O(r) \Big) \|\dot{\gamma}\|_{L^2}.\end{aligned}$$

Once (3.2) holds and $r > 0$ is small, we have

$$\varepsilon(1-\varepsilon)^{-1} + \max_t \|d^2(K^s_t)_p\| + O(r) < \frac{2\pi}{T}.$$

Therefore, $\dot{\gamma} = 0$. In other words, γ is a constant loop, and thus a critical point of K^s_t for $t \in S^1_T$. Then, by (3.3), $dF(\gamma) = 0$. As a consequence, $\gamma \equiv p$ since p is an isolated critical point of F. This shows that p is a uniformly isolated T-periodic orbit of K^s.

By (LF1), the local Floer homology $\mathrm{HF}^{\mathrm{loc}}_*\big((K^s)^{(T)}, p\big)$ is constant throughout the family K^s, and $\mathrm{HF}^{\mathrm{loc}}_*\big(K^{(T)}, p\big) = \mathrm{HF}^{\mathrm{loc}}_*\big(F^{(T)}, p\big)$. As a consequence of (3.2), the condition (3.1) of Example 3.5 is satisfied. Applying this example, we conclude that $\mathrm{HF}^{\mathrm{loc}}_*\big(K^{(T)}, p\big) = \mathrm{HF}^{\mathrm{loc}}_*\big(F^{(T)}, p\big) = \mathrm{HM}^{\mathrm{loc}}_{*+n}(F, p)$. This proves (a).

By (LM2), p is an isolated local maximum of $F = K^1_t$, and hence, as is easy to see from the first condition of the lemma, p is a uniformly isolated critical point of K^s_t for $s \in [0, 1]$ and every fixed $t \in S^1_T$. Now, by (LM1) and (LM2) applied to $f_s = K^s_t$, all functions K^s_t, and, in particular, $K_t = K^0_t$, have a (strict) local maximum at p. This proves (b) and concludes the proof of the lemma. □

4. **Proof of Theorem** 1.1

As has been pointed out above, it is sufficient to prove the theorem for (contractible!) periodic orbits of a Hamiltonian H generating φ rather than for all periodic points of φ. Let $H: S^1 \times W \to \mathbb{R}$ be a one-periodic Hamiltonian with finitely many one-periodic orbits α. Then, these orbits are isolated and the action spectrum of H consists of finitely many points.

For every one-periodic orbit α of H denote by $d_1(\alpha), \dots, d_{m_\alpha}(\alpha)$ the degrees of roots of unity, different from 1, among the Floquet multipliers of α.

Arguing by contradiction, assume that for every sufficiently large integer τ, all τ-periodic orbits of H are iterated or, in other words, φ_H has only finitely many *simple periods*, i.e., periods of simple, noniterated, orbits. In particular, every periodic orbit of H with sufficiently large period is iterated. Let $m_1, \dots, m_k$ be the finite collection of integers comprising all simple periods (greater than 1) and the degrees $d_j(\alpha)$ for all one-periodic orbits α. Then, in particular, every τ-periodic orbit is an iterated one-periodic orbit when τ is not divisible by any of the integers m_j. Moreover, all τ-periodic orbits are isolated and $\mathcal{S}(H^{(\tau)}) = \tau\mathcal{S}(H)$.

Recall also that, when α is a weakly nondegenerate one-periodic orbit of H and τ is a sufficiently large integer, not divisible by $d_1(\alpha), \dots, d_{m_\alpha}(\alpha)$, we have

$$\mathrm{HF}_n^{\mathrm{loc}}\left(H^{(\tau)}, \alpha^{(\tau)}\right) = 0. \tag{4.1}$$

Indeed, as is shown in [SZ92], for a generic perturbation of H supported near α, the orbit $\alpha^{(\tau)}$ splits into nondegenerate orbits with Conley-Zehnder index different from n.

Next observe that there exists a strongly degenerate one-periodic orbit γ of H such that $\gamma^{(\tau_i)}$ is an isolated τ_i-periodic orbit for some sequence $\tau_i \to \infty$ and

$$\mathrm{HF}_n^{\mathrm{loc}}\left(H^{(\tau_i)}, \gamma^{(\tau_i)}\right) \neq 0, \tag{4.2}$$

where all τ_i are divisible by τ_1 and none of τ_i is divisible by $m_1, \dots, m_k$.

To prove this, first note that by (4.1) for any sufficiently large integer τ, not divisible by $m_1, \dots, m_k$, there exists a strongly degenerate one-periodic orbit η such that $\mathrm{HF}_n^{\mathrm{loc}}\left(H^{(\tau)}, \eta^{(\tau)}\right) \neq 0$. (Otherwise, (4.1) held for all τ-periodic orbits, and we would have $\mathrm{HF}_n\left(H^{(\tau)}\right) = 0$ by (LF2).) Pick an infinite sequence $\tau_1' < \tau_2' < \dots$ of such integers satisfying the additional requirement that τ_{i+1}' is divisible by τ_i' for all $i \geq 1$. (For instance, we can take $\tau_i' = q^i$, where q is a sufficiently large prime.) As we have observed, for every τ_i' there exists a strongly degenerate one-periodic orbit η_i such that $\mathrm{HF}_n^{\mathrm{loc}}\left(H^{(\tau_i')}, \eta_i^{(\tau_i')}\right) \neq 0$. Since there are only finitely many distinct one-periodic orbits, one of the orbits γ among the orbits η_i and some infinite subsequence τ_i in τ_i' satisfy (4.2). (We also re-index the subsequence τ_i to make it begin with τ_1.)

Let $a = A_H(\gamma)$. We will use the orbit γ and the sequence τ_i to prove

CLAIM. *For every $\varepsilon > 0$ there exists T_0 such that for any $T > T_0$ and some δ_T in the range $(0, \varepsilon)$, depending on T, we have*

$$\mathrm{HF}_{n+1}^{(T\tau_1 a + \delta_T,\, T\tau_1 a + \varepsilon)}\left(H^{(T\tau_1)}\right) \neq 0. \tag{4.3}$$

The theorem readily follows from the claim. Indeed, set $\mathcal{S}(H) = \{c_1, \dots, c_s\}$. Then, if $T > T_0$ is such that $T\tau_1$ is not divisible by $m_1, \dots, m_k$, we have $\mathcal{S}\big(H^{(T\tau_1)}\big) = \{T\tau_1 c_1, \dots, T\tau_1 c_s\}$. Thus, for any fixed $\varepsilon > 0$ and $0 < \delta_T < \varepsilon$, the interval $(T\tau_1 a + \delta_T,\ T\tau_1 a + \varepsilon)$ contains no action values of $H^{(T\tau_1)}$ when T is sufficiently large. This contradicts the claim. (Note that we have used the assumption that φ has finitely many simple periods twice: the first time to find the orbit γ and the sequence τ_i and the second time to arrive at the contradiction with the claim.)

To establish the claim, it is convenient to adopt the following

Definition 4.1. A one-periodic orbit γ of a one-periodic Hamiltonian H is said to be a *symplectically degenerate* maximum if there exists a sequence of loops η_i of Hamiltonian diffeomorphisms such that $\gamma(t) = \eta_i^t(p)$, i.e., η_i sends p to γ, for some point $p \in W$ and all i and t, and such that the Hamiltonians K^i given by

$$\varphi_H^t = \eta_i^t \circ \varphi_{K^i}^t$$

and the loops η_i have the following properties:

(K1) the point p is a strict local maximum of K_t^i for all $t \in S^1$ and all i,

(K2) there exist symplectic bases Ξ^i in T_pW such that

$$\|d^2(K_t^i)_p\|_{\Xi^i} \to 0 \text{ uniformly in } t \in S^1, \text{ and}$$

(K3) the linearization of the loop $\eta_i^{-1} \circ \eta_j$ at p is the identity map for all i and j (i.e., $d\big((\eta_i^t)^{-1} \circ \eta_j^t\big)_p = I$ for all $t \in S^1$) and, moreover, the loop $(\eta_i^t)^{-1} \circ \eta_j^t$ is contractible to id in the class of loops fixing p and having the identity linearization at p.

Remark 4.2. Regarding (K1) and (K3) note that since $\gamma(t) = \eta_i^t(p)$, the point p is a fixed point of the flow

$$\varphi_{K^i}^t = \big(\eta_i^t\big)^{-1} \circ \varphi_H^t$$

of K^i, and thus a critical point of K_t^i for all t. Furthermore, p is also a fixed point of the loop $\eta_i^{-1} \circ \eta_j$ for all i and j, for $\eta_i^t = \varphi_H^t \circ (\varphi_{K^i}^t)^{-1}$, and hence $(\eta_i^t)^{-1} \circ \eta_j^t = \varphi_{K^i}^t \circ (\varphi_{K^j}^t)^{-1}$. We refer the reader to Section 2.1.3 for the definition and discussion of the norm with respect to a basis, used in (K2).

The Hamiltonians K^i and H have the same time-one map and there is a natural one-to-one correspondence between (contractible) one-periodic orbits of the Hamiltonians. The Hamiltonians K^i can be chosen so that $K_t^i(p)$ is constant and equal to $c = A_H(\gamma)$. In what follows, we will always assume that K^i is normalized in this way. Then the corresponding orbits of K^i and H have equal actions and, in particular, all Hamiltonians K^i have the same action spectrum and action filtration; see Section 2.3. Symplectically degenerate maxima are further investigated in [GG07]. In particular, it is shown there that condition (K3) is superfluous; see [GG07, Remark 5.5]. This fact is not used in the present paper.

Example 4.3. Assume that H_t has a strict local maximum at p (and $H_t(p) \equiv$ const) and $d^2(H_t)_p = 0$ for all t. Then p is a symplectically degenerate maximum of H. Indeed, we can take $K^i = H$ and $\eta_i = \mathrm{id}$ and any fixed symplectic basis as Ξ^i. More generally, vanishing of the Hessian may be replaced by the condition that $\|d^2(H_t)_p\|_\Xi$ can be made arbitrarily small by a suitable choice of Ξ; cf. [Hin09]. This condition is satisfied, for instance, when H is autonomous and all eigenvalues of the linearization of X_H at p are equal to zero; see Lemma 5.1.

Example 4.4. Assume that γ is a symplectically degenerate maximum of H. Let $\widetilde{H}$ be a Hamiltonian generating the flow $\psi^t \circ \varphi_H^t$, where ψ is a loop of Hamiltonian diffeomorphisms. Then, the periodic orbit $\psi(\gamma)(t) := \psi^t(\gamma(t))$ of $\widetilde{H}$ is a symplectically degenerate maximum of $\widetilde{H}$ as is easy to verify. (In other words, symplectic degeneracy is a property of the fixed point $\gamma(0)$ of the time-one map φ_H.)

For instance, in the notation of Definition 4.1, the constant orbit p is a symplectically degenerate maximum of each Hamiltonian K^i.

Now we are in a position to state the two results that we need to complete the proof of Theorem 1.1. The first result gives a Floer homological criterion for an isolated, strongly degenerate orbit γ to be a symplectically degenerate maximum, and thus translates local Floer homological properties of γ to geometrical features of a constant orbit p of Hamiltonians K^i. The second one asserts nonvanishing of the filtered Floer homology of an iterated Hamiltonian $H^{(T)}$ for an interval of actions just above the action $T \cdot A_H(\gamma)$, provided that γ is a symplectically nondegenerate maximum of H. When applied to the Hamiltonian $H^{(\tau_1)}$ in place of H, where τ_1 is as in the claim, these results will yield the claim.

PROPOSITION 4.5. *Let γ be a strongly degenerate isolated one-periodic orbit of H such that its l-th iteration $\gamma^{(l)}$ is also isolated and*

$$\mathrm{HF}_n^{\mathrm{loc}}(H,\gamma) \neq 0 \textit{ and } \mathrm{HF}_n^{\mathrm{loc}}\left(H^{(l)},\gamma^{(l)}\right) \neq 0 \textit{ for some } l \geq n+1. \tag{4.4}$$

Then γ is a symplectically degenerate maximum of H.

Remark 4.6. Note that, similarly to Definition 4.1, requirement (4.4) is a condition on the fixed point $\gamma(0)$ of φ_H, independent of a particular choice of H.

PROPOSITION 4.7. *Let γ be a symplectically degenerate maximum of H and let $c = A_H(\gamma)$. Then for every $\varepsilon > 0$ there exists T_0 such that*

$$\mathrm{HF}_{n+1}^{(Tc+\delta_T,\,Tc+\varepsilon)}\left(H^{(T)}\right) \neq 0 \textit{ for all } T > T_0 \textit{ and some } \delta_T \textit{ with } 0 < \delta_T < \varepsilon.$$

Combining the propositions, we conclude that whenever a strongly degenerate one-periodic orbit γ of H satisfies the hypotheses of Proposition 4.5, for every $\varepsilon > 0$ there exists T_0 such that

$$\mathrm{HF}_{n+1}^{(Tc+\delta_T,\,Tc+\varepsilon)}\left(H^{(T)}\right) \neq 0 \text{ for all } T > T_0, \tag{4.5}$$

where $c = A_H(\gamma)$ and $0 < \delta_T < \varepsilon$.

To prove the claim, first note that although Propositions 4.5 and 4.7 are stated for one-periodic Hamiltonians, similar results hold, of course, for Hamiltonians and orbits of any period. Thus, consider the Hamiltonian $H^{(\tau_1)}$ in place of H and the isolated orbit $\gamma^{(\tau_1)}$ in place of γ in (4.4) and (4.5). Then the requirement (4.4) is met due to (4.2): $l = \tau_i/\tau_1 \geq n+1$ if i is large enough, since $\tau_i \to \infty$. Furthermore, $c = A_{H^{(\tau_1)}}(\gamma^{(\tau_1)}) = \tau_1 a$ and (4.3) follows immediate from (4.5).

It remains to establish Propositions 4.5 and 4.7 to complete the proof of the theorem.

