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Duality via cycle complexes

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Dedicated to the memory of Hermann Braun

Abstract

We show that Bloch's complex of relative zero-cycles can be used as a dualizing complex over perfect fields and number rings. This leads to duality theorems for torsion sheaves on arbitrary separated schemes of finite type over algebraically closed fields, finite fields, local fields of mixed characteristic, and rings of integers in number rings, generalizing results which so far have only been known for smooth schemes or in low dimensions, and unifying the p -adic and l -adic theory. As an application, we generalize Rojzman's theorem to normal, projective schemes.

1. Introduction

If $f : X \rightarrow S$ is separated and of finite type, then in order to obtain duality theorems from the adjointness

$$R \operatorname{Hom}_X(\mathcal{G}, Rf^! \mathcal{F}) \cong R \operatorname{Hom}_S(Rf_! \mathcal{G}, \mathcal{F})$$

for torsion étale sheaves \mathcal{G} on X and \mathcal{F} on S , one has to identify the complex $Rf^! \mathcal{F}$. For example, if f is smooth of relative dimension d and if m is invertible on S , then Poincaré duality of [1, Exp. XVIII] states that $Rf^! \mathcal{F} \cong f^* \mathcal{F} \otimes \mu_m^{\otimes d}[2d]$ for m -torsion sheaves \mathcal{F} . We show that if S is the spectrum of a perfect field, or a Dedekind ring of characteristic 0 with perfect residue fields, then Bloch's complex of zero-cycles can be used to explicitly calculate $Rf^! \mathcal{F}$.

For a scheme X essentially of finite type over S , let \mathbb{Z}_X^c be the complex of étale sheaves which in degree $-i$ associates to $U \rightarrow X$ the free abelian group generated by cycles of relative dimension i over S on $U \times_S \Delta^i$ which meet all faces properly, and alternating sum of intersection with faces as differentials [5]. Over a field, the higher Chow group of zero-cycles $\operatorname{CH}_0(X, i)$ is by definition the $-i$ th cohomology of the global sections $\mathbb{Z}_X^c(X)$, and if X is smooth of relative dimension d over a perfect field k of characteristic p , then $\mathbb{Z}_X^c/m \cong \mu_m^{\otimes d}[2d]$ for m prime to p , and $\mathbb{Z}_X^c/p^r \cong \nu_r^d[d]$, the logarithmic de Rham-Witt sheaf [15], [14].

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Our main result is that if $f : X \rightarrow Y$ is separated and of finite type over a perfect field k , and if \mathcal{F} is a torsion sheaf on X , then there is a quasi-isomorphism

$$(1) \quad R \operatorname{Hom}_X(\mathcal{F}, \mathbb{Z}_X^c) \cong R \operatorname{Hom}_Y(Rf_! \mathcal{F}, \mathbb{Z}_Y^c).$$

If k is algebraically closed and \mathcal{F} is constructible, then this yields perfect pairings of finite groups

$$\operatorname{Ext}_X^{1-i}(\mathcal{F}, \mathbb{Z}_X^c) \times H_c^i(X_{\text{et}}, \mathcal{F}) \rightarrow \mathbb{Q}/\mathbb{Z}.$$

In particular, we obtain an isomorphism $\operatorname{CH}_0(X, i, \mathbb{Z}/m) \cong H_c^i(X_{\text{et}}, \mathbb{Z}/m)^*$ of finite groups, generalizing a theorem of Suslin [45] to arbitrary m , and an isomorphism of the abelianized (pro-finite) fundamental group $\pi_1^{\text{ab}}(X)$ with $\operatorname{CH}_0(X, 1, \hat{\mathbb{Z}})$ for any proper scheme X over k . As an application, we generalize Rojtman’s theorem [33], [38] to normal schemes X , projective over an algebraically closed field: The albanese map induces an isomorphism

$$\operatorname{tor} \operatorname{CH}_0(X) \cong \operatorname{tor} \operatorname{Alb}_X(k)$$

between the torsion points of the Chow group of zero-cycles on X and the torsion points of the albanese variety in the sense of Serre [42]. This is a homological version of Rojtman’s theorem which differs from the cohomological version of Levine and Krishna-Srinivas [28], [26] relating the albanese variety in the sense of Lang-Weil to the Chow group defined by Levine-Weibel [30]. We give an example to show that for nonnormal schemes, the torsion elements of $\operatorname{CH}_0(X)$ cannot be parametrized by an abelian variety in general.

If k is finite, and X is separated and of finite type over k , we obtain for constructible sheaves \mathcal{F} perfect pairings of finite groups

$$\operatorname{Ext}_X^{2-i}(\mathcal{F}, \mathbb{Z}_X^c) \times H_c^i(X_{\text{et}}, \mathcal{F}) \rightarrow \mathbb{Q}/\mathbb{Z}.$$

This generalizes results of Deninger [9] for curves, Spieß [44] for surfaces, and Milne [35] and Moser [36] for the p -part in characteristic p . In fact, the dualizing complex of Deninger is quasi-isomorphic to \mathbb{Z}_X^c by a result of Nart [37], and the niveau spectral sequence of \mathbb{Z}_X^c/p^r degenerates to the dualizing complex of Moser. If $\pi_1^{\text{ab}}(X)^0$ is the kernel of $\pi_1^{\text{ab}}(X) \rightarrow \operatorname{Gal}(k)$, then for X proper and $\bar{X} = X \times_k \bar{k}$, we obtain a short exact sequence

$$0 \rightarrow \operatorname{CH}_0(\bar{X}, 1)_G^\wedge \rightarrow \pi_1^{\text{ab}}(X)^0 \rightarrow \operatorname{CH}_0(\bar{X})^G \rightarrow 0.$$

We obtain a similar duality theorem for schemes over local fields of characteristic 0.

If $f : X \rightarrow S$ is a scheme over the spectrum of a Dedekind ring of characteristic 0 with perfect residue fields, then, assuming the Beilinson-Lichtenbaum conjecture, there is a quasi-isomorphism

$$(2) \quad R \operatorname{Hom}_X(\mathcal{F}, \mathbb{Z}_X^c) \cong R \operatorname{Hom}_S(Rf_! \mathcal{F}, \mathbb{Z}_S^c).$$

Even though the duality theorem over a field of characteristic p could in principle be formulated by treating the prime to p -part and p -part separately, it is clear that over a Dedekind ring one needs a complex which treats both cases uniformly.

If S is the ring of integers in a number ring, and if we define cohomology with compact support $H_c^i(X_{\text{et}}, \mathcal{F})$ as the cohomology of the complex $R\Gamma_c(S_{\text{et}}, Rf_!\mathcal{F})$, where $R\Gamma_c(S_{\text{et}}, -)$ is cohomology with compact support of S [31], [34], then combining (2) with Artin-Verdier duality, we get perfect pairings of finite groups

$$\text{Ext}_X^{2-i}(\mathcal{F}, \mathbb{Z}_X^c) \times H_c^i(X_{\text{et}}, \mathcal{F}) \rightarrow \mathbb{Q}/\mathbb{Z}$$

for constructible \mathcal{F} . This generalizes results of Artin-Verdier [31] for $\dim X = 1$, Milne [34] for $\dim X = 1$ and X possibly singular, or X smooth over S , and Spieß [44] for $\dim X = 2$.

If S is a henselian discrete valuation ring of mixed characteristic with closed point $i : s \rightarrow S$, and if we define cohomology with compact support in the closed fiber $H_{X_s, c}^i(X_{\text{et}}, \mathcal{F})$ to be the cohomology of $R\Gamma(S_{\text{et}}, i_* Ri^! Rf_!\mathcal{F})$, then there are perfect pairings of finite groups

$$\text{Ext}_X^{2-i}(\mathcal{F}, \mathbb{Z}_X^c) \times H_{X_s, c}^i(X_{\text{et}}, \mathcal{F}) \rightarrow \mathbb{Q}/\mathbb{Z}$$

for constructible \mathcal{F} .

We outline the proof of our main theorem (1). The key observation is that for $i : Z \rightarrow X$ a closed embedding over a perfect field, we have a quasi-isomorphism of complexes of étale sheaves $Ri^! \mathbb{Z}_X^c \cong \mathbb{Z}_Z^c$ (purity). In order to prove purity, we show that \mathbb{Z}_X^c has étale hypercohomological descent over algebraically closed fields (i.e., its cohomology and étale hypercohomology agree), and then use purity for the cohomology of \mathbb{Z}_X^c proved by Bloch [4] and Levine [29]. To prove étale hypercohomological descent, we use the argument of Thomason [46] to reduce to finitely generated fields over k , and in this case use results of Suslin [45] for the prime to p -part, and Geisser-Levine [14] and Bloch-Kato [6] for the p -part.

Having purity, an induction and devissage argument is used to reduce to the case of a constant sheaf on a smooth and proper scheme, in which case we check that our pairing agrees with the classical pairing of [1, XVIII] and Milne [35] for the prime to p and p -primary part, respectively.

Throughout the paper, scheme over S denotes a separated scheme of finite type over S . We always work on the small étale site of a scheme. For an abelian group A , we denote by $A^* = \text{Hom}(A, \mathbb{Q}/\mathbb{Z})$ its Pontrjagin dual, by $A^\wedge = \varprojlim A/m$ its pro-finite completion, by ${}_m A$ the m -torsion of A , and by $TA = \varprojlim_m A$ its Tate-module.

2. The dualizing complex

We recall some properties of Bloch’s higher Chow complex [5], see [12] for a survey and references. For a fixed regular scheme S of finite Krull dimension d , and an integral scheme X essentially of finite type over S , the relative dimension

of X over S is

$$\dim_S X = \text{trdeg}(k(X) : k(\mathfrak{p})) - \text{ht } \mathfrak{p} + d.$$

Here $k(X)$ is the function field of X , \mathfrak{p} is the image of the generic point of X in S , and $k(\mathfrak{p})$ its residue field. For example, if K is an extension field of transcendence degree r over k , then the dimension of K over k is r . For a scheme X essentially of finite type over S , we define $z_n(X, i)$ to be the free abelian group generated by closed integral subschemes of relative dimension $n + i$ over S on $X \times_S \Delta^i$ which meet all faces properly. If $z_n(X, *)$ is the complex of abelian groups obtained by taking the alternating sum of intersection with face maps as differentials, then $z_n(-, *)$ is a (homological) complex of sheaves for the étale topology [11]. We define $\mathbb{Z}_X^c(n) = z_n(-, *)[2n]$ to be the (cohomological) complex with the étale sheaf $z_n(-, -i - 2n)$ in degree i . If $n = 0$, then we sometimes write \mathbb{Z}_X^c instead of $\mathbb{Z}_X^c(0)$, and we sometimes omit X if there is no ambiguity. For a quasi-finite, flat map $f : X \rightarrow Y$, we have a pull-back $f^*\mathbb{Z}_Y^c(n) \rightarrow \mathbb{Z}_X^c(n)$ because $z_n(-, *)$ is contravariant for such maps, and for a proper map $f : X \rightarrow Y$ we have a push-forward $f_*\mathbb{Z}_X^c(n) \rightarrow \mathbb{Z}_Y^c(n)$ because $z_n(-, *)$ is covariant for such maps. We frequently use that $\mathbb{Z}_X^c(n)$ is a complex of flat sheaves, hence tensor product and derived tensor product with $\mathbb{Z}_X^c(n)$ agree.

From now on we assume that the base S is the spectrum of a field or of a Dedekind ring. Then for a closed embedding $i : Z \rightarrow X$ over S , we have a quasi-isomorphism $\mathbb{Z}_Z^c(n) \cong Ri^!\mathbb{Z}_X^c(n)$ on the Zariski-site [4], [29]. We will refer to this fact as purity or the localization property; the fact that the analog statement holds on the étale site if $n \leq 0$ and the residue fields of S are perfect is a key result of this paper. If $p : X \times \mathbb{A}^r \rightarrow X$ is the projection, we have a quasi-isomorphism of complexes of Zariski-sheaves

$$(3) \quad p_*\mathbb{Z}_{X \times \mathbb{A}^r}^c(n) \cong \mathbb{Z}_X^c(n - r)[2r].$$

For a Grothendieck topology t , we define

$$H_i(X_t, \mathbb{Z}^c(n)) = H^{-i} R\Gamma(X_t, \mathbb{Z}^c(n)).$$

We sometimes omit the t when we use the Zariski topology, and note that for X over a field or discrete valuation ring [5], [29]

$$\text{CH}_n(X, i - 2n) \cong H_i(X_{\text{Zar}}, \mathbb{Z}^c(n)).$$

If $\mathbb{Z}_X(n)$ is the motivic complex of Voevodsky [47], then on a smooth scheme X of dimension d over a field k ,

$$(4) \quad \mathbb{Z}_X^c(n) \cong \mathbb{Z}_X(d - n)[2d].$$

The Beilinson-Lichtenbaum conjecture (translated into homological notation by (4)) states that for a scheme X as above, and m prime to the characteristic of k , the change of topology map induces an isomorphism

$$H_i(X_{\text{Zar}}, \mathbb{Z}_X^c/m(n)) \xrightarrow{\sim} H_i(X_{\text{et}}, \mathbb{Z}_X^c/m(n))$$

for $i \geq d + n$. If m is a power of the characteristic of p , then the analog statement is known [14], and with \mathbb{Q} -coefficients, Zariski and étale hypercohomology agree for smooth schemes and all i . The Beilinson-Lichtenbaum conjecture is implied by the Bloch-Kato conjecture stating that for any field F of characteristic prime to m , the norm residue homomorphism $K_n^M(F)/m \rightarrow H^n(F_{\text{et}}, \mu_m^{\otimes n})$ is an isomorphism for all n [15]. A proof of the Bloch-Kato conjecture is announced by Rost and Voevodsky, but since there is no published account at this time, we will point out any use of the Beilinson-Lichtenbaum conjecture.

We define $H_i(F_t, \mathbb{Z}^c(n)) = * \text{colim}_U H_i(U_t, \mathbb{Z}^c(n))$ for F a finitely generated field over S , where the colimit runs through U of finite type over S with field of functions F . Then $H_i(F_t, \mathbb{Z}^c(n)) \cong H^{2d-i}(F_t, \mathbb{Z}(d-n))$ if F has dimension d over S by (4). In particular, this group vanishes for the Zariski topology if $i < d + n$, and agrees with $K_{d-n}^M(F)$ for $i = d + n$. The Beilinson-Lichtenbaum conjecture implies that $H_i(F, \mathbb{Z}^c(n)) \cong H_i(F_{\text{et}}, \mathbb{Z}^c(n))$ for $i \geq d + n$.

