ANNALS OF MATHEMATICS

Elliptic functions, Green functions and the mean field equations on tori

By Chang-Shou Lin and Chin-Lung Wang

SECOND SERIES, VOL. 172, NO. 2

September, 2010

ANMAAH

Annals of Mathematics, **172** (2010), 911–954

Elliptic functions, Green functions and the mean field equations on tori

By CHANG-SHOU LIN and CHIN-LUNG WANG

Abstract

We show that the Green functions on flat tori can have either three or five critical points only. There does not seem to be any direct method to attack this problem. Instead, we have to employ sophisticated nonlinear partial differential equations to study it. We also study the distribution of the number of critical points over the moduli space of flat tori through deformations. The functional equations of special theta values provide important inequalities which lead to a solution for all rhombus tori.

1. Introduction and statement of results

The study of geometric or analytic problems on two dimensional tori is the same as the study of problems on $\mathbb{R}^2$ with doubly periodic data. Such situations occur naturally in sciences and mathematics since early days. The mathematical foundation of elliptic functions was subsequently developed in the 19th century. It turns out that these special functions are rather deep objects by themselves. Tori of different shapes may result in very different behavior of the elliptic functions and their associated objects. Arithmetic on elliptic curves is perhaps the eldest and the most vivid example.

In this paper, we show that this is also the case for certain nonlinear partial differential equations. Indeed, researches on doubly periodic problems in mathematical physics or differential equations often restrict the study to rectangular tori for simplicity. This leaves the impression that the theory for general tori may be much the same as for the rectangular case. However this turns out to be false. We will show that the solvability of the *mean field equation* depends on the shape of the Green function, which in turn depends on the geometry of the tori in an essential way.

Recall that the Green function $G(z, w)$ on a flat torus $T = \mathbb{C}/\mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$ is the unique function on $T \times T$ which satisfies

$$-\triangle_z G(z, w) = \delta_w(z) - \frac{1}{|T|}$$

and $\int_T G(z,w)\,dA = 0$, where δ_w is the Dirac measure with singularity at $z = w$. Because of the translation invariance of $\triangle_z$, we have $G(z,w) = G(z-w,0)$ and it is enough to consider *the Green function* $G(z) := G(z,0)$.

Not surprisingly, G can be explicitly solved in terms of elliptic functions. For example, using theta functions we have (cf. Lemmas 2.1 and 7.1)

$$G(z) = -\frac{1}{2\pi}\log|\vartheta_1(z)| + \frac{1}{2b}y^2 + C(\tau)$$

where $z = x + iy$ and $\tau := \omega_2/\omega_1 = a + ib$. The structure of G, especially its critical points and critical values, will be the fundamental objects that interest us. The critical point equation $\nabla G(z) = 0$ is given by

$$\frac{\partial G}{\partial z} \equiv \frac{-1}{4\pi}\left((\log\vartheta_1)_z + 2\pi i\,\frac{y}{b}\right) = 0.$$

In terms of Weierstrass' elliptic functions $\wp(z)$, $\zeta(z) := -\int^z \wp$ and by the relation $(\log\vartheta_1)_z = \zeta(z) - \eta_1 z$ and with $\eta_i = \zeta(z+\omega_i) - \zeta(z)$ the quasi-periods, the equation takes the simpler form: $z = t\omega_1 + s\omega_2$ is a critical point of G if and only if the following linear relation (Lemma 2.3) holds:

$$\zeta(t\omega_1 + s\omega_2) = t\eta_1 + s\eta_2. \tag{1.1}$$

Since G is even, it is elementary to see that half-periods $\frac{1}{2}\omega_1$, $\frac{1}{2}\omega_2$ and $\frac{1}{2}\omega_3 = (\omega_1+\omega_2)/2$ are the three obvious critical points and other critical points must appear in pairs. The question is: *Are there other critical points*? or *How many critical points might G have*? It turns out that this is a delicate question and cannot be attacked easily from the simple looking equation (1.1). One of our chief purposes in this paper is to understand the geometry of the critical point set over the moduli space of flat tori $\mathcal{M}_1 = \mathcal{H}/\mathrm{SL}(2,\mathbb{Z})$ and to study its interaction with the nonlinear mean field equation.

The mean field equation on a flat torus T takes the form ($\rho \in \mathbb{R}_+$)

$$\triangle u + \rho e^u = \rho\delta_0. \tag{1.2}$$

This equation has its origin in the prescribed curvature problem in geometry like the Nirenberg problem, cone metrics etc. It also comes from statistical physics as the mean field limits of the Euler flow, hence the name. Recently it was shown to be related to the self dual condensation of the Chern-Simons-Higgs model. We refer to [3], [4], [2], [5], [6], [8], [9] and [10] for the recent development of this subject.

When $\rho \neq 8m\pi$ for any $m \in \mathbb{Z}$, it has been recently proved in [4], [2], [5] that the Leray-Schauder degree is nonzero; so the equation always has solutions, regardless of the actual shape of T.

The first interesting case remaining is when $\rho = 8\pi$ where the degree theory fails completely. Instead of the topological degree, precise knowledge on the Green function plays a fundamental role in the investigation of (1.2). The first main result of this paper is the following existence criterion whose proof is given in Section 3 by a detailed manipulation on elliptic functions:

THEOREM 1.1 (Existence). *For $\rho = 8\pi$, the mean field equation on a flat torus has solutions if and only if the Green function has critical points other than the three half-period points. Moreover, each extra pair of critical points corresponds to a one-parameter scaling family of solutions.*

It is known that for rectangular tori $G(z)$ has precisely the three obvious critical points; hence for $\rho = 8\pi$ equation (1.2) has no solutions. However we will show in Section 2 that for the case $\omega_1 = 1$ and $\tau = \omega_2 = e^{\pi i/3}$ there are at least five critical points and the solutions of (1.2) exist.

Our second main result is the uniqueness theorem.

THEOREM 1.2 (Uniqueness). *For $\rho = 8\pi$, the mean field equation on a flat torus has at most one solution up to scaling.*

In view of the correspondence in Theorem 1.1, an equivalent statement of Theorem 1.2 is the following result:

THEOREM 1.3. *The Green function has at most five critical points.*

Unfortunately we were unable to find a direct proof of Theorem 1.3 from the critical point equation (1.1). Instead, we will prove uniqueness theorem first, and then Theorem 1.3 is an immediate corollary. Our proof of Theorem 1.2 is based on the method of symmetrization applied to the linearized equation at a particularly chosen, even solution in the scaling family. In fact we study in Section 4 the one parameter family

$$\triangle u + \rho e^u = \rho \delta_0, \quad \rho \in [4\pi, 8\pi]$$

on T within *even solutions*. This extra assumption allows us to construct a double cover $T \to S^2$ via the Weierstrass $\wp$ function and to transform equation (1.2) into a similar one on S^2 but with three more delta singularities with negative coefficients. The condition $\rho \geq 4\pi$ is to guarantee that the original singularity at 0 still has a nonnegative coefficient of delta singularity.

The uniqueness is proved for this family via the method of continuity. For the starting point $\rho = 4\pi$, by a construction similar to the proof of Theorem 1.1 we sharpen the result on the nontrivial degree to the existence and uniqueness of solution (Theorem 3.2). For $\rho \in [4\pi, 8\pi]$, the symmetrization reduces the problem on the nondegeneracy of the linearized equation to the isoperimetric inequality on domains in $\mathbb{R}^2$ with respect to certain singular measure:

THEOREM 1.4 (Symmetrization Lemma). *Let $\Omega \subset \mathbb{R}^2$ be a simply connected domain and let v be a solution of*

$$\triangle v + e^v = \sum_{j=1}^{N} 2\pi\alpha_j \delta_{p_j}$$

in Ω. Suppose that the first eigenvalue of $\triangle + e^v$ is zero on Ω with φ the first eigenfunction. If the isoperimetric inequality with respect to $ds^2 = e^v|dx|^2$:

$$2\ell^2(\partial\omega) \geq m(\omega)(4\pi - m(\omega)) \tag{1.3}$$

holds for all level domains $\omega = \{\varphi > t\}$ with $t > 0$, then

$$\int_\Omega e^v\, dx \geq 2\pi.$$

Moreover, (1.3) *holds if there is only one negative α_j and $\alpha_j = -1$.*

The proof of the number of critical points appears to be one of the very few instances that one needs to study a simple analytic equation, here the critical point equation (1.1), by way of sophisticated nonlinear analysis.

To get a deeper understanding of the underlying structure of solutions, we first notice that for $\rho = 8\pi$, (1.2) is the Euler-Lagrange equation of the nonlinear functional

$$J_{8\pi}(v) = \frac{1}{2}\int_T |\nabla v|^2\, dA - 8\pi \log \int_T e^{v-8\pi G(z)}\, dA \tag{1.4}$$

on $H^1(T) \cap \{v \mid \int_T v = 0\}$, the Sobolev space of functions with L^2-integrable first derivative. From this viewpoint, the nonexistence of minimizers for rectangular tori was shown in [5]. Here we sharpen the result to the nonexistence of solutions. Also for $\rho \in (4\pi, 8\pi)$ we sharpen the result on the nontrivial degree of equation (1.2) in [4] to the uniqueness of solutions within even functions. We expect the uniqueness holds true without the even assumption, but our method only achieves this at $\rho = 4\pi$. Obviously, uniqueness without the even assumption fails at $\rho = 8\pi$ due to the existence of scaling.

Naturally, the next question after Theorem 1.2 is to determine those tori whose Green functions have five critical points. It is the case if the three half-periods are all saddle points. A strong converse is proved in [7]:

THEOREM A. *If the Green function has five critical points then the extra pair of critical points are minimum points and the three half-periods are all saddle points.*

Together with Theorem 1.1, this implies that *a minimizer of $J_{8\pi}$ exists if and only if the Green function has more than three critical points.* In fact we show in [7] that any solution of equation (1.2) must be a minimizer of the nonlinear functional $J_{8\pi}$. Thus we completely solve the existence problem on minimizers, a question raised by Nolasco and Tarantello in [9].

By Theorem A, we have reduced the question on detecting a given torus to have five critical points to the technically much simpler criterion on (non)-local minimality of the three half-period points. In this paper, however, no reference to Theorem A is needed. Instead, it motivates the following comparison result, which also simplifies the criterion further:

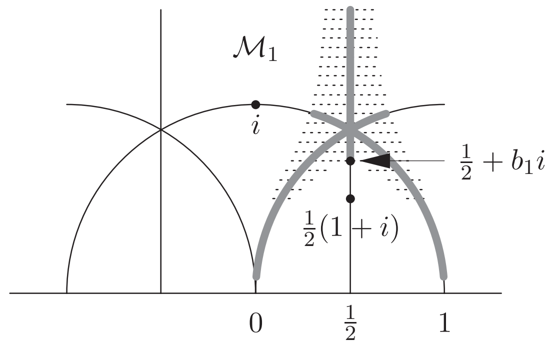

Figure 1. Ω_5 contains a neighborhood of $e^{\pi i/3}$.

THEOREM 1.5. *Let z_0 and z_1 be two half-period points. Then*

$$G(z_0) \geq G(z_1) \quad \textit{if and only if} \quad |\wp(z_0)| \geq |\wp(z_1)|.$$

For general flat tori, a computer simulation suggests the following picture: Let Ω_3 (resp. Ω_5) be the subset of the moduli space $\mathcal{M}_1 \cup \{\infty\} \cong S^2$ which corresponds to tori with three (resp. five) critical points. Then $\Omega_3 \cup \{\infty\}$ is a closed subset containing i, Ω_5 is an open subset containing $e^{\pi i/3}$, both of them are simply connected and their common boundary $C := \partial\Omega_3 = \partial\Omega_5$ is a curve homeomorphic to S^1 containing ∞. Moreover, the extra critical points are split out from some half-period point when the tori move from Ω_3 to Ω_5 across C.

We propose to prove the experimental observation by the method of deformations in $\mathcal{M}_1$. The degeneracy analysis of critical points, especially the half-period points, is a crucial step. In this direction we have the following partial result on tori corresponding to the line $\operatorname{Re}\tau = 1/2$. These are equivalent to the rhombus tori and $\tau = \frac{1}{2}(1+i)$ is equivalent to the square torus where there are only three critical points.

THEOREM 1.6 (Moduli Dependence). *Let $\omega_1 = 1$ and $\omega_2 = \tau = \frac{1}{2} + ib$ with $b > 0$. Then*

(1) *There exists $b_0 < \frac{1}{2} < b_1 < \sqrt{3}/2$ such that $\frac{1}{2}\omega_1$ is a degenerate critical point of $G(z;\tau)$ if and only if $b = b_0$ or $b = b_1$. Moreover, $\frac{1}{2}\omega_1$ is a local minimum point of $G(z;\tau)$ if $b_0 < b < b_1$ and it is a saddle point if $b < b_0$ or $b > b_1$.*

(2) *Both $\frac{1}{2}\omega_2$ and $\frac{1}{2}\omega_3$ are nondegenerate saddle points of G.*

(3) *$G(z;\tau)$ has two more critical points $\pm z_0(\tau)$ when $b < b_0$ or $b > b_1$. They are nondegenerate global minimum points of G and in the former case,*

$$\operatorname{Re} z_0(\tau) = \frac{1}{2}; \quad 0 < \operatorname{Im} z_0(\tau) < \frac{b}{2}.$$

Part (1) gives a strong support of the conjectural shape of the decomposition $\mathcal{M}_1 = \Omega_3 \cup \Omega_5$. Part (3) implies that minimizers of $J_{8\pi}$ exist for tori with $\tau = \frac{1}{2} + ib$ where $b < b_0$ or $b > b_1$.

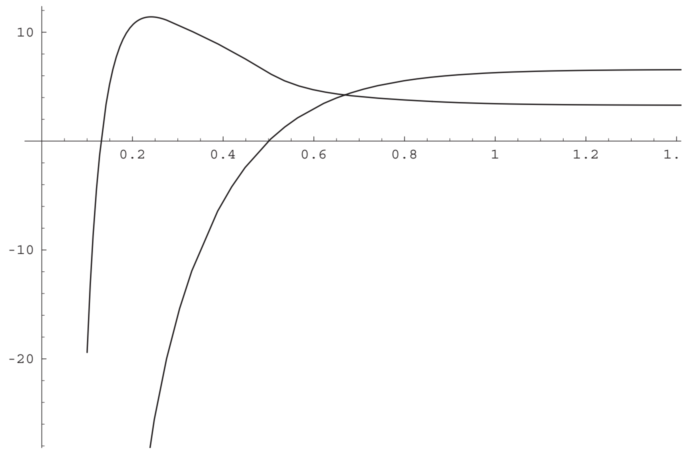

Figure 2. Graphs of η_1 (the left one) and e_1 in b where e_1 is increasing.

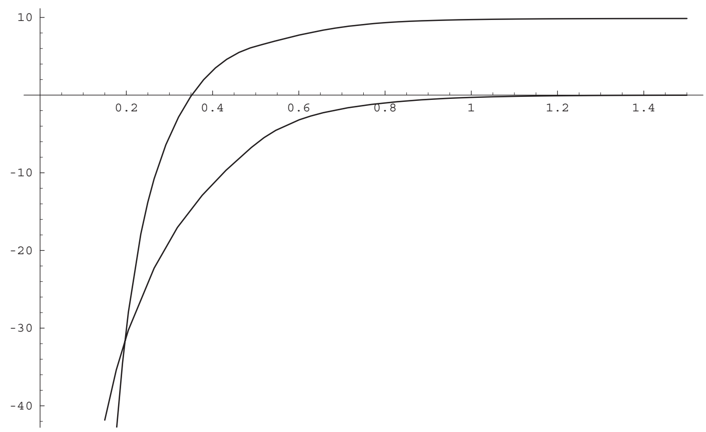

Figure 3. Graphs of $e_1 + \eta_1$ (the upper one) and $\frac{1}{2}e_1 - \eta_1$. Both functions are increasing in b and $\frac{1}{2}e_1 - \eta_1 \nearrow 0$.

The proofs are given in Section 6, notably in Lemmas 6.1, 6.2, 6.6 and Theorem 6.7. They rely on two fundamental inequalities of special values of elliptic functions and we would like to single out the statements (recall that $e_i = \wp(\frac{1}{2}\omega_i)$ and $\eta_i = 2\zeta(\frac{1}{2}\omega_i)$):

THEOREM 1.7 (Fundamental Inequalities). *Let $\omega_1 = 1$ and $\omega_2 = \tau = \frac{1}{2} + ib$ with $b > 0$. Then*

(1) $\frac{d}{db}(e_1 + \eta_1) = -4\pi \frac{d^2}{db^2} \log |\vartheta_2(0)| > 0$.

(2) $\frac{1}{2}e_1 - \eta_1 = 4\pi \frac{d}{db} \log |\vartheta_3(0)| < 0$ *and* $\frac{d^2}{db^2} \log |\vartheta_3(0)| > 0$. *The same holds for $\vartheta_4(0) = \overline{\vartheta_3(0)}$. In particular, e_1 increases in b.*

These modular functions come into play due to the explicit computation of Hessian at half-periods along $\operatorname{Re}\tau = \frac{1}{2}$ (cf. (6.3) and (6.21)):

$$4\pi^2 \det D^2 G\Big(\frac{\omega_1}{2}\Big) = (e_1 + \eta_1)\Big(e_1 + \eta_1 - \frac{2\pi}{b}\Big),$$
$$4\pi^2 \det D^2 G\Big(\frac{\omega_2}{2}\Big) = \Big(\eta_1 - \frac{1}{2}e_1\Big)\Big(\frac{2\pi}{b} + \frac{1}{2}e_1 - \eta_1\Big) - |\operatorname{Im} e_2|^2.$$

Although e_i's and η_i's are classical objects, we were unable to find an appropriate reference where these inequalities were studied. Part of (2), namely $\frac{1}{2}e_1 - \eta_1 < 0$, can be proved within the Weierstrass theory (cf. (6.22)). The whole theorem, however, requires theta functions in an essential way. Theta functions are recalled in Section 7 and the theorem is proved in Theorems 8.1 and 9.1. The proofs make use of the modularity of special values of theta functions (Jacobi's imaginary transformation formula) as well as the Jacobi triple product formula. Notice that the geometric meaning of these two inequalities has not yet been fully explored. For example, the variation on signs from ϑ_2 to ϑ_3 is still mysterious to us.

