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Compactifications of smooth families and of moduli spaces of polarized manifolds

By ECKART VIEHWEG

Abstract

Let M_h be the moduli scheme of canonically polarized manifolds with Hilbert polynomial h. We construct for $\nu \geq 2$ with $h(\nu) > 0$ a projective compactification $\overline{M}_h$ of the reduced moduli scheme $(M_h)_{\mathrm{red}}$ such that the ample invertible sheaf λ_ν, corresponding to $\det(f_*\omega^\nu_{X_0/Y_0})$ on the moduli stack, has a natural extension $\overline{\lambda}_\nu \in \mathrm{Pic}(\overline{M}_h)_{\mathbb{Q}}$. A similar result is shown for moduli of polarized minimal models of Kodaira dimension zero. In both cases "natural" means that the pullback of $\overline{\lambda}_\nu$ to a curve $\varphi : C \to \overline{M}_h$, induced by a family $f_0 : X_0 \to C_0 = \varphi^{-1}(M_h)$, is isomorphic to $\det(f_*\omega^\nu_{X/C})$ whenever f_0 extends to a semistable model $f : X \to C$.

Besides of the weak semistable reduction of Abramovich-Karu and the extension theorem of Gabber there are new tools, hopefully of interest by themselves. In particular we will need a theorem on the flattening of multiplier sheaves in families, on their compatibility with pullbacks and on base change for their direct images, twisted by certain semiample sheaves.

Contents

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Introduction

Let $h_0 : S_0 \to C_0$ be a smooth family of complex projective manifolds over a nonsingular curve C_0. Replacing C_0 by a finite covering $\widehat{C}_0$ one can extend the family $\hat{h}_0 : \widehat{S}_0 = S_0 \times_{C_0} \widehat{C}_0 \to \widehat{C}_0$ to a semistable family $\hat{h} : \widehat{S} \to \widehat{C}$. The model $\widehat{S}$ is not unique, but the sheaves $\mathcal{F}^{(\nu)}_{\widehat{C}} = \hat{h}_* \omega^{\nu}_{\widehat{S}/\widehat{C}}$ are independent of $\widehat{S}$ and compatible with further pullback. For a smooth family $f_0 : X_0 \to Y_0$ of n-folds over a higher dimensional base the existence of flat semistable extension over a compactification Y of Y_0 is not known, not even the existence of a flat Cohen-Macaulay family, except if the fibers are curves or surfaces of general type.

It is the aim of this article to perform such constructions on the sheaf level. So we fix a finite set I of positive integers, and construct a finite covering W_0 of Y_0, and a compactification W of W_0 such that for $\nu \in I$ the pullbacks of $f_{0*}\omega^{\nu}_{X_0/Y_0}$ extend to natural locally free and numerically effective (nef) sheaves $\mathcal{F}^{(\nu)}_{\widehat{Y}}$. The word "natural" means, that one has compatibility with pullback for certain morphisms $\widehat{Y} \to W$. The precise statements are:

THEOREM 1. *Let $f_0 : X_0 \to Y_0$ be a smooth projective morphism of quasi-projective reduced schemes such that ω_F is semiample for all fibers F of f_0. Let I be a finite set of positive integers. Then there exists a projective compactification Y of Y_0, a finite covering $\phi : W \to Y$ with a splitting trace map, and for $\nu \in I$ a locally free sheaf $\mathcal{F}^{(\nu)}_W$ on W with:*

(i) *For $W_0 = \phi^{-1}(Y_0)$ and $\phi_0 = \phi|_{W_0}$ one has $\phi_0^* f_{0*}\omega^{\nu}_{X_0/Y_0} = \mathcal{F}^{(\nu)}_W|_{W_0}$.*

(ii) *Let $\xi : \widehat{Y} \to W$ be a morphism from a nonsingular projective variety $\widehat{Y}$ with $\widehat{Y}_0 = \xi^{-1}(W_0)$ dense in $\widehat{Y}$. Assume either that $\widehat{Y}$ is a curve, or that $\widehat{Y} \to W$ is dominant. For some $r \geq 1$ let $X^{(r)}$ be a nonsingular projective model of the r-fold product family $\widehat{X}^r_0 = (X_0 \times_{Y_0} \cdots \times_{Y_0} X_0) \times_{Y_0} \widehat{Y}_0$ which admits a morphism $f^{(r)} : X^{(r)} \to \widehat{Y}$. Then $f^{(r)}_* \omega^{\nu}_{X^{(r)}/\widehat{Y}} = \bigotimes^r \xi^* \mathcal{F}^{(\nu)}_W$.*

The formulation of Theorem 1 is motivated by what is needed to prove positivity properties of direct image sheaves.

THEOREM 2. *Conditions* (i) *and* (ii) *in Theorem* 1 *imply*:

(iii) *The sheaf* $\mathcal{F}_W^{(\nu)}$ *is nef.*

(iv) *Assume that for some* $\eta_1, \dots, \eta_s \in I$ *and for some* $a_1, \dots, a_s \in \mathbb{N}$ *the sheaf* $\bigotimes_{i=1}^{s} \det(\mathcal{F}_W^{(\eta_i)})^{a_i}$ *is ample with respect to* W_0. *Then*, *if* $\nu \geq 2$ *and if* $\mathcal{F}_W^{(\nu)}$ *is nonzero*, *it is ample with respect to* W_0.

In Section 1 we recall the definition of the positivity properties "nef", "ample with respect to W_0" and of "weakly positive over W_0" for locally free sheaves $\mathcal{F}$ on W. Obviously $\mathcal{F}$ is nef if and only if its pullback under a surjective morphism $\varphi : W' \to W$ is nef. For "nef and ample with respect to W_0" the same holds if φ is finite over W_0. "Weakly positive over W_0" is compatible with finite coverings with a splitting trace map, i.e., if $\mathcal{O}_W$ is a direct factor of $\varphi_* \mathcal{O}_{W'}$.

In Section 2 we show that part (iii) of Theorem 2 follows from Theorem 1. Unfortunately, as we will explain in 2.5, the verification of property (iv) is much harder. Here multiplier ideals will enter the scene. Whereas in a neighborhood of a smooth fiber F one can bound the threshold, introduced in 2.1, in terms of invariants of F, a similar result fails close to the boundary. So we need a variant of parts (i) and (ii), allowing certain multiplier sheaves, introduced in Section 9, as well as the Flattening Theorem 6.5 for multiplier ideal sheaves on total spaces of morphisms, and their compatibility with alterations of the base and fiber products. So the proof of part (iv) will only be given at the end of Section 13.

There are two main ingredients which will allow the construction of W and $\mathcal{F}_W^{(\nu)}$ in Section 12. The first one is the Weak Semistable Reduction Theorem [AK00] recalled in Section 4. Roughly speaking it says that a given morphism $f : X \to Y$ between projective varieties, with a smooth general fiber can be flattened over some nonsingular alteration of Y without allowing horrible singularities of the total space. However one pays a price, having to modify the smooth fibers as well. As explained in Section 5 this theorem has some strong consequences for the compatibility of certain sheaves on the total space of a family with base change and products, similar to those stated in part (ii) of Theorem 1.

The second ingredient is Gabber's Extension Theorem, stated (and proved) in [Vie95, §5.1], which we will recall in Section 12.

The comments made in 2.5 and in 5.9 could serve as a "Leitfaden" for the second part of the article. Here we try to indicate why certain constructions contained in Sections 6–11 are needed for the proof of Theorem 2(iv).

From Theorem 2 one finds by Lemma 1.6 that the restriction of $\mathcal{F}_W^{(\nu)}$ to W_0 is weakly positive over W_0. We will show in Section 13 that part (iv), in a slightly modified version, also restricts to W_0. Since $W_0 \to Y_0$ has a splitting trace map, one obtains by Lemma 1.7 the "weak positivity" and "weak stability" for the direct images of powers of the dualizing sheaf, already shown in [Vie95, §6.4].

COROLLARY 3. *Under the assumptions made in Theorem* 1 *one has*:

(a) *The sheaves* $\mathcal{F}_{Y_0}^{(\nu)} = f_{0*}\omega_{X_0/Y_0}^{\nu}$ *are weakly positive over* Y_0.

(b) *Assume that for some positive integers* $\eta_1, \dots, \eta_s$ *and* $a_1, \dots, a_s \in \mathbb{N}$ *the sheaf* $\bigotimes_{i=1}^{s} \det(\mathcal{F}_{Y_0}^{(\eta_i)})^{a_i}$ *is ample. Then for all* $\nu \geq 2$ *the sheaf* $\mathcal{F}_{Y_0}^{(\nu)}$ *is either ample or zero.*

As explained in [Vie95] this is just what is needed for the construction of quasi-projective moduli schemes M_h for families of canonically polarized manifolds with Hilbert polynomial h. At the time [Vie95] was written, the Weak Semistable Reduction Theorem of Abramovich and Karu was not known. So we were only able to use Gabber's Extension Theorem to construct W and $\mathcal{F}_W^{(\nu)}$ for $\nu = 1$, and correspondingly to prove the weak positivity just for $\mathcal{F}_{Y_0}^{(1)}$. A large part of [Vie95] is needed to reduce the proof of Corollary 3 to this case. Having W and $\mathcal{F}_W^{(\nu)}$ for all ν clarifies this part considerably. We could not resist recalling in Section 3 how to apply Corollary 3 to construct M_h together with an ample invertible sheaf.

There are several ways. One can first construct the moduli scheme as an algebraic space, and then show the existence of an ample sheaf. Or one can use geometric invariant theory, and stability criteria. Guided by personal taste, we restrict ourselves to the second method in Section 3, applying the Stability Criterion [Vie95, Th. 4.25].

If one uses instead the first method, starting from the existence of M_h as an algebraic space, it has been shown in [Vie95] how to deduce from Corollary 3 the quasi-projectivity of the normalization of M_h. The starting point is Seshadri's Theorem on the elimination of finite isotropies (see [Vie95, Th. 3.49]) or the direct construction in [Kol90]. Both allow us to get a universal family $f_0 : X_0 \to Y_0$ over some reduced covering $\gamma_0 : Y_0 \to M_h$. Then one can try to apply arguments similar to those used in the proof of Lemma 1.9 and in Section 14 to get the quasi-projectivity of $(M_h)_{\mathrm{red}}$, hence of M_h itself.

As stated in the report on [ST04] in *Mathematical Reviews*, J. Kollár pointed out that the proof of the quasi-projectivity of the algebraic moduli space M_h seems to contain a gap, even in the canonically polarized case. The authors claim without any justification that for a certain line bundle, which descends to a quotient of the Hilbert scheme, the curvature current descends as well. In a more recent attempt to handle moduli of canonically polarized manifolds Tsuji avoids this point by claiming that a certain determinant sheaf extends to some compactification in a natural way, again without giving an argument. Suitable variants of Theorems 1 and 2 could allow one to fill those gaps, and to get another proof of the quasi-projectivity of M_h, replacing the GIT-approach in Section 3 by the analytic methods presented in the second part of [ST04].

Either one of the constructions of moduli schemes mentioned above gives an explicit ample sheaf on M_h.

Notation 4. For $\nu,\ p \in \mathbb{N}$ we write $\lambda_{0,\nu}^{(p)}$ for an invertible sheaf satisfying

($\ast$) If a morphism $\varphi : Y_0 \to M_h$ factors through the moduli stack, hence if it is induced by a family $f_0 : X_0 \to Y_0$, one has $\varphi^* \lambda_{0,\nu}^{(p)} = \det(f_{0*}\omega_{X_0/Y_0}^{\nu})^p$.

Of course, $\lambda_{0,\nu}^{(p)}$ can only exist if $H^0(F,\omega_F^\nu)\neq 0$ for all manifolds F parametrized by M_h, for example in the canonically polarized case if $\nu\geq 2$ and $h(\nu)\neq 0$. As we will recall in Addendum 3.1 for those values of ν the sheaf $\lambda_{0,\nu}^{(p)}$ is ample.

As indicated in the title of this article we want to construct compactification of moduli schemes M_h. Assume for a moment that M_h is reduced and a fine moduli scheme, hence that there is a universal family $\mathcal{X}_0\to M_h$ with $\mathcal{X}_0$ reduced. Here one may choose $p=1$ and applying Theorems 1 and 2, and Lemma 1.9 it is easy to see that $\lambda_{0,\nu}^{(1)}=\det(g_{0*}\omega_{\mathcal{X}/M_h}^\nu)$ extends to an invertible sheaf $\lambda_\nu^{(1)}$ on $\overline{M}_h$, which is nef and ample with respect to M_h for $\nu\geq 2$. It is compatible with the restriction to curves, provided the induced family has a smooth general fiber and a semistable model. In Section 14 we will use a variant of Theorems 1 and 2 to obtain a similar result for coarse moduli schemes, using the Seshadri-Kollár construction mentioned above.

THEOREM 5. *Let M_h be the coarse moduli scheme of canonically polarized manifolds with Hilbert polynomial h. Given a finite set I of integers $\nu\geq 2$ with $h(\nu)>0$, one finds a projective compactification $\overline{M}_h$ of $(M_h)_{\mathrm{red}}$ and for $\nu\in I$ and some $p>0$ invertible sheaves $\lambda_\nu^{(p)}$ on $\overline{M}_h$ with*:

(1) $\lambda_\nu^{(p)}$ *is nef, and it is ample with respect to $(M_h)_{\mathrm{red}}$.*

(2) *The restrictions of $\lambda_\nu^{(p)}$ and of $\lambda_{0,\nu}^{(p)}$ to $(M_h)_{\mathrm{red}}$ coincide.*

(3) *Let $\varsigma:C\to\overline{M}_h$ be a morphism from a nonsingular curve with $C_0=\varsigma^{-1}(M_h)$ dense in C and such that $C_0\to M_h$ is induced by a family $h_0:S_0\to C_0$. If h_0 extends to a semistable family $h:S\to C$, then $\varsigma^*\lambda_\nu^{(p)}=\det(h_*\omega_{S/C}^\nu)^p$.*

It would be nicer to have an extension of $\lambda_{0,\nu}^{(p)}$ to an invertible sheaf $\lambda_\nu^{(p)}$ on a compactification of M_h itself, but we were not able to get hold of it. On the other hand, since the compatibility condition in part (3) only sees the reduced structure of M_h, such an extension would not really be of help for possible applications of Theorem 5.

The compactification $\overline{M}_h$ depends on the set I and the points in $\overline{M}_h\setminus M_h$ have no interpretation as moduli of geometric objects. Shortly after a first version of this article was submitted there was a "quantum leap" in the minimal model program due to [BCHM10] (see also [Siu06]). By [Kar00] the existence of a minimal model in dimension $\deg(h)+1$ should allow the construction of a compactification $\overline{M}_h$ which has an interpretation as a moduli scheme. Unfortunately at the present time a proper explanation of this implication is not in the literature and there is not even a conjectural picture explaining how to construct geometrically meaningful compactifications of moduli of polarized manifolds.

Only part of what was described up to now carries over to families or moduli of smooth minimal models with an arbitrary polarization. Theorems 1 and 2 apply, but even if $f_0:X_0\to Y_0$ is the universal family over a fine moduli scheme, the sheaf $\det(\mathcal{F}^{(\nu)}_{Y_0})$ might not be ample. Theorem 12.12 is a generalization of Theorem 1 for direct images of the form $f_{0*}(\omega^{\nu}_{X_0/Y_0} \otimes \mathcal{L}^{\mu}_0)$ with $\mathcal{L}_0$ semiample over Y_0. The corresponding variant of Corollary 3 is stated in Lemma 3.2 and we will sketch how to use it to show the existence of quasi-projective moduli schemes in the second half of Section 3. However we are not able to generalize Theorem 2. Thus, we are not able to apply Lemma 1.9 which will be essential for the proof of Theorem 5 in Section 14 and we are not able to extend the natural ample sheaf to some compactification.

The situation is better for the moduli functor $\mathfrak{M}_h$ of polarized minimal manifolds $(F, \mathcal{H})$ of Kodaira dimension zero and with Hilbert polynomial h. Replacing the corresponding moduli scheme M_h by a connected component, we may assume that for some $\nu > 0$ and for all $(F, \mathcal{H}) \in \mathfrak{M}_h$ one has $\omega^{\nu}_F = \mathcal{O}_F$. Notation 4 carries over and there exists a sheaf $\lambda^{(p)}_{0,\nu}$ with the property $(*)$ or equivalently with $f_0^* \varphi^* \lambda^{(p)}_{0,\nu} = \omega^{p\cdot\nu}_{X_0/Y_0}$. As we will recall in Addendum 3.4 the sheaf $\lambda^{(p)}_{0,\nu}$ is again ample. In fact, the natural ample invertible sheaf first looks quite different, but an easy calculation identifies it with some power of $\lambda^{(p)}_{0,\nu}$. This calculation only extends to boundary points if the polarization is *saturated*, as explained in Remark 8.1. This together with the need to consider multiplier ideals makes the notation even more unpleasant, but the general line of arguments remain as in the canonically polarized case.

THEOREM 6. *Let M_h be the coarse moduli scheme of polarized manifolds $(F, \mathcal{H})$ with $\omega^{\nu}_F = \mathcal{O}_F$, for some $\nu > 0$ and with Hilbert polynomial $h(\mu) = \chi(\mathcal{H}^{\mu})$. Then there exists a projective compactification $\overline{M}_h$ of $(M_h)_{\mathrm{red}}$ and for some $p > 0$ an invertible sheaf $\lambda^{(p)}_{\nu}$ on $\overline{M}_h$ with*:

(1) *$\lambda^{(p)}_{\nu}$ is nef and ample with respect to $(M_h)_{\mathrm{red}}$.*

(2) *Let Y_0 be reduced and $\varphi : Y_0 \to M_h$ induced by a family $f_0 : X_0 \to Y_0$ in $\mathfrak{M}_h(Y_0)$. Then $\varphi^* \lambda^{(p)}_{\nu} = f_{0*} \omega^{p\cdot\nu}_{X_0/Y_0}$.*

(3) *Let $\varsigma : C \to \overline{M}_h$ be a morphism from a nonsingular curve C, with $C_0 = \varsigma^{-1}(M_h)$ dense in C and such that $C_0 \to M_h$ is induced by a family $h_0 : S_0 \to C_0$. If h_0 extends to a semistable family $h : S \to C$, then $\varsigma^* \lambda^{(p)}_{\nu} = h_* \omega^{p\cdot\nu}_{S/C}$.*

Consider, for example, the moduli schemes $\mathcal{A}_g$ of g-dimensional polarized Abelian varieties. To stay close to the usual notation we write $\overline{\mathcal{A}}_g$ for the compactification in Theorem 6 in this case. One may assume that there is a morphism $\overline{\mathcal{A}}_g \to \mathcal{A}^*_g$ to the Baily-Borel compactification $\mathcal{A}^*_g$. With $\nu = 1$ the ample sheaf $\lambda^{(p)}_{0,1}$ extends to an ample sheaf on $\mathcal{A}^*_g$. So the sheaf $\lambda^{(p)}_1$ in Theorem 6 is semiample and $\mathcal{A}^*_g$ is the image under the morphism defined by a high power of $\lambda^{(p)}_1$.

In general we are not able to verify in Theorem 6 the semiampleness of $\lambda^{(p)}_{\nu}$. One of the obstacles is the missing geometric interpretation of the boundary points as moduli of certain varieties. So the theorem can only be seen as a very weak substitute for the Baily-Borel compactification. Nevertheless, since $\lambda_\nu^{(p)}$ is nef and ample with respect to M_h the degree of $\varsigma^*\lambda_\nu^{(p)} = h_*\omega_{S/C}^{p\cdot\nu}$ can serve as a height function for curves in the moduli stack. An upper bound for this height function in terms of the genus of C and $\#(C \setminus C_0)$ was given in [VZ02], and it played its role in the proof of the Brody hyperbolicity of the moduli stack of canonically polarized manifolds in [VZ03]. In both articles we had to use unpleasant ad hoc arguments to control the positivity along the boundary of the moduli schemes and some of those arguments were precursors of methods used here.

A second motivation for this article was the hope that compactifications could help to generalize the uniform boundedness, obtained in [Cap02] for families of curves, to families of higher dimensional manifolds. The missing point was the construction of moduli of morphisms from curves to the corresponding moduli stacks, as was done in [AV02] for compact moduli problems. In between, this has been achieved in [KL] for families of canonically polarized manifolds, using Theorem 5. It is likely that referring to Theorem 6 instead, their methods allow one to handle polarized manifolds of Kodaira dimension zero, as well.

I was invited to lecture on the construction of moduli schemes at the workshop "Compact moduli spaces and birational geometry" (American Institute of Mathematics, 2004), an occasion to reconsider some of the constructions in [Vie95] in view of the Weak Semistable Reduction Theorem. A preliminary version of this article, handling just the canonically polarized case, was written during a visit to the I.H.E.S., Bures sur Yvette, September and October 2005. I thank the members of the Institute for their hospitality.

I am grateful for the referee's suggestions on how to improve the presentation of the results and the methods leading to their proofs.

Conventions 7. All schemes and varieties will be defined over the field $\mathbb{C}$ of complex numbers (or over an algebraically closed field K of characteristic zero). A quasi-projective variety Y is a reduced quasi-projective scheme. In particular we do not require Y to be irreducible or connected. A locally free sheaf on Y will always be locally free of constant finite rank and a finite covering will denote a finite surjective morphism.

- If Π is an effective divisor and $\iota : Y \setminus \Pi_{\mathrm{red}} \to Y$, then $\mathcal{O}(* \cdot \Pi) = \iota_*\mathcal{O}_{Y \setminus \Pi_{\mathrm{red}}}$.
- An *alteration* $\Psi : \widehat{Y} \to Y$ is a proper, surjective, generically finite morphism between quasi-projective varieties.
- An alteration Ψ is called a *modification* if it is birational. If $U \subset Y$ is an open subscheme with $\Psi|_{\Psi^{-1}(U)}$ an isomorphism, we say that the center of Ψ lies in $Y \setminus U$.
- For a nonsingular (or normal) alteration or modification we require in addition that $\widehat{Y}$ be nonsingular (or normal).
- A modification Ψ will be called a *desingularization* (*or resolution of singularities*), if $\widehat{Y}$ is nonsingular and if the center of Ψ lies in the singular locus of Y.

- Given a Cartier divisor D on Y we call Ψ a *log-resolution* (for D) if it is a nonsingular modification and if $\Psi^* D$ is a normal crossing divisor.
- If $f_0 : X_0 \to Y_0$ is a projective morphism, we call $f : X \to Y$ a *projective model* of f_0 if X and Y are projective, Y_0 open in Y and $X_0 \cong f^{-1}(Y_0)$ over Y_0.
- If $f : X \to Y$ is a projective morphism, we call $f_0 : X_0 \to Y_0$ *the smooth part of* f if $Y_0 \subset Y$ is the largest open subscheme with $X_0 = f^{-1}(Y_0) \to Y_0$ smooth. In particular, if X and Y are nonsingular, or if f is a mild morphism, as defined in 4.1, Y_0 is dense in Y.

Finally the numbering of displayed formulas follows the numbering of the theorems, lemmas etc. Hence (2.1.1) is after Definition 2.1 and before Lemma 2.2.

Note to the reader. Eckart Viehweg died on January 30, 2010. He had not yet received the proofs of this article from the journal. We, Dan Abramovich, Hélène Esnault and Sándor Kovács, corrected them. We apologize if we introduced any inaccuracies.

1. Numerically effective and weakly positive sheaves

Definition 1.1. Let $\mathcal{G}$ be a locally free sheaf on a projective reduced variety W. Then $\mathcal{G}$ is *numerically effective* (nef) if for all morphisms $\tau : C \to W$ from a projective curve C and for all invertible quotients $\tau^*\mathcal{G} \to \mathcal{L}$ one has $\deg(\mathcal{L}) \geq 0$.

Definition 1.2. Let $\mathcal{G}$ be a locally free sheaf on a quasi-projective reduced variety W and let $W_0 \subset W$ be an open dense subvariety. Let $\mathcal{H}$ be an ample invertible sheaf on W.

(a) $\mathcal{G}$ is *globally generated over* W_0 if the natural morphism $H^0(W, \mathcal{G}) \otimes \mathcal{O}_W \to \mathcal{G}$ is surjective over W_0.

(b) $\mathcal{G}$ is *weakly positive over* W_0 if for all $\alpha > 0$ there exists some $\beta > 0$ such that $S^{\alpha\cdot\beta}(\mathcal{G}) \otimes \mathcal{H}^{\beta}$ is globally generated over W_0.

(c) $\mathcal{G}$ is *ample with respect to* W_0 if for some $\eta > 0$ the sheaf $S^{\eta}(\mathcal{G}) \otimes \mathcal{H}^{-1}$ is weakly positive over W_0, or equivalently, if for some $\eta' > 0$ one has a morphism $\bigoplus \mathcal{H} \to S^{\eta'}(\mathcal{G})$, which is surjective over W_0.

It is obvious, that "nef" is related to "weakly positive" and that it is compatible with pullbacks.

LEMMA 1.3. *For a locally free sheaf* $\mathcal{G}$ *on a projective variety* W *the following conditions are equivalent*:

(1) $\mathcal{G}$ *is nef.*

(2) $\mathcal{G}$ *is weakly positive over* W.

(3) *There exists a projective surjective morphism* $\xi : \widehat{Y} \to W$ *with* $\xi^*\mathcal{G}$ *nef.*

(4) *The sheaf* $\mathcal{O}_{\mathbb{P}(\mathcal{G})}(1)$ *on* $\mathbb{P}(\mathcal{G})$ *is nef.*

(5) *There exists some integer* $\mu > 0$ *such that for all projective surjective morphisms* $\xi : \widehat{Y} \to W$ *and for all ample invertible sheaves* $\widehat{\mathcal{H}}$ *on* $\widehat{Y}$ *the sheaf* $\widehat{\mathcal{H}}^{\mu} \otimes \xi^*\mathcal{G}$ *is nef.*

Remark 1.4. As we will see in the proof it is sufficient in Lemma 1.3(5) to require the existence of a tower of finite maps $\xi : \widehat{Y} \to W$, such that for each $N > 0$ there is some $\xi : \widehat{Y} \to W$ with $\xi^*\mathcal{H}$ the N-th power of an invertible sheaf. Such coverings exist by [Vie95, Lemma 2.1], and one may even assume that they have splitting trace maps.

Proof. The equivalence of the first four conditions has been shown in [Vie95, Prop. 2.9], and of course they imply (5). The equivalence of (5) and (2) is a special case of [Vie95, Lemma 2.15, 3)]. Nevertheless let us give the argument. Let $\mathcal{H}$ be ample and invertible on W. Let $\pi : C \to W$ be a curve and $\mathcal{N}$ an invertible quotient of $\pi^*\mathcal{G}$ of degree d. By [Vie95, Lemma 2.1] for all N there exist a finite covering $\xi : \widehat{Y} \to W$ and an invertible sheaf $\widehat{\mathcal{H}}$ with $\xi^*\mathcal{H} = \widehat{\mathcal{H}}^N$. By assumption $\widehat{\mathcal{H}}^{\mu} \otimes \xi^*\mathcal{G}$ is nef, hence if $\tau : \widehat{C} \to C$ is a finite covering such that π lifts to $\pi' : \widehat{C} \to \widehat{Y}$ one has

$$0 \leq \deg(\tau) \cdot d + \mu \cdot \deg(\pi'^*\widehat{\mathcal{H}}) = \deg(\tau) \cdot \Big(d + \frac{\mu}{N} \cdot \deg(\pi^*\mathcal{H})\Big).$$

This being true for all N, the degree d cannot be negative. □

Obviously the notion "nef" is compatible with tensor products, direct sums, symmetric products and wedge products. For the corresponding properties for weakly positive sheaves, one has to work a bit more, or to refer to [Vie95, §2.3].

LEMMA 1.5. *Let* $\mathcal{F}$ *and* $\mathcal{G}$ *be locally free sheaves on* W.

(1) *Let* $\mathcal{L}$ *be an invertible sheaf. Assume that for all* $\alpha > 0$ *there exists some* $\beta > 0$ *such that* $S^{\alpha\cdot\beta}(\mathcal{G}) \otimes \mathcal{L}^{\beta}$ *is globally generated over* W_0. *Then* $\mathcal{G}$ *is weakly positive over* W_0. *In particular Definition* 1.2(b), *is independent of* $\mathcal{H}$.

(2) *If* $\mathcal{G}$ *is weakly positive over* W_0 *and if* $\xi : \widehat{Y} \to W$ *is a dominant morphism, then* $\xi^*\mathcal{G}$ *is weakly positive over* $\xi^{-1}(W_0)$.

(3) *If* $\mathcal{G}$ *is weakly positive over* W_0 *and if* $\mathcal{G} \to \mathcal{F}$ *is a morphism, surjective over* W_0, *then* $\mathcal{F}$ *is weakly positive over* W_0.

(4) *If* $\mathcal{F}$ *and* $\mathcal{G}$ *are weakly positive over* W_0, *the same holds for* $\mathcal{F} \oplus \mathcal{G}$, *for* $\mathcal{F} \otimes \mathcal{G}$, *for* $S^{\nu}(\mathcal{G})$ *and for* $\bigwedge^{\mu}(\mathcal{G})$, *where* ν *and* $\mu \leq \mathrm{rk}(\mathcal{G})$ *are natural numbers.*

The equivalence of (1) and (3) in Lemma 1.3 does not carry over to "weakly positive over W_0"; one needs in addition that the morphism is finite with a splitting trace map.

LEMMA 1.6. *For a locally free sheaf* $\mathcal{G}$ *on* W *and an open and dense subscheme* $W_0 \subset W$ *the following conditions are equivalent*:

(1) $\mathcal{G}$ *is weakly positive over* W_0.

(2) $\bigotimes^r \mathcal{G}$ *is weakly positive over* W_0 *for some* $r > 0$.

(3) *$S^r\mathcal{G}$ is weakly positive over W_0 for some $r>0$.*

(4) *There exists an invertible sheaf $\mathcal{A}$ on W such that $\mathcal{A}\otimes S^r(\mathcal{G})$ is weakly positive over W_0, for all $r>0$.*

(5) *For all (or some) ample invertible sheaves $\mathcal{A}$ on W and for all $r>0$ the sheaf $\mathcal{A}\otimes S^r(\mathcal{G})$ is ample with respect to W_0.*

(6) *There exists an alteration $\phi:\widetilde{W}\to W$ such that $\phi^*\mathcal{G}$ is weakly positive over $\phi^{-1}(W_0)$, and such that for $\widetilde{W}_0=\phi^{-1}(W_0)$ the restriction $\phi_0:\widetilde{W}_0\to W_0$ is finite with a splitting trace map (i.e., with a splitting of $\mathcal{O}_{W_0}\to\phi_{0*}\mathcal{O}_{\widetilde{W}_0}$).*

(7) *There exists a constant $\mu>0$ such that for all $\xi:\widehat{Y}\to W$ and for all ample invertible sheaves $\mathcal{H}'$ on $\widehat{Y}$ the sheaf $\mathcal{H}'^{\mu}\otimes\xi^*\mathcal{G}$ is weakly positive over $\xi^{-1}(W_0)$.*

Remark 1.4 applies to Lemma 1.6(7) as well, if one assumes that for all $\xi:\widehat{Y}\to W$ the trace map splits.

Proof. The equivalence of the first three conditions has been shown in [Vie95, Lemma 2.16]. The equivalence of (1), (4) and (5) follows directly from the definition, and the equivalence of (1), (6) and (7) is in [Vie95, Lemma 2.15]. □

Let us consider next the condition "ample with respect to W_0".

LEMMA 1.7. *Let $\mathcal{G}$ and $\mathcal{F}$ be locally free sheaves on W and let $W_0\subset W$ be open and dense.*

(1) *$\mathcal{G}$ is ample with respect to W_0 if and only if there exists an ample invertible sheaf $\mathcal{H}$ on W and a finite morphism $\sigma:W'\to W$ with a splitting trace map, and with $\sigma^*\mathcal{H}=\mathcal{H}'^{\eta}$, for some positive integer η, such that $\sigma^*(\mathcal{G})\otimes\mathcal{H}'^{-1}$ is weakly positive over $\sigma^{-1}(W_0)$.*

(2) *If $\mathcal{F}$ is ample with respect to W_0 and if $\mathcal{G}$ is weakly positive over W_0, then $\mathcal{F}\otimes\mathcal{G}$ is ample with respect to W_0. In particular, Definition* 1.2(c) *is independent of the ample invertible sheaf $\mathcal{H}$.*

(3) *If $\mathcal{F}$ is invertible and ample with respect to W_0, and if $S^{\eta}(\mathcal{G})\otimes\mathcal{F}^{-1}$ is weakly positive over W_0, then $\mathcal{G}$ is ample over W_0.*

(4) *The following conditions are equivalent*:
   (a) *$\mathcal{G}$ is ample with respect to W_0.*
   (b) *There exists an alteration $\phi:\widetilde{W}\to W$ with $\widetilde{W}_0=\phi^{-1}(W_0)\to W_0$ finite and with a splitting trace map, such that $\phi^*\mathcal{G}$ is ample with respect to $\widetilde{W}_0$.*

(5) *If $\mathcal{G}$ is ample with respect to W_0 and if $\mathcal{G}\to\mathcal{F}$ is a morphism, surjective over W_0, then $\mathcal{F}$ is ample with respect to W_0.*

(6) *If $\mathcal{F}$ and $\mathcal{G}$ are both ample with respect to W_0, then the same holds for $\mathcal{F}\oplus\mathcal{G}$, for $S^{\nu}(\mathcal{G})$ and for $\bigwedge^{\mu}(\mathcal{G})$, where ν and $\mu\le\mathrm{rk}(\mathcal{G})$ are natural numbers.*

(7) *If $\mathcal{F}$ is an invertible sheaf, then $\mathcal{F}$ is ample with respect to W_0, if and only if for some $\beta > 0$ the sheaf $\mathcal{F}^\beta$ is globally generated over W_0 and the induced morphism $\tau : W_0 \to \mathbb{P}(H^0(W, \mathcal{F}^\beta))$ is finite over its image.*

Proof. If in (1) the sheaf $\mathcal{G}$ is ample with respect to W_0 there is some η such that $S^\eta(\mathcal{G}) \otimes \mathcal{H}^{-1}$ is weakly positive. By [Vie95, Lemma 2.1] there is a covering $\sigma : W' \to W$ with a splitting trace map, such that $\sigma^*\mathcal{H}$ is the η-th power of an invertible sheaf $\mathcal{H}'$, necessarily ample. Then $\sigma^*(S^\eta(\mathcal{G}) \otimes \mathcal{H}^{-1})$ is weakly positive over $\sigma^{-1}(W_0)$, hence by Lemma 1.6 the sheaf $\sigma^*\mathcal{G} \otimes \mathcal{H}'^{-1}$ is as well. On the other hand, the weak positivity of $\sigma^*(\mathcal{G}) \otimes \mathcal{H}'^{-1}$ in (1) implies that $\sigma^* S^\eta(\mathcal{G}) \otimes \sigma^*\mathcal{H}^{-1}$ is weakly positive over $\sigma^{-1}(W_0)$, hence $S^\eta(\mathcal{G}) \otimes \mathcal{H}^{-1}$ is weakly positive over W_0, again by Lemma 1.6.