PROPOSITION 2.1. *Let X be a scheme over a field or Dedekind ring. Then there are spectral sequences*

$$(5) \quad E_{s,t}^1 = \bigoplus_{x \in X_{(s)}} H^{s-t}(k(x), \mathbb{Z}(s-n)) \Rightarrow H_{s+t}(X, \mathbb{Z}^c(n)).$$

In particular, $H_i(X, \mathbb{Z}^c(n)) = 0$ for $i < n$.

Proof. If we let $F_s \mathbb{Z}^c(n)$ be the subcomplex generated by cycles of dimension $n + i$ on $X \times \Delta^i$ such that the projection to X has dimension at most s over S , then we get the spectral sequence

$$E_{s,t}^1 = H_{s+t}(X, F_s/F_{s-1} \mathbb{Z}^c(n)) \Rightarrow H_{s+t}(X, \mathbb{Z}^c(n)).$$

By the localization property, we get

$$\begin{aligned} H_{s+t}(X, F_s/F_{s-1} \mathbb{Z}^c(n)) &\cong \bigoplus_{x \in X_{(s)}} H_{s+t}(k(x), \mathbb{Z}^c(n)) \\ &\cong \bigoplus_{x \in X_{(s)}} H^{s-t}(k(x), \mathbb{Z}(s-n)). \end{aligned}$$

An inspection shows that $E_{s,t}^1 = 0$ for $t < n$, hence the vanishing. □

Recall that $v_r^i = W_r \Omega_{X, \log}^i$ is the logarithmic de Rham-Witt sheaf. The following proposition is a finite coefficient-version of (4):

PROPOSITION 2.2. *Let X be smooth of dimension d over a perfect field k , and let $n \leq d$. Then there are quasi-isomorphisms of complexes of étale sheaves*

$$\begin{aligned} \mathbb{Z}_X^c/m(n) &\cong \mu_m^{\otimes d-n}[2d] && \text{if } \text{char } k \nmid m; \\ \mathbb{Z}_X^c/p^r(n) &\cong v_r^{d-n}[d+n] && \text{if } p = \text{char } k. \end{aligned}$$

This is compatible with the Gysin maps $H_j(Z_{\text{et}}, \mathbb{Z}_Z^c(n)) \rightarrow H_j(X_{\text{et}}, \mathbb{Z}_X^c(n))$, for closed embeddings $i : Z \rightarrow X$ of pure codimension between smooth schemes.

Proof. The prime to p -part has been proved in [15, Th. 4.14, Prop. 4.5(2)]. For the p -primary part, the quasi-isomorphism is given by [14]

$$\begin{aligned} \mathbb{Z}^c/p^r(n)[-d-n] &\xleftarrow{\sim} \tau_{\leq 0}(\mathbb{Z}^c/p^r(n)[-d-n]) \xrightarrow{\sim} \mathcal{H}^{-d-n}(\mathbb{Z}^c/p^r(n)) \\ &\xrightarrow{\sim} G(\mathcal{H}^{-d-n}(\mathbb{Z}^c/p^r(n))) \cong G(v_r^{d-n}) \xleftarrow{\sim} v_r^{d-n}, \end{aligned}$$

where $G(v_r^{d-n})$ is the Gersten resolution arising as the E^1 -complex of the niveau spectral sequence (5)

$$\bigoplus_{x \in X^{(0)}} (i_x)_* v_{r,k(x)}^{d-n} \rightarrow \bigoplus_{x \in X^{(1)}} (i_x)_* v_{r,k(x)}^{d-n-1} \rightarrow \dots,$$

and similarly for $\mathcal{H}^{-d-n}(\mathbb{Z}^c/p^r(n))$. The Gersten resolutions identify via the isomorphisms $H_{d+n-i}(k(x), \mathbb{Z}^c/p^r(n)) \xleftarrow{\sim} K_{d-n-i}^M(k(x))/p^r \xrightarrow{\sim} v_{r,k(x)}^{d-n-i}$ for a field $k(x)$ of codimension i , i.e. transcendence degree $d-i$. The compatibility of cohomology with proper push-forward and flat equidimensional pull-back follows from the corresponding property of the Gersten resolution [16, Prop. 1.18]. \square

The p -primary part of Proposition 2.2 can be generalized to singular schemes. Let $\tilde{v}_r(0)$ be the complex of étale sheaves

$$\dots \rightarrow \bigoplus_{x \in X_{(1)}} (i_x)_* v_{r,k(x)}^1 \rightarrow \bigoplus_{x \in X_{(0)}} (i_x)_* v_{r,k(x)}^0 \rightarrow 0$$

used by Moser [36, 1.5] (loc. cit. indexes by codimension, which makes the treatment more complicated).

PROPOSITION 2.3. *Let X be a separated scheme of finite type over a perfect field k of characteristic p . Then $\mathbb{Z}^c/p(n) \cong 0$ for $n < 0$, and there are isomorphisms of étale sheaves $\mathcal{H}_i(\mathbb{Z}^c/p^r(0)) \cong \mathcal{H}_i(\tilde{v}_r(0))$, compatible with proper push-forward. In particular, $H_i(X_{\text{ét}}, \mathbb{Z}^c/p^r(0)) \cong H_i(X_{\text{ét}}, \tilde{v}_r(0))$.*

Proof. If $n < 0$, let R be a finitely generated algebra over k . Write R as a quotient of a smooth algebra A , and let $U = \text{Spec } A - \text{Spec } R$. Then the localization sequence for higher Chow groups

$$\dots \rightarrow H_{i+1}(U, \mathbb{Z}^c/p(n)) \rightarrow H_i(\text{Spec } R, \mathbb{Z}^c/p(n)) \rightarrow H_i(\text{Spec } A, \mathbb{Z}^c/p(n)) \rightarrow \dots$$

together with the fact that $H_i(X_{\text{Zar}}, \mathbb{Z}^c/p(n)) = 0$ for smooth X and $n < 0$ [14] shows that $H_i(\text{Spec } R, \mathbb{Z}^c/p(n)) \cong H_i(\mathbb{Z}^c/p(n)(\text{Spec } R)) \cong 0$. For $n = 0$, we can assume that k is algebraically closed. Consider the spectral sequence (5) with mod p^r -coefficients. The $E_{s,t}^1$ -terms vanish for $t < 0$, and according to [14], they also vanish for $t > 0$. Since $H^s(k(x), \mathbb{Z}/p^r(s)) \cong H^0(k(x)_{\text{ét}}, v_r^s)$, the cohomology of $\mathbb{Z}^c/p^r(0)(X)$ agrees with the cohomology of the complex $\tilde{v}_r(0)(X)$ in a functorial way. \square

It would be interesting to write down a map $\mathbb{Z}^c/p^r(0) \rightarrow \tilde{v}_r(0)$ of complexes inducing the isomorphism on cohomology of Proposition 2.3. We will see below

that there is a quasi-isomorphism $\mathbb{Z}^c/p^r(0) \xrightarrow{\sim} Rf^1\mathbb{Z}/p^r$. On the other hand, Jannsen, Saito, and Sato [20] show that there is a quasi-isomorphism $\tilde{v}_r(0) \xrightarrow{\sim} Rf^1\mathbb{Z}/p^r$.

LEMMA 2.4. *If \mathcal{F} is an m -torsion sheaf, then we have a quasi-isomorphism*

$$R\mathrm{Hom}_{X,\mathbb{Z}/m}(\mathcal{F}, \mathbb{Z}_X^c/m)[-1] \cong R\mathrm{Hom}_X(\mathcal{F}, \mathbb{Z}_X^c).$$

Proof. The exact, fully faithful inclusion functor $F : \mathrm{Shv}_{\mathbb{Z}/m}(X) \rightarrow \mathrm{Shv}_{\mathbb{Z}}(X)$ from étale sheaves of \mathbb{Z}/m -modules to étale sheaves of abelian groups has the left adjoint $- \otimes_{\mathbb{Z}} \mathbb{Z}/m$ and the right adjoint “ m -torsion” $T_m = \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/m, -)$; in particular, T_m preserves injectives. Moreover, the left derived functor $- \otimes_{\mathbb{Z}}^L \mathbb{Z}/m$ of the tensor product agrees with the shift $RT_m[1]$ of the right derived functor of the m -torsion functor as functors $D(\mathrm{Shv}_{\mathbb{Z}}(X)) \rightarrow D(\mathrm{Shv}_{\mathbb{Z}/m}(X))$. Indeed, both are quasi-isomorphic to the double complex $C^\cdot \xrightarrow{\cdot m} C^\cdot$. Since \mathbb{Z}_X^c consists of flat sheaves, we have $\mathbb{Z}_X^c/m \cong \mathbb{Z}_X^c \otimes^L \mathbb{Z}/m \cong RT_m\mathbb{Z}_X^c[1]$. The lemma follows from

$$\begin{aligned} R\mathrm{Hom}_X(F\mathcal{F}, \mathbb{Z}_X^c) &\cong R\mathrm{Hom}_{X,\mathbb{Z}/m}(\mathcal{F}, RT_m\mathbb{Z}_X^c) \\ &\cong R\mathrm{Hom}_{X,\mathbb{Z}/m}(\mathcal{F}, \mathbb{Z}_X^c/m[-1]). \end{aligned} \quad \square$$

We also use frequently that for a complex of torsion abelian groups C^\cdot we have $\mathrm{Hom}(C^\cdot, \mathbb{Q}/\mathbb{Z})[-1] \cong R\mathrm{Hom}(C^\cdot, \mathbb{Z})$, and in particular $H^i R\mathrm{Hom}(C^\cdot, \mathbb{Z}) \cong H^{1-i}(C^\cdot)^*$.

3. Étale descent

The main result of this section is purity and a trace map for $\mathbb{Z}^c(0)$. We give a conceptual proof assuming the Beilinson-Lichtenbaum conjecture, and an ad-hoc proof of a weaker, but for our purposes sufficient result, avoiding the use of the Beilinson-Lichtenbaum conjecture. We first use an argument of Thomason [46] to show that for $n \leq 0$, $\mathbb{Z}^c(n)$ has étale cohomological descent.

THEOREM 3.1. *Assume that the Beilinson-Lichtenbaum conjecture holds for schemes over the algebraically closed field k . If X is a scheme over k and $n \leq 0$, then $\mathbb{Z}^c(n)(X) \cong R\Gamma(X_{\mathrm{et}}, \mathbb{Z}^c(n))$.*

Proof. Since $\mathbb{Z}^c(n)$ satisfies the localization property, we can apply the argument of Thomason [46, Prop. 2.8] using induction on the dimension of X , to reduce to showing that for an artinian local ring R , essentially of finite type over k , we have $\mathbb{Z}^c(n)(\mathrm{Spec} R) \cong R\Gamma(\mathrm{Spec} R_{\mathrm{et}}, \mathbb{Z}^c(n))$ for $n \leq 0$. Since $\mathbb{Z}^c(n)(U) = \mathbb{Z}^c(n)(U^{\mathrm{red}})$, we can assume that R is reduced, in which case it is the spectrum of a field F of finite transcendence degree d over k . We have to show that the canonical map $H_i(F, \mathbb{Z}^c(n)) \rightarrow H_i(F_{\mathrm{et}}, \mathbb{Z}^c(n))$ is an isomorphism for all i and $n \leq 0$. Rationally, Zariski and étale hypercohomology of the motivic complex agree. With prime to p -coefficients, both sides agree for $i \geq d+n$ by the Beilinson-Lichtenbaum conjecture. For $i < d$ both sides vanish because by Proposition 2.2, $H_i(F_{\mathrm{et}}, \mathbb{Z}^c/m(n)) = H^{2d-i}(F_{\mathrm{et}}, \mu_m^{\otimes(d-n)})$ and because the cohomological

dimension of F is d . With mod p -coefficients, both sides agree for $n = 0$ because $\mathbb{Z}^c/p^r(0) \cong v_r^d[d] \cong R\varepsilon_*v_r^d[d] \cong R\varepsilon_*\mathbb{Z}^c/p^r(0)$ from Proposition 2.2 and [23], where ε denotes the map from the small étale site to the Zariski site. Finally, $\mathbb{Z}/p^r(n) = 0$ for $n < 0$. \square

We remark that the same argument gives an unconditional result for the cycle complex $\mathbb{Z}_{(p)}^c(n)$ localized at p .

COROLLARY 3.2. *Let $f : X \rightarrow Y$ be a map over the perfect field k , and $n \leq 0$. If f is proper, then there is a functorial push-forward $f_* : Rf_*\mathbb{Z}_X^c(n) \rightarrow \mathbb{Z}_Y^c(n)$ in the derived category of étale sheaves. For arbitrary f , we obtain for every torsion sheaf \mathcal{G} on Y a functorial map*

$$(6) \quad Rf_!(f^*\mathcal{G} \otimes \mathbb{Z}_X^c(n)) \rightarrow \mathcal{G} \otimes \mathbb{Z}_Y^c(n).$$

Proof. For proper f , the map is given as

$$Rf_*\mathbb{Z}_X^c(n) \xleftarrow{\sim} \varepsilon^*Rf_*^{\text{Zar}}\mathbb{Z}_X^c(n)^{\text{Zar}} \rightarrow \varepsilon^*\mathbb{Z}_Y^c(n)^{\text{Zar}} \cong \mathbb{Z}_Y^c(n).$$

The second map is the map induced by the proper push-forward of higher Chow groups. The first map is the base-change map between push-forward on the Zariski-site and push-forward on the étale site $\varepsilon^*f_*^{\text{Zar}} \rightarrow f_*^{\text{ét}}\varepsilon^*$. The push-forward on the Zariski site $\varepsilon^*Rf_*^{\text{Zar}}\mathbb{Z}_X^c(n)^{\text{Zar}}$ and on the étale site $Rf_*\mathbb{Z}_X^c(n)$ are the complexes of étale sheaves on Y associated to the complexes of presheaves $U \mapsto R\Gamma_{\text{Zar}}(f^{-1}U, \mathbb{Z}^c(n))$ and $U \mapsto R\Gamma_{\text{ét}}(f^{-1}U, \mathbb{Z}^c(n))$, respectively. Showing that these complexes are quasi-isomorphic is a problem local for the étale topology, hence we can assume that k is algebraically closed. In this case, the base change map is a quasi-isomorphism by Theorem 3.1.