2. **Green functions and periods integrals**

We start with some basic properties of the Green functions that will be used in the proof of Theorem 1.1. Detailed behavior of the Green functions and their critical points will be studied in later sections.

Let $T = \mathbb{C}/\mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$ be a flat torus. As usual we let $\omega_3 = \omega_1 + \omega_2$. The Green function $G(z, w)$ is the unique function on T which satisfies

$$-\triangle_z G(z, w) = \delta_w(z) - \frac{1}{|T|} \tag{2.1}$$

and $\int_T G(z, w)\, dA = 0$. It has the property that $G(z, w) = G(w, z)$ and it is smooth in (z, w) except along the diagonal $z = w$, where

$$G(z, w) = -\frac{1}{2\pi} \log|z - w| + O(|z - w|^2) + C \tag{2.2}$$

for a constant C which is independent of z and w. Moreover, due to the translation invariance of T we have $G(z, w) = G(z - w, 0)$. Hence it is also customary to call $G(z) := G(z, 0)$ the Green function. It is an even function with the only singularity at 0.

There are explicit formulae for $G(z, w)$ in terms of elliptic functions, either in terms of the Weierstrass $\wp$ function or the Jacobi-Riemann theta functions ϑ_j. Both are developed in this paper since they have different advantages. We adopt the first approach in this section.

LEMMA 2.1. *There exists a constant $C(\tau)$, $\tau = \omega_2/\omega_1$, such that*

$$8\pi G(z) = \frac{2}{|T|} \int_T \log|\wp(\xi) - \wp(z)|\, dA + C(\tau). \tag{2.3}$$

It is straightforward to verify that the function of z, defined in the right-hand side of (2.3), satisfies the equation for the Green function. By comparing with the behavior near 0 we obtain Lemma 2.1. Since the proof is elementary, also an equivalent form in theta functions will be proved in Lemma 7.1, we skip the details here.

In view of Lemma 2.1, in order to analyze critical points of $G(z)$, it is natural to consider the following periods integral

$$F(z) = \int_L \frac{\wp'(z)}{\wp(\xi) - \wp(z)}\, d\xi, \tag{2.4}$$

where L is a line segment in T which is parallel to the ω_1-axis.

Fix a fundamental domain $T^0 = \{\, s\omega_1 + t\omega_2 \mid -\frac{1}{2} \le s, t \le \frac{1}{2}\,\}$ and set $L^* = -L$. Then $F(z)$ is an analytic function, except at 0, in each region of T^0 divided by $L \cup L^*$. Clearly, $\wp'(z)/(\wp(\xi) - \wp(z))$ has residue ± 1 at $\xi = \pm z$. Thus for any fixed z, $F(z)$ may change its value by $\pm 2\pi i$ if the integration lines cross z. Let $T^0 \backslash (L \cup L^*) = T_1 \cup T_2 \cup T_3$, where T_1 is the region above $L \cup L^*$, T_2 is the region bounded by L and L^* and T_3 is the region below $L \cup L^*$. Recall that $\zeta'(z) = -\wp(z)$ and $\eta_i = \zeta(z + \omega_i) - \zeta(z)$ for $i \in \{1, 2, 3\}$. Then we have

LEMMA 2.2. *Let $C_1 = 2\pi i$, $C_2 = 0$ and $C_3 = -2\pi i$. Then for $z \in T_k$,*

$$F(z) = 2\omega_1 \zeta(z) - 2\eta_1 z + C_k. \tag{2.5}$$

Proof. For $z \in T_1 \cup T_2 \cup T_3$, we have

$$F'(z) = \int_L \frac{d}{dz}\left(\frac{\wp'(z)}{\wp(\xi) - \wp(z)}\right) d\xi.$$

Clearly, z and $-z$ are the only (double) poles of $\frac{d}{dz}\left(\frac{\wp'(z)}{\wp(\xi)-\wp(z)}\right)$ as a meromorphic function of ξ and $\frac{d}{dz}\left(\frac{\wp'(z)}{\wp(\xi)-\wp(z)}\right)$ has zero residues at $\xi = z$ and $-z$. Thus the value of $F'(z)$ is independent of L and it is easy to see $F'(z)$ is a meromorphic function with the only singularity at 0.

By fixing L such that $0 \notin L \cup L^*$, a straightforward computation shows that

$$F(z) = \frac{2\omega_1}{z} - 2\eta_1 z + O(z^2)$$

in a neighborhood of 0. Therefore

$$F'(z) = -2\omega_1 \wp(z) - 2\eta_1.$$

By integrating F', we get

$$F(z) = 2\omega_1 \zeta(z) - 2\eta_1 z + C_k.$$

Since $F(\omega_2/2) = 0$, $F(\omega_1/2) = 0$ and $F(-\omega_2/2) = 0$, C_k is as claimed. Here we have used the Legendre relation $\eta_1\omega_2 - \eta_2\omega_1 = 2\pi i$. $\square$

From (2.3), we have

$$8\pi G_z = \frac{1}{|T|}\int_T \frac{-\wp'(z)}{\wp(\xi)-\wp(z)}\,dA. \tag{2.6}$$

LEMMA 2.3. *Let G be the Green function. Then for $z = t\omega_1 + s\omega_2$,*

$$G_z = -\frac{1}{4\pi}(\zeta(z) - \eta_1 t - \eta_2 s). \tag{2.7}$$

In particular, z is a critical point of G if and only if

$$\zeta(t\omega_1 + s\omega_2) = t\eta_1 + s\eta_2. \tag{2.8}$$

Proof. We shall prove (2.7) by applying Lemma 2.2. Since critical points appear in pair, without loss of generality we may assume that $z = t\omega_1 + s\omega_2$ with $s \ge 0$. We first integrate (2.6) along the ω_1 direction and obtain

$$\begin{aligned} f(s_0) :&= \int_{L_1(s_0)} \frac{-\wp'(z)}{\wp(\xi)-\wp(z)}\,d\xi \\ &= \begin{cases} -2\omega_1\zeta(z) + 2\eta_1 z & \text{if } \; s_0 > s, \\ -2\omega_1\zeta(z) + 2\eta_1 z - 2\pi i & \text{if } \; -s < s_0 < s, \\ -2\omega_1\zeta(z) + 2\eta_1 z & \text{if } \; s_0 < -s, \end{cases} \end{aligned}$$

where $L_1(s_0) = \{\, t_0\omega_1 + s_0\omega_2 \mid |t_0| \le \frac{1}{2} \,\}$. Thus,

$$\begin{aligned} 8\pi G_z &= \int_{-\frac{1}{2}}^{\frac{1}{2}} \int_{L_1(s_0)} \frac{-\wp'(z)}{\wp(\xi)-\wp(z)}\,dt_0\,ds_0 = \omega_1^{-1}\int_{-\frac{1}{2}}^{\frac{1}{2}} f(s_0)\,ds_0 \\ &= \omega_1^{-1}((-2\omega_1\zeta(z) + 2\eta_1 z)(1-2s) + (-2\omega_1\zeta(z) + 2\eta_1 z - 2\pi i)2s) \\ &= \omega_1^{-1}(-2\omega_1\zeta(z) + 2\eta_1 z - 4\pi s i) \\ &= \omega_1^{-1}(-2\omega_1\zeta(z) + 2\eta_1 t\omega_1 + 2s(\eta_1\omega_2 - 2\pi i)) \\ &= -2\zeta(z) + 2\eta_1 t + 2s\eta_2, \end{aligned}$$

where the Legendre relation is used again. □

COROLLARY 2.4. *Let $G(z)$ be the Green function. Then $\frac{1}{2}\omega_k$, $k \in \{1,2,3\}$ are critical points of $G(z)$. Furthermore, if z is a critical point of G then both periods integrals*

$$\begin{aligned} F_1(z) :&= 2(\omega_1\zeta(z) - \eta_1 z) \quad \textit{and} \\ F_2(z) :&= 2(\omega_2\zeta(z) - \eta_2 z) \end{aligned} \tag{2.9}$$

are purely imaginary numbers.

Proof. The half-periods $\frac{1}{2}\omega_1$, $\frac{1}{2}\omega_2$ and $\frac{1}{2}\omega_3$ are obvious solutions of (2.8). Alternatively, the half-periods are critical points of any even functions. Indeed for $G(z) = G(-z)$, we get $\nabla G(z) = -\nabla G(-z)$. Let $p = \frac{1}{2}\omega_i$ for some $i \in \{1,2,3\}$, then $p = -p$ in T and so $\nabla G(p) = -\nabla G(p) = 0$.

If $z = t\omega_1 + s\omega_2$ is a critical point, then by Lemma 2.3,

$$\omega_1\zeta(z) - \eta_1 z = \omega_1(t\eta_1 + s\eta_2) - \eta_1(t\omega_1 + s\omega_2) = s(\omega_1\eta_2 - \omega_2\eta_1) = -2s\pi i.$$

The proof for F_2 is similar. □

Example 2.5. When $\tau = \omega_2/\omega_1 \in i\,\mathbb{R}$, by symmetry considerations it is known (cf. [5, Lemma 2.1]) that the half-periods are all the critical points.

Example 2.6. There are tori such that equation (2.8) has more than three solutions. One such example is the torus with $\omega_1 = 1$ and $\omega_2 = \frac{1}{2}(1 + \sqrt{3}i)$. In this case, the multiplication map $z \mapsto \omega_2 z$ is simply the counterclockwise rotation by angle $\pi/3$, which preserves the lattice $\mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$; hence $\wp$ satisfies

$$\wp(\omega_2 z) = \wp(z)/\omega_2^2. \tag{2.10}$$

Similarly in $\wp'^2 = 4\wp^3 - g_2\wp - g_3$,

$$g_2 = 60\sum{}' \frac{1}{\omega^4} = 60\sum{}' \frac{1}{(\omega_2\omega)^4} = \omega_2^2 g_2,$$

which implies that $g_2 = 0$ and

$$\wp'' = 6\wp^2. \tag{2.11}$$

Let z_0 be a zero of $\wp(z)$. Then $\wp''(z_0) = 0$ too. By (2.10), $\wp(\omega_2 z_0) = 0$; hence either $\omega_2 z_0 = z_0$ or $\omega_2 z_0 = -z_0$ on T since $\wp(z) = 0$ has zeros at z_0 and $-z_0$ only. From here, it is easy to check that either z_0 is one of the half-periods or $z_0 = \pm\frac{1}{3}\omega_3$. But z_0 cannot be a half-period because $\wp''(z_0) \neq 0$ at any half-period. Therefore, we conclude that $z_0 = \pm\frac{1}{3}\omega_3$ and $\wp''(\pm\frac{1}{3}\omega_3) = \wp(\pm\frac{1}{3}\omega_3) = 0$.

We claim that $\frac{1}{3}\omega_3$ is a critical point. Indeed from the addition formula

$$\zeta(2z) = 2\zeta(z) + \frac{1}{2}\frac{\wp''(z)}{\wp'(z)} \tag{2.12}$$

we have

$$\zeta\Big(\frac{2\omega_3}{3}\Big) = 2\zeta\Big(\frac{\omega_3}{3}\Big). \tag{2.13}$$

On the other hand,

$$\zeta\Big(\frac{2\omega_3}{3}\Big) = \zeta\Big(\frac{-\omega_3}{3}\Big) + (\eta_1 + \eta_2) = -\zeta\Big(\frac{\omega_3}{3}\Big) + \eta_1 + \eta_2.$$

Together with (2.13) we get

$$\zeta\Big(\frac{\omega_3}{3}\Big) = \frac{1}{3}(\eta_1 + \eta_2).$$

That is, $\frac{1}{3}\omega_3$ satisfies the critical point equation.

Thus $G(z)$ has at least five critical points at $\frac{1}{2}\omega_k$, $k = 1, 2, 3$ and $\pm\frac{1}{3}\omega_3$ when $\tau = \omega_2/\omega_1 = \frac{1}{2}(1 + \sqrt{3}i)$.

By way of Theorem 1.3, these are precisely the five critical points, though we do not know how to prove this directly.

To conclude this section, let u be a solution of (1.2) with $\rho = 8\pi$ and set

$$v(z) = u(z) + 8\pi G(z). \tag{2.14}$$

Then $v(z)$ satisfies

$$\triangle v(z) + 8\pi\Big(e^{-8\pi G(z)}e^{v(z)} - \frac{1}{|T|}\Big) = 0 \tag{2.15}$$

in T. By (2.2), it is obvious that $v(z)$ is a smooth solution of (2.15). An important fact which we need is the following: Assume that there is a blow-up sequence of solutions $v_j(z)$ of (2.15). That is,

$$v_j(p_j) = \max_T v_j \to +\infty \quad \text{as} \quad j \to \infty.$$

Then the limit $p = \lim_{j\to\infty} p_j$ is the only blow-up point of $\{v_j\}$ and p is in fact a critical point of $G(z)$:

$$\nabla G(p) = 0. \tag{2.16}$$

We refer the reader to [3, p. 739, Estimate B] for a proof of (2.16).

3. **The criterion for existence via monodromies**

Consider the mean field equation

$$\triangle u + \rho e^u = \rho\delta_0, \quad \rho \in \mathbb{R}^+ \tag{3.1}$$

in a flat torus T, where δ_0 is the Dirac measure with singularity at 0 and the volume of T is normalized to be 1. A well known theorem due to Liouville says that any solution u of $\triangle u + \rho e^u = 0$ in a simply connected domain $\Omega \subset \mathbb{C}$ must be of the form

$$u = c_1 + \log\frac{|f'|^2}{(1+|f|^2)^2}, \tag{3.2}$$

where f is holomorphic in Ω. Conventionally f is called a developing map of u. Given a torus $T = \mathbb{C}/\mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$, by gluing the f's among simply connected domains it was shown in [5] that for $\rho = 4\pi l$, $l \in \mathbb{N}$, (3.2) holds on the whole $\mathbb{C}$ with f a meromorphic function. (The statement there is for rectangular tori with $l = 2$, but the proof works for the general case.)

It is straightforward to show that u and f satisfy

$$u_{zz} - \frac{1}{2}u_z^2 = \frac{f'''}{f'} - \frac{3}{2}\left(\frac{f''}{f'}\right)^2. \tag{3.3}$$

The right-hand side of (3.3) is the Schwartz derivative of f. Thus for any two developing maps f and $\tilde{f}$ of u, there exists $S = \left(\begin{smallmatrix} p & -\bar{q} \\ q & \bar{p}\end{smallmatrix}\right) \in \mathrm{PSU}(1)$ (i.e. $p, q \in \mathbb{C}$ and $|p|^2 + |q|^2 = 1$) such that

$$\tilde{f} = Sf := \frac{pf - \bar{q}}{qf + \bar{p}}. \tag{3.4}$$

Now we look for the constraints. The first type of constraint is imposed by the double periodicity of the equation. By applying (3.4) to $f(z+\omega_1)$ and $f(z+\omega_2)$, we find S_1 and S_2 in PSU(1) with

$$
\begin{aligned}
f(z+\omega_1) &= S_1 f,\\
f(z+\omega_2) &= S_2 f.
\end{aligned} \tag{3.5}
$$

These relations also force that $S_1S_2 = S_2S_1$ (up to a sign, as $A \equiv -A$ in PSU(1)).

The second type of constraint is imposed by the Dirac singularity of (3.1) at 0. A straightforward local computation with (3.2) shows that

LEMMA 3.1. (1) *If $f(z)$ has a pole at $z_0 \equiv 0 \pmod{\omega_1, \omega_2}$, then the order $k = l + 1$.*

(2) *If $f(z) = a_0 + a_k z^k + \cdots$ is regular at $z_0 \equiv 0 \pmod{\omega_1, \omega_2}$ with $a_k \neq 0$ then $k = l + 1$.*

(3) *If $f(z)$ has a pole at $z_0 \not\equiv 0 \pmod{\omega_1, \omega_2}$, then the order is* 1.

(4) *If $f(z) = a_0 + a_k(z - z_0)^k + \cdots$ is regular at $z_0 \not\equiv 0 \pmod{\omega_1, \omega_2}$ with $a_k \neq 0$ then $k = 1$.*

Now we are in a position to prove Theorem 1.1, namely the case $l = 2$.

Proof. We first prove the "only if" part. Let u be a solution and f be a developing map of u. By the above discussion, we may assume, after conjugating a matrix in PSU(1), that $S_1 = \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}$ for some $\theta \in \mathbb{R}$. Let $S_2 = \begin{pmatrix} p & -\bar{q} \\ q & \bar{p} \end{pmatrix}$ and then (3.5) becomes

$$
\begin{aligned}
f(z+\omega_1) &= e^{2i\theta} f(z),\\
f(z+\omega_2) &= S_2 f(z).
\end{aligned} \tag{3.6}
$$

Since $S_1S_2 = S_2S_1$, a direct computation shows that there are three possibilities:

(1) $p = 0$ and $e^{i\theta} = \pm i$;

(2) $q = 0$;

(3) $e^{i\theta} = \pm 1$.

Case 1. By assumption we have

$$
f(z+\omega_1) = -f(z), \qquad f(z+\omega_2) = -\frac{(\bar{q})^2}{f(z)}. \tag{3.7}
$$

For any $l \in \mathbb{N}$, the logarithmic derivative

$$
g = (\log f)' = \frac{f'}{f}
$$

is a nonconstant elliptic function on $\widetilde{T} = \mathbb{C}/\mathbb{Z}\omega_1 + \mathbb{Z}2\omega_2$ which has a simple pole at each zero or pole of f. By Lemma 3.1, if f or $1/f$ is singular at $z = 0$ then g

has no zero, which is not possible. So f must be regular at $z = 0$ with $f(0) \neq 0$. Moreover g has two zeros of order l at 0 and ω_2.