Remark 4.8. It is illuminating to compare the above proof with the argument due to Salamon and Zehnder from [SZ92] asserting that every large prime is a simple period whenever all one-periodic orbits of H are weakly nondegenerate. (In particular, the number of simple periods less than or equal to k is of order at least $k/\log k$.) In the context of the present paper relying, of course, on [SZ92], this is an immediate consequence of (4.1). To be more specific, if τ is a large prime and all τ-periodic orbits are iterated, (4.1) holds for all weakly nondegenerate one-periodic orbits and $\mathrm{HF}_n\left(H^{(\tau)}\right) = 0$ by (LF2), if there are no strongly degenerate one-periodic orbits. When such one-periodic orbits exist, we can no longer use the Salamon-Zehnder argument to conclude that every large prime is a simple period or even to establish the existence of infinitely many simple periods. The reason is that in this case the argument implies that for every large prime τ there is a one-periodic orbit γ such that $\mathrm{HF}_n^{\mathrm{loc}}\left(H^{(\tau)}, \gamma^{(\tau)}\right) \neq 0$. It is unclear, however, if $\mathrm{HF}_n^{\mathrm{loc}}(H, \gamma) \neq 0$, and hence whether or not γ is a symplectically degenerate maximum.

5. **Proof of Proposition** 4.5

Our goal in this section is to construct the Hamiltonians K^i and the loops η_i meeting requirements (K1)–(K3). This construction relies on two technical lemmas, proved in Section 6, and is carried out in several steps.

First, in Section 5.1, we reduce the problem to the case where γ is a fixed point p of the flow φ_H^t.

In Section 5.2, we construct the Hamiltonians K^i and the loops η_i near p. We begin by proving in Section 5.2.1 that the time-one map $\varphi = \varphi_H^1$ can be made C^1-close to id by an appropriate choice of a canonical coordinate system ξ near p. This is essentially an elementary linear algebra fact (Lemma 5.1, proved in §5.4), asserting that a strongly degenerate linear symplectomorphism can be made arbitrarily close to the identity by conjugation within the linear symplectic group.

As a consequence, near p, the map φ is given by a generating function F in the coordinate system ξ. In Section 5.2.2, we show that on a neighborhood of p there exists a Hamiltonian K with time-one map φ, which is in a certain sense close to F. Here, the key result is Lemma 5.2 spelling out the relation between F and K and established in Section 6. Choosing a sequence of coordinate systems ξ^i so that $\|\varphi - \mathrm{id}\|_{C^1(\xi^i)} \to 0$, we obtain a sequence of Hamiltonians K^i defined near p and meeting requirement (K2). Then, again near p, the loop η_i is defined by $\eta_i^t = \varphi_H^t \circ (\varphi_{K^i}^t)^{-1}$.

Utilizing condition (4.4), we show in Section 5.2.3 that the Maslov index of η_i is zero. This enables us to relate homological properties of $\gamma \equiv p$ to the geometrical properties of K^i near p and prove (K1) as a consequence of Lemma 3.6. (Assertion (K2) easily follows from the construction of K^i.)

Property (K3) is proved in Section 5.3. At this stage, we further specialize our choice of canonical coordinate systems ξ^i to ensure that all flows $\varphi_{K^i}^t$ have the same linearization at p. Then, the first part of assertion (K3) is obvious. By Lemma 2.8, the loops η_i extend to W, for $\mu(\eta_i) = 0$. This, in turn, gives an extension of K^i to W. Carrying out these extensions with some care, we can guarantee that (K3) holds in its entirety.

5.1. *Reduction to the case of a constant orbit.* In this section, we reduce the proposition to the case where

- *$\gamma \equiv p$ is a constant, strongly degenerate one-periodic orbit of H and $H_t(p) = c$ for all $t \in S^1$*

by constructing a loop of Hamiltonian diffeomorphisms ψ^t, $t \in S^1$, of W such that $\gamma(t) = \psi^t(p)$ with $p = \gamma(0) \in W$.

First recall that for any contractible, closed curve $\gamma: S^1 \to W$ there exists a contractible loop of Hamiltonian symplectomorphisms ψ^t for which γ is an integral curve, i.e., $\gamma(t) = \psi^t(\gamma(0))$; cf. [SZ92, §9].

For the sake of completeness, let us outline a proof of this fact. Consider a smooth family of closed curves $\gamma_s: S^1 \to W$, $s \in [0, 1]$, connecting the constant loop $\gamma_0 \equiv \gamma(0)$ to $\gamma_1 = \gamma$. It is easy to show that there exists a smooth family of Hamiltonians G_t^s such that for every t, the curve $s \mapsto \gamma_s(t)$ is an integral curve of G_t^s with respect to s, i.e., $\gamma_s(t) = \varphi_G^s(\gamma(0))$. Let $\psi^t = \varphi_G^1$ be the time-one map (in s) of this family, parametrized by $t \in S^1$. Then $\gamma(t) = \psi^t(\gamma(0))$. The family of Hamiltonians G_t^s can be chosen so that $G_{s,0} \equiv 0 \equiv G_{s,1}$. Then ψ^t is a loop of Hamiltonian diffeomorphisms with $\psi^0 = \mathrm{id} = \psi^1$. As readily follows from the construction, the loop ψ is contractible.

Composing φ_H^t with the loop $(\psi^t)^{-1}$ and adding, if necessary, a time-dependent constant function to the resulting Hamiltonian $\widehat{H}$, we may assume without loss of generality that $\gamma(t) \equiv p$ is a fixed point of the flow $\varphi_{\widehat{H}}^t = (\psi^t)^{-1}\varphi_H^t$ for all $t \in S^1$ and $\widehat{H}_t(p) \equiv c$. Then $\widehat{H}$ has the same time-one map and the same filtered Floer homology as H. By Example 4.4 and Remark 4.6, it is sufficient to prove the proposition for $\widehat{H}$. Thus, we will assume from now on that $\gamma \equiv p$ and keep the notation H for the modified Hamiltonian $\widehat{H}$.

5.2. *The construction of the Hamiltonians K^i and the loops η_i near p.* Our main objective in this section is to show that for every $\sigma > 0$, there exists a symplectic basis Ξ in T_pW and a Hamiltonian K on a neighborhood of p such that the time-one map of K is φ, condition (K1) is satisfied, and

$$\|d\varphi_K^t|_{T_pW} - I\|_\Xi < \sigma \text{ for all } t \in S^1.$$

Then, clearly, there exists a sequence of symplectic bases Ξ^i and a sequence of Hamiltonians K^i meeting requirements (K1) and (K2). The loop η_i is defined near p by $\eta_i^t = \varphi_H^t \circ (\varphi_{K_i}^t)^{-1}$.

5.2.1. *Making φ close to the identity.* Our first goal is to show that for any $\sigma > 0$ there exists a symplectic basis Ξ in T_pW such that $\|d\varphi_p - I\|_\Xi < \sigma$. As a consequence (cf. Example 2.2), for any $\sigma > 0$ there exists a canonical system of coordinates ξ on a neighborhood U of p such that the $C^1(\xi)$-distance from φ to the identity is less than σ.

This fact is an immediate consequence of

LEMMA 5.1. *Let $\Phi\colon V \to V$ be a linear symplectic map of a finite-dimensional symplectic vector space (V, ω) such that all eigenvalues of Φ are equal to one. Then Φ is conjugate in $\mathrm{Sp}(V, \omega)$ to a linear map which is arbitrarily close to the identity.*

Indeed, since p is a strongly degenerate fixed point of H, all eigenvalues of $d\varphi_p$ are equal to one. Thus, the desired statement follows from this lemma applied to $\Phi = d\varphi_p$. The proof of the lemma is elementary and provided for the sake of completeness in Section 5.4. Here we only mention that Φ is given by an upper triangular matrix in some basis Ξ and, by scaling the elements of Ξ appropriately, one can make Φ arbitrarily close to the identity; cf. Example 2.2. Hence, we only need to show that Ξ and the scaling can be made symplectic.

5.2.2. *The Hamiltonian K near p.* Pick a system ξ of canonical coordinates near p such that φ is $C^1(\xi)$-close to the identity. In particular, $\|d\varphi_p - I\|_{\xi_p}$ is small. Furthermore, the map φ is given, near p, by a generating function F. The precise definition of F and the relation between F and φ and ξ are immaterial at the moment and these issues will be discussed in Section 6. At this stage, we only need to know that F is defined on a neighborhood of p and uniquely determined by ξ and φ. (To make this statement accurate, let us agree that a canonical coordinate system is formed by *ordered* pairs of functions $(x_1, y_1), \dots, (x_n, y_n)$ such that $\omega = \sum dx_i \wedge dy_i$. Thus, each coordinate function is assigned to either x_i- or y_i-group.) Moreover, F has the following properties:

(GF1) p is an isolated critical point of F,

(GF2) $\|F\|_{C^2(\xi)} = O(\|\varphi - \mathrm{id}\|_{C^1(\xi)})$ and $\|d^2F_p\|_{\xi_p} = \|d\varphi_p - I\|_{\xi_p}$.

Item (GF2) requires, perhaps, a clarification. First note that $\|\varphi - \mathrm{id}\|_{C^1(\xi)}$ stands here for the $C^1(\xi)$-distance from φ to id; see Section 2.1.3. Furthermore, F and $\|\varphi - \mathrm{id}\|_{C^1(\xi)}$ depend on ξ. Therefore, in (GF2), we view both $\|F\|_{C^2(\xi)}$ and $\|\varphi - \mathrm{id}\|_{C^1(\xi)}$ as functions of ξ with φ fixed and the second item asserts that

$\|F\|_{C^2(\xi)} \leq \text{const} \cdot \|\varphi - \mathrm{id}\|_{C^1(\xi)}$, where const is independent of ξ, provided that $\|\varphi - \mathrm{id}\|_{C^1(\xi)}$ is small enough.

More generally, let f and g be nonnegative functions of ξ and some (numerical) variables. We write $f = O(g)$, when $f \leq \text{const} \cdot g$ pointwise, where const is independent of ξ. The notation $f = O_\xi(g)$ will be used when $f \leq \text{const}(\xi) \cdot g$ pointwise as functions of other variables, with const(ξ) depending on ξ and possibly becoming arbitrarily large. Furthermore, we denote by $B_r(\xi)$ the ball of radius r with respect to ξ centered at p.

We will prove

LEMMA 5.2 ([Hin09]). *Let ξ be a coordinate system near p such that the norm $\|\varphi - \mathrm{id}\|_{C^1(\xi)}$ is small. Then for every sufficiently small $r > 0$ (depending on ξ), there exists a one-periodic Hamiltonian K_t on $B_r(\xi)$ such that*

(i) *the time-one map φ_K of K is φ,*

(ii) *p is an isolated critical point of K_t and $K_t(p) \equiv c$,*

(iii) $\|d^2(K_t)_p\|_{\xi_p} = O(\|d\varphi_p - I\|_{\xi_p})$,

(iv) *the following estimates hold pointwise near p:*

$$\|X_K - X_F\|_\xi \leq \big(O(\|d^2F_p\|_{\xi_p}) + O_\xi(r)\big) \cdot \|X_F\|_\xi$$

and

$$\|\dot{X}_K\|_\xi \leq \big(O(\|d^2F_p\|_{\xi_p}) + O_\xi(r)\big) \cdot \|X_F\|_\xi,$$

where the dot denotes the time derivative of a vector field.

Note that in (iv) we could have written $\|d\varphi_p - I\|_{\xi_p}$ in place of $\|d^2F_p\|_{\xi_p}$ by (GF2). The important point here is that $\|d\varphi_p - I\|_{\xi_p}$ and $\|d^2F_p\|$ can be made arbitrarily small by choosing an appropriate coordinate system ξ. Then, shrinking the domain of K, we can also make the right hand sides in the estimates (iii) and (iv) arbitrarily small.

A proof of Lemma 5.2 can be extracted from [Hin09]. However, to make our proof of Theorem 1.1 self-contained, we provide a detailed argument. Deferring this to Section 6, we proceed with the proof of Proposition 4.5.

5.2.3. *Properties* (K1) *and* (K2). Let K be a Hamiltonian on a neighborhood of p, such that (i)–(iv) of Lemma 5.2 are satisfied and $\|\varphi - \mathrm{id}\|_{C^1(\xi)}$ is small. Our first goal is to prove that K meets requirements (K1) and (K2).

Since $\|d\varphi_p - I\|_{\xi_p} \leq \|\varphi - \mathrm{id}\|_{C^1(\xi)}$, by (iii), we have

$$\|d^2(K_t)_p\|_{\xi_p} = O(\|\varphi - \mathrm{id}\|_{C^1(\xi)}), \tag{5.1}$$

and hence (K2) is satisfied when $\|\varphi - \mathrm{id}\|_{C^1(\xi)}$ is sufficiently small.

To establish (K1), consider the loop $\eta^t = \varphi_H^t (\varphi_K^t)^{-1}$, where $t \in \mathbb{R}$. Thus,

$$\varphi_H^t = \eta^t \varphi_K^t \text{ for all } t \in \mathbb{R}.$$

Note that $\eta^1 = \mathrm{id}$, i.e., η^t with $t \in S^1$ is a loop of Hamiltonian symplectomorphisms near p. We denote this loop by $\eta|_{S^1}$. The T-th iteration $\eta|_{S^1_T}$ of $\eta|_{S^1}$ is simply η^t with $t \in S^1_T$.

First, let us prove that the Maslov index $\mu = \mu(\eta|_{S^1})$ of $\eta|_{S^1}$ is necessarily zero, when $\|\varphi - \mathrm{id}\|_{C^1(\xi)}$ is small.

Let $\bar{K}$ be the time-average of K_t, i.e.,

$$\bar{K} = \int_0^1 K_t \, dt.$$

A straightforward calculation utilizing Lemma 5.2 and (5.1) and detailed in Section 5.2.4 shows that the requirements of Lemma 3.6 are met, for any fixed T, by the pair $(\bar{K}, K)$, provided that $\|\varphi - \mathrm{id}\|_{C^1(\xi)}$ is small enough. In other words, these requirements are satisfied when φ is $C^1(\xi)$-close to the identity and $\bar{K}$ is taken as the reference function in Lemma 3.6 (denoted there by F). In particular, p is an isolated l-periodic orbit. We set, $T = l$, where l is as in (4.4).