For (2) one can use (1), assume that $\mathcal{G} \otimes \mathcal{H}^{-1}$ is weakly positive, and then apply Lemma 1.5(4). In the same way one obtains (6). Part (3) is a special case of (2) and (5) follows from Lemma 1.5(3).

Let us next verify (7). If $\mathcal{F}$ is ample with respect to W_0, one has for a very ample invertible sheaf $\mathcal{H}$ on W and for some η', a morphism $\bigoplus^s \mathcal{H} \to \mathcal{F}^{\eta'}$, surjective over W_0. Let V denote the image of $H^0(W, \bigoplus^s \mathcal{H})$ in $H^0(W, \mathcal{F}^{\eta'})$. Then $\mathcal{F}^{\eta'}$ is generated by V over W_0 and one has embeddings

$$W \longrightarrow \bigtimes^s \mathbb{P}(H^0(W, \mathcal{H})) \longrightarrow \mathbb{P}\Big(\bigotimes^s H^0(W, \mathcal{H})\Big).$$

The restriction of the composite to W_0 factors through

$$W_0 \to \mathbb{P}(V) \subset \mathbb{P}\Big(\bigotimes^s H^0(W, \mathcal{H})\Big),$$

and $W_0 \to \mathbb{P}(V)$, hence $W_0 \to \mathbb{P}(H^0(W, \mathcal{F}^{\eta'}))$ are embeddings.

If on the other hand $\mathcal{F}^\beta$ is globally generated over W_0 and if

$$\tau : W_0 \to \mathbb{P} = \mathbb{P}\big(H^0\big(W, \mathcal{F}^\beta\big)\big)$$

is finite over its image, consider a blowing up $\phi : \widetilde{W} \to W$ with centers outside of W_0 such that τ extends to a morphism $\tau' : W \to \mathbb{P}$. We may choose ϕ such that for some effective exceptional divisor E the sheaf $\mathcal{O}_{\widetilde{W}}(-E)$ is τ'-ample. For α sufficiently large $\mathcal{A} = \mathcal{O}_{\widetilde{W}}(-E) \otimes \tau'^*\mathcal{O}_{\mathbb{P}}(\alpha)$ will be ample. Replacing E and α by some multiple, one may assume that for a given ample sheaf $\mathcal{H}$ on W the sheaf $\phi^*\mathcal{H}^{-1} \otimes \mathcal{A}$ is globally generated, hence nef. Since one has an inclusion $\mathcal{A} \to \phi^*\mathcal{F}^{\eta' \cdot \alpha}$, which is an isomorphism over $\phi^{-1}(W_0)$, the sheaf $\phi^*\mathcal{F}^{\eta' \cdot \alpha} \otimes \mathcal{H}^{-1}$ is weakly positive over $\phi^{-1}(W_0)$, and by Lemma 1.6 one obtains the weak positivity of $\mathcal{F}^{\eta' \cdot \alpha} \otimes \mathcal{H}^{-1}$.

For (4) we use (7). Consider in (4a), an ample invertible sheaf $\mathcal{F}$ on W. Obviously the condition (7) holds for $\phi^*\mathcal{F}$; hence this sheaf is again ample with respect to $\phi^{-1}(W_0)$. If $\mathcal{G}$ is ample with respect to W_0, by definition $S^\nu(\mathcal{G}) \otimes \mathcal{F}^{-1}$

is weakly positive over W_0. Then by Lemma 1.7 (6) the sheaf $\phi^* S^\nu(\mathcal{G}) \otimes \phi^* \mathcal{F}^{-1}$ is weakly positive over $\phi^{-1}(W_0)$ and (4b), follows from (3).

So assume that the condition (b) in (4) holds. Let $\mathcal{H}$ and $\mathcal{A}$ be ample invertible sheaves on W and Y. Then $\mathcal{A} \otimes \phi^*\mathcal{H}$ is ample. By definition we find some β such that $S^\beta(\phi^*\mathcal{G}) \otimes \mathcal{A}^{-1} \otimes \phi^*\mathcal{H}^{-1}$ is weakly positive over Y_0. Then $S^\beta(\phi^*\mathcal{G}) \otimes \phi^*\mathcal{H}^{-1}$ has the same property, and by Lemma 1.6, $S^\beta(\mathcal{G}) \otimes \mathcal{H}^{-1}$ is weakly positive over W_0. □

LEMMA 1.8. *A locally free sheaf $\mathcal{G}$ on W is ample with respect to W_0 if and only if on the projective bundle $\pi : \mathbb{P}(\mathcal{G}) \to W$ the sheaf $\mathcal{O}_{\mathbb{P}(\mathcal{G})}(1)$ is ample with respect to $\mathbb{P}_0 = \pi^{-1}(W_0)$.*

Proof. If $\mathcal{G}$ is ample with respect to W_0 choose a very ample invertible sheaf $\mathcal{H}$ on W and for some $\eta' > 0$ the morphism

$$\bigoplus^s \mathcal{H} \longrightarrow S^{\eta'}(\mathcal{G}) = \pi_* \mathcal{O}_{\mathbb{P}(\mathcal{G})}(\eta'),$$

surjective over W_0. The composite

$$\bigoplus^s \pi^*\mathcal{H} \longrightarrow S^{\eta'}(\pi^*\mathcal{G}) \longrightarrow \mathcal{O}_{\mathbb{P}(\mathcal{G})}(\eta')$$

induces a rational map $\iota : \mathbb{P}(\mathcal{G}) \to \mathbb{P}^{s-1}$, whose restriction to $\mathbb{P}_0 = \pi^{-1}(W_0)$ is an embedding, and $\mathcal{O}_{\mathbb{P}(\mathcal{G})}(\eta')$ is globally generated over $\mathbb{P}_0$. So by Lemma 1.7(7) $\mathcal{O}_{\mathbb{P}(\mathcal{G})}(1)$ is ample with respect to $\mathbb{P}_0$.

Assume now that $\mathcal{O}_{\mathbb{P}(\mathcal{G})}(1)$ is ample with respect to $\mathbb{P}_0$. Choose ample invertible sheaves $\mathcal{H}$ on W and $\mathcal{A}$ on $\mathbb{P}(\mathcal{G})$ such that $\pi^*\mathcal{H}^{-1} \otimes \mathcal{A}$ is globally generated. Then for some η' and for all $\alpha > 0$ one has morphisms

$$\bigoplus \pi^*\mathcal{H}^\alpha \xrightarrow{\Psi} \bigoplus \mathcal{A}^\alpha \xrightarrow{\Phi} \mathcal{O}_{\mathbb{P}(\mathcal{G})}(\eta' \cdot \alpha)$$

with Ψ surjective and Φ surjective over $\mathbb{P}_0$. For α sufficiently large, this defines a rational map $\mathbb{P}(\mathcal{G}) \to \mathbb{P}^M \times W$ whose restriction to $\mathbb{P}_0$ is an embedding. For $\beta \gg 1$ the multiplication map

$$S^\beta\Big(\bigoplus \mathcal{H}^\alpha\Big) \longrightarrow \pi_* \mathcal{O}_{\mathbb{P}(\mathcal{G})}(\eta' \cdot \beta \cdot \alpha) = S^{\eta' \cdot \beta \cdot \alpha}(\mathcal{G})$$

will be surjective over W_0; hence $\mathcal{G}$ is ample with respect to W_0. □

For the compatibility of "ample with respect to W_0" under arbitrary finite morphisms one either needs that the nonnormal locus of W_0 is proper (see [Vie95, Prop. 2.22] and the references given there) or one has to add the condition "nef":

LEMMA 1.9. *For a locally free sheaf $\mathcal{G}$ on a projective variety W, and for an open dense subscheme $W_0 \subset W$ the following conditions are equivalent*:

(1) *$\mathcal{G}$ is nef and ample with respect to W_0.*

(2) *There exists a finite morphism $\sigma : W' \to W$ such that $\mathcal{G}' = \sigma^*\mathcal{G}$ is nef and ample with respect to $W_0' = \sigma^{-1}(W_0)$.*

(3) *There exists an alteration* $\phi : \widetilde{W} \to W$ *with* $\phi^{-1}(W_0) \to W_0$ *finite, such that* $\phi^*\mathcal{G}$ *is nef and ample with respect to* $\widetilde{W}_0 = \phi^{-1}(W_0)$.

Proof. Of course (1) implies (2) and (2) implies (3). In order to see that (3) implies (2) choose for $\sigma : W' \to W$ the Stein factorization of $\phi : \widetilde{W} \to W$. Since $\widetilde{W} \to W'$ is an isomorphism over W'_0, Lemma 1.7(4) says that $\mathcal{G}' = \sigma^*\mathcal{G}$ is ample with respect to W'_0 if and only if $\phi^*\mathcal{G}$ is ample with respect to $\widetilde{W}_0$. Since by Lemma 1.3 the same holds for nef, one obtains (2).

Note that (2) implies that the sheaf $\mathcal{G}$ is nef, as well as the sheaf $\mathcal{O}_{\mathbb{P}(\mathcal{G})}(1)$ on $\mathbb{P}(\mathcal{G})$. Consider the induced morphism $\sigma' : \mathbb{P}(\mathcal{G}') \to \mathbb{P}(\mathcal{G})$. Lemma 1.8 implies that $\mathcal{O}_{\mathbb{P}(\mathcal{G}')}(1) = \sigma'^*\mathcal{O}_{\mathbb{P}(\mathcal{G})}(1)$ is ample with respect to the preimage of W'_0 if and only if $\mathcal{G}'$ is ample with respect to W'_0, and that the same holds for $\mathcal{G}$ instead of $\mathcal{G}'$.

Now it will be sufficient to consider an invertible nef sheaf $\mathcal{G}$ on W, and a finite covering $\sigma : W' \to W$, such that $\mathcal{G}' = \sigma^*\mathcal{G}$ is ample with respect to W'_0, and we have to show that $\mathcal{G}$ is ample with respect to W_0.

As we have already seen, that (1) implies (2), we may replace W' by any dominating finite covering. In particular we may assume W' to be normal. By [Vie95, Lemma 2.2] the morphism $\sigma : W' \to W$ factors like $W' \xrightarrow{\gamma} W'' \xrightarrow{\rho} W$, where ρ has a splitting trace map, and where γ is birational.

At this point we could also apply Lemma 12.3, replacing W' by a larger normal covering. In any case $\gamma^*\rho^*\mathcal{G}$ is again ample with respect to $\gamma^{-1}\rho^{-1}(W_0)$ and by Lemma 1.7(4) one knows the equivalence of (1) and (2) with W' replaced by W''. Hence it is sufficient to study $V \to W''$, and by abuse of notation we may assume that W' is normal and σ birational.

Let $\xi : \widehat{Y} \to W'$ be a desingularization, $\delta = \sigma \circ \xi : \widehat{Y} \to W$ and let $U \subset W$ be the complement of the center of δ. Choose a sheaf of ideals $\mathcal{J}$ on W with $\mathcal{O}_W/\mathcal{J}$ supported in $W \setminus U$ and such that $\sigma_*\sigma^*\mathcal{J}$ maps to $\mathcal{O}_W$. One can assume that $\delta^*\mathcal{J}$/torsion is invertible hence of the form $\mathcal{O}_{\widehat{Y}}(-E)$ for an effective divisor supported in $\widehat{Y} \setminus \delta^{-1}(U)$. Then $\delta_*\mathcal{O}_{\widehat{Y}}(-E)$ is contained in $\mathcal{O}_W$. One may assume in addition that $\mathcal{O}_{\widehat{Y}}(-E)$ is δ-ample. Finally choose an ample invertible sheaf $\mathcal{H}$ on W, such that $\delta^*\mathcal{H} \otimes \mathcal{O}_{\widehat{Y}}(-E)$ is ample and such that $\mathcal{H} \otimes \sigma_*\mathcal{O}_{W'}$ is generated by global sections.

By assumption, for some η there are morphisms

$$\bigoplus \sigma^*\mathcal{H} \longrightarrow \sigma^*\mathcal{G}^\eta \quad \text{and hence} \quad \bigoplus \delta^*\mathcal{H} \longrightarrow \delta^*\mathcal{G}^\eta, \tag{1.9.1}$$

surjective over W'_0 and $\delta^{-1}(W_0)$, respectively. Blowing up a bit more, we can assume that the image of the second map is of the form $\delta^*\mathcal{G}^\eta \otimes \mathcal{O}_{\widehat{Y}}(-\Delta)$ for a divisor Δ. Then $\delta^*\mathcal{G}^\eta \otimes \mathcal{O}_{\widehat{Y}}(-\Delta - E)$ as a quotient of an ample sheaf will be ample. Replacing η, Δ and E by some multiple, one may also assume that

$$\delta^*\mathcal{G}^\eta \otimes \mathcal{O}_{\widehat{Y}}(-\Delta - E) \otimes \omega_{\widehat{Y}}^{-1} \otimes \delta^*\mathcal{H}^{-1}$$

is ample. Define $\mathcal{I}' = \xi_*\mathcal{O}_{\widehat{Y}}(-\Delta - E))$ on W' and $\mathcal{I} = \sigma_*\mathcal{I}'$.

Since $\mathcal{G}$ is nef, for all $\alpha \geq \eta$ and for all $\beta \geq -1$ the sheaf

$$\delta^* \mathcal{G}^\alpha \otimes \mathcal{O}_{\hat{Y}}(-\Delta - E) \otimes \delta^* \mathcal{H}^\beta$$

has no higher cohomology. For $\beta \gg 1$ this can only hold if for all $i > 0$

$$R^i \delta_* (\delta^* \mathcal{G}^\alpha \otimes \mathcal{O}_{\hat{Y}}(-\Delta - E)) = 0.$$

For $\beta = -1$ one finds that $H^i(W, \mathcal{G}^\alpha \otimes \mathcal{H}^{-1} \otimes \mathcal{I}) = 0$.

For some $\beta \gg 1$ the sheaf $\sigma^* \mathcal{H}^{\beta - 2} \otimes \mathcal{I}'$ is generated by global sections. Using the left-hand side of (1.9.1) one obtains a morphism

$$\bigoplus \sigma^* \mathcal{H}^\beta \otimes \mathcal{I}' \longrightarrow \sigma^* \mathcal{G}^{\eta \cdot \beta} \otimes \mathcal{I}',$$

surjective over W_0'. Therefore the sheaf $\sigma^*(\mathcal{G}^{\eta \cdot \beta} \otimes \mathcal{H}^{-2}) \otimes \mathcal{I}'$ will be globally generated over W_0'; hence there are morphisms $\bigoplus \sigma^* \mathcal{H} \to \sigma^* \mathcal{H}^{\beta - 1} \otimes \mathcal{I}'$ and

$$\bigoplus \mathcal{H} \otimes \sigma_* \mathcal{O}_{W'} \longrightarrow \mathcal{G}^{\eta \cdot \beta} \otimes \mathcal{H}^{-1} \otimes \mathcal{I}, \tag{1.9.2}$$

surjective over W_0' and W_0. By the choice of $\mathcal{H}$ the left-hand side of (1.9.2) is globally generated over W_0; hence the right-hand side as well. For all positive multiples α of $\eta \cdot \beta$, one has an exact sequence

$$0 \to H^0(W, \mathcal{G}^\alpha \otimes \mathcal{H}^{-1} \otimes \mathcal{I}) \longrightarrow H^0(W, \mathcal{G}^\alpha \otimes \mathcal{H}^{-1}) \longrightarrow H^0(W, \mathcal{G}^\alpha \otimes \mathcal{H}^{-1}|_{T'}) \to 0,$$

where T' denotes the subscheme of W defined by $\mathcal{I}$. If $T' \cap W_0 = \varnothing$ we are done. Otherwise let T be the closure of $T'_{\mathrm{red}} \cup W_0$ in W. So there is a coherent sheaf $\mathcal{F}$, supported on T and an inclusion $\mathcal{F} \to \mathcal{O}_{T'}$ which is an isomorphism on $W_0 \cap T' = W_0 \cap T$.

By induction on the dimension of W we may assume that $\mathcal{G}|_T$ is ample with respect to $T \cap W_0$. Then for each $\beta' > 0$ one finds η' and morphisms

$$\bigoplus \mathcal{H}^{\beta' - 1}|_T \longrightarrow (\mathcal{G}^{\eta' \cdot \beta'} \otimes \mathcal{H}^{-1})|_T,$$

surjective over $Z \cap W_0$. Choose β', such that $\mathcal{F} \otimes \mathcal{H}|_T^{\beta' - 1}$ is globally generated, and $\alpha = \eta' \cdot \beta'$ a multiple of $\eta \cdot \beta$. Then the sheaf $(\mathcal{G}^\alpha \otimes \mathcal{H}^{-1})|_T \otimes \mathcal{F}$ is globally generated over $T \cap W_0$, as well as $\mathcal{G}^\alpha \otimes \mathcal{H}^{-1}|_{T'}$. Since all global sections of this sheaf lift to $H^0(W, \mathcal{G}^\alpha \otimes \mathcal{H}^{-1})$ we find that $\mathcal{G}^\alpha \otimes \mathcal{H}^{-1}$ is globally generated over W_0. □

2. Positivity of direct images I

Examples of nef sheaves are direct images of powers of dualizing sheaves. We will see in this section that Theorem 1 is just what is needed to verify this property in Theorem 2 for $\mathcal{F}^{(\nu)}_{\hat{Y}}$. Then the compatibility $\mathcal{F}^{(\nu)}_{\hat{Y}} = \xi^* \mathcal{F}^{(\nu)}_W$ allows us to deduce the nefness of $\mathcal{F}^{(\nu)}_W$. At the end of the section we will make a first step towards Theorem 2(iv), assuming that W_0 is nonsingular. This will allow us in 2.5 to explain why we have to include the study of multiplier ideals in Sections 6–11.

Let us recall the definitions of multiplier ideal sheaves and of the corresponding threshold.

Definition 2.1. Let Z be a normal projective variety with at most rational Gorenstein singularities; let $\mathcal{N}$ be an invertible sheaf on Z and let D be the zero divisor of a section of $\mathcal{N}$.

(i) For $a \in \mathbb{Q}$ the *multiplier ideal* is defined as $\mathcal{J}(-a\cdot D) = \tau_*\omega_{\tilde{Z}/Z} \otimes \mathcal{O}_{\tilde{Z}}(-[a\cdot\tilde{D}])$, where $\tau : \tilde{Z} \to Z$ is a log-resolution, $\tilde{D} = \tau^* D$, and where $[a \cdot \tilde{D}] = \llcorner a \cdot \tilde{D} \lrcorner$ denotes the integral part of the $\mathbb{Q}$-divisor $a \cdot \tilde{D}$.

(ii) For $b > 0$ one defines the *threshold*

$$e(b\cdot D) := \mathrm{Min}\Big\{a \in \mathbb{Z}_{>0};\ \mathcal{J}\Big(-\frac{b}{a}\cdot D\Big) = \mathcal{O}_Z\Big\}.$$

(iii) Finally $e(\mathcal{N}) := \mathrm{Max}\big\{e(D);\ D \text{ the zero divisor of a section of } \mathcal{N}\big\}$.

Most authors write $\mathcal{J}(a\cdot D)$ instead of $\mathcal{J}(-a\cdot D)$. We prefer the second notion, since, for a smooth divisor $\mathcal{O}_Z(-D) = \mathcal{J}(-D)$, and since it is closer to the classical notion $\omega_Z\{-a\cdot D\} = \omega_Z \otimes \mathcal{J}(-a\cdot D)$ used in [Vie95] and [EV92].

One easily shows that the multiplier ideal is independent of the log resolution. In [EV92] and [Vie95] one finds a long list of properties of multiplier ideals and of $e(b\cdot D)$ and $e(\mathcal{N})$. In particular, if $\mathcal{N}$ is a globally generated invertible sheaf on Z, then

$$\mathcal{J}(-a\cdot D) = \mathcal{J}(-a\cdot(D+H)) \tag{2.1.1}$$

for the divisor H of a general section of $\mathcal{N}$ and for $0 \le a < 1$. In fact, using the notation introduced above, $\tilde{H} = \tau^* H$ will be nonsingular and it intersects $\tilde{D}$ transversely. Then $[a\cdot\tilde{D}] = [a\cdot(\tilde{D}+\tilde{H})]$.

Let $f_0 : X_0 \to Y_0$ be a flat morphism over a nonsingular variety Y_0, with irreducible normal fibers with at most rational singularities. Then for a $\mathbb{Q}$-divisor Δ_0 on X_0 not containing fibers, the threshold $e(\Delta|_{f_0^{-1}(y)})$ is upper semi-continuous for the Zariski topology (see [Vie95, Prop. 5.17]). This implies in [Vie95, Cor. 5.21] that for Z and $\mathcal{N}$ as in Definition 2.1 one has:

$$e(\mathcal{N}) = e(\mathrm{pr}_1^*\mathcal{N} \otimes \cdots \otimes \mathrm{pr}_r^*\mathcal{N}) \quad \text{for} \quad Z^r = Z \times \cdots \times Z. \tag{2.1.2}$$

If one replaces $\mathbb{Z}_{>0}$ in Definition 2.1(ii) by $\mathbb{Q}_{>0}$ one obtains the inverse of the logarithmic threshold.

The multiplier ideals occur in a natural way as direct images of relative dualizing sheaves for certain alterations:

LEMMA 2.2. *Let Z' and Z be normal with rational Gorenstein singularities and let $\phi : Z' \to Z$ be an alteration. If $\mathcal{O}_Z(D) = \mathcal{L}^N$ for an invertible sheaf $\mathcal{L}$ and if $\phi^* D$ is divisible by N, then $\mathcal{J}\big(-\frac{1}{N}\cdot D\big)$ is a direct factor of $\mathcal{L}^{-1} \otimes \phi_*\omega_{Z'/Z}$.*

Proof. The sheaf $\phi_*\omega_{Z'/Z}$ does not change, if we replace Z' by a nonsingular modification. So we may assume that Z' is nonsingular and that it dominates a log resolution $\tau : \widetilde{Z} \to Z$ for D. When $\pi : Z' \to \widetilde{Z}$ for the induced morphism, $\pi^*(\tau^* D)$ is still divisible by N. So π factors through the cyclic covering $\widetilde{\pi} : \widetilde{Z}' \to \widetilde{Z}$, obtained by taking the N-th root out of $\tau^* D$. By [EV92, §3] the sheaf

$$\tau^* \mathcal{L} \otimes \omega_{\widetilde{Z}/Z} \otimes \mathcal{O}_{\widetilde{Z}}\Big(- \Big[\frac{1}{N} \cdot \tau^* D \Big] \Big)$$

is a direct factor of $\widetilde{\pi}_* \omega_{\widetilde{Z}'/Z}$. The latter is a direct factor of $\pi_* \omega_{Z'/Z}$. Applying τ_* one obtains $\mathcal{L} \otimes \mathcal{J}\big(- \frac{1}{N} \cdot D \big)$ as a direct factor of $\phi_* \omega_{Z'/Z}$. □

One starting point for the study of positivity of direct image sheaves is the following corollary of Kollár's Vanishing Theorem.

LEMMA 2.3. *Let X be a projective normal variety with at most rational Gorenstein singularities, let $f : X \to Y$ be a surjection to a projective m-dimensional variety Y, and let $U \subset Y$ be open and dense. Let $\mathcal{A}$ be a very ample invertible sheaf on Y, let $\mathcal{M}$ be an invertible sheaf on X, let Γ be an effective divisor, and let $\mathcal{E}$ be a locally free sheaf on Y, weakly positive over U. Assume that for some $N > 0$ there is a morphism $\mathcal{E} \to f_* \mathcal{M}^N(-\Gamma)$ for which the composite*

$$f^* \mathcal{E} \longrightarrow f^* f_* \mathcal{M}^N(-\Gamma) \longrightarrow \mathcal{M}^N(-\Gamma)$$

is surjective over $V = f^{-1}(U)$. Then for all β the sheaf

$$\mathcal{A}^{m+2} \otimes f_* \Big(\mathcal{M}^\beta \otimes \omega_X \otimes \mathcal{J}\Big(- \frac{\beta}{N} \Gamma \Big) \Big)$$

is globally generated over U.

Proof. We can replace X by a desingularization. The sheaf $\mathcal{A}^N \otimes \mathcal{E}$ is ample with respect to U; hence for some $M > 0$ the sheaf $\mathcal{A}^{N \cdot M} \otimes S^M(\mathcal{E})$ is globally generated over U. Blowing up X with centers outside of V we may assume that the image $\mathcal{B}$ of the evaluation map $f^* S^M(\mathcal{E}) \to \mathcal{M}^{N \cdot M}(-M \cdot \Gamma)$ is invertible. Let D be the divisor, supported in $X \setminus V$ with $\mathcal{B} \otimes \mathcal{O}_X(D) = \mathcal{M}^{N \cdot M}(-M \cdot \Gamma)$. Then

$$\mathcal{B} \otimes f^*\big(\mathcal{A}^{N \cdot M}\big) = \mathcal{M}^{N \cdot M}(-M \cdot \Gamma - D) \otimes f^*\big(\mathcal{A}^{N \cdot M}\big)$$

is generated by global sections over V. Blowing up again, we find a divisor Δ supported in $X \setminus V$ such that $\mathcal{M}^{N \cdot M}(-M \cdot \Gamma - D - \Delta) \otimes f^*(\mathcal{A}^{N \cdot M})$ is generated by global sections, and such that $\Gamma + D + \Delta$ is a normal crossing divisor.

Now, $\mathcal{M}^{N \cdot M}(-M \cdot \Gamma - D - \Delta) \otimes f^* \mathcal{A}^{N \cdot M}$ is semiample. As in [Vie95, Cor. 2.37, 2)], Kollár's Vanishing Theorem implies that the sheaf

$$\mathcal{A}^\iota \otimes f_* \Big(\mathcal{M}^\beta \Big(- \Big[\frac{\beta}{N \cdot M} (M \cdot \Gamma - D - \Delta) \Big] \Big) \otimes \omega_X \otimes \hat{f}^* \mathcal{A} \Big)$$

has no higher cohomology for $\iota > 1$. Then by an argument due to N. Nakayama (see [Kaw99, Lemma 2.11]),

$$\mathcal{P} = \mathcal{A}^{m+1} \otimes f_* \Big(\mathcal{M}^\beta \Big(- \Big[\frac{\beta}{N \cdot M} (M \cdot \Gamma - D - \Delta) \Big] \Big) \otimes \omega_X \otimes \hat{f}^* \mathcal{A} \Big)$$

is generated by global sections. On the other hand, $\mathcal{P}$ is contained in

$$\mathcal{A}^{m+2} \otimes f_*\Big(\mathcal{M}^{\beta}\big(-\big[\frac{\beta}{N}\Gamma\big]\big) \otimes \omega_X\Big),$$

and since $(D+\Delta) \cap V = \varnothing$, both coincide over U. □

Kawamata's Semipositivity Theorem, saying that the direct image sheaves $\mathcal{F}^{(1)}_{\widehat{Y}}$ in Theorem 2(iii) are nef, can be shown by using Lemma 2.3. This in turn implies part (iii) for all ν. We will give a slightly different argument:

Proof of Theorem 2(iii). Let $\widehat{Y} \to W$ be any nonsingular alteration of W. In Theorem 1(ii) the sheaf $f^{(r)}_* \omega^{\nu}_{X^{(r)}/\widehat{Y}}$ remains the same if one replaces $X^{(r)}$ by a nonsingular modification. Hence by abuse of notation one may assume that for some normal crossing divisor Π on $\widehat{X}$ the evaluation map induces a surjection

$$f^{(r)*}\mathcal{F}^{(\nu)\otimes r}_{\widehat{Y}} \longrightarrow \omega^{\nu}_{X^{(r)}/\widehat{Y}} \otimes \mathcal{O}_{X^{(r)}}(-\Pi).$$

Let $\mathcal{H}$ be an ample invertible sheaf on $\widehat{Y}$ and define

$$s(\nu) = \mathrm{Min}\big\{\mu > 0; \mathcal{F}^{(\nu)}_{\widehat{Y}} \otimes \mathcal{H}^{\nu\cdot\mu} \text{ is nef}\big\}.$$

Then $(\mathcal{H}^{s(\nu)\cdot\nu} \otimes \mathcal{F}^{(\nu)}_{\widehat{Y}})^{\otimes r} = \mathcal{H}^{s(\nu)\cdot\nu\cdot r} \otimes \hat{f}_*\omega^{\nu}_{X^{(r)}/\widehat{Y}}$ is nef. Let $\mathcal{A}$ be a very ample invertible sheaf on $\widehat{Y}$. By Lemma 2.3

$$\mathcal{A}^{m+2} \otimes f^{(r)}_*\Big(\omega_{X^{(r)}} \otimes \big(\omega_{X^{(r)}/\widehat{Y}} \otimes f^{(r)*}\mathcal{H}^{s(\nu)\cdot r}\big)^{\nu-1} \otimes \mathcal{O}_{\widehat{Y}}\Big(-\Big[\frac{(\nu-1)\cdot\Pi}{\nu}\Big]\Big)\Big)$$

is generated by global sections. It is a subsheaf of $\mathcal{A}^{m+2} \otimes \omega_{\widehat{Y}} \otimes \mathcal{F}^{(\nu)\otimes r}_{\widehat{Y}} \otimes \mathcal{H}^{s(\nu)\cdot r\cdot(\nu-1)}$ and it contains the sheaf

$$\mathcal{A}^{m+2} \otimes \omega_{\widehat{Y}} \otimes \mathcal{H}^{s(\nu)\cdot r\cdot(\nu-1)} \otimes f^{(r)}_*(\omega^{\nu}_{X^{(r)}/\widehat{Y}} \otimes \mathcal{O}_{\widehat{Y}}(-\Pi))$$
$$= \mathcal{A}^{m+2} \otimes \omega_{\widehat{Y}} \otimes \mathcal{H}^{s(\nu)\cdot r\cdot(\nu-1)} \otimes \mathcal{F}^{(\nu)\otimes r}_{\widehat{Y}}.$$

Thus, the three sheaves are equal, and the quotient sheaf

$$\mathcal{A}^{m+2} \otimes \omega_{\widehat{Y}} \otimes S^r\big(\mathcal{H}^{s(\nu)(\nu-1)} \otimes \mathcal{F}^{(\nu)}_{\widehat{Y}}\big)$$

is generated by global sections as well. Hence $\mathcal{H}^{s(\nu)\cdot(\nu-1)} \otimes \mathcal{F}^{(\nu)}_{\widehat{Y}}$ is weakly positive over $\widehat{Y}$. Since $\mathcal{H}^{(s(\nu)-1)\cdot\nu} \otimes \mathcal{F}^{(\nu)}_{\widehat{Y}}$ does not have this property, one obtains

$$s(\nu)\cdot(\nu-1) > (s(\nu)-1)\cdot\nu \quad \text{or} \quad s(\nu) < \nu.$$

Now, $\mathcal{H}^{\nu^2} \otimes \mathcal{F}^{(\nu)}_{\widehat{Y}}$ is weakly positive over $\widehat{Y}$, hence nef.

Since the same exponent ν^2 works for all $\widehat{Y}$ mapping to W and for all ample invertible sheaves $\widehat{\mathcal{H}}$ on $\widehat{Y}$, the nefness of $\mathcal{F}^{(\nu)}_W$ follows from the equivalence of (1) and (5) in Lemma 1.3. □

VARIANT 2.4. *Assume in Theorem* 1 *that the normalization of* W_0 *is nonsingular and that for some* $\eta > 0$ *the sheaf* $\det(\mathcal{F}_W^{(\eta)})$ *is ample with respect to* W_0. *Then for* $\nu \geq 2$ *the sheaf* $\mathcal{F}_W^{(\nu)}$ *is ample with respect to* W_0 *or zero.*

Proof. We sketch the argument, knowing that $\mathcal{F}_W^{(\nu)}$ is nef; Lemma 1.9 allows us to replace W by a desingularization $\hat{Y}$. Consider for some γ and for $r = \gamma \cdot \operatorname{rk}(\mathcal{F}_{\hat{Y}}^{(\eta)})$ the tautological map

$$\Xi : \det\left(\mathcal{F}_{\hat{Y}}^{(\eta)}\right)^{\gamma} \longrightarrow \bigotimes^{r} \mathcal{F}_{\hat{Y}}^{(\eta)} = f_*^{(r)} \omega_{X^{(r)}/\hat{Y}}^{\eta}.$$

Assume that $\det(\mathcal{F}_{\hat{Y}}^{(\eta)}) = \mathcal{N}^{\alpha}$ for some invertible sheaf $\mathcal{N}$ and for some $\alpha > 0$. Then Ξ induces a section of $\omega_{X^{(r)}/\hat{Y}}^{\eta} \otimes f^{(r)*}\mathcal{N}^{-\gamma\cdot\alpha}$ with zero divisor Γ.