Let $\wp(z)$, $\zeta(z)$ and $\sigma(z) = \exp \int^z \zeta(w)\,dw = z + \cdots$ be the Weierstrass elliptic functions on $\widetilde{T}$. Recall that σ is odd with a simple zero at each lattice point. Moreover, for $\widetilde{\omega}_1 = \omega_1$, $\widetilde{\omega}_2 = 2\omega_2$ and $\widetilde{\omega}_3 = \omega_1 + 2\omega_2$,

$$\sigma(z \pm \widetilde{\omega}_i) = -e^{\mp\eta_i(z \pm \frac{1}{2}\widetilde{\omega}_i)}\sigma(z). \tag{3.8}$$

Now $l = 2$. By the standard representation of elliptic functions,

$$g(z) = A\frac{\sigma^2(z)\sigma^2(z-\omega_2)}{\sigma(z-a)\sigma(z-b)\sigma(z-c)\sigma(z-d)} \tag{3.9}$$

with four distinct simple poles such that $a + b + c + d = 2\omega_2$. We will show that such a function $g(z)$ does not exist.

By (3.7), we have $g(z+\omega_2) = -g(z)$. Hence we may assume that $c \equiv a+\omega_2$ and $d \equiv b + \omega_2$ modulo $\omega_1, 2\omega_2$. Thus $a + b \equiv 0$ modulo $\frac{1}{2}\omega_1, \omega_2$ and we arrive at two inequivalent cases. (i) $(a, b, c, d) = (a, -a, a + \omega_2, -a + \omega_2)$. (ii) $(a, b, c, d) = (a, -a + \frac{1}{2}\omega_1, a + \omega_2, -a - \frac{1}{2}\omega_1 + \omega_2)$. Using (3.8), it is easily checked that (i) leads to $g(z + \omega_2) = g(z)$ and (ii) leads to $g(z + \omega_2) = -g(z)$. Hence we are left with (ii).

The residues of g at a, b, c and d are equal to $-Ar$, Ar', Ar and $-Ar'$ respectively, where

$$r = \frac{\sigma^2(a+\omega_2)\sigma^2(a)}{\sigma(\omega_2)\sigma(2a - \frac{1}{2}\omega_1 + \omega_2)\sigma(2a + \frac{1}{2}\omega_1)}$$

and

$$r' = \frac{\sigma^2(a - \frac{1}{2}\omega_1)\sigma^2(a - \frac{1}{2}\omega_1 + \omega_2)}{\sigma(2a - \frac{1}{2}\omega_1)\sigma(2a - \frac{1}{2}\omega_1 + \omega_2)\sigma(\omega_1 - \omega_2)}.$$

We claim that $Ar = \pm 1$ and $Ar' = \pm 1$: Since

$$f(z) = \exp \int^z g(w)\,dw$$

is well-defined, by the residue theorem, we must have $Ar = m$ for some $m \in \mathbb{Z}$. Moreover one of a, b is a pole of order $|m|$ of f and then by Lemma 3.1 we conclude that $m = \pm 1$. Similarly $Ar' = \pm 1$.

In particular we must have $r/r' = \pm 1$. Using (3.8), this is equivalent to

$$\frac{\sigma^2(a+\omega_2)\sigma^2(a)}{\sigma^2(a + \frac{1}{2}\omega_1)\sigma^2(a - \frac{1}{2}\omega_1 + \omega_2)} = \mp e^{\eta_1(-\frac{1}{2}\omega_1 + \omega_2)}. \tag{3.10}$$

To solve a from this equation, we first recall that

$$\wp(z) - \wp(y) = -\frac{\sigma(z+y)\sigma(z-y)}{\sigma^2(z)\sigma^2(y)}.$$

By substituting $y = \frac{1}{2}\widetilde{\omega}_i$ into it and using (3.8), we get

$$\wp(z) - e_i = \frac{\sigma^2(z + \frac{1}{2}\widetilde{\omega}_i)}{\sigma^2(z)\sigma^2(\frac{1}{2}\widetilde{\omega}_i)}\, e^{-\eta_i z}. \tag{3.11}$$

With (3.11), the "+" case in equation (3.10) simplifies to

$$\wp\Big(a - \frac{\omega_1}{2} + \omega_2\Big) - e_1 = \wp(a) - e_1.$$

That is, $2a \equiv \frac{1}{2}\omega_1 - \omega_2$. But this implies that $b \equiv c$, a contradiction.

Similarly, the "–" case in (3.10) simplifies to (using the Legendre relation)

$$\wp\Big(a + \frac{\omega_1}{2}\Big) - e_3 = \wp(a) - e_3.$$

That is, $2a + \frac{1}{2}\omega_1 \equiv 0$. This leads to $c \equiv d$, which is again a contradiction.

Case 2. In this case we have

$$\begin{aligned} f(z+\omega_1) &= e^{2i\theta_1} f(z), \\ f(z+\omega_2) &= e^{2i\theta_2} f(z). \end{aligned} \tag{3.12}$$

The logarithmic derivative $g = (\log f)' = f'/f$ is now elliptic on T which has a simple pole at each zero or pole of f. As in Case 1, it suffices to investigate the situation when f is regular at 0 and $f(0) \neq 0$. Since g has 0 as its only zero (of order 2), we have

$$g(z) = A\frac{\sigma^2(z)}{\sigma(z-z_0)\sigma(z+z_0)} \tag{3.13}$$

where $\sigma(z)$ is the Weierstrass sigma function on T. Without loss of generality we may assume that f has a zero at z_0 and a pole at $-z_0$. In particular $z_0 \neq -z_0$ in T and we conclude that $z_0 \neq \omega_k/2$ for any $k \in \{1, 2, 3\}$.

Notice that if a meromorphic function f satisfies (3.12), then $e^{\lambda} f$ also satisfies (3.12) for any $\lambda \in \mathbb{R}$. Thus

$$u_\lambda(z) = c_1 + \log\frac{e^{2\lambda}|f'(z)|^2}{(1 + e^{2\lambda}|f(z)|^2)^2} \tag{3.14}$$

is a scaling family of solutions of (3.1).

Clearly $u_\lambda(z) \to -\infty$ as $\lambda \to +\infty$ for any z such that $f(z) \neq 0$ and $u_\lambda(z_0) \to +\infty$ as $\lambda \to +\infty$. Hence z_0 is the blow-up point and we have by (2.16) that

$$\nabla G(z_0) = 0.$$

Namely, it is a critical point other than the half-periods.

Case 3. In this case we get that S_1 is the identity. So by another conjugation in PSU(1) we may assume that S_2 is in diagonal form. But this case is then reduced to Case 2. The proof of the "only if" part of Theorem 1.1 is completed.

Now we prove the "if" part. Suppose that there is a critical point z_0 of $G(z)$ with $z_0 \neq \frac{1}{2}\omega_k$ for any $k \in \{1, 2, 3\}$.

For any closed curve C such that z_0 and $-z_0 \notin C$, the residue theorem implies that

$$\int_C \frac{\wp'(z_0)}{\wp(\xi) - \wp(z_0)}\, d\xi = 2\pi m i \tag{3.15}$$

for some $m \in \mathbb{Z}$. Hence

$$f(z) := \exp\left(\int_0^z \frac{\wp'(z_0)}{\wp(\xi) - \wp(z_0)}\, d\xi\right) \tag{3.16}$$

is well-defined as a meromorphic function. Notice that f is nonconstant since $\wp'(z_0) \neq 0$.

Let L_1 and L_2 be lines in T which are parallel to the ω_1-axis and ω_2-axis respectively and with $\pm z_0 \notin L_1, L_2$. Then for $j = 1, 2$,

$$f(z + \omega_j) = f(z) \exp\left(\int_{L_j} \frac{\wp'(z_0)}{\wp(\xi) - \wp(z_0)}\, d\xi\right). \tag{3.17}$$

By Lemma 2.2,

$$\int_{L_j} \frac{\wp'(z_0)}{\wp(\xi) - \wp(z_0)}\, d\xi = F_j(z_0) + C_k$$

for some $C_k \in \{2\pi i, 0, -2\pi i\}$. Also, by Corollary 2.4,

$$F_j(z_0) = 2(\omega_j \zeta(z_0) - \eta_j z_0) = 2i\theta_j \in i\,\mathbb{R}.$$

Hence for $j = 1, 2$,

$$f(z + \omega_j) = f(z) e^{2i\theta_j} \tag{3.18}$$

holds. Set

$$u_\lambda(x) = c_1 + \log \frac{e^{2\lambda}|f'(z)|^2}{(1 + e^{2\lambda}|f(z)|^2)^2}.$$

Then $u_\lambda(x)$ satisfies (3.1) for any $\lambda \in \mathbb{R}$ and u_λ is doubly periodic by (3.18). Therefore, solutions have been constructed and the proof of Theorem 1.1 is completed. □

A similar argument leads to

THEOREM 3.2. *For $\rho = 4\pi$, there exists a unique solution of* (3.1).

Proof. By the same procedure of the previous proof, there are three cases to be discussed. For Case 2, the subcases that f or $1/f$ is singular at $z = 0$ leads to contradiction as before. For the subcase that f is regular at $z = 0$ and $f(0) \neq 0$ we see that $f(z)/f'(z)$ is an elliptic function with 0 as its only simple pole (since now $k - 1 = l = 1$). Hence Case 2 does not occur. Similarly Case 3 is not possible.

Now we consider Case 1. By (3.7), the function $g = (\log f)' = f'/f$ is elliptic on $\widetilde{T} = \mathbb{C}/\mathbb{Z}\omega_1 + \mathbb{Z}2\omega_2$ and g has a simple pole at each zero or pole of f.

By Lemma 3.1, if f or $1/f$ is singular at $z = 0$ then g has no zero and we get a contradiction. So f is regular at $z = 0$, $f(0) \neq 0$ and g has two simple zeros at 0 and ω_2.

Let σ be the Weierstrass sigma function on $\widetilde{T}$. Then

$$g(z) = A\frac{\sigma(z)\sigma(z-\omega_2)}{\sigma(z-a)\sigma(z-b)} \tag{3.19}$$

for some a, b with $a + b = \omega_2$.

From (3.7), we have $g(z + \omega_2) = -g(z)$. So $a + \omega_2 = b \pmod{\omega_1, 2\omega_2}$. Since the representation of g in terms of sigma functions is unique up to the lattice $\{\omega_1, 2\omega_2\}$, there is a unique solution of (a, b):

$$a = -\frac{\omega_1}{2}, \qquad b = \frac{\omega_1}{2} + \omega_2. \tag{3.20}$$

Notice that the residues of g at a and b are equal to $-Ar$ and Ar respectively, where

$$r = \frac{\sigma(\frac{1}{2}\omega_1)\sigma(\frac{1}{2}\omega_1 + \omega_2)}{\sigma(\omega_1 + \omega_2)}.$$

We claim that $Ar = \pm 1$. Since

$$f(z) = f(0)\exp\int_0^z g(w)\,dw$$

is well-defined, by the residue theorem, we must have $Ar = m$ for some $m \in \mathbb{Z}$. Moreover one of a, b is a pole of order $|m|$ and then by Lemma 3.1 we conclude that $m = \pm 1$.

Conversely, by picking up a, b and $A = 1/r$ as above, $f(z)$ is a uniquely defined meromorphic function up to a factor $f(0)$. There is an unique choice of $f(0)$ up to a norm one factor such that $c := f(\omega_2)f(0)$ has $|c| = 1$. Thus by integrating $g(z + \omega_2) = -g(z)$ we get $f(z + \omega_2) = c/f(z)$.

By integrating $g(z + \omega_1) = g(z)$ we get $f(z + \omega_1) = c' f(z)$ where

$$c' = \frac{f(\omega_1)}{f(0)} = \exp\int_0^{\omega_1} g(z)\,dz.$$

To evaluate the period integral, notice that $g(\frac{1}{2}\omega_1 + u) = -g(\frac{1}{2}\omega_1 - u)$. By using the Cauchy principal value integral and the fact that the residue of g at $\frac{1}{2}\omega$ is ± 1, we get

$$\int_0^{\omega_1} g(z)\,dz = \pm\frac{1}{2} \times 2\pi i = \pm\pi i \tag{3.21}$$

and so $c' = -1$.

Thus f gives rise to a solution of (3.1) for $\rho = 4\pi$. The developing map for the other choice $A = -1/r$ is $1/f(z)$ which leads to the same solution. The proof is completed. $\square$

Since (3.1) is invariant under $z \mapsto -z$, the unique solution is necessarily even.

4. A uniqueness theorem for $\rho \in [4\pi, 8\pi]$ via symmetrizations

From the previous section, for $\rho = 8\pi$, solutions to the mean field equation exist in a one-parameter scaling family in λ with developing map f and centered at a critical point other than the half-periods. By choosing $\lambda = -\log|f(0)|$ we may assume that $f(0) = 1$. Since $g = (\log f)'$ is even by (3.13), we have $\log f(-z) = -\log f(z)$ and then

$$f(-z) = \frac{1}{f(z)}.$$

Consider the particular solution

$$u(z) = c_1 + \log \frac{|f'(z)|^2}{(1 + |f(z)|^2)^2}.$$

It is easy to verify that $u(-z) = u(z)$ and u is the unique even function in this family of solutions. In order to prove uniqueness up to scaling, it is equivalent to prove uniqueness within the class of even functions.

The idea is to consider the following equations

$$\begin{cases} \triangle u + \rho e^u = \rho\delta_0 \quad \text{and} \\ u(-z) = u(z) \quad \text{on } T \end{cases} \tag{4.1}$$

where $\rho \in [4\pi, 8\pi]$. We will use the *method of symmetrization* to prove

THEOREM 4.1. *For $\rho \in [4\pi, 8\pi]$, the linearized equation of* (4.1) *is nondegenerate. That is, the linearized equation has only trivial solutions.*

Together with the uniqueness of solution in the case $\rho = 4\pi$ (Theorem 3.2), we conclude the proof of Theorem 1.2 by the inverse function theorem.

We first prove Theorem 1.4, the Symmetrization Lemma. The proof will consist of several lemmas. The first step is an extension of the classical isoperimetric inequality of Bol for domains in $\mathbb{R}^2$ with metric $e^w|dx|^2$ to the case when the metric becomes singular.

Let $\Omega \subset \mathbb{R}^2$ be a domain and $w \in C^2(\Omega)$ satisfy

$$\begin{cases} \triangle w + e^w \geq 0 \quad \text{in } \Omega \text{ and} \\ \displaystyle\int_\Omega e^w \, dx \leq 8\pi. \end{cases} \tag{4.2}$$

This is equivalent to saying that the Gaussian curvature of $e^w|dx|^2$ is

$$K \equiv -\frac{1}{2}e^{-w}\triangle w \leq \frac{1}{2}.$$

For any domain $\omega \Subset \Omega$, we set

$$m(\omega) = \int_\omega e^w \, dx \quad \text{and} \quad \ell(\partial\omega) = \int_{\partial\omega} e^{w/2} \, ds. \tag{4.3}$$

Bol's isoperimetric inequality says that if Ω is simply connected then

$$2\ell^2(\partial\omega) \geq m(\omega)(8\pi - m(\omega)). \tag{4.4}$$

We first extend it to the case when w acquires singularities:

$$\begin{cases} \triangle w + e^w = \sum_{j=1}^{N} 2\pi\alpha_j\delta_{p_j} & \text{in } \Omega \text{ and} \\ \displaystyle\int_\Omega e^w\,dx \leq 8\pi, \end{cases} \tag{4.5}$$

with $\alpha_j > 0$, $j = 1, 2, \ldots, N$.