Then, by Lemma 3.6(a),

$$\mathrm{HF}^{\mathrm{loc}}_*(K^{(l)}, p) = \mathrm{HM}^{\mathrm{loc}}_{*+n}(\bar{K}, p),$$

and hence

$$\mathrm{HF}^{\mathrm{loc}}_k(K^{(l)}, p) = 0 \text{ whenever } |k| > n. \tag{5.2}$$

Next note that $\mu(\eta|_{S^1_T}) = \mu T$ for any $T \in \mathbb{Z}$. By (LF4),

$$\mathrm{HF}^{\mathrm{loc}}_{n-2\mu T}\big(K^{(T)}, p\big) = \mathrm{HF}^{\mathrm{loc}}_n\big(H^{(T)}, p\big),$$

as long as p is an isolated one-periodic orbit of $H^{(T)}$. Applying this identity to $T = l$, we conclude from (4.4) that

$$\mathrm{HF}^{\mathrm{loc}}_{n-2\mu l}\big(K^{(l)}, p\big) = \mathrm{HF}^{\mathrm{loc}}_n\big(H^{(l)}, p\big) \neq 0.$$

In particular, since $l \geq n + 1$,

$$\mathrm{HF}^{\mathrm{loc}}_k\big(K^{(l)}, p\big) \neq 0 \text{ for some } k \text{ with } |k| > n \text{ if } \mu \neq 0. \tag{5.3}$$

Combining (5.2) and (5.3), we conclude that $\mu = 0$. A different proof of this fact, relying on the properties of the mean Conley-Zehnder index (see [SZ92]), can be found in [GG09, §5.2].

Recall that the condition $\mathrm{HF}^{\mathrm{loc}}_n(H, p) \neq 0$ is a part of the assumption (4.4) in Proposition 4.5. Using (LF4) again — this time for $t \in [0, 1]$ — and taking into account that $\mu = 0$, we see that

$$\mathrm{HF}^{\mathrm{loc}}_n(K, p) = \mathrm{HF}^{\mathrm{loc}}_n(H, p) \neq 0.$$

Furthermore, when φ is sufficiently $C^1(\xi)$-close to the identity, the requirements of Lemma 3.6 with $T = 1$ and the Hamiltonians K and F as in Lemma 5.2

are obviously met due to (GF2), (5.1), and (iv). Thus, by Lemma 3.6(b), the function K_t has strict local maximum at p. This proves (K1).

Applying this construction to a sequence of symplectic bases Ξ^i in T_pW such that $\|d\varphi_p - I\|_{\Xi^i} \to 0$, we obtain a sequence of Hamiltonians K^i, meeting requirements (K1) and (K2), and also the loops η_i. We emphasize that K^i and η^i have so far been defined only on a neighborhood of p.

5.2.4. *The pair* $(\bar{K}, K)$. The goal of this auxiliary section, which is included for the sake of completeness, is to show that, as stated above, the pair $(\bar{K}, K)$ satisfies the hypotheses of Lemma 3.6 with T fixed.

To this end, note first that by Lemma 5.2(iv), we have

$$\|X_K - X_F\| \le \big(O(\|d^2F_p\|) + O(r)\big)\|X_F\| \text{ and } \|\dot{X}_K\| \le \big(O(\|d^2F_p\|) + O(r)\big)\|X_F\|$$

pointwise near p. (Here the coordinate system ξ is suppressed in the notation.) Let us integrate the first of these inequalities with respect to t over S^1_T. Then, since F is independent of time, we have, again pointwise near p,

$$\|X_{\bar{K}} - X_F\| \le \int_{S^1} \|X_K - X_F\|\, dt \le a\|X_F\|$$

with $a = O(\|d^2F_p\|) + O(r)$. Thus,

$$\|X_{\bar{K}} - X_F\| \le a\|X_F\|$$

and, as a consequence,

$$\|X_{\bar{K}}\| \ge \|X_F\| - a\|X_F\| = (1-a)\|X_F\|.$$

Then

$$\begin{aligned}\|X_K - X_{\bar{K}}\| &\le \|X_K - X_F\| + \|X_F - X_{\bar{K}}\| \\ &\le a\|X_F\| + a\|X_F\| \\ &\le 2a(1-a)^{-1}\|X_{\bar{K}}\|\end{aligned}$$

Likewise,

$$\|\dot{X}_K\| \le a(1-a)^{-1}\|X_{\bar{K}}\|.$$

Therefore,

$$\|X_K - X_{\bar{K}}\| \le \varepsilon\|X_{\bar{K}}\| \text{ and } \|\dot{X}_K\| \le \varepsilon\|X_{\bar{K}}\|,$$

where $\varepsilon = 2a(1-a)^{-1}$ and all inequalities are pointwise.

Recall now that we can make $a > 0$ arbitrarily small (with T fixed) by making a suitable choice of ξ and then requiring $r > 0$ to be sufficiently small. It follows that we can also make $\varepsilon > 0$ arbitrarily small. In the same vein, the left hand side of (3.2) can be made arbitrarily small. Furthermore, since p is an isolated critical point of F, it is also an isolated critical point of $\bar{K}$. Therefore, the pair $(\bar{K}, K)$ satisfies the hypotheses of Lemma 3.6.

Remark 5.3. As has been pointed out in Section 5.2.3, a pair of function satisfying the hypotheses of Lemma 5.2 also satisfies the hypotheses of Lemma 3.6. We have shown that $(\bar{K}, K)$ satisfies the conditions of Lemma 3.6 whenever (F, K) meets the requirements of Lemma 5.2. Moreover, by arguing as in this section, it is not hard to show that $(\bar{K}, K)$ satisfies the conditions of Lemma 5.2 (and hence of Lemma 3.6) once (F, K) does. We omit this (straightforward) calculation, for it is never used in the proof.

5.3. *Property* (K3) *and the extension to* W. To ensure that (K3) holds, we need to impose an addition requirement on the bases Ξ^i. We will prove

LEMMA 5.4. *There exists a sequence of symplectic bases* Ξ^i *in* T_pW *such that* $\|d\varphi_p - I\|_{\Xi^i} \to 0$ *and the flows* $\varphi^t_{K^i}$ *have the same linearization at* p.

Here K^i is the sequence of Hamiltonians constructed in Section 5.2.3 using Lemma 5.2. We prove Lemma 5.4 in Section 6 along with Lemma 5.2. At this point, we only note that, as will become clear in Section 6, the linearized flow $d(\varphi^t_{K^i})_p$ is completely determined by φ and the basis Ξ^i. In particular, the linearization is independent of the extension of Ξ^i to a canonical coordinate system ξ^i near p. (Here, we use a convention similar to that of Section 5.2.2 for canonical coordinate systems: a symplectic basis is divided into two groups of n vectors spanning Lagrangian subspaces and this division is a part of the structure of a symplectic basis.)

Since $\eta^t_i = \varphi^t_H(\varphi^t_{K_i})^{-1}$, we conclude from Lemma 5.4 that

$$d\big((\eta^t_i)^{-1} \circ \eta^t_j\big)_p = d\big(\varphi^t_{K_i}\big)^{-1}_p \circ d\big(\varphi^t_{K_j}\big)_p = I.$$

Let us now extend the loops η_i and the Hamiltonians K^i to W so that the remaining part of requirement (K3) is met: the loop $\eta_i^{-1} \circ \eta_j$ is contractible to id in the class of loops with identity linearization at p.

Recall that the Maslov index of the loop η_i is zero, as is shown in Section 5.2.3. Hence, by Lemma 2.8, each of these loops extends to a loop of Hamiltonian diffeomorphisms of W, contractible in the class of loops fixing p. Let us fix such an extension for η_1. For the sake of simplicity we denote this extension by η_1 again. Consider now the loop $\psi^t_i = \big(\eta^t_1\big)^{-1}\eta^t_i$. Then $d(\psi^t_i)_p = I$. Hence, by Lemma 2.8 and Remark 2.9, ψ_i extends to a loop of Hamiltonian diffeomorphisms of W, contractible in the class of loops with identity linearization at p. Keeping the notation ψ_i for this extension, we set $\eta^t_i = \eta^t_1\psi^t_i$. It is clear that η_i is contractible in the class of loops with identity linearization at p.

5.4. *Proof of Lemma* 5.1. Lemma 5.1 is an immediate consequence of the following stronger result which is also used in the proof of Lemma 5.4.

LEMMA 5.5. *Let* $\Phi\colon V \to V$ *be a linear symplectic map of a finite-dimensional symplectic vector space* (V, ω). *Assume that all eigenvalues of* Φ *are equal to one.*

Then V can be decomposed as a direct sum of two Lagrangian subspaces L and L' with $\Phi(L) = L$. Moreover, by a suitable choice of $\Psi \in \mathrm{Sp}(V, \omega)$ preserving the subspaces L and L', the map $\Psi\Phi\Psi^{-1}$ can be made arbitrarily close to the identity.

Proof. We prove the lemma by induction in $\dim V$. The statement is obvious when V is two-dimensional. When $\dim V > 2$, we have the following alternative:

- either $K = \ker(\Phi - I)$ contains a symplectic subspace V_0
- or $K = \ker(\Phi - I)$ is isotropic.

In the former case, we decompose V as $V_0 \oplus V_0^\omega$, where the superscript ω denotes the symplectic orthogonal. It is easy to see that this decomposition is preserved by Φ and $\Phi|_{V_0} = I_{V_0}$. Now the assertion follows from the induction hypothesis applied to $\Phi|_{V_0^\omega}$.

In the latter case, pick a symplectic subspace V_0 complementary to $K = \ker(\Phi - I)$ in K^ω and an isotropic subspace N complementary to K^ω in V. (We are assuming at the moment that $V_0 \neq \{0\}$, i.e., L is not Lagrangian.) Thus, $V = K^\omega \oplus N$ and $K^\omega = K \oplus V_0$. Furthermore, K and K^ω are preserved by Φ; the spaces V_0 and N can be canonically identified with K^ω/K and $K^* = V/K^\omega$, respectively; and $\Phi|_K = I_K$. Note that Φ induces a symplectic linear map $\Phi_0\colon V_0 \to V_0$ with all eigenvalues equal to one and the identity map I_N on $N = V/K^\omega$. Hence, using the decomposition

$$V = K \oplus V_0 \oplus N,$$

we can write Φ in the block upper-triangular form

$$\Phi = \begin{bmatrix} I_K & A & C \\ 0 & \Phi_0 & B \\ 0 & 0 & I_N \end{bmatrix},$$

where $A\colon V_0 \to K$ and $C\colon N \to K$ and $B\colon N \to V_0$. (There are relations between these operators, resulting from the fact that Φ is symplectic.)

Consider a block-diagonal symplectic linear transformation of the form

$$\Psi = \begin{bmatrix} \Lambda & 0 & 0 \\ 0 & \Psi_0 & 0 \\ 0 & 0 & (\Lambda^*)^{-1} \end{bmatrix},$$

where $\Psi_0\colon V_0 \to V_0$ is symplectic, $\Lambda\colon K \to K$ is invertible, and we have identified N with K^*. Then

$$\Psi\Phi\Psi^{-1} = \begin{bmatrix} I_K & \Lambda A \Psi_0^{-1} & \Lambda C \Lambda^* \\ 0 & \Psi_0 \Phi_0 \Psi_0^{-1} & \Psi_0 B \Lambda^* \\ 0 & 0 & I_N \end{bmatrix}.$$

By the induction assumption, there exists a decomposition $V_0 = L_0 \oplus L_0'$, where $\Phi_0(L_0) = L_0$, and transformations Ψ_0 preserving this decompositions and making

$\Psi_0\Phi_0\Psi_0^{-1}$ arbitrarily close to I_{V_0}. Set $L = K \oplus L_0$ and $L = L_0' \oplus N$. Then $\Phi(L) = L$ and the decomposition $V = L \oplus L'$ is preserved by Ψ. Furthermore, noticing that Λ^* is close to zero when Λ is close to zero, we can pick Λ to make the off-diagonal entries in Φ arbitrarily small. With this choice of Ψ, the map $\Psi\Phi\Psi^{-1}$ is close to I_V if $\Psi_0\Phi_0\Psi_0^{-1}$ is close to I_{V_0}.

When $K = \ker(\Phi - I)$ is Lagrangian (i.e., $V_0 = \{0\}$), no induction reasoning is needed. We simply set $L = K$ and let $L' = N$ be an arbitrary complementary Lagrangian subspace. Then the map Φ is decomposed as a two-by-two block upper-triangular matrix, and, similarly to the argument above, Λ is chosen to make the off-diagonal block arbitrarily small. $\square$

6. The generating function F and the proofs of Lemmas 5.2 and 5.4

6.1. *Generating functions.* In this section, we recall the definition of a generating function on $\mathbb{R}^{2n}$ and set the stage for proving Lemma 5.2. The material reviewed here is absolutely standard — it goes back to Poincaré — and we refer the reader to [Arn74, App. 9] and [Wei71], [Wei77] for a more detailed discussion of generating functions.

Let us identify $\mathbb{R}^{2n}$ with the Lagrangian diagonal $\Delta \subset \mathbb{R}^{2n} \times \bar{\mathbb{R}}^{2n}$ via the projection to the first factor, where $\mathbb{R}^{2n} \times \bar{\mathbb{R}}^{2n}$ is equipped with the symplectic structure $(\omega, -\omega)$, and fix a Lagrangian complement N to Δ. Thus, $\mathbb{R}^{2n} \times \bar{\mathbb{R}}^{2n}$ can now be treated as $T^*\Delta$.