We know already that $\mathcal{F}_{\hat{Y}}^{(\rho)}$ is nef. So we can apply Lemma 2.3 to the sheaf

$$\omega_{X^{(r)}/\hat{Y}}^{\rho} = \omega_{X^{(r)}/\hat{Y}}^{\rho+\eta} \otimes f^{(r)*}\mathcal{N}^{-\gamma\cdot\alpha} \otimes \mathcal{O}_{X^{(r)}}(-\Gamma).$$

If α is divisible by $\rho + \eta$, for $\mathcal{A}$ very ample on $\hat{Y}$, the sheaf

$$\mathcal{A}^{\dim(\hat{Y})+2} \otimes \omega_{\hat{Y}} \otimes f_*^{(r)}\left(\omega_{X^{(r)}/\hat{Y}}^{\nu} \otimes f^{(r)*}\mathcal{N}^{-\frac{(\nu-1)\gamma\cdot\alpha}{\rho+\eta}} \otimes \mathcal{J}\Big(-\frac{\nu-1}{\rho+\eta}\Gamma\Big)\right)$$

is globally generated over the preimage $\hat{Y}_0$ of W_0. For ρ sufficiently large one may assume that $\frac{\rho+\eta}{\nu-1} \geq e(\omega_F^{\eta})$, for all smooth fibers of f_0. Then by [Vie95, Lemma 5.14 and Cor. 5.21] one finds that $\mathcal{J}\big(-\frac{\nu-1}{\rho+\eta}\Gamma\big)$ is trivial over $f^{(r)^{-1}}(\hat{Y}_0)$. Then the inclusion of sheaves

$$\mathcal{G}(r) := f_*^{(r)}\left(\omega_{X^{(r)}/\hat{Y}}^{\nu} \otimes \mathcal{J}\Big(-\frac{\nu-1}{\rho+\eta}\Gamma\Big)\right) \subset \bigotimes^{r} \mathcal{F}_{\hat{Y}}^{(\nu)} \tag{2.4.1}$$

is an isomorphism over $\hat{Y}_0$ and

$$\begin{aligned}\mathcal{A}^{\dim(\hat{Y})+2} &\otimes \omega_{\hat{Y}} \otimes \mathcal{N}^{-\frac{(\nu-1)\gamma\cdot\alpha}{\rho+\eta}} \otimes \bigotimes^{r} \mathcal{F}_{\hat{Y}}^{(\nu)} \\ &= \mathcal{A}^{\dim(\hat{Y})+2} \otimes \omega_{\hat{Y}} \otimes \bigotimes^{\gamma}\left(\mathcal{N}^{-\frac{(\nu-1)\alpha}{\rho+\eta}} \otimes \bigotimes^{\operatorname{rk}(\mathcal{F}_{\hat{Y}}^{(\nu)})} \mathcal{F}_{\hat{Y}}^{(\nu)}\right)\end{aligned}$$

is globally generated over $\hat{Y}_0$. This being true for all γ, Lemmas 1.6 and 1.5 imply that $\mathcal{F}_{\hat{Y}}^{(\nu)}$ is ample over $\hat{Y}_0$.

By [Vie95, Lemma 2.1] the assumption that $\det(\mathcal{F}_{\hat{Y}}) = \mathcal{N}^{\alpha}$, with α divisible by $\rho + \eta$, will always be true over sufficiently large finite coverings of $\hat{Y}$, and by Lemma 1.7 we are done. $\square$

Comments 2.5. We repeated the well-known proof of Variant 2.4 just to point out the difficulties we will encounter, trying to get rid of the additional assumption "W_0 nonsingular". The notion "ample with respect to W_0" is not compatible with blowing ups of W_0, if the center meets W_0. We may assume that $\mathcal{N}$ is the pullback of an invertible sheaf $\mathcal{N}_W$ on W. In addition we have to construct a sheaf $\mathcal{G}_W(r)$ whose pullback to a desingularization is the sheaf $\mathcal{G}(r)$ considered in the proof of Variant 2.4. In order to be allowed to use the functorial property Lemma 1.3 the sheaf

$$\mathcal{G}(r) \otimes \mathcal{N}^{-\frac{(\nu-1)\gamma\cdot\alpha}{\rho+\eta}}$$

must be nef, and not just weakly positive over $\widehat{Y}_0$. This would hold, if the inclusion (2.4.1) is an isomorphism, but giving bounds for the threshold in bad fibers of a morphism does not seem to work.

So as a way out we will modify the construction of W in such a way, that Theorem 1 remains true for the direct images $\mathcal{G}_\bullet(r)$ of invertible sheaves tensored by multiplier ideals.

3. **On the construction of moduli schemes**

The weak positivity and ampleness of the direct image sheaves in Corollary 3, together with the stability criterion [Vie95, Th. 4.25], allows the construction of a quasi-projective moduli scheme of canonically polarized manifolds. Following a suggestion of the referee, we sketch the argument before entering the quite technical details needed for the construction of W, hence for the proof of Corollary 3(b).

Let $\mathfrak{M}_h$ be the moduli functor of canonically polarized manifolds with Hilbert polynomial h. As in [Vie95, Exs. 1.4] we consider for a scheme Y_0 the set

$$\mathfrak{M}_h(Y_0) = \big\{ f_0 : X_0 \to Y_0;\ f_0 \text{ smooth, projective, } \omega_{X_0/Y_0}\ f_0\text{-ample} \\ \text{and } h(\nu) = \mathrm{rk}(f_{0*}\omega^{\nu}_{X_0/Y_0}),\ \text{for } \nu \geq 2 \big\}/\sim.$$

In order to allow the canonical models of surfaces we could also consider

$$\mathfrak{M}'_h(Y_0) = \big\{ f_0 : X_0 \to Y_0;\ f_0 \text{ flat, projective; all fibers } F \text{ normal} \\ \text{with at most rational Gorenstein singularities, } \omega_{X_0/Y_0} \\ f_0\text{-ample and } h(\nu) = \mathrm{rk}(f_{0*}\omega^{\nu}_{X_0/Y_0}),\ \text{for } \nu \geq 2 \big\}/\sim.$$

We leave the necessary changes of the arguments to the reader.

Outline of the construction of a coarse quasi-projective moduli scheme M_h for $\mathfrak{M}_h$. One first has to verify that the functor $\mathfrak{M}_h$ is a nice moduli functor, i.e., locally closed, separated and bounded (see [Vie95, Lemma 1.18]). This implies that for some $\eta \gg 1$ one has the Hilbert scheme H of η-canonically embedded manifolds in $\mathfrak{M}_h(\mathrm{Spec}(\mathbb{C}))$, together with the universal family $g : \mathcal{X} \to H$.

The universal property gives an action of $G = \mathbb{P}\mathrm{Gl}(h(\eta))$ on H and, as explained in [Mum65] or [Vie95, Lemma 7.6], the separatedness of the moduli functor implies that this action is proper and with finite stabilizers. The sheaves $\lambda_\eta = \det(g_*\omega^{\eta}_{\mathcal{X}/H})$ are all G-linearized.

The moduli scheme M_h, if it exists, should be a good quotient H/G. So one has to verify that all points in H are stable for the group action and for a suitable ample sheaf. At this point one is allowed to replace H by $H_{\rm red}$; the set of stable points will not change. So by abuse of notation we will assume that H (and hence M_h) is reduced.

In order to apply the stability criterion [Vie95, Th. 4.25] one has to verify that the invertible sheaf λ_η on H is ample on H, and that for a certain family $f_0 : X_0 \to Y_0$ in $\mathfrak{M}_h(Y_0)$ the sheaf $f_{0*}\omega^{\eta}_{X_0/Y_0}$ is weakly positive over Y_0.

The second statement follows from Corollary 3(a). For the first one we start with the Plücker embedding showing that the invertible sheaves $\lambda^{h(\eta)}_{\eta\cdot\mu} \otimes \lambda_\eta^{-h(\eta\cdot\mu)\cdot\mu}$ are ample, for all μ sufficiently large. By Corollary 3(a) the sheaf λ_η is weakly positive over H, hence by Lemma 1.3(3) $\lambda_{\eta\cdot\mu}$ is ample. Using Corollary 3(b) one finds that the sheaf $g_*\omega^{\eta}_{\mathcal{X}/H}$ is ample on H; hence its determinant λ_η is also. □

Let us express what we have shown in terms of stability of Hilbert points. On H the sheaf λ_η is G linearized and ample. The stability criterion says that all the points in H are stable with respect to the polarization λ_η of H. This in turn shows the ampleness of the sheaf $\lambda^{(p)}_{0,\eta}$ for $\eta \gg 1$.

One can consider the sheaf λ_ν on H for all $\nu \geq 2$ with $h(\nu) > 0$. Those sheaves are G-linearized and for some $p > 0$ the p-th power of λ_ν descends to an invertible sheaf $\lambda^{(p)}_{0,\nu}$ on M_h. Using a slightly different stability criterion, stated in [Vie95, Addendum 4.26], one obtains:

Addendum 3.1. For all $\nu \geq 2$ with $h(\nu) > 0$ and for some $p > 0$ there exists an ample invertible sheaf $\lambda^{(p)}_{0,\nu}$ on M_h whose pullback to H is $\lambda_\nu = \det(h_*\omega^{\nu}_{\mathcal{X}/H})$. In particular the sheaf $\lambda^{(p)}_{0,\nu}$ will satisfy the condition $(*)$ stated in Notation 4.

In Section 14 we will even show that $\lambda^{(p)}_{0,\nu}$ extends to an invertible sheaf $\lambda^{(p)}_{\nu}$ on a suitable compactification of M_h and that this sheaf is ample with respect to M_h.

For points of the Hilbert scheme of η-canonically embedded curves or surfaces of general type the stability has been verified with respect to the Plücker embedding (see [Mum65] and [Gie77]). So one obtains on M_h the ampleness of $\lambda^{h(\eta)}_{0,\eta\cdot\mu} \otimes \lambda^{-\mu\cdot h(\eta\cdot\mu)}_{0,\eta}$.

Before turning our attention to moduli schemes of polarized minimal models, let us formulate the generalization of Corollary 3, needed for their construction. The proof will be given at the end of Section 13. Here we use again the threshold $e(\mathcal{N})$ defined in 2.1.

LEMMA 3.2. *Let $f_0 : X_0 \to Y_0$ be a smooth family of minimal models, and let $\mathcal{L}_0$ be an f_0-ample invertible sheaf. Assume that for some $\kappa > 0$ the direct image $f_{0*}(\mathcal{L}_0^{\kappa})$ is nonzero, locally free and compatible with arbitrary base change. Choose some $\epsilon > e(\mathcal{L}_0^{\kappa}|_F)$, for all fibers F of f_0. Then:*

(1) *For all positive integers η the sheaf*
$$S^{\mathrm{rk}(f_{0*}(\mathcal{L}_0^{\kappa}))}\big(f_{0*}\big(\omega_{X_0/Y_0}^{\epsilon\cdot\eta}\otimes\mathcal{L}_0^{\kappa\cdot\eta}\big)\big)\otimes\det\big(f_{0*}(\mathcal{L}_0^{\kappa})\big)^{-\eta}$$
is weakly positive over W_0 or zero.

(2) *If for some $\eta'>0$ the sheaf*
$$\det\big(f_{0*}\big(\omega_{X_0/Y_0}^{\epsilon\cdot\eta'}\otimes\mathcal{L}_0^{\kappa\cdot\eta'}\big)\big)^{\mathrm{rk}(f_{0*}(\mathcal{L}_0^{\kappa}))}\otimes\det\big(f_{0*}(\mathcal{L}_0^{\kappa})\big)^{-\eta'\cdot\mathrm{rk}(f_{0*}(\omega_{X_0/Y_0}^{\epsilon\cdot\eta'}\otimes\mathcal{L}_0^{\kappa\cdot\eta'}))}$$
is ample, then $S^{\mathrm{rk}(f_{0*}(\mathcal{L}_0^{\kappa}))}(f_{0*}(\omega_{X_0/Y_0}^{\epsilon\cdot\eta}\otimes\mathcal{L}_0^{\kappa\cdot\eta}))\otimes\det(f_{0*}(\mathcal{L}_0^{\kappa}))^{-\eta}$ is ample, if not zero.

The moduli functor $\mathfrak{M}_h$ of minimal polarized manifolds is given by
$$\mathfrak{M}_h(Y_0)=\big\{(f_0:X_0\longrightarrow Y_0,\mathcal{L}_0);\ f_0\text{ smooth, projective; }\omega_{X_0/Y_0}\ f_0\text{-semiample;}\\ \mathcal{L}_0\ f_0\text{-ample, with Hilbert polynomial }h\big\}/\sim.$$
Recall that $(f_0:X_0\to Y_0,\mathcal{L}_0)\sim(\widetilde{f}_0:\widetilde{X}_0\to Y_0,\widetilde{\mathcal{L}}_0)$ if there are a Y_0-isomorphism $\iota:X_0\to\widetilde{X}_0$ and an invertible sheaf $\mathcal{A}$ on Y_0 with $\iota^*\widetilde{\mathcal{L}}_0=\mathcal{L}_0\otimes f_0^*\mathcal{A}$.

As we will see, it is easier to study the moduli functor $\mathfrak{M}'_h$ with
$$\mathfrak{M}'_h(Y_0)=\big\{(f_0:X_0\longrightarrow Y_0,\mathcal{L}_0)\in\mathfrak{M}_h;\mathcal{L}_0\ f_0\text{-very ample}\\ \text{with Hilbert polynomial }h;\ R^if_{0*}\mathcal{L}_0^{\mu}=0\text{ for }i>0,\text{ and }\mu>0,\big\}/\sim.$$
For families of minimal varieties F of Kodaira dimension zero the second condition will hold automatically. In fact, if $\omega_F^{\nu}=\mathcal{O}_F$ and if $\mathcal{A}$ is ample, $\mathcal{A}\otimes\omega_F^{\nu-1}$ is ample and Kodaira's Vanishing Theorem implies that $H^i(F,\mathcal{A})=H^i(F,\mathcal{A}\otimes\Omega^{\nu})=0$, for $i>0$. So here we should consider the functors $\mathfrak{M}_h^{(\nu)}$ with
$$\mathfrak{M}_h^{(\nu)}(Y_0)=\big\{(f_0:X_0\longrightarrow Y_0,\mathcal{L}_0)\in\mathfrak{M}_h;\ f_0^*f_{0*}\omega_{X_0/Y_0}^{\nu}=\omega_{X_0/Y_0};\\ \mathcal{L}_0\ f_0\text{-very ample with Hilbert polynomial }h\big\}/\sim.$$

LEMMA 3.3.

(1) *Assume that for all $\hat{h}$ the moduli functor $\mathfrak{M}'_{\hat{h}}$ has a coarse quasi-projective moduli scheme $M'_{\hat{h}}$. Then the same holds true for $\mathfrak{M}_h$.*

(2) *To prove Theorem* 6 *it is sufficient to consider the moduli functors $\mathfrak{M}_{\hat{h}}^{(\nu)}$.*

Proof. The boundedness of the moduli functor $\mathfrak{M}_h$ allows us to find some γ_0 such that for all $(F,\mathcal{A})\in\mathfrak{M}_h(\mathbb{C})$ and for all $\gamma\geq\gamma_0$ the sheaf $\mathcal{A}^{\gamma}$ is very ample and without higher cohomology. For suitable polynomials h_1 and h_2 one defines a map $\mathfrak{M}_h\longrightarrow\mathfrak{M}'_{h_1}\times\mathfrak{M}'_{h_2}$ by
$$(f_0:X_0\to Y_0,\mathcal{L}_0)\mapsto\big[\big(f_0:X_0\to Y_0,\mathcal{L}_0^{\gamma_0}\big),\big(f_0:X_0\to Y_0,\mathcal{L}_0^{\gamma_0+1}\big)\big].$$
It is easy to see that the image is locally closed. Hence if one is able to construct the corresponding moduli schemes M'_{h_1} and M'_{h_2} as quasi-projective schemes, M_h is a

locally closed subscheme. And if one finds nice projective compactifications $\overline{M}'_{h_1}$ and $\overline{M}'_{h_2}$ of M'_{h_1} and M'_{h_2}, one chooses $\overline{M}_h$ as the closure of M_h in $\overline{M}'_{h_1} \times \overline{M}'_{h_2}$.

The additional condition $\omega_F^\nu = \mathcal{O}_F$ considered in Theorem 6 just signals certain irreducible components of M_h. So by abuse of notation let M_h be one of those. Then the image of M_h lies in the product $M_{h_1}^{(\nu)} \times M_{h_2}^{(\nu)}$. If one has constructed the compactifications $\overline{M}_{h_1}^{(\nu)}$ and $\overline{M}_{h_2}^{(\nu)}$ according to Theorem 6 one can choose $\overline{M}_h$ as the closure of M_h and for $\lambda_\nu^{(2\cdot p)}$ the restriction of the exterior tensor product of the corresponding sheaves on $\overline{M}'_{h_1}$ and $\overline{M}'_{h_2}$ for p instead of $2 \cdot p$. □

Outline of the construction of coarse quasi-projective moduli scheme M'_h for $\mathfrak{M}'_h$. The construction is parallel to the one in the canonically polarized case. One constructs the Hilbert scheme H parametrizing the elements $(F, \mathcal{A})$ of $\mathfrak{M}'_h(\mathbb{C})$ together with an isomorphism $\mathbb{P}(H^0(F, \omega_F^\epsilon \otimes \mathcal{A})) \cong \mathbb{P}^N$. Here ν is chosen such that ω_F^ν is globally generated and ϵ should be a multiple of ν, larger than the threshold $e(\mathcal{A})$.

The Plücker embedding provides us with an ample invertible sheaf of the form $\varpi_\mu^{r(1)} \otimes \varpi_1^{-\mu\cdot r(\mu)}$, where

$$\varpi_\nu = \det\big(g_*\big(\omega_{\mathcal{X}/H}^{\epsilon\cdot\nu} \otimes \mathcal{L}_{\mathcal{X}}^\nu\big)\big) \quad \text{and} \quad r(\nu) = \operatorname{rk}\big(g_*\big(\omega_{\mathcal{X}/H}^{\epsilon\cdot\nu} \otimes \mathcal{L}_{\mathcal{X}}^\nu\big)\big)$$

for the universal family $(g : \mathcal{X} \to H, \mathcal{L}_{\mathcal{X}}) \in \mathfrak{M}'(H)$.

By 3.2(1) the sheaf $\varpi_1^{h(1)} \otimes \det(g_*\mathcal{L}_{\mathcal{X}})^{-r(1)}$ is weakly positive over H, hence

$$\varpi_\mu^{h(1)\cdot r(1)} \otimes \det(g_*\mathcal{L}_{\mathcal{X}})^{-r(1)\cdot\mu\cdot r(\mu)} = \big(\varpi_\mu^{h(1)} \otimes \det(g_*\mathcal{L}_{\mathcal{X}})^{-\mu\cdot r(\mu)}\big)^{r(1)}$$

is ample. Using 3.2(2) one finds that $\varpi_1^{h(1)} \otimes \det(g_*\mathcal{L}_{\mathcal{X}})^{-r(1)}$ must be ample.

In order to apply the stability criterion [Vie95, Th. 4.25] to obtain the stability of all points of H with respect to the sheaf $\varpi_1^{h(1)} \otimes \det(g_*\mathcal{L}_{\mathcal{X}})^{-r(1)}$, it remains to show that for a special family $(f_0 : X_0 \to Y_0, \mathcal{L}_0)$ the rigidified direct image sheaf is weakly positive over Y_0. This is exactly the sheaf

$$S^{h(1)}\big(f_{0*}\big(\omega_{X_0/Y_0}^\epsilon \otimes \mathcal{L}_0\big)\big) \otimes \det(f_{0*}(\mathcal{L}_0))^{-1}$$

considered in Lemma 3.2(1). □

Addendum 3.4. If $\omega_F^\nu = \mathcal{O}_F$ for all $(F, \mathcal{A}) \in \mathfrak{M}_h(\mathbb{C})$, and if $\nu > 0$ then for some $p \gg 1$ there exists an ample invertible sheaf $\lambda_{0,\nu}^{(p)}$ satisfying the property $(\ast)$ in Notation 4 in the introduction.

Proof. The existence of the sheaf $\lambda_{0,\nu}^{(p)}$ satisfying the property $(\ast)$ follows from the construction of moduli schemes as a quotient of the Hilbert scheme. In order to verify the ampleness, write (for the universal family $G : \mathcal{X} \to H$ over the Hilbert scheme) $\omega_{\mathcal{X}/H}^\nu = g^*\lambda_\nu$. One has $r(1) = h(1)$ and the ample sheaf $\varpi_1^{h(1)} \otimes \det(g_*\mathcal{L}_{\mathcal{X}})^{-r(1)}$ is

$$\det\big(g_*\big(\omega_{\mathcal{X}/H}^\epsilon \otimes \mathcal{L}_{\mathcal{X}}\big)\big)^{r(1)} \otimes \det(g_*\mathcal{L}_{\mathcal{X}})^{-r(1)} = \lambda_\nu^{\frac{\epsilon\cdot r(1)^2}{\nu}}. \tag{3.4.1}$$

□

Remark 3.5. Trying in the following sections to extend the polarization to degenerate fibers, we have to keep the equality stated in (3.4.1). As explained in Remark 8.1 this will force us to choose "saturated extensions" of the polarizations.

Remark 3.6. So for polarized minimal models we verified the stability of the points of H for the polarization given by $\det(h_*(\omega^{\epsilon}_{\mathcal{X}/H}\otimes\mathcal{L}))^{\hat{h}(1)}\otimes\det(h_*\mathcal{L})^{-r(1)}$. Let us assume for a moment, that ω_F^{η} is very ample for all $F\in\mathfrak{M}^1_{\hat{h}}$. One can replace H by the locally closed subscheme given by the condition that $\mathcal{L}\sim\omega^{\eta}_{\mathcal{X}/H}$. Of course this can only happen if for the Hilbert polynomial h of ω_F one has $\hat{h}(t)=h(\eta\cdot t)$. We assume that ϵ is divisible by η and write $\mu=\frac{\epsilon}{\eta}+1$. Then

$$\det\big(h_*\big(\omega^{\epsilon}_{\mathcal{X}/H}\otimes\mathcal{L}\big)\big)^{\hat{h}(1)}\otimes\det(h_*\mathcal{L})^{-r(1)} \\ =\det\big(h_*\omega^{\mu\cdot\eta}_{\mathcal{X}/H}\big)^{h(\eta)}\otimes\det\big(h_*\omega^{\eta}_{\mathcal{X}/H}\big)^{-h(\mu\cdot\eta)}.$$

Thus, we are still missing a factor μ on the right-hand side, compared with the ample sheaf obtained by Mumford and Gieseker for moduli of curves or surfaces.

4. **Weak semistable reduction**

Let us recall the Weak Semistable Reduction Theorem in [AK00] and some of the steps used in its proof. The presentation is influenced by [VZ03] and [VZ02], but all the concepts and results are due to D. Abramovich and K. Karu.

Definition 4.1. A projective morphism $\hat{g}:\widehat{Z}\to\widehat{Y}$ between quasi-projective varieties is called *mild* if:

(i) $\hat{g}$ is flat, Gorenstein, and all fibers are reduced.

(ii) $\widehat{Y}$ is nonsingular and $\widehat{Z}$ is normal with at most rational singularities. There exists an open dense subscheme $\widehat{Y}_g\subset\widehat{Y}$ with $\hat{g}^{-1}(\widehat{Y}_g)\to\widehat{Y}_g$ smooth.

(iii) Given a dominant morphism $\widehat{Y}_1\to\widehat{Y}$ from a normal quasi-projective variety $\widehat{Y}_1$ with at most rational Gorenstein singularities, $\widehat{Z}\times_{\widehat{Y}}\widehat{Y}_1$ is normal with at most rational Gorenstein singularities.

(iv) Given a nonsingular curve $\widehat{C}$ and a morphism $\tau:\widehat{C}\to\widehat{Y}$ whose image meets $\widehat{Y}_g$, the fibered product $\widehat{Z}\times_{\widehat{Y}}\widehat{C}$ is normal, Gorenstein and with at most rational singularities.

For a curve $\widehat{Y}$ an example of a mild morphism is a semistable one, i.e., a morphism $\hat{g}:\widehat{Z}\to\widehat{Y}$ with $\widehat{Z}$ a manifold and with all fibers reduced normal crossing divisors.

Obviously property (iii) implies that for two mild morphisms $\hat{g}_i:\widehat{Z}_i\to\widehat{Y}$ the fiber product $\widehat{Z}_1\times_{\widehat{Y}}\widehat{Z}_2\to\widehat{Y}$ is again mild. So one has:

LEMMA 4.2. *If $\hat{g}_i:\widehat{Z}_i\to\widehat{Y}$ are mild morphisms, for $i=1,\dots,s$, then the fiber product $\widehat{Z}^r=\widehat{Z}_1\times_{\widehat{Y}}\cdots\times_{\widehat{Y}}\widehat{Z}_s\to\widehat{Y}$ is mild.*

Definition 4.3. Let $\widehat{Y}$ be a projective manifold, $\widehat{Y}_0 \subset \widehat{Y}$ open and dense, and let $\hat{f}_0 : \widehat{X}_0 \to \widehat{Y}_0$ be a dominant morphism. Then $\hat{f}_0$ has a *mild model* if there exists a mild morphism $\hat{g} : \widehat{Z} \to \widehat{Y}$, with $\widehat{Z}$ birational to some compactification of $\widehat{X}$ over $\widehat{Y}$.

The Weak Semistable Reduction Theorem implies that after a nonsingular alteration of the base, every morphism $f_0 : X_0 \to Y_0$ has a mild model:

Construction 4.4. *Start.* Let $f_0 : X_0 \to Y_0$ be a flat surjective projective morphism between quasi-projective varieties of pure dimension $n+m$ and m, respectively, and with a geometrically integral generic fiber.

We will consider two cases. Either f_0 is smooth, or Y_0 is nonsingular and f_0 a flat morphism.

Step I. Choose a flat projective model $f : X \to Y$ of f_0. If $\tilde{f} : \widetilde{X} \to \widetilde{Y}$ is any projective model of f_0 one may choose Y and X to be modifications of $\widetilde{Y}$ and $\widetilde{X}$, respectively.

Start with any compactification $\tilde{f} : \widetilde{X} \to \widetilde{Y}$ and with an embedding $\widetilde{X} \to \mathbb{P}^\ell$. Then f_0 defines a morphism $\vartheta : Y_0 \to \mathfrak{Hilb}$ to the Hilbert scheme of subvarieties of $\mathbb{P}^\ell$. We choose a modification Y of $\widetilde{Y}$ such that the morphism ϑ extends to $\vartheta : Y \to \mathfrak{Hilb}$. The family $f : X \to Y$ is defined as the pullback of the universal family.

Step II. There exist modifications σ and σ' and a diagram

$$\begin{array}{ccc} X' & \xrightarrow{\sigma'} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow{\sigma} & Y \end{array} \tag{4.4.1}$$

with Y' nonsingular, such that for some open dense subschemes $U_Y \subset Y'$ and $U_X \subset X'$ the morphism $f' : (U_X \subset X') \to (U_Y \subset Y')$ is equidimensional, toroidal, and where X' is without horizontal divisors, i.e. where none of the irreducible components of $X' \setminus U_X$ is dominant over Y'.

The construction is done in [AK00] in several steps. Replacing Y by its normalization and X by the pullback family one may assume that Y is integral. Theorem 2.1 (loc. cit.) allows us to find the diagram (4.4.1) with f' toroidal for suitable subsets $U_X \subset X'$ and $U_Y \subset Y'$, and with X' and Y' nonsingular. Next, Section 3 (loc. cit.) explains how to get rid of horizontal divisors in X', without changing f'.

In Proposition 4.4 (loc. cit.) the authors construct a nonsingular projective modification of Y' and a projective modification of X' such that the induced rational map is in fact an equidimensional toroidal morphism.

Step III. For each component D_i of $Y' \setminus U_Y$ there exists a positive integer m_i with the following property.

For a "Kawamata covering package" $(D_i, m_i, H_{i,j})$ (defined on page 261 (loc. cit.)) consider the diagram

$$\begin{tikzcd} \widehat{Z} \arrow[r, "\pi'"] \arrow[d, "\hat{g}"'] & X' \arrow[r, "\sigma'"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ \widehat{Y} \arrow[r, "\pi"] & Y' \arrow[r, "\sigma"] & Y \end{tikzcd}$$

where $\pi : \widehat{Y} \to Y'$ is the covering given by $(D_i, m_i, H_{i,j})$, and where $\widehat{Z}$ is the normalization of $X' \times_{Y'} \widehat{Y}$. Then $\hat{g} : \widehat{Z} \to \widehat{Y}$ is mild.

The definition of the numbers m_i is given in [AK00, p. 264], and the rest is contained in Propositions 5.1 and 6.4 (loc. cit.). There however the authors define a mild morphism as one satisfying conditions 4.1(i)–(iii). As pointed out by K. Karu in [Kar00, proof of 2.12], the arguments used to verify property 4.1 (iii) carry over "word by word" to show property (iv). So there is no harm in adding this condition.

Summing up what was obtained in Construction 4.4:

PROPOSITION 4.5. *Starting with a flat projective morphism $f : X \to Y$ as in Step* I, *one finds a commutative diagram*

(4.5.1)
$$\begin{tikzcd} X \arrow[d, "f"'] & \widehat{Z} \arrow[l, "\hat{\varphi}"'] \arrow[d, "\hat{g}"] \\ Y & \widehat{Y} \arrow[l, "\varphi"'] \end{tikzcd}$$

of projective morphisms such that:

(a) *$\widehat{Y}$ is nonsingular and φ is an alteration. In particular if $f_0 : X_0 \to Y_0$ is smooth, then $X_0 \times_{Y_0} \varphi^{-1}(Y_0)$ is nonsingular.*

(a$'$) *If Y_0 is nonsingular and if $f_0 : X_0 \to Y_0$ is a mild morphism, then the variety $X_0 \times_{Y_0} \varphi^{-1}(Y_0)$ is normal with at most rational Gorenstein singularities.*

(b) *$\hat{g} : \widehat{Z} \to \widehat{Y}$ is mild.*

(c) *The induced morphism $\widehat{Z} \to X \times_Y \widehat{Y}$ is a modification.*

A more natural object to study is a desingularization $\widehat{X}$ of the pullback family $\mathrm{pr}_2 : X \times_Y \widehat{Y} \to \widehat{Y}$. Although the resulting morphism not necessarily flat we will use both constructions joint by a nonsingular modification Z:

Set-up 4.6. Assume that $f_0 : X_0 \to Y_0$ is smooth. Starting with the diagram (4.5.1), one can find projective morphisms

(4.6.1)
$$\begin{tikzcd} X \arrow[dr, "f"'] & \widehat{Z} \arrow[l, "\hat{\varphi}"'] \arrow[drr, "\hat{g}"'] & & Z \arrow[ll, "\hat{\delta}"'] \arrow[r, "\delta"] \arrow[d, "g"] & \widehat{X} \arrow[r, "\rho"] \arrow[dl, "\hat{f}"] & X \arrow[dl, "f"] \\ & Y & & \widehat{Y} \arrow[ll, "\varphi"] \arrow[r, "\varphi"'] & Y, & \end{tikzcd}$$

(i) such that $\rho : \widehat{X} \to X$ factors through a desingularization $\rho' : \widehat{X} \to X \times_Y \widehat{Y}$,
(ii) $\hat{\delta}$ and δ are modifications, and Z is nonsingular.

Notation 4.7. We will denote from 4.6, by $\widehat{Y}_0$, $\widehat{Z}_0$ $\widehat{X}_0$ (and so on) the preimages of the open subscheme $Y_0 \subset Y$, and by $\hat{\varphi}_0$, $\hat{g}_0$, $\hat{f}_0$ (and so on) the restriction of the corresponding morphisms. Condition (i) implies in particular that $\widehat{X}$ contains $\widehat{X}_0 = X_0 \times_{Y_0} \widehat{Y}_0$ as an open dense subscheme. Later we will also consider a "good" dense open subscheme $Y_g \subset Y_0$ and correspondingly its preimages will be denoted by $\widehat{Y}_g$, $\widehat{Z}_g$, $\widehat{X}_g$ (and so on).

Obviously the properties in 4.5 are compatible with replacing $\widehat{Y}$ by any nonsingular alteration $\widehat{Y}_1 \to \widehat{Y}$. We will do so several times, in order to add additional conditions on the morphism $\hat{g}$. We write $\widehat{Z}_1 = \widehat{Z} \times_{\widehat{Y}} \widehat{Y}_1$ and $\hat{g}_1$ for the second projection. For $\widehat{X}_1$ and Z_1 choose desingularizations of the main components of $\widehat{X} \times_{\widehat{Y}} \widehat{Y}_1$ and $Z \times_{\widehat{Y}} \widehat{Y}_1$, respectively. All the morphisms in the diagram corresponding to (4.6.1) will keep their names, decorated by a little $_1$. Once the additional property is verified, we usually will change notation back and drop the lower index $_1$.