LEMMA 4.2. *Let Ω be a simply connected domain and w be a solution of* (4.5), *or more generally a sub-solution with prescribed singularities*:

$$w(x) - \sum_j \alpha_j \log|x - p_j| \in C^2(\Omega).$$

Then for any domain $\omega \Subset \Omega$,

$$2\ell^2(\partial\omega) \geq m(\omega)(8\pi - m(\omega)). \tag{4.6}$$

Proof. Define v and w_ε by

$$w(x) = \sum_j \alpha_j \log|x - p_j| + v(x),$$
$$w_\varepsilon(x) = \sum_j \frac{\alpha_j}{2} \log(|x - p_j|^2 + \varepsilon^2) + v(x).$$

By straightforward computations, we have

$$\begin{aligned} &\triangle w_\varepsilon(x) + e^{w_\varepsilon(x)} \\ &= \sum_j \frac{2\alpha_j\varepsilon^2}{(|x - p_j|^2 + \varepsilon^2)^2} + e^v\Big(\prod_j (|x - p_j|^2 + \varepsilon^2)^{\alpha_j/2} - \prod_j |x - p_j|^{\alpha_j}\Big) \geq 0. \end{aligned}$$

Let ℓ_ε and m_ε be defined as in (4.3) with respect to the metric $e^{w_\varepsilon(x)}|dx|^2$. Then we have

$$2\ell_\varepsilon^2(\partial\omega) \geq m_\varepsilon(\omega)(8\pi - m_\varepsilon(\omega)).$$

By letting $\varepsilon \to 0$ we obtain (4.6). $\square$

Next we consider the case that some of the α_j's are negative. For our purpose, it suffices to consider the case with only one singularity p_1 with negative α_1 (and we only need the case that $\alpha_1 = -1$). In view of (the proof of) Lemma 4.2, the problem is reduced to the case with only one singularity p_1. In other words, let w satisfy

$$\triangle w + e^w = -2\pi\delta_{p_1} \quad \text{in } \Omega. \tag{4.7}$$

LEMMA 4.3. *Let w satisfy* (4.7) *with Ω simply connected. Suppose that*

$$\int_{\Omega} e^{w}\, dx \le 4\pi; \tag{4.8}$$

then

$$2\ell^2(\partial\omega) \ge m(\omega)(4\pi - m(\omega)). \tag{4.9}$$

Proof. We may assume that $p_1 = 0$. If $0 \notin \omega$ then

$$2\ell^2(\partial\omega) \ge m(\omega)(8\pi - m(\omega)) > m(\omega)(4\pi - m(\omega))$$

by Bol's inequality, trivially. If $0 \in \omega$, we consider the double cover $\widetilde{\Omega}$ of Ω branched at 0. Namely we set $\widetilde{\Omega} = f^{-1}(\Omega)$ where

$$x = f(z) = z^2 \quad \text{for } z \in \mathbb{C}.$$

The induced metric $e^{v}|dz|^2$ on $\widetilde{\Omega}$ satisfies

$$e^{v(z)}|dz|^2 = e^{w(x)}|dx|^2 = e^{w(z^2)}4|z|^2|dz|^2.$$

That is, the metric potential v is the regular part

$$v(z) := w(x) + \log|x| + \log 4 = w(z^2) + 2\log|z| + \log 4.$$

By construction, v satisfies

$$\triangle v + e^{v} = 0 \quad \text{in } \widetilde{\Omega}\backslash\{0\}.$$

Since v is bounded in a neighborhood of 0, by the regularity of elliptic equations, $v(z)$ is smooth at 0. Hence v satisfies

$$\triangle v + e^{v} = 0 \quad \text{in } \widetilde{\Omega}.$$

Let $\widetilde{\omega} = f^{-1}(\omega)$. Clearly $\partial\widetilde{\omega} \subseteq f^{-1}(\partial\omega)$. Also

$$l(\partial\widetilde{\omega}) \le 2l(\partial\omega) \quad \text{and} \quad m(\tilde{w}) = 2m(\omega),$$

where

$$l(\partial\widetilde{\omega}) = \int_{\partial\widetilde{\omega}} e^{v/2}\, ds \quad \text{and} \quad m(\widetilde{\omega}) = \int_{\widetilde{\omega}} e^{v}\, dx.$$

By Bol's inequality, we have

$$4\ell^2(\partial\omega) \ge \ell^2(\partial\widetilde{\omega}) \ge \frac{1}{2}m(\widetilde{\omega})(8\pi - m(\widetilde{\omega})) = m(\omega)(8\pi - 2m(\omega)).$$

Thus $2\ell^2(\partial\omega) \ge m(\omega)(4\pi - m(\omega))$. $\square$

LEMMA 4.4 (Symmetrization I). *Let $\Omega \subset \mathbb{R}^2$ be a simply connected domain with $0 \in \Omega$ and let v be a solution of*

$$\triangle v + e^{v} = -2\pi\delta_0$$

in Ω. If the first eigenvalue of $\triangle + e^v$ is zero on Ω then

$$\int_\Omega e^v\,dx \ge 2\pi. \tag{4.10}$$

Proof. Let ψ be the first eigenfunction of $\triangle + e^v$:

$$\begin{cases} \triangle\psi + e^v\psi = 0 \quad \text{in } \Omega \text{ and} \\ \psi = 0 \quad \text{on } \partial\Omega. \end{cases} \tag{4.11}$$

In $\mathbb{R}^2$, let U and φ be the radially symmetric functions

$$\begin{cases} U(x) = \log \dfrac{2}{(1+|x|)^2|x|} \quad \text{and} \\ \varphi(x) = \dfrac{1-|x|}{1+|x|}. \end{cases} \tag{4.12}$$

From $\triangle = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}$, it is easy to verify that U and φ satisfy

$$\begin{cases} \triangle U + e^U = -2\pi\delta_0 \quad \text{and} \\ \triangle\varphi + e^U\varphi = 0 \quad \text{in } \mathbb{R}^2. \end{cases} \tag{4.13}$$

For any $t > 0$, set $\Omega_t = \{x \in \Omega \mid \psi(x) > t\}$ and $r(t) > 0$ such that

$$\int_{B_{r(t)}} e^{U(x)}\,dx = \int_{\{\psi > t\}} e^{v(x)}\,dx, \tag{4.14}$$

where $B_{r(t)}$ is the ball with center 0 and radius $r(t)$. Clearly $r(t)$ is strictly decreasing in t for $t \in (0, \max\psi)$. In fact, $r(t)$ is Lipschitz in t. Denote by $\psi^*(r)$ the symmetrization of ψ with respect to the measure $e^{U(x)}\,dx$ and $e^{v(x)}\,dx$. That is,

$$\psi^*(r) = \sup\{t \mid r < r(t)\}.$$

Obviously $\psi^*(r)$ is decreasing in r and for $t \in (0, \max\psi)$, $\psi^*(r) = t$ if and only if $r(t) = r$. Thus by (4.14), we have a decreasing function

$$f(t) := \int_{\{\psi^* > t\}} e^{U(x)}\,dx = \int_{\{\psi > t\}} e^{v(x)}\,dx. \tag{4.15}$$

By Lemma 4.3, for any $t > 0$,

$$2\ell^2(\{\psi = t\}) \ge f(t)(4\pi - f(t)). \tag{4.16}$$

We will use inequality (4.16) in the following computation: For any $t > 0$, by the Co-Area formula,

$$-\frac{d}{dt}\int_{\Omega_t} |\nabla\psi|^2\,dx = \int_{\partial\Omega_t} |\nabla\psi|\,ds \quad \text{and}$$
$$-f'(t) = -\frac{d}{dt}\int_{\Omega_t} e^v\,dx = \int_{\partial\Omega_t} \frac{e^v}{|\nabla\psi|}\,ds$$

hold almost everywhere in t. Thus

$$
\begin{aligned}
(4.17)\qquad -\frac{d}{dt}\int_{\Omega_t}|\nabla\psi|^2\,dx &= \int_{\{\psi=t\}}|\nabla\psi|\,ds \\
&\geq \left(\int_{\{\psi=t\}} e^{v/2}\,ds\right)^2\left(\int_{\{\psi=t\}}\frac{e^v}{|\nabla\psi|}\,ds\right)^{-1} \\
&= -\ell^2(\{\psi=t\})f'(t)^{-1} \\
&\geq -\frac{1}{2}f(t)(4\pi-f(t))f'(t)^{-1}.
\end{aligned}
$$

It is known that $\psi^* \in H_0^1(B_{r(0)})$ and the same procedure for ψ^* leads to

$$
\begin{aligned}
(4.18)\quad -\frac{d}{dt}\int_{\{\psi^*>t\}}|\nabla\psi^*|^2\,dx &= \int_{\{\psi^*=t\}}|\nabla\psi^*|\,ds \\
&= \left(\int_{\{\psi^*=t\}} e^{U/2}\,ds\right)^2\left(\int_{\{\psi^*=t\}}\frac{e^U}{|\nabla\psi^*|}\,ds\right)^{-1} \\
&= -\ell^2(\{\psi^*=t\})f'(t)^{-1} \\
&= -\frac{1}{2}f(t)(4\pi-f(t))f'(t)^{-1},
\end{aligned}
$$

with all inequalities being equalities. Hence

$$
-\frac{d}{dt}\int_{\{\psi>t\}}|\nabla\psi|^2\,dx \geq -\frac{d}{dt}\int_{\{\psi^*>t\}}|\nabla\psi^*|^2 ds
$$

holds almost everywhere in t. By integrating along t, we get

$$
\int_\Omega |\nabla\psi|^2\,dx \geq \int_{B_{r(0)}}|\nabla\psi^*|^2\,dx.
$$

Since ψ and ψ^* have the same distribution (or by looking at $-\int f'(t)t^2\,dt$ directly), we have

$$
\int_{B_{r(0)}} e^{U(x)}\psi^{*2}\,dx = \int_\Omega e^{v(x)}\psi^2\,dx.
$$

Therefore

$$
0 = \int_\Omega |\nabla\psi|^2\,dx - \int_\Omega e^v\psi^2\,dx \geq \int_{B_{r(0)}}|\nabla\psi^*|^2\,dx - \int_{B_{r(0)}} e^U\psi^{*2}\,dx.
$$

This implies that the linearized equation $\triangle + e^{U(x)}$ has nonpositive first eigenvalue. By (4.12), this happens if and only if $r(0) \geq 1$. Thus

$$
\int_\Omega e^{v(x)}\,dx = \int_{B_{r(0)}} e^{U(x)}\,dx \geq 2\pi. \qquad \square
$$

Remark 4.5 (see [1]). By applying symmetrization to $\triangle v + e^v = 0$ in Ω with $\lambda_1(\triangle + e^v) = 0$, the corresponding radially symmetric functions are

$$U(x) = -2\log\left(1 + \frac{1}{8}|x|^2\right) \quad \text{and} \quad \varphi(x) = \frac{8 - |x|^2}{8 + |x|^2}.$$

The same computations leads to $\int_\Omega e^v\,dx \geq 4\pi$.

A closer look at the proof of Lemma 4.4 shows that it works for more general situations as long as the isoperimetric inequality holds:

LEMMA 4.6 (Symmetrization II). *Let $\Omega \subset \mathbb{R}^2$ be a simply connected domain and let v be a solution of*

$$\triangle v + e^v = \sum_{j=1}^{N} 2\pi\alpha_j\delta_{p_j}$$

in Ω. Suppose that the first eigenvalue of $\triangle + e^v$ is zero on Ω with φ the first eigenfunction. If the isoperimetric inequality with respect to $ds^2 = e^v|dx|^2$,

$$2\ell^2(\partial\omega) \geq m(\omega)(4\pi - m(\omega))$$

holds for all level domains $\omega = \{\varphi \geq t\}$ with $t \geq 0$, then

$$\int_\Omega e^v\,dx \geq 2\pi.$$

Notice that we do not need any further constraint on the sign of α_j.

For the last statement of Theorem 1.4, the limiting procedure of Lemma 4.2 implies that the isoperimetric inequality (Lemma 4.3) and symmetrization I (Lemma 4.4) both hold regardless on the presence of singularities with nonnegative α.

Indeed the proof of Lemma 4.3 can be adapted to the case

$$\triangle w + e^w = -2\pi\delta_{p_1} + \sum_{j=2}^{N} 2\pi\alpha_j\delta_{p_j}$$

with $\alpha_j > 0$ for $j = 2, \ldots, N$. On the double cover $\bar{\Omega} \to \Omega$ branched over $p_1 = 0$, the metric potential $v(z)$ again extends smoothly over $z = 0$ and satisfies

$$\triangle v + e^v = \sum_{j=2}^{N} 2\pi\alpha_j(\delta_{q_j} + \delta_{q'_j}),$$

where $q_j, q'_j \in \bar{\Omega}$ are points lying over p_j. The remaining argument works by using Lemma 4.2 and we still conclude $2\ell^2(\partial\omega) \geq m(\omega)(4\pi - m(\omega))$.

Thus the proof of Theorem 1.4 is complete.

Proof of Theorem 4.1. Let u be a solution of equation (4.1). It is clear that we must have $\int_T e^u = 1$. Suppose that $\varphi(x)$ is a nontrivial solution of the linearized equation at u:

$$\begin{cases} \triangle\varphi + \rho e^u\varphi = 0 & \text{and} \\ \varphi(z) = \varphi(-z) & \text{in } T. \end{cases} \tag{4.19}$$

We will derive from this a contradiction.

Since both u and φ are even functions, by using $x = \wp(z)$ as a two-fold covering map of T onto $S^2 = \mathbb{C} \cup \{\infty\}$, we may require that since $\wp$ is an isometry:

$$e^{u(z)}|dz|^2 = e^{v(x)}|dx|^2 = e^{v(x)}|\wp'(z)|^2|dz|^2.$$

Namely, we set

$$v(x) := u(z) - \log|\wp'(z)|^2 \quad \text{and} \quad \psi(x) := \varphi(z). \tag{4.20}$$

There are four branch points on $\mathbb{C} \cup \{\infty\}$, namely $p_0 = \wp(0) = \infty$ and $p_j = e_j := \wp(\omega_j/2)$ for $j = 1, 2, 3$. Since $\wp'(z)^2 = 4\prod_{j=1}^3 (x - e_j)$, by construction $v(x)$ and $\psi(x)$ then satisfy

$$\begin{cases} \triangle v + \rho e^v = \sum_{j=1}^3 (-2\pi)\delta_{p_j} \quad \text{and} \\ \triangle \psi + \rho e^v \psi = 0 \quad \text{in } \mathbb{R}^2. \end{cases} \tag{4.21}$$

To take care of the point at infinity, we use coordinate $y = 1/x$ or equivalently we consider $T \to S^2$ via $y = 1/\wp(z) \sim z^2$. The isometry condition reads as

$$e^{u(z)}|dz|^2 = e^{w(y)}|dy|^2 = e^{w(y)}\frac{|\wp'(z)|^2}{|\wp(z)|^4}|dz|^2.$$

Near $y = 0$ we get

$$w(y) = u(z) - \log\frac{|\wp'(z)|^2}{|\wp(z)|^4} \sim \Big(\frac{\rho}{4\pi} - 1\Big)\log|y|.$$

Thus $\rho \geq 4\pi$ implies that p_0 is a singularity with nonnegative α_0:

$$\triangle w + \rho e^w = \alpha_0\delta_0 + \sum_{j=1}^3 (-2\pi)\delta_{1/p_j}.$$

In dealing with equation (4.1) and the above resulting equations, by replacing u by $u + \log\rho$ etc., we may (and will) replace the ρ on the left-hand side by 1 for simplicity. The total measures on T and $\mathbb{R}^2$ are then given by

$$\int_T e^u\,dz = \rho \leq 8\pi \quad \text{and} \quad \int_{\mathbb{R}^2} e^v\,dx = \frac{\rho}{2} \leq 4\pi.$$

The nodal line of ψ decomposes S^2 into at least two connected components and at least two of them are simply connected. If there is a simply connected component Ω which contains no p_j's, then the symmetrization (Remark 4.5) leads to

$$\int_\Omega e^v\,dx \geq 4\pi,$$

which is a contradiction because $\mathbb{R}^2 \backslash \Omega \neq \varnothing$. If every simply connected component Ω_i, $i = 1, \ldots, m$, contains only one p_j, then Lemma 4.4 implies that

$$\int_{\Omega_i} e^v\,dx \geq 2\pi \quad \text{for } i = 1, \ldots, m.$$

The sum is at least $2m\pi$, which is again impossible unless $m=2$ and $\mathbb{R}^2 = \overline{\Omega}_1 \cup \overline{\Omega}_2$. So without lost of generality we are left with one of the following two situations:

$$\mathbb{R}^2 \cup \{\infty\} \backslash \{\psi = 0\} = \Omega_+ \cup \Omega_-$$

where

$$\Omega_+ \subset \{x \mid \psi(x) > 0\} \quad \text{and} \quad \Omega_- \supset \{x \mid \psi(x) < 0\}.$$

Both Ω_+ and Ω_- are simply connected.

(1) Either Ω_- contains p_1, p_2 and $p_3 \in \Omega_+$ or

(2) $p_1 \in \Omega_-$, $p_3 \in \Omega_+$ and $p_2 \in C = \partial\Omega_+ = \partial\Omega_-$.

Assume that we are in case (1). By Lemma 4.3, we have on Ω_+

$$2\ell^2(\{\psi = t\}) \geq m(\{\psi \geq t\})(4\pi - m(\{\psi \geq t\})) \tag{4.22}$$

for $t \geq 0$.

We will show that the similar inequality

$$2\ell^2(\{\psi = t\}) \geq m(\{\psi \leq t\})(4\pi - m(\{\psi \leq t\})) \tag{4.23}$$

holds on Ω_- for all $t \leq 0$.

Let $t \leq 0$ and ω be a component of $\{\psi \leq t\}$.

If ω contains at most one point of p_1 and p_2, then Lemma 4.3 implies that

$$2\ell^2(\partial\omega) \geq (4\pi - m(\omega))m(\omega).$$

If ω contains both p_1 and p_2, then $\mathbb{R}^2 \cup \{\infty\} \backslash \omega$ is simply connected which contains p_3 only. Thus by Lemma 4.3

$$\begin{aligned} 2\ell^2(\partial\omega) &\geq (4\pi - m(\mathbb{R}^2\backslash\omega))m(\mathbb{R}^2\backslash\omega) \\ &= (4\pi - \rho/2 + m(\omega))(\rho/2 - m(\omega)) \\ &= (4\pi - m(\omega))m(\omega) + (4\pi - \rho/2)(\rho/2 - 2m(\omega)). \end{aligned} \tag{4.24}$$

Since $\rho \leq 8\pi$ and $\int_{\Omega_+} e^v dx \geq 2\pi$, we get

$$\frac{\rho}{2} = \int_{\mathbb{R}^2} e^v \geq \int_{\Omega_+} e^v + \int_\omega e^v \geq 2\pi + m(\omega) \geq 2m(\omega).$$

Then again

$$2\ell^2(\partial\omega) \geq (4\pi - m(\omega))m(\omega)$$

with equality hold only when $\rho = 8\pi$ and $m(\omega) = 2\pi$.

Now it is a simple observation that domains which satisfy the isoperimetric inequality (4.9) have the addition property. Indeed, if $2a^2 \geq (4\pi - m)m$ and $2b^2 \geq (4\pi - n)n$, then

$$\begin{aligned} 2(a+b)^2 &= 2a^2 + 4ab + 2b^2 > (4\pi - m)m + (4\pi - n)n \\ &= 4\pi(m+n) - (m+n)^2 + 2mn > (4\pi - (m+n))(m+n). \end{aligned}$$

Hence (4.23) holds for all $t \leq 0$.

Now we can apply Lemma 4.6 to Ω_- to conclude that

$$\int_{\Omega_-} e^v\,dx = 2\pi$$

(which already leads to a contradiction if $\rho < 8\pi$) and the equality

$$2\ell^2(\{\psi = t\}) = (4\pi - m(\{\psi \le t\}))m(\{\psi \le t\})$$

in (4.23) holds for all $t \in (\min\psi, 0)$. This implies that $\{\psi \le t\}$ has only one component and it contains p_1 and p_2 for all $t \in (\min\psi, 0)$. But then ψ attains its minimum along a connected set $\psi^{-1}(\min\psi)$ containing p_1 and p_2, which is impossible.