Let φ be a Hamiltonian diffeomorphism defined on a neighborhood of the origin p in $\mathbb{R}^{2n}$ and such that $\|\varphi - \mathrm{id}\|_{C^1}$ is sufficiently small. Then the graph Γ of φ is close to Δ, and hence Γ can be viewed as the graph in $T^*\Delta$ of an exact form dF near $p \in \Delta = \mathbb{R}^{2n}$. (We normalize F by $F(p) = 0$.) The function F, called the generating function of φ, has the following properties:

(GF1′) p is an isolated critical point of F if and only if p is an isolated fixed point of φ,

(GF2′) $\|F\|_{C^2} = O(\|\varphi - \mathrm{id}\|_{C^1})$ and $\|d^2F_p\| = \|d\varphi_p - I\|$.

For instance, it is clear that the critical points of F are in one-to-one correspondence with the fixed points of φ. If p (the origin) is an isolated fixed point of φ, the origin is also an isolated critical point of F. Hence, (GF1′) holds. The second property of F, (GF2′), is also easy to check; see the references above.

The function F depends on the choice of the Lagrangian complement N to Δ. To be specific, we take as N the linear subspace of vectors of the form $((x,0),(0,y))$ in $\mathbb{R}^{2n} \times \bar{\mathbb{R}}^{2n}$, where $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_n)$ are the standard canonical coordinates on $\mathbb{R}^{2n}$, i.e., $\omega = \sum dy_i \wedge dx_i$.

In the setting of Section 5.2.2, let ξ be a coordinate system near $p \in W$. Using ξ, we identify a neighborhood of p in W with a neighborhood of the origin in $\mathbb{R}^{2n}$, keeping the notation p for the origin. With this identification, φ defined near $p \in W$ turns into a Hamiltonian diffeomorphism $\xi\varphi\xi^{-1}$ defined near the origin

$p \in \mathbb{R}^{2n}$. By definition, $\|\varphi - \mathrm{id}\|_{C^1(\xi)} = \|\xi\varphi\xi^{-1} - \mathrm{id}\|_{C^1}$. Abusing notation, we denote the resulting Hamiltonian diffeomorphism $\xi\varphi\xi^{-1}$ near $p \in \mathbb{R}^{2n}$ by φ again. By our background assumptions, p is an isolated fixed point of φ, and thus (GF1) and (GF2) follow immediately from (GF1′) and (GF2′), respectively.

Furthermore, Lemma 5.2 is an immediate consequence of

LEMMA 6.1 ([Hin09]). *Let φ be a Hamiltonian diffeomorphism of a neighborhood of the origin $p \in \mathbb{R}^{2n}$. Assume that p is an isolated fixed point of φ and $\|\varphi - \mathrm{id}\|_{C^1}$ is so small that the generating function F is defined. Then for every sufficiently small $r > 0$ (depending on φ), there exists a one-periodic Hamiltonian K_t on the ball B_r of radius r centered at p such that*

(i) *the time-one map φ_K of K is φ,*

(ii) *p is an isolated critical point of K_t and $K_t(p) = 0$ for all $t \in S^1$,*

(iii) $\|d^2(K_t)_p\| = O(\|d^2F_p\|)$,

(iv) *the upper bounds*

$$\|X_K - X_F\| \le \big(O(\|d^2F_p\|) + O_\varphi(r)\big) \cdot \|X_F\| \tag{6.1}$$

and

$$\|\dot{X}_K\| \le \big(O(\|d^2F_p\|) + O_\varphi(r)\big) \cdot \|X_F\| \tag{6.2}$$

hold pointwise near p.

The notation used here is similar to that of Section 5.2.2. For instance, (6.1) should be read as that its left hand side is pointwise bounded from above by $\big(C_1\|d^2F_p\| + C_2(\varphi)r\big) \cdot \|X_F\|$, where C_1 is independent of φ and $C_2(\varphi)$ depends on φ and can be arbitrarily large.

Although the proof of Lemma 6.1 is essentially contained in [Hin09], for the sake of completeness we give a detailed argument here.

6.2. *Proof of Lemma* 6.1. The proof of the lemma is organized as follows. First we consider the time-dependent Hamiltonian $\widetilde{K}$ generating the flow φ^t given by the family of generating functions $F_t = tF$, $t \in [0, 1]$, and verify (i)–(iv) for $\widetilde{K}$. The time-one map of $\widetilde{K}$ is φ. However, in general, the Hamiltonian $\widetilde{K}$ is *not* periodic in time. Hence, as the next step, we modify $\widetilde{K}$ to obtain the required periodic Hamiltonian K and then again check that the new Hamiltonian K satisfies (i)–(iv).

6.2.1. *The Hamiltonian $\widetilde{K}$; properties* (i) *and* (ii). Consider the family of generating functions $F_t = tF$ with $t \in [0, 1]$. The family of graphs Γ_t of dF_t in $T^*\Delta$ beginning with $\Gamma_0 = \Delta$ and ending with $\Gamma_1 = \Gamma$ can be viewed as a family of graphs of Hamiltonian diffeomorphisms φ^t near p with $\varphi^0 = \mathrm{id}$ and $\varphi^1 = \varphi$. Thus, φ^t is a time-dependent Hamiltonian flow with the time-one map φ, defined near p. Let $\widetilde{K}_t$ be the Hamiltonian generating this flow, normalized by $\widetilde{K}_t(p) = 0$.

Condition (i) is satisfied for $\widetilde{K}$ by definition, and (ii) is an immediate consequence of (GF1′) and (6.1). Below, we will also give a direct proof of (ii).

6.2.2. *The Hamiltonian vector field* $X_{\widetilde{K}}$. Set

$$x = (x_1, \dots, x_n) \text{ and } y = (y_1, \dots, y_n),$$

and

$$\partial_1 = (\partial_{x_1}, \dots, \partial_{x_n}) \text{ and } \partial_2 = (\partial_{y_1}, \dots, \partial_{y_n}).$$

Let $(\bar{x}^t, \bar{y}^t) = \varphi^t(x, y)$. Then, as is well-known and can be checked by a simple calculation, we have

$$\begin{cases} \bar{x}^t - x = -\partial_2 F_t(\bar{x}^t, y) \\ \bar{y}^t - y = \partial_1 F_t(\bar{x}^t, y). \end{cases} \tag{6.3}$$

Differentiating with respect to time, we obtain the following expression for the Hamiltonian $X_{\widetilde{K}}$ (cf. [Hin09]):

$$X_{\widetilde{K}_t}(\bar{x}^t, \bar{y}^t) = A_t(\bar{x}^t, y) X_F(\bar{x}^t, y), \tag{6.4}$$

where

$$A_t(x, y) = \begin{bmatrix} (I + \partial_{12} F_t(x, y))^{-1} & 0 \\ (I + \partial_{12} F_t(x, y))^{-1} \partial_{11} F_t(x, y) & I \end{bmatrix}.$$

Here $\partial_{12} F_t$ stands for the matrix of partial derivatives $\partial^2 F_t / (\partial x_i \partial y_j)$ and, similarly, $\partial_{22} F_t$ is the matrix $\partial^2 F_t / (\partial y_i \partial y_j)$.

In other words, introducing the auxiliary diffeomorphism κ^t sending $(\bar{x}^t, \bar{y}^t)$ to $(\bar{x}^t, y)$, we can rewrite (6.4) as

$$X_{\widetilde{K}_t}(z) = A_t(\kappa^t(z)) X_F(\kappa^t(z)) \tag{6.5}$$

for every z near the origin.

Clearly,

$$\|\varphi^t - \mathrm{id}\|_{C^1} = O(\|\varphi - \mathrm{id}\|_{C^1}) \text{ and } \|\kappa^t - \mathrm{id}\|_{C^1} = O(\|\varphi - \mathrm{id}\|_{C^1}) \tag{6.6}$$

uniformly in t. In particular, κ^t is indeed a diffeomorphism, fixes p, and is, moreover, C^1-close to the identity when φ is close to id. Furthermore,

$$\|A_t - I\| = O(\|\varphi - \mathrm{id}\|_{C^1}) \tag{6.7}$$

pointwise near p and uniformly in t. Hence, A is invertible near p. Since p is an isolated critical point of F by (GF1′), p is also an isolated zero of X_F, and thus an isolated zero of $X_{\widetilde{K}_t}$. This gives a direct proof of (ii) for $\widetilde{K}$.

6.2.3. *Property* (iii) *for* $\widetilde{K}$. Since $X_F(p) = 0$, the linearization of (6.5) at p yields

$$d(X_{\widetilde{K}_t})_p = A_t(p) \circ d(X_F)_p \circ d(\kappa^t)_p.$$

Here, $d(X_{\widetilde{K}_t})_p$ and $d(X_F)_p$ are the linear Hamiltonian vector fields on $T_p\mathbb{R}^{2n} = \mathbb{R}^{2n}$ with quadratic Hamiltonians $d^2(\widetilde{K}_t)_p$ and, respectively, $d^2 F_p$. Furthermore, it is easy to see from (6.7) and (6.6) that $A_t(p)$ and $d(\kappa^t)_p$ are both close to I, with error $O(\|d^2F_p\|)$, and hence are small. Combining these observations, we conclude that

$$\|d^2(\widetilde{K}_t)_p\| = O(\|d^2F_p\|)$$

proving (iii) for $\widetilde{K}$.

6.2.4. *The upper bound* (6.1) *for* $\widetilde{K}$. Turning to the proof of (iv) for $\widetilde{K}$, observe that for every small $R > 0$ there exists $r > 0$ such that $(\bar{x}_t, \bar{y}_t)$ and $(\bar{x}_t, y)$ are in B_R for all $t \in [0, 1]$ and all (x, y) in B_r. Furthermore, it is clear that

$$R = O(\|\varphi - \mathrm{id}\|_{C^1}) \cdot r = O_\varphi(r). \tag{6.8}$$

To establish the upper bound (6.1) of (iv), let us first show that

$$\|X_{\widetilde{K}_t}(z) - X_{\widetilde{K}_t}(\kappa^t(z))\| = \big(O(\|d^2F_p\|) + O_\varphi(r)\big) \cdot \|X_F(\kappa^t(z))\| \tag{6.9}$$

for every z in B_r and all $t \in [0, 1]$. We have

$$\begin{aligned}
\|X_{\widetilde{K}_t}(z) - X_{\widetilde{K}_t}(\kappa^t(z))\| &= \left\| \int_0^1 \frac{d}{ds} X_{\widetilde{K}_t}\big(sz + (1-s)\kappa^t(z)\big)\, ds \right\| \\
&\leq \int_0^1 \|dX_{\widetilde{K}_t}\big(sz + (1-s)\kappa^t(z)\big)\|\, ds \cdot \|z - \kappa^t(z)\| \\
&\leq \max_{w \in B_R} \|dX_{\widetilde{K}_t}(w)\| \cdot \|z - \kappa^t(z)\| \\
&\leq \max_{w \in B_R} \|dX_{\widetilde{K}_t}(w)\| \cdot \|X_F(\kappa^t(z))\|,
\end{aligned}$$

where in the last inequality we used the fact that, by (6.3),

$$\|z - \kappa^t(z)\| = \|y - \bar{y}^t\| = \|\partial_1 F_t(\kappa^t(z))\| \leq \|X_F(\kappa^t(z))\|.$$

Thus, we only need to show that

$$\max_{B_R} \|dX_{\widetilde{K}_t}\| = O(\|d^2F_p\|) + O_\varphi(r).$$

By (6.5), we have

$$\begin{aligned}
\max_{w \in B_R} \|dX_{\widetilde{K}_t}(w)\| \leq{}& \max_{w \in B_R} \left\| \big(dA_t(\kappa^t(w)) d\kappa^t(w)\big)\, X_F(\kappa^t(w)) \right\| \\
&+ \max_{w \in B_R} \left\| A_t(\kappa^t(w))\, \big(dX_F(\kappa^t(w)) d\kappa^t(w)\big) \right\|.
\end{aligned}$$

Since $\kappa^t(p) = p$ and $X_F(p) = 0$, the first summand is obviously $O_\varphi(r)$. (When $z \in B_r$, both $w = \kappa^t(z)$ and $\kappa^t(w)$ are, by (6.8), in the ball of radius $O_\varphi(r)$.) The second summand is bounded as

$$\begin{aligned}
&\max_{w \in B_R} \left\| A_t(\kappa^t(w))\, \big(dX_F(\kappa^t(w)) d\kappa^t(w)\big) \right\| \\
&\qquad \leq \max_{w \in B_R} \left\| A_t(\kappa^t(w)) \right\| \cdot \max_{w \in B_R} \left\| dX_F(\kappa^t(w)) \right\| \cdot \max_{w \in B_R} \left\| d\kappa^t(w) \right\|.
\end{aligned}$$

Here, the first and the last factors are $O(\|\varphi - \mathrm{id}\|_{C^1})$, and hence bounded from above by a constant independent of φ, when φ is sufficiently close to id. The middle factor is $O(\|d^2F_p\|) + O_\varphi(r)$, for $\|dX_F(p)\| = \|d^2F_p\|$ and, as a consequence,

$$\left\|dX_F(\kappa^t(w))\right\| = O(\|d^2F_p\|) + O_\varphi(\|\kappa^t(w)\|).$$

Thus, the second summand is $O(\|d^2F_p\|) + O_\varphi(r)$, which completes the proof of (6.9).

Then

$$\|X_{\widetilde{K}_t}(\kappa^t(z)) - X_F(\kappa^t(z))\| \le \|X_{\widetilde{K}_t}(z) - X_{\widetilde{K}_t}(\kappa^t(z))\| + \|X_{\widetilde{K}_t}(z) - X_F(\kappa^t(z))\|.$$

By (6.5), the second term is bounded as

$$\begin{aligned}\|X_{\widetilde{K}_t}(z) - X_F(\kappa^t(z))\| &\le \|A_t(\kappa^t(z)) - I\| \cdot \|X_F(\kappa^t(z))\| \\ &= \big(O(\|d^2F_p\|) + O_\varphi(r)\big) \cdot \|X_F(\kappa^t(z))\|\end{aligned}$$

and the first term is $\big(O(\|d^2F_p\|) + O_\varphi(r)\big) \cdot \|X_F(\kappa^t(z))\|$ by (6.9). This proves the pointwise estimate (6.1) at $\kappa^t(z)$ in place of z. Since κ^t is a diffeomorphism fixing p, the upper bound (6.1) in its original form (at z) follows.