For arbitrary $f : X \rightarrow Y$, factor f through a compactification $X \xrightarrow{j} T \xrightarrow{g} Y$. Writing \mathcal{G} as a direct limit of m -torsion sheaves we can assume that \mathcal{G} is m -torsion and replace $\mathcal{G} \otimes \mathbb{Z}_Y^c(n)$ by $\mathcal{G} \otimes_{\mathbb{Z}/m} \mathbb{Z}_Y^c/m(n)$. Then using the proper base-change theorem, we obtain a map

$$\begin{aligned} Rf_!(f^*\mathcal{G} \otimes_{\mathbb{Z}/m} \mathbb{Z}_X^c/m(n)) &= Rg_*j_!j^*(g^*\mathcal{G} \otimes_{\mathbb{Z}/m} \mathbb{Z}_T^c/m(n)) \\ &\rightarrow Rg_*(g^*\mathcal{G} \otimes_{\mathbb{Z}/m} \mathbb{Z}_T^c/m(n)) \xleftarrow{\sim} \mathcal{G} \otimes_{\mathbb{Z}/m}^{\mathbb{L}} Rg_*\mathbb{Z}_T^c/m(n) \rightarrow \mathcal{G} \otimes_{\mathbb{Z}/m} \mathbb{Z}_Y^c/m(n). \end{aligned}$$

The usual argument comparing compactifications shows that the composition is independent of the compactification. \square

Under the identification of Proposition 2.3, the trace map with mod p^r -coefficients and for $n = 0$ was constructed by Jannsen, Saito, and Sato [20]. If $f : X \rightarrow k$ is proper over a perfect field, then for $n = 0$, the trace map agrees on the stalk $\text{Spec } \bar{k}$ with the map sending a complex to its highest cohomology group, composed with the degree map,

$$\text{tr} : Rf_*\mathbb{Z}_X^c(\bar{k}) \xleftarrow{\sim} \mathbb{Z}_X^c(X_{\bar{k}}) \rightarrow \text{CH}_0(X_{\bar{k}}) \xrightarrow{\text{deg}_{\bar{k}}} \mathbb{Z}.$$

COROLLARY 3.3. a) Let $i : Z \rightarrow X$ be a closed embedding with open complement U over a perfect field k . Then for every $n \leq 0$, we have a quasi-isomorphism $\mathbb{Z}_Z^c(n) \xrightarrow{\sim} Ri^!\mathbb{Z}_X^c(n)$, or equivalently a distinguished triangle

$$\cdots \rightarrow R\Gamma(Z_{\text{et}}, \mathbb{Z}_Z^c(n)) \rightarrow R\Gamma(X_{\text{et}}, \mathbb{Z}_X^c(n)) \rightarrow R\Gamma(U_{\text{et}}, \mathbb{Z}_U^c(n)) \rightarrow \cdots .$$

b) If $p : X \times \mathbb{A}^r \rightarrow X$ is the projection and $n \leq 0$, then we have a quasi-isomorphism of complexes of étale sheaves $Rp_*\mathbb{Z}_{X \times \mathbb{A}^r}^c(n) \cong \mathbb{Z}_X^c(n-r)[2r]$.

Proof. a) Over an algebraically closed field, we get $\mathbb{Z}_Z^c(n) \xrightarrow{\sim} Ri^!\mathbb{Z}_X^c(n)$ because $\mathbb{Z}^c(n)(X) \cong R\Gamma(X_{\text{et}}, \mathbb{Z}^c(n))$, and higher Chow groups satisfy localization [4]. The general case follows by applying $R\Gamma_G$ to the distinguished triangle over the algebraic closure of k , for G the absolute Galois group of k .

b) This follows from the homotopy formula (3) for $\mathbb{Z}^c(n)$ by Theorem 3.1. \square

As in Proposition 2.1, localization gives

COROLLARY 3.4. If X is a scheme over a perfect field k and $n \leq 0$, then there are spectral sequences

$$(7) \quad E_{s,t}^1 = \bigoplus_{x \in X(s)} H^{s-t}(k(x)_{\text{et}}, \mathbb{Z}(s-n)) \Rightarrow H_{s+t}(X_{\text{et}}, \mathbb{Z}^c(n)).$$

In particular, $H_i(X_{\text{et}}, \mathbb{Z}^c(n)) = 0$ for $i < n$.

3.1. *An alternate argument.* We deduce a version of the previous results without using the Beilinson-Lichtenbaum conjecture. Instead, we use a theorem of Suslin [45], who proves that for a scheme X of dimension d over an algebraically closed field k , and an integer m not divisible by the characteristic of k , there is an isomorphism of finite groups $\text{CH}_0(X, i, \mathbb{Z}/m) \cong H_c^{2d-i}(X_{\text{et}}, \mathbb{Z}/m)^*$. If X is smooth, then this implies by Poincaré duality that the groups $H^j(X_{\text{Zar}}, \mathbb{Z}/m(d))$ and $H^j(X_{\text{et}}, \mathbb{Z}/m(d))$ are abstractly isomorphic, but this is weaker than the Beilinson-Lichtenbaum conjecture, which states that the canonical (change of topology) map is an isomorphism.

PROPOSITION 3.5. Let k be a perfect field and $n \leq 0$.

- a) For a closed embedding $i : Z \rightarrow X$, the canonical map $\mathbb{Z}_Z^c(n) \rightarrow Ri^!\mathbb{Z}_X^c(n)$ is a quasi-isomorphism. In particular, we obtain the spectral sequence (7).
- b) If $p : X \times \mathbb{A}^r \rightarrow X$ is the projection, then we have a quasi-isomorphism of complexes of étale sheaves $Rp_*\mathbb{Z}_{X \times \mathbb{A}^r}^c(n) \cong \mathbb{Z}_X^c(n-r)[2r]$.
- c) If $f : X \rightarrow k$ is proper, then we have a trace map $Rf_*\mathbb{Z}_X^c \rightarrow \mathbb{Z}$ in the derived category of étale sheaves on k .

Proof. We first prove a) for $n = 0$. The statement is étale local, so we can assume that k is algebraically closed and that Z and X are strictly henselian. Furthermore, it suffices to show the statement for smooth X . Indeed, if we embed $j : X \rightarrow T$ into a smooth scheme, and if $\mathbb{Z}_X^c \rightarrow Rj^!\mathbb{Z}_T^c$ as well as the composition $\mathbb{Z}_Z^c \rightarrow Ri^!\mathbb{Z}_X^c \rightarrow Ri^!Rj^!\mathbb{Z}_T^c$ are quasi-isomorphisms, then so is the map

$\mathbb{Z}_Z^c \rightarrow Ri^! \mathbb{Z}_X^c$. By the remark after Theorem 3.1 and the proof of Corollary 3.3a), the statement is known with $\mathbb{Z}_{(p)}^c$ -coefficients, hence it suffices to prove it for mod m -coefficients, with m not divisible by the characteristic of k .

Consider the following commutative diagram of maps of long sequences, where the upper vertical maps are the change of topology maps, and the map g is induced by the base-change $\varepsilon^* Ri_{Zar}^! \mathcal{G} \rightarrow Ri_{\text{et}}^! \varepsilon^* \mathcal{G}$ together with localization $Ri_{Zar}^! \mathbb{Z}_X^c \cong \mathbb{Z}_Z^c$,

$$(8) \quad \begin{array}{ccccccc} \longrightarrow & H_i(Z_{Zar}, \mathbb{Z}_Z^c/m) & \longrightarrow & H_i(X_{Zar}, \mathbb{Z}_X^c/m) & \longrightarrow & H_i(U_{Zar}, \mathbb{Z}_U^c/m) & \longrightarrow \\ & \parallel & & \parallel & & f_i \downarrow & \\ \longrightarrow & H_i(Z_{\text{et}}, \mathbb{Z}_Z^c/m) & \longrightarrow & H_i(X_{\text{et}}, \mathbb{Z}_X^c/m) & \longrightarrow & H_i(U_{\text{et}}, \mathbb{Z}_U^c/m) & \longrightarrow \\ & g_i \downarrow & & \parallel & & \parallel & \\ \longrightarrow & H_i(Z_{\text{et}}, Ri^! \mathbb{Z}_X^c/m) & \longrightarrow & H_i(X_{\text{et}}, \mathbb{Z}_X^c/m) & \longrightarrow & H_i(U_{\text{et}}, \mathbb{Z}_U^c/m) & \longrightarrow . \end{array}$$

The left two upper maps are isomorphisms because Z and X are strictly henselian. The upper row is exact by localization [4], and the lower row is exact by definition. By Proposition 2.2, the groups $H_i(X_{Zar}, \mathbb{Z}_X^c/m)$ vanish for $i \neq 2d$ because X is strictly henselian. Hence for $i \neq 2d, 2d - 1$ a diagram chase in the commutative diagram

$$\begin{array}{ccc} H_{i+1}(U_{Zar}, \mathbb{Z}_U^c/m) & \xlongequal{\quad} & H_i(Z_{Zar}, \mathbb{Z}_Z^c/m) \\ f \downarrow & & \parallel \\ H_{i+1}(U_{\text{et}}, \mathbb{Z}_U^c/m) & \longrightarrow & H_i(Z_{\text{et}}, \mathbb{Z}_Z^c/m) \\ \parallel & & g \downarrow \\ H_{i+1}(U_{\text{et}}, \mathbb{Z}_U^c/m) & \xlongequal{\quad} & H_i(Z_{\text{et}}, Ri^! \mathbb{Z}_X^c/m) \end{array}$$

shows that all groups are isomorphic. If $i = 2d$, then by Proposition 2.2, the map $H_{2d}(X_{\text{et}}, \mathbb{Z}_X^c/m) \rightarrow H_{2d}(U_{\text{et}}, \mathbb{Z}_U^c/m)$ is the identity of the group \mathbb{Z}/m , and $H_{2d+1}(U_{\text{et}}, \mathbb{Z}_U^c/m) = 0$. Consequently the lower row of (8) shows that $H_{2d}(Z_{\text{et}}, Ri^! \mathbb{Z}_X^c/m) = H_{2d-1}(Z_{\text{et}}, Ri^! \mathbb{Z}_X^c/m) = 0$. The map $H_{2d}(Z_{\text{et}}, \mathbb{Z}_Z^c/m) \rightarrow H_{2d}(X_{\text{et}}, \mathbb{Z}_X^c/m)$ factors through $H_{2d}(Z_{\text{et}}, Ri^! \mathbb{Z}_X^c/m) = 0$, hence is the zero map. Similarly, the map $H_{2d+1}(U_{Zar}, \mathbb{Z}_U^c/m) \rightarrow H_{2d}(Z_{Zar}, \mathbb{Z}_Z^c/m)$ factors through $H_{2d+1}(U_{\text{et}}, \mathbb{Z}_U^c/m) = 0$, hence is the zero map as well and we can conclude that $H_{2d}(Z_{Zar}, \mathbb{Z}_Z^c/m) = 0$. Finally, a part of the upper two rows of (8) gives the commutative diagram with exact rows

$$\begin{array}{ccccccc} \mathbb{Z}/m & \longrightarrow & H_{2d}(U_{Zar}, \mathbb{Z}_U^c/m) & \longrightarrow & H_{2d-1}(Z_{Zar}, \mathbb{Z}_Z^c/m) & \longrightarrow & 0 \\ & & \parallel & & f_i \downarrow & & \\ \mathbb{Z}/m & \xlongequal{\quad} & H_{2d}(U_{\text{et}}, \mathbb{Z}_U^c/m) & & & & \end{array}$$

We now invoke Suslin’s theorem, which implies that the source and target of the surjection f_i have the same finite order to conclude that $H_{2d-1}(Z_{\text{Zar}}, \mathbb{Z}_Z^c/m) = 0$. This finishes the proof that the map $\mathbb{Z}_Z^c/m \rightarrow Ri^!\mathbb{Z}_X^c/m$ is a quasi-isomorphism.

We now prove b) for $n = 0$. Rationally and with p -primary coefficients, we have étale hypercohomological descent, and the claim follows from (3). With prime to p -coefficients, we embed X into a smooth scheme and use localization a) to reduce to the case that X is smooth. Then we apply Proposition 2.2, and use the homotopy invariance of étale cohomology of $\mu_m^{\otimes(d+t-n)}$ (a consequence of the smooth base-change theorem and the calculation of étale cohomology of the affine line).

To obtain a) for arbitrary $n \leq 0$, we let $r = -n$, consider the diagram

$$\begin{array}{ccc} Z \times \mathbb{A}^r & \xrightarrow{i_1} & X \times \mathbb{A}^r \\ p' \downarrow & & p \downarrow \\ Z & \xrightarrow{i} & X \end{array}$$

and get that

$$\begin{aligned} Ri^!\mathbb{Z}_X^c(n) &\cong Ri^!Rp_*\mathbb{Z}_{X \times \mathbb{A}^r}^c(0)[2n] \\ &\cong Rp'_*Ri_1^!\mathbb{Z}_{X \times \mathbb{A}^r}^c(0)[2n] \cong Rp'_*\mathbb{Z}_{Z \times \mathbb{A}^r}^c(0)[2n] \cong \mathbb{Z}_Z^c(n). \end{aligned}$$

Now b) for arbitrary $n \leq 0$ follows from this as above.