In case (2), $\rho < 8\pi$ again leads to a contradiction via the same argument. For $\rho = 8\pi$, we have

$$\int_{\Omega_+} e^v\,dx = \int_{\Omega_-} e^v\,dx = 2\pi$$

and all inequalities in (4.17) are equalities. So under the notation there

$$\left(\int_{\partial\Omega_t} e^{v/2}\,dx\right)^2 = \int_{\partial\Omega_t} |\nabla\psi|\,ds \int_{\partial\Omega_t} \frac{e^v}{|\nabla\psi|}\,ds$$

for all $t \in (\min_{\Omega_-}\psi, \max_{\Omega_+}\psi)$. This implies that $|\nabla\psi|^2(x) = C(t)e^{v(x)}$ almost everywhere in $\psi^{-1}(t)$ for some constant C which depends only on t. By continuity we have

$$|\nabla\psi|^2(x) = C(\psi(x))e^{v(x)} \tag{4.25}$$

for all x except when $\psi(x) = \max_{\Omega_+}\psi$ or $\psi(x) = \min_{\Omega_-}\psi$.

By letting $x = p_2 \in \psi^{-1}(0)$, we find $C(0) = |\nabla\psi|^2(p_2)e^{-v(p_2)} = 0$. By (4.25) this implies that $|\nabla\psi(x)| = 0$ for all $x \in \psi^{-1}(0)$, which is clearly impossible. Hence the proof of Theorem 4.1 is completed. $\square$

Since equation (4.1) has a unique solution at $\rho = 4\pi$, by the continuation from $\rho = 4\pi$ to 8π and Theorem 4.1, we conclude that (4.1) has at most a solution at $\rho = 8\pi$, and this implies that the mean field equation (1.2) has at most one solution up to scaling. Thus, Theorem 1.2 is proved and then Theorem 1.3 follows immediately.

5. **Comparing critical values of Green functions**

For simplicity, from now on we normalize all tori to have $\omega_1 = 1$, $\omega_2 = \tau$. In Section 3 we showed that the existence of solutions of equation (1.2) is equivalent to the existence of nonhalf-period critical points of $G(z)$. The main goal of this and the next sections is to provide criteria for detecting minimum points of $G(z)$. The following theorem is useful in this regard.

THEOREM 5.1. *Let z_0 and z_1 be two half-periods. Then $G(z_0) \ge G(z_1)$ if and only if $|\wp(z_0)| \ge |\wp(z_1)|$.*

Proof. By integrating (2.7), the Green function $G(z)$ can be represented by

$$G(z) = -\frac{1}{2\pi}\mathrm{Re}\int(\zeta(z)-\eta_1 z)\,dz + \frac{1}{2b}y^2 + C(\tau). \tag{5.1}$$

Thus

$$G\Big(\frac{\omega_2}{2}\Big) - G\Big(\frac{\omega_3}{2}\Big) = \frac{1}{2\pi}\mathrm{Re}\int_{\frac{\omega_2}{2}}^{\frac{\omega_3}{2}}(\zeta(z)-\eta_1 z)\,dz. \tag{5.2}$$

Set $F(z) = \zeta(z) - \eta_1 z$. We have $F(z+\omega_1) = F(z)$ and

$$\begin{aligned}\zeta\Big(t+\frac{\omega_2}{2}\Big) - \eta_1\Big(t+\frac{\omega_2}{2}\Big) &= -\zeta\Big(\frac{-\omega_2}{2}-t\Big) - \eta_1\Big(t+\frac{\omega_2}{2}\Big)\\ &= -\zeta\Big(\frac{\omega_2}{2}-t\Big) + \eta_2 - \eta_1\Big(t+\frac{\omega_2}{2}\Big)\\ &= -\Big[\zeta\Big(\frac{\omega_2}{2}-t\Big) - \eta_1\Big(\frac{\omega_2}{2}-t\Big)\Big] + \eta_2 - \eta_1\omega_2\\ &= -\Big[\zeta\Big(\frac{\omega_2}{2}-t\Big) - \eta_1\Big(\frac{\omega_2}{2}-t\Big)\Big] - 2\pi i;\end{aligned}$$

hence $\mathrm{Re}\, F(\frac{1}{2}\omega_2 + t)$ is antisymmetric in $t \in \mathbb{C}$.

To calculate the integral in (5.2), we use the addition theorem to get

$$\begin{aligned}\frac{\wp'(z)}{\wp(z)-e_1} &= \zeta\Big(z-\frac{\omega_1}{2}\Big) + \zeta\Big(z+\frac{\omega_1}{2}\Big) - 2\zeta(z)\\ &= F\Big(z-\frac{\omega_1}{2}\Big) + F\Big(z+\frac{\omega_1}{2}\Big) - 2F(z).\end{aligned}$$

Integrating along the segment from $\frac{1}{2}\omega_2$ to $\frac{1}{2}\omega_3$, we get

$$\log\frac{e_3-e_1}{e_2-e_1} = \int_{L_1} F(z)\,dz + \int_{L_2} F(z)\,dz - 2\int_{L_3} F(z)\,dz,$$

where L_1 is the line from $\frac{1}{2}(\omega_2-\omega_1)$ to $\frac{1}{2}\omega_2$, L_2 is the line from $\frac{1}{2}\omega_3$ to $\frac{1}{2}\omega_2+\omega_1$ and L_3 is the line from $\frac{1}{2}\omega_2$ to $\frac{1}{2}\omega_3$. Since $F(z) = F(z+\omega_1)$ and $\mathrm{Re}\,F(z)$ is antisymmetric with respect to $\frac{1}{2}\omega_2$, we have

$$\log\frac{e_3-e_1}{e_2-e_1} = 2\int_L F(z)\,dz - 4\int_{L_3} F(z)\,dz = -2\pi i - 4\int_{L_3} F(z)\,dz,$$

where L is the line from $\frac{1}{2}(\omega_2-\omega_1)$ to $\frac{1}{2}\omega_3$. Thus we have

$$\log\left|\frac{e_3-e_1}{e_2-e_1}\right| = -4\,\mathrm{Re}\int_{\frac{\omega_2}{2}}^{\frac{\omega_3}{2}} F(z)\,dz = -8\pi\left(G\Big(\frac{\omega_2}{2}\Big) - G\Big(\frac{\omega_3}{2}\Big)\right).$$

That is,

$$G\Big(\frac{\omega_2}{2}\Big) - G\Big(\frac{\omega_3}{2}\Big) = \frac{1}{8\pi}\log\left|\frac{e_2-e_1}{e_3-e_1}\right|. \tag{5.3}$$

Similarly, by integrating (2.7) in the ω_2 direction, we get

$$G\Big(\frac{\omega_1}{2}\Big) - G\Big(\frac{\omega_3}{2}\Big) = \frac{1}{2\pi}\mathrm{Re}\int_{\frac{\omega_1}{2}}^{\frac{\omega_3}{2}} (\zeta(z) - \eta_2 z)\,dz. \tag{5.4}$$

The same proof then gives rise to

$$G\Big(\frac{\omega_1}{2}\Big) - G\Big(\frac{\omega_3}{2}\Big) = \frac{1}{8\pi}\log\left|\frac{e_1 - e_2}{e_3 - e_2}\right|. \tag{5.5}$$

By combining the above two formulae we get also that

$$G\Big(\frac{\omega_1}{2}\Big) - G\Big(\frac{\omega_2}{2}\Big) = \frac{1}{8\pi}\log\left|\frac{e_1 - e_3}{e_2 - e_3}\right|. \tag{5.6}$$

In order to compare, say, $G(\frac{1}{2}\omega_1)$ and $G(\frac{1}{2}\omega_3)$, we may use (5.5). Let

$$\lambda = \frac{e_3 - e_2}{e_1 - e_2}.$$

By using $e_1 + e_2 + e_3 = 0$, we get

$$\frac{e_3}{e_1} = \frac{2\lambda - 1}{2 - \lambda}. \tag{5.7}$$

It is easy to see that $|2\lambda - 1| \geq |2 - \lambda|$ if and only if $|\lambda| \geq 1$. Hence

$$\left|\frac{e_3}{e_1}\right| \geq 1 \quad \text{if and only if} \quad |\lambda| \geq 1.$$

The same argument applies to the other two cases too and the theorem follows. $\square$

It remains to make the criterion effective in τ. Recall the modular function

$$\lambda(\tau) = \frac{e_3 - e_2}{e_1 - e_2}.$$

By (5.3), we have

$$G\Big(\frac{\omega_3}{2}\Big) - G\Big(\frac{\omega_2}{2}\Big) = \frac{1}{4\pi}\log|\lambda(\tau) - 1|. \tag{5.8}$$

Therefore, it is important to know when $|\lambda(\tau) - 1| = 1$.

LEMMA 5.2. $|\lambda(\tau) - 1| = 1$ *if and only if* $\mathrm{Re}\,\tau = \frac{1}{2}$.

Proof. Let $\wp(z;\tau)$ be the Weierstrass $\wp$ function with periods 1 and τ; then

$$\overline{\wp(z;\tau)} = \wp(\bar{z};\bar{\tau}).$$

For $\tau = \frac{1}{2} + ib$, $\bar{\tau} = 1 - \tau$ and then $\wp(z;\bar{\tau}) = \wp(z;\tau)$. Thus

$$\wp(z;\tau) = \overline{\wp(\bar{z};\tau)} \quad \text{for} \quad \tau = \frac{1}{2} + ib. \tag{5.9}$$

Note that if $z = \frac{1}{2}\omega_2$ then $\bar{z} = \frac{1}{2}\bar{\omega}_2 = \frac{1}{2}(1 - \omega_2) = \frac{1}{2}\omega_3 \pmod{\omega_1, \omega_2}$. Therefore

$$\bar{e}_2 = e_3 \quad \text{and} \quad \bar{e}_1 = e_1.$$

Since $e_1 + e_2 + e_3 = 0$, we have

$$\operatorname{Re} e_2 = -\frac{1}{2} e_1 \quad \text{and} \quad \operatorname{Im} e_2 = -\operatorname{Im} e_3. \tag{5.10}$$

Thus

$$|\lambda(\tau) - 1| = \left| \frac{e_3 - e_1}{e_2 - e_1} \right| = 1. \tag{5.11}$$

A classic result says that $\lambda'(\tau) \neq 0$ for all τ. By this and (5.11), it follows that λ maps $\{ \tau \mid \operatorname{Re} \tau = \frac{1}{2} \}$ bijectively onto $\{ \lambda(\tau) \mid |\lambda(\tau) - 1| = 1 \}$. □

Let Ω be the fundamental domain for $\lambda(\tau)$, i.e.,

$$\Omega = \{ \tau \in \mathbb{C} \mid |\tau - 1/2| \geq 1/2,\ 0 \leq \operatorname{Re} \tau \leq 1,\ \operatorname{Im} \tau > 0 \},$$

and let Ω' be the reflection of Ω with respect to the imaginary axis.

Since $G(\frac{1}{2}\omega_3) < G(\frac{1}{2}\omega_2)$ for $\tau = ib$, we conclude that for $\tau \in \Omega' \cup \Omega$, $|\lambda(\tau) - 1| < 1$ if and only if $|\operatorname{Re} \tau| < \frac{1}{2}$. Therefore for $\tau \in \Omega' \cup \Omega$,

$$|\operatorname{Re} \tau| < \frac{1}{2} \quad \text{if and only if} \quad G\Big(\frac{\omega_3}{2}\Big) < G\Big(\frac{\omega_2}{2}\Big).$$

For $|\tau| = 1$, using suitable Möbius transformations we may obtain similar results. For example, from the definition of $\wp$, (5.9) implies that

$$\bar{\wp}(z) = \Big(\frac{\tau + 1}{\bar{\tau} + 1}\Big)^2 \wp\Big(\frac{\tau + 1}{\bar{\tau} + 1} \bar{z}\Big) \tag{5.12}$$

and so (compare (2.3))

$$G(z) = G\Big(\frac{\tau + 1}{\bar{\tau} + 1} \bar{z}\Big). \tag{5.13}$$

Clearly, for $z = \frac{1}{2}\tau$, (5.13) implies that $G(\frac{1}{2}\omega_2) = G(\frac{1}{2}\omega_1)$. So

$$|\tau| = 1 \quad \text{if and only if} \quad \left| \frac{e_2 - e_3}{e_1 - e_3} \right| = 1, \tag{5.14}$$

$$|\tau| < 1 \quad \text{if and only if} \quad G\Big(\frac{\omega_1}{2}\Big) < G\Big(\frac{\omega_2}{2}\Big). \tag{5.15}$$

Similarly,

$$|\tau - 1| < 1 \quad \text{if and only if} \quad G\Big(\frac{\omega_1}{2}\Big) < G\Big(\frac{\omega_3}{2}\Big). \tag{5.16}$$

6. Degeneracy analysis of critical points along $\operatorname{Re} \tau = \frac{1}{2}$

By (2.7), the derivatives of G can be computed by

$$2\pi G_x = \operatorname{Re}(\eta_1 t + \eta_2 s - \zeta(z)), \quad -2\pi G_y = \operatorname{Im}(\eta_1 t + \eta_2 s - \zeta(z)). \tag{6.1}$$

When $\tau = \frac{1}{2} + ib$, since $\wp(z)$ is real for $z \in \mathbb{R}$, η_1 is real and (6.1) becomes

$$2\pi G_x = \eta_1 t + \frac{1}{2}\eta_1 s - \operatorname{Re}\zeta(z), \quad 2\pi G_y = \operatorname{Im}\zeta(z) + (2\pi - \eta_1 b)s. \tag{6.2}$$

Thus the Hessian of G is given by

$$\begin{aligned} 2\pi G_{xx} &= \operatorname{Re}\wp(z) + \eta_1, \\ 2\pi G_{xy} &= -\operatorname{Im}\wp(z), \\ 2\pi G_{yy} &= -\Big(\operatorname{Re}\wp(z) + \eta_1 - \frac{2\pi}{b}\Big). \end{aligned} \tag{6.3}$$

We first consider the point $\frac{1}{2}\omega_1$. The degeneracy condition of G at $\frac{1}{2}\omega_1$ reads as

$$e_1 + \eta_1 = 0 \quad \text{or} \quad e_1 + \eta_1 - \frac{2\pi}{b} = 0.$$

We will use the following two inequalities (Theorem 1.7) whose proofs will be given in Sections 8 and 9 through theta functions:

$$e_1(b) + \eta_1(b) \quad \text{is increasing in } b \tag{6.4}$$

and

$$e_1(b) \quad \text{is increasing in } b. \tag{6.5}$$

LEMMA 6.1. *There exists $b_0 < \frac{1}{2} < b_1 < \sqrt{3}/2$ such that $\frac{1}{2}\omega_1$ is a degenerate critical point of $G(z;\tau)$ if and only if $b = b_0$ or $b = b_1$. Moreover, $\frac{1}{2}\omega_1$ is a local minimum point of $G(z;\tau)$ if $b \in (b_0, b_1)$ and is a saddle point of $G(z;\tau)$ if $b \in (0, b_0)$ or $b \in (b_1, \infty)$.*

Proof. Let b_0 and b_1 be the zero of $e_1 + \eta_1 = 0$ and $e_1 + \eta_1 - 2\pi/b = 0$ respectively. Then Lemma 6.1 follows from the explicit expression of the Hessian of G by (6.4). □

Numerically we know that $b_1 \approx 0.7 < \sqrt{3}/2$. Now we analyze the behavior of G near $\frac{1}{2}\omega_1$ for $b > b_1$.

LEMMA 6.2. *When $b > b_0$, then $\frac{1}{2}\omega_1$ is the only critical point of G along the x-axis.*

Proof. $\wp(t;\tau)$ is real if $t \in \mathbb{R}$. Since $\wp'(t;\tau) \neq 0$ for $t \neq \frac{1}{2}\omega_1$, $\wp'(t;\tau) < 0$ for $0 < t < \frac{1}{2}\omega_1$. Since $b > b_0$, by (6.3), (6.4) and Lemma 6.1,

$$2\pi G_{xx}(t) = \wp(t) + \eta_1 > e_1 + \eta_1 > 0,$$

which implies that $G_x(t) < G_x(\frac{1}{2}\omega_1) = 0$ if $0 < t < \frac{1}{2}\omega_1$. Hence G has no critical points on $(0, \frac{1}{2}\omega_1)$. Since $G(z) = G(-z)$, G cannot have any critical point on $(-\frac{1}{2}\omega_1, 0)$. □

By Lemma 6.1 and the conservation of local Morse indices, we know that $G(z;\tau)$ has two more critical points near $\frac{1}{2}\omega_1$ when b is close to b_1 and $b > b_1$. We denote these two extra points by $z_0(\tau)$ and $-z_0(\tau)$. In this case, $\frac{1}{2}\omega_1$ becomes a saddle point and $z_0(\tau)$ and $-z_0(\tau)$ are local minimum points. From Lemma 5.2, (5.15) and (5.16) we know that

$$G\Big(\frac{\omega_1}{2}\Big) < G\Big(\frac{\omega_2}{2}\Big) = G\Big(\frac{\omega_3}{2}\Big) \quad \text{if} \quad b_1 < b < \sqrt{3}/2.$$

Thus in this region $\pm z_0(\tau)$ must exist and they turn out to be the minimum point of G since there are at most five critical points. In fact we will see below that this is true for all $b > b_1$ and $\frac{1}{2}\omega_2$, $\frac{1}{2}\omega_3$ are all saddle points.

LEMMA 6.3. *The critical point $z_0(\tau)$ is on the line* $\operatorname{Re} z = \frac{1}{2}$. *Moreover, the Green function $G(z;\tau)$ is symmetric with respect to the line* $\operatorname{Re} z = \frac{1}{2}$.

Proof. Representing the torus T in question by the rhombus torus with sides τ and $\bar\tau = 1-\tau$, we see that the obvious symmetries $z \mapsto \bar z$, $z \mapsto 1-z$ of T and Theorem 1.2 show that $z_0(\tau)$ must be on the x-axis or the line $\operatorname{Re}\tau = \frac{1}{2}$. The former case is excluded by Lemma 6.2. □

Let $G_{\frac{1}{2}}$ be the restriction of G on the line $\operatorname{Re} z = \frac{1}{2}$; i.e., $G_{\frac{1}{2}}(y) = G(\frac{1}{2} + iy)$. Thus, Lemma 6.3 implies that any critical point of $G_{\frac{1}{2}}$ is automatically a critical point of G.