6.2.5. *The upper bound* (6.2) *for* $\widetilde{K}$. Arguing exactly as in the proof of (6.9), it is easy to show that

$$\|X_F(z) - X_F(\kappa^t(z))\| = \big(O(\|d^2F_p\|) + O_\varphi(r)\big) \cdot \|X_F(\kappa^t(z))\|$$

and, as a consequence,

$$\big[1 - \big(O(\|d^2F_p\|) + O_\varphi(r)\big)\big] \cdot \|X_F(z)\| \le \|X_F(\kappa^t(z))\|.$$

Therefore, to establish (6.2) for $\widetilde{K}$, it is sufficient to prove the upper bound

$$\|\dot{X}_{\widetilde{K}_t}(z)\| = \big(O(\|d^2F_p\|) + O_\varphi(r)\big) \cdot \|X_F(\kappa^t(z))\| \tag{6.10}$$

for $z \in B_r$.

Differentiating (6.5) with respect to t and setting $w = \kappa^t(z)$, we obtain

$$\begin{aligned}\dot{X}_{\widetilde{K}_t}(z) &= \dot{A}_t(w)X_F(w) \\ &\quad + \big(dA_t(w)\dot{\kappa}^t(w)\big)\, X_F(w) \\ &\quad + A_t(w)\,\big(dX_F(w)\dot{\kappa}^t(w)\big)\,.\end{aligned} \tag{6.11}$$

To prove (6.10), we will estimate all three terms in this identity. As a straightforward calculation shows,

$$\dot{A}_t = -(I + \partial_{12}F_t)^{-2} \begin{bmatrix} \partial_{12}F & 0 \\ \partial_{22}F_t\partial_{12}F - (I + \partial_{12}F_t)\partial_{11}F & 0 \end{bmatrix}.$$

Thus, $\|\dot{A}_t\| = O(\|d^2F_p\|) + O_\varphi(r)$ and

$$\|\dot{A}_t(w)X_F(w)\| = \big(O(\|d^2F_p\|) + O_\varphi(r)\big) \cdot \|X_F(w)\|.$$

Furthermore,

$$\dot{\kappa}^t(w) = \Big(0, -\big(I - \partial_{12}F_t(w)\big)^{-1}\partial_1 F(w)\Big)$$

as follows from the definition of κ^t and (6.3). Using again the inequality $\|\partial_1 F(w)\| \leq \|X_F(w)\|$, we see that

$$\begin{aligned}\|\dot{\kappa}^t(w)\| &\leq O(\|\varphi - \mathrm{id}\|_{C^1}) \cdot \|X_F(w)\| \\ &\leq \mathrm{const} \cdot \|X_F(w)\|,\end{aligned}$$

where const is independent of φ, when φ is sufficiently close to id. Then, since $\|X_F(w)\| = O_\varphi(r)$, we have

$$\big\|\big(dA_t(w)\dot{\kappa}^t(w)\big)\, X_F(w)\big\| = O_\varphi(r) \cdot \|X_F(w)\|$$

and

$$\big\|A_t(w)\,\big(dX_F(w)\dot{\kappa}^t(w)\big)\big\| = \big(O(\|d^2F_p\|) + O_\varphi(r)\big) \cdot \|X_F(w)\|,$$

for $\|dX_F(w)\| = O(\|d^2F_p\|) + O_\varphi(r)$.

These estimates combined with (6.11) yield (6.10), and hence (6.2) for $\widetilde{K}$.

6.2.6. *The Hamiltonian* K; *properties* (i), (ii), *and* (iii). Fix a monotone increasing function $\lambda\colon [0, 1] \to [0, 1]$ such that $\lambda(t) \equiv 0$ when t is near 0 and $\lambda(t) \equiv 1$ when t is near 1. This function is independent of φ, and hence $|\lambda'|$ and $|\lambda''|$ are bounded from above by constants independent of φ.

The Hamiltonian K is the one generating the flow

$$\varphi_K^t = \varphi_F^{t-\lambda(t)}\varphi_{\widetilde{K}}^{\lambda(t)}. \tag{6.12}$$

Explicitly, since F is autonomous,

$$K_t(z) = (1 - \lambda'(t))F(z) + \lambda'(t)\widetilde{K}_{\lambda(t)}\big(\varphi_F^{\lambda(t)-t}(z)\big). \tag{6.13}$$

It is clear that $K_t \equiv F$ when t is close to 0 and 1 and hence K can be viewed as a Hamiltonian one-periodic in t. Also, $\varphi_K^1 = \varphi_{\widetilde{K}}^1 = \varphi$, i.e., requirement (i) is satisfied. As has been pointed out, the second condition, (ii), follows from (GF1$'$) and (iv) which is proved below.

Passing to the Hessians of the Hamiltonians in (6.13) at p, we have

$$d^2(K_t)_p = (1 - \lambda'(t))d^2F_p + \lambda'(t)d^2(\widetilde{K}_{\lambda(t)})_p \circ d\big(\varphi_F^{\lambda(t)-t}\big)_p.$$

By (iii) for $\widetilde{K}$ and (6.6), $\|d^2(K_t)_p\| = O(\|d^2F_p\|)$, which proves (iii) for K.

6.2.7. *The upper bound* (6.1) *for* K. The Hamiltonian vector field of K is

$$X_{K_t}(z) = (1 - \lambda'(t))X_F(z) + \lambda'(t)d\varphi_F^{t-\lambda(t)}(w)\big(X_{\widetilde{K}_{\lambda(t)}}(w)\big), \tag{6.14}$$

where $w = \varphi_F^{\lambda(t)-t}(z)$. Since F is autonomous, $X_F(z) = d\varphi_F^{t-\lambda(t)}(w)\,(X_F(w))$. In particular,

$$\|X_F(w)\| \leq \mathrm{const} \cdot \|X_F(z)\|, \tag{6.15}$$

when z is close to p. Also note that when $z \in B_r$, the point $w = \varphi_F^{\lambda(t)-t}(z)$ is in B_R, where the radius R satisfies (6.8).

With these facts in mind, we have

$$\begin{aligned}
\|X_{K_t}(z) - X_F(z)\| &\le |\lambda'(t)| \left\| d\varphi_F^{t-\lambda(t)}(w)\Big(X_{\widetilde{K}_{\lambda(t)}}(w)\Big) - X_F(z)\right\| \\
&\le |\lambda'(t)| \left\| d\varphi_F^{t-\lambda(t)}(w)\Big(X_{\widetilde{K}_{\lambda(t)}}(w)\Big) - d\varphi_F^{t-\lambda(t)}(w)(X_F(w))\right\| \\
&\le |\lambda'(t)| \left\| d\varphi_F^{t-\lambda(t)}(w)\right\| \cdot \left\| X_{\widetilde{K}_{\lambda(t)}}(w) - X_F(w)\right\| \\
&= \big(O(\|d^2F_p\|) + O_\varphi(r)\big)\|X_F(w)\| \\
&= \big(O(\|d^2F_p\|) + O_\varphi(r)\big)\|X_F(z)\|,
\end{aligned}$$

when $z \in B_r$. Here, the next to the last estimate follows from (6.1) for $\widetilde{K}$. This proves (6.1) for K.

6.2.8. *The upper bound* (6.2) *for K.* Differentiating (6.14) with respect to t, we obtain

$$\begin{aligned}
\dot{X}_{K_t}(z) = \lambda''(t) &\Big(d\varphi_F^{t-\lambda(t)}(w)\big(X_{\widetilde{K}_{\lambda(t)}}(w)\big) - X_F(z)\Big) \\
&+ \lambda'(t)^2 d\varphi_F^{t-\lambda(t)}(w)\big(\dot{X}_{\widetilde{K}_{\lambda(t)}}(w)\big) \\
&+ \lambda'(t)(\lambda'(t)-1) d\varphi_F^{t-\lambda(t)}(w)[X_F, X_{\widetilde{K}_t}](w).
\end{aligned}$$

Arguing as in Section 6.2.7, we see that the norm of the first term in this sum is $\big(O(\|d^2F_p\|) + O_\varphi(r)\big)\|X_F(z)\|$. Similarly, the same holds for the second term by (6.2) for $\widetilde{K}$. (In both cases we use (6.15) to relate $X_F(z)$ and $X_F(w)$ and also the fact that $\|d\varphi_F^{t-\lambda(t)}(w)\| \le \text{const}$ when $z \in B_r$.)

To estimate the third term, it is sufficient to show that

$$\|[X_F, X_{\widetilde{K}_t}](w)\| = \big(O(\|d^2F_p\|) + O_\varphi(r)\big)\|X_F(w)\|, \tag{6.16}$$

for then, by (6.15), this term is $\big(O(\|d^2F_p\|) + O_\varphi(r)\big)\|X_F(z)\|$ when $z \in B_r$. To prove (6.16), observe that

$$\begin{aligned}
\big\|[X_F, X_{\widetilde{K}_t}](w)\big\| &= \big\|[X_F, X_{\widetilde{K}_t} - X_F](w)\big\| \\
&\le \alpha(w)\|X_F(w)\| + \beta(w)\|X_{\widetilde{K}_t}(w) - X_F(w)\|,
\end{aligned}$$

where the functions $\alpha(w) \ge 0$ and $\beta(w) \ge 0$ are bounded from above by the partial derivatives of $X_{\widetilde{K}_t} - X_F$ and, respectively, X_F at w. Hence, both of these functions are $O(\|d^2F_p\|) + O_\varphi(r)$ and (6.16) follows from (6.1) for $\widetilde{K}$.

This completes the proof of (6.2) for K and the proof of the lemma.

6.3. *Proof of Lemma* 5.4. Let $\Phi = d\varphi_p$ and let $V = L \oplus L'$ be the decomposition of $V = T_pW$ from Lemma 5.5. Pick a linear canonical coordinate system (x, y) on T_pW, which is compatible with the decomposition, i.e., such that the x-coordinates span L and the y-coordinates span L'. By Lemma 5.5, we can do this so that $\|\Phi - I\|$ is small in this coordinate system, and thus φ is given by the generating function F. Denote by Q the Hessian of F at p and let X_Q be the linear Hamiltonian vector field of Q on V.

Linearizing (6.3) at p, we see that Φ and Q are related via the equation

$$\Phi - I = X_Q P(\Phi). \tag{6.17}$$

Here $P(\Phi)\colon V \to V$ is obtained from Φ by replacing its y-component by the identity map, i.e., $P(\Phi)(x, y) = (\bar{x}, y)$ in the decomposition $V = L \oplus L'$, where $\Phi(x, y) = (\bar{x}, \bar{y})$. Note that (6.17) uniquely determines X_Q.

Furthermore, let $\widetilde{\Phi}_t$ be the linearization of $\varphi^t_{\widetilde{K}}$ at p. This family of linear symplectic transformations satisfies the equation

$$\widetilde{\Phi}_t = I + tX_Q P(\widetilde{\Phi}_t), \tag{6.18}$$

which again uniquely determines $\widetilde{\Phi}_t$.

It is clear from (6.17) and (6.18) that X_Q and $\widetilde{\Phi}_t$ depend only on the decomposition $V = L \oplus L'$. Hence, any other coordinate system compatible with this decomposition will give rise to the same quadratic form Q and the same maps $\widetilde{\Phi}_t$.

Due to (6.12), the linearization $d(\varphi^t_K)_p$ is equal to

$$\Phi_t = \exp\big((t - \lambda(t))X_Q\big)\widetilde{\Phi}_{\lambda(t)}.$$

Hence, Φ_t also depends only on the decomposition, but not on the coordinate system as long as the latter is compatible with the decomposition. In other words, all such coordinate systems result in the flows φ^t_K with linearization Φ_t.

Lemma 5.4 follows now from Lemma 5.5, which guarantees that there exist symplectic bases Ξ (or, equivalently, linear canonical coordinate systems) compatible with $V = L \oplus L'$ and making $\|\Phi - I\|_\Xi$ arbitrarily small.

Remark 6.2. Recall that $Q \le 0$ due to (K1) and that all eigenvalues of Φ are equal to one. Combining these facts with the normal forms of quadratic Hamiltonians (see [Arn74, Appendix 7] and [Wil36]), it is not hard to show that $Q = -(y_1^2 + \cdots + y_k^2)$ in some symplectic basis compatible with the decomposition $T_pW = L \oplus L'$. Then it is straightforward to write down an explicit expression for $\widetilde{\Phi}_t$ and Φ_t. This, however, does not lead to any simplification in the line of reasoning used here, for the required result readily follows from (6.17) and (6.18).

7. **Proof of Proposition** 4.7

7.1. *Outline of the proof.* First note that it is sufficient to prove the proposition for the Hamiltonian K^1 in place of H and the constant orbit p of K^1 in place of γ.

Indeed, p is a symplectically degenerate maximum of K^1 as is pointed out in Example 4.4. The Hamiltonians K^1 and H have the same time-T flow and there is a natural one-to-one correspondence between (contractible) T-periodic orbits of the Hamiltonians, for $\varphi_H^t = \eta^t \circ \varphi_{K^1}^t$ with $t \in S_T^1$. Due to our normalization of K^1, the corresponding T-periodic orbits of K^1 and H have equal actions and, in particular, $(K^1)^{(T)}$ has the same action spectrum and action filtration in the Floer complex as $H^{(T)}$; see Section 2.3. As a consequence,

$$\mathrm{HF}_{n+1}^{(Tc+\delta_T,\, Tc+\varepsilon)}\big(H^{(T)}\big) = \mathrm{HF}_{n+1}^{(Tc+\delta_T,\, Tc+\varepsilon)}\big((K^1)^{(T)}\big).$$

Thus, the proposition holds for H if (and only if) it holds for K^1. Furthermore, when H is replaced by K^1, the loops η_i^t get replaced by the loops $\eta_i^t \circ (\eta_1^t)^{-1}$ which have the identity linearization at p by (K3).