We are also allowed to replace Y by a modification with center in $Y \setminus Y_0$, provided we modify the other schemes in the diagram (4.5.1) accordingly.

As said in the introduction, we are also interested in the polarized case, starting with a morphism $f_0 : X_0 \to Y_0$ and an f_0-ample invertible sheaf $\mathcal{L}_0$. In order to have a reference sheaf one starts with the extension of $\mathcal{L}_0$ to some projective compactification.

VARIANT 4.8. *Assume in Construction* 4.4 *that $\mathcal{L}_0$ is an f_0-ample invertible sheaf. Then one may choose X such that the sheaf $\mathcal{L}_0$ extends to an invertible sheaf $\mathcal{L}$ on X. Moreover, given $\tilde{f} : \widetilde{X} \to \widetilde{Y}$ with $Y_0 \subset \widetilde{Y}$ open and dense and with $\tilde{f}^{-1}(Y_0)$ isomorphic to X_0 over Y_0, one may choose Y and X to be modifications of $\widetilde{Y}$ and $\widetilde{X}$, respectively.*

Proof of 4.8. In fact, one just has to modify the first step in the construction 4.4. Start with any compactification $\tilde{f} : \widetilde{X} \to \widetilde{Y}$. Blowing up $\widetilde{X}$ one may assume that $\mathcal{L}_0$ extends to an invertible sheaf $\widetilde{\mathcal{L}}$. Choose an invertible sheaf $\mathcal{A}$ on $\widetilde{X}$ with $\mathcal{A}$ and $\mathcal{A} \otimes \widetilde{\mathcal{L}}$ very ample. Those two sheaves define embeddings $\iota : \widetilde{X} \to \mathbb{P}^{\ell}$ and $\iota' : \widetilde{X} \to \mathbb{P}^{\ell'}$. The restriction of the diagram

$$\begin{tikzcd} \widetilde{X} \arrow[rr, "(\iota, \iota', \tilde{f})"] \arrow[dr, "\tilde{f}"'] & & \mathbb{P}^{\ell} \times \mathbb{P}^{\ell'} \times \widetilde{Y} \arrow[dl, "\mathrm{pr}_3"] \\ & \widetilde{Y} & \end{tikzcd}$$

to Y_0 gives rise to a morphism $\vartheta : Y_0 \to \mathfrak{Hilb}$ to the Hilbert scheme of subvarieties of $\mathbb{P}^{\ell} \times \mathbb{P}^{\ell'}$. We choose a projective compactification Y of Y_0 such that the morphism

ϑ extends to $\vartheta : Y \to \mathfrak{Hilb}$. The family $f : X \to Y$ is defined as the pullback of the universal family, and $\mathcal{L}$ as the pullback of $\mathcal{O}_{\mathbb{P}^\ell \times \mathbb{P}^{\ell'}}(-1, 1)$. □

5. **Direct images and base change**

We start by recalling some well-known corollaries of "Cohomology and Base Change" for projective morphisms.

LEMMA 5.1. *Let Y be quasiprojective, let $f : X \to Y$ be a projective morphism and let $\mathcal{N}$ be a coherent sheaf on X, flat over Y.*

(i) *There exists a maximal, open, dense subscheme $Y_m \subset Y$ such that the sheaf $f_*\mathcal{N}|_{Y_m}$ is locally free and compatible with base change for morphisms $T \to Y$, factoring through Y_m.*

(ii) *If $f_*\mathcal{N}$ is locally free and compatible with base change for all modifications $\theta : Y' \to Y$, then it is compatible with base change for all morphisms $\varrho : T \to Y$ with $\varrho^{-1}(Y_m)$ dense in T.*

(iii) *There exists a modification $Y' \to Y$ with center in $Y \setminus Y_m$ such that for*

$$\begin{tikzcd} X' = X \times_Y Y' \arrow[r, "\theta'"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ Y' \arrow[r, "\theta"'] & Y \end{tikzcd}$$

the sheaf $f'_*(\theta'^*\mathcal{N})$ is locally free and compatible with base change for morphisms $\varrho : T \to Y'$ with $\varrho^{-1}\theta^{-1}(Y_m)$ dense in T.

Proof. One can assume that Y is affine. By "Cohomology and Base Change" there is a complex

$$E_0 \xrightarrow{\delta_0} E_1 \xrightarrow{\delta_1} \cdots \xrightarrow{\delta_{m-1}} E_m \tag{5.1.1}$$

of locally free sheaves, whose i-th cohomology calculates $R^i f_*\mathcal{N}$, as well as its base change. We choose Y_m to be the open dense subscheme, where the image $\mathcal{C}$ of δ_0 locally splits in E_1. One has an exact sequence on Y,

$$0 \longrightarrow \mathcal{K} = \mathrm{Ker}(\delta_0) \longrightarrow E_0 \longrightarrow \mathcal{C} \longrightarrow 0. \tag{5.1.2}$$

Part (ii) can be extended in the following way:

CLAIM 5.2. *The following conditions are equivalent:*

(a) *$\mathcal{C}$ is locally free.*

(b) *$f_*\mathcal{N}$ is locally free and compatible with base change for all modifications $\varrho : T \to Y$.*

(c) *$f_*\mathcal{N}$ is locally free and compatible with base change for all morphisms $\varrho : T \to Y$ with $\varrho^{-1}(Y_m)$ dense in T.*

Proof. Of course (c) implies (b). If $\mathcal{C}$ is locally free $\mathcal{H} = f_*\mathcal{N}$ is locally free, and for all morphisms $\varrho : T \to Y$ the sequence

$$0 \longrightarrow \varrho^*\mathcal{H} \longrightarrow \varrho^* E_0 \longrightarrow \varrho^*\mathcal{C} \longrightarrow 0$$

remains an exact sequence of locally free sheaves. If $\varrho^{-1}(Y_m)$ is dense in T the morphism $\varrho^*\mathcal{C} \to \varrho^* E_1$ is injective on some open dense subset, hence injective. Recall that the complex

$$\varrho^* E_0 \xrightarrow{\delta'_0} \varrho^* E_1 \xrightarrow{\delta'_1} \cdots \xrightarrow{\delta'_{m-1}} \varrho^* E_m \tag{5.2.1}$$

calculates the higher direct images of $\mathrm{pr}_1^*\mathcal{N}$ on the pullback family $X \times_Y T \to T$. As we have just seen, $\varrho^*\mathcal{H}$ is the kernel of δ'_0, hence equal to $\mathrm{pr}_{2*}\mathrm{pr}_1^*\mathcal{N}$.

So (a) implies (c) and it remains to show that (b) implies (a). By assumption $\mathcal{H} = f_*\mathcal{N}$ is locally free, so that $\mathcal{C}$ is the cokernel of a morphism between locally free sheaves of rank $\ell = \mathrm{rk}(\mathcal{H})$ and $e = \mathrm{rk}(E_0)$, and $r = e - \ell = \mathrm{rk}(\mathcal{C})$. Thus, $\mathcal{C}$ is not locally free if and only if the r-th Fitting ideal is nontrivial (see for example [Eis95, Prop. 20.6]). Choose for $\varrho : T \to Y$ a blowing up, such that $\varrho^*\mathcal{C}/_{\text{torsion}}$ is locally free. The fitting ideal is compatible with pullback (see [Eis95, Cor. 20.5]), hence $\varrho^*\mathcal{C}$ itself is not locally free. Then, by the notation from (5.2.1), $\varrho^*\mathcal{H} \subsetneq \mathrm{Ker}(\delta'_0) = \mathrm{pr}_{2*}\mathrm{pr}_1^*\mathcal{N}$, violating (b). □

The argument used at the end of the proof of 5.2 also implies that the subscheme Y_m is maximal with the property asked for in (ii). In fact, if the image $\mathcal{C}$ does not split locally in a neighborhood of a general point of $\varrho(T)$, the map $\varrho^*\mathcal{C} \to \varrho^* E_1$ cannot be injective and one finds again that $\varrho^*\mathcal{H} \subsetneq \mathrm{Ker}(\delta'_0)$.

By the choice of Y_m the sequence (5.1.2) locally splits on Y_m, and there is a blowing up $\theta : Y' \to Y$ with center in $Y \setminus Y_m$, such that $\theta^*(\mathcal{C})/_{\text{torsion}}$ is locally free. This sheaf is a subsheaf of $\theta^*(E_1)$, hence it is the image of $\theta^*(\delta_0)$. So the latter is locally free, and by Claim 5.2 we found the modification we are looking for in (iii). □

For relatively semiample sheaves on the total space of a mild morphism the modification of the base in 5.1(iii) is not needed. Let us recall the following base change criterion, essentially due to Kollár:

LEMMA 5.3. *Let $\hat{g} : \widehat{Z} \to \widehat{Y}$ be a mild morphism, and let $\hat{\mathcal{L}}$ be a $\hat{g}$-semiample invertible sheaf on $\widehat{Z}$. Then for all $i \geq 0$ the sheaves $R^i \hat{g}_*(\omega_{\widehat{Z}/\widehat{Y}} \otimes \hat{\mathcal{L}})$ are locally free and compatible with arbitrary base change.*

Proof. By "Cohomology and Base Change", i.e., using the complex $E_\bullet$ in (5.1.1) for $\hat{g} : \widehat{Z} \to \widehat{Y}$ instead of $f : X \to Y$, one finds that it is sufficient to show that the sheaves $R^i \hat{g}_*(\omega_{\widehat{Z}/\widehat{Y}} \otimes \hat{\mathcal{L}})$ are locally free, or equivalently that the cohomology sheaves $\mathcal{H}^i(E_\bullet)$ are all locally free.

As recalled in [EV92, Cor. 6.12] Kollár's vanishing theorem (loc.sit. Cor. 5.6) implies that the sheaves $R^i \hat{g}_*(\omega_{\widehat{Z}/\widehat{Y}} \otimes \hat{\mathcal{L}})$ are torsion-free. In particular, if $\dim(\widehat{Y}) = 1$ we are done.

In general consider the largest open subscheme $\widehat{Y}_g$ of $\widehat{Y}$ with $\hat{g}^{-1}(\widehat{Y}_g) \to \widehat{Y}_g$ smooth. Let $\iota : C \to \widehat{Y}$ be a morphism from a projective curve to $\widehat{Y}$ whose image meets $\widehat{Y}_g$. Then $h : S = \widehat{Z} \times_{\widehat{Y}} C \to C$ is again mild, in particular S is again normal with rational Gorenstein singularities. Hence $R^i h_*(\omega_{S/C} \otimes \mathrm{pr}_1^* \hat{\mathscr{L}})$ is locally free. This implies that for all points $y \in \iota(C)$ the dimensions

$$h^i(y) = \dim H^i(\hat{g}^{-1}(y), \omega_{\hat{g}^{-1}(y)} \otimes \hat{\mathscr{L}}|_{\hat{g}^{-1}(y)})$$

are the same. Since $\widehat{Y}$ is covered by such curves $h^i(y)$ is constant on $\widehat{Y}$. Hence $\mathscr{H}^i(E_\bullet)$ is locally free. □

The proof of 5.3 gives a first indication why we need weak semistable models. In general even if X has at most rational Gorenstein singularities, and if $f : X \to Y$ is flat, the arguments used in the proof of 5.3 fail. Given $\iota : C \to Y$ and $S = X \times_Y C$ we would not know that S again has rational Gorenstein singularities.

Let us return to the notation introduced in the last section. Starting from a smooth morphism $f_0 : X_0 \to Y_0$ consider again morphisms $\varphi : \widehat{Y} \to Y$ and $\hat{g} : \widehat{Z} \to \widehat{Y}$ satisfying conditions (a)–(c) in 4.5. We choose the diagram (4.6.1) in 4.6 in such a way that conditions (i) and (ii) hold.

Set-up 5.4. Let $\mathscr{L}_0$ be an invertible sheaf on X_0, either equal to $\mathscr{O}_{X_0}$ or f_0-ample sheaf. In the first case we write $\mathscr{L} = \mathscr{O}_X$; in the second one we fix an invertible extension $\mathscr{L}$ of $\mathscr{L}_0$ to X, as constructed in Variant 4.8. Assume that $\mathscr{M}_Z$, $\mathscr{M}_{\widehat{Z}}$ and $\mathscr{M}_{\widehat{X}}$ are invertible sheaves on Z, $\widehat{Z}$ and $\widehat{X}$, respectively, with

$$\hat{\delta}_* \mathscr{M}_Z = \mathscr{M}_{\widehat{Z}}, \;\; \delta_* \mathscr{M}_Z = \mathscr{M}_{\widehat{X}}, \;\; \hat{\varphi}^* \mathscr{L} \subset \mathscr{M}_{\widehat{Z}},$$
$$\mathscr{M}_{\widehat{Z}_0} = \mathscr{M}_{\widehat{Z}}|_{\widehat{Z}_0} = \hat{\varphi}_0^* \mathscr{L}_0 \quad \text{and} \quad \mathscr{M}_{\widehat{X}_0} = \mathscr{M}_{\widehat{X}}|_{\widehat{X}_0} = \rho_0^* \mathscr{L}_0.$$

We fix some finite set I of tuples (ν, μ) of nonnegative integers and define

$$\mathscr{F}_{\widehat{Y}}^{(\nu,\mu)} = \hat{g}_*\big(\omega_{\widehat{Z}/\widehat{Y}}^{\nu} \otimes \mathscr{M}_{\widehat{Z}}^{\mu}\big).$$

We choose for $\widehat{Y}_g$ an open dense subscheme of $\widehat{Y}_0$ such that $\hat{g}^{-1}(\widehat{Y}_g) \to \widehat{Y}_g$ is smooth and such that the sheaves $R^i \hat{g}_*(\omega_{\widehat{Z}/\widehat{Y}}^{\nu} \otimes \mathscr{M}_{\widehat{Z}}^{\mu})$ are locally free and compatible with base change for morphisms $\varrho : T \to \widehat{Y}_g$, for all $(\nu, \mu) \in I$ and for all i.

If $\mathscr{L}_0 = \mathscr{O}_{X_0}$ we choose $\mathscr{M}_\bullet = \mathscr{O}_\bullet$. In this case $I = I' \times \{0\}$ for some finite set of natural numbers I'.

Given an invertible sheaf $\mathscr{L}$ on X one could define $\mathscr{M}_Z$, $\mathscr{M}_{\widehat{Z}}$ and $\mathscr{M}_{\widehat{X}}$ as the pullbacks of $\mathscr{L}$. In particular this choice seems to be the most natural one if $\mathscr{L}$ is f-ample. For families of polarized minimal models we will define in Section 8 other extensions of $\mathscr{M}_{\widehat{Z}_0} = \hat{\varphi}_0^* \mathscr{L}_0$ and $\mathscr{M}_{\widehat{X}_0} = \rho_0^* \mathscr{L}_0$, better suited for a generalization of Addendum 3.4.

If $\widehat{Y}_1 \to \widehat{Y}$ is a nonsingular alteration (see 4.7 for our standard notation), the sheaves $\mathcal{M}_{\widehat{Z}_1}$, $\mathcal{M}_{\widehat{X}_1}$ and $\mathcal{M}_{Z_1}$ are defined by pullback, and they obviously satisfy again the properties asked for in 5.4, with $\widehat{Y}_g$ replaced by its preimage in $\widehat{Y}_1$.

COROLLARY 5.5. *In* 5.4 *one may choose* $\widehat{Y}$ *and* $\widehat{Z}$ *in Proposition* 4.5 *such that in addition to conditions* (a)–(c) *one has*:

(d) *For* $(\nu, \mu) \in I$ *the sheaves* $\mathcal{F}_{\widehat{Y}}^{(\nu,\mu)}$ *are locally free and compatible with base change for morphisms* $\varrho : T \to \widehat{Y}$ *with* $\varrho^{-1}(\widehat{Y}_g)$ *dense in* T. *Hence writing*

$$\begin{array}{ccc} \widehat{Z}_T & \xrightarrow{\varrho'} & \widehat{Z} \\ \downarrow{\scriptstyle \hat{g}_T} & & \downarrow{\scriptstyle \hat{g}} \\ T & \xrightarrow{\varrho} & \widehat{Y} \end{array}$$

for the fiber product, one has $\mathcal{F}_T^{(\nu,\mu)} := \varrho^* \mathcal{F}_{\widehat{Y}}^{(\nu,\mu)} = \hat{g}_{T*}(\omega^{\nu}_{\widehat{Z}_T/T} \otimes \varrho'^* \mathcal{M}^{\mu}_{\widehat{Z}})$.

Proof. Properties (a)–(c) in Proposition 4.5 are compatible with base change by nonsingular alterations $\widehat{Y}_1 \to \widehat{Y}$. So using Lemma 5.1(iii), we may assume that for a given tuple (ν, μ) and $\mathcal{N} = \omega^{\nu}_{\widehat{Z}/\widehat{Y}} \otimes \mathcal{M}^{\mu}_{\widehat{Z}}$ condition (ii) in 5.1, holds true on $\widehat{Y}$ itself. Again (d) is compatible with base change for alterations, and repeating the construction for the other tuples in I one obtains Corollary 5.5. □

One important example are the r-fold fiber products. Recall that by Lemma 4.2 the morphism

$$\hat{g}^r : \widehat{Z}^r = \widehat{Z} \times_{\widehat{Y}} \cdots \times_{\widehat{Y}} \widehat{Z} \longrightarrow \widehat{Y}$$

is mild. One has $\omega_{\widehat{Z}^r/\widehat{Y}} = \mathrm{pr}_1^* \omega_{\widehat{Z}/\widehat{Y}} \otimes \cdots \otimes \mathrm{pr}_r^* \omega_{\widehat{Z}/\widehat{Y}}$. For $\mathcal{M}_{\widehat{Z}^r} = \mathrm{pr}_1^* \mathcal{M}_{\widehat{Z}} \otimes \cdots \otimes \mathrm{pr}_r^* \mathcal{M}_{\widehat{Z}}$ flat base change and the projection formula give:

COROLLARY 5.6. *Condition* (*d*) *in* 5.5 *implies that* $\hat{g}^r_*(\omega^{\nu}_{\widehat{Z}^r/\widehat{Y}} \otimes \mathcal{M}^{\mu}_{\widehat{Z}^r}) = \bigotimes^r \mathcal{F}_{\widehat{Y}}^{(\nu,\mu)}$, *for* $(\nu, \mu) \in I$. *In particular those sheaves are again locally free and compatible with base change for morphisms* $\varrho : T \to \widehat{Y}$ *with* $\varrho^{-1}(\widehat{Y}_g)$ *dense in* T.

In order to define the sheaves $\mathcal{F}_{\widehat{Y}}^{(\nu,\mu)}$ and to study their behavior under base change and products, we used the mild model $\hat{g} : \widehat{Z} \to \widehat{Y}$. However since we might have blown up the smooth fibers of $X_0 \to Y_0$ in order to find the mild model, this is not really the right object to study. As a next step we will use the right-hand side of the diagram (4.6.1) in 4.6 to derive properties of the geometrically more meaningful morphism $\hat{f} : \widehat{X} \to \widehat{Y}$. Jumping from one side of (4.6.1) to the other is possible by:

LEMMA 5.7. *For all $\nu, \mu \geq 0$ the natural maps*

$$g_*\big(\omega^{\nu}_{Z/\widehat{Y}} \otimes \mathcal{M}^{\mu}_{Z}\big) \longrightarrow \hat{f}_*\big(\omega^{\nu}_{\widehat{X}/\widehat{Y}} \otimes \mathcal{M}^{\mu}_{\widehat{X}}\big) \quad \text{and}$$
$$g_*\big(\omega^{\nu}_{Z/\widehat{Y}} \otimes \mathcal{M}^{\mu}_{Z}\big) \longrightarrow \hat{g}_*\big(\omega^{\nu}_{\widehat{Z}/\widehat{Y}} \otimes \mathcal{M}^{\mu}_{\widehat{Z}}\big) = \mathcal{F}^{(\nu,\mu)}_{\widehat{Y}}$$

are both isomorphisms.

Proof. The morphisms δ and $\hat{\delta}$ are both birational. Since $\widehat{X}$ is smooth and $\widehat{Z}$ is Gorenstein with rational singularities one can find effective divisors $E_{\widehat{Z}}$ and $E_{\widehat{X}}$, contained in the exceptional loci of $\hat{\delta}$ and δ, with

$$\omega_{Z/\widehat{Y}} = \hat{\delta}^*\omega_{\widehat{Z}/\widehat{Y}} \otimes \mathcal{O}_Z(E_{\widehat{Z}}) = \delta^*\omega_{\widehat{X}/\widehat{Y}} \otimes \mathcal{O}_Z(E_{\widehat{X}}).$$

On the other hand, $\mathcal{M}_{\widehat{X}} = \delta_*\mathcal{M}_Z$ and $\mathcal{M}_{\widehat{Z}} = \hat{\delta}_*\mathcal{M}_Z$; hence for some effective divisors $F_{\widehat{Z}}$ and $F_{\widehat{X}}$, contained again in the exceptional loci of $\hat{\delta}$ and δ, one has

$$\mathcal{M}_Z = \hat{\delta}^*\mathcal{M}_{\widehat{Z}} \otimes \mathcal{O}_Z(F_{\widehat{Z}}) = \delta^*\mathcal{M}_{\widehat{X}} \otimes \mathcal{O}_Z(F_{\widehat{X}}).$$

The projection formula implies that

$$\hat{\delta}_*\big(\omega^{\nu}_{Z/Y} \otimes \mathcal{M}^{\mu}_{Z}\big) = \omega^{\nu}_{\widehat{Z}/\widehat{Y}} \otimes \mathcal{M}^{\mu}_{\widehat{Z}} \otimes \hat{\delta}_*\mathcal{O}_Z(\nu \cdot E_{\widehat{Z}} + \mu \cdot F_{\widehat{Z}}) = \omega^{\nu}_{\widehat{Z}/\widehat{Y}} \otimes \mathcal{M}_{\widehat{Z}}$$

and $$\delta_*\big(\omega^{\nu}_{Z/Y} \otimes \mathcal{M}^{\mu}_{Z}\big) = \omega^{\nu}_{\widehat{X}/\widehat{Y}} \otimes \mathcal{M}^{\mu}_{\widehat{X}} \otimes \delta_*\mathcal{O}_Z(\nu \cdot E_{\widehat{X}} + \mu \cdot F_{\widehat{X}}) = \omega^{\nu}_{\widehat{X}/\widehat{Y}} \otimes \mathcal{M}_{\widehat{X}},$$

hence 5.7. □

As we just have seen, the isomorphisms of sheaves in 5.7 are given over some open dense subscheme by the birational maps $\hat{\delta}$ and δ. We will write in a sloppy way $=$ instead of $\cong$ for all such isomorphisms and for those induced by base change.

Since $f : \widehat{X} \to \widehat{Y}$ is not necessarily flat, we cannot apply "Cohomology and Base Change" to the right-hand side of the diagram (4.6.1), except if the (unnatural) assumptions of the next lemma hold true, for example for embedded semistable reductions over curves considered in Section 7.

LEMMA 5.8. *Assume in 5.5 that for $(0, \mu) \in I$ the sheaves $f_{0*}\mathcal{M}^{\mu}_{X_0}$ are locally free and compatible with arbitrary base change. Let $U \subset \widehat{Y}$ be an open subscheme, such that $V = \hat{f}^{-1}(U) \to U$ is flat. Let $T \subset U$ be a curve, meeting $\widehat{Y}_0$, and assume that for all coverings $T' \to T$ the variety $V \times_U T'$ is normal with at most rational Gorenstein singularities. Then for $(\nu, \mu) \in I$ the direct image $\hat{f}_*(\omega^{\nu}_{\widehat{X}/\widehat{Y}} \otimes \mathcal{M}^{\mu}_{\widehat{X}})$ is compatible with base change for all $T' \to T \subset U$.*

Proof. By Lemma 5.3 the sheaves $f_{0*}(\omega^{\nu}_{X_0/Y_0} \otimes \mathcal{M}^{\mu}_{X_0})$ are locally free and compatible with arbitrary base change for all $(\nu, \mu) \in I$ with $\nu > 0$, and by assumption the same holds for $\nu = 0$.

Let $\theta : \widehat{Y}_1 \to \widehat{Y}$ be a modification. By the choice of I in Corollary 5.5 one knows that $\theta^* \mathcal{F}^{(\nu,\mu)}_{\widehat{Y}} = \mathcal{F}^{(\nu,\mu)}_{\widehat{Y}_1}$, and by Lemma 5.7

$$\mathcal{F}^{(\nu,\mu)}_{\widehat{Y}} = \hat{f}_*\big(\omega^{\nu}_{\widehat{X}/\widehat{Y}} \otimes \mathcal{M}^{\mu}_{\widehat{X}}\big), \quad \text{and} \quad \mathcal{F}^{(\nu,\mu)}_{\widehat{Y}_1} = \hat{f}_{1*}\big(\omega^{\nu}_{\widehat{X}_1/\widehat{Y}_1} \otimes \mathcal{M}^{\mu}_{\widehat{X}_1}\big).$$

Cutting $\widehat{Y}_1$ with hyperplanes one finds, through any point $p \in \theta^{-1}(T)$, a curve T' mapping surjectively to T. Then, as we will see in Lemma 7.3, $\widehat{X} \times_{\widehat{Y}} \widehat{Y}_1$ will be normal with at most rational Gorenstein singularities in a neighborhood of $\theta^{-1}(T)$. Replacing $\widehat{Y}$ by U, we may assume that this is the case for $\widehat{X} \times_{\widehat{Y}} \widehat{Y}_1$ itself. So $\hat{f}_*(\omega^{\nu}_{\widehat{X}/\widehat{Y}} \otimes \mathcal{M}^{\mu}_{\widehat{X}})$ is locally free and compatible with base change for modifications. On the other hand by assumption the sheaves $\hat{f}_{0*}(\omega^{\nu}_{\widehat{X}_0/\widehat{Y}_0} \otimes \mathcal{M}^{\mu}_{\widehat{X}_0})$ are locally free and compatible with arbitrary base change; hence the open subscheme $\widehat{Y}_m$ in Lemma 5.1(ii) contains $\widehat{Y}_0$, and 5.8 follows from Lemma 5.1(ii). $\square$

Note that Lemma 5.8 does not imply that $\hat{g}_*(\omega^{\nu}_{\widehat{Z}/\widehat{Y}} \otimes \mathcal{M}^{\mu}_{\widehat{Z}})$ is compatible with base change for morphisms $\varrho : T \to \widehat{Y}$ with $\varrho^{-1}(\widehat{Y}_0)$ dense. If $\varrho^{-1}(\widehat{Y}_g)$ is not dense, we do not know that $\widehat{Z} \times_{\widehat{Y}} T \to T$ is mild; hence we cannot use Lemma 5.7.

Comments 5.9. The proof of Theorem 1 could be finished at this stage. Let us sketch the line of arguments, hoping that it will serve as an introduction to the remaining part of the article.

The Extension Theorem of Gabber starts with a projective scheme Y, an open dense subscheme Y_0, a nonsingular alteration $\varphi : \widehat{Y} \to Y$. Write $\widehat{Y}_0 = \varphi^{-1}(Y_0)$ and $\varphi_0 = \varphi|_{\widehat{Y}_0}$. Next, one considers locally free sheaves $\mathcal{F}_{Y_0}$ and $\mathcal{F}_{\widehat{Y}}$ with $\varphi_0^*(\mathcal{F}_{Y_0}) = \mathcal{F}_{\widehat{Y}}|_{\widehat{Y}_0}$. In addition one has a finite covering $\phi : W \to Y$ with a splitting trace map, whose normalization is the Stein factorization of φ.

In addition one needs a sheaf $\mathcal{F}_C$ for each curve $\pi : C \to W$ whose image meets $W_0 = \phi^{-1}(Y_0)$. If π factors through $\chi : C \to \widehat{Y}$ one requires that $\chi^* \mathcal{F}_{\widehat{Y}} = \mathcal{F}_C$, and $\mathcal{F}_C$ must be compatible with replacing C by a covering. The conclusion is the existence of the sheaf $\mathcal{F}_W$, perhaps after one replaces W by a modification with center in $W \setminus W_0$.

Let us try to verify those conditions for $\mathcal{F}^{(\nu)}_{Y_0}$. The alteration $\widehat{Y}$ and the sheaf $\mathcal{F}^{(\nu)}_{\widehat{Y}}$ have been constructed in Sections 4 and 5. For $\mathcal{F}^{(\nu)}_C$ there is little choice. It has to be the direct image $h_* \omega^{\nu}_{S/C}$ for a desingularization $h : S \to C$ of the pullback family. The compatibility with finite coverings enforces the assumption that $h : S \to C$ has a semistable or a mild model. So we have to verify two conditions:

(1) If $\pi_0 : C_0 \to W_0$ is a morphism from a nonsingular curve, then the pullback family $h_0 : S_0 \to C_0$ allows a mild model $f : S \to C$ over the smooth compactification C of C_0.

(2) If the morphism π_0 factors through the restriction of $\chi : C \to \widehat{Y}$ to C_0, then $\chi^* \mathcal{F}^{(\nu)}_{\widehat{Y}} = h_* \omega^{\nu}_{S/C}$.

For (2), Lemma 5.8 will be of help. Its application is made possible by the embedded weak semistable reduction over curves, discussed in the first part of Section 7 and stated in Proposition 7.8. There we first make flat the morphism in a neighborhood of a given curve. Replacing this neighborhood by an alteration we may assume that the pull-back family is semistable over C. Using this construction we will verify (2) in Proposition 10.5.

Note that (1) holds for curves $\pi_0 : C_0 \to W_0$ whose image meets the open dense subscheme W_g where $\widehat{Y} \to W$ is an isomorphism, and whose lifting to $\widehat{Y}$ meets the open set $\widehat{Y}_g$, defined in 4.1. This allows us to verify (1) in Section 12 on coverings of certain locally closed subschemes of W. The necessary gluing is made possible by studying embeddings of W into projective spaces, which at the same time will take care of the splitting trace map.

So both sides of the diagram 4.3 play their role. The left-hand side is needed for the definition of $\mathcal{F}^{(\nu)}_{\widehat{Y}}$, for its compatibility with alterations and for the verification of (1). The right-hand side is needed to control the compatibility of the sheaves to curves, as stated in (2).

As indicated in the comments in 2.5, the properties of W and $\mathcal{F}^{(\nu)}_{W}$ stated in Theorem 1 are not sufficient to get condition (iv) in Theorem 2. This forces us to allow multiplier ideals, which will also help to extend Corollary 3 to sheaves of the form $\mathcal{F}^{(\nu,\mu)}_{\bullet}$ with $\mu > 0$. Since we do not want to repeat the same construction in two slightly different set-ups, we will first try to understand multiplier ideals in families in Section 6, or to be more precise, base change properties of "multiplier sheaves", i.e., of the tensor product of a multiplier ideal with an invertible sheaf. As we will see in 6.3 one does not even have a reasonable "base change morphism" to start with. And in general such multiplier sheaves will not be flat over the base. If they occur as in Definition 2.1 as a direct factor of the direct image of an alteration $Z' \to Z$, and if Z is the total space of a family $Z \to Y$ we will perform the weakly stable reduction for $Z' \to Y$, in order to find a nice model.

The corresponding application of the embedded weak semistable reduction is given in the second half of Section 7. In Section 9 we define certain multiplier sheaves on fiber products, depending on certain tautological maps, similar to the determinant Ξ in the proof of Variant 2.4. Here we have to give the definitions for both sides of the diagram 4.3 and to verify certain compatibilities. In Section 11 we verify similar compatibilities for the restriction to curves, and we extends the proof of (2) to multiplier sheaves.

We invite the reader to jump directly to Section 12, stopping shortly at Sections 7 and 10, and to fill in the details needed for the ampleness property later. As we said before in (3.4.1), Section 8 is mainly needed for the identification of the natural ample sheaf on the compactification of the moduli scheme in Theorem 6.

6. **Flattening and pullbacks of multiplier ideals**

If multiplier ideals occur as in Definition 2.1 as a direct factor of the direct image of an alteration $Z' \to Z$, and if Z is the total space of a family $Z \to Y$, the weakly stable reduction for $Z' \to Y$ will allow us to verify certain functorial properties. This will later be applied to the mild morphism $\hat{g} : \hat{Z} \to \hat{Y}$. The constructions will force us to replace the base by a new alteration, an excuse to drop the ˆ and to start with:

Assumptions and Notation 6.1. Let $g : Z \to Y$ be a flat, projective, surjective, Gorenstein morphism over a nonsingular variety Y. Assume that the r-fold fiber product $Z^r = Z \times_Y \cdots \times_Y Z$ is normal with at most rational singularities.

Let $\mathcal{N}$ be an invertible sheaf on Z, let Δ be an effective Cartier divisor on Z and let $N > 1$ be a natural number. Assume that there is a locally free sheaf $\mathcal{E}$ together with a morphism $\mathcal{E} \to g_*\mathcal{N}^N$ on Y with $g^*\mathcal{E} \to \mathcal{N}^N \otimes \mathcal{O}_Z(-\Delta)$ surjective.

Let $\mathfrak{C}$ be a set of morphisms from normal varieties T with at most rational Gorenstein singularities to Y, such that for all $(\theta : T \to Y) \in \mathfrak{C}$ and for all $r > 0$ the variety $Z^r_T = Z^r \times_Y T$ is again normal with at most rational Gorenstein singularities. We will need in addition that $g^{-1}(\theta(T))$ is not contained in the support of Δ.