LEMMA 6.4. *For $b > b_1$, $G_{\frac{1}{2}}(y)$ has exactly one critical point in* $(0, b)$. *This point is necessarily a nondegenerate minimal point.*

Proof. Let $z = t\omega_1 + s\omega_2$. Then $\operatorname{Re} z = \frac{1}{2}$ is equivalent to $2t + s = 1$, which implies $\bar z = (t+s)\omega_1 - s\omega_2 = -z$. By (5.9)

$$\wp(z;\tau) = \overline{\wp(\bar z;\tau)} = \overline{\wp(-z;\tau)} = \overline{\wp(z;\tau)}.$$

Hence $\wp(z;\tau)$ is real for $\operatorname{Re} z = \frac{1}{2}$. By (6.3),

$$2\pi G_{yy} = -\Big(\wp + \eta_1 - \frac{2\pi}{b}\Big).$$

Since $\partial\wp/\partial y = i\wp'(z;\tau) \neq 0$ for $z \neq \frac{1}{2}\omega_1$ and $\operatorname{Re} z = \frac{1}{2}$, $G_{yy}(z)$ has at most one zero. Let $z_0(\tau)$ be the critical point above (which exists at least for $b > b_1$ and close to b_1). Then $G_y(z_0(\tau)) = G_y(\frac{1}{2}\omega_1) = 0$ and so $G_{yy}(\hat z_0) = 0$ for some $\hat z_0$ in the open line segment $(\frac{1}{2}\omega_1, z_0(\tau))$. Since $2\pi G_{yy}(\frac{1}{2}\omega_1) = -(e_1 + \eta_1 - 2\pi/b) < 0$, we have $G_{yy}(z_0(\tau)) > 0$. Hence $z_0(\tau)$ is a nondegenerate minimum point of $G_{\frac{1}{2}}$ as long as it exists with $b > b_1$.

By the stability of nondegenerate minimal points (here for one variable function), we conclude that $z_0(\tau)$ exists for all $b > b_1$. □

LEMMA 6.5. *If $\wp''(z_0(\tau);\tau) = 0$ then $\tau = (1+\sqrt{3}i)/2$.*

Proof. Let $z_0 = t_0\omega_1 + s_0\omega_2$. If $\wp''(z_0;\tau) = 0$, by the addition theorem (2.12)

$$\zeta(2z_0) = 2\zeta(z_0) = 2(t_0\eta_1 + s_0\eta_2),$$

so that $2z_0$ is also a critical point. Note that $\operatorname{Re} 2z_0 = 1$. Since $2z_0 - 1 + \omega_2$ is also a critical point with $\operatorname{Re}(2z_0 - 1 + \omega_2) = \frac{1}{2}$, we have either $2z_0 - 1 + \omega_2 = -z_0$ or

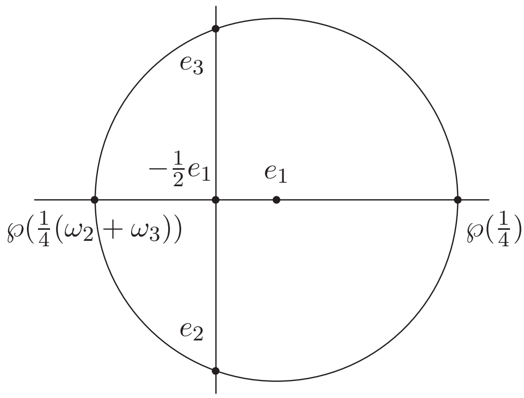

Figure 4. The image of the mapping $z \mapsto \wp(z)$.

$2z_0 - 1 + \omega_2 = z_0$. The latter leads to $z_0 = 1 - \omega_2 = \omega_3$, which is not impossible. Thus we have $2z_0 = -z_0$ in T and $\wp(2z_0) = \wp(-z_0) = \wp(z_0)$. By the addition theorem for $\wp$,

$$\wp(z_0) = \wp(2z_0) = -2\wp(z_0) + \frac{1}{4}\left(\frac{\wp''(z_0)}{\wp'(z_0)}\right)^2 = -2\wp(z_0). \tag{6.6}$$

Therefore $\wp(z_0) = 0$. Together with $2\wp'' = 12\wp^2 - g_2$ we find $g_2 = 0$, which is equivalent to $\tau = (1 + \sqrt{3}i)/2$. □

We need also the following technical lemma:

LEMMA 6.6. *$\wp$ maps $[\frac{1}{2}\omega_2, \frac{1}{2}\omega_3] \cup [\frac{1}{2}(1-\omega_2), \frac{1}{2}\omega_2]$ one-to-one and onto the circle $\{w \mid |w - e_1| = |e_2 - e_1|\}$, where $e_2 = \bar{e}_3$, $\wp(\frac{1}{4}(\omega_2 + \omega_3)) = e_1 - |e_2 - e_1| < 0$ and $\wp(\frac{1}{4}) = e_1 + |e_2 - e_1| > 0$.*

Here $[\frac{1}{2}\omega_2, \frac{1}{2}\omega_3]$ means segment $\{\frac{1}{2}\omega_2 + t \mid 0 \le t \le \frac{1}{2}\}$, and $[\frac{1}{2}(1-\omega_2), \frac{1}{2}\omega_2]$ is $\{\frac{1}{4} + it \mid |t| \le \frac{b}{2}\}$. Thus, the image of $[\frac{1}{2}\omega_2, \frac{1}{2}\omega_3]$ is the arc connecting e_2 and e_3 through $\wp(\frac{1}{4}(\omega_2 + \omega_3))$ and the image of $[\frac{1}{2}(1-\omega_2), \frac{1}{2}\omega_2]$ is the arc connecting e_3 and e_2 through $\wp(\frac{1}{4})$. See Figure 4. Note our figure is for the case $e_1 > 0$, i.e., $b > \frac{1}{2}$. In this case, the angle $\angle e_3 e_1 e_2$ is less than π. For the case $e_1 < 0$, the angle is greater than π.

Proof. First we have

$$4\pi\left(G(z) - G\left(z, \frac{\omega_1}{2}\right)\right) = \log\left|\wp(z) - \wp\left(\frac{\omega_1}{2}\right)\right| + C. \tag{6.7}$$

Since $G(\frac{1}{2}\omega_2, \frac{1}{2}\omega_1) = G(\frac{1}{2}\omega_2 - \frac{1}{2}\omega_1) = G(\frac{1}{2}\omega_2)$,

$$4\pi\left(G(z) - G\left(z, \frac{\omega_1}{2}\right)\right) = \log\left|\frac{\wp(z) - e_1}{e_2 - e_1}\right|. \tag{6.8}$$

Let $z = \frac{1}{2}\omega_2 + t$, $t \in \mathbb{R}$. We have

$$G\Big(z, \frac{\omega_1}{2}\Big) = G\Big(\frac{\omega_2}{2} + t - \frac{\omega_1}{2}\Big) = G\Big(\frac{\omega_1}{2} - \frac{\omega_2}{2} - t\Big)$$
$$= G\Big(\overline{\frac{\omega_2}{2} + t}\Big) = G(z).$$

By (6.8),

$$\left|\frac{\wp(z) - e_1}{e_2 - e_1}\right| = 1 \quad \text{for} \quad z = \frac{\omega_2}{2} + t. \tag{6.9}$$

Since $\wp(z)$ is decreasing in y for $z = \frac{1}{2} + iy$,

$$\wp\Big(\frac{\omega_2 + \omega_3}{4}\Big) = \wp\Big(\frac{1}{2} + \frac{ib}{2}\Big) < \wp(1/2) = e_1.$$

Thus $\wp(\frac{1}{4}(\omega_2 + \omega_3)) = e_1 - |e_2 - e_1|$ and the image of $[\frac{1}{2}\omega_2, \frac{1}{2}\omega_3]$ is exactly the arc on the circle $\{\, w \mid |w - e_1| = |e_2 - e_1| \,\}$ connecting e_2 and e_3 through $\wp(\frac{1}{4}(\omega_2 + \omega_3))$. It is one-to-one since $\wp'(z) \neq 0$ for $z = \frac{1}{2}\omega_2 + t$, $t \neq 0$.

Next let $z = \frac{1}{4} + it$. Then we have

$$G\Big(z; \frac{\omega_1}{2}\Big) = G\Big(z - \frac{\omega_1}{2}\Big) = G\Big(-\frac{1}{4} + it\Big) = G\Big(-\frac{1}{4} - it\Big)$$
$$= G\Big(\frac{1}{4} + it\Big) = G(z).$$

Thus by (6.8) again,

$$|\wp(z) - e_1| = |e_2 - e_1| \quad \text{for} \quad z = \frac{1}{4} + it.$$

Since $\wp(t)$ is decreasing in t for $t \in (0, \frac{1}{2})$, $\wp(\frac{1}{4}) > \wp(\frac{1}{2})$. So $\wp(\frac{1}{4}) = e_1 + |e_2 - e_1|$ and the image of $[\frac{1}{2}(1-\omega_2), \frac{1}{2}\omega_2]$ is the arc of $\{\, w \mid |w - e_1| = |e_2 - e_1| \,\}$ connecting e_3 and e_2 through $e_1 + |e_2 - e_1|$. □

We are ready to prove the main results of this section:

THEOREM 6.7. *For $b > b_1$, $\pm z_0(\tau)$ are nondegenerate (local) minimum points of G. Furthermore,*

$$0 < \operatorname{Im} z_0(\tau) < \frac{b}{2}. \tag{6.10}$$

Proof. We want to prove $G_{xx}(z_0(\tau); \tau) > 0$ and $\operatorname{Im} z_0(\tau) < b/2$. By (6.3),

$$2\pi G_{xx}(z_0(\tau); \tau) = \wp(z_0(\tau); \tau) + \eta_1.$$

Note that $e_1 e_2 + e_2 e_3 + e_1 e_3 = |e_2|^2 - e_1^2$ and

$$\wp''(z) = 2(3\wp^2(z) + |e_2|^2 - |e_1|^2). \tag{6.11}$$

As in (5.7), in terms of $\lambda(\tau)$,

$$\frac{e_2}{e_1} = \frac{\lambda+1}{\lambda-2}. \tag{6.12}$$

Thus

$$\frac{d}{d\lambda}\left(\frac{e_2}{e_1}\right) = \frac{-3}{(\lambda-2)^2} \neq 0 \quad \text{for} \quad b \neq \frac{1}{2}.$$

From here, we have $\frac{d}{db}\left|\frac{e_2}{e_1}\right|^2 \neq 0$ for $b \neq \frac{1}{2}$, which implies that $\frac{d}{db}\left|\frac{e_2}{e_1}\right|^2 < 0$ for $b > \frac{1}{2}$. Therefore at $\tau = \frac{1}{2}(1+\sqrt{3}i)$ (where $\wp(z_0(\tau)) = 0$),

$$\frac{d}{db}\wp''(z_0(\tau);\tau) = 2|e_1|^2\frac{d}{db}\left|\frac{e_2}{e_1}\right|^2 < 0. \tag{6.13}$$

Since $\wp''(z_0(\tau);\tau) = 0$ at $\tau = (1+\sqrt{3}i)/2$, we have $\wp''(z_0(\tau),\tau) < 0$ for $b > \sqrt{3}/2$ and, sufficiently close to $\sqrt{3}/2$. By Lemma 6.5, we then have that $\wp''(z_0(\tau);\tau) < 0$ for all $b > \sqrt{3}/2$. Thus

$$|\wp(z_0(\tau);\tau)| \le \sqrt{\tfrac{1}{3}(|e_1|^2 - |e_2|^2)}, \tag{6.14}$$

and

$$\begin{aligned} \eta_1 + \wp(z_0(\tau);\tau) &\ge \eta_1 - \sqrt{\tfrac{1}{3}(|e_1|^2 - |e_2|^2)} \\ &> \eta_1 - \sqrt{\tfrac{1}{4}|e_1|^2} = \eta_1 - \frac{1}{2}e_1, \end{aligned} \tag{6.15}$$

where $|e_2|^2 = |e_1|^2/4 + |\operatorname{Im} e_2|^2 > |e_1|^2/4$ is used (cf. (5.10)).

Later we will show that $\eta_1 > \frac{1}{2}e_1$ always holds (this is part of Theorem 1.7 to be proved in §9; another direct proof will be given in (6.22)). Thus the nondegeneracy of $z_0(\tau)$ for $b > \sqrt{3}/2$ follows.

For $b_1 < b < \sqrt{3}/2$, write $z_0(\tau) = t_0(\tau)\cdot 1 + (1-2t_0(\tau))\tau$. It is clear that when $b \to b_1^+$, $t_0 \to 1/2$. We claim that $t_0(\tau) > 1/3$ if $b_1 < b < \sqrt{3}/2$. For if $t_0(\tau) = 1/3$, i.e. $z_0(\tau) = \frac{1}{3}\omega_3$, then $2z_0(\tau) = -z_0(\tau)$ is also a critical point. By the addition formula of ζ we get $\wp''(\frac{1}{3}\omega_3) = 0$, and then Lemma 6.5 implies that $b = \sqrt{3}/2$, which is a contradiction.

Note that by (6.6), it is easy to see for $\tau = \frac{1}{2} + ib$,

$$12\wp\Big(\frac{\omega_3}{3};\tau\Big) = \left(\frac{\wp''(\frac{1}{3}\omega_3;\tau)}{\wp'(\frac{1}{3}\omega_3;\tau)}\right)^2 = -\left(\frac{\partial_y^2\wp(\frac{1}{3}\omega_3;\tau)}{\partial_y\wp(\frac{1}{3}\omega_3;\tau)}\right)^2 < 0. \tag{6.16}$$

Hence by the monotone property of $\wp(\frac{1}{2}+iy;\tau)$ in $y \in (0,b)$, it decreases to $-\infty$ when $y \to b$ and

$$\wp(z_0(\tau);\tau) > \wp\Big(\frac{\omega_3}{3};\tau\Big). \tag{6.17}$$

Let $f(b) := \wp(\frac{1}{3}\omega_3;\tau) + \frac{1}{2}e_1(\tau)$. We have $f(\frac{1}{2}) = \wp(\frac{1}{3}\omega_3; \frac{1}{2}(1+i)) < 0$ and $f(\sqrt{3}/2) = \frac{1}{2}e_1(\frac{1}{2}(1+\sqrt{3}i)) > 0$. Therefore there exists a $\tau_0 = \frac{1}{2} + ib_0$ such that $f(b_0) = 0$. For this τ_0 and at $z = \frac{1}{3}\omega_3$ we compute

$$
\begin{aligned}
\wp - e_1 &= -\frac{3}{2}e_1, \quad \wp - e_2 = -\Big(\frac{e_1}{2} + e_2\Big), \quad \wp - e_3 = \frac{e_1}{2} + e_2, \\
\wp'^2 &= 4(\wp - e_1)(\wp - e_2)(\wp - e_3) = 6e_1\Big(\frac{e_1}{2} + e_2\Big)^2, \\
\wp'' &= 2\sum\nolimits_{1\le i<j\le 3}(\wp - e_i)(\wp - e_j) = -2\Big(\frac{e_1}{2} + e_2\Big)^2.
\end{aligned}
\tag{6.18}
$$

Plugging these into (6.16) we solve

$$
\frac{e_2}{e_1} = -\frac{1}{2} \pm 3i \quad \text{and then} \quad \Big|\frac{e_2}{e_1}\Big|^2 = \frac{37}{4}.
\tag{6.19}
$$

Numerically at $b = b_1$, $|e_2/e_1|^2 \approx 3.126 < 37/4$. By the decreasing property of $|e_2/e_1|$ (cf. (6.13)) we find that τ_0 is unique with $b_1 > b_0$. Thus $\wp(\frac{1}{3}\omega_3) + \frac{1}{2}e_1 = f(b) > 0$ for $b > b_1$. Together with (6.17),

$$
\wp(z_0(\tau), \tau) + \eta_1 > \eta_1 - \frac{1}{2}e_1 > 0
$$

for $b_1 < b < \sqrt{3}/2$. This completes the proof of $G_{xx}(z_0(\tau);\tau) > 0$.

It remains to show that $\operatorname{Im} z_0(\tau) < b/2$. This has already been proved in the case $b_1 < b < \sqrt{3}/2$ since $t_0(\tau) > 1/3$. For $b \ge \sqrt{3}/2$, from the continuity of $z_0(\tau)$ in b and $\wp''(z_0(\tau),\tau) \le 0$, it is enough to show that

$$
\wp''\Big(\frac{1}{2} + \frac{1}{2}bi;\tau\Big) = \wp''\Big(\frac{1}{4}(\omega_2 + \omega_3)\Big) > 0.
$$

Since

$$
\wp'' = \frac{1}{2}\wp'^2\left(\frac{1}{\wp - e_1} + \frac{1}{\wp - e_2} + \frac{1}{\wp - e_3}\right),
$$

the positivity at $\frac{1}{4}(\omega_2 + \omega_3)$ follows from $\wp'^2 < 0$ and the negativity of the right-hand side via Lemma 6.6, Figure 4. □

Now we discuss the nondegeneracy of G at $\frac{1}{2}\omega_2$ and $\frac{1}{2}\omega_3$. The local minimum property of $z_0(\tau)$ is in fact global by

THEOREM 6.8. *For $\tau = \frac{1}{2} + ib$, both $\frac{1}{2}\omega_2$ and $\frac{1}{2}\omega_3$ are nondegenerate saddle points of G.*

Proof. By (6.3), we have

$$
\begin{aligned}
2\pi G_{xx}\Big(\frac{\omega_2}{2}\Big) &= \operatorname{Re} e_2 + \eta_1 = \eta_1 - \frac{1}{2}e_1, \\
2\pi G_{xy}\Big(\frac{\omega_2}{2}\Big) &= -\operatorname{Im} e_2, \\
2\pi G_{yy}\Big(\frac{\omega_2}{2}\Big) &= \frac{2\pi}{b} + \frac{1}{2}e_1 - \eta_1.
\end{aligned}
\tag{6.20}
$$

Hence the nondegeneracy of $\frac{1}{2}\omega_2$ holds if

$$|\mathrm{Im}\, e_2|^2 > \Big(\eta_1 - \frac{1}{2}e_1\Big)\Big(\frac{2\pi}{b} + \frac{1}{2}e_1 - \eta_1\Big). \tag{6.21}$$

First we claim that

$$\eta_1 - \frac{1}{2}e_1 > 0 \tag{6.22}$$

and

$$\frac{2\pi}{b} + \frac{1}{2}e_1 - \eta_1 > 0. \tag{6.23}$$

To prove (6.22), we have

$$-\eta_1 = 2\int_0^{\frac{1}{2}} \mathrm{Re}\,\wp\Big(\frac{\omega_2}{2} + t\Big)\, dt, \tag{6.24}$$

where $\wp(\frac{1}{2}\omega_2 + t) = \wp(\frac{1}{2}\omega_2 - t)$ is used. By Lemma 6.6,

$$\mathrm{Re}\,\wp\Big(\frac{\omega_2}{2} + t\Big) \le -\frac{1}{2}e_1, \quad \text{for all } t \in (0, \tfrac{1}{2}),$$

hence $-\eta_1 < -\frac{1}{2}e_1$. To prove (6.23), we have

$$\begin{aligned}\int_{\frac{1-\omega_2}{2}}^{\frac{\omega_2}{2}} \wp(z)\, dz &= \zeta\Big(\frac{1-\omega_2}{2}\Big) - \zeta\Big(\frac{\omega_2}{2}\Big)\\ &= \frac{1}{2}(\eta_1 - 2\eta_2) = i(2\pi - b\eta_1).\end{aligned}$$

Therefore,

$$(2\pi - b\eta_1) = \int_{-\frac{b}{2}}^{\frac{b}{2}} \mathrm{Re}\,\wp\Big(\frac{1}{4} + it\Big)\, dt > -\frac{1}{2}e_1 b$$

and the inequality (6.23) follows.