We now prove c). Since it suffices to find a map over the algebraic closure of k compatible with the Galois action, we can suppose that k is algebraically closed; in this case $Rf_*\mathbb{Z}_X^c$ can be identified with $R\Gamma(X_{\text{et}}, \mathbb{Z}_X^c)$. From localization a), we obtain a spectral sequence (7),

$$E_{s,t}^1 = \bigoplus_{x \in X(s)} H^{s-t}(k(x)_{\text{et}}, \mathbb{Z}(s)) \Rightarrow H_{s+t}(X_{\text{et}}, \mathbb{Z}_X^c).$$

The terms $E_{s,t}^1$ vanish for $t < 0$ as in Theorem 3.1 by reasons of cohomological dimension. Hence $R^{-t}f_*\mathbb{Z}_X^c = H_t(X_{\text{et}}, \mathbb{Z}_X^c) = 0$ for $t < 0$, and the map $E_{0,0}^2 \rightarrow R^0f_*\mathbb{Z}_X^c$ is an isomorphism. On the other hand, comparing spectral sequences (5) and (7), we obtain a diagram

$$\begin{array}{ccccc} \bigoplus_{x \in X(1)} H^1(k(x), \mathbb{Z}(1)) & \longrightarrow & \bigoplus_{x \in X(0)} H^0(k(x), \mathbb{Z}(0)) & \longrightarrow & \text{CH}_0(X) \rightarrow 0 \\ \downarrow & & \downarrow & & \downarrow \\ \bigoplus_{x \in X(1)} H^1(k(x)_{\text{et}}, \mathbb{Z}(1)) & \longrightarrow & \bigoplus_{x \in X(0)} H^0(k(x)_{\text{et}}, \mathbb{Z}(0)) & \longrightarrow & E_{0,0}^2 \rightarrow 0. \end{array}$$

Since the left vertical maps are isomorphisms, so is the right vertical map. We get the trace map as the map of complexes of Galois modules

$$\mathrm{tr} : Rf_*\mathbb{Z}_X^c \xleftarrow{\sim} \tau_{\leq 0} Rf_*\mathbb{Z}_X^c \rightarrow R^0 f_*\mathbb{Z}_X^c \xleftarrow{\sim} E_{0,0}^2 \xleftarrow{\sim} \mathrm{CH}_0(X) \xrightarrow{\mathrm{tr}} \mathbb{Z}. \quad \square$$

4. The main theorem

THEOREM 4.1. *Let $f : X \rightarrow k$ be separated and of finite type over a perfect field. Then for every constructible sheaf \mathcal{F} on X , there is a canonical quasi-isomorphism*

$$R\mathrm{Hom}_X(\mathcal{F}, \mathbb{Z}_X^c) \cong R\mathrm{Hom}_k(Rf_!\mathcal{F}, \mathbb{Z}).$$

Proof. We can assume that k is algebraically closed, because if G is the Galois group of \bar{k}/k , then $R\mathrm{Hom}_X(\mathcal{F}, \mathcal{G}) \cong R\Gamma_G R\mathrm{Hom}_{\bar{X}}(\mathcal{F}, \mathcal{G})$. Indeed, for an injective sheaf \mathcal{G} , $\mathrm{Hom}_{\bar{X}}(\mathcal{F}, \mathcal{G})$ is flabby [32, III Cor. 2.13c)]. Note that $Rf_!\mathcal{G} = R\Gamma_c(X_{\mathrm{et}}, \mathcal{G})$ over an algebraically closed field. If $X \xrightarrow{j} T \xrightarrow{g} k$ is a compactification, then the trace map of Proposition 3.5c) induces by adjointness the pairing

$$\begin{aligned} \alpha(\mathcal{F}) : R\mathrm{Hom}_X(\mathcal{F}, \mathbb{Z}_X^c) &\cong R\mathrm{Hom}_T(j_!\mathcal{F}, \mathbb{Z}_T^c) \\ &\rightarrow R\mathrm{Hom}_k(Rg_*j_!\mathcal{F}, Rg_*\mathbb{Z}_T^c) \rightarrow R\mathrm{Hom}_k(Rf_!\mathcal{F}, \mathbb{Z}). \end{aligned}$$

The standard argument comparing compactifications shows that this does not depend on the compactification. We proceed by induction on the dimension d of X , and assume that the theorem is known for schemes of dimension less than d .

LEMMA 4.2. *If $f : U \rightarrow X$ is étale, then $\alpha(f_!\mathcal{F})$ is a quasi-isomorphism on X if and only if $\alpha(\mathcal{F})$ is a quasi-isomorphism on U . In particular, if f is finite and étale, then $\alpha(\mathcal{F})$ is a quasi-isomorphism if and only if $\alpha(f_*\mathcal{F})$ is a quasi-isomorphism.*

Proof. If $f : U \rightarrow X$ is étale, then since $Rf^!\mathbb{Z}_X^c = f^*\mathbb{Z}_X^c = \mathbb{Z}_U^c$, $\alpha(f_!\mathcal{F})$ can be identified with $\alpha(\mathcal{F})$ by adjointness. If f is also finite, then $f_* = f_!$. \square

LEMMA 4.3. *Let $j : U \rightarrow X$ be a dense open subscheme of a scheme of dimension d . Then $\alpha(\mathcal{F})$ is a quasi-isomorphism if and only if $\alpha(j^*\mathcal{F}) \cong \alpha(j_!j^*\mathcal{F})$ is a quasi-isomorphism.*

Proof. This follows by a 5-Lemma argument and induction on the dimension from the map of distinguished triangles

$$\begin{array}{ccccc} R\mathrm{Hom}_Z(i^*\mathcal{F}, \mathbb{Z}_Z^c) & \rightarrow & R\mathrm{Hom}_X(\mathcal{F}, \mathbb{Z}_X^c) & \rightarrow & R\mathrm{Hom}_U(j^*\mathcal{F}, \mathbb{Z}_U^c) \\ \downarrow & & \downarrow & & \downarrow \\ R\mathrm{Hom}(R\Gamma_c(Z_{\mathrm{et}}, i^*\mathcal{F}), \mathbb{Z}) & \rightarrow & R\mathrm{Hom}(R\Gamma_c(X_{\mathrm{et}}, \mathcal{F}), \mathbb{Z}) & \rightarrow & R\mathrm{Hom}(R\Gamma_c(U_{\mathrm{et}}, j^*\mathcal{F}), \mathbb{Z}) \end{array}$$

arising by adjointness from the short exact sequence $0 \rightarrow j_!j^*\mathcal{F} \rightarrow \mathcal{F} \rightarrow i_*i^*\mathcal{F} \rightarrow 0$ and purity for \mathbb{Z}_X^c , Proposition 3.5a). \square

LEMMA 4.4. *Let X be a scheme of dimension d over an algebraically closed field k . Then for any constructible sheaf \mathcal{F} , we have $\text{Ext}_X^i(\mathcal{F}, \mathbb{Z}^c) = 0$ for $i > 2d + 1$.*

Proof. If \mathcal{F} is locally constant, then the stalk of $\mathcal{E}xt_X^i(\mathcal{F}, \mathbb{Z}^c)_x$ agrees with $\text{Ext}_{\text{Ab}}^i(\mathcal{F}_x, \mathbb{Z}_x^c)$ by [32, III Ex. 1.31b], hence vanishes for $i > 1$ because \mathbb{Z}^c is concentrated in nonpositive degrees. Since X has cohomological dimension $2d$, we conclude by the spectral sequence

$$(9) \quad E_2^{s,t} = H^s(X_{\text{et}}, \mathcal{E}xt_X^t(\mathcal{F}, \mathcal{G})) \Rightarrow \text{Ext}_X^{s+t}(\mathcal{F}, \mathcal{G})$$

in this case. In general, we proceed by induction on the dimension of X . Let $j : U \rightarrow X$ be a dense open subset with complement $i : Z \rightarrow X$ such that $\mathcal{F}|_U$ is locally constant. The statement follows with the long exact Ext-sequence arising from the short exact sequence $0 \rightarrow j_! j^* \mathcal{F} \rightarrow \mathcal{F} \rightarrow i_* i^* \mathcal{F} \rightarrow 0$ by purity,

$$\dots \rightarrow \text{Ext}_Z^i(\mathcal{F}, \mathbb{Z}_Z^c(n)) \rightarrow \text{Ext}_X^i(\mathcal{F}, \mathbb{Z}_X^c(n)) \rightarrow \text{Ext}_U^i(\mathcal{F}, \mathbb{Z}_U^c(n)) \rightarrow \dots \quad \square$$

LEMMA 4.5. *If $\alpha(\mathcal{F})$ is a quasi-isomorphism for all constant constructible sheaves \mathcal{F} on smooth and projective schemes, then $\alpha(\mathcal{F})$ is a quasi-isomorphism for all constructible sheaves \mathcal{F} on all schemes X .*

Proof. Replacing X by a compactification $j : X \rightarrow T$ and \mathcal{F} by $j_! \mathcal{F}$, we can assume that X is proper. We fix X and show by descending induction on i that the map

$$\alpha_i(\mathcal{F}) : \text{Ext}_X^i(\mathcal{F}, \mathbb{Z}_X^c) \rightarrow R\text{Hom}_k^i(R\Gamma(X, \mathcal{F}), \mathbb{Z}) \cong H^{1-i}(X_{\text{et}}, \mathcal{F})^*$$

induced by α is an isomorphism for all constructible sheaves \mathcal{F} on X . By Lemma 4.4, both sides vanish for large i . We can find an alteration $\pi : Y \rightarrow X$ with Y smooth and projective, together with a dense open subset $j : U \rightarrow X$ such that the restriction $p : V = \pi^{-1}U \rightarrow U$ is étale and $p^* j^* \mathcal{F} = C_V$ is constant:

$$\begin{array}{ccc} V & \longrightarrow & Y \\ p \downarrow & & \pi \downarrow \\ U & \xrightarrow{j} & X. \end{array}$$

Indeed, find an open subset U of X and a finite étale cover V' of U such that $\mathcal{F}|_{V'}$ is constant. Let Y' be the closure of V' in $X \times T$ for some compactification T of V' . Now let Y be a generically étale alteration of Y' which is smooth and projective, and shrink U such that $V = V' \times_{Y'} Y = U \times_X Y$ is finite and étale over U .

By hypothesis, $\alpha(C_Y)$ is a quasi-isomorphism, hence $\alpha(C_V)$ is a quasi-isomorphism by Lemma 4.3 (note that $C_Y|_V = C_V$ because Y is smooth), and so is $\alpha(p_* C_V)$ by Lemma 4.2. By the proper base change theorem, $\pi_* \pi^* \mathcal{F}$ is constructible, and $j^* \pi_* \pi^* \mathcal{F} = p_* p^* j^* \mathcal{F} = p_* C_V$. Thus $\alpha(\pi_* \pi^* \mathcal{F})$ is a quasi-isomorphism by Lemma 4.3. Let \mathcal{F}' be the cokernel of the adjoint inclusion

$\mathcal{F} \rightarrow \pi_*\pi^*\mathcal{F}$, and consider the map of long exact sequences

$$(10) \quad \begin{array}{ccccc} \mathrm{Ext}_X^i(\mathcal{F}', \mathbb{Z}_X^c) & \longrightarrow & \mathrm{Ext}_X^i(\pi_*\pi^*\mathcal{F}, \mathbb{Z}_X^c) & \longrightarrow & \mathrm{Ext}_X^i(\mathcal{F}, \mathbb{Z}_X^c) \\ \alpha_i(\mathcal{F}') \downarrow & & \parallel & & \alpha_i(\mathcal{F}) \downarrow \\ H^{1-i}(X_{\mathrm{et}}, \mathcal{F}')^* & \longrightarrow & H^{1-i}(X_{\mathrm{et}}, \pi_*\pi^*\mathcal{F})^* & \longrightarrow & H^{1-i}(X_{\mathrm{et}}, \mathcal{F})^*. \end{array}$$

If $\alpha_j(\mathcal{F})$ is an isomorphism for $j > i$ and all constructible sheaves \mathcal{F} , then a 5-Lemma argument in (10) shows that $\alpha_i(\mathcal{F})$ is surjective. Since this holds for all constructible sheaves, in particular \mathcal{F}' , another application of the 5-Lemma shows that $\alpha_i(\mathcal{F})$ is an isomorphism. \square

It remains to prove

PROPOSITION 4.6. *Let X be smooth and projective over an algebraically closed field k . Then for every positive integer m , the map $\alpha(\mathbb{Z}/m)$ is a quasi-isomorphism*

$$R\mathrm{Hom}_X(\mathbb{Z}/m, \mathbb{Z}_X^c) \xrightarrow{\sim} R\mathrm{Hom}_k(Rf_*\mathbb{Z}/m, \mathbb{Z}).$$

Proof. We can assume that X is irreducible of dimension d . By Lemma 2.4, the pairing agrees up to a shift with

$$\begin{array}{ccc} R\mathrm{Hom}_{X, \mathbb{Z}/m}(\mathbb{Z}/m, \mathbb{Z}_X^c/m) & \longrightarrow & R\mathrm{Hom}_{k, \mathbb{Z}/m}(Rf_*\mathbb{Z}/m, \mathbb{Z}/m) \\ \parallel & & \parallel \\ R\Gamma(X_{\mathrm{et}}, \mathbb{Z}_X^c/m) & \longrightarrow & \mathrm{Hom}_{\mathrm{Ab}}(R\Gamma(X_{\mathrm{et}}, \mathbb{Z}/m), \mathbb{Z}/m). \end{array}$$

By induction on the number of prime factors of m we can assume that m is a prime number. If m is prime to the characteristic of k , then we claim that the lower row of the diagram agrees with Poincaré duality [32, VI Th. 11.1]

$$R\Gamma(X_{\mathrm{et}}, \mu_m^{\otimes d}[2d]) \rightarrow \mathrm{Hom}_{\mathrm{Ab}}(R\Gamma(X_{\mathrm{et}}, \mathbb{Z}/m), \mathbb{Z}/m).$$

Since $Rf_*\mathcal{G} = R\mathrm{Hom}_{X, \mathbb{Z}/m}(\mathbb{Z}/m, \mathcal{G})$, it is easy to see that our pairing agrees with the Yoneda pairing, and it remains to show that the trace map agrees with the map of Proposition 3.5c), i.e., the following diagram commutes:

$$\begin{array}{ccc} R\Gamma(X_{\mathrm{et}}, \mathbb{Z}_X^c/m) & \xrightarrow{f_*} & R\Gamma(k_{\mathrm{et}}, \mathbb{Z}_k^c/m) \\ \parallel & & \parallel \\ R\Gamma(X_{\mathrm{et}}, \mu_m^{\otimes d}[2d]) & \xrightarrow{\mathrm{tr}'} & \mathbb{Z}/m. \end{array}$$

The trace map tr' is characterized by the property that it sends the class $i_*(1) \in H^{2d}(X_{\mathrm{et}}, \mu_m^{\otimes d})$ of a closed point $i : p \rightarrow X$ to 1 [32, VI Th. 11.1], a property which is clear for the map of Proposition 3.5c) by functoriality. Hence it suffices

to show the commutativity of the diagram

$$\begin{CD} \mathbb{Z}/m \cong R\Gamma(k_{\text{et}}, \mathbb{Z}_k^c/m) @>i_*>> R\Gamma(X_{\text{et}}, \mathbb{Z}_X^c/m) \\ @| @| \\ \mathbb{Z}/m \cong R\Gamma(k_{\text{et}}, \mathbb{Z}/m) @>i_*>> R\Gamma(X_{\text{et}}, \mu_m^{\otimes d}[2d]), \end{CD}$$

which follows from Proposition 2.2.