For $(\varrho : T \to Y) \in \mathfrak{C}$ we will write $\varrho_T : Z_T \to Z$ and $g_T : Z_T \to T$ for the induced morphisms. On the products the corresponding morphisms will be denoted by

$$\varrho^r_T : Z^r_T \to Z^r \quad \text{and} \quad g^r_T : Z^r_T \to T.$$

We consider $\Delta^r = \mathrm{pr}_1^*\Delta + \cdots + \mathrm{pr}_r^*\Delta$ on Z^r and Δ_T or Δ^r_T denotes the pullbacks of those divisors to Z_T or Z^r_T. We write

$$\mathcal{N}_{Z^r} = \mathrm{pr}_1^*\mathcal{N} \otimes \cdots \otimes \mathrm{pr}_r^*\mathcal{N} \quad \text{and} \quad \mathcal{A}_{Z^r} = \mathrm{pr}_1^*\mathcal{A} \otimes \cdots \otimes \mathrm{pr}_r^*\mathcal{A}$$

for an invertible auxiliary sheaf $\mathcal{A}$ on $\hat{Z}$, usually ample or semiample.

If $g : Z \to Y$ is a mild morphism, smooth over a dense open subscheme Y_g, and if Δ does not contain $g^{-1}(y)$ for $y \in Y_g$, then $\mathfrak{C}$ can be chosen as the set of morphisms $\varrho : T \to Y$ with T a normal variety with at most rational Gorenstein singularities, where either ϱ is dominant, or $\dim(T) = 1$ and $\varrho^{-1}(Y_g)$ is dense in T.

Conditions 6.2. In 6.1 write $\varepsilon = \frac{1}{N}$ and consider for $\varrho \in \mathfrak{C}$ the following statements:

(a) $\mathcal{J}(-\varepsilon \cdot \Delta)$ is compatible with r-th products, i.e.

$$\mathcal{J}(-\varepsilon \cdot \Delta^r) \cong \left[\mathrm{pr}_1^*\mathcal{J}(-\varepsilon \cdot \Delta) \otimes \cdots \otimes \mathrm{pr}_r^*\mathcal{J}(-\varepsilon \cdot \Delta)\right]/_{\text{torsion}}.$$

(b) For all $r \geq 1$ there is a natural isomorphism

$$\varrho^{r*}_T \mathcal{J}(-\varepsilon \cdot \Delta^r)/_{\text{torsion}} \xrightarrow{\cong} \mathcal{J}\big(-\varepsilon \cdot \Delta^r_T\big).$$

(c) For all g-semiample invertible sheaves $\mathcal{A}$ on Z the direct image

$$g^r_*(\omega_{Z^r/Y} \otimes \mathcal{A}_{Z^r} \otimes \mathcal{N}_{Z^r} \otimes \mathcal{J}(-\varepsilon \cdot \Delta^r))$$

is locally free and the composite

$$\begin{aligned} &\varrho^* g^r_*(\omega_{Z^r/Y} \otimes \mathcal{A}_{Z^r} \otimes \mathcal{N}_{Z^r} \otimes \mathcal{J}(-\varepsilon \cdot \Delta^r)) \\ &\qquad \xrightarrow{\gamma} g^r_{T*}(\omega_{Z^r_T/T} \otimes \varrho^{r*}_T(\mathcal{A}_{Z^r} \otimes \mathcal{N}_{Z^r} \otimes \mathcal{J}(-\varepsilon \cdot \Delta^r))) \\ &\qquad \xrightarrow{\eta} g^r_{T*}(\omega_{Z^r_T/T} \otimes \varrho^{r*}_T(\mathcal{A}_{Z^r} \otimes \mathcal{N}_{Z^r} \otimes \mathcal{J}(-\varepsilon \cdot \Delta^r))/_{\text{torsion}}) \\ &\qquad \xrightarrow{\cong} g^r_{T*}(\omega_{Z^r_T/T} \otimes \varrho^{r*}_T(\mathcal{A}_{Z^r} \otimes \mathcal{N}_{Z^r}) \otimes \mathcal{J}(-\varepsilon \cdot \Delta^r_T)) \end{aligned}$$

of the base change morphism and the quotient map in (b) is an isomorphism.

(d) One then has an isomorphism

$$\bigotimes^r g_*(\omega_{Z/Y} \otimes \mathcal{A} \otimes \mathcal{N} \otimes \mathcal{J}(-\varepsilon \cdot \Delta)) \cong g^r_*\big(\omega_{Z^r/Y} \otimes \mathcal{A}_{Z^r} \otimes \mathcal{N}_{Z^r} \otimes \mathcal{J}(-\varepsilon \cdot \Delta^r)\big).$$

Example 6.3. In general, multiplier ideals behave badly under base change. Even if $T \subset Y$ is a smooth divisor on a surface, $\mathcal{J}(-\varepsilon \cdot \Delta)|_{Z_T}$ might be larger than $\mathcal{J}(-\varepsilon \cdot \Delta_T)$. So in general one cannot even expect the existence of a map

$$\varrho^*_T \mathcal{J}(-\varepsilon \cdot \Delta) \longrightarrow \mathcal{J}(-\varepsilon \cdot \Delta_T)$$

in 6.2(b).

LEMMA AND DEFINITION 6.4. *Under the assumptions made in* 6.1 *we say that $\mathcal{J}(-\varepsilon \cdot \Delta)$ is compatible with pullback, base change and products for $\varrho \in \mathfrak{C}$ if conditions* (a)–(d) *in* 6.2 *hold true, or if equivalently*:

(i) *For all $r > 0$ the sheaves*

$$\mathrm{pr}_1^* \mathcal{J}(-\varepsilon \cdot \Delta) \otimes \cdots \otimes \mathrm{pr}_r^* \mathcal{J}(-\varepsilon \cdot \Delta) \quad \textit{and} \quad \varrho^{r*}_T \mathcal{J}(-\varepsilon \cdot \Delta^r)$$

are torsion-free and isomorphic to $\mathcal{J}(-\varepsilon \cdot \Delta^r)$ and $\mathcal{J}(-\varepsilon \cdot \Delta^r_T)$, respectively.

(ii) *For all g-semiample invertible sheaves $\mathcal{A}$ on Z the direct image*

$$g^r_*(\omega_{Z^r/Y} \otimes \mathcal{A}_{Z^r} \otimes \mathcal{N}_{Z^r} \otimes \mathcal{J}(-\varepsilon \cdot \Delta^r))$$

is locally free and compatible with base change for $\varrho \in \mathfrak{C}$.

Moreover conditions (i) *and* (ii) *imply*:

(iii) *The multiplier ideal $\mathcal{J}(-\varepsilon \cdot \Delta^r)$ is flat over Y.*

Proof. Let us remark first, that by Grothendieck's cohomological criterion for flatness [GD61, Prop. 7.9.14] the local freeness of the sheaves

$$g^r_*(\omega_{Z^r/Y} \otimes \mathcal{A}_{Z^r} \otimes \mathcal{N}_{Z^r} \otimes \mathcal{J}(-\varepsilon \cdot \Delta^r))$$

for all powers of a given ample sheaf implies that $\mathcal{J}(-\varepsilon \cdot \Delta^r))$ is flat over Y. Hence either one of condition (ii) and condition (c) of 6.2 implies condition (iii).

Let us assume that (i) and (ii) hold true. Then (a) and (b) follow from (i), and condition (iii) allows us to get (d) in 6.2 by flat base change. By (i) the morphism η in (c) is the identity, and γ is the usual base change map, hence an isomorphism by (ii).

For the other direction we have already seen that 6.4(iii) holds. So for $\mathcal{A}$ sufficiently ample, the base change morphism γ in 6.2(c) is an isomorphism. Since its composite with η is assumed to be an isomorphism as well, η must be an isomorphism. This being true for all ample sheaves $\mathcal{A}$ one finds that $\varrho_T^{r*}\mathcal{J}(-\varepsilon\cdot\Delta^r)$ is torsion-free, and (b) implies

$$\varrho_T^{r*}\mathcal{J}(-\varepsilon\cdot\Delta^r) \xrightarrow{\cong} \varrho_T^{r*}\mathcal{J}(-\varepsilon\cdot\Delta^r)/_{\text{torsion}} \xrightarrow{\cong} \mathcal{J}\big(-\varepsilon\cdot\Delta_T^r\big).$$

Using (iii) for $r=1$, one finds by flat base change and by the projection formula that

$$\begin{aligned}(6.4.1)\qquad & g_*^r(\omega_{Z^r/Y}\otimes\mathcal{A}_{Z^r}\otimes\mathcal{N}_{Z^r}\otimes \mathrm{pr}_1^*\mathcal{J}(-\varepsilon\cdot\Delta)\otimes\cdots\otimes\mathrm{pr}_r^*\mathcal{J}(-\varepsilon\cdot\Delta))\\ &\cong \bigotimes^r g_*(\omega_{Z/Y}\otimes\mathcal{A}\otimes\mathcal{N}\otimes\mathcal{J}(-\varepsilon\cdot\Delta)).\end{aligned}$$

In particular both sheaves are locally free, and the cohomological criterion for flatness implies that $\mathrm{pr}_1^*\mathcal{J}(-\varepsilon\cdot\Delta)\otimes\cdots\otimes\mathrm{pr}_r^*\mathcal{J}(-\varepsilon\cdot\Delta)$ is flat over Y. Condition (d) tells us that the direct images in (6.4.1) are isomorphic to

$$g_*^r(\omega_{Z^r/Y}\otimes\mathcal{A}_{Z^r}\otimes\mathcal{N}_{Z^r}\otimes\mathcal{J}(-\varepsilon\cdot\Delta^r)),$$

hence that $\mathrm{pr}_1^*\mathcal{J}(-\varepsilon\cdot\Delta)\otimes\cdots\otimes\mathrm{pr}_r^*\mathcal{J}(-\varepsilon\cdot\Delta)$ is isomorphic to $\mathcal{J}(-\varepsilon\cdot\Delta^r)$ and torsion-free. Condition (ii) now follows from (i) and (c). □

The main result of this section is a complement to the Weak Semistable Reduction Theorem.

THEOREM 6.5. *Assume in* 6.1 *that* $g: Z\to Y$ *is mild, that* $Y_g\subset Y$ *is open with* $g^{-1}(Y_g)\to Y_g$ *smooth, and that* Δ *does not contain any fibers* $g^{-1}(y)$ *for* $y\in Y_g$. *Then there exists a fiber product diagram of morphisms*

$$\begin{tikzcd} Z_1 \arrow[r, "\theta'"] \arrow[d, "g_1"'] & Z \arrow[d, "g"] \\ Y_1 \arrow[r, "\theta"] & Y, \end{tikzcd}$$

with θ *a nonsingular alteration, and an open dense subscheme* Y_{1g} *of* $\theta^{-1}(Y_g)$, *such that for*

$\mathfrak{C}_1 = \big\{\varrho: T\to Y_1$ *with either* ϱ *dominant and* T *normal with at most rational Gorenstein singularities, or* T *a nonsingular curve and* $\varrho_T^{-1}(Y_{1g})$ *dense in* $T\big\}$

and for $\Delta_1=\theta'^*\Delta$, *the sheaf* $\mathcal{J}(-\varepsilon\cdot\Delta_1)$ *is compatible with pullback, base change and products for all* $(\varrho: T\to Y_1)\in\mathfrak{C}_1$.

Proof. We will verify conditions (a)–(d) stated in 6.2.

Step I. As a first step, under the additional assumption

$$\mathcal{N}^N \otimes \mathcal{O}_Z(-\Delta) = \mathcal{O}_Z \tag{6.5.1}$$

we will construct a nonsingular alteration $Y_1 \to Y$ such that the pullback family $g_1 : Z_1 \to Y_1$ satisfies condition 6.2(b), for $r = 1$.

Consider the cyclic covering $W \to Z$ obtained by taking the N-th root out of Δ and a log-resolution $\delta' : \widetilde{Z} \to Z$ for Δ. One has a diagram

$$\begin{tikzcd} \widetilde{W} \arrow[r] \arrow[d, "\widetilde{\pi}"'] \arrow[dr, "\pi"] & W \arrow[d] \\ \widetilde{Z} \arrow[r, "\delta'"'] & Z \end{tikzcd} \tag{6.5.2}$$

where $\widetilde{W}$ is a desingularization of the fiber product. Since $\varepsilon = \frac{1}{N}$, Lemma 2.2 implies that $\mathcal{N} \otimes \delta'_*(\omega_{\widetilde{Z}/Y} \otimes \mathcal{O}_{\widetilde{Z}}(-[\varepsilon \cdot \delta'^* \Delta])) = \mathcal{N} \otimes \omega_{Z/Y} \otimes \mathcal{J}(-\varepsilon \cdot \Delta)$ is a direct factor of $\pi_* \omega_{\widetilde{W}/Y}$. As we have seen there, the assumption that $\widetilde{W} \to Z$ factors through $\widetilde{Z}$ is not needed. Similarly it is sufficient to require $\widetilde{W}$ to have rational Gorenstein singularities.

Nevertheless let us start with $\widetilde{W}$ as in (6.5.2). We choose $Y_1 \to Y$ to be a nonsingular alteration, such that $\mathrm{pr}_2 : \widetilde{W} \times_Y Y_1 \to Y_1$ has a mild model $h_1 : W_1 \to Y_1$. By construction, one has a morphism $W_1 \to \widetilde{W}$ and hence $\pi_1 : W_1 \to Z_1$. Note that the divisor $\pi_1^* \theta'^* \Delta$ is divisible by N.

Let us formulate what we know up to now and what will be used in the next step:

Set-up 6.6. $Y_1 \to Y$ is a nonsingular alteration, $h_1 : W_1 \to Y_1$ is a flat Gorenstein morphism factoring through an alteration $\pi_1 : W_1 \to Z_1$. The morphism h_1 has reduced fibers and it is smooth over an open dense subscheme Y_{1g} of Y_1.

Moreover $g_1 : Z_1 \to Y_1$ is mild and for all g_1-semiample sheaves $\mathcal{A}$ on Z_1 by Lemma 5.3 the sheaf $h_{1*}(\pi_1^* \mathcal{A} \otimes \omega_{W_1/Y_1})$ is locally free and compatible with arbitrary base change.

Given $\varrho : T \to Y_1$, as in Theorem 6.5, one has

$$\begin{tikzcd} W_T \arrow[r, "\varrho'"] \arrow[d, "\pi_T"'] & W_1 \arrow[d, "\pi_1"] \\ Z_T \arrow[r, "\varrho_T"] \arrow[d, "g_T"'] & Z_1 \arrow[d, "g_1"] \\ T \arrow[r, "\varrho"] & Y_1. \end{tikzcd}$$

So g_1 and $h_1 = g_1 \circ \pi_1$, as well as g_T and $h_T = g_T \circ \pi_T$ are flat.

Let us write $\Delta_T = \varrho_T^* \Delta_1$, let $\mathcal{A}$ be an invertible sheaf on Z_1 and $\mathcal{A}_T = \varrho_T^* \mathcal{A}$. One has compatible base change morphisms

$$\varrho_T^* \pi_{1*}(\pi_1^* \mathcal{A} \otimes \omega_{W_1/Y_1}) = \mathcal{A}_T \otimes \varrho_T^* \pi_{1*}\omega_{W_1/Y_1} \xrightarrow{\alpha} \mathcal{A}_T \otimes \pi_{T*}\omega_{W_T/T},$$

$$\varrho^* h_{1*}(\pi_1^* \mathcal{A} \otimes \omega_{W_1/Y_1}) = \varrho^* g_{1*}(\mathcal{A} \otimes \pi_{1*}\omega_{W_1/Y_1}) \xrightarrow{\gamma} g_{T*}(\mathcal{A}_T \otimes \varrho_T^* \pi_{1*}\omega_{W_1/Y_1}),$$

and

$$\varrho^* h_{1*}(\pi_1^* \mathcal{A} \otimes \omega_{W_1/Y_1}) \xrightarrow{\beta = (g_{T*}(\alpha))\circ\gamma} h_{T*}(\pi_T^* \mathcal{A}_T \otimes \omega_{W_T/T}).$$

CLAIM 6.7. *In* 6.6 *for all invertible sheaves $\mathcal{A}$ on Z_1 the morphism α is surjective and induces an isomorphism*

$$[\mathcal{A}_T \otimes \varrho_T^* \pi_{1*}\omega_{W_1/Y_1}]/_{\text{torsion}} \longrightarrow \mathcal{A}_T \otimes \pi_{T*}\omega_{W_T/T}.$$

Proof. Note that $h_T : W_T \to T$ is flat, Gorenstein, with reduced fibers and with a nonsingular general fiber. So the singular locus of W_T lies in codimension at least two, and W_T has to be normal; hence it is a disjoint union of irreducible schemes, each one flat over an irreducible components of T. So $\pi_{T*}\omega_{W_T/T}$ will be a torsion-free $\mathcal{O}_T$ module.

It is sufficient to prove Claim 6.7 for one invertible sheaf $\mathcal{A}$. So we may assume that $\mathcal{A}$ is ample, hence $\pi_1^* \mathcal{A}$ semiample. By assumption β is an isomorphism, and

$$g_{T*}(\alpha) : g_{T*}(\mathcal{A}_T \otimes \varrho_T^* \pi_{1*}\omega_{W_1/Y_1}) \longrightarrow g_{T*}(\mathcal{A}_T \otimes \pi_{T*}\omega_{W_T/Y_T})$$

has to be surjective. For $\mathcal{A}$ sufficiently ample, the evaluation map induces a surjection

$$g_T^* g_{T*}\big(\mathcal{A}_T \otimes \varrho_T^* \pi_{1*}\omega_{W_1/Y_1}\big) \xrightarrow{g_T^*(g_{T*}(\alpha))} g_T^* g_{T*}(\mathcal{A}_T \otimes \pi_{T*}\omega_{W_T/Y_T}) \longrightarrow \mathcal{A}_T \otimes \pi_{T*}\omega_{W_T/Y_T}.$$

Since it factors through $\alpha : \mathcal{A}_T \otimes \varrho_T^* \pi_{1*}\omega_{W_1/Y_1} \to \mathcal{A}_T \otimes \pi_{T*}\omega_{W_T/Y_T}$, the latter must be surjective as well. By flat base change α is an isomorphism over some open dense subscheme of Z_T; hence its kernel is exactly the torsion subsheaf. $\square$

Let us return to the notation used in the beginning, where $\widetilde{W}$ is a desingularization of the cyclic covering obtained by taking the N-th root out of Δ and Y_1 is chosen, such that $\widetilde{W} \to Y$ has a mild reduction $h_1 : W_1 \to Y_1$. Thus, the conditions in 6.6 hold true by the definition of a mild morphism and by Lemma 5.3.

Since W_T has at most rational Gorenstein singularities one obtains $\mathcal{J}(-\varepsilon \cdot \Delta_T)$ as a direct factor of

$$\varrho_T^* \theta'^* \mathcal{N}^{-1} \otimes \pi_{T*}\omega_{W_T/Z_T} = \varrho_T^* \theta'^* \mathcal{N}^{-1} \otimes \omega_{Z_T/T}^{-1} \otimes \pi_{T*}\omega_{W_T/Y_T}.$$

By flat base change this factor coincides with $\varrho_T^* \mathcal{J}(-\varepsilon \cdot \Delta_1)$ on some open dense subscheme of Z_1. Applying 6.7 for $\mathcal{A} = \theta'^* \mathcal{N}^{-1} \otimes \omega_{Z_1/Y_1}^{-1}$, the morphism α induces an isomorphism $\varrho_T^* \mathcal{J}(-\varepsilon \cdot \Delta)/_{\text{torsion}} \xrightarrow{\cong} \mathcal{J}(-\varepsilon \cdot \varrho_T^* \Delta)$.

Step II. Next we will verify (b) for $r = 1$ without the additional assumption (6.5.1), by gluing local models constructed in the first step.

To construct a nonsingular alteration Y_1 such that property (b) in 6.2 holds true for the family $g_1 : Z_1 \to Y_1$, one may replace $\mathcal{N}$ by $\mathcal{N} \otimes g^*\mathcal{H}$ and correspondingly $\mathcal{E}$ by $\mathcal{E} \otimes \mathcal{H}^N$.

Now, choosing $\mathcal{H}$ sufficiently ample, one may assume that $\mathcal{E}$ is generated by global sections, as well as $\mathcal{N}^N \otimes \mathcal{O}_Z(-\Delta)$. Next choose $H_1, \ldots, H_\ell$ to be zero divisors of general global sections of $\mathcal{N}^N \otimes \mathcal{O}_Z(-\Delta)$ and $U_i = Z \setminus H_i$, with

$$\bigcap_{i=1}^{\ell} H_i = \varnothing \quad \text{or} \quad \bigcup_{i=1}^{\ell} U_i = Z. \tag{6.7.1}$$

By Step I, for $H_i + \Delta$ instead of Δ and for each i, one has a nonsingular alteration $Y_1^{[i]} \to Y$ and a fiber product

$$\begin{tikzcd} Z_1^{[i]} \arrow[r, "\theta^{[i]}"] \arrow[d, "g_1^{[i]}"'] & Z \arrow[d, "g"] \\ Y_1^{[i]} \arrow[r] & Y \end{tikzcd}$$

such that $\mathcal{J}(-\varepsilon \cdot \theta^{[i]*}(H_i + \Delta))$ is compatible with pullback up to torsion. Fix a nonsingular alteration $\theta : Y_1 \to Y$ dominating all the $Y_1^{[i]}$. For $Y_{1,g}$ choose the intersection of the preimages of the different good loci $Y_{1,g}^{[i]}$ and for Z_1 the pullback family.

By construction $\mathcal{J}(-\varepsilon \cdot (\Delta + H_i))|_{U_i} = \mathcal{J}(-\varepsilon \cdot \Delta)|_{U_i}$ and

$$\mathcal{J}(-\varepsilon \cdot (\Delta_1 + \theta'^* H_i))|_{\theta^{-1}(U_i)} = \mathcal{J}(-\varepsilon \cdot \Delta_1)|_{\theta^{-1}(U_i)}.$$

Since $\mathcal{J}(-\varepsilon \cdot (\Delta_1 + \theta'^* H_i))$ is compatible with pullback up to torsion, the sheaf of ideals $\mathcal{J}(-\varepsilon \cdot \Delta_1)$ has the same property over U_i. Since $\{U_i;\ i = 1, \ldots, \ell\}$ is an open covering of Z, condition 6.2(b) follows for Δ_1 and for $r = 1$.

Step III. For the model $Z_1 \to Y_1$ constructed in Step II we will verify property (b) for $r > 1$ and the compatibility with products, stated in 6.2(a). Let us formulate again the set-up we will refer to at this point.

Set-up 6.8. $\pi_1^{[i]} : W_1^{[i]} \to Z_1$ are alterations such that the induced morphisms $h_1^{[i]} : W_1^{[i]} \to Y_1$ satisfy the assumptions stated in 6.6.

Choose a tuple $\underline{i}$ consisting of r elements $i_1, \ldots, i_r \in \{1, \ldots, \ell\}$ and the induced morphisms $h_1^r : W^r = W_1^{[i_1]} \times_{Y_1} \cdots \times_{Y_1} W_1^{[i_r]} \to Y_1$ and $\pi_1^r : W^r \to Z^r$. Let

(6.8.1)

$$\mathcal{A}_{Z^r} \otimes \bigotimes_{\iota=1}^{r} \mathrm{pr}_{i_\iota}^* \pi_*^{[i_\iota]} \omega_{W^{[i_\iota]}/Y_1} \xrightarrow{\alpha^r} \mathcal{A}_{Z^r} \otimes \pi_*^r \omega_{W^r/Y_1} = \mathcal{A}_{Z^r} \otimes \pi_*^r \bigotimes^{r} \mathrm{pr}_{i_\iota}^* \omega_{W^{[i_\iota]}/Y_1}$$

be induced by the tensor products of the base change maps

$$\mathrm{pr}_{i_\iota}^* \pi_*^{[i_\iota]} \omega_{W^{[i_\iota]}/Y_1} \longrightarrow \pi_*^r \mathrm{pr}_{i_\iota}^* \omega_{W^{[i_\iota]}/Y_1}.$$

By assumption, for $\mathcal{A}$ ample, the sheaves $h^{[i]}_{1*}\pi^{[i]*}_1\mathcal{A}\otimes\omega_{W^{[i]}/Y_1}$ are locally free. By flat base change and the projection formula, one has an isomorphism

$$\bigotimes_{\iota=1}^{r} h^{[i_\iota]}_{1*}\big(\pi^{[i_\iota]*}\mathcal{A}\otimes\omega_{W^{[i_\iota]}/Y_1}\big)\xrightarrow{\beta^r} h^r_{1*}\big(\pi^{r*}_1\mathcal{A}_{Z^r}\otimes\omega_{W^r/Y_1}\big).$$

CLAIM 6.9. *There is a natural morphism*

$$\bigotimes_{\iota=1}^{r} h^{[i_\iota]}_{1*}\big(\pi^{[i_\iota]*}\mathcal{A}\otimes\omega_{W^{[i_\iota]}/Y_1}\big)=\bigotimes_{\iota=1}^{r} g_{1*}\big(\mathcal{A}\otimes\pi^{[i_\iota]}{}_*\omega_{W^{[i_\iota]}/Y_1}\big)$$
$$\xrightarrow{\gamma^r} g^r_{1*}\Big(\mathcal{A}_{Z^r}\otimes\bigotimes_{\iota=1}^{r}\mathrm{pr}^*_{i_\iota}\pi^{[i_\iota]}_*\omega_{W^{[i_\iota]}/Y_1}\Big).$$

Proof. Let $p_1 : Z^r_1\to Z_1$ and $p_2 : Z^r_1\to Z^{r-1}_1$ denote the projection to the first and the last $r-1$ factors of the fiber product. Assume one has constructed γ^{r-1}, hence the morphism

$$\bigotimes_{\iota=1}^{r} g_{1*}\big(\mathcal{A}\otimes\pi^{[i_\iota]}{}_*\omega_{W^{[i_\iota]}/Y_1}\big)$$
$$\xrightarrow{\gamma^{r-1}\otimes\mathrm{id}} g_{1*}\big(\mathcal{A}\otimes\pi^{[i_r]}{}_*\omega_{W^{[i_r]}/Y_1}\big)\otimes g^{r-1}_{1*}\Big(\mathcal{A}_{Z^{(r-1)}}\otimes\bigotimes_{\iota=1}^{r-1}\mathrm{pr}^*_{i_\iota}\pi^{[i_\iota]}_*\omega_{W^{[i_\iota]}/Y_1}\Big).$$

The right-hand side maps to

$$g^{r-1}_{1*}g^{r-1*}_1g_{1*}\big(\mathcal{A}\otimes\pi^{[i_r]}{}_*\omega_{W^{[i_r]}/Y_1}\big)\otimes g_{1*}g^*_1g^{r-1}_{1*}\Big(\mathcal{A}_{Z^{(r-1)}}\otimes\bigotimes_{\iota=1}^{r-1}\mathrm{pr}^*_{i_\iota}\pi^{[i_\iota]}_*\omega_{W^{[i_\iota]}/Y_1}\Big).$$

The tensor product of the two base change maps and the multiplication of sections map this sheaf to

$$g^r_{1*}p^*_2\Big(\mathcal{A}_{Z^{(r-1)}}\otimes\bigotimes_{\iota=1}^{r-1}\mathrm{pr}^*_{i_\iota}\pi^{[i_\iota]}_*\omega_{W^{[i_\iota]}/Y_1}\Big)\otimes g^r_{1*}p^*_1\big(\mathcal{A}\otimes\pi^{[i_r]}{}_*\omega_{W^{[i_r]}/Y_1}\big)$$
$$\xrightarrow{\text{mult}} g^r_{1*}\Big(\mathcal{A}_{Z^r}\otimes\bigotimes_{\iota=1}^{r}\mathrm{pr}^*_{i_\iota}\pi^{[i_\iota]}_*\omega_{W^{[i_\iota]}/Y_1}\Big).\qquad\square$$

Again the isomorphism β^r is equal to $g^r_{1*}(\alpha^r)\circ\gamma^r$; hence $g^r_{1*}(\alpha^r)$ has to be surjective. As in the proof of 6.7, for $\mathcal{A}$ sufficiently ample, one finds that α^r is surjective. We obtained

CLAIM 6.10. *In* 6.8 *the base change map* β^r *in* (6.8.1) *is an isomorphism for all* g_1*-semiample sheaves* $\mathcal{A}$. *The morphism* α^r *is surjective and its kernel is a torsion sheaf.*

Let us return to the situation considered in Step II. We have chosen alterations $\pi_1^{[i]} : W_1^{[i]} \to Z_1$, dominating the cyclic covering obtained by taking the N-th root out of $\Delta_1 + \theta'^* H_i$, such that the induced morphisms $h_1^{[i]} : W_1^{[i]} \to Y_1$ are mild. By Lemma 4.2 the morphism h_1^r is again mild and W^r has rational Gorenstein singularities. W^r dominates the cyclic covering obtained by taking the N-th root out of $\Delta_1 + \theta'^* H_i$. So $\pi_1^r : W^r \to Z^r$ is again an alteration, dominating the cyclic covering obtained by taking the N-th root out of

$$\Gamma = \mathrm{pr}_{i_1}^*(\Delta_1 + \theta'^* H_{i_1}) + \cdots + \mathrm{pr}_{i_r}^*(\Delta_1 + \theta'^* H_{i_r}).$$

By Step I, up to the tensor product with an invertible sheaf, $\mathcal{J}(-\varepsilon \cdot \Gamma)$ is a direct factor of

$$\pi_{1*}^r \omega_{W^r/Y_1} = \mathrm{pr}_{i_1}^* \pi_*^{[i_1]} \omega_{W^{[i_1]}/Y_1} \otimes \cdots \otimes \mathrm{pr}_{i_r}^* \pi_*^{[i_r]} \omega_{W^{[i_r]}/Y_1}.$$

On some open dense subscheme this factor is isomorphic to

$$\mathrm{pr}_{i_1}^* \mathcal{J}(-\varepsilon \cdot (\Delta_1 + \theta'^* H_{i_1})) \otimes \cdots \otimes \mathrm{pr}_{i_r}^* \mathcal{J}(-\varepsilon \cdot (\Delta_1 + \theta'^* H_{i_r})).$$

So the first part of Claim 6.10 implies that α^r induces an isomorphism

$$[\mathrm{pr}_{i_1}^* \mathcal{J}(-\varepsilon \cdot (\Delta_1 + \theta'^* H_{i_1})) \otimes \cdots \otimes \mathrm{pr}_{i_r}^* \mathcal{J}(-\varepsilon \cdot (\Delta_1 + \theta'^* H_{i_r}))]/_{\mathrm{torsion}} \xrightarrow{\cong} \mathcal{J}(-\varepsilon \cdot \Gamma).$$

For $U_i = Z \setminus H_i$ and $U_{\underline{i}} = U_{i_1} \times \cdots \times U_{i_r}$ one has

$$\mathcal{J}(-\varepsilon \cdot (\Delta_1 + \theta'^* H_{i_\iota}))|_{U_{i_\iota}} = \mathcal{J}(-\varepsilon \cdot \Delta_1)|_{U_{i_\iota}}$$

and $\mathcal{J}(-\varepsilon \cdot \Gamma)|_{U_{\underline{i}}} = \mathcal{J}(-\varepsilon \cdot \Delta^r)|_{U_{\underline{i}}}$. Since by (6.7.1) each point of Z^r lies in $U_{\underline{i}}$ for some choice of the tuple $\underline{i}$, one obtains property 6.2(a).

The same construction gives the proof of property (b) for $r > 1$. One just has to note that $\pi_1^r : W_1^r \to Z_1^r$ and $\hat{h}_1^r : W_1^r \to Y$ satisfy again the assumption made in 6.6. We replace the ample sheaf $\mathcal{A}$ by a sufficiently high power of $\mathcal{A}_{Z^r}$ and obtain an isomorphism

$$\varrho_T^{r*} \mathcal{J}(-\varepsilon \cdot \Gamma)/_{\mathrm{torsion}} \xrightarrow{\cong} \mathcal{J}(-\varepsilon \cdot \varrho_T^{r*} \Gamma),$$

for the divisor Γ introduced above. So 6.2(b) holds for Δ^r on $U_{\underline{i}}$, hence everywhere.

Step IV. It remains to verify properties 6.2(c) and (d). To simplify notation, let us drop the lower index $_1$ and assume that properties (a) and (b) in 6.2 hold true for $g : Z \to Y$ itself.

Let us first remark that we know (c) and (d) if $\mathcal{N}^N \otimes \mathcal{O}_Z(-\Delta)$ is the pullback of an invertible sheaf on Y. In fact, the base change morphisms in 6.2(c) and (d) are just direct factors of the base change morphisms β in step I or β^r in Claim 6.10 in Step III. So we will reduce everything to this case.

As we have seen this can be done by adding the zero divisor H of a general section of $\mathcal{N}^N \otimes \mathcal{O}_Z(-\Delta)$ to Δ. There is a problem with the term "general". We can choose H to be general for a fiber of $g_1 : Z_1 \to Y_1$; hence (2.1.1) holds for F

and for F replaced by a small neighborhood. However we cannot choose H such that this remains true for neighborhoods of fibers of all $g_T : Z_T \to T$ and for the pullback of H. So we will argue in a different way.

Let us assume that the construction in Step II was possible over Y. In particular $\mathcal{E}$ and hence $\mathcal{N}^N \otimes \mathcal{O}_Z(-\Delta)$ are generated by global sections, and for some section of $\mathcal{N}^N \otimes \mathcal{O}_Z(-\Delta)$ with zero divisor H_1 the cyclic covering obtained by taking the N-th root out of $\Delta + H_1$ has a mild model $h^{[1]} : W^{[1]} \to Y$ factoring through $\pi^{[1]} : W^{[1]} \to Z$.