To prove (6.21), we need two more inequalities. By (6.24),

$$-\eta_1 > \wp\Big(\frac{\omega_2 + \omega_3}{4}\Big) = e_1 - |e_2 - e_1|, \tag{6.25}$$

and by Lemma 6.6,

$$(2\pi - b\eta_1) < \wp(1/4)b = (e_1 + |e_2 - e_1|)b. \tag{6.26}$$

Thus

$$\begin{aligned}\Big(\eta_1 - \frac{1}{2}e_1\Big)\Big(\frac{2\pi}{b} + \frac{1}{2}e_1 - \eta_1\Big) &< \Big(|e_2 - e_1| - \frac{3}{2}e_1\Big)\Big(|e_2 - e_1| + \frac{3}{2}e_1\Big)\\ &= |e_2 - e_1|^2 - \frac{9}{4}e_1^2 = |\mathrm{Im}\, e_2|^2,\end{aligned}$$

Therefore, the nondegeneracy of G at $\frac{1}{2}\omega_2$ is proved. By (6.21), $\frac{1}{2}\omega_2$ is always a saddle point. Representing the torus T by the rhombus torus with sides τ and $\bar{\tau} = 1 - \tau$, then the case for $\frac{1}{2}\omega_3$ follows from the case for $\frac{1}{2}\omega_2$ for symmetry reasons. □

7. **Green functions via theta functions**

The purpose of Sections 7–9 is to prove the two fundamental inequalities (Theorem 1.7) that have been used in previous sections. The natural setup is based on theta functions (we take [11] as our general reference). This is easy to explain since the moduli variable τ is explicit in theta functions and differentiations in τ are much easier than in the Weierstrass theory. To avoid cross references, the discussions here are independent of previous sections.

Consider a torus $T = \mathbb{C}/\Lambda$ with $\Lambda = (\mathbb{Z}+\mathbb{Z}\tau)$, a lattice with $\tau = a+bi$, $b>0$. Let $q = e^{\pi i\tau}$ with $|q| = e^{-\pi b} < 1$. The theta function $\vartheta_1(z;\tau)$ is the exponentially convergent series

$$(7.1)\qquad \vartheta_1(z;\tau) = -i\sum_{n=-\infty}^{\infty}(-1)^n q^{(n+\frac{1}{2})^2}e^{(2n+1)\pi iz} = 2\sum_{n=0}^{\infty}(-1)^n q^{(n+\frac{1}{2})^2}\sin(2n+1)\pi z.$$

For simplicity we also write $\vartheta_1(z;\tau)$ as $\vartheta_1(z)$. It is entire with

$$(7.2)\qquad \vartheta_1(z+1) = -\vartheta_1(z), \qquad \vartheta_1(z+\tau) = -q^{-1}e^{-2\pi iz}\vartheta_1(z),$$

which has simple zeros at the lattice points (and no others). The following heat equation is clear from the definition

$$(7.3)\qquad \frac{\partial^2\vartheta_1}{\partial z^2} = 4\pi i\,\frac{\partial\vartheta_1}{\partial\tau}.$$

As usual we use $z = x+iy$. Here comes the starting point:

LEMMA 7.1. *Up to a constant $C(\tau)$, the Green function $G(z,w)$ for the Laplace operator $\triangle$ on T is given by*

$$(7.4)\qquad G(z,w) = -\frac{1}{2\pi}\log|\vartheta_1(z-w)| + \frac{1}{2b}(\operatorname{Im}(z-w))^2 + C(\tau).$$

Proof. Let $R(z,w)$ be the right-hand side. Clearly for $z\neq w$ we have $\triangle_z R(z,w) = 1/b$ which integrated over T gives 1. Near $z=w$, $R(z,w)$ has the correct behavior. So it remains to show that $R(z,w)$ is indeed a function on T. From the quasi-periodicity, $R(z+1,w) = R(z,w)$ is obvious. Also

$$R(z+\tau,w) - R(z,w) = -\frac{1}{2\pi}\log e^{\pi b+2\pi y} + \frac{1}{2b}((y+b)^2 - y^2) = 0.$$

These properties characterize the Green function up to a constant. □

By the translation invariance of G, it is enough to consider $w=0$. Let

$$G(z) = G(z,0) = -\frac{1}{2\pi}\log|\vartheta_1(z)| + \frac{1}{2b}y^2 + C(\tau).$$

If we represent the torus T as centered at 0, then the symmetry $z\mapsto -z$ shows that $G(z) = G(-z)$. By differentiation, we get $\nabla G(z) = -\nabla G(-z)$. If $-z_0 = z_0$

in T, that is $2z_0 = 0 \pmod{\Lambda}$, then we get $\nabla G(z_0) = 0$. Hence we obtain the half-periods $\frac{1}{2}$, $\frac{1}{2}\tau$ and $\frac{1}{2}(1+\tau)$ as three obvious critical points of $G(z)$ for any T. By computing $\partial G/\partial z = \frac{1}{2}(G_x - iG_y)$ we find

COROLLARY 7.2. *The equation of critical points $z = x+iy$ of $G(z)$ is given by*

$$\frac{\partial G}{\partial z} \equiv \frac{-1}{4\pi}\left((\log\vartheta_1)_z + 2\pi i\frac{y}{b}\right) = 0. \tag{7.5}$$

Remark 7.3. With $\zeta(z) - \eta_1 z = (\log\vartheta_1(z))_z$ understood (cf. (9.9)), this leads to alternative simple proofs of Lemma 2.3 and Corollary 2.4.

We compute easily

$$\begin{aligned}
G_x &= -\frac{1}{2\pi}\mathrm{Re}\,(\log\vartheta_1)_z,\\
G_y &= -\frac{1}{2\pi}\mathrm{Re}\,(\log\vartheta_1)_z i + \frac{y}{b} = \frac{1}{2\pi}\mathrm{Im}\,(\log\vartheta_1)_z + \frac{y}{b},\\
G_{xx} &= -\frac{1}{2\pi}\mathrm{Re}\,(\log\vartheta_1)_{zz},\\
G_{xy} &= +\frac{1}{2\pi}\mathrm{Im}\,(\log\vartheta_1)_{zz},\\
G_{yy} &= -\frac{1}{2\pi}\mathrm{Re}\,(\log\vartheta_1)_{zz} i^2 + \frac{1}{b} = \frac{1}{2\pi}\mathrm{Re}\,(\log\vartheta_1)_{zz} + \frac{1}{b},
\end{aligned} \tag{7.6}$$

and the Hessian

$$\begin{aligned}
H &= \begin{vmatrix} G_{xx} & G_{xy} \\ G_{yx} & G_{yy} \end{vmatrix}\\
&= \frac{-1}{4\pi^2}\left[(\mathrm{Re}\,(\log\vartheta_1)_{zz})^2 + \frac{2\pi}{b}(\mathrm{Re}\,(\log\vartheta_1)_{zz}) + (\mathrm{Im}\,(\log\vartheta_1)_{zz})^2\right]\\
&= \frac{-1}{4\pi^2}\left[\left|(\log\vartheta_1)_{zz} + \frac{\pi}{b}\right|^2 - \left(\frac{\pi}{b}\right)^2\right].
\end{aligned} \tag{7.7}$$

To analyze the critical point of $G(z)$ in general, we use the methods of continuity to connect τ to a standard model like the square torus, that is $\tau = i$, which under the modular group $\mathrm{SL}(2,\mathbb{Z})$ is equivalent to the point $\tau = \frac{1}{2}(1+i)$ by $\tau \mapsto 1/(1-\tau)$. On this special torus, there are precisely three critical points given by the half-periods (cf. [5, Lemma 2.1]).

The idea is, new critical points should be born only at certain half-period points when they degenerate under the deformation in τ. The heat equation provides a bridge between the degeneracy condition and deformations in τ. In the following, we focus on the critical point $z = \frac{1}{2}$ and analyze its degeneracy behavior along the half line L given by $\frac{1}{2} + ib$, $b \in \mathbb{R}$.

8. First inequality along the line $\mathrm{Re}\,\tau = \frac{1}{2}$

When $\tau = \frac{1}{2} + ib \in L$,

$$\vartheta_1(z) = 2\sum_{n=0}^{\infty}(-1)^n e^{\pi i/8} e^{\pi i\frac{n(n+1)}{2}} e^{-\pi b(n+\frac{1}{2})^2}\sin(2n+1)\pi z;$$

hence we have the important observation that

$$e^{-\pi i/8}\vartheta_1(z) \in \mathbb{R} \quad \text{when} \quad z \in \mathbb{R}.$$

Similarly this holds for any derivatives of $\vartheta_1(z)$ in z. In particular,

$$(8.1) \qquad \begin{aligned}(\log\vartheta_1)_{zz} &= \frac{\vartheta_{1zz}\vartheta_1 - (\vartheta_{1z})^2}{\vartheta_1^2} \\ &= 4\pi i\frac{\vartheta_{1\tau}}{\vartheta_1} - (\log\vartheta_1)_z^2 = 4\pi(\log\vartheta_1)_b - (\log\vartheta_1)_z^2\end{aligned}$$

is real-valued for all $z \in \mathbb{R}$ and $\tau \in L$. Here the heat equation and the holomorphicity of $(\log\vartheta_1)$ have been used.

Now we focus on the critical point $z = \frac{1}{2}$ which is fixed until the end of this section. The critical point equation implies that $(\log\vartheta_1)_z(\frac{1}{2}) = -2\pi i y/b = 0$ since now $y = 0$. Thus

$$(8.2) \qquad (\log\vartheta_1)_{zz} = 4\pi(\log\vartheta_1)_b$$

as real functions in b. In this case, the point $\frac{1}{2}$ is a degenerate critical point ($H(b) = 0$) if and only if

$$(8.3) \qquad 4\pi(\log\vartheta_1)_b = 0 \quad \text{or} \quad 4\pi(\log\vartheta_1)_b + \frac{2\pi}{b} = 0.$$

Notice that as functions in $b > 0$,

$$|\vartheta_1| = e^{-\pi i/8}\vartheta_1\Big(\frac{1}{2}; \frac{1}{2} + ib\Big) = 2\sum_{n=0}^{\infty}(-1)^{\frac{n(n+1)}{2}} e^{-\frac{1}{4}\pi b(2n+1)^2} \in \mathbb{R}^+.$$

To see this, notice that the right-hand side is nonzero, real and positive for large b, hence positive for all b. Clearly $(\log|\vartheta_1|)_b = (\log\vartheta_1)_b$.

THEOREM 8.1. *Over the line L, $(\log\vartheta_1)_{bb} = (\log|\vartheta_1|)_{bb} < 0$. Namely, $(\log\vartheta_1)_b$ is decreasing from positive infinity to $-\pi/4$. Hence, $G_{xx} = 0$ and $G_{yy} = 0$ occur exactly once on L, respectively.*

Proof. Denote $e^{-\pi b/4}$ by h and $r = h^8 = e^{-2\pi b}$. Since $(2n+1)^2 - 1 = 4n(n+1)$, we get

$$(8.4) \qquad \begin{aligned}|\vartheta_1|_b &= -2\frac{\pi}{4}\sum_{n=0}^{\infty}(-1)^{\frac{n(n+1)}{2}}(2n+1)^2 h^{(2n+1)^2} \\ &= -2h\frac{\pi}{4}\sum_{n=0}^{\infty}(-1)^{\frac{n(n+1)}{2}}(2n+1)^2 r^{\frac{n(n+1)}{2}}, \\ |\vartheta_1|_{bb} &= 2\frac{\pi^2}{4^2}\sum_{n=0}^{\infty}(-1)^{\frac{n(n+1)}{2}}(2n+1)^4 h^{(2n+1)^2} \\ &= 2h\frac{\pi^2}{4^2}\sum_{n=0}^{\infty}(-1)^{\frac{n(n+1)}{2}}(2n+1)^4 r^{\frac{n(n+1)}{2}}.\end{aligned}$$

Denote the arithmetic sum $n(n+1)/2$ by A_n, then

(8.5)
$$
\begin{aligned}
(\log|\vartheta_1|)_{bb} &= \frac{|\vartheta_1|_{bb}|\vartheta_1| - |\vartheta_1|_b^2}{|\vartheta_1|^2} \\
&= h^2\frac{\pi^2}{4}|\vartheta_1|^{-2}\sum_{n,m=0}^{\infty}(-1)^{A_n+A_m}((2n+1)^4-(2n+1)^2(2m+1)^2)r^{A_n+A_m} \\
&= h^2\frac{\pi^2}{4}|\vartheta_1|^{-2}\sum_{n>m}(-1)^{A_n+A_m}((2n+1)^2-(2m+1)^2)^2r^{A_n+A_m} \\
&= 16h^2\pi^2|\vartheta_1|^{-2}\sum_{n>m}(-1)^{A_n+A_m}(A_n-A_m)^2r^{A_n+A_m} \\
&= 16h^2\pi^2|\vartheta_1|^{-2}(-r-9r^3+4r^4+36r^6-25r^7-9r^9+100r^{10}+\cdots).
\end{aligned}
$$

We will prove $(\log|\vartheta_1|)_{bb} < 0$ in two steps. First we show by direct estimate that this is true for $b \ge \frac{1}{2}$ (indeed the argument holds for $b > 0.26$). Then we derive a functional equation for $(\log|\vartheta_1|)_{bb}$ which implies that the case with $0 < b \le \frac{1}{2}$ is equivalent to the case $b \ge \frac{1}{2}$.

Step 1 (Direct Estimate). The point is to show that in the above expression the sum of positive (even degree) terms is small. So let $2k \in 2\mathbb{N}$. The number of terms with degree $2k$ is certainly no more than $2k$, so a trivial upper bound for the positive part is given by

$$
A = \sum_{k=2}^{\infty}(2k)^3r^{2k} = 8r^4\frac{8-5r^2+4r^4-r^6}{(1-r^2)^4}, \tag{8.6}
$$

where the last equality is an easy exercise in power series calculations in calculus. For $r \le 1/5$ we compute

(8.7)
$$
\begin{aligned}
&(-r-9r^3+A)(1-r^2)^4 \\
&= -r-5r^3+64r^4+30r^5-40r^6-50r^7+32r^8+35r^9-8r^{10}-9r^{11} \\
&< -r-5r^3+64r^4+30r^5 \\
&< -5r^3 - r\Big(1-\frac{64}{125}-\frac{30}{625}\Big) = -5r^3-\frac{11}{25}r < 0.
\end{aligned}
$$

So $(\log|\vartheta_1|)_{bb} < 0$ for $b = -(\log r)/2\pi > (\log 5)/2\pi \sim 0.25615$.

Step 2 (Functional Equation). By the lemma to be proved below, we have for $\hat\tau = (\tau-1)/(2\tau-1) = \hat a + i\hat b$, that

$$
(\log\vartheta_1)_{\hat b}(1/2;\hat\tau) = -i(1-2\tau) + (1-2\tau)^2(\log\vartheta_1)_b(1/2;\tau). \tag{8.8}
$$

When $\tau = \frac{1}{2}+ib$, we have $\hat\tau = \frac{1}{2}+\frac{i}{4b}$. As before we may then replace ϑ_1 by $|\vartheta_1|$. Under $\tau \to \hat\tau$, $[\frac{1}{2},\infty)$ is mapped onto $(0,\frac{1}{2}]$ with directions reversed. Let

$f(b) = (\log|\vartheta_1|)_b(\frac{1}{2}, \frac{1}{2} + ib)$. Then we get

$$f(1/4b) = -2b - 4b^2 f(b). \tag{8.9}$$

Plugging in $b = \frac{1}{2}$ we get that $f(\frac{1}{2}) = -\frac{1}{2}$. So $-\frac{1}{2} = f(\frac{1}{2}) > f(b)$ for $b > \frac{1}{2}$. Then

$$\begin{aligned} f\Big(\frac{1}{4b}\Big) &= -2b + 2b^2 + 4b^2\Big[-\frac{1}{2} - f(b)\Big] \\ &= 2\Big[b - \frac{1}{2}\Big]^2 - \frac{1}{2} + 4b^2\Big[-\frac{1}{2} - f(b)\Big] \end{aligned} \tag{8.10}$$

is strictly increasing when $b > \frac{1}{2}$. That is, $f(b)$ is strictly decreasing when $b \in (0, \frac{1}{2}]$. The remaining statements are all clear. □

Now we prove the functional equation. For this we need to use Jacobi's imaginary transformation formula, which explains the modularity for certain special theta values (cf. [11, p. 475]). It reads that for $\tau\tau' = -1$,

$$\vartheta_1(z;\tau) = -i(i\tau')^{\frac{1}{2}} e^{\pi i \tau' z^2} \vartheta_1(z\tau';\tau'). \tag{8.11}$$

Recall that the two generators of $\mathrm{SL}(2,\mathbb{Z})$ are $S\tau = -1/\tau$ and $T\tau = \tau + 1$. Since $\vartheta_1(z;\tau+1) = e^{\pi i/4}\vartheta_1(z;\tau)$, T plays no role in $(\log\vartheta_1(z;\tau))_\tau$.