If $m = p$ is the characteristic, then we claim that our pairing agrees with Milne’s duality [35]

$$(11) \quad R\Gamma(X_{\text{et}}, v^d)[d] \rightarrow \text{Hom}_{\text{Ab}}(R\Gamma(X_{\text{et}}, \mathbb{Z}/p), \mathbb{Z}/p).$$

More precisely, consider the complexes of commutative algebraic perfect p -torsion group schemes $\underline{H}^i(X, v^d)$ and $\underline{H}^i(X, \mathbb{Z}/p)$ over k ; see [35]. Finite generation of $H^i(X_{\text{et}}, \mathbb{Z}/p)$ implies that the unipotent part of $\underline{H}^i(X, \mathbb{Z}/p)$ is trivial, and the same then holds for the unipotent part of $\underline{H}^i(X, v^d)$ by [35, Th. 1.11]. Hence the same theorem shows that the Yoneda pairing induces a duality of étale group schemes

$$\underline{H}^i(X, v^d) \rightarrow \text{Hom}(\underline{H}^i(X, \mathbb{Z}/p), \mathbb{Z}/p)[-d],$$

and (11) are the global sections over k . By the argument above, it suffices to show that the following diagram commutes

$$\begin{CD} R\Gamma(X_{\text{et}}, \mathbb{Z}_X^c/p) @>f_*>> R\Gamma(k_{\text{et}}, \mathbb{Z}_k^c/p) \\ @| @| \\ R\Gamma(X_{\text{et}}, v^d[d]) @>tr'>> \mathbb{Z}/p. \end{CD}$$

Again the trace map is characterized by the property that it sends the class of a closed point $i : p \rightarrow X$ to 1 [35, p. 308], and it suffices to show the commutativity of the diagram

$$\begin{CD} \mathbb{Z}/p \cong H^0(k_{\text{et}}, \mathbb{Z}_k^c/p) @>i_*>> H^0(X_{\text{et}}, \mathbb{Z}_X^c/p) \\ @| @| \\ \mathbb{Z}/p \cong H^0(k_{\text{et}}, \mathbb{Z}/p) @>i_*>> H^d(X_{\text{et}}, v^d), \end{CD}$$

which again is Proposition 2.2. □

Remark. If X is smooth of dimension d over a perfect field of characteristic p , and \mathcal{F} a locally constant m -torsion sheaf, then we can identify the left-hand side of Theorem 4.1,

$$\text{Ext}_X^i(\mathcal{F}, \mathbb{Z}_X^c) \cong H^{2d+i-1}(X_{\text{et}}, \mathcal{F}^D),$$

where $\mathcal{F}^D = \mathcal{H}om_X(\mathcal{F}, \mu_m^{\otimes d})$ if $p \nmid m$, and $\mathcal{F}^D = \mathcal{H}om_X(\mathcal{F}, v_r^d)[d]$ if $m = p^r$. Indeed, consider the spectral sequence (9). The calculation of the Ext-groups is

local for the étale topology, and since \mathcal{F} is locally constant, we can calculate them at stalks [32, III Ex. 1.31b] and assume that $\mathcal{F} = \mathbb{Z}/m$. Then by Lemma 2.4 and Proposition 2.2,

$$\mathcal{E}rt_X^q(\mathbb{Z}/m, \mathbb{Z}_X^c) = \mathcal{H}^{q-1}(\mathbb{Z}_X^c/m) \cong \begin{cases} \mu_m^{\otimes d}, & p \nmid m, q = 1 - 2d \\ v_r^d, & m = p^r, q = 1 - d \\ 0, & \text{otherwise.} \end{cases}$$

COROLLARY 4.7. *Let $f : X \rightarrow Y$ be a map of schemes over a perfect field k and $n \leq 0$.*

a) *For every locally constant constructible sheaf \mathcal{G} on Y , the map (6) induces a quasi-isomorphism*

$$\mathbb{Z}_X^c(n) \otimes f^*\mathcal{G} \cong Rf^!(\mathbb{Z}_Y^c(n) \otimes \mathcal{G}).$$

b) *For every torsion sheaf \mathcal{F} on X and finitely generated, locally constant sheaf \mathcal{G} on Y , we have a functorial quasi-isomorphism*

$$R\text{Hom}_X(\mathcal{F}, \mathbb{Z}_X^c(n) \otimes f^*\mathcal{G}) \cong R\text{Hom}_Y(Rf_!\mathcal{F}, \mathbb{Z}_Y^c(n) \otimes \mathcal{G}).$$

It was pointed out to us by A. Abbes that a) is false for general constructible sheaves \mathcal{G} . For example, if i is the inclusion of a point into the affine line, $\mathcal{G} = i_*\mathbb{Z}/m$ and $n = 0$, then the left-hand side is \mathbb{Z}/m , but the right-hand side is $Ri^!(\mathbb{Z}_{\mathbb{A}^1}^c \otimes i_*\mathbb{Z}/m) \cong Ri^!(i_*\mu_m[2]) \cong \mu_m[2]$ by Proposition 2.2.

Proof. a) Since the statement is local for the étale topology on Y , we can assume that \mathcal{G} is of the form \mathbb{Z}/m . If $Y = \text{Spec } k$ and $n = 0$, let $j : U \rightarrow X$ be an étale map and $g = f \circ j$. Then it suffices to show that the upper row in the following diagram is an isomorphism

$$\begin{array}{ccc} R\text{Hom}_U(\mathbb{Z}/m, \mathbb{Z}_U^c/m) & \xrightarrow{\text{ad}(\text{tr})} & R\text{Hom}_U(\mathbb{Z}/m, j^*Rf^!\mathbb{Z}_k^c/m) \\ \downarrow & & \parallel \\ R\text{Hom}_k(Rg_!\mathbb{Z}/m, Rg_!\mathbb{Z}_X^c/m) & \xrightarrow{\text{tr}} & R\text{Hom}_k(Rg_!\mathbb{Z}/m, \mathbb{Z}_k^c/m). \end{array}$$

The diagram commutes by the property of adjoints, and the lower left composition is a quasi-isomorphism by Theorem 4.1. For arbitrary Y , if $t : Y \rightarrow k$ is the structure map, then by the previous case, the second map and the composition in

$$\mathbb{Z}_X^c/m \rightarrow Rf^!\mathbb{Z}_Y^c/m \rightarrow Rf^!Rt^!\mathbb{Z}_k^c/m$$

are quasi-isomorphisms, hence so is the first map. If $n < 0$, let $r = -n$ and consider the following commutative diagram

$$\begin{array}{ccc} X \times \mathbb{A}^r & \xrightarrow{p'} & X \\ g \downarrow & & f \downarrow \\ Y \times \mathbb{A}^r & \xrightarrow{p} & Y. \end{array}$$

By homotopy invariance $Rp_*\mathbb{Z}_{Y \times \mathbb{A}^r}^c(0)[2n] \cong \mathbb{Z}_Y^c(n)$ and similar for X , and we obtain

$$\begin{aligned} \mathbb{Z}_X^c(n) &\cong Rp'_*\mathbb{Z}_{X \times \mathbb{A}^r}^c[2n] \cong Rp'_*Rg^!\mathbb{Z}_{Y \times \mathbb{A}^r}^c[2n] \\ &\cong Rf^!Rp_*\mathbb{Z}_{Y \times \mathbb{A}^r}^c[2n] \cong Rf^!\mathbb{Z}_Y^c(n). \end{aligned}$$

b) If \mathcal{F} is a constructible m -torsion sheaf, we can replace \mathcal{G} by \mathcal{G}/m and use a). For general \mathcal{F} , write $\mathcal{F} = *colim \mathcal{F}_i$ as a filtered colimit of constructible sheaves [34, II Prop. 0.9]. Then the canonical map

$$R *lim RHom_X(\mathcal{F}_i, \mathbb{Z}_X^c(n) \otimes f^*\mathcal{G}) \rightarrow R *lim RHom_Y(Rf_!\mathcal{F}_i, \mathbb{Z}_Y^c(n) \otimes \mathcal{G})$$

induces a map of spectral sequences [39]

$$\begin{array}{ccc} E_2^{s,t} = *lim^s Ext_X^t(\mathcal{F}_i, \mathbb{Z}_X^c(n) \otimes f^*\mathcal{G}) & \Rightarrow & Ext_X^{s+t}(*colim \mathcal{F}_i, \mathbb{Z}_X^c(n) \otimes f^*\mathcal{G}) \\ \downarrow & & \downarrow \\ E_2^{s,t} = *lim^s Ext_Y^t(Rf_!\mathcal{F}_i, \mathbb{Z}_Y^c(n) \otimes \mathcal{G}) & \Rightarrow & Ext_Y^{s+t}(*colim Rf_!\mathcal{F}_i, \mathbb{Z}_Y^c(n) \otimes \mathcal{G}). \end{array}$$

The map on E_2 -terms is an isomorphism by the above, and by the following lemma the spectral sequences converge. Hence the map on abutments is an isomorphism. Finally, étale cohomology with compact support commutes with filtered colimits. \square

LEMMA 4.8. *Let \mathcal{F} be a torsion sheaf and \mathcal{G} be finitely generated locally constant sheaf on a scheme X of dimension d over an algebraically closed field. Then $Ext_X^i(\mathcal{F}, \mathbb{Z}_X^c(n) \otimes \mathcal{G}) = 0$ for $i < -2d$ and $n \leq 0$.*

Proof. By (9) it suffices to show that $\mathcal{E}xt_X^i(\mathcal{F}, \mathbb{Z}_X^c(n) \otimes \mathcal{G}) = 0$ for $i < -2d$, and since this is étale local, we can assume that $\mathcal{G} = \mathbb{Z}$ or $\mathcal{G} = \mathbb{Z}/m$. By the long exact coefficient sequence, both cases follows from $\mathcal{E}rt_X^i(\mathcal{F}, \mathbb{Z}_X^c(n)) = 0$ for $i < -2d + 1$. Since \mathcal{F} is torsion, this will follow if $\mathcal{E}rt_X^i(\mathcal{F}, \mathbb{Q}/\mathbb{Z}_X^c(n)) = 0$ for $i < -2d$, which by [32, III Rem. 1.24] will follow from $Ext_V^i(\mathcal{F}|_V, \mathbb{Q}/\mathbb{Z}_V^c(n)) = 0$ for all V étale over X . Let U be a dense smooth open subscheme of V with complement Z . Then we conclude by induction on the dimension of V , Proposition 2.2 and the long exact localization sequence arising from purity

$$\rightarrow Ext_Z^i(\mathcal{F}|_Z, \mathbb{Q}/\mathbb{Z}_Z^c(n)) \rightarrow Ext_V^i(\mathcal{F}|_V, \mathbb{Q}/\mathbb{Z}_V^c(n)) \rightarrow Ext_U^i(\mathcal{F}|_U, \mathbb{Q}/\mathbb{Z}_U^c(n)) \rightarrow .$$

\square

COROLLARY 4.9. *Let \mathcal{F} be a torsion sheaf on a scheme X over a perfect field k , and let $n \leq 0$. Then for any constructible sheaf \mathcal{F} , $Ext_X^i(\mathcal{F}, \mathbb{Z}_X^c(n))$ and $\mathcal{E}rt_X^i(\mathcal{F}, \mathbb{Z}_X^c(n))$ vanish for $i > cd k + 1$. In particular, $\mathbb{Z}^c(n)$ has quasi-injective dimension $cd k + 1$ in the sense of [2, I, Def. 1.4].*

Proof. If k is algebraically closed, then

$$Ext_X^i(\mathcal{F}, \mathbb{Z}_X^c(n)) \cong Ext_{Ab}^i(R\Gamma_c(X_{et}, \mathcal{F}), \mathbb{Z}_k^c(n)) \cong Hom(H_c^{1-i}(X_{et}, \mathcal{F}), \mathbb{Q}/\mathbb{Z}_k^c(n)).$$

By Proposition 2.2, the complex $\mathbb{Q}/\mathbb{Z}_k^c(n)$ is concentrated in degree zero, hence this vanishes for $i > 1$. In the general case, we use the spectral sequence

$$H^s(\text{Gal}(k), \text{Ext}_{\bar{X}}^t(\mathcal{F}, \mathbb{Z}_X^c(n))) \Rightarrow \text{Ext}_X^{s+t}(\mathcal{F}, \mathbb{Z}_X^c(n)).$$

The statement for the extension sheaves follows because $\mathcal{E}rt_X^i(\mathcal{F}, \mathbb{Z}^c(n))$ is the sheaf associated to the presheaf $U \mapsto \text{Ext}_U^i(\mathcal{F}|_U, \mathbb{Z}_U^c(n))$. \square

5. Duality

If a perfect field k has duality for Galois cohomology with a dualizing sheaf that is related to some $\mathbb{Z}_k^c(n)$, then Theorem 4.1 gives a duality theorem over k . For example, if k is algebraically closed, then we immediately obtain from Theorem 4.1 a quasi-isomorphism

$$R \text{Hom}_X(\mathcal{F}, \mathbb{Z}_X^c) \cong R \text{Hom}_{\text{Ab}}(R\Gamma_c(X_{\text{et}}, \mathcal{F}), \mathbb{Z})$$

for every constructible sheaf \mathcal{F} on X . In particular, we get perfect pairings of finitely generated groups

$$\text{Ext}_X^{1-i}(\mathcal{F}, \mathbb{Z}_X^c) \times H_c^i(X_{\text{et}}, \mathcal{F}) \rightarrow \mathbb{Q}/\mathbb{Z}.$$

From Lemma 2.4 and Theorem 3.1 this gives

$$(12) \quad \text{CH}_0(X, i, \mathbb{Z}/m) \cong H_c^i(X_{\text{et}}, \mathbb{Z}/m)^*,$$

generalizing Suslin’s theorem [45] to arbitrary m .