To summarize, keeping the notation H for the Hamiltonian K^1, we may assume throughout the proof that the Hamiltonian H is such that

- the point p is a strict local maximum of H_t for all $t \in S^1$, and
- $d(\eta_i^t)_p = I$ for all $t \in S^1$.

With these observations in mind, we establish the proposition by using the squeezing method of [BPS03], [GG04]. Namely, closely following [GG04], we construct functions $H_\pm$ such that $H_- \le H \le H_+$ (see Fig. 1) and such that the map Ψ_{H_+,H_-} in the filtered Floer homology for the interval $(Tc+\delta_T,\ Tc+\varepsilon)$ induced by a monotone homotopy from H_+ to H_- is nonzero. This map factors as

$$\mathrm{HF}_{n+1}^{(Tc+\delta_T,Tc+\varepsilon)}\big(H_+^{(T)}\big) \to \mathrm{HF}_{n+1}^{(Tc+\delta_T,\, Tc+\varepsilon)}\big(H^{(T)}\big) \to \mathrm{HF}_{n+1}^{(Tc+\delta_T,\, Tc+\varepsilon)}\big(H_-^{(T)}\big),$$

and, therefore, $\mathrm{HF}_{n+1}^{(Tc+\delta_T,\, Tc+\varepsilon)}\big(H^{(T)}\big) \neq 0$ as required.

The Hamiltonian H_+ depends only on H and ε. Outside a ball B_R of radius $R > 0$, centered at p, the function H_+ is constant and equal to $\max H_+$. (Here the distance is taken with respect to some fixed metric compatible with ω.) Within B_R, the Hamiltonian H_+ is a function of the distance to p, equal to $c = H(p)$ when the distance is small, dropping to some constant $a < c$, and then increasing to $\max H_+$ near the boundary of B_R.

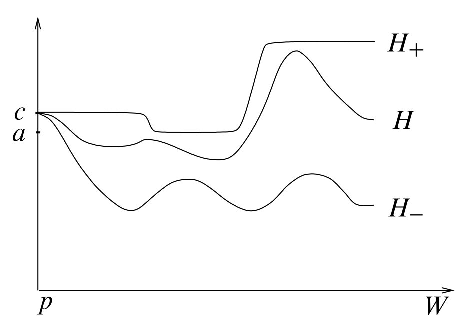

Figure 1. The functions H and $H_\pm$

The period T is required to be large enough, i.e., $T \ge T_0$, where T_0 is determined by H_+ (see §7.4) and ultimately by how fast the function H decreases on a neighborhood of p. A larger variation of H on a neighborhood of p results in a smaller period T_0. The function H_- and the constant $\delta_T > 0$ depend on $T \geq T_0$. The condition that p is a symplectically degenerate maximum of H is used in the construction of H_- and also in proving that $\Psi_{H_+,H_-} \neq 0$.

The function H_- is constructed as follows. Pick i so that $T \cdot \|d^2(K_t^i)_p\|_{\Xi^i}$ is small. (Here, as above, K_i^t is normalized by $K_t^i(p) \equiv c$.) There exists a bump function $F \leq K_i$, supported near p, with nondegenerate maximum at p and $F(p) = c$ and such that $T \cdot \|d^2 F_p\|_{\Xi^i}$ is also small. Then $\Psi_{H_+,F} \neq 0$. Setting H_- to be the Hamiltonian generating the flow $\eta_i^t \circ \varphi_F^t$, normalized by $H_-(p) \equiv c$, we note that $H_- \leq H$. Hence, $H_- \leq H \leq H_+$. The Hamiltonian H_- has the same filtered Floer homology as F, and we show that $\Psi_{H_+,H_-} \neq 0$ using (K3).

7.2. *Bump functions.* In this section we recall a few standards facts needed in the proof, concerning the filtered Floer complex of a bump function.

7.2.1. *Bump functions on* $\mathbb{R}^{2n}$. Set $\rho = (x_1^2 + \cdots + x_n^2 + y_1^2 + \cdots + y_n^2)/2$ in the standard canonical coordinates $(x_1, \dots, x_n, y_1, \dots, y_n)$ on $\mathbb{R}^{2n}$. All orbits of φ_ρ^t are closed and have period 2π. Fix a ball $B_r \subset \mathbb{R}^{2n}$ of radius $r > 0$, centered at the origin p.

Consider a rotationally symmetric function F on $\mathbb{R}^{2n}$ supported in B_r. The function F depends only on the distance to p and it will be convenient in our analysis to also view F as a function of ρ. Assume, in addition, that F has the following properties (see Fig. 2):

- F is decreasing as a function of ρ;
- $|F'| < \pi$ and $|F'|$ is increasing on some closed ball $\bar{B}_{r_-} \subset B_r$; and
- on the shell $B_r \smallsetminus B_{r_-}$,
  - F is concave, i.e., $F'' \leq 0$, on $[r_-^2/2,\ (r')^2/2]$, where $r_- < r' < r$,
  - F is convex ($F'' \geq 0$) on $[(r'')^2/2,\ r^2/2]$, where $r' < r'' < r$,
  - F has constant slope ($F' = \text{const}$) on the interval $[(r')^2/2,\ (r'')^2/2]$, where const/π is irrational.

We will refer to F as a *standard bump function* on $\mathbb{R}^{2n}$ and we will call $C := F(p)$ and r_- and r and other constants from the construction of F the parameters of F. In what follows, it will also be convenient to assume that on the interval $[0,\ (r')^2/2]$ the function F is C^0-close to the constant C, or equivalently the difference $F((r')^2/2) - C$ is small, and that F is C^0-close to zero on $[(r'')^2/2,\ r^2/2]$, i.e., $F((r'')^2/2)$ is small.

The trivial one-periodic orbits of F (i.e., its critical points) are either contained in $\bar{B}_{r'}$ or in the complement to $B_{r''}$. The orbits from the first group form a closed ball (possibly of zero radius) centered at p and have action C; the orbits from the second group are exactly the points where $F = 0$.

Nontrivial one-periodic orbits fill in spheres of radii $r_i^{\pm}$ with

$$r_- < r_1^- < r_2^- < \cdots < r' \text{ and } r'' < \cdots < r_2^+ < r_1^+ < r.$$

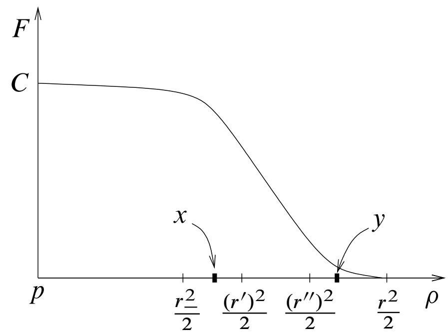

Figure 2. The bump function F

Let $\tilde{F}$ be the standard C^2-small (periodic in time) perturbation of F as the ones considered in, e.g., [FHW94], [GG04], and still supported in B_r. For such a perturbation each sphere filled in by one-periodic orbits of F breaks down into $2n$ nondegenerate orbits. Within $\bar{B}_{r_-}$, we may assume that $\tilde{F}$ is still autonomous and rotationally symmetric and $0 < |\tilde{F}'| < \pi$. (Hence the only one-periodic orbit of $\tilde{F}$ in $\bar{B}_{r_-}$ is the trivial orbit p of Conley-Zehnder index n.)

As is well-known, the filtered Floer complex of $\tilde{F}$ and the filtered Floer homology of $\tilde{F}$ (and F) are still defined, say, for any positive interval of actions $0 \leq a < b$ even though $\mathbb{R}^{2n}$ is not compact; see, e.g., [FH94], [FHW94], [FS07], [CGK04], [Gin07], [GG04]. Here we adopt the conventions of [CGK04], [Gin07], [GG04].

Pick $\varepsilon > \delta > 0$ so that

$$\varepsilon > \pi r^2, \text{ and } C - \delta > 2\pi r^2 \text{ and } \delta < \pi r_-^2. \tag{7.1}$$

We are interested in the periodic orbits of $\tilde{F}$ with indices $n-1$ or n or $n+1$ and action in the range $(C-\delta,\ C+\varepsilon)$.

It is not hard to see that $\tilde{F}$ has no one-periodic orbits of index $n-1$. Furthermore, it has exactly two one-periodic orbits of index n. One of these is the constant orbit p. The second orbit y arises from the sphere of periodic orbits farthest from the origin. This sphere has radius r_1^+ and $A_{\tilde{F}}(y) = \pi(r_1^+)^2 + \ldots$, where the dots denote an error which can be made arbitrarily small by a suitable choice of F and $\tilde{F}$; cf. [GG04, §5.2]. (There are two terms contributing to this error. The first term reflects the fact that the value of F on this sphere is not exactly zero but can be made arbitrarily close to zero. This term has order $O(r - r'')$. The second term has order $O(\|F - \tilde{F}\|_{C^1})$ and is due to the fact that x is a periodic orbit of $\tilde{F}$ rather than of F.) By (7.1), y is outside the range of action.

Finally, $\tilde{F}$ has two one-periodic orbits of index $n+1$, but only one of them, x, has action in $(C-\delta,\ C+\varepsilon)$. (The action of the second orbit is approximately equal to $\pi(r_1^+)^2$.) The orbit x arises from the sphere of periodic orbits closest to the origin. This is the sphere of radius r_1^- and $A_{\tilde{F}}(x) = C + \pi(r_1^-)^2 + \ldots$, where again the dots denote an error which can be made arbitrarily small by a suitable choice of F and $\widetilde{F}$. (To be more precise, the error is bounded from above by $(F((r')^2/2) - C) + O(\|F - \widetilde{F}\|_{C^1})$. See [CGK04], [GG04] or §7.3, where we analyze in detail periodic orbits of a function similar to F.)

It is well-known that $\partial x = y + p$ in the Floer complex of $\widetilde{F}$ for the interval $(0, \infty)$; see, e.g., [Gin07, pp. 138–139]. Summarizing these observations, we conclude that

- $\widetilde{F}$ has no one-periodic orbits of index $n - 1$;
- p is the only one-periodic orbit of $\widetilde{F}$ with index n and action in $(C - \delta,\ C + \varepsilon)$;
- x is the only one-periodic orbit of $\widetilde{F}$ with index $n + 1$ and action in $(C - \delta,\ C + \varepsilon)$;
- the connecting map from the long exact sequence
$$\mathbb{Z}_2 \cong \mathrm{HF}_{n+1}^{(C+\delta,\, C+\varepsilon)}(F) \to \mathrm{HF}_{n}^{(C-\delta,\, C+\delta)}(F) \cong \mathbb{Z}_2$$
is an isomorphism sending $[x]$ to $[p]$, and hence $\mathrm{HF}_{n}^{(C-\delta,\, C+\varepsilon)}(F) = 0$.

7.2.2. *Bump functions on a closed manifold.* Let U be a small neighborhood of $p \in W$. Fixing a canonical coordinate system on U, denote the open ball of radius $r > 0$ in U, centered at p, by B_r and let S_r be the boundary of B_r.

We define a bump function F on W exactly as for $\mathbb{R}^{2n}$ by using the coordinate system in U. Furthermore, since W is compact, now we need not assume that F is supported in B_r. Instead we just require F to be constant outside B_r. In other words, we allow F to be shifted up and down.

The description of periodic orbits of $\widetilde{F}$ and the Floer homology of F given in Section 7.2.1 extends word-for-word to this case, provided that B_r is sufficiently small (e.g., displaceable) and the variation $C - \min F$ is sufficiently large. The requirement $C > \pi r^2$ is replaced by that $C - \min F > h(B_r)$, where $h(B_r)$ depends only on B_r and goes to zero as $r \to 0$; see, e.g., [Gin07]. (Hypothetically, $h(B_r)$ is equal to the displacement energy of B_r, although the estimate we have been able to prove is somewhat weaker.) The requirement (7.1) carries over to this case unchanged when F is supported in B_r, and is, in general, replaced by

$$\varepsilon > \pi r^2, \text{ and } C - \delta > \min F + 2\pi r^2 \text{ and } \delta < \pi r_-^2. \tag{7.2}$$

7.2.3. *Connecting trajectories from x to p.* Let us show that by making r sufficiently small, we can ensure that the Floer gradient trajectories of F from x to p are close to p. (When F is a bump function on $\mathbb{R}^{2n}$, all such trajectories are contained in $\bar{B}_r$ by the maximum principle.)