LEMMA 8.2. *Let $\hat\tau = ST^{-2}ST^{-1}\tau = (\tau-1)/(2\tau-1)$. Then*

$$(\log\vartheta_1)_{\hat\tau}(1/2;\hat\tau) = -(1-2\tau) + (1-2\tau)^2(\log\vartheta_1)_\tau(1/2;\tau). \tag{8.12}$$

Proof. Let $\hat\tau = S\tau_1 = -1/\tau_1$, $\tau_1 = T^{-2}\tau_2 = \tau_2 - 2$, $\tau_2 = S\tau_3 = -1/\tau_3$ and finally $\tau_3 = T^{-1}\tau = \tau - 1$. Notice that for $\tau\tau' = -1$ we have $d/d\tau = \tau'^2 d/d\tau'$. Then

$$\begin{aligned} &\frac{d}{d\hat\tau}\log\vartheta_1(1/2;\hat\tau) \\ &\quad= \tau_1^2\frac{d}{d\tau_1}\Big[\log(-i\tau_1)^{\frac{1}{2}} + \pi i\tau_1(1/2)^2 + \log\vartheta_1(\tau_1/2;\tau_1)\Big] \\ &\quad= \frac{\tau_1}{2} + \frac{\pi i\tau_1^2}{4} + \tau_1^2\frac{d}{d\tau_1}\log\vartheta_1(\tau_1/2;\tau_1) \\ &\quad= \frac{\tau_2-2}{2} + \frac{\pi i(\tau_2-2)^2}{4} + (\tau_2-2)^2\frac{d}{d\tau_2}\log\vartheta_1(\tau_2/2;\tau_2) \\ &\quad= \frac{1}{2}\Big[\frac{-1}{\tau_3} - 2\Big] + \frac{\pi i}{4}\Big[\frac{-1}{\tau_3} - 2\Big]^2 + \Big[\frac{-1}{\tau_3} - 2\Big]^2 \\ &\qquad\times\bigg(\frac{\tau_3}{2} + \pi i\tau_3^2\frac{d}{d\tau_3}\big[\tau_3(\tau_2/2)^2\big] + \tau_3^2\frac{d}{d\tau_3}\log\vartheta_1(\tau_2\tau_3/2;\tau_3)\bigg). \end{aligned} \tag{8.13}$$

We plug in $\tau_2\tau_3 = -1$ and $\tau_3 = \tau - 1$. It is clear that the second and the fourth terms are canceled out, the first and the third terms are combined into

$$\frac{1}{2}\frac{1-2\tau}{\tau-1}\Big[1 + \frac{1-2\tau}{\tau-1}(\tau-1)\Big] = -(1-2\tau). \tag{8.14}$$

This proves the statement. □

By the previous explicit computations,

$$
\begin{aligned}
G_{xx}\Big(\frac{1}{2};\frac{1}{2}+ib\Big) &= -2(\log\vartheta_1(1/2))_b,\\
G_{xy}\Big(\frac{1}{2};\frac{1}{2}+ib\Big) &= 0,\\
G_{yy}\Big(\frac{1}{2};\frac{1}{2}+ib\Big) &= 2(\log\vartheta_1(1/2))_b+\frac{1}{b}.
\end{aligned}
\tag{8.15}
$$

A numerical computation shows that $G_{xx}(b_0)=0$ for $b_0=0.35\cdots$ and $G_{yy}(b_1)=0$ for $b_1=0.71\cdots$. Hence the sign of $(G_{xx}(b),G_{yy}(b))$ is $(-,+)$, $(+,+)$ and $(+,-)$ for $b<b_0$, $b_0<b<b_1$ and $b_1<b$ respectively. That is, $\frac{1}{2}$ is a saddle point, local minimum point and saddle point respectively. This implies that for $b=b_1+\varepsilon>b_1$, there are more critical points near $\frac{1}{2}$ which come from the degeneracy of $\frac{1}{2}$ at $b=b_1$ and the conservation of local Morse index.

9. **Second inequality along the line** $\operatorname{Re}\tau=\frac{1}{2}$

The analysis of extra critical points split from $\frac{1}{2}$ also relies on other, though similar, inequalities. Recall the three other theta functions:

$$
\begin{aligned}
\vartheta_2(z;\tau) &:= \vartheta_1\Big(z+\frac{1}{2};\tau\Big),\\
\vartheta_4(z;\tau) &:= \sum_{n=-\infty}^{\infty}(-1)^n q^{n^2}e^{2\pi i n z} = 1+2\sum_{n=1}^{\infty}(-1)^n q^{n^2}\cos 2n\pi z,\\
\vartheta_3(z;\tau) &:= \vartheta_4\Big(z+\frac{1}{2};\tau\Big) = 1+2\sum_{n=1}^{\infty}q^{n^2}\cos 2n\pi z = \sum_{n=-\infty}^{\infty}q^{n^2}e^{2\pi i n z}.
\end{aligned}
\tag{9.1}
$$

It is readily seen that $\vartheta_1(z)=-ie^{\pi i z+\pi i\tau/4}\vartheta_4(z+\frac{1}{2}\tau)$. So the four theta functions are translates of others by half-periods.

We had seen previously that $(\log|\vartheta_1(\frac{1}{2})|)_{bb}<0$ over the line $\operatorname{Re}\tau=\frac{1}{2}$. This is equivalent to the fact that $(\log|\vartheta_2(0)|)_{bb}<0$. We now discuss the case for $\vartheta_3(0)$ and $\vartheta_4(0)$ where the situation is reversed(!) and it turns out the proof is easier and purely algebraic.

THEOREM 9.1. *Over the line* $\operatorname{Re}\tau=\frac{1}{2}$, *we have* $\vartheta_4(0)=\overline{\vartheta_3(0)}$. *Moreover,* $(\log|\vartheta_3(0)|)_b<0$ *and* $(\log|\vartheta_3(0)|)_{bb}>0$.

Proof. Since $q=e^{\pi i\tau}=e^{\pi i/2}e^{-\pi b}=ie^{-\pi b}$, $q^{n^2}=i^{n^2}e^{-\pi b n^2}$, we see that

$$
\begin{aligned}
\vartheta_3(0) &= \sum_{n\in 2\mathbb{Z}} r^{n^2}+i\sum_{m\in 2\mathbb{Z}+1} r^{m^2},\\
\vartheta_4(0) &= \sum_{n\in 2\mathbb{Z}} r^{n^2}-i\sum_{m\in 2\mathbb{Z}+1} r^{m^2},
\end{aligned}
\tag{9.2}
$$

where $r = e^{-\pi b} < 1$. Then we compute directly that

$$(9.3)\qquad |\vartheta_3(0)|^2 = \Big(\sum\nolimits_{n\in 2\mathbb{Z}} r^{n^2}\Big)^2 + \Big(\sum\nolimits_{m\in 2\mathbb{Z}+1} r^{m^2}\Big)^2 = \sum_{n,n'\in 2\mathbb{Z}} r^{n^2+n'^2} + \sum_{m,m'\in 2\mathbb{Z}+1} r^{m^2+m'^2} = \sum_{k\in 2\mathbb{Z}_{\geq 0}} p_2(k) r^k,$$

where $p_2(k)$ is the number of ways to represent k as the (ordered) sum of two squares of integers. Then

$$(9.4)\qquad (|\vartheta_3(0)|^2)_b = -\pi \sum_{k\in 2\mathbb{Z}_{\geq 0}} p_2(k) k r^k$$

and in particular $(\log|\vartheta_3(0)|^2)_b < 0$.

We also have

$$(9.5)\qquad (|\vartheta_3(0)|^2)_{bb} = \pi^2 \sum_{k\in 2\mathbb{Z}_{\geq 0}} p_2(k) k^2 r^k.$$

Hence,

$$(9.6)\qquad (|\vartheta_3(0)|^2)_{bb}|\vartheta_3(0)|^2 - (|\vartheta_3(0)|^2)_b^2 = \pi^2 \sum_{k,l\in 2\mathbb{Z}_{\geq 0}} p_2(k)p_2(l)(k^2-kl)r^{k+l} = \pi^2 \sum_{k<l} p_2(k)p_2(l)(k-l)^2 r^{k+l} > 0.$$

This implies that $(\log|\vartheta_3(0)|)_{bb} > 0$. The proof is complete. $\square$

Now we relate these to Weierstrass' elliptic functions. From $(\log\sigma(z))' = \zeta(z)$, $\sigma(z)$ is entire, odd with a simple zero on lattice points. Moreover,

$$(9.7)\qquad \sigma(z+\omega_i) = -e^{\eta_i(z+\frac{1}{2}\omega_i)}\sigma(z).$$

This is similar to the theta function transformation law; indeed,

$$(9.8)\qquad \sigma(z) = e^{\eta_1 z^2/2}\frac{\vartheta_1(z)}{\vartheta_1'(0)}.$$

Hence

$$(9.9)\qquad \zeta(z) - \eta_1 z = \left[\log\frac{\vartheta_1(z)}{\vartheta_1'(0)}\right]_z = (\log\vartheta_1(z))_z$$

and

$$(9.10)\qquad \wp(z) + \eta_1 = -(\log\vartheta_1(z))_{zz} = -4\pi i(\log\vartheta_1(z))_\tau + [(\log\vartheta_1(z))_z]^2.$$

For $z = \frac{1}{2}$, this simplifies to $e_1 + \eta_1 = -4\pi i(\log\vartheta_1(\frac{1}{2}))_\tau$. Thus our first inequality simply says that on the line $\operatorname{Re}\tau = \frac{1}{2}$,

$$(9.11)\qquad (e_1+\eta_1)_b = -4\pi\left[\log\vartheta_1\Big(\frac{1}{2}\Big)\right]_{bb} = -4\pi(\log\vartheta_2(0))_{bb} > 0.$$

From the Taylor expansion of $\sigma(z)$ and $\vartheta_1(z)$, it is known that

$$\eta_1 = -\frac{2}{3!}\frac{\vartheta_1'''(0)}{\vartheta_1'(0)} = -\frac{4\pi i}{3}(\log \vartheta_1'(0))_\tau, \tag{9.12}$$

hence

$$e_1 = -4\pi i(\log \vartheta_2(0))_\tau + \frac{4\pi i}{3}(\log \vartheta_1'(0))_\tau. \tag{9.13}$$

The Jacobi Triple Product Formula (cf. [11, p. 490]) asserts that

$$\vartheta_1'(0) = \pi\vartheta_2(0)\vartheta_3(0)\vartheta_4(0).$$

So

$$\begin{aligned} \frac{1}{2}e_1 - \eta_1 &= 2\pi i\left[\log\frac{\vartheta_1'(0)}{\vartheta_2(0)}\right]_\tau \\ &= 2\pi i(\log\vartheta_3(0)\vartheta_4(0))_\tau = 4\pi(\log|\vartheta_3(0)|)_b. \end{aligned} \tag{9.14}$$

Our second inequality then says that on the line $\operatorname{Re}\tau = \frac{1}{2}$, $\frac{1}{2}e_1 - \eta_1 < 0$, $(\frac{1}{2}e_1 - \eta_1)_b > 0$ and $\frac{1}{2}e_1 - \eta_1$ increases to zero in b. Together with the first inequality (9.11), we find also that e_1 increases in b.

Acknowledgements. C.-S. Lin is partially supported by the National Science Council. C.-L. Wang is partially supported by the Shiing-shen Chern Fellowship of the National Center for Theoretic Sciences at Hsinchu, Taiwan.

References

[1] S.-Y. A. Chang, C.-C. Chen, and C.-S. Lin, Extremal functions for a mean field equation in two dimension, in *Lectures on Partial Differential Equations*, *New Stud. Adv. Math.* **2**, Int. Press, Somerville, MA, 2003, pp. 61–93. MR 2005d:35061 Zbl 1071.35040

[2] C.-C. Chen and C.-S. Lin, Topological degree for a mean field equation with singular data, in preparation.

[3] ———, Sharp estimates for solutions of multi-bubbles in compact Riemann surfaces, *Comm. Pure Appl. Math.* **55** (2002), 728–771. MR 2003d:53056 Zbl 1040.53046

[4] ———, Topological degree for a mean field equation on Riemann surfaces, *Comm. Pure Appl. Math.* **LVI** (2003), 1667–1727. MR 2004h:35065 Zbl 1032.58010

[5] C.-C. Chen, C.-S. Lin, and G. Wang, Concentration phenomena of two-vortex solutions in a Chern-Simons model, *Ann. Sc. Norm. Super. Pisa Cl. Sci.* **3** (2004), 367–397. MR 2005m:58033 Zbl 1170.35413

[6] J. Jost, C.-S. Lin, and G. Wang, Analytic aspects of the Toda system. II. Bubbling behavior and existence of solutions, *Comm. Pure Appl. Math.* **59** (2006), 526–558. MR 2007h:35099 Zbl pre05023695

[7] C.-S. Lin and C.-L. Wang, On the minimality of extra critical points of Green functions on flat tori, in preparation.

[8] M. Nolasco and G. Tarantello, On a sharp Sobolev-type inequality on two-dimensional compact manifolds, *Arch. Ration. Mech. Anal.* **145** (1998), 161–195. MR 2000j:58021 Zbl 0980.46022

[9] M. NOLASCO and G. TARANTELLO, Double vortex condensates in the Chern-Simons-Higgs theory, *Calc. Var. Partial Differential Equations* **9** (1999), 31–94. MR 2001e:58022 Zbl 0951.58030

[10] ———, Vortex condensates for the SU(3) Chern-Simons theory, *Comm. Math. Phys.* **213** (2000), 599–639. MR 2001j:58025 Zbl 0998.81047

[11] E. T. WHITTAKER and G. N. WATSON, *A Course of Modern Analysis*, *Cambridge Mathematical Library*, Cambridge Univ. Press, Cambridge, 1996, reprint of the fourth (1927) edition. MR 97k:01072 Zbl 0951.30002

(Received June 1, 2006)
(Revised April 25, 2008)

E-mail address: cslin@math.ntu.edu.tw
DEPARTMENT OF MATHEMATICS AND TAIDA INSTITUTE OF MATHEMATICAL SCIENCES (TIMS), NATIONAL TAIWAN UNIVERSITY, TAIPEI, TAIWAN
http://www.math.ntu.edu.tw/people/bio.php?PID=30

E-mail address: dragon@math.ntu.edu.tw
DEPARTMENT OF MATHEMATICS, NATIONAL TAIWAN UNIVERSITY, NO. 1, SEC. 4, ROOSEVELT ROAD, TAIPEI, 10617, TAIWAN

and

DEPARTMENT OF MATHEMATICS, NATIONAL CENTRAL UNIVERSITY, CHUNG-LI, TAIWAN
http://www.math.ntu.edu.tw/∼dragon/

ISSN 0003-486X

ANNALS OF MATHEMATICS

This periodical is published bimonthly by the Department of Mathematics at Princeton University with the cooperation of the Institute for Advanced Study. Annals is typeset in TEX by Sarah R. Warren and produced by Mathematical Sciences Publishers. The six numbers each year are divided into two volumes of three numbers each.

Editorial correspondence

Papers submitted for publication and editorial correspondence should be addressed to Maureen Schupsky, Annals of Mathematics, Fine Hall-Washington Road, Princeton University, Princeton, NJ, 08544-1000 U.S.A. The e-mail address is annals@math.princeton.edu.

Preparing and submitting papers

The Annals requests that all papers include an abstract of about 150 words which explains to the nonspecialist mathematician what the paper is about. It should not make any reference to the bibliography. Authors are encouraged to initially submit their papers electronically and in PDF format. Please send the file to: annals@math.princeton.edu or to the Mathematics e-print arXiv: front.math.ucdavis.edu/submissions. If a paper is submitted through the arXiv, then please e-mail us with the arXiv number of the paper.

Proofs

A PDF file of the galley proof will be sent to the corresponding author for correction. If requested, a paper copy will also be sent to the author.

Offprints

Authors of single-authored papers will receive 30 offprints. (Authors of papers with one co-author will receive 15 offprints, and authors of papers with two or more co-authors will receive 10 offprints.) Extra offprints may be purchased through the editorial office.

Subscriptions

The price for a print and online subscription, or an online-only subscription, is $390 per year for institutions. In addition, there is a postage surcharge of $40 for print subscriptions that are mailed to countries outside of the United States. Individuals interested in subscriptions for their own personal use should contact the publisher at the address below. Subscriptions and changes of address should be sent to Mathematical Sciences Publishers, Department of Mathematics, University of California, Berkeley, CA 94720-3840 (e-mail: contact@mathscipub.org; phone: 1-510-643-8638; fax: 1-510-295-2608). (Checks should be made payable to "Mathematical Sciences Publishers".)

Back issues and reprints

Orders for missing issues and back issues should be sent to Mathematical Sciences Publishers at the above address. Claims for missing issues must be made within 12 months of the publication date. Online versions of papers published five or more years ago are available through JSTOR (www.jstor.org).

Microfilm

Beginning with Volume 1, microfilm may be purchased from NA Publishing, Inc., 4750 Venture Drive, Suite 400, PO Box 998, Ann Arbor, MI 48106-0998; phone: 1-800-420-6272 or 734-302-6500; email: info@napubco.com, website: www.napubco.com/contact.html.

Printed in U.S.A. by Sheridan Printing Company, Inc., Alpha, NJ

TABLE OF CONTENTS