5.1. *Finite fields.* Let G be the absolute Galois group of $\bar{\mathbb{F}}_q/\mathbb{F}_q$ with $q = p^r$. The following theorem has been proved by Jannsen, Saito, and Sato [20] and Moser [36] for p -power torsion sheaves, by Deninger [9, Ths. 1.4, 2.3] for curves, and smooth schemes over curves and coefficients of order prime to p , and by Spieß [44] for surfaces.

THEOREM 5.1. *Let X be a scheme over a finite field, and \mathcal{F} be a torsion sheaf on X . Then there is a quasi-isomorphism*

$$R \text{Hom}_X(\mathcal{F}, \mathbb{Z}_X^c) \xrightarrow{\sim} R \text{Hom}_{\text{Ab}}(R\Gamma_c(X_{\text{et}}, \mathcal{F}), \mathbb{Z})[-1].$$

In particular, if \mathcal{F} is constructible, there are perfect pairings of finite groups

$$\text{Ext}_X^{2-i}(\mathcal{F}, \mathbb{Z}_X^c) \times H_c^i(X_{\text{et}}, \mathcal{F}) \rightarrow \mathbb{Q}/\mathbb{Z}.$$

Proof. If $\bar{f} : X \times_{\mathbb{F}_q} \bar{\mathbb{F}}_q \rightarrow \bar{\mathbb{F}}_q$ is the structure map, and if we apply $R\Gamma_G$ to the pairing over $\bar{\mathbb{F}}_q$, then we get

$$R \text{Hom}_X(\mathcal{F}, \mathbb{Z}_X^c) = R\Gamma_G R \text{Hom}_{\bar{X}}(\mathcal{F}, \mathbb{Z}_{\bar{X}}^c) \cong R\Gamma_G R \text{Hom}_{\bar{\mathbb{F}}_q}(R\bar{f}_! \mathcal{F}, \mathbb{Z}).$$

By duality for Galois cohomology, this is quasi-isomorphic to

$$R \text{Hom}_{\text{Ab}}(R\Gamma_G R\bar{f}_! \mathcal{F}, \mathbb{Z})[-1] \cong R \text{Hom}_{\text{Ab}}(R\Gamma_c(X_{\text{et}}, \mathcal{F}), \mathbb{Z})[-1]. \quad \square$$

Let $H_i^K(X, \mathbb{Z}/m)$ be the i th homology group of the Kato complex [22]

$$(13) \quad \bigoplus_{X_{(0)}} H^1(k(x)_{\text{et}}, \mathbb{Z}/m) \leftarrow \bigoplus_{X_{(1)}} H^2(k(x)_{\text{et}}, \mathbb{Z}/m(1)) \leftarrow \dots$$

Kato [22, Conj. 0.3] conjectures that $H_i^K(X, \mathbb{Z}/m) = 0$ for $i > 0$ and X smooth and proper. Kato’s conjecture has been proved in low degrees by Colliot-Thélène [7], and in general by Jannsen and Saito [18], [19] assuming resolution of singularities. One important application of Kato homology is, in view of $H^1(X_{\text{et}}, \mathbb{Z}/m)^* \cong \pi_1^{\text{ab}}(X)/m$, the following

COROLLARY 5.2. *Assuming the Beilinson-Lichtenbaum conjecture, there is, for every scheme over a finite field, an exact sequence*

$$(14) \quad \dots \rightarrow \text{CH}_0(X, i, \mathbb{Z}/m) \rightarrow H_c^{i+1}(X_{\text{et}}, \mathbb{Z}/m)^* \rightarrow H_{i+1}^K(X, \mathbb{Z}/m) \rightarrow \dots$$

Proof. By Theorem 5.1 and Lemma 2.4, we have

$$\begin{aligned} H_c^{i+1}(X, \mathbb{Z}/m)^* &= \text{Ext}_X^{1-i}(\mathbb{Z}/m, \mathbb{Z}_X^c) \\ &\cong \text{Ext}_{X, \mathbb{Z}/m}^{-i}(\mathbb{Z}/m, \mathbb{Z}_X^c/m) \cong H_i(X_{\text{et}}, \mathbb{Z}_X^c/m). \end{aligned}$$

The corollary follows by comparing the niveau spectral sequences (5) and (7) with \mathbb{Z}/m -coefficients, and identifying the terms $E_{s,t}^1$ with $t \geq n$ by the Beilinson-Lichtenbaum conjecture. □

5.2. Local fields. Let k be the field of fractions of a henselian discrete valuation ring of characteristic 0 with finite residue field, for example a local field of mixed characteristic, and G the Galois group of k .

THEOREM 5.3. *If $f : X \rightarrow k$ is a scheme over k , and \mathcal{F} a torsion sheaf, then there is a quasi-isomorphism*

$$R \text{Hom}_X(\mathcal{F}, \mathbb{Z}_X^c(-1)) \cong R \text{Hom}_{\text{Ab}}(R\Gamma_c(X, \mathcal{F}), \mathbb{Z})[-2].$$

In particular, for constructible \mathcal{F} , we have perfect pairings of finite groups

$$\text{Ext}_X^{3-i}(\mathcal{F}, \mathbb{Z}_X^c(-1)) \times H_c^i(X_{\text{et}}, \mathcal{F}) \rightarrow \mathbb{Q}/\mathbb{Z}.$$

Proof. From Corollary 4.7 we get the following quasi-isomorphisms

$$\begin{aligned} R \text{Hom}_X(\mathcal{F}, \mathbb{Z}_X^c(-1)) &\cong R\Gamma_G \text{Hom}_{\bar{X}}(\mathcal{F}, \mathbb{Z}_{\bar{X}}^c(-1)) \\ &\cong R\Gamma_G R \text{Hom}_{\bar{k}}(R\Gamma_c(\bar{X}, \mathcal{F}), \mathbb{Z}_{\bar{k}}^c(-1)) \cong R \text{Hom}_G(R\Gamma_c(\bar{X}, \mathcal{F}), \mathbb{Z}_{\bar{k}}^c(-1)). \end{aligned}$$

The claim follows with the following lemma. □

LEMMA 5.4 (Duality for Galois cohomology). *If G is the Galois group of \bar{k}/k , and C^\cdot a complex of torsion G -modules, then there is a quasi-isomorphism*

$$R \text{Hom}_G(C^\cdot, \mathbb{Z}_{\bar{k}}^c(-1)) \cong R \text{Hom}_{\text{Ab}}(R\Gamma(G, C^\cdot), \mathbb{Z})[-2].$$

Proof. Since $\mathbb{Z}_{\bar{k}}^c(-1) \cong \mathbb{G}_m[-1]$, this is local duality

$$R \text{Hom}_G(C^\cdot, \mathbb{G}_m)[-1] \cong R \text{Hom}_{\text{Ab}}(R\Gamma(G, C^\cdot), \mathbb{Q}/\mathbb{Z})[-3].$$

Indeed, the Yoneda pairing induces an isomorphism [34, I Th. 2.14]

$$\text{Ext}_G^i(M, \mathbb{G}_m) \cong \text{Hom}_{\text{Ab}}(H^{2-i}(G, M), \mathbb{Q}/\mathbb{Z})$$

for every G -module M , and the result for complexes follows by a spectral sequence argument. \square

Remark. If X is smooth of dimension d over a local field k of characteristic 0, \mathcal{F} is a locally constant m -torsion sheaf, and $\mathcal{F}^D := \mathcal{H}om(\mathcal{F}, \mu_m^{d+1})$, then we recover the perfect pairing of finite groups

$$H^{2d+2-i}(X_{\text{et}}, \mathcal{F}^D) \times H_c^i(X_{\text{et}}, \mathcal{F}) \rightarrow \mathbb{Q}/\mathbb{Z}.$$

This follows from Theorem 5.3, using Proposition 2.2, and the degeneration of the local-to-global spectral sequence of Ext-groups.

Let $H_i^K(X, \mathbb{Z}/m)$ be the i th homology of the Kato complex

$$(15) \quad \oplus_{X_{(0)}} H^2(k(x)_{\text{et}}, \mathbb{Z}/m(1)) \leftarrow \oplus_{X_{(1)}} H^3(k(x)_{\text{et}}, \mathbb{Z}/m(2)) \leftarrow \dots$$

Kato conjectures [22, Conj. 5.1] that the terms $H_i^K(X, \mathbb{Z}/m)$ vanish for X smooth and proper and $i > 0$. The same proof as for (14) gives

COROLLARY 5.5. *Assuming the Beilinson-Lichtenbaum conjecture, there is, for every scheme over a local field of characteristic 0, a long exact sequence,*

$$(16) \quad \dots \rightarrow \text{CH}_{-1}(X, i, \mathbb{Z}/m) \rightarrow H_c^i(X_{\text{et}}, \mathbb{Z}/m)^* \rightarrow H_i^K(X, \mathbb{Z}/m) \rightarrow \dots$$

Remark. If K is a r -local field of characteristic 0 such that the finite field in the definition of K has characteristic p , then one can use local duality for its Galois cohomology to prove a quasi-isomorphism

$$R \text{Hom}_X(\mathcal{F}, \mathbb{Z}_X^c(-r)) \cong R \text{Hom}_{\text{Ab}}(R\Gamma_c(X, \mathcal{F}), \mathbb{Z})[-r-1]$$

for torsion sheaves \mathcal{F} of order not divisible by p , generalizing a result of Deninger and Wingberg [10].

6. Applications

6.1. Rojtman’s theorem. We generalize Rojtman’s theorem [33], [38] to normal projective varieties. Our proof follows the line of Bloch [3] and Milne [33]. Let X be a proper scheme over an algebraically closed field. Then by [32, Prop. 4.16] there is an isomorphism $H^1(X_{\text{et}}, \mathbb{Z}/m) \cong \text{Hom}_k(\mu_m, \text{Pic}_X)$, where Hom_k is the group of homomorphisms of flat group schemes over k . Let Pic_X^τ be the scheme representing line bundles, such that a power is algebraically equivalent to 0. Then the quotient $\text{Pic}_X / \text{Pic}_X^\tau$ is a finitely generated free group, hence $\text{Hom}_k(\mu_m, \text{Pic}_X) \cong \text{Hom}_k(\mu_m, \text{Pic}_X^\tau)$. The quotient $C = \text{Pic}_X^\tau / \text{Pic}_X^{0,\text{red}}$ by the reduced part of the connected component is a finite group scheme, and $\text{Pic}_X^{0,\text{red}}$ is a smooth commutative algebraic group, which by Chevalley’s theorem is an extension of an abelian variety

If X is an arbitrary proper scheme, we define $M^1(X)$ to be the quotient of $\text{Pic}_X^{0,\text{red}}$ by its largest unipotent subgroup. Then we have a short exact sequence

$$(18) \quad 0 \rightarrow T_X \rightarrow M^1(X) \rightarrow A_X \rightarrow 0,$$

where T_X is a torus and A_X an abelian variety. We let $M_1(X)$ be the dual of $M^1(X)$ in the category of 1-motives; see [8, §10] for the definition and basic properties.

Concretely, $M_1(X)$ is the complex $\chi(T_X) \xrightarrow{f} A_X^t(k)$, where $\chi(T_X)$ is the character group of T_X , A_X^t the dual abelian variety of A_X , and f is given by pushing-out (18) along a given map $T_X \rightarrow \mathbb{G}_m$ to obtain an element in $\text{Ext}^1(A_X, \mathbb{G}_m) = A_X^t(k)$. The Tate realization $T_m(M)$ of a 1-motive $M = [F \rightarrow G]$ is the cone of multiplication by m . Since abelian varieties and tori are divisible, $T_m(M)$ is concentrated in a single degree, and there is a short exact sequence $0 \rightarrow {}_mG(k) \rightarrow T_m(M) \rightarrow F/m \rightarrow 0$. In our case, we obtain a short exact sequence

$$(19) \quad 0 \rightarrow {}_{\text{tor}}A_X^t(k) \rightarrow {}^* \text{colim}_m T_m M_1(X) \rightarrow \chi(T_X) \otimes \mathbb{Q}/\mathbb{Z} \rightarrow 0.$$

If M and M^\vee are dual 1-motives, then Cartier duality gives a perfect pairing $T_m(M) \times T_m(M^\vee) \rightarrow \mu_m$.

PROPOSITION 6.2. *Let X be a proper scheme over an algebraically closed field. Then there is an isomorphism*

$$\text{CH}_0(X, 1, \mathbb{Q}/\mathbb{Z}) \cong {}^* \text{colim}_m T_m M_1(X).$$

Proof. There are no homomorphisms and extensions between μ_m and a unipotent group, hence

$$\text{Hom}_k(\mu_m, \text{Pic}_X^{0,\text{red}}) \cong \text{Hom}_k(\mu_m, M^1(X)) \cong \text{Hom}_k(\mu_m, T_m M^1(X)),$$

and the argument which leads to (17) gives a short exact sequence

$$0 \rightarrow C^\vee(k)/m \rightarrow \text{CH}_0(X, 1, \mathbb{Z}/m) \rightarrow T_m M_1(X) \rightarrow 0.$$

The result follows because, in the colimit, the first term vanishes. □

Example. Let $C = E/0 \sim P$ be an elliptic curve E with the points 0 and P identified to the point $Q \in C$. Then from the long exact sequences for higher Chow groups and étale cohomology arising from the blow-up diagram

$$\begin{array}{ccc} \{P, 0\} & \longrightarrow & E \\ \downarrow & & \downarrow \\ Q & \longrightarrow & C \end{array}$$

we obtain $H^1(C_{\text{ét}}, \mathbb{Z}) = \mathbb{Z}$, hence $\chi(T_C) \cong \mathbb{Z}$ [13], and $\text{CH}_0(C, 1, \mathbb{Q}_l/\mathbb{Z}_l)$ has corank 3 for $l \neq \text{char } k$. If P is not a torsion point, then $\text{CH}_0(C, 1) \otimes \mathbb{Q}_l/\mathbb{Z}_l = 0$

and $l_\infty\text{CH}_0(C)$ has corank 3. Hence the maps

$$\begin{aligned} \text{tor}A_C^t(k) &\xrightarrow{f} \text{torCH}_0(C) \\ \text{CH}_0(C, 1) \otimes \mathbb{Q}/\mathbb{Z} &\xrightarrow{g} \chi(T_C) \otimes \mathbb{Q}/\mathbb{Z}, \end{aligned}$$

obtained from (19) have infinite cokernels, and $\text{torCH}_0(C)$ cannot be parametrized by an abelian variety because it has odd corank. If P is an m -torsion point, then $\text{CH}_0(C, 1) \otimes \mathbb{Q}/\mathbb{Z} = \mathbb{Q}/\mathbb{Z}$ and f and g are surjective with kernel \mathbb{Z}/m .