As before $\delta' : \tilde{Z} \to Z$ denotes a log-resolution for Δ. Fix a point $y \in Y$. For the zero set H of a general section of $\mathcal{N}^N \otimes \mathcal{O}_Z(-\Delta)$ the divisor $\delta'^* H$ will be smooth meeting $\delta'^* \Delta$ transversely. Thus, $\mathcal{J}(-a \cdot \Delta) = \mathcal{J}(-a \cdot (\Delta + H))$ for $0 \leq a < 1$. Moreover, $\pi^{[1]^*} H$ will not contain any component of $h^{[1]^{-1}}(y)$.

On $W^{[1]}$ the divisor $\pi^{[1]^*} \Delta$ is divisible by N. Hence the sheaf

$$\pi^{[1]^*}(\mathcal{N}^N \otimes \mathcal{O}_Z(-\Delta)) = (\pi^{[1]^*} \mathcal{N}^N) \otimes \mathcal{O}_Z(-\pi^{[1]^*} \Delta)) = \mathcal{O}_{W^{[1]}}(\pi^{[1]^*} H)$$

is the N-th power of an invertible sheaf $\mathcal{L}$. We choose $\phi : W \to W^{[1]}$ to be the cyclic covering obtained by taking the N-th root out of $\pi^{[1]^*} H$ and $\pi = \pi^{[1]} \circ \phi$.

CLAIM 6.11. *For H sufficiently general, replacing Y by a neighborhood of y, one has*:

(i) $\phi_* \omega_{W/Y} = \bigoplus_{\iota=0}^{N-1} \omega_{W^{[1]}/Y} \otimes \mathcal{L}^{\iota}$.

(ii) *The induced morphism $h : W \to Y$ is flat and Gorenstein.*

(iii) *The fibers of h are reduced and the general fiber is nonsingular.*

(iv) *If $\mathcal{A}$ is g-semiample the direct image sheaves $h_*(\pi^* \mathcal{A} \otimes \omega_{W/Y})$ are locally free and compatible with arbitrary base change.*

(v) *The sheaf $\mathcal{J}(-\varepsilon \cdot \Delta) \otimes \mathcal{N} \otimes \omega_{Z/Y}$ is a direct factor of $\pi_* \omega_{W/Y}$.*

Proof. The first part follows from [EV92, §3]. However, there cyclic coverings over a nonsingular base were considered and we have to explain, how to reduce the statement to this case.

Let $\tau : V \to W^{[1]}$ be a desingularization. For H sufficiently general, $\tau^* H$ is nonsingular. The normalization V' of V in the function field of W is nonsingular and isomorphic to

$$\mathrm{Spec}(\mathcal{F}) \quad \text{for} \quad \mathcal{F} = \bigoplus_{\iota=0}^{N-1} \tau^* \mathcal{L}^{-\iota}.$$

The canonical sheaf $\omega_{V'}$ is the invertible sheaf corresponding to

$$\mathcal{F} \otimes \mathcal{L}^{N-1} = \bigoplus_{\iota=0}^{N-1} \omega_V \otimes \tau^* \mathcal{L}^{\iota}.$$

Since $W^{[1]}$ is Gorenstein with rational singularities, $\phi_* \mathcal{O}_W = \tau_* \mathcal{F} = \bigoplus_{\iota=0}^{N-1} \mathcal{L}^{-\iota}$.

So $\phi_*\omega_W$ contains $\tau_*\mathcal{F}\otimes\mathcal{L}^{N-1}$ and both are isomorphic outside of a codimension two subset. The second sheaf is a locally free $\tau_*\mathcal{F}$ module of rank 1, hence equal to $\phi_*\omega_W$. In particular $\phi : W \to W^{[1]}$ is flat, and ω_W is invertible.

For (iii), note that $g^{[1]}$ is smooth over some open dense subset Y_g of Y. The restriction of a general divisor H to one fiber will be nonsingular, and thereby g has at least one nonsingular fiber. Choosing Y small enough, we may assume that H does not contain components of any fiber of $g^{[1]}$. Since the fibers of $g^{[1]}$ are reduced, the fibers of h have the same property.

Part (iv) follows from 5.3, applied to the sheaves $g^{[1]}_*(\omega_{W^{[1]}/Y}\otimes\mathcal{L}^\iota\otimes\mathcal{A})$, and by the direct sum decomposition in (i). So it remains to verify (v).

Let $\pi' : W' \to Z$ be the cyclic covering obtained by taking the N-th root out of $\Delta + H$. Then W is just the normalization of the fiber product $W'\times_Z W^{[1]}$. In fact, the latter is the cyclic covering of $W^{[1]}$, obtained by taking the N-th root out of $\pi^{[1]^*}\Delta + \pi^{[1]^*}H$. However, $\pi^{[1]^*}\Delta$ is divisible by N; hence it is the same to take the N-th root out of $\pi^{[1]^*}H$.

So $\pi'_*\omega_{W'/Y}$ is a direct factor of $\pi_*\omega_{W/Y}$, and

$$\mathcal{J}(-\varepsilon\cdot(\Delta+H))\otimes\mathcal{N}\otimes\omega_{Z/Y} = \mathcal{J}(-\varepsilon\cdot\Delta)\otimes\mathcal{N}\otimes\omega_{Z/Y}$$

is a direct factor of both of them. $\square$

Parts (ii), (iii) and (iv) of 6.11 imply that the assumptions stated in 6.6 hold. Hence by Claim 6.7 for all $\varrho : T \to Y$ considered in 6.5 the morphism

$$\alpha : \varrho_T^*\pi_*\omega_{W/Y} \to \pi_{T*}\omega_{W_T/T}$$

is a surjection with torsion kernel. Moreover the composite

$$\beta : \varrho^* g_*(\mathcal{A}\otimes\pi_*\omega_{W/Y}) \xrightarrow{\gamma} g_{T*}(\mathcal{A}_T\otimes\varrho_T^*\pi_*\omega_{W/Y}) \xrightarrow{g_*\alpha} g_{T*}(\mathcal{A}_T\otimes\pi_{T*}\omega_{W_T/T})$$

is an isomorphism for all g-semiample sheaves $\mathcal{A}$ on Z. By 6.11(v) the sheaf

$$\varrho^* g_*(\mathcal{A}\otimes\mathcal{J}(-\varepsilon\cdot\Delta)\otimes\mathcal{N}\otimes\omega_{Z/Y})$$

is a direct factor of the left-hand side, and by property (b), which we verified in Steps I and II, the corresponding direct factor of the right-hand side is

$$g_{T*}(\omega_{Z_T/T}\otimes\varrho_T^*(\mathcal{A}\otimes\mathcal{N})\otimes\mathcal{J}(-\varepsilon\cdot\Delta_T)).$$

So we obtained property (c) for $r=1$. For $r>1$ the argument is the same. Using the notation from Step II for $\underline{i}=(1,\ldots,1)$ we just have to replace Z by Z^r and the divisor H_1 by $\mathrm{pr}_1^*H_1+\cdots+\mathrm{pr}_r^*H_1$.

For (d) we choose for the morphisms $h^{[i]} : W^{[i]} \to Z$ in Step III the same morphism $h : W \to Y$. By 6.11(ii), (iii), and (iv) the assumptions made in 6.8 hold true, and by Claim 6.10 the composite

$$\bigotimes_{\iota=1}^{r} g_*(\mathcal{A}\otimes\pi_*\omega_{W/Y}) = \bigotimes_{\iota=1}^{r} h_*(\pi^*\mathcal{A}\otimes\omega_{W/Y}) \xrightarrow{\beta^r} h^r_*(\pi^*\mathcal{A}_{Z^r}\otimes\omega_{W^r/Y})$$

$$= g^r_*(\mathcal{A}_{Z^r}\otimes\pi^r_*\omega_{W^r/Y}) \underset{\alpha^r}{\xleftarrow{\cong}} g^r_*\Big(\mathcal{A}_{Z^r}\otimes\Big[\bigotimes_{\iota=1}^{r}\mathrm{pr}_\iota^*\pi_*\omega_{W/Y}\Big]/_{\text{torsion}}\Big)$$

is an isomorphism. The left-hand side contains $\bigotimes_{\iota=1}^{r} g_*(\omega_{Z/Y}\otimes\mathcal{A}\otimes\mathcal{N}\otimes\mathcal{J}(-\varepsilon\cdot\Delta))$ as a direct factor, and the corresponding direct factor of the right-hand side is

$$g^r_*\Big(\omega_{Z^r/Y}\otimes\mathcal{A}_{Z^r}\otimes\mathcal{N}_{Z^r}\otimes\Big[\bigotimes_{\iota=1}^{r}\mathrm{pr}_\iota^*\mathcal{J}(-\varepsilon\cdot\Delta)\Big]/_{\text{torsion}}\Big).$$

By part (a) of 6.2 this is $g^r_*(\omega_{Z^r/Y}\otimes\mathcal{A}_{Z^r}\otimes\mathcal{N}_{Z^r}\otimes\mathcal{J}(-\varepsilon\cdot\Delta^r))$ and we obtain (d). □

Remark 6.12. Even if one adds to 6.1 the additional condition $N > e(\Delta)$ one cannot expect in Theorem 6.5 that $\mathcal{J}(-\varepsilon\cdot\Delta_1)$ is isomorphic to $\mathcal{O}_{Z_1}$. At least we were not able to compare $e(\Delta_1)$ and $e(\Delta)$. So for the equality $\mathcal{J}(-\varepsilon_1\cdot\Delta_1)=\mathcal{O}_{Z_1}$ with $\varepsilon_1=\frac{1}{N_1}$ we have to choose N_1 larger than N and we lose the compatibility with base change and products.

7. **Embedded weak semistable reduction over curves**

For a morphism to a curve with smooth general fiber, a semistable model is mild. The existence of such a model over some covering of the base has been shown by Kempf, Knudsen, Mumford, and Saint-Donat in [KKMSD73]. Applying it to a family over a discrete valuation ring one obtains the semistable reduction theorem in codimension one:

THEOREM 7.1. *Let U and V be quasi-projective manifolds and let $E\subset U$ be a submanifold of codimension one. Let $f: V\to U$ be a surjective projective morphism with connected general fiber. Then there exists a finite covering $\theta: U'\to U$, a desingularization V' of the main component of $V\times_U U'$, and an open neighborhood $\widetilde{U}$ of the general points of $\theta^{-1}(E)$ such that for the induced morphism $f': V'\to U'$ the restriction $f'^{-1}(\widetilde{U})\to\widetilde{U}$ is flat and $f'^{-1}(\widetilde{U}\cap\theta^{-1}(E))$ is a reduced, relative, normal crossing divisor over $\widetilde{U}\cap\theta^{-1}(E)$.*

As indicated in 5.9 we will need some "embedded version" of the semistable reduction in a neighborhood of a given curve. This will allow us to apply the base change criterion in Lemma 5.8.

Since as in Section 6 we have to allow multiplier ideals the notation again gets a bit complicated in the second half of the section. The multiplier ideal of the restriction of a divisor is contained in the restriction of the multiplier ideal. As we will see, for total spaces of families of varieties one can enforce an isomorphism after an alteration of the base.

LEMMA 7.2. *Let $f : X \to Y$ be a projective surjective morphism between quasi-projective manifolds with smooth part $f_0 = f|_{X_0} : X_0 \to Y_0$. Let $\pi : C \to Y$ be a morphism from a nonsingular curve C with $C_0 = \pi^{-1}(Y_0)$ dense in C. Then there exist a nonsingular alteration $\theta : Y_1 \to Y$ and a desingularization $\theta' : X_1 \to X \times_Y Y_1$ of the main component such that for the induced morphism $f_1 : X_1 \to Y_1$ the following holds*:

(a) *$C \to Y$ lifts to an embedding $C \subset Y_1$.*

(b) *There exists a neighborhood U_1 of C in Y_1 with $f_1^{-1}(U) \to U$ flat.*

(c) *$S = f_1^{-1}(C)$ is nonsingular and $f_1^{-1}(C \setminus C_0)$ a normal crossing divisor in S.*

Proof. Replacing Y by a hyperplane in $C \times Y$, containing the graph of $\pi : C \to Y$, one may assume that $C \to Y$ is an embedding. Next, replace X by an embedded log-resolution of the closure S of $f^{-1}(C) \cap X_0$ for the divisor $f^{-1}(C \setminus C_0)$. Now, we may assume that the closure S of $f^{-1}(C_0)$ is nonsingular and that the singular fibers of $S \to C$ are normal crossing divisors. Consider for a very ample invertible sheaf $\mathcal{A}$ on X, the induced embedding $\iota : X \to \mathbb{P}^M$, and the diagram

$$\begin{tikzcd} X \arrow[rr, "(\iota, f)"] \arrow[dr, "f"'] & & \mathbb{P}^M \times Y \arrow[dl, "\mathrm{pr}_2"] \\ & Y. & \end{tikzcd}$$

Since $X_0 \to Y_0$ is flat, it gives rise to a morphism $\vartheta_0 : Y_0 \to \mathfrak{Hilb}$ to the Hilbert scheme of subvarieties of $\mathbb{P}^M$. Since $S \to C$ is also flat the restriction of ϑ_0 to $C \cap Y_0$ extends to a morphism $\varrho : C \to \mathfrak{Hilb}$, and the pullback of the universal family over $\mathfrak{Hilb}$ to C coincides with S.

We choose a modification $\theta : Y_1 \to Y$ with center outside of Y_0 such that ϑ_0 extends to a morphism $\vartheta : Y_1 \to \mathfrak{Hilb}$. For $f_1 : X_1 \to Y_1$ we choose the pullback of the universal family. Note that f_1 satisfies conditions (a), (b) and (c); however X_1 might be singular. Since we are allowed to modify X_1 outside of a neighborhood of S it remains to verify that X_1 is nonsingular in such a neighborhood. This will be done in the next lemma. $\square$

LEMMA 7.3. *Let $f : V \to U$ be a flat morphism, with U nonsingular. Let $C \subset U$ be a nonsingular curve and $S = f^{-1}(C)$. Then there is an open neighborhood U_0 of C in U with*:

(i) *If S is nonsingular, $f^{-1}(U_0)$ is nonsingular.*

(ii) *If S is reduced, normal, Gorenstein with at most rational singularities then $f^{-1}(U_0)$ is normal, Gorenstein with at most rational singularities.*

(iii) *If S is reduced, and Gorenstein, and if for some open subscheme U_g of U, meeting C, the preimage $f^{-1}(U_g)$ is nonsingular, then V is normal and Gorenstein.*

Proof. C is a smooth curve in U. For a point $p \in C$ choose local parameter $t_1, \dots, t_\ell$ such that C is the zero-set of $(t_1, \dots, t_{\ell-1})$. The parameters $(t_1, \dots, t_{\ell-1})$ define a smooth morphism $\mathrm{Spec}\mathcal{O}_{p,U} \to \mathrm{Spec}\mathcal{O}_{0,\mathbb{A}^{\ell-1}}$. The composite flat morphism $\Phi : V \times_U \mathrm{Spec}\mathcal{O}_{p,U} \longrightarrow \mathrm{Spec}\mathcal{O}_{p,U} \longrightarrow \mathrm{Spec}\mathcal{O}_{0,\mathbb{A}^{\ell-1}}$ has $S_0 = S \times_C \mathrm{Spec}\mathcal{O}_{p,C}$ as closed fiber. If the latter is smooth, Φ is smooth and one obtains (i).

Assume that S is Gorenstein. Then S_0 is Gorenstein, and Φ is a Gorenstein morphism.

If in addition S is reduced and normal, it is smooth outside of a codimension one subset, hence $V \times_U \mathrm{Spec}\mathcal{O}_{p,U}$ will be normal. And if S has at most rational singularities, the same holds true for $V \times_U \mathrm{Spec}\mathcal{O}_{p,U}$.

In (iii) the assumptions imply that the singular locus Υ of $V \times_U \mathrm{Spec}\mathcal{O}_{p,U}$ does not meet the general fiber of Φ. On the other hand, since the special fiber S_0 is reduced, Υ contains no component of S_0. So again Υ is of codimension two and since $V \times_U \mathrm{Spec}\mathcal{O}_{p,U}$ is Gorenstein it is normal. □

VARIANT 7.4. *Under the assumptions made in* 7.2 *one can find a finite covering $\widehat{C} \to C$, a nonsingular alteration $\theta : Y_1 \to Y$ and a desingularization $\theta' : X_1 \to X \times_Y Y_1$ such that for the induced morphism $f_1 : X_1 \to Y_1$ in addition to the properties* (*a*), (*b*) *and* (*c*) (*for $\widehat{C}$ instead of C*) *in* 7.2 *one has*:

(d) $f_1^{-1}(\widehat{C} \setminus \widehat{C}_0)$ *is a reduced normal crossing divisor in* $\widehat{S} = f_1^{-1}(\widehat{C})$.

Proof. We use the notation from the proof of 7.2, except that we assume that conditions (a)–(c) hold true for Y itself. So $C \subset Y$ and the morphism f is flat in a neighborhood of $S = f^{-1}(C)$. The latter is nonsingular and the fibers of $S \to C$ are normal crossing divisors.

Choose $\widehat{C} \to C$ to be a covering, such that $S \times_C \widehat{C} \to \widehat{C}$ has a semistable model $\widehat{S} \to S$. In particular there is a morphism $\widehat{S} \to S$ inducing $\tau : \widehat{S} \to S \times_C \widehat{C}$. As in the proof of 7.2 we can choose Y_1 such that $\widehat{C} \to C \to Y$ lifts to an embedding $\widehat{C} \to Y_1$. Consider the fiber product $X \times_Y Y_1$. It contains $S \times_C \widehat{C}$. Since τ is birational and projective, it is given by the blowing up of a sheaf of ideals $\mathcal{I}$ on $S \times_C \widehat{C}$. Let $\mathcal{J}$ be a sheaf of ideals on $X \times_Y Y_1$, whose restriction to $\widehat{S} \to S \times_C \widehat{C}$ is $\mathcal{I}$, and let $\delta : X_1 \to X \times_Y Y_1$ be the blowing up of $\mathcal{J}$. Then one obtains a closed immersion $\widehat{S} \to X_1$, whose image is contained in $f_1^{-1}(\widehat{C})$.

Repeating the argument in the proof of 7.2 we replace X_1 by some modification and $X_1 \to Y_1$ by the pullback of a universal family over a Hilbert scheme, with $f_1^{-1}(\widehat{C}) = \widehat{S}$. □

Definition 7.5. Let U be a quasi-projective manifold, let C be a smooth curve and $\pi : C \to U$ a morphism. We call $\theta : U_1 \to U$ a *local alteration for* C if θ is the restriction of a nonsingular alteration to some open subscheme, and if there is a smooth curve $C_1 \subset \theta^{-1}(C)$ with $C_1 \to C$ finite. We call such a curve C_1 a lifting of C.

LEMMA 7.6. *Assume that $C \subset Y$ is a smooth curve, that $S = f^{-1}(C)$ is a nonsingular variety, semistable over C, that f is flat over a neighborhood U of C, and that $V = f^{-1}(U)$ is nonsingular. Let $\theta : U_1 \to U$ be a local alteration for C, let $C_1 \in U_1$ be a lifting of C and $f_1 : V_1 = X \times_U U_1 \to U_1$ the pullback family. Write $f_1^r : V_1^r = V_1 \times_{U_1} \cdots \times_{U_1} V_1 \to U_1$ for the r-fold fiber product. Then*

($\diamond$) *For each $r > 0$ there exists a neighborhood $\widetilde{U}$ of C_1 in U_1 such that $\widetilde{V}^r = (f_1^r)^{-1}(\widetilde{U})$ is normal, Gorenstein, with at most rational singularities and the induced morphism $\widetilde{f}^r : \widetilde{V}^r \to \widetilde{U}$ is flat and projective. Moreover $S_1^r = (\widetilde{f}^r)^{-1}(C_1)$ is normal with at most rational Gorenstein singularities, and $S_1^r \to C_1$ has reduced fibers.*

Proof. The pullback of a semistable family $S_1 = F_1^{-1}(C_1) = S \times_C C_1$ is normal, Gorenstein with rational singularities. The same holds true for the r-fold product $S_1^r = S_1 \times_{C_1} \cdots \times_{C_1} S_1$. So one can apply Lemma 7.3. □

Definition 7.7. Let $f : X \to Y$ be a projective surjective morphism between quasi-projective manifolds with smooth part $f_0 = f|_{X_0} : X_0 \to Y_0$. Let $\pi : C \to Y$ be a morphism from a nonsingular curve C with $C_0 = \pi^{-1}(Y_0)$ dense in C; let $\theta : U_1 \to Y$ be a morphism, and let $V_1 \to X \times_Y U_1$ be a modification of the main component with center outside of the preimage of Y_0. We call the induced family $f_1 : V_1 \to U_1$ an *embedded, weak, semistable reduction (of $X \to Y$) over C* if $\theta : U_1 \to Y$ is a local alteration for C and if for some lifting $C_1 \in U_1$ the condition ($\diamond$) 7.6 holds true.

We call $f_1 : V_1 \to U_1$ an *embedded semistable reduction over C* if in addition $S_1 = f_1^{-1}(C_1)$ is nonsingular and semistable over C_1.

Usually we will replace U_1 by some neighborhood $\widetilde{U}$ and assume that the condition in ($\diamond$) holds for $\widetilde{U}$. Let us restate what we obtained:

PROPOSITION 7.8. *Let $f : X \to Y$ be a projective surjective morphism between quasi-projective manifolds with smooth part $f_0 = f|_{X_0} : X_0 \to Y_0$ and let $\pi : C \to Y$ be a morphism from a nonsingular curve C with $C_0 = \pi^{-1}(Y_0)$ dense in C.*

(a) *There exists an embedded, semistable reduction $V_1 \to U_1$ over C.*

(b) *Let $Y_1 \to Y$ be a nonsingular alteration. Then there exist a scheme U_2 and a morphism $U_2 \to Y_1$ such that the image of the composed morphism $U_2 \to Y$ is in U_1 and such that $V_2 = V_1 \times_{U_1} U_2 \to U_2$ is a weak semistable reduction over C.*

Proposition 7.8 will allow us to apply the base change criterion in Lemma 5.8. As in Section 6 we will need a similar criterion for multiplier sheaves. We start with a variant of Theorem 6.5 replacing the mild morphism by an embedded, weak, semistable reduction over a curve.

Assumptions 7.9. $f : V \to U$ is an embedded, weak, semistable reduction for $C \subset U$, with smooth part $f_0 : V_0 \to U_0$ for U_0 dense in U. There exists a mild morphism $g : Z \to U$ factoring through a modification $\tau : Z \to V$. Let $\mathcal{N}$ be an invertible sheaf on V, and let Δ be an effective Cartier divisor on V not containing fibers of f_0 and let $N > 1$ be a natural number. There is a morphism $\mathcal{E} \to f_*\mathcal{N}^N$ on U with $\mathcal{E}$ locally free and with $f^*\mathcal{E} \to \mathcal{N}^N \otimes \mathcal{O}_V(-\Delta)$ surjective. As in the last section we write $\varepsilon = \frac{1}{N}$.

Assume that $\mathcal{J}(-\varepsilon \cdot \tau^*\Delta)$ is compatible with pullback, base change and products, for all alterations of U, as defined in 6.4, and (for simplicity) that on the general fiber of $S \to C$ the multiplier sheaf $\mathcal{J}(-\varepsilon \cdot \Delta|_S)$ is isomorphic to $\mathcal{O}_S$.

LEMMA 7.10. *In* 7.9 *let $\mathfrak{C}$ be the set of local alterations $\theta : U_1 \to U$ such that $f_1 : V_1 = V \times_U U_1 \to U_1$ is an embedded weak semistable reduction for $f : V \to U$ over C. Then $\mathcal{J}(-\varepsilon \cdot \Delta)$ is flat over U and compatible with pullback, base change and products for $(\varrho : U_1 \to U) \in \mathfrak{C}$ in a neighborhood of each lifting C_1 of C, i.e., conditions* (i) *and* (ii) *in Lemma and Definition* 6.4 *hold true over a neighborhood $\widetilde{U} \subset U_1$ of C_1, possibly depending on r.*

Proof. Choose a log-resolution $\widetilde{\delta} : \widetilde{Z} \to Z$. For $\delta = \tau \circ \widetilde{\delta} : \widetilde{Z} \to V$,

$$\begin{aligned}\mathcal{J}(-\varepsilon \cdot \Delta) &= \delta_*(\omega_{\widetilde{Z}/V} \otimes \mathcal{O}_{\widetilde{Z}}(-[\varepsilon \cdot \delta^*\Delta])) \\ &= \tau_*\widetilde{\delta}_*(\omega_{\widetilde{Z}/V} \otimes \mathcal{O}_{\widetilde{Z}}(-[\varepsilon \cdot \widetilde{\delta}^*\tau^*\Delta])) = \tau_*(\omega_{Z/V} \otimes \mathcal{J}(-\varepsilon \cdot \tau^*\Delta)).\end{aligned}$$

Then

$$g_*(\tau^*\mathcal{A} \otimes \omega_{Z/V} \otimes \tau^*\mathcal{N} \otimes \mathcal{J}(-\varepsilon \cdot \tau^*\Delta)) = f_*(\mathcal{A} \otimes \omega_{V/U} \otimes \mathcal{N} \otimes \mathcal{J}(-\varepsilon \cdot \Delta)),$$

and by 6.4(ii) both are locally free, and the left-hand side is compatible with pullbacks. The cohomological criterion [GD61, Prop. 7.9.14] implies that $\mathcal{J}(-\varepsilon \cdot \Delta)$ is flat over U.

For the compatibility with base change for $\varrho : U_1 \to U$ consider the induced fiber products

$$\begin{array}{ccc} Z_1 & \xrightarrow{\hat{\varrho}} & Z \\ {\scriptstyle \tau_1}\downarrow & & \downarrow{\scriptstyle \tau} \\ V_1 & \xrightarrow{\varrho'} & V \\ {\scriptstyle f_1}\downarrow & & \downarrow{\scriptstyle f} \\ U_1 & \xrightarrow{\varrho} & U. \end{array}$$

One has for $\mathcal{A}$ ample on Z the base change map

$$\begin{aligned}\varrho'^*(\omega_{V/U} \otimes \mathcal{N} \otimes \mathcal{A} \otimes \mathcal{J}(-\varepsilon \cdot \Delta)) &= \varrho'^*\tau_*(\omega_{Z/U} \otimes \tau^*(\mathcal{N} \otimes \mathcal{A}) \otimes \mathcal{J}(-\varepsilon \cdot \tau^*\Delta)) \\ &\xrightarrow{\alpha} \tau_{1*}(\omega_{Z_1/U_1} \otimes \tau_1^*\varrho'^*(\mathcal{N} \otimes \mathcal{A}) \otimes \mathcal{J}(-\varepsilon \cdot \tau_1^*\varrho'^*\Delta)).\end{aligned}$$

The base change map for $g_*(\tau^*\mathcal{A}\otimes\omega_{Z/V}\otimes\tau^*\mathcal{N}\otimes\mathcal{J}(-\varepsilon\cdot\tau^*\Delta))$ factors through $f_{1*}(\alpha)$, so that the latter must be surjective. This being true for all ample sheaves $\mathcal{A}$, as in the proof of 6.7 one finds that α is surjective. By flat base change, α is an isomorphism on some open dense subscheme.

By assumption on the general fiber of $S\to C$ the multiplier sheaf $\mathcal{J}(-\varepsilon\cdot\Delta|_S)$ is trivial. By [Vie95, §5.4] or [EV92, Properties 7.5] this implies that $\mathcal{J}(-\varepsilon\cdot\Delta)$ is isomorphic to $\mathcal{O}_V$ in a neighborhood of a general fiber of f. Since the latter is flat over U, the sheaf $\varrho'^*\mathcal{J}(-\varepsilon\cdot\Delta)$ is torsion-free, hence isomorphic to

$$\tau_{1*}\omega_{Z_1/V_1}\otimes\mathcal{J}(-\varepsilon\cdot\tau_1^*\varrho'^*\Delta)=\mathcal{J}(-\varepsilon\cdot\varrho'^*\Delta).$$

In addition $f_{1*}(\alpha)$ is an isomorphism, hence $f_*(\mathcal{A}\otimes\omega_{V/U}\otimes\mathcal{N}\otimes\mathcal{J}(-\varepsilon\cdot\Delta))$ is compatible with base change for $\varrho\in\mathfrak{C}$.

A similar argument allows us to identify the multiplier ideals on the r-fold fiber products, for $r>1$. Let us write $\tau^r:Z^r\to V^r$ for the modification, $\mathrm{pr}_\iota:V^r\to V$ and $\mathrm{p}_\iota:Z^r\to Z$ for the projections. By flat base change one has a natural isomorphism

$$\mathrm{pr}_\iota^*\mathcal{J}(-\varepsilon\cdot\Delta)\longrightarrow\tau_*^r(p_\iota^*\omega_{Z/V}\otimes\mathcal{J}(-\varepsilon\cdot\mathrm{p}_\iota^*\tau^*\Delta)).$$

Since the multiplier ideal on Z is compatible with products, as formulated in 6.4(i) multiplication of sections induces a morphism α^r from $\bigotimes_{\iota=1}^r\mathrm{pr}_\iota^*\mathcal{J}(-\varepsilon\cdot\Delta)$ to

$$\tau_*^r(\omega_{Z^r/V^r}\otimes\mathcal{J}(-\varepsilon\cdot(\mathrm{p}_1^*\tau^*\Delta+\cdots+\mathrm{p}_r^*\tau^*\Delta)))=\mathcal{J}(-\varepsilon\cdot(\mathrm{pr}_1^*\Delta+\cdots+\mathrm{pr}_r^*\Delta)).$$

By flat base change

$$f_*^r\Big(\bigotimes_{\iota=1}^r\mathrm{pr}_\iota^*(\omega_{V/U}\otimes\mathcal{A}\otimes\mathcal{N}\otimes\mathcal{J}(-\varepsilon\cdot\Delta))\Big)=\bigotimes_{\iota=1}^r f_*(\omega_{V/U}\otimes\mathcal{A}\otimes\mathcal{N}\otimes\mathcal{J}(-\varepsilon\cdot\Delta))$$

is locally free; hence on V^r the sheaf $\bigotimes_{\iota=1}^r\mathrm{pr}_\iota^*\mathcal{J}(-\varepsilon\cdot\Delta)$ is flat over U and torsion-free. So

$$\bigotimes_{\iota=1}^r\mathrm{pr}_\iota^*\mathcal{J}(-\varepsilon\cdot\Delta)\xrightarrow{\alpha^r}\mathcal{J}(-\varepsilon\cdot(\mathrm{pr}_1^*\Delta+\cdots+\mathrm{pr}_r^*\Delta))$$

is injective. Finally, writing again $\mathcal{A}_{V^r}$ for the exterior tensor product and $\mathcal{A}_{Z^r}$ for the pullback to Z^r, the composite

$$\begin{aligned}
&f_*^r\Big(\bigotimes_{\iota=1}^r\mathrm{pr}_\iota^*(\omega_{V/U}\otimes\mathcal{A}\otimes\mathcal{N}\otimes\mathcal{J}(-\varepsilon\cdot\Delta))\Big)\\
&\qquad\xrightarrow{f_*^r(\alpha^r)} f_*^r(\omega_{V^r/U}\otimes\mathcal{A}_{V^r}\otimes\mathcal{N}_{V^r}\otimes\mathcal{J}(-\varepsilon\cdot(\mathrm{pr}_1^*\Delta+\cdots+\mathrm{pr}_r^*\Delta)))\\
&\qquad= f_*^r\tau_*^r(\omega_{Z^r/U}\otimes\mathcal{A}_{Z^r}\otimes\mathcal{N}_{Z^r}\otimes\mathcal{J}(-\varepsilon\cdot(\mathrm{p}_1^*\tau^*\Delta+\cdots+\mathrm{p}_r^*\tau^*\Delta)))\\
&\qquad=\bigotimes f_*\tau_*(\omega_{Z/U}\otimes\tau^*\mathcal{A}\otimes\tau^*\mathcal{N}\otimes\mathcal{J}(-\varepsilon\cdot\tau^*\Delta))
\end{aligned}$$

is an isomorphism. For $\mathcal{A}$ sufficiently ample, as in the proof of 6.7, this implies that α^r is an isomorphism.

Since $Z^r \to U$ is again mild, one may replace, in the first part of the proof, Z and V by Z^r and V^r, respectively, and may obtain the compatibility with pullbacks, required in 6.4(ii), for all r. □

As promised we can now formulate and prove the compatibility of multiplier ideal sheaves in total spaces of families with the restriction to subfamilies over curves. This will lead for suitable models to the compatibility of certain direct image sheaves with restriction to curves.

PROPOSITION 7.11. *Under the assumptions made in* 7.9 *there exists a local alteration* $\theta : U_1 \to U$ *for* C *such that*:

(1) $f_1 : V_1 = V \times_U U_1 \to U_1$ *is an embedded weak semistable reduction of* f *over* C.

(2) *For a lifting* $C_1 \subset U_1$ *of* C, *for* $S_1 = f_1^{-1}(C_1)$ *denote the induced morphisms by*

$$\begin{array}{ccc} S_1 & \xrightarrow{\chi'} & V \\ \zeta \downarrow & & \downarrow f \\ C_1 & \xrightarrow{\chi} & U. \end{array}$$

Then there is an isomorphism $\mathcal{J}(-\varepsilon \cdot \chi'^* \Delta) \cong \chi'^* \mathcal{J}(-\varepsilon \cdot \Delta)$.