To this end, pick a ball $B_R \supset \bar{B}_r$ contained in U and fix once and forever a compatible with ω almost complex structure J_0 on W coinciding with the standard complex structure on a neighborhood of $\bar{B}_R$. Consider holomorphic curves v in $\bar{B}_R \smallsetminus B_r$ with boundary in $S_r \cup S_R$ and such that the part of the boundary of v lying in S_r is nonempty. (Then the part of the boundary of v in S_R is also nonempty due to the maximum principle.) Denote by $A(r, R) > 0$ the infimum of the areas of such curves v. It is easy to see that $A(r, R)$ remains separated from zero as $r \to 0$. (Otherwise we would have $A(r, R) = 0$ for some fixed $r > 0$ as is clear from considering the intersections with $\bar{B}_R \smallsetminus B_r$ of holomorphic curves whose areas approach zero.) In other words, $\liminf_{r\to 0+} A(r, R) > 0$. Replacing R by $R/2$, we see that there exists $r_0(R, J_0) > 0$ such that

$$\pi r^2 < A(r, R/2) \text{ for all positive } r < r_0(R, J_0) < R/2. \tag{7.3}$$

LEMMA 7.1. *Let F be an arbitrary bump function F such that* (7.3) *holds and $C - \min F > h(B_r)$. Assume that $\varepsilon > 0$ and $\delta > 0$ satisfy* (7.2). *Then for a perturbation $\widetilde{F}$ of F as above and any regular perturbation of J of J_0 all Floer anti-gradient trajectories from x to p are contained in B_R.*

Proof. Assume the contrary. Then for some $\widetilde{F}$ close to F and for some sequence of regular perturbations $J_l \to J_0$, there exists a sequence of connecting trajectories u_l from x to p, leaving a neighborhood of $\bar{B}_{R/2}$. Observe that the part of u_l contained in $\bar{B}_{R/2} \smallsetminus B_r$ is a J_l-holomorphic curve. By the compactness theorem, in the limit we have a J_0-holomorphic curve v in $\bar{B}_{R/2} \smallsetminus B_r$ with nonempty boundary in S_r. By the definition of $A(r, R/2)$, the area of v is greater than $A(r, R/2)$. Therefore, the same is true for the part of u_l contained in $\bar{B}_{R/2} \smallsetminus B_r$ when J_l is close to J_0. Thus, $E(u_l) > A(r, R/2) > \pi r^2$ by (7.3). This is impossible, for

$$E(u_l) = A_{\widetilde{F}}(x) - A_{\widetilde{F}}(p) = \pi (r_1^-)^2 + \cdots < \pi r^2,$$

where as in Section 7.2.1 the dots denote an error which can be made arbitrarily small by a suitable choice of F and $\widetilde{F}$. □

7.3. *The function H_+.* Without loss of generality, we may assume that $H \geq 0$. Furthermore, throughout this section we will keep the notation and convention of Section 7.2. In particular, we fix a system of canonical coordinates on a neighborhood U of p and let, as in Section 7.2, the function ρ on U be one half of the square of the distance to p with respect to this coordinate system.

7.3.1. *The description of H_+.* Pick four balls centered at p in U:

$$B_{r_-} \subset B_r \subset B_R \subset B_{R_+} \Subset U.$$

Let H_+ be a function of ρ, also treated as a function on U, with the following properties (see Fig. 1):

- $H_+ \geq H$;
- $H_+|_{B_{r_-}} \equiv c = H(p)$;
- on the shell $B_r \smallsetminus B_{r_-}$ the function H_+ is monotone decreasing, as a function of ρ, and
  - H_+ is concave ($H_+'' \leq 0$) on $[r_-^2/2,\ (r')^2/2]$, where $r_- < r' < r$,
  - H_+ is convex ($H_+'' \geq 0$) on $[(r'')^2/2,\ r^2/2]$, where $r' < r'' < r$,

  - H_+ has constant slope ($H'_+ = \text{const}$) on the interval $[(r')^2/2,\ (r'')^2/2]$, where const/π is irrational;
- $H_+ \equiv a$ on the shell $B_R \smallsetminus B_r$, where the constant a is to be specified latter;
- H_+ is monotone increasing on the shell $B_{R_+} \smallsetminus B_R$, and
  - H_+ is convex ($H''_+ \geq 0$) on $[R^2/2,\ (R')^2/2]$, where $R < R' < R_+$,
  - H_+ is concave ($H''_+ \leq 0$) on $[(R'')^2/2,\ R_+^2/2]$, where $R' < R'' < R_+$,
  - H_+, as a function of ρ, has constant slope ($H'_+ = \text{const}$) on the interval $[(R')^2/2,\ (R'')^2/2]$, where const/π is irrational;
- $H_+ \equiv \max H_+$ on $U \smallsetminus B_{R_+}$, with the constant $\max H_+ > c$ to be specified.

Furthermore, we extend H_+ to W by setting it to be constant and equal to $\max H_+$ on the complement of U. The constant $\max H_+$ is chosen so that $H \leq H_+$ on W and $\max H_+ > c$.

Note that within B_R, the function H_+ is a standard bump function of Section 7.2. This bump function has variation $c - a$ which, due to the requirement $H_+ > H$, may be very small.

7.3.2. *The parameters of* H_+. Let us now specify the parameters of H_+. The main, but not the only, requirement on H_+ is that $H_+ \geq H$.

The neighborhood U is chosen so that $c = H(p)$ is a strict global maximum of H on U and U is displaceable in W by a Hamiltonian diffeomorphism. The values $R_+ > R > 0$ are chosen arbitrarily, with the only restriction that $B_{R_+} \Subset U$.

To pick r, we fix $\varepsilon > 0$ and also fix a compatible with ω almost complex structure J_0 on W coinciding with the standard complex structure on a neighborhood of the closed ball $\bar{B}_R$. The radius r is chosen so that

$$0 < r < r_0(R, J_0) \text{ and } \pi r^2 < \varepsilon, \tag{7.4}$$

where the upper bound $r_0(R, J_0)$ is as in Section 7.2.3. The radius $r_- > 0$ is chosen arbitrarily with the only restriction that $0 < r_- < r$.

The constants a and $\max H_+$ are picked so that $a < c$ and $H_+ \geq H$ on $[r_-^2/2,\ R^2/2]$ and on $W \smallsetminus B_{R_+}$. (This may require a to be very close to c.) Likewise, on the intervals $[r_-^2/2,\ r^2/2]$ and $[R^2/2,\ R_+^2/2]$, the behavior of H_+ is specified to guarantee that $H_+ \geq H$. Finally, we will also impose the condition that

$$\max H_+ > c + \varepsilon. \tag{7.5}$$

At this stage we fix H_+ satisfying the above requirements.

7.3.3. *Periodic orbits of* $H_+^{(T)}$. In this section, we analyze the relevant T-periodic orbits of H_+ when T is sufficiently large. Since H_+ is autonomous, its T-periodic orbits can simply be treated as one-periodic orbits of $T \cdot H_+$. Furthermore, it is clear that all T-periodic orbits of H_+ outside U are trivial. Those in U are either trivial or fill in spheres of certain radii. Replacing H_+ by its standard time-dependent C^2-small perturbation $\widetilde{H}_+$ as in [CGK04], [FHW94], [GG04] and Section 7.2 results in each of these spheres splitting into $2n$ nondegenerate orbits.

Here, as in Section 7.2, we are primarily interested in the orbits with index $n-1$ or n or $n+1$ and action in the interval $(Tc-\delta,\ Tc+\varepsilon)$ for some small $\delta>0$. We will show that these orbits are essentially the same as for a bump function with (large) variation $T\cdot(c-a)>\pi r^2$.

The perturbation $\widetilde{H}_+$ is similar to $\widetilde{F}$ from Section 7.2. We emphasize that H_+ is perturbed not only within the shells $\bar{B}_{R_+}\smallsetminus B_R$ and $\bar{B}_r\smallsetminus B_{r_-}$ where nontrivial periodic orbits are, but also within the ball B_{r_-} where $H_+\equiv c$. On this ball, $\widetilde{H}_+$ is a monotone decreasing function of ρ with a nondegenerate maximum at p equal to c. This function is C^2-close to the constant function H_+ so that (for a fixed T) the function $T\cdot\widetilde{H}_+$ is C^2-close to Tc on B_{r_-}. In particular, the eigenvalues of the Hessian $d^2(T\cdot\widetilde{H}_+)_p$ are close to zero and the Conley-Zehnder index of the constant T-periodic orbit p of $\widetilde{H}_+$ is n. In what follows, we will always assume that $\widetilde{H}_+$ is as close to H_+ as necessary. In the shell $B_R\smallsetminus B_r$ and in the complement to B_{R_+} we keep $\widetilde{H}_+$ constant and equal to H_+.

With H_+ and $\varepsilon>0$ fixed, assume throughout this subsection that T is sufficiently large and $\delta>0$ is small or, more specifically, that

$$T\cdot(c-a)>2\pi r^2+\delta,\ \text{ where } \delta<\pi r_-^2 \text{ and } \delta<c-a. \tag{7.6}$$

The trivial T-periodic orbits of H_+ are the points of $\bar{B}_{r_-}$ (with action Tc), the points of $\bar{B}_R\smallsetminus B_r$ (with action Ta), and the points of $W\smallsetminus B_R$ (with action $T\cdot\max H_+$). Here, only the points of $\bar{B}_{r_-}$ have action within the range in question. Indeed, $T\cdot\max H_+>T(c+\varepsilon)$ by (7.5) and $Ta<Tc-\delta$, for $0<\delta<c-a$ by (7.6). Thus, p is the only trivial T-periodic orbit of $\widetilde{H}_+$ with action in $(Tc-\delta,\ Tc+\varepsilon)$; it has index n.

We divide nontrivial T-periodic orbits of H_+ and $\widetilde{H}_+$ into four groups.

The first group is formed by the T-periodic orbits in the shell $\bar{B}_{r'}\smallsetminus B_{r_-}$. These orbits fill in a finite number of spheres $S_{r_l^-}$ of radii r_l^- with

$$r_-<r_1^-<r_2^-<\cdots<r'.$$

The orbits on $S_{r_l^-}$ have action $Tc+\pi(r_l^-)^2\cdot l+\dots$. Here and throughout this section, the dots denote, as in Section 7.2.1, an error which can be made arbitrarily small. Once the Hamiltonian H_+ is replaced by $\widetilde{H}_+$, a sphere $S_{r_l^-}$ breaks down into $2n$ nondegenerate orbits. The Conley-Zehnder indices of these orbits are $(2l-1)n+1,\dots,(2l+1)n$ as is proved in [CGK04], [GG04]. Only one of the orbits in this group has index from $n-1$ to $n+1$ and action in the range $(Tc-\delta,\ Tc+\varepsilon)$. This is a periodic orbit, denoted by x, of index $n+1$ and action $Tc+\pi(r_1^-)^2+\dots$, arising from the sphere $S_{r_1^-}$. The Conley-Zehnder indices of the remaining orbits are greater than $n+1$ although some of these orbits may have action within the range $(Tc-\delta,\ Tc+\varepsilon)$.

The second group consists of the T-periodic orbits in the shell $\bar{B}_r\smallsetminus B_{r''}$. These orbits fill in the spheres $S_{r_l^+}$ of radii r_l^+ with

$$r''<\cdots<r_2^+<r_1^+<r.$$

The orbits on $S_{r_l^+}$ have action $Ta + \pi(r_l^+)^2 \cdot l + \dots$. Again, once the Hamiltonian H_+ is replaced by $\widetilde{H}_+$, each sphere $S_{r_l^+}$ breaks down into $2n$ nondegenerate orbits. The Conley-Zehnder indices of these orbits are $(2l-1)n, \dots, (2l+1)n - 1$; see [CGK04], [GG04]. Only the orbits arising from $S_{r_1^+}$ and $S_{r_2^+}$ can have index $n-1$ or n or $n+1$. (Other spheres give rise to orbits of index greater than $5n - 1 > n + 1$.) However, the orbits coming from the spheres $S_{r_1^+}$ and $S_{r_2^+}$ have action not exceeding $Ta + 2\pi r^2$ if $\widetilde{H}_+$ is close to H_+. By (7.6), these orbits are outside the action range $(Tc - \delta,\ Tc + \varepsilon)$.

The T-periodic orbits in the shell $\bar{B}_{R'} \setminus B_R$ are in the third (possibly empty) group. These orbits fill in the spheres $S_{R_l^-}$ of radii R_l^- with

$$R < R_1^- < R_2^- < \cdots < R',$$

and the orbits on $S_{R_l^-}$ have action $Ta - \pi(R_l^-)^2 \cdot l + \dots$. Hence, all of these orbits are outside of the range of action $(Tc - \delta,\ Tc + \varepsilon)$.

The fourth group, which may also be empty, is formed by the T-periodic orbits in the shell $\bar{B}_{R_+} \setminus B_{R''}$. These orbits fill in a finite collection of spheres $S_{R_l^+}$ of radii R_l^+ such that

$$R'' < \cdots < R_2^+ < R_1^+ < R_+,$$

and the orbits on $S_{R_l^+}$ have action $T \cdot \max H_+ - \pi(R_l^+)^2 \cdot l + \dots$, which can be in the interval $(Tc - \delta,\ Tc + \varepsilon)$. However, calculating the Conley-Zehnder indices of the resulting orbits of $\widetilde{H}_+$ as in [CGK04], [GG04], it is easy to see that the sphere $S_{R_l^+}$ breaks down into nondegenerate orbits of $\widetilde{H}_+$ with indices $-(2l+1)n + 1, \dots, -(2l-1)n$. In particular, all resulting orbits have indices not exceeding $-n$, and none of the orbits has index $n-1$ or n or $n+1$.

To summarize, the perturbation $\widetilde{H}_+$ has only one T-periodic orbit of index n with action in $(Tc - \delta,\ Tc + \varepsilon)$ — this is the trivial orbit p — and only one orbit, namely x, of index $n+1$ with action within this range. The action of p is Tc and the action of x is $Tc + \pi(r_1^-)^2 + \dots$. There are no orbits with index $n-1$ and action in the range $(Tc - \delta,\ Tc + \varepsilon)$.

7.3.4. *The Floer homology of $H_+^{(T)}$.* As in the previous section, assume that T is sufficiently large and $\delta > 0$ is small (independently of T). Explicitly, now we require in addition to (7.6) that

$$T(c - a) > h(B_R), \tag{7.7}$$

where $h(B_R)$ is defined in Section 7.2.2. In this section we prove

LEMMA 7.2. *Under the above assumptions on the function H_+, the period T, and ε and δ, we have*

$$\mathrm{HF}_n^{(Tc-\delta,\, Tc+\delta)}\left(H_+^{(T)}\right) \cong \mathbb{Z}_2 \text{ and } \mathrm{HF}_{n+1}^{(Tc+\delta,\, Tc+\varepsilon)}\left(H_+^{(T)}\right) \cong \mathbb{Z}_2$$

with generators $[p]$ *and*, *respectively*, $[x]$. *Moreover*, *the connecting map*

$$\mathbb{Z}_2 \cong \mathrm{HF}_{n+1}^{(Tc+\delta,\, Tc+\varepsilon)}\left(H_+^{(T)}\right) \to \mathrm{HF}_n^{(Tc-\delta,\, Tc+\delta)}\left(H_+^{(T)}\right) \cong \mathbb{Z}_2 \tag{7.8}$$

is an isomorphism.