6.2. *The fundamental group.* Let X be a proper scheme over an algebraically closed field and consider the (pro-finite) fundamental group. Then we have an isomorphism

$$\pi_1^{\text{ab}}(X) = H^1(X_{\text{et}}, \mathbb{Q}/\mathbb{Z})^* \cong \text{CH}_0(X, 1, \hat{\mathbb{Z}}),$$

in particular, a short exact sequence

$$(20) \quad 0 \rightarrow \text{CH}_0(X, 1)^\wedge \rightarrow \pi_1^{\text{ab}}(X) \rightarrow T\text{CH}_0(X) \rightarrow 0.$$

If X is normal, then taking the inverse limit in (17), and using Theorem 6.1, we see that this sequence agrees with the sequence

$$(21) \quad 0 \rightarrow C^\vee(k) \rightarrow \pi_1^{\text{ab}}(X) \rightarrow T \text{Alb}_X(k) \rightarrow 0,$$

in particular, $\text{CH}_0(X, 1)^\wedge$ is finite. The latter sequence can be found in Milne [32, Cor. III 4.19], and in Katz-Lang [24, Lemma 5].

For geometrically connected X over a perfect field k , the dual of the Hochschild-Serre spectral sequence $H^s(k, H^t(\bar{X}_{\text{et}}, \mathbb{Q}/\mathbb{Z})) \Rightarrow H^{s+t}(X_{\text{et}}, \mathbb{Q}/\mathbb{Z})$ gives an exact sequence

$$(22) \quad H^2(k_{\text{et}}, \mathbb{Q}/\mathbb{Z})^* \rightarrow \pi_1^{\text{ab}}(X \times_k \bar{k})_{\text{Gal}(k)} \rightarrow \pi_1^{\text{ab}}(X) \rightarrow \text{Gal}(k)^{\text{ab}} \rightarrow 0,$$

which is short exact if X has a k -rational point [24, Lemma 1], or if k is finite, local, or global, because then $H^2(k_{\text{et}}, \mathbb{Q}/\mathbb{Z}) = 0$. By the argument in [24], the sequence (21) implies that $\pi_1^{\text{ab}}(X \times_k \bar{k})_{\text{Gal}(k)}$ is finite if k is absolutely finitely generated and X is proper and normal.

Let X be a proper scheme over a finite field k with Galois group G . Let $\pi_1^{\text{ab}}(X)^0$ be the kernel of $\pi_1^{\text{ab}}(X) \rightarrow G$, and $\text{CH}_0(X)^0$ the subgroup of cycles of degree zero of $\text{CH}_0(X)$.

PROPOSITION 6.3. *If X is proper and geometrically connected over a finite field, then we have a short exact sequence*

$$0 \rightarrow \text{CH}_0(\bar{X}, 1)_G^\wedge \rightarrow \pi_1^{\text{ab}}(X)^0 \rightarrow (\text{CH}_0(\bar{X})^0)^G \rightarrow 0.$$

If X is normal, then these groups are finite, and the right-hand term agrees with $\text{Alb}_X(k)$.

Proof. By (22), it suffices to calculate the cokernel of $1 - F$ on the sequence (20), for F the Frobenius, and to show that $T\text{CH}_0(\bar{X})_{1-F} \cong (\text{CH}_0(\bar{X})^0)^G$. We

can replace $\text{CH}_0(\bar{X})$ by the divisible group $\text{CH}_0(\bar{X})^0$ in (20). Since the finitely generated group $\text{CH}_0(X)^0$ surjects onto $(\text{CH}_0(\bar{X})^0)^G$, the cokernel of $F - 1$ on $\text{CH}_0(\bar{X})^0$ is divisible and finite, hence vanishes. The short exact sequence

$$0 \rightarrow (\text{CH}_0(\bar{X})^0)^G \rightarrow \text{CH}_0(\bar{X})^0 \xrightarrow{1-F} \text{CH}_0(\bar{X})^0 \rightarrow 0$$

gives rise to the exact sequence

$$0 \rightarrow {}_m(\text{CH}_0(\bar{X})^0)^G \rightarrow {}_m\text{CH}_0(\bar{X})^0 \xrightarrow{1-F} {}_m\text{CH}_0(\bar{X})^0 \rightarrow (\text{CH}_0(\bar{X})^0)^G / m \rightarrow 0.$$

The result follows by taking limits. □

6.3. *Duality theory.* For a complex \mathcal{G} of torsion sheaves on a scheme X over a perfect field k , we consider the functor

$$\mathcal{D}(\mathcal{G}) = R\mathcal{H}om_X(\mathcal{G}, \mathbb{Z}_X^c)$$

(if \mathcal{G} is unbounded, this is defined using a K-injective resolution as in [43, Prop. 6.1]).

PROPOSITION 6.4. *The functor \mathcal{D} sends bounded above complexes to bounded below complexes and conversely.*

Proof. Since the statement is étale local, we can assume that k is algebraically closed. By Lemma 4.8 and Corollary 4.9, $\mathcal{E}rt_X^i(\mathcal{F}, \mathbb{Z}_X^c)$ vanishes for any torsion sheaf \mathcal{F} unless $-2d \leq i \leq 1$. Hence the spectral sequence

$$E_2^{p,q} = \mathcal{E}rt_X^p(\mathcal{H}^{-q}(\mathcal{G}), \mathbb{Z}_X^c) \Rightarrow \mathcal{E}rt_X^{p+q}(\mathcal{G}, \mathbb{Z}_X^c),$$

converges, and the claim follows. □

THEOREM 6.5 (Exchange formulas). *Let $f : X \rightarrow Y$ be a map between schemes over a perfect field and \mathcal{G} and \mathcal{F} be constructible sheaves on X . Then the following formulas hold*

$$\mathcal{D}(\mathcal{F} \otimes \mathcal{G}) \cong R\mathcal{H}om(\mathcal{F}, \mathcal{D}(\mathcal{G})),$$

$$Rf_*\mathcal{D}(\mathcal{G}) \cong \mathcal{D}(Rf_!\mathcal{G}),$$

$$Rf^!\mathcal{D}(\mathcal{G}) \cong \mathcal{D}(f^*\mathcal{G}).$$

Proof. The first formula is adjointness of Hom and \otimes . For the second formula, in view of

$$R\mathcal{H}om(\mathcal{F}, \mathcal{G})(U) \cong R\text{Hom}_U(\mathcal{F}|_U, \mathcal{G}|_U) \cong R\text{Hom}_X(j_{!}j^*\mathcal{F}, \mathcal{G}),$$

it suffices to prove that $R\text{Hom}_X(\mathcal{F}, \mathbb{Z}_X^c) \cong R\text{Hom}_Y(Rf_!\mathcal{F}, \mathbb{Z}_Y^c)$, for constructible \mathcal{F} , which is Corollary 4.7. The last formula holds by [1, XVIII Cor. 3.1.12.2]. □

THEOREM 6.6 (Biduality). *Let \mathcal{G} be a constructible m -torsion sheaf for some integer m not divisible by the characteristic of k . Then*

$$\mathcal{G} \cong \mathcal{D}(\mathcal{D}(\mathcal{G})).$$

In particular,

$$\begin{aligned} \mathcal{D}(\mathcal{F} \otimes \mathcal{D}(\mathcal{G})) &\cong R\mathcal{H}om(\mathcal{F}, \mathcal{G}), \\ Rf_! \mathcal{D}(\mathcal{G}) &\cong \mathcal{D}(Rf_* \mathcal{G}), \\ f^* \mathcal{D}(\mathcal{G}) &\cong \mathcal{D}(Rf^! \mathcal{G}). \end{aligned}$$

Proof. For a closed embedding $i : Z \rightarrow X$ with open complement $j : U \rightarrow X$, $R\mathcal{H}om(j_! \mathcal{F}, \mathcal{G}) \cong j_! R\mathcal{H}om(\mathcal{F}, j^* \mathcal{G})$, hence $\mathcal{D}(\mathcal{D}(j_! j^* \mathcal{G})) \cong j_! \mathcal{D}(\mathcal{D}(j^* \mathcal{G}))$ on the one hand, and $R\mathcal{H}om(i_* \mathcal{F}, \mathcal{G}) \cong i_* R\mathcal{H}om(\mathcal{F}, Ri^! \mathcal{G})$ hence $\mathcal{D}(\mathcal{D}(i_* i^* \mathcal{G})) \cong i_* \mathcal{D}(\mathcal{D}(i^* \mathcal{G}))$ on the other hand. Thus we can use devissage to reduce to the case that $\mathcal{G} \cong \mathbb{Z}/m$ on a smooth and proper scheme X of dimension d over k . In this case, by Lemma 2.4, the statement reduces to $\mathbb{Z}/m \cong R\mathcal{H}om_{\mathbb{Z}/m}(\mu_m^{\otimes d}, \mu_m^{\otimes d})$. Since μ_m is locally constant, this can be checked at stalks, where it is clear. The other formulas follow from this by substitution in Theorem 6.5; see [2, I, Prop. 1.12]. \square

Remark. The biduality map at the characteristic is not an isomorphism in general. For example, if k is algebraically closed and U is an open subset of the projective line \mathbb{P}_k^1 , then the localization sequence for cohomology with compact support gives a short exact sequence

$$0 \rightarrow \bigoplus_{x \in \mathbb{P}^1 - U} i_x^* v^1 \rightarrow H_c^1(U_{\text{et}}, v^1) \rightarrow \mathbb{Z}/p \rightarrow 0$$

and it follows that the dual

$$\mathcal{H}om_{\mathbb{Z}/p}(v^1, v^1)(U) = \text{Hom}_{\mathbb{P}^1, \mathbb{Z}/p}(j_! v^1, v^1)$$

of $H_c^1(U_{\text{et}}, v^1)$ is very large, hence $\mathbb{Z}/p \neq R\mathcal{H}om_{\mathbb{Z}/p}(v^1, v^1)$. The small étale site does not give well-behaved extension groups for nonconstructible sheaves. See Kato [21] for a good duality using the relative perfect site.

7. One dimensional bases

For the remainder of the paper we assume the validity of the Beilinson-Lichtenbaum conjecture. Let S be a connected one-dimensional regular scheme such that all closed points have perfect residue fields, and the generic point η has characteristic 0 (the case that S is a curve over a finite field is covered by Theorem 5.1). The dimension of an irreducible scheme is the relative dimension over S ; in particular, an irreducible scheme of dimension d over η has dimension $d + 1$ over S , and the complex $\mathbb{Z}_X^c(n)$ restricted to the generic fiber would be the complex $\mathbb{Z}_{X_\eta}^c(n - 1)[2]$ if it were viewed relative to the generic point.

THEOREM 7.1. *Let S be a strictly henselian discrete valuation ring of mixed characteristic with algebraically closed residue field. If X is essentially of finite type over S and $n \leq 0$, then $\mathbb{Z}_X^c(n)(X) \cong R\Gamma(X_{\text{et}}, \mathbb{Z}_X^c(n))$.*

Proof. As in the proof of Theorem 3.1, it suffices to show that $\mathbb{Z}^c(n)(\text{Spec } F) \cong R\Gamma(\text{Spec } F_{\text{et}}, \mathbb{Z}^c(n))$ for an extension of finite transcendence degree d of the residue

field of the closed point or the generic point of S , because $\mathbb{Z}_X^c(n)$ has the localization property for schemes of finite type over a discrete valuation ring by Levine [29]. The case that F lies over the closed point was treated in the proof of Theorem 3.1, hence we can assume that F lies over η . We have to show that the map $H_i(F, \mathbb{Z}^c(n)) \rightarrow H_i(F_{\text{et}}, \mathbb{Z}^c(n))$ is an isomorphism for all i . (It is important to remember that we use dimension relative to S here, so that F is a limit of schemes of dimension $d + 1$.) Rationally, Zariski and étale hypercohomology of the motivic complex agree. With mod m -coefficients, we get the isomorphism from the Beilinson-Lichtenbaum conjecture for $i \geq d + 1 + n$. On the other hand, by Proposition 2.2, $H_i(F_{\text{et}}, \mathbb{Z}^c/m(n)) \cong H^{2d+2-i}(F_{\text{et}}, \mu_m^{\otimes(d+1-n)})$, and this vanishes for $i < d + 1$ because F has cohomological dimension $d + 1$. \square

COROLLARY 7.2. *Let S be a Dedekind ring of characteristic 0 with perfect residue fields, and let $n \leq 0$.*

a) *If $i : Z \rightarrow X$ is a closed embedding of schemes over S with open complement U , then we have a quasi-isomorphism $\mathbb{Z}_Z^c(n) \cong Ri^! \mathbb{Z}_X^c(n)$, hence a distinguished triangle*

$$\cdots \rightarrow R\Gamma(Z_{\text{et}}, \mathbb{Z}_Z^c(n)) \rightarrow R\Gamma(X_{\text{et}}, \mathbb{Z}_X^c(n)) \rightarrow R\Gamma(U_{\text{et}}, \mathbb{Z}_U^c(n)) \rightarrow \cdots .$$

b) *If $f : X \rightarrow Y$ is a proper map between schemes over S , we have a push-forward map $f_* : Rf_* \mathbb{Z}_X^c(n) \rightarrow \mathbb{Z}_Y^c(n)$.*