(3) *Let* $\mathcal{A}$ *be an* f*-semiample sheaf on* V. *Then*

$$\chi^* f_*(\mathcal{A} \otimes \mathcal{N} \otimes \omega_{V/U} \otimes \mathcal{J}(-\varepsilon \cdot \Delta)) = \zeta_*(\chi'^*(\mathcal{A} \otimes \mathcal{N}) \otimes \omega_{S_1/C_1} \otimes \mathcal{J}(-\varepsilon \cdot \chi^* \Delta)).$$

Proof. Let us first show that (1) and (2) imply (3). By Lemma 7.10 the sheaf $\mathcal{J}(-\varepsilon \cdot \Delta)$ is flat over U and compatible with pullbacks and base change for $\theta : U_1 \to U$. So by abuse of notation it is sufficient in (3) to consider the case $U_1 = U$, and to assume that $C \subset U$. On a general fiber of $S \to C$ the multiplier ideal sheaf is isomorphic to the structure sheaf; hence by [EV92, Properties 7.5] the same holds over a neighborhood of the general point of C in U. As in the proof of 5.3, Kollár's Vanishing Theorem implies that over this neighborhood the direct image of $\mathcal{A} \otimes \mathcal{N} \otimes \omega_{V/U} \otimes \mathcal{J}(-\varepsilon \cdot \Delta)$ is locally free and compatible with arbitrary base change. Hence, if we apply 5.1 to this sheaf the open dense subscheme U_m in part (i) contains a general point of C. Then (3) follows from 5.1(ii).

To construct U_1 with the properties (1) and (2), we may assume that $\mathcal{E}$, hence $\mathcal{N}^N \otimes \mathcal{O}_V(-\Delta)$ is globally generated. Since the question is local on V, as in the second step in the proof of 6.5 we can cover V by the complements of divisors of general sections of $\mathcal{N}^N \otimes \mathcal{O}_V(-\Delta)$. Hence we may replace Δ by $\Delta + H$ and assume that $\mathcal{N}^N = \mathcal{O}_V(\Delta)$.

Choose a desingularization of the cyclic covering, obtained by taking the N-th root out of Δ. Over some alteration, this desingularization will have a mild model. Since this property is compatible with pullbacks, we may choose a local alteration for C, dominating the alteration, and we find some U_1 such that (1) holds and such that $V_1 \to U_1$ has a mild model. The compatibility for local alterations, shown in Lemma 7.10 allows us to assume that $U_1 = U$, hence that the mild model exists over U itself. Let us call it $T \to U$, and the induced morphism $\psi : T \to V$. So $\psi^* \Delta$ is the N-th power of a Cartier divisor.

Next we want to construct a desingularization W of T, which is flat over a general point of the curve C. To this aim, let $\widetilde{U} \to U$ be the blowing up of C, or a finite covering of such a blowing up. Let $\widetilde{V} \to \widetilde{U}$ be the pullback family. The preimage of the exceptional divisor E in $\widetilde{U}$ is covered by curves $\widetilde{C}$, finite over C. Lemma 7.3 allows us to shrink $\widetilde{U}$ such that the total space $\widetilde{V}$ is still normal with at most rational Gorenstein singularities.

Let $\widetilde{\phi} : \widetilde{W} \to \widetilde{T} = T \times_U \widetilde{U}$ be a desingularization. It dominates the finite covering obtained by taking the N-th root out of $\widetilde{\Delta} = \mathrm{pr}_1^* \Delta$. If $\widetilde{h} : \widetilde{W} \to \widetilde{U}$ denotes the induced map, we also assume that $\widetilde{h}^{-1}(E)$ is a normal crossing divisor. Over the complement $\widetilde{U}_g$ of a codimension two subset of $\widetilde{U}$ the morphism $\widetilde{h}$ will be flat and $\widetilde{h}^{-1}(E) \cap \widetilde{h}^{-1}(\widetilde{U}_g)$ will be equisingular over $E \cap \widetilde{U}_g$.

The divisor $\widetilde{h}^{-1}(E \cap \widetilde{U}_g)$ might be nonreduced. If so we perform the semistable reduction in codimension one, described in Theorem 7.1. Replacing $\widetilde{U}$ by some alteration and choosing $\widetilde{U}_g$ sufficiently small, we can assume that $\widetilde{h}^{-1}(E \cap \widetilde{U}_g)$ is a reduced, relative, normal crossing divisor.

For a curve $\widehat{C} \subset E$ meeting $\widetilde{U}_g$ choose a neighborhood U' in $\widetilde{U}$. By construction $\widetilde{h}^{-1}(\widehat{C} \cap \widetilde{U}_g)$ has nonsingular components, meeting transversely. For W' choose an embedded desingularization of the components of $\widetilde{h}^{-1}(\widehat{C})$, and assume that the closure Σ of $\widetilde{h}^{-1}(\widehat{C} \cap \widetilde{U}_g)$ is the union of manifolds, meeting transversely. Note that the induced morphism $h' : W' \to U'$ is still flat over some open subscheme U'_g, meeting $\widehat{C}$, and that there are morphisms

$$\psi' : T' = T \times_U U' \to V' = V \times_U U' \quad \text{and} \quad \phi : W' \to T'.$$

For $\widehat{C}$ sufficiently general, ϕ' is birational and ψ' an alteration.

As in the proof of 7.2 one obtains a morphism $\vartheta_0 : U'_g \to \mathfrak{Hilb}$ to the Hilbert scheme of subvarieties of some $\mathbb{P}^M$, parametrizing the fibers of h'.

Since $\Sigma \to \widehat{C}$ is flat the restriction of ϑ_0 to $\widehat{C} \cap U'_g$ extends to a morphism $\widehat{C} \to \mathfrak{Hilb}$, and the pullback of the universal family over $\mathfrak{Hilb}$ to $\widehat{C}$ coincides with Σ.

Blowing up U' with centers in $U' \setminus U'_g$ we obtain a new family, again denoted by $h' : W' \to U'$, which is flat and such that $h'^{-1}(\widehat{C}) = \Sigma$. By 7.3(ii), choosing the neighborhood U' of $\widehat{C}$ small enough, W' will be normal and Gorenstein.

Let us drop again all the $'$ and assume that the morphisms just constructed exist over V itself. Now, assume the alterations $W \xrightarrow{\phi} T \xrightarrow{\psi} V$, $\pi = \psi \circ \phi$, and $\gamma : \Sigma = \pi^*(S) \to S$ such that:

(i) $T \to U$ is mild and $\psi^* \Delta$ is divisible by N.

(ii) W is normal and Gorenstein, flat over U and ϕ is birational.

(iii) Σ is reduced, and the union of manifolds, meeting transversely.

The multiplier ideal $\mathcal{J}(-\varepsilon\cdot\Delta)$ is a direct factor of $\psi_*\omega_{T/V}\otimes\mathcal{N}^{-1}$. Let $\delta:\widetilde{W}\to W$ be a desingularization. Then one has

$$\delta_*\omega_{\widetilde{W}}\overset{\subset}{\to}\omega_W \quad\text{and}\quad \phi_*\delta_*\omega_{\widetilde{W}}\overset{\subset}{\to}\phi_*\omega_W\overset{\subset}{\to}\omega_T.$$

Since T has rational singularities, $\phi_*\delta_*\omega_{\widetilde{W}}=\omega_T$ and $\mathcal{N}\otimes\mathcal{J}(-\varepsilon\cdot\Delta)$ is a direct factor of $\pi_*\omega_{W/V}$.

The base change map induces a morphism

$$\eta:\mathcal{N}\otimes\mathcal{J}(-\varepsilon\cdot\Delta)|_S\longrightarrow\pi_*\omega_{W/V}|_S\longrightarrow\gamma_*\omega_{\Sigma/S}.$$

Recall that the sheaf $\mathcal{J}(-\varepsilon\cdot\Delta)|_S$ is flat over C. By [EV92, Properties 7.5], it contains $\mathcal{J}(-\varepsilon\cdot\Delta|_S)$, and by assumption both coincide on the general fiber of $S\to C$. Hence $\mathcal{J}(-\varepsilon\cdot\Delta)|_S$ is torsion-free and η is injective.

Choose $\hat{\Sigma}$ as the union of all components of Σ which dominate the irreducible variety S, and R the union of the other irreducible components $R_1,\ldots,R_\ell$. By construction, the components of Σ are nonsingular, and meet transversely. So one has the exact sequences

$$0\longrightarrow\omega_{\hat{\Sigma}}\longrightarrow\omega_\Sigma\longrightarrow\omega_R\otimes\mathcal{O}_R(R\cap\hat{\Sigma})\longrightarrow 0\quad\text{and}$$
$$0\longrightarrow\gamma_*\omega_{\hat{\Sigma}}\longrightarrow\gamma_*\omega_\Sigma\longrightarrow\gamma_*(\omega_R\otimes\mathcal{O}_R(R\cap\hat{\Sigma})).$$

The nonsingular alteration $\hat{\Sigma}\to S$ dominates the covering obtained by taking the N-th root out of $\Delta|_S$. By Lemma 2.2 the multiplier ideal $\mathcal{J}(-\varepsilon\cdot\Delta|_S)$ is a direct factor of $\mathcal{N}^{-1}|_S\otimes\gamma_*\omega_{\hat{\Sigma}}$. On the other hand, the sheaf $\gamma_*(\omega_R\otimes\mathcal{O}_R(R\cap\hat{\Sigma}))$ is contained in

$$\bigoplus_{\iota=1}^{\ell}\gamma_*(\omega_{R_\iota}\otimes\mathcal{O}_{R_\iota}(\Gamma_\iota))$$

where Γ_ι is the intersection of R_ι with the other components. Each of the sheaves $\gamma_*(\omega_{R_\iota}\otimes\mathcal{O}_{R_\iota}(\Gamma_\iota))$ is torsion-free over its support $\pi(R_\iota)$. By construction $\pi(R_\iota)$ is dominant over C. By assumption the composite

$$\eta:\omega_V\otimes\mathcal{N}\otimes\mathcal{J}(-\varepsilon\cdot\Delta)|_S\overset{\eta}{\to}\gamma_*\omega_\Sigma\longrightarrow\bigoplus_{\iota=1}^{\ell}\gamma_*(\omega_{R_\iota}\otimes\mathcal{O}_{R_\iota}(\Gamma_\iota))$$

is zero along the general fiber of $S\to C$; hence it is zero. So $\mathcal{J}(-\varepsilon\cdot\Delta)|_S$ maps to $\mathcal{J}(-\varepsilon\cdot\Delta|_S)$, and both must be equal. $\square$

8. Saturated extensions of polarizations

As in Section 4, $f:X\to Y$ will denote a projective morphism with smooth part $f_0:X_0\to Y_0$ for $Y_0\subset Y$ dense and with ω_{X_0/Y_0} relative semiample over Y_0.

If $\mathcal{L}_0$ is f_0-ample, we choose X and $\mathcal{L}$ as in Variant 4.8. Recall the diagram (4.6.1)

(8.0.1)
$$\begin{tikzcd} X \arrow[dr, "f"'] & \widehat{Z} \arrow[l, "\hat{\varphi}"'] \arrow[dr, "\hat{g}"] & Z \arrow[l, "\hat{\delta}"'] \arrow[r, "\delta"] \arrow[d, "g"] & \widehat{X} \arrow[r, "\rho"] \arrow[dl, "\hat{f}"] & X \arrow[dl, "f"] \\ & Y & \widehat{Y} \arrow[l, "\varphi"] \arrow[r, "\varphi"'] & Y, & \end{tikzcd}$$

with $\hat{g}$ mild, $\widehat{X}$ nonsingular, and such that the left- and right-hand diagrams are birational to fiber products. Z is a modification of both $\widehat{Z}$ and $\widehat{X}$. Recall moreover, that starting from an invertible sheaf $\mathcal{L}$ on X with $\mathcal{L}_0 = \mathcal{L}|_{X_0}$ we considered in 5.4 invertible sheaves $\mathcal{M}_\bullet$ on $\bullet$, where $\bullet$ stands for Z, $\widehat{Z}$ and $\widehat{X}$. They should satisfy the compatibilities $\hat{\delta}_*\mathcal{M}_Z = \mathcal{M}_{\widehat{Z}}$, $\delta_*\mathcal{M}_Z = \mathcal{M}_{\widehat{X}}$, $\hat{\varphi}^*\mathcal{L} \subset \mathcal{M}_{\widehat{Z}}$, $\mathcal{M}_{\widehat{Z}_0} = \mathcal{M}_{\widehat{Z}}|_{\widehat{Z}_0} = \hat{\varphi}_0^*\mathcal{L}_0$ and $\mathcal{M}_{\widehat{X}_0} = \mathcal{M}_{\widehat{X}}|_{\widehat{X}_0} = \rho_0^*\mathcal{L}_0$.

In this section we will impose more conditions on those sheaves. First of all, if the evaluation map for $\omega^\nu_{X_0/Y_0} \otimes \mathcal{L}_0^\mu$ is surjective, we may replace $\widehat{X}$ by a modification with center in $\widehat{X} \setminus \widehat{X}_0$ and assume that the image of the evaluation map

$$\hat{f}^*\hat{f}_*\big(\omega^\nu_{\widehat{X}/\widehat{Y}} \otimes \mathcal{M}^\mu_{\widehat{X}}\big) \longrightarrow \omega^\nu_{\widehat{X}/\widehat{Y}} \otimes \mathcal{M}^\mu_{\widehat{X}}$$

is invertible. As a first step we have to show that the same is possible with $\widehat{X}$ replaced by $\widehat{Z}$ without losing the mildness of $\hat{g}$. This will simplify some of the constructions in the next sections, but mainly it will be needed to define *saturated extensions*:

Remark 8.1. Assume for a moment that $\dim Y = 1$, hence that we can choose $\widehat{Z} = Z = \widehat{X} \to \widehat{Y}$ to be the semistable model; assume moreover that the smooth fibers of f_0 are of Kodaira dimension zero. Then $\omega^\nu_{\widehat{X}/\widehat{Y}} = \mathcal{O}_{\widehat{X}}(\Pi) \otimes \hat{f}^*\hat{f}_*\omega^\nu_{\widehat{X}/\widehat{Y}}$; for some ν and for some effective divisor contained in the singular fibers of f. For $\mathcal{M} = \mathcal{M}_{\widehat{X}}$ the sheaf corresponding to the left-hand side in (3.4.1) is the $r(1)$-th power of

$$\det\big(\hat{f}_*\big(\omega^\epsilon_{\widehat{X}/\widehat{Y}} \otimes \mathcal{M}\big)\big) \otimes \det(\hat{f}_*\mathcal{M})^{-1} = \det\left(\hat{f}_*\left(\mathcal{O}_{\widehat{X}}\left(\frac{\epsilon}{\nu}\Pi\right) \otimes \mathcal{M}\right)\right) \otimes \det(\hat{f}_*\mathcal{M})^{-1} \otimes \lambda_\nu^{\frac{\epsilon}{\nu}}.$$

Roughly speaking $\mathcal{M}$ will be a saturated extension of the polarization $\mathcal{M}|_{\widehat{X}_0}$ if

$$\hat{f}_*\left(\mathcal{O}_{\widehat{X}}\left(\frac{\epsilon}{\nu}\cdot\Pi\right) \otimes \mathcal{M}\right) = \hat{f}_*(\mathcal{O}_{\widehat{X}}(*\cdot\Pi) \otimes \mathcal{M}) = \hat{f}_*\mathcal{M}.$$

Lemma and Notation 8.2. Consider in Corollary 5.5 for a given tuple $(\nu, \mu) \in I$ a locally free sheaf $\mathcal{E}_{\widehat{Y}}$ and a morphism $\mathcal{E}_{\widehat{Y}} \to \hat{f}_*(\omega^\nu_{\widehat{X}/\widehat{Y}} \otimes \mathcal{M}^\mu_{\widehat{X}})$ such that the evaluation map $\hat{f}^*\mathcal{E}_{\widehat{Y}} \to \omega^\nu_{\widehat{X}/\widehat{Y}} \otimes \mathcal{M}^\mu_{\widehat{X}}$ is surjective over $\widehat{X}_0$. Then, replacing $\widehat{Y}$ by some nonsingular alteration, $\widehat{Z}$ by a modification of the pullback family and $\mathcal{E}_{\widehat{Y}}$

by its pullback, one can assume that beside conditions (a)–(c) in 4.5 and beside condition (d) in 5.5 one has:

(e) The images of the evaluation maps

$$\hat{g}^*\mathcal{E}_{\hat{Y}} \to \omega^{\nu}_{\hat{Z}/\hat{Y}} \otimes \mathcal{M}^{\mu}_{\hat{Z}} \quad \text{and} \quad \hat{f}^*\mathcal{E}_{\hat{Y}} \to \omega^{\nu}_{\hat{X}/\hat{Y}} \otimes \mathcal{M}^{\mu}_{\hat{X}}$$

are invertible sheaves. So for some divisors $\Sigma_{\hat{Z}}$ and $\Sigma_{\hat{X}}$ those images are of the form

$$\mathcal{B}_{\hat{Z}} = \omega^{\nu}_{\hat{Z}/\hat{Y}} \otimes \mathcal{M}^{\mu}_{\hat{Z}} \otimes \mathcal{O}_{\hat{Z}}(-\Sigma_{\hat{Z}}) \quad \text{and} \quad \mathcal{B}_{\hat{X}} = \omega^{\nu}_{\hat{X}/\hat{Y}} \otimes \mathcal{M}^{\mu}_{\hat{X}} \otimes \mathcal{O}_{\hat{X}}(-\Sigma_{\hat{X}}).$$

On the common modification Z one has $\hat{\delta}^*\mathcal{B}_{\hat{Z}} = \delta^*\mathcal{B}_{\hat{X}}$. We denote this sheaf by $\mathcal{B}_Z$.

Proof. Consider a blowing up $\tau : Z' \to \hat{Z}$ such that the image $\mathcal{B}_{Z'}$ of

$$\tau^*\hat{g}^*\mathcal{E}_{\hat{Y}} \longrightarrow \omega^{\nu}_{Z'/\hat{Y}} \otimes \tau^*\mathcal{M}^{\mu}_{\hat{Z}}$$

is invertible.

We perform the weak semistable reduction 4.4 a second time, starting from a flattening of the morphism $Z' \to \hat{Y}$ as explained in 4.4, Step I. By 4.5 we obtain a mild morphism $\tilde{g} : \tilde{Z}_1 \to \hat{Y}_1$ and a diagram

$$\begin{tikzcd}
\hat{Z} \arrow[dr, "\hat{g}"'] & Z' \arrow[l, "\tau"'] \arrow[d, "g'"] & \tilde{Z}_1 \arrow[l, "\tilde{\varphi}_1"'] \arrow[d, "\tilde{g}_1"] \\
 & \hat{Y} & \hat{Y}_1. \arrow[l, "\varphi_1"']
\end{tikzcd}$$

So over $\hat{Y}_1$ we have two different mild models, $\tilde{g}_1 : \tilde{Z}_1 \to \hat{Y}_1$ and $g_1 : \hat{Z}_1 \to \hat{Y}_1$, and a morphism $\tau' : \tilde{Z}_1 \to \hat{Z}_1$. We define $\mathcal{M}_{\tilde{Z}_1}$ as the pullback of $\mathcal{M}_{\hat{Z}_1}$.

The sheaf $\mathcal{F}^{(\nu,\mu)}_{\hat{Y}_1}$ is independent of the mild model, and Corollary 5.5 implies that $\varphi_1^*\mathcal{F}^{(\nu,\mu)}_{\hat{Y}} = \mathcal{F}^{(\nu,\mu)}_{\hat{Y}_1}$. So for $\mathcal{E}_{\hat{Y}_1} = \varphi^*\mathcal{E}_{\hat{Y}}$ the pullback $\tilde{g}_1^*\mathcal{E}_{\hat{Y}_1} = \tilde{\varphi}_1^*\tau^*\hat{g}^*\mathcal{E}_{\hat{Y}}$ maps surjectively to the invertible sheaf $\mathcal{B}_{\tilde{Z}} = \tilde{\varphi}_1^*\mathcal{B}_{Z'}$.

Since the evaluation map $\hat{f}_0^*\mathcal{E}_{\hat{Y}_0} \to \omega^{\nu}_{\hat{X}_0/\hat{Y}_0} \otimes \mathcal{M}^{\mu}_{\hat{X}_0}$ is surjective, the same holds true for the pullback family, and the image sheaf $\mathcal{B}_{\hat{X}_1}$ is locally free over the preimage of $\hat{Y}_0$. So replacing $\hat{X}_1$ by a suitable nonsingular modification, we may assume that it is invertible.

Replacing $\tilde{Z}_1$ by $\hat{Z}$ and dropping the index $_1$ we found the invertible sheaf $\mathcal{B}_{\hat{Z}}$ and $\mathcal{B}_{\hat{X}}$. Both, $\hat{\delta}^*\mathcal{B}_{\hat{Z}}$ and $\delta^*\mathcal{B}_{\hat{X}}$ are the images of the evaluation map $g^*\mathcal{E}_{\hat{Y}} \to \omega^{\nu}_{Z/\hat{Y}} \otimes \mathcal{M}^{\mu}_Z$; hence they coincide. □

Note that the divisor $\Sigma_{\hat{X}}$ is supported in the boundary, whereas in general the divisor $\Sigma_{\hat{Z}}$ can meet $\hat{Z}_0$.

For dominant morphisms $\theta : \hat{Y}_1 \to \hat{Y}$ or for morphisms from curves, whose images meet $\hat{Y}_g$, the sheaves $\mathcal{B}_{\hat{Z}}$ and $\mathcal{B}_{\hat{X}}$ are compatible with base change in the following sense.

Consider $\hat{Z}_1 = \hat{Z} \times_{\hat{Y}} \hat{Y}_1$ and a desingularization $\iota : \hat{X}_1 \to \hat{X} \times_{\hat{Y}} \hat{Y}_1$ of the main component. Writing $\mathcal{E}_{\hat{Y}_1} = \theta^*\mathcal{E}_{\hat{Y}}$, the evaluation maps factor through surjections

$$\text{(8.2.1)} \qquad \hat{g}_1^*\mathcal{E}_{\hat{Y}_1} \longrightarrow \mathrm{pr}_1^*\mathcal{B}_{\hat{Z}} \quad \text{and} \quad \hat{f}_1^*\mathcal{E}_{\hat{Y}_1} \longrightarrow \iota^*\mathrm{pr}_1^*\mathcal{B}_{\hat{X}}.$$

On the other hand, $\mathcal{M}_{\hat{Z}_1} = \mathrm{pr}_1^*\mathcal{M}_{\hat{Z}}$ and $\omega_{\hat{Z}_1/\hat{Y}_1} = \mathrm{pr}_1^*\omega_{\hat{Z}/\hat{Y}}$. So $\mathrm{pr}_1^*\mathcal{B}_{\hat{Z}}$ is a subsheaf of $\omega^{\nu}_{\hat{Z}_1/\hat{Y}_1} \otimes \mathcal{M}^{\mu}_{\hat{Z}_1}$, and we write $\mathcal{B}_{\hat{Z}_1} = \mathrm{pr}_1^*\mathcal{B}_{\hat{Z}}$. By Corollary 5.5, $\mathcal{F}^{(\nu,\mu)}_{\hat{Y}_1} = \theta^*\mathcal{F}^{(\nu,\mu)}_{\hat{Y}}$ and Lemma 5.7 implies that the images of the second evaluation maps in (8.2.1) lie in $\omega^{\nu}_{\hat{X}_1/\hat{Y}_1} \otimes \mathcal{M}^{\mu}_{\hat{X}_1}$. Then $\mathcal{B}_{\hat{Z}_1}$ and $\mathcal{B}_{\hat{X}_1} = \iota^*\mathrm{pr}_1^*\mathcal{B}_{\hat{X}}$ satisfy again the conditions stated in 8.2.

However in 8.2 we also changed the mild model. Using the notation from the proof of 8.2 we replaced $\hat{Z}_1 \to \hat{Y}_1$ by a new mild model $\tilde{Z}_1 \to \hat{Y}_1$. One is allowed to do so, if there is a birational morphism $\tau' : \tilde{Z}_1 \to \hat{Z}_1$, as is the case in 8.2. One chooses $\mathcal{M}_{\tilde{Z}}$ as the pullback of $\mathcal{M}_{\hat{Z}_1}$. Then $\mathcal{B}_{\tilde{Z}} = \tau'^*\mathcal{B}_{\hat{Z}_1}$ satisfies again the conditions stated in 8.2.

Addendum 8.3. Assume that $\hat{Y}$ and $\hat{Z}$ are chosen such that the conclusion of 8.2 holds true. Then we may replace $\hat{Y}$ by a nonsingular alteration $\hat{Y}_1$ and the pullback of the given mild model $\hat{Z}_1 \to \hat{Y}_1$ by any mild morphism $\tilde{Z}_1 \to \hat{Y}_1$ provided there is a morphism $\tau' : \tilde{Z}_1 \to \hat{Z}_1$, birational over $\hat{Y}_1$.

In particular, given a finite number of $(\nu, \mu) \in I$, and a finite number of sheaves $\mathcal{E}_{\hat{Y}}$, one can apply 8.2 successively. Since we assumed that $\mathcal{F}^{(\nu,\mu)}_{\hat{Y}}$ is locally free, one possible choice for $\mathcal{E}_{\hat{Y}}$ is the sheaf $\mathcal{F}^{(\nu,\mu)}_{\hat{Y}}$ itself.

Notation 8.4. Consider in 5.5 a subset $\tilde{I} \subset I$ and assume that for $(\nu, \mu) \in \tilde{I}$ the evaluation map $f_0^* f_{0*}\big(\omega^{\nu}_{X_0/Y_0} \otimes \mathcal{L}_0^{\mu}\big) \longrightarrow \omega^{\nu}_{X_0/Y_0} \otimes \mathcal{L}_0^{\mu}$ is surjective. If one chooses, in 8.2, $\mathcal{E}_{\hat{Y}} = \mathcal{F}^{(\nu,\mu)}_{\hat{Y}}$, we will write $\Sigma^{(\nu,\mu)}_{\bullet}$ and $\mathcal{B}^{(\nu,\mu)}_{\bullet}$ instead of $\Sigma_{\bullet}$ and $\mathcal{B}_{\bullet}$, where $\bullet$ stands for $\hat{Z}$, $\hat{X}$ or Z. In particular

$$\mathcal{B}^{(\nu,\mu)}_{\bullet} = \omega^{\nu}_{\bullet/\hat{Y}} \otimes \mathcal{M}^{\mu}_{\bullet} \otimes \mathcal{O}_{\bullet}(-\Sigma^{(\nu,\mu)}_{\bullet}).$$

If $\mu = 0$ we will write $\varpi^{(\nu)}_{\bullet}$ and $\Pi^{(\nu)}_{\bullet}$ instead of $\mathcal{B}^{(\nu,0)}_{\bullet}$ and $\Sigma^{(\nu,0)}_{\bullet}$.

Let us collect the properties we require for a well chosen nonsingular alteration $\hat{Y} \to Y$ and for the morphisms in the diagram (8.0.1).

Conclusion and Notation 8.5. We start with a finite set I of tuples (ν, μ) of natural numbers, and with a subset $\tilde{I}$ of I. We assume that for some $\eta_0 > 0$ with $(\eta_0, 0) \in \tilde{I}$ the evaluation map $f_0^* f_{0*}\omega^{\eta_0}_{X_0/Y_0} \to \omega^{\eta_0}_{X_0/Y_0}$ is surjective, and that for all other $(\eta, 0) \in \tilde{I}$ the natural number η is divisible by η_0.

Then we can find $\widehat{Y}$ and the diagram (4.6.1) (recalled in (8.0.1)) such that:

(i) Conditions (a), (b) and (c) in Proposition 4.5 hold true, as well as the conditions i) and (ii) in 4.6.

(ii) For $(\eta, 0) \in \widetilde{I}$ there are invertible sheaves $\varpi_{\widehat{Z}}^{(\eta)}$, $\varpi_Z^{(\eta)}$, and $\varpi_{\widehat{X}}^{(\eta)}$ on $\widehat{Z}$, Z and on $\widehat{X}$, respectively, with surjective evaluation maps,

$$\varpi_Z^{(\eta)} = \hat{\delta}^* \varpi_{\widehat{Z}}^{(\eta)} = \delta^* \varpi_{\widehat{X}}^{(\eta)} \quad \text{and with}$$
$$\mathcal{F}_{\widehat{Y}}^{(\eta)} := \mathcal{F}_{\widehat{Y}}^{(\eta,0)} = \hat{g}_* \omega_{\widehat{Z}/\widehat{Y}}^{\eta} = \hat{g}_* \varpi_{\widehat{Z}}^{(\eta)} = \hat{f}_* \varpi_{\widehat{X}}^{(\eta)}.$$

(iii) For all $(\nu, 0) \in I$ the sheaves $\mathcal{F}_{\widehat{Y}}^{(\nu)} := \mathcal{F}_{\widehat{Y}}^{(\nu,0)} = \hat{g}_* \omega_{\widehat{Z}/\widehat{Y}}^{\nu}$ are locally free.

(iv) There is an open dense subscheme $\widehat{Y}_g$ with $\hat{g}^{-1}(\widehat{Y}_g) \to \widehat{Y}_g$ smooth such that for all $(\nu, 0) \in I$ the sheaves $\mathcal{F}_{\widehat{Y}}^{(\nu)} = \hat{g}_* \omega_{\widehat{Z}/\widehat{Y}}^{\nu}$ are compatible with base change for morphisms $\varrho : T \to \widehat{Y}$ with $\varrho^{-1}(\widehat{Y}_g)$ dense in T.

(v) In (ii) $\Pi_Z^{(\eta)}$, $\Pi_{\widehat{Z}}^{(\eta)}$ and $\Pi_{\widehat{X}}^{(\eta)}$ denote the divisors with

$$\omega_{Z/\widehat{Y}}^{\eta} = \varpi_Z^{(\eta)} \otimes \mathcal{O}_Z\big(\Pi_Z^{(\eta)}\big), \qquad \omega_{\widehat{Z}/\widehat{Y}}^{\eta} = \varpi_{\widehat{Z}}^{(\eta)} \otimes \mathcal{O}_{\widehat{Z}}\big(\Pi_{\widehat{Z}}^{(\eta)}\big)$$
$$\text{and} \quad \omega_{\widehat{X}/\widehat{Y}}^{\eta} = \varpi_{\widehat{X}}^{(\eta)} \otimes \mathcal{O}_{\widehat{X}}\big(\Pi_{\widehat{X}}^{(\eta)}\big).$$

Conclusion and Notation 8.6 (canonical polarizations). All we need in this case is collected in 8.5. We will of course choose $\widetilde{I}$ and I as subsets of $\mathbb{N} \times \{0\}$. If $\mathcal{L}_0 \neq \mathcal{O}_{X_0}$, i.e., if we consider polarized manifolds, we will need more:

Conclusion and Notation 8.7 (polarizations). We consider in 8.5 an invertible sheaf $\mathcal{L}$ on X with $\mathcal{L}_0 = \mathcal{L}|_{X_0}$ f_0-ample, and we choose $\gamma_0 > 0$ such that the evaluation map $f_0^* f_{0*} \mathcal{L}_0^{\gamma_0} \to \mathcal{L}_0^{\gamma_0}$ is surjective. We fix some subset $\widetilde{I}$ of I consisting of tuples (β, α) of natural numbers with α divisible by γ_0 and with β divisible by η_0. By Lemma 5.3 the direct images $f_{0*}(\omega_{X_0/Y_0}^{\nu} \otimes \mathcal{L}_0^{\mu})$ are locally free and compatible with arbitrary base change, whenever $\nu > 0$ and $\mu \geq 0$. For $(0, \mu) \in I$ we have to add the corresponding statement to the list of assumptions.

Then we can find $\widehat{Y}$ and the diagram (4.6.1) such that conditions (i)–(v) in 8.5 hold true and in addition:

(vi) $\mathcal{M}_{\widehat{Z}}$, $\mathcal{M}_Z$, and $\mathcal{M}_{\widehat{X}}$ are the pullback of $\mathcal{L}$.

(vii) For $(\beta, \alpha) \in \widetilde{I}$ there are invertible sheaves $\mathcal{B}_{\widehat{Z}}^{(\beta,\alpha)}$, $\mathcal{B}_Z^{(\beta,\alpha)}$, and $\mathcal{B}_{\widehat{X}}^{(\beta,\alpha)}$ on $\widehat{Z}$, Z and on $\widehat{X}$, respectively, with surjective evaluation maps, with

$$\mathcal{B}_Z^{(\beta,\alpha)} = \hat{\delta}^* \mathcal{B}_{\widehat{Z}}^{(\beta,\alpha)} = \delta^* \mathcal{B}_{\widehat{X}}^{(\beta,\alpha)} \quad \text{and with}$$
$$\mathcal{F}_{\widehat{Y}}^{(\beta,\alpha)} = \hat{g}_*\big(\omega_{\widehat{Z}/\widehat{Y}}^{\beta} \otimes \mathcal{M}_{\widehat{Z}}^{\alpha}\big) = \hat{g}_* \mathcal{B}_{\widehat{Z}}^{(\beta,\alpha)} = \hat{f}_* \mathcal{B}_{\widehat{X}}^{(\beta,\alpha)}.$$

(viii) For all $(\nu, \mu) \in I$ the sheaves $\mathcal{F}_{\widehat{Y}}^{(\nu,\mu)} = \hat{g}_*(\omega_{\widehat{Z}/\widehat{Y}}^{\nu} \otimes \mathcal{M}_{\widehat{Z}}^{\mu})$ are locally free.