Proof. Since (7.6) is satisfied, the results of the previous section apply, and x and p are the only T-periodic orbits of $\widetilde{H}_+$ of index $n-1$ or n on $n+1$ with action in the range $(Tc-\delta,\ Tc+\varepsilon)$. It is clear that $[p]$ is the generator of $\mathrm{HF}_n^{(Tc-\delta,\, Tc+\delta)}\left(H_+^{(T)}\right) \cong \mathbb{Z}_2$. Furthermore, x is the only T-periodic orbit of index $n+1$ with action in $(Tc+\delta,\ Tc+\varepsilon)$ and there are no T-periodic orbits of index n with action in this interval. Hence, the homology $\mathrm{HF}_{n+1}^{(Tc+\delta,\, Tc+\varepsilon)}\left(H_+^{(T)}\right)$, generated by $[x]$, is either zero or $\mathbb{Z}_2$. (The former is *a priori* possible, for in fact there exists a T-periodic orbit of index $n+2$ with action in $(Tc+\delta,\ Tc+\varepsilon)$.) To finish the proof of the lemma, it is sufficient now to show that the connecting map (7.8) is onto or, equivalently,

$$\mathrm{HF}_n^{(Tc-\delta,\, Tc+\varepsilon)}\left(H_+^{(T)}\right) = 0, \tag{7.9}$$

i.e., the number of Floer anti-gradient trajectories for $\widetilde{H}_+^{(T)}$ from x to p is odd.

Within B_R, the Hamiltonian $T \cdot H_+$ coincides with a standard bump function F whose variation $C - \min F = T(c-a)$ is greater than $h(B_R)$ by (7.7). Thus, the assumptions of Section 7.2.2 are satisfied, and $\mathrm{HF}_n^{(C-\delta,\, C+\varepsilon)}(F) = 0$. Furthermore, $\widetilde{H}_+^{(T)}$ agrees with $\widetilde{F}$ on B_R. Due to our choice of r, the condition (7.3) holds and Lemma 7.1 is applicable. Therefore, $\widetilde{F}$, and hence $\widetilde{H}_+^{(T)}$, have an odd number of Floer anti-gradient trajectories from x to p contained in B_R. Moreover, every Floer anti-gradient trajectory for $\widetilde{H}_+^{(T)}$ from x to p is automatically in B_R. This is established by arguing exactly as in the proof of Lemma 7.1 with F replaced by $\widetilde{H}_+^{(T)}$ and using again (7.4). As a consequence, $\widetilde{H}_+^{(T)}$ and F have the same Floer anti-gradient trajectories from x to p, and the total number of such trajectories is odd. This concludes the proof of (7.9) and of the lemma. $\square$

7.4. *The function H_-.* Recall that the function H_+ and the parameter $\varepsilon > 0$ were fixed above, while T and δ have been variable. At this point, we also fix a large period T meeting the requirement (7.7) and such that $T \cdot (c-a) > 2\pi r^2$. Then, condition (7.6) is satisfied if $\delta > 0$ is small, and hence Lemma 7.2 applies.

In this section, we construct a Hamiltonian $H_- \geq 0$, depending on T, such that $H_- \leq H$ and $H_-(p) = c$, and the connecting map

$$\mathbb{Z}_2 \cong \mathrm{HF}_{n+1}^{(Tc+\delta_T,\, Tc+\varepsilon)}\left(H_-^{(T)}\right) \to \mathrm{HF}_n^{(Tc-\delta_T,\, Tc+\delta_T)}\left(H_-^{(T)}\right) \cong \mathbb{Z}_2 \tag{7.10}$$

is an isomorphism if $\delta_T > 0$ is sufficiently small.

Recall from Section 7.1 (see also Definition 4.1) that there exist

- a loop $\eta^t = \eta_i^t$, $t \in S^1$, of Hamiltonian diffeomorphisms fixing p and
- a system of canonical coordinates $\xi = \xi^i$ on a neighborhood V of p

such that the Hamiltonian $K = K^i$ generating the flow $(\eta^t)^{-1} \circ \varphi_H^t$ has a strict local maximum at p and

$$\max_t \|d^2(K_t)_p\|_{\xi_p} < \frac{\pi}{T}. \tag{7.11}$$

Moreover, the loop η has identity linearization at p, i.e., $d(\eta^t)_p = I$ for all $t \in S^1$, and is contractible to id in the class of loops with identity linearization at p. (See (K3) and §7.1.) Let η_s be a homotopy from η to the identity such that $d(\eta_s^t)_p \equiv I$ and let G_t^s be the one-periodic Hamiltonian generating η_s^t and normalized by $G_t^s(p) \equiv 0$. The condition $d(\eta_s^t)_p = I$ is equivalent to that $d^2(G_t^s)_p = 0$.

As usual, we normalize K by requiring that $K_t(p) \equiv c$ or, equivalently, by $H = G\#K$. Without loss of generality, we may also assume that $\bar{V} \subset B_{r_-}$, where V is the domain of the coordinate system ξ and B_{r_-} is the ball from the construction of K_+; see Section 7.3. Note that this ball is taken with respect to the original metric and is not related to ξ.

Let F be a bump function, "centered" at p, with respect to the coordinate system ξ. As in Section 7.2.2, we do not require F to be supported in V, but only constant outside V. Thus, $F \equiv \min F$ on $W \smallsetminus V$. We may assume that $\min F < a$. It is also clear that F can be chosen so that

- $F(p) = c = K(p)$ and $F \le K$ and, by (7.11),

$$\|d^2 F_p\|_{\xi_p} < \frac{2\pi}{T}. \tag{7.12}$$

Furthermore, utilizing the condition $d^2(G_t^s)_p = 0$ and the flexibility in the choice of $\min F$ (e.g., making $\min F$ large negative), we can ensure that

- $F^s := G^s\#F \le H_+$ for all s.

Then F^s is an isospectral homotopy (cf. Example 2.4) beginning with

$$H_- := G^0\#F \le G^0\#K = H \le H_+ \tag{7.13}$$

and ending with $F^1 = F$. Throughout the homotopy, $F^s(p) = c$ and $F^s \le H_+$.

The variation of $T \cdot F$, equal to $T(c - \min F)$, is much larger than $T(c-a) \ge h(B_{r_-}) \ge h(V)$. Hence, as shown in Section 7.2, we have the isomorphism

$$\mathbb{Z}_2 \cong \mathrm{HF}_{n+1}^{(Tc+\delta_T,\, Tc+\varepsilon)}(T \cdot F) \to \mathrm{HF}_n^{(Tc-\delta_T,\, Tc+\delta_T)}(T \cdot F) \cong \mathbb{Z}_2,$$

provided that $\delta_T > 0$ is sufficiently small. Finally note that $A(G^s) = 0$ for all s, for $G_t^s(p) \equiv 0$. Therefore, the functions F^s have equal filtered Floer homology for any period T. In particular, the filtered Floer homology of $H_-^{(T)}$ and of $F^{(T)}$ are the same and the latter is identical to the (one-periodic) filtered Floer homology of $T \cdot F$. Thus, we obtain the desired isomorphism (7.10).

7.5. *The monotone homotopy map.* By construction, $H_+ \ge H_-$. A monotone decreasing homotopy from H_+ to H_- induces maps of filtered Floer homology, which commute with the maps from the long exact sequence; see, e.g., [Sch00,

§2.4] or [BPS03, §4.4] and references therein. In particular, combining the monotone homotopy maps with the connecting maps (7.8) for H_+ and (7.10) for H_-, we obtain the following commutative diagram:

(7.14)
$$\begin{array}{ccc}
\mathbb{Z}_2 \cong \mathrm{HF}_{n+1}^{(Tc+\delta_T,\,Tc+\varepsilon)}\big(H_+^{(T)}\big) & \xrightarrow{\ \cong\ } & \mathrm{HF}_{n}^{(Tc-\delta_T,\,Tc+\delta_T)}\big(H_+^{(T)}\big) \cong \mathbb{Z}_2 \\
\big\downarrow & & \big\downarrow \Psi \\
\mathbb{Z}_2 \cong \mathrm{HF}_{n+1}^{(Tc+\delta_T,\,Tc+\varepsilon)}\big(H_-^{(T)}\big) & \xrightarrow{\ \cong\ } & \mathrm{HF}_{n}^{(Tc-\delta_T,\,Tc+\delta_T)}\big(H_-^{(T)}\big) \cong \mathbb{Z}_2.
\end{array}$$

To prove the theorem, it is sufficient to show that the right vertical arrow Ψ in the diagram (7.14), i.e., the homotopy map

$$\Psi\colon \mathrm{HF}_{n}^{(Tc-\delta_T,\,Tc+\delta_T)}\big(H_+^{(T)}\big) \to \mathrm{HF}_{n}^{(Tc-\delta_T,\,Tc+\delta_T)}\big(H_-^{(T)}\big)$$

is an isomorphism. Indeed, the rows of (7.14) are isomorphisms, and hence the left vertical arrow is an isomorphism whenever Ψ is an isomorphism. Since $H_- \leq H \leq H_+$ by (7.13), the left vertical arrow factors as

$$\mathrm{HF}_{n+1}^{(Tc+\delta_T,\,Tc+\varepsilon)}\big(H_+^{(T)}\big) \to \mathrm{HF}_{n+1}^{(Tc+\delta_T,\,Tc+\varepsilon)}\big(H^{(T)}\big) \to \mathrm{HF}_{n+1}^{(Tc+\delta_T,\,Tc+\varepsilon)}\big(H_-^{(T)}\big),$$

and, as a consequence, the middle group is nonzero as desired.

To show that Ψ is an isomorphism, first observe that since $F^s \leq H_+$ for all s and $F^0 = H_-$ and $F^1 = F$, the diagram

$$\begin{tikzcd}
\mathrm{HF}_{n}^{(Tc-\delta_T,\,Tc+\delta_T)}\big(H_+^{(T)}\big) \arrow[d, "\Psi"'] \arrow[dr] & \\
\mathrm{HF}_{n}^{(Tc-\delta_T,\,Tc+\delta_T)}\big(H_-^{(T)}\big) \arrow[r, "\cong"] & \mathrm{HF}_{n}^{(Tc-\delta_T,\,Tc+\delta_T)}\big(F^{(T)}\big)
\end{tikzcd}$$

is commutative, where the horizontal isomorphism is induced by the isospectral homotopy F^s and the remaining two arrows are monotone homotopy maps. (See Section 2.2.2 and, in particular, (2.5).) Recall also that H_+ and F are autonomous. It remains to prove that the diagonal arrow, which can be identified with

$$\mathbb{Z}_2 \cong \mathrm{HF}_{n}^{(Tc-\delta_T,\,Tc+\delta_T)}(T\cdot H_+) \to \mathrm{HF}_{n}^{(Tc-\delta_T,\,Tc+\delta_T)}(T\cdot F) \cong \mathbb{Z}_2$$

is an isomorphism.

Consider a C^2-small autonomous perturbation $\widehat{H}_+ \geq F$ of H_+ such that $d^2(\widehat{H}_+)_p$ is negative definite and $\mathcal{S}(\widehat{H}_+) = \mathcal{S}(H_+)$. (It is straightforward to construct $\widehat{H}_+$ by modifying H_+ on a neighborhood of $\bar{B}_{r_-}$.) Then $\widehat{H}_+(p) \equiv c$ and $\|d^2(\widehat{H}_+)_p\|$ can be made arbitrarily small, for $d^2(H_+)_p = 0$. We take $\widehat{H}_+$ such that $T\cdot\widehat{H}_+$ is C^2-close to $T\cdot H_+$, and, in particular, $\|d^2(T\cdot\widehat{H}_+)_p\|$ is small. Essentially by definition, the filtered Floer homology of $T\cdot\widehat{H}_+$ is isomorphic to the filtered Floer homology of $T\cdot H_+$, and it suffices to show that

$$\mathbb{Z}_2 \cong \mathrm{HF}_{n}^{(Tc-\delta_T,\,Tc+\delta_T)}(T\cdot\widehat{H}_+) \to \mathrm{HF}_{n}^{(Tc-\delta_T,\,Tc+\delta_T)}(T\cdot F) \cong \mathbb{Z}_2 \tag{7.15}$$

is an isomorphism for some small $\delta_T > 0$ independent of the choice of $\widehat{H}_+$.

Recall that Tc is an isolated action value of $T \cdot F$ and $T \cdot H_+$, and hence of $T \cdot \widehat{H}_+$, for a generic choice of parameters of these functions. Fix $\delta_T > 0$ meeting the requirement (7.6) and such that Tc is the only action value of these Hamiltonians in $(Tc - \delta_T,\ Tc + \delta_T)$. Consider the linear decreasing homotopy $\widehat{H}^s = (1-s)\widehat{H}_+ + sF$. Since both of the Hessians $d^2(\widehat{H}_+)_p$ and $d^2 F_p$ are negative definite, $d^2(T \cdot \widehat{H}^s_+)_p$ is also negative definite. Thus, p is a nondegenerate critical point of $\widehat{H}^s_+$ for all $s \in [0, 1]$ with $\widehat{H}^s_+(p) = c$. Furthermore, by (7.12) and since $\|d^2(T \cdot \widehat{H}_+)_p\|$ is small, $\|d^2(T \cdot \widehat{H}^s_+)_p\| < 2\pi$. As a consequence, p is a uniformly isolated one-periodic orbit of $T \cdot \widehat{H}^s_+$; see, e.g., [SZ92, pp. 184–185] or Section 3.3. By Lemma 2.5, the homotopy map (7.15) is nonzero, and hence an isomorphism.

This concludes the proof of Proposition 4.7 and of Theorem 1.1. A slightly different proof of the proposition, although based on the same ideas as the present argument and following the same line of reasoning, can be found in [GG09, §5].

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