Proof. a) Since the statement is local for the étale topology, we can assume that S is strictly henselian local. Let $s = \text{Spec } k$ be the closed point, η be the generic point, and X_s and X_η be the closed and generic fiber, respectively. We first treat the case $Z = X_s, U = X_\eta$. Consider the following map of distinguished triangles, where the global section functor is with respect to the étale topology,

$$\begin{array}{ccccc} \mathbb{Z}^c(n)(X_s) & \longrightarrow & \mathbb{Z}^c(n)(X) & \longrightarrow & \mathbb{Z}^c(n)(X_\eta) \\ \downarrow & & \downarrow & & \downarrow \\ R\Gamma(X_s, \mathbb{Z}_{X_s}^c(n)) & \longrightarrow & R\Gamma(X, \mathbb{Z}_X^c(n)) & \longrightarrow & R\Gamma(X_\eta, \mathbb{Z}_{X_\eta}^c(n)). \end{array}$$

The vertical maps are quasi-isomorphisms by Theorem 7.1, and the upper triangle is exact by [29], hence the lower triangle is exact. For arbitrary Z , we consider the diagram

$$\begin{array}{ccccc} R\Gamma(Z_s, \mathbb{Z}^c(n)) & \longrightarrow & R\Gamma(Z, \mathbb{Z}^c(n)) & \longrightarrow & R\Gamma(Z_\eta, \mathbb{Z}^c(n)) \\ \downarrow & & \downarrow & & \downarrow \\ R\Gamma(X_s, \mathbb{Z}^c(n)) & \longrightarrow & R\Gamma(X, \mathbb{Z}^c(n)) & \longrightarrow & R\Gamma(X_\eta, \mathbb{Z}^c(n)) \\ \downarrow & & \downarrow & & \downarrow \\ R\Gamma(U_s, \mathbb{Z}^c(n)) & \longrightarrow & R\Gamma(U, \mathbb{Z}^c(n)) & \longrightarrow & R\Gamma(U_\eta, \mathbb{Z}^c(n)). \end{array}$$

The horizontal triangles are distinguished by the above, and the outer vertical triangles are distinguished by Corollary 3.3. Hence the middle vertical triangle is distinguished as well. Part b) is proved exactly like Corollary 3.2, using Theorem 7.1. \square

THEOREM 7.3. *Let $f : X \rightarrow S$ be a scheme over a Dedekind ring of mixed characteristic with perfect residue fields. Then for every torsion sheaf \mathcal{F} on X , and $n \leq 0$, there is a quasi-isomorphism*

$$R \operatorname{Hom}_X(\mathcal{F}, \mathbb{Z}_X^c(n)) \cong R \operatorname{Hom}_S(Rf_! \mathcal{F}, \mathbb{Z}_S^c(n)).$$

Proof. Replacing X by a compactification $j : X \rightarrow T$ and \mathcal{F} by $j_! \mathcal{F}$, we can assume that X is proper. Writing \mathcal{F} as a colimit of constructible sheaves, we can assume that \mathcal{F} is constructible; see the proof of Corollary 4.7. The quasi-isomorphism is induced by the map $Rf_* \mathbb{Z}_X^c(n) \rightarrow \mathbb{Z}_S^c(n)$ of Corollary 7.2b). We can assume that \mathcal{F} is m -torsion for some integer m , and it suffices to show that the adjoint map $\mathbb{Z}_X^c/m(n) \rightarrow Rf^! \mathbb{Z}_S^c/m(n)$ is an isomorphism. We can check this at stalks and assume that the base is a henselian discrete valuation ring; the case that the base is an algebraically closed field is Corollary 4.7. Consider the following commutative diagram, coming from Corollary 7.2, and the quasi-isomorphisms $Rj_* Rf^! = Rf^! Rj_*$ and $i_* Rf^! = Rf^! i_*$:

$$\begin{array}{ccccc} i_* \mathbb{Z}_{X_s}^c/m(n) & \longrightarrow & \mathbb{Z}_X^c/m(n) & \longrightarrow & Rj_* \mathbb{Z}_{X_\eta}^c/m(n) \\ \downarrow & & \downarrow & & \downarrow \\ i_* Rf^! \mathbb{Z}_S^c/m(n) & \longrightarrow & Rf^! \mathbb{Z}_S^c/m(n) & \longrightarrow & Rj_* Rf^! \mathbb{Z}_\eta^c/m(n). \end{array}$$

The outer maps are quasi-isomorphisms by Corollary 4.7a), hence so is the middle map. \square

LEMMA 7.4. *Under the conditions of Theorem 7.3, we have $\mathbb{Z}_S^c \cong \mathbb{G}_m[1]$ on S .*

Proof. By [27, Lemma 11.2], \mathbb{Z}_S^c is acyclic (as a complex of étale sheaves) except in degree -1 . The quasi-isomorphism is induced by the map $\mathbb{G}_m \rightarrow \mathcal{H}^{-1}(\mathbb{Z}_S^c)$, sending a unit u on U to the subscheme $(\frac{1}{1-u}, \frac{-u}{1-u})$. \square

7.1. Local duality. Let $f : X \rightarrow S$ be a scheme over a discrete valuation ring and $i : s \rightarrow X$ be the closed point. For a torsion sheaf \mathcal{F} , we define cohomology with compact support in the closed fiber $R\Gamma_{X_s,c}(X_{\text{et}}, \mathcal{F})$ to be $R\Gamma(S_{\text{et}}, i_* Ri^! Rf_! \mathcal{F})$.

THEOREM 7.5. *Let $X \rightarrow S$ be a scheme over a henselian discrete valuation ring of characteristic 0 with finite residue field. Then for every torsion sheaf \mathcal{F} on X , there is a quasi-isomorphism*

$$R \operatorname{Hom}_X(\mathcal{F}, \mathbb{Z}_X^c) \cong R \operatorname{Hom}_{\text{Ab}}(R\Gamma_{X_s,c}(X_{\text{et}}, \mathcal{F}), \mathbb{Z})[-1].$$

In particular, for constructible \mathcal{F} , there are perfect pairings of finite groups

$$\mathrm{Ext}_X^{2-i}(\mathcal{F}, \mathbb{Z}_X^c) \times H_{X_s, c}^i(X_{\mathrm{et}}, \mathcal{F}) \rightarrow \mathbb{Q}/\mathbb{Z}.$$

Proof. From Theorem 7.3 we get

$$R \mathrm{Hom}_X(\mathcal{F}, \mathbb{Z}_X^c) \cong R \mathrm{Hom}_S(Rf_! \mathcal{F}, \mathbb{Z}_S^c),$$

hence the result follows from

LEMMA 7.6. *For every complex of constructible sheaves \mathcal{G} on S , we have a quasi-isomorphism*

$$R \mathrm{Hom}_S(\mathcal{G}, \mathbb{Z}_S^c) \cong R \mathrm{Hom}_{\mathrm{Ab}}(R\Gamma_S(S_{\mathrm{et}}, \mathcal{G}), \mathbb{Z})[-1].$$

Proof. This follows using Lemma 7.4 from the local duality quasi-isomorphism [34, II Th. 1.8]

$$R \mathrm{Hom}_S(\mathcal{G}, \mathbb{Z}_S^c)[-1] \cong R \mathrm{Hom}_S(\mathcal{G}, \mathbb{G}_m) \cong R \mathrm{Hom}_{\mathrm{Ab}}(R\Gamma_S(S_{\mathrm{et}}, \mathcal{G}), \mathbb{Q}/\mathbb{Z})[-3].$$

□

Kato-homology $H_i^K(X, \mathbb{Z}/m)$ over a henselian discrete valuation ring is defined as in (13). If X is proper and regular over S , with generic fiber X_η and closed fiber X_s , then Kato conjectures [22, Conj. 5.1] that the Kato-homology $H_i^K(X_s, \mathbb{Z}/m)$ of the closed and $H_i^K(X_\eta, \mathbb{Z}/m)$ of the generic fiber agree, or equivalently that the Kato-homology of X vanishes for all i . The same proof as for (14) gives

COROLLARY 7.7. *For every scheme over a henselian discrete valuation ring of characteristic 0 with finite residue fields, there is a long exact sequence*

$$(23) \quad \cdots \rightarrow \mathrm{CH}_0(X, i, \mathbb{Z}/m) \rightarrow H_{X_s, c}^{i+1}(X_{\mathrm{et}}, \mathbb{Z}/m)^* \rightarrow H_{i+1}^K(X, \mathbb{Z}/m) \rightarrow \cdots .$$

Note that the exact sequences (14), (16), and (23) fit together into a double-complex

$$\begin{array}{ccccc} \mathrm{CH}_{-1}(X_\eta, i, \mathbb{Z}/m) & \longrightarrow & \mathrm{CH}_0(X_s, i-1, \mathbb{Z}/m) & \longrightarrow & \mathrm{CH}_0(X, i-1, \mathbb{Z}/m) \\ \downarrow & & \downarrow & & \downarrow d \\ H_c^i((X_\eta)_{\mathrm{et}}, \mathbb{Z}/m)^* & \longrightarrow & H_c^i((X_s)_{\mathrm{et}}, \mathbb{Z}/m)^* & \longrightarrow & H_{X_s, c}^i(X_{\mathrm{et}}, \mathbb{Z}/m)^* \\ \downarrow & & \downarrow & & \downarrow \\ H_i^K(X_\eta, \mathbb{Z}/m) & \longrightarrow & H_i^K(X_s, \mathbb{Z}/m) & \longrightarrow & H_i^K(X, \mathbb{Z}/m). \end{array}$$

7.2. *Number rings.* Let B be the spectrum of a number ring. For a torsion sheaf \mathcal{G} on B , the cohomology with compact supports $R\Gamma_c(B_{\mathrm{et}}, \mathcal{G})$ is defined, for example, in [34, II §2] and differs from $R\Gamma(B_{\mathrm{et}}, \mathcal{G})$ only at the prime 2 and only for those B having a real embedding. For a torsion sheaf \mathcal{F} on X , we define cohomology with compact support $H_c^i(X_{\mathrm{et}}, \mathcal{F})$ to be the cohomology of the complex $R\Gamma_c(B_{\mathrm{et}}, Rf_! \mathcal{F})$. The following generalizes and unifies [34, II Thms. 6.2, 7.16].

THEOREM 7.8. *For every scheme $f : X \rightarrow B$ and torsion sheaf \mathcal{F} on X , we have a quasi-isomorphism*

$$R \operatorname{Hom}_X(\mathcal{F}, \mathbb{Z}_X^c) \cong R \operatorname{Hom}_{\text{Ab}}(R\Gamma_c(X_{\text{et}}, \mathcal{F}), \mathbb{Z})[-1],$$

which induces perfect pairings of finite groups for constructible \mathcal{F} ,

$$\operatorname{Ext}_X^{2-i}(\mathcal{F}, \mathbb{Z}_X^c) \times H_c^i(X_{\text{et}}, \mathcal{F}) \rightarrow \mathbb{Q}/\mathbb{Z}.$$

Proof. For a complex of constructible sheaves \mathcal{G} on B , we have by Artin-Verdier duality [31][34, II Th. 3.1b)] a quasi-isomorphism

$$R \operatorname{Hom}_B(\mathcal{G}, \mathbb{G}_m) \cong R \operatorname{Hom}_{\text{Ab}}(R\Gamma_c(B, \mathcal{G}), \mathbb{Q}/\mathbb{Z})[-3].$$

If we apply this to the quasi-isomorphism of Theorem 7.3, we get

$$\begin{aligned} R \operatorname{Hom}_X(\mathcal{F}, \mathbb{Z}_X^c) &\cong R \operatorname{Hom}_B(Rf_! \mathcal{F}, \mathbb{G}_m)[1] \\ &\cong R \operatorname{Hom}_{\text{Ab}}(R\Gamma_c(X_{\text{et}}, \mathcal{F}), \mathbb{Q}/\mathbb{Z})[-2]. \quad \square \end{aligned}$$

Over the spectrum of a number ring, higher Chow groups $\operatorname{CH}_i(X, n)$ are defined as the Zariski-hypercohomology of the complex $z_n(-, *)$, and Kato-homology $H_i^K(X, \mathbb{Z}/m)$ is defined [22, Conj. 05] as the homology of the cone of the canonical map of the complex (13) to the direct sum of the complexes (15) for $X \times_B F_\nu$, where F_ν runs through the real places of B . Kato conjectures that $H_i^K(X, \mathbb{Z}/m) = 0$ for $i > 0$ and X regular, flat and proper over B .

COROLLARY 7.9. *For every scheme over a number ring, there is a long exact sequence*

$$\cdots \rightarrow \operatorname{CH}_0(X, i, \mathbb{Z}/m) \rightarrow H_c^{i+1}(X_{\text{et}}, \mathbb{Z}/m)^* \rightarrow H_{i+1}^K(X, \mathbb{Z}/m) \rightarrow \cdots .$$

For completeness we give the following analog of Corollary 4.7 and Theorem 6.5.

PROPOSITION 7.10. *Let $f : X \rightarrow Y$ be a map of schemes over the spectrum of a number ring S , and let $n \leq 0$.*

a) *For every locally constant constructible sheaf \mathcal{G} on Y , the map of Corollary 7.2b) induces a quasi-isomorphism*

$$\mathbb{Z}_X^c(n) \otimes f^* \mathcal{G} \cong Rf^!(\mathbb{Z}_Y^c(n) \otimes \mathcal{G}).$$

b) *For every torsion sheaf \mathcal{F} on X and every finitely generated, locally constant sheaf \mathcal{G} on Y , we have a functorial quasi-isomorphism*

$$\operatorname{Hom}_Y(Rf_! \mathcal{F}, \mathbb{Z}_Y^c(n) \otimes \mathcal{G}) \cong \operatorname{Hom}_X(\mathcal{F}, \mathbb{Z}_X^c(n) \otimes f^* \mathcal{G}).$$

c) *(Exchange formulas) If \mathcal{F} and \mathcal{G} are constructible, then*

$$\mathcal{D}(\mathcal{F} \otimes \mathcal{G}) \cong R \mathcal{H}om(\mathcal{F}, \mathcal{D}(\mathcal{G})),$$

$$Rf_* \mathcal{D}(\mathcal{G}) \cong \mathcal{D}(Rf_! \mathcal{G}),$$

$$Rf^! \mathcal{D}(\mathcal{G}) \cong \mathcal{D}(f^* \mathcal{G}).$$

Proof. a) Since the statement is local for the étale topology, we can assume that S is strictly henselian and that \mathcal{G} is a constant sheaf of the form \mathbb{Z}/m . Then the proof of Theorem 7.3 works in this situation. b) follows from a) as in Corollary 4.7, and c) is proved as in Theorem 6.5. \square

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