(ix) There is an open dense subscheme $\widehat{Y}_g$ with $\hat{g}^{-1}(\widehat{Y}_g) \to \widehat{Y}_g$ smooth such that for all $(\nu, \mu) \in I$ the sheaves $\mathcal{F}_{\widehat{Y}}^{(\nu,\mu)} = \hat{g}_*(\omega^{\nu}_{\widehat{Z}/\widehat{Y}} \otimes \mathcal{M}^{\mu}_{\widehat{Z}})$ are compatible with base change for morphisms $\varrho : T \to \widehat{Y}$ with $\varrho^{-1}(\widehat{Y}_g)$ dense in T.

(x) $\Sigma_Z^{(\beta,\alpha)}$, $\Sigma_{\widehat{Z}}^{(\beta,\alpha)}$ and $\Sigma_{\widehat{X}}^{(\beta,\alpha)}$ denote the divisors with

$$\omega^{\beta}_{Z/\widehat{Y}} \otimes \mathcal{M}^{\alpha}_Z = \mathcal{B}_Z^{(\beta,\alpha)} \otimes \mathcal{O}_Z\big(\Sigma_Z^{(\beta,\alpha)}\big), \qquad \omega^{\beta}_{\widehat{Z}/\widehat{Y}} \otimes \mathcal{M}^{\alpha}_{\widehat{Z}} = \mathcal{B}_{\widehat{Z}}^{(\beta,\alpha)} \otimes \mathcal{O}_{\widehat{Z}}\big(\Sigma_{\widehat{Z}}^{(\beta,\alpha)}\big)$$
$$\text{and} \quad \omega^{\beta}_{\widehat{X}/\widehat{Y}} \otimes \mathcal{M}^{\alpha}_{\widehat{X}} = \mathcal{B}_{\widehat{X}}^{(\beta,\alpha)} \otimes \mathcal{O}_{\widehat{X}}\big(\Sigma_{\widehat{X}}^{(\beta,\alpha)}\big).$$

Allowed Constructions 8.8. The conditions stated in 8.5 and 8.7 and the sheaves $\mathcal{F}_{\bullet}^{(\nu,\mu)}$ for $(\nu, \mu) \in I$ are compatible with the following constructions:

I. Replace $\widehat{Y}$ by a nonsingular alteration, $\widehat{Z}$ by its pullback, and $\widehat{X}$ by a desingularization of the main component of its pullback.

II. Replace $\widehat{Z}$ by a mild morphism $\widetilde{Z} \to \widehat{Y}$, for which there is a birational $\widehat{Y}$-morphism $\tau : \widetilde{Z} \to \widehat{Z}$.

In particular assume that for some open set $U \subset \widehat{Y}$ containing $\widehat{Y}_0$ the morphism $f^{-1}(U) \to \widehat{Y}$ is flat. Then one can choose a mild morphism $\widetilde{Z}_1 \to \widehat{Y}_1$ factoring through $\tau_1 : \widetilde{Z}_1 \to \widehat{X}_1$, and still assume that 8.5 and 8.7 hold true.

Proof. This has been shown in Addendum 8.3. For the last part, one performs the weak semistable reduction, starting with $\widehat{X} \to \widehat{Y}$ instead of $\widetilde{X} \to \widetilde{Y}$ in Step I of 4.4. □

Next we will start to construct the saturated extensions of the polarizations. Although this will only be applied for families of Kodaira dimension zero, we will allow ω_{X_0/Y_0} to be f_0-semiample.

LEMMA 8.9. *Let $\mathcal{M}_{\widehat{Z}}$, $\mathcal{M}_{\widehat{X}}$ and $\mathcal{M}_Z$ be invertible sheaves on $\widehat{Z}$, $\widehat{X}$ and Z, respectively, satisfying the compatibility conditions in* 5.4. *Assume that κ is a positive integer with $(0, \kappa) \in I$. Using the notation and conditions in* 8.5 *one has*:

(1) *For all $\varepsilon \geq 0$ and for all alterations $\widehat{Y}_1$ of $\widehat{Y}$*

$$\hat{\delta}_*\Big(\mathcal{M}^{\kappa}_{Z_1} \otimes \mathcal{O}_{Z_1}\big(\varepsilon \cdot \Pi^{(\eta_0)}_{Z_1}\big)\Big) = \mathcal{M}^{\kappa}_{\widehat{Z}_1} \otimes \mathcal{O}_{\widehat{Z}_1}\big(\varepsilon \cdot \Pi^{(\eta_0)}_{\widehat{Z}_1}\big) \quad \textit{and}$$
$$\delta_*\Big(\mathcal{M}^{\kappa}_{Z_1} \otimes \mathcal{O}_{Z_1}\big(\varepsilon \cdot \Pi^{(\eta_0)}_{Z_1}\big)\Big) = \mathcal{M}^{\kappa}_{\widehat{X}_1} \otimes \mathcal{O}_{\widehat{X}_1}\big(\varepsilon \cdot \Pi^{(\eta_0)}_{\widehat{X}_1}\big).$$

(2) *For each $\kappa > 0$ there exists some $\varepsilon_0 \geq 0$ such that*

$$\iota : \hat{g}_{1*}\mathcal{M}^{\kappa}_{\widehat{Z}_1} \otimes \mathcal{O}_{\widehat{Z}_1}\big(\varepsilon_0 \cdot \Pi^{(\eta_0)}_{\widehat{Z}_1}\big) \longrightarrow \hat{g}_{1*}\mathcal{M}^{\kappa}_{\widehat{Z}_1} \otimes \mathcal{O}_{\widehat{Z}_1}\big(\varepsilon \cdot \Pi^{(\eta_0)}_{\widehat{Z}_1}\big)$$

are isomorphisms for all $\varepsilon \geq \varepsilon_0$, and for all alterations $\widehat{Y}_1$ of $\widehat{Y}$.

Note that (1) and (2) imply that for all $\varepsilon \geq \varepsilon_0$ one also has

$$\hat{f}_{1*}\mathcal{M}^{\kappa}_{\widehat{X}_1} \otimes \mathcal{O}_{\widehat{X}_1}\big(\varepsilon_0 \cdot \Pi^{(\eta_0)}_{\widehat{X}_1}\big) \xrightarrow{\cong} \hat{f}_{1*}\mathcal{M}^{\kappa}_{\widehat{X}_1} \otimes \mathcal{O}_{\widehat{X}_1}\big(\varepsilon \cdot \Pi^{(\eta_0)}_{\widehat{X}_1}\big).$$

Proof of 8.9. We replace $\mathcal{M}_\bullet^\kappa$ by $\mathcal{M}_\bullet$ and assume that $\kappa = 1$. For (1) consider the common modification Z. By 8.5(ii)

$$\varpi_Z^{(\eta_0)} = \hat{\delta}^* \varpi_{\widehat{Z}}^{(\eta_0)} = \delta^* \varpi_{\widehat{X}}^{(\eta_0)}, \quad \text{and}$$
$$\Pi_Z^{(\eta_0)} = \hat{\delta}^* \Pi_{\widehat{Z}}^{(\eta_0)} + \eta_0 \cdot E_{\widehat{Z}} = \delta^* \Pi_{\widehat{X}}^{(\eta_0)} + \eta_0 \cdot E_{\widehat{X}},$$

where $E_\bullet$ are effective relative canonical divisors for $Z/\bullet$. The assumptions $\delta_* \mathcal{M}_Z = \mathcal{M}_{\widehat{X}}$ and $\delta_* \mathcal{M}_Z = \mathcal{M}_{\widehat{X}}$ imply that

$$\mathcal{M}_Z = \hat{\delta}^* \mathcal{M}_{\widehat{Z}} \otimes \mathcal{O}_Z(F_{\widehat{Z}}) = \delta^* \mathcal{M}_{\widehat{X}} \otimes \mathcal{O}_Z(F_{\widehat{X}})$$

for effective exceptional divisors $F_{\widehat{Z}}$ and $F_{\widehat{X}}$, and (1) for $\widehat{Y}_1 = \widehat{Y}$ follows from the projection formula. The same argument works over any alteration.

For (2), note that one may replace $\widehat{Y}_1$ by a modification $\theta : \widehat{Y}_2$ and $\widehat{Z}_1$ by the pullback family $\widehat{Z}_2 = \widehat{Z}_1 \times_{\widehat{Y}_1} \widehat{Y}_2 \to \widehat{Y}_2$. In fact, the divisor $\Pi_{\widehat{Z}_1}$ is compatible with pullback, and for all $\varepsilon \geq 0$ one has

$$\mathrm{pr}_{1*}\Big(\mathcal{M}_{\widehat{Z}_2} \otimes \mathcal{O}_{\widehat{Z}_2}(\epsilon \cdot \Pi_{\widehat{Z}_2})\Big) = \mathcal{M}_{\widehat{Z}_1} \otimes \mathcal{O}_{\widehat{Z}_1}(\varepsilon \cdot \Pi_{\widehat{Z}_1}).$$

Hence

$$\theta_* \hat{g}_{2*} \mathcal{M}_{\widehat{Z}_2}^\kappa \otimes \mathcal{O}_{\widehat{Z}_2}(\varepsilon \cdot \Pi_{\widehat{Z}_2}^{(\eta_0)}) = \hat{f}_{1*} \mathcal{M}_{\widehat{X}_1}^\kappa \otimes \mathcal{O}_{\widehat{X}_1}(\varepsilon \cdot \Pi_{\widehat{X}_1}^{(\eta_0)})$$

and if the first sheaf is independent of ε, for ε sufficiently large, the same holds for the second one.

The fibers of $\widehat{Z} \to \widehat{Y}$ are reduced. Then the compatibility of $\mathcal{F}_{\widehat{Y}}^{(\eta_0)}$ with pullback under alterations and the surjectivity of the evaluation map for $\omega_{\widehat{Z}/\widehat{Y}}^{\eta_0} \otimes \mathcal{O}_{\widehat{Z}}(-\Pi_{\widehat{Z}}^{(\eta_0)})$ imply that $\Pi_{\widehat{Z}}^{(\eta_0)}$ cannot contain a whole fiber. Otherwise, for some sheaf of ideals $\mathcal{J}$ on $\widehat{Y}$ one would have $\varpi_{\widehat{Z}}^{(\eta_0)} \subset \hat{g}^* \mathcal{J} \otimes \omega_{\widehat{Z}/\widehat{Y}}^{\eta_0}$. Blowing up $\widehat{Y}$ one gets the same, with $\mathcal{J} = \mathcal{O}_{\widehat{Y}}(-\Gamma)$, for an effective divisor Γ. Then the projection formula implies that $\hat{g}_* \varpi_{\widehat{Z}}^{(\eta_0)} \subset \mathcal{J} \otimes \hat{g}_* \omega_{\widehat{Z}/\widehat{Y}}^{\eta_0}$, contradicting 8.5(ii).

By flat base change, the question whether ι is an isomorphism is local for the étale topology. So by abuse of notation we may replace $\widehat{Y}$ by any étale neighborhood. Hence given $y \in \widehat{Y}$ we may assume that $\hat{g}$ has a section $\sigma : \widehat{Y} \to \widehat{Z}$ whose image does not meet $\Pi_{\widehat{Z}}^{(\eta_0)}$, but meets the open set V_0 where $\hat{\varphi}_0 : \widehat{Z}_0 \to \widehat{X}_0$ is an isomorphism. Let $\mathcal{I}$ be the ideal sheaf of $\sigma(\widehat{Y})$. For a general fiber F of $\hat{f}$ and for υ sufficiently large $H^0(F, (\varphi_* \mathcal{I}^\upsilon) \otimes \mathcal{M}_{\widehat{X}}|_F) = 0$. Then

$$\hat{g}_{0*}\bigg(\Big(\mathcal{I}^\upsilon \otimes \mathcal{M}_{\widehat{Z}} \otimes \mathcal{O}_{\widehat{Z}}(\varepsilon \cdot \Pi_{\widehat{Z}}^{(\eta_0)})\Big)\Big|_{\widehat{Z}_0}\bigg)$$
$$= \hat{f}_{0*}\bigg(\hat{\varphi}_{0*}\Big(\mathcal{I}^\upsilon \otimes \mathcal{O}_{\widehat{Z}}(\varepsilon \cdot \Pi_{\widehat{Z}}^{(\eta_0)})\Big)\Big|_{\widehat{Z}_0} \otimes \mathcal{M}_{\widehat{X}_0}\bigg) = 0,$$

and $\hat{g}_*\mathcal{M}_{\widehat{Z}} \otimes \mathcal{O}_{\widehat{Z}}(\varepsilon \cdot \Pi_{\widehat{Z}}^{(\eta_0)})$ is a subsheaf of

$$\hat{g}_*\mathcal{M}_{\widehat{Z}}/\mathcal{I}^{\nu} = \hat{g}_*\Big(\mathcal{O}_{\widehat{Z}}\big(\varepsilon \cdot \Pi_{\widehat{Z}}^{(\eta_0)}\big) \otimes \mathcal{M}_{\widehat{Z}}/\mathcal{I}^{\nu}\Big).$$

So $\mathcal{C} = \hat{g}_*\mathcal{M}_{\widehat{Z}} \otimes \mathcal{O}_{\widehat{Z}}(* \cdot \Pi_{\widehat{Z}}^{(\eta_0)})$ as a subsheaf of a fixed, locally free sheaf is isomorphic to $\hat{g}_*\mathcal{M}_{\widehat{Z}} \otimes \mathcal{O}_{\widehat{Z}}(\varepsilon_1 \cdot \Pi_{\widehat{Z}}^{(\eta_0)})$ for some ε_1.

Let $\theta : \widehat{Y}_2 \to \widehat{Y}$ be a modification, such that $\mathcal{C}_2 = \theta^*\mathcal{C}/_{\text{torsion}}$ is locally free, and contained in a locally free, locally splitting subsheaf $\mathcal{C}'$ of $\theta^*\hat{g}_{1*}(\mathcal{M}_{\widehat{Z}_1}/\mathcal{I}_1^{\nu})$ with $\mathrm{rk}(\mathcal{C}') = \mathcal{C}_2$. Writing $\mathcal{I}_2$ for the pullback of the sheaf of ideals $\mathcal{I}$, the latter is of the form $\hat{g}_{2*}(\mathcal{M}_{\widehat{Z}_2}/\mathcal{I}_2^{\nu})$. For some effective divisor D one has an inclusion $\mathcal{C}' \subset \mathcal{C}_2 \otimes \mathcal{O}_{\widehat{Y}_2}(D)$. The base change morphism

$$\theta^*\hat{g}_*\mathcal{M}_{\widehat{Z}} \otimes \mathcal{O}_{\widehat{Z}}\big(\varepsilon \cdot \Pi_{\widehat{Z}}^{(\eta_0)}\big) \longrightarrow \hat{g}_{2*}\mathcal{M}_{\widehat{Z}_2} \otimes \mathcal{O}_{\widehat{Z}_2}\big(\varepsilon \cdot \Pi_{\widehat{Z}_2}^{(\eta_0)}\big)$$

implies that for all $\varepsilon \geq \varepsilon_1$

$$\begin{aligned}\mathcal{C}_2 \subset \hat{g}_{2*}\mathcal{M}_{\widehat{Z}_2} \otimes \mathcal{O}_{\widehat{Z}_2}\big(\varepsilon \cdot \Pi_{\widehat{Z}_2}^{(\eta_0)}\big) \subset \mathcal{C}' \subset \mathcal{C}_2 \otimes \mathcal{O}_{\widehat{Y}_2}(D)& \\ \subset \hat{g}_{2*}\mathcal{M}_{\widehat{Z}_2} \otimes \mathcal{O}_{\widehat{Z}_2}\big(\varepsilon_1 \cdot \Pi_{\widehat{Z}_2}^{(\eta_0)} + \hat{g}_2^* D\big).&\end{aligned}$$

Let us choose $\varepsilon_0 \geq \varepsilon_1$ so that for an irreducible Weil divisors Π the multiplicity in $(\varepsilon_0 - \varepsilon_1) \cdot \Pi_{\widehat{Z}_2}^{(\eta_0)}$ is either zero, or larger that its multiplicity in $g_2^* D$. Note that this choice of ε_0 is compatible with further pullback.

For $\varepsilon \geq \varepsilon_0$ the image of the evaluation map

$$\hat{g}_2^*\hat{g}_{2*}\mathcal{M}_{\widehat{Z}_2} \otimes \mathcal{O}_{\widehat{Z}_2}\big(\varepsilon \cdot \Pi_{\widehat{Z}_2}^{(\eta_0)}\big) \longrightarrow \mathcal{M}_{\widehat{Z}_2} \otimes \mathcal{O}_{\widehat{Z}_2}\big(* \cdot \Pi_{\widehat{Z}_2}^{(\eta_0)}\big)$$

is contained in the image of $\hat{g}_2^*\mathcal{C}' \to \mathcal{M}_{\widehat{Z}_2} \otimes \mathcal{O}_{\widehat{Z}_2}(* \cdot \Pi_{\widehat{Z}_2}^{(\eta_0)})$ hence in

$$\mathcal{M}_{\widehat{Z}_2} \otimes \mathcal{O}_{\widehat{Z}_2}\big(\varepsilon_1 \cdot \Pi_{\widehat{Z}_2}^{(\eta_0)} + \hat{g}_2^* D\big) \cap \mathcal{M}_{\widehat{Z}_2} \otimes \mathcal{O}_{\widehat{Z}_2}\big(* \cdot \Pi_{\widehat{Z}_2}^{(\eta_0)}\big) \subset \mathcal{M}_{\widehat{Z}_2} \otimes \mathcal{O}_{\widehat{Z}_2}\big(\varepsilon_0 \cdot \Pi_{\widehat{Z}_2}^{(\eta_0)}\big).$$

We found ε_0 after replacing $\widehat{Y}$ by some nonsingular modification $\widehat{Y}_2$; hence as remarked above the same ε_0 works for $\widehat{Y}$ itself. Moreover, the same ε_0 works for all alterations dominating $\widehat{Y}_2$. Since for any alteration $\widehat{Y}_1$ of $\widehat{Y}$ one can find a nonsingular modification, dominating $\widehat{Y}_2$, one obtains the same for $\widehat{Y}_1$. □

Definition 8.10. Assume that $\mathcal{L}$ is an invertible sheaf on X, and let κ be a positive integer. Assume that $f_{0*}\mathcal{L}_0^{\kappa}$ is locally free and compatible with arbitrary base change.

(1) An invertible sheaf $\mathcal{M}_{\widehat{Z}}$ on $\widehat{Z}$ is a *κ-saturated extension* of $\mathcal{L}$ if

$$\hat{\varphi}^*\mathcal{L} \subset \mathcal{M}_{\widehat{Z}} \subset \hat{\varphi}^*\mathcal{L} \otimes \Big(\mathcal{O}_{\widehat{Z}}\big(* \cdot \Pi_{\widehat{Z}}^{(\eta_0)}\big) \cap \mathcal{O}_{\widehat{Z}}\big(* \cdot \hat{g}^{-1}(\widehat{Y} \setminus \widehat{Y}_0)\big)\Big), \tag{8.10.1}$$

$$\text{and if} \quad \hat{g}_{1*}\mathcal{M}_{\widehat{Z}_1}^{\kappa} = \hat{g}_{1*}\Big(\mathcal{M}_{\widehat{Z}_1}^{\kappa} \otimes \mathcal{O}_{\widehat{Z}_1}\big(\varepsilon \cdot \Pi_{\widehat{Z}_1}^{(\eta_0)}\big)\Big)$$

for all $\varepsilon \geq 0$ and for all alterations $\widehat{Y}_1 \to \widehat{Y}$. Moreover we require 5.5(d) to hold for $(\nu, \mu) = (0, \kappa)$, i.e., that there exists an open dense subscheme $\widehat{Y}_g$ of $\widehat{Y}$ such that $\hat{g}_* \mathcal{M}^{\kappa}_{\widehat{Z}}$ is locally free and compatible with pullback for morphisms $\theta : T \to \widehat{Y}$ with $\theta^{-1}(\widehat{Y}_g)$ dense in T.

(2) We call a tuple of invertible sheaves $\mathcal{M}_{\widehat{Z}}$, $\mathcal{M}_{\widehat{X}}$ and $\mathcal{M}_Z$ on $\widehat{Z}$, $\widehat{X}$ and Z a *κ-saturated extension of the polarization $\mathcal{L}$*, if $\mathcal{M}_{\widehat{Z}}$ is κ saturated and if (as in 5.4) $\hat{\delta}_* \mathcal{M}_Z = \mathcal{M}_{\widehat{Z}}$, $\delta_* \mathcal{M}_Z = \mathcal{M}_{\widehat{X}}$, $\mathcal{M}_{\widehat{Z}_0} = \hat{\varphi}_0^* \mathcal{L}_0$ and $\mathcal{M}_{\widehat{X}_0} = \rho_0^* \mathcal{L}_0$.

LEMMA 8.11. *Assume that the conditions in* 8.5 *hold true.*

(a) *If $\mathcal{M}_{\widehat{Z}}$ is a κ-saturated extension of $\mathcal{L}$, one can always find $\mathcal{M}_{\widehat{X}}$ and $\mathcal{M}_Z$ such that $(\mathcal{M}_{\widehat{Z}}, \mathcal{M}_{\widehat{X}}, \mathcal{M}_Z)$ is κ-saturated.*

(b) *The condition* (8.10.1) *in* (1) *is equivalent to the existence of an effective Cartier divisor $\hat{\Pi}$, supported in $\hat{g}^{-1}(\widehat{Y} \setminus \widehat{Y}_0) \cap (\Pi^{(\eta_0)}_{\widehat{Z}})_{\mathrm{red}}$, and with*

$$\mathcal{M}_{\widehat{Z}} = \hat{\varphi}^* \mathcal{L} \otimes \mathcal{O}_{\widehat{Z}}(\hat{\Pi}).$$

(c) *If $(\mathcal{M}_{\widehat{Z}}, \mathcal{M}_{\widehat{X}}, \mathcal{M}_Z)$ is κ-saturated*

$$\hat{f}_{1*} \mathcal{M}^{\kappa}_{\widehat{X}_1} = \hat{f}_{1*}\Big(\mathcal{M}^{\kappa}_{\widehat{X}_1} \otimes \mathcal{O}_{\widehat{X}_1}\big(\varepsilon \cdot \Pi^{(\eta_0)}_{\widehat{X}_1}\big)\Big) = \hat{f}_*\Big(\rho^* \mathcal{L}^{\kappa} \otimes \mathcal{O}_{\widehat{X}_1}\big(* \cdot \Pi^{(\eta_0)}_{\widehat{X}}\big)\Big)$$

for all $\varepsilon \geq 0$ and for all alterations $\widehat{Y}_1 \to \widehat{Y}$.

(d) *Let $\tilde{g} : \widetilde{Z} \to \widehat{Y}$ be a second mild morphism and $\tau' : \widetilde{Z}_1 \to \widehat{Z}_1$ a birational morphism over $\widehat{Y}$. If $\mathcal{M}_{\widehat{Z}}$ is κ-saturated the same holds for $\mathcal{M}_{\widetilde{Z}} = \tau'^* \mathcal{M}_{\widehat{Z}}$.*

(e) *If $\mathcal{M}_{\widehat{Z}}$ (or $(\mathcal{M}_{\widehat{Z}}, \mathcal{M}_{\widehat{X}}, \mathcal{M}_Z)$) is κ-saturated, and if κ' divides κ then $\mathcal{M}_{\widehat{Z}}$ (or $(\mathcal{M}_{\widehat{Z}}, \mathcal{M}_{\widehat{X}}, \mathcal{M}_Z)$) is also κ'-saturated, provided that $\hat{g}_* \mathcal{M}^{\kappa'}_{\widehat{Z}}$ is locally free and compatible with base change for morphisms $\theta : T \to \widehat{Y}$ with $\theta^{-1}(\widehat{Y}_g)$ dense in T.*

Proof. (b) is just a translation and the first equality in (c) follows directly from 8.9. For the second one, apply 8.9 first to the pullback of $\mathcal{L}$ and then to $\mathcal{M}_{\bullet}$. One finds that $\hat{f}_{1*} \mathcal{M}^{\kappa}_{\widehat{X}_1}$ is given by

$$\begin{aligned} \hat{f}_{1*}\Big(\rho_1^* \mathcal{L}^{\kappa} \otimes \mathcal{O}_{\widehat{X}_1}\big(* \cdot \Pi^{(\eta_0)}_{\widehat{X}_1}\big)\Big) &= \hat{g}_{1*} \hat{\varphi}_1^* \mathcal{L}^{\kappa} \otimes \mathcal{O}_{\widehat{Z}_1}\big(* \cdot \Pi^{(\eta_0)}_{\widehat{Z}_1}\big) \\ &= \hat{g}_{1*} \mathcal{M}^{\kappa}_{\widehat{Z}_1} \otimes \mathcal{O}_{\widehat{Z}_1}\big(* \cdot \Pi^{(\eta_0)}_{\widehat{Z}_1}\big). \end{aligned}$$

For (a) consider $\Pi = \hat{\delta}^* \hat{\Pi}$ and the divisor $\delta_* \Pi$ on $\widehat{X}$. Define $\mathcal{M}_{\widehat{X}} = \rho^* \mathcal{L} \otimes \mathcal{O}_{\widehat{X}}(\delta_* \Pi)$.

Since δ is a modification of a manifold, $\Pi - \delta^* \delta_* \Pi$ is supported in exceptional divisors for δ, and

$$\delta^* \mathcal{M}_{\widehat{X}} \subset \delta^* \rho^* \mathcal{L} \otimes \mathcal{O}_Z(\Pi) = \hat{\delta}^* \varphi^* \mathcal{L} \otimes \mathcal{O}_Z(\hat{\delta}^* \hat{\Pi}) = \hat{\delta}^* \mathcal{M}_{\widehat{Z}},$$

and $\mathcal{M}_{\widehat{X}} = \delta_* \hat{\delta}^* \mathcal{M}_{\widehat{Z}}$. So we can choose $\mathcal{M}_Z = \hat{\delta}^* \mathcal{M}_{\widehat{Z}}$.

In (d) note that $\varpi^{(\eta_0)}_{\widehat{Z}}$ is invertible and its pullback is $\varpi^{(\eta_0)}_{\widetilde{Z}}$. So $\Pi^{(\eta_0)}_{\widetilde{Z}} - \tau'^{*}\Pi^{(\eta_0)}_{\widehat{Z}}$ is an effective divisor, supported in the exceptional locus of τ. By the projection formula, for all $\varepsilon \geq 0$,

$$\tau'_{*}\mathcal{M}^{\kappa}_{\widetilde{Z}} \otimes \mathcal{O}_{\widetilde{Z}}\big(\varepsilon \cdot \Pi^{(\eta_0)}_{\widetilde{Z}}\big) = \mathcal{M}^{\kappa}_{\widehat{Z}} \otimes \mathcal{O}_{\widehat{Z}}\big(\varepsilon \cdot \Pi^{(\eta_0)}_{\widehat{Z}}\big);$$

hence $\widetilde{g}_{*}(\mathcal{M}^{\kappa}_{\widetilde{Z}} \otimes \mathcal{O}_{\widetilde{Z}}(\varepsilon \cdot \Pi^{(\eta_0)}_{\widetilde{Z}})) = \hat{g}_{*}(\mathcal{M}^{\kappa}_{\widehat{Z}} \otimes \mathcal{O}_{\widehat{Z}}(\varepsilon \cdot \Pi^{(\eta_0)}_{\widehat{Z}}))$. Since the right-hand side is independent of ε the polarization $\mathcal{M}_{\widetilde{Z}}$ is again κ-saturated.

For (e) note first that condition (2) in Definition 8.10 is independent of κ, as well as (8.10.1) in (1). If for some $\widehat{Y}_1 \to \widehat{Y}$ and some $\varepsilon > 0$ the sheaf

$$\hat{g}_{1*}\mathcal{M}^{\kappa'}_{\widehat{Z}_1} \neq \hat{g}_{1*}\Big(\mathcal{M}^{\kappa'}_{\widehat{Z}_1} \otimes \mathcal{O}_{\widehat{Z}_1}\big(\varepsilon \cdot \Pi^{(\eta_0)}_{\widehat{Z}_1}\big)\Big),$$

then the multiplication map shows that the same holds for all multiples of κ', in particular for κ. $\square$

LEMMA 8.12. *Given a natural number κ one may choose $\widehat{Y}$ and $\widehat{Z}$ in* 5.4 *and the sheaf $\mathcal{M}_{\widehat{Z}}$ such that $\mathcal{M}_{\widehat{Z}}$ is a κ-saturated extension of $\mathcal{L}$.*

Proof. Start with any $\widehat{Y}$ as in 8.5 and with $\mathcal{M}_{\widehat{Z}}$ the pullback of the invertible sheaf $\mathcal{L}$ in 4.8. Apply 8.9 to the polarization $\mathcal{M}^{\kappa}_{\widehat{Z}}$, and replace ε_0 by some larger natural number, divisible by κ.

Define $\hat{\Pi}$ to be the sum over all components of $\Pi^{(\eta_0)}_{\widehat{Z}}$ whose image in $\widehat{Y}$ does not meet $\varphi_1^{-1}(Y_0)$, and choose

$$\widetilde{\mathcal{M}}_{\widehat{Z}} = \mathcal{M}_{\widehat{Z}} \otimes \mathcal{O}_{\widehat{Z}}\Big(\frac{\varepsilon_0}{\kappa} \cdot \hat{\Pi}\Big).$$

Note that $\hat{\Pi}$ might be just a Weil divisor; hence $\widetilde{\mathcal{M}}_{\widehat{Z}}$ is reflexive, but not necessarily invertible. So choose a modification $\sigma : W \to \widehat{Z}$, such that $\mathcal{M}_W = \sigma^{*}\widetilde{\mathcal{M}}_{\widehat{Z}}/_{\text{torsion}}$ is invertible. By Proposition 4.5 there exists a nonsingular alteration $\theta : \widehat{Y}_1 \to \widehat{Y}$ such that $W \otimes_{\widehat{Y}} \widehat{Y}_1$ has a mild model $W' \to \widehat{Y}_1$. Again we may assume that the conditions in 8.5 hold for $W' \to \widehat{Y}_1$. One has a factorization $W' \to W \to \widehat{Z}$ of σ, inducing a birational morphism $\sigma' : W' \to \widehat{Z}_1 = \widehat{Z} \times_{\widehat{Y}} \widehat{Y}_1$. By 8.9(2) we know that the evaluation map

$$\hat{g}_1^{*}\hat{g}_{1*}\mathcal{M}^{\kappa}_{\widehat{Z}_1}\big(*\cdot\Pi^{(\eta_0)}_{\widehat{Z}_1}\big) \longrightarrow \mathcal{M}^{\kappa}_{\widehat{Z}_1}\big(*\cdot\Pi^{(\eta_0)}_{\widehat{Z}_1}\big)$$

has image $\mathcal{C}$ in $\mathcal{M}^{\kappa}_{\widehat{Z}_1}(\varepsilon_0 \cdot \Pi^{(\eta_0)}_{\widehat{Z}_1})$. On the other hand, on $\hat{g}^{-1}(\theta^{-1}(\widehat{Y}_0))$ the sheaf $\mathcal{C}$ is equal to $\mathcal{M}^{\kappa}_{\widehat{Z}_1}$ and $\mathcal{C}$ lies in the reflexive hull $\widetilde{\mathcal{M}}^{(\kappa)}_{\widehat{Z}_1}$ of $\mathrm{pr}_1^{*}\widetilde{\mathcal{M}}^{\kappa}_{\widehat{Z}}$. By construction $\mathcal{M}_{W'} = \sigma^{*}\widetilde{\mathcal{M}}_{\widehat{Z}_1}/_{\text{torsion}}$ is invertible and $\sigma^{*}\widetilde{\mathcal{M}}^{(\kappa)}_{\widehat{Z}_1}/_{\text{torsion}} = \mathcal{M}^{\kappa}_{W'}$.

Writing again $\Pi^{(\eta_0)}_{W'}$ for the relatively fixed locus of $\omega^{\eta_0}_{W'/\widehat{Y}_1}$ one has

$$\varpi^{(\eta_0)}_{W'} = \omega^{\eta_0}_{W'/\widehat{Y}_1} \otimes \mathcal{O}_{W'}\big(-\Pi^{(\eta_0)}_{W'}\big) = \sigma'^{*}\varpi^{(\eta_0)}_{\widehat{Z}_1}.$$

For all $\varepsilon \geq 0$ one obtains

$$\sigma_*\Big(\mathcal{M}^{\kappa}_{W'} \otimes \mathcal{O}_{W'}\big(\varepsilon \cdot \Pi^{(\eta_0)}_{W'}\big)\Big) = \widetilde{\mathcal{M}}^{(\kappa)}_{\hat{Z}_1} \otimes \mathcal{O}_{\hat{Z}_1}\big(\varepsilon \cdot \Pi^{(\eta_0)}_{\hat{Z}_1}\big),$$

and

$$\begin{aligned}\hat{g}_{1*}\sigma_*\mathcal{M}^{\kappa}_{W'} = \hat{g}_{1*}\widetilde{\mathcal{M}}^{(\kappa)}_{\hat{Z}_1} &= \hat{g}_{1*}\Big(\widetilde{\mathcal{M}}^{(\kappa)}_{\hat{Z}_1} \otimes \mathcal{O}_{\hat{Z}_1}\big(\varepsilon \cdot \Pi^{(\eta_0)}_{\hat{Z}_1}\big)\Big) \\ &= \hat{g}_{1*}\sigma_*\Big(\mathcal{M}^{\kappa}_{W'} \otimes \mathcal{O}_{W'}\big(\varepsilon \cdot \Pi^{(\eta_0)}_{W'}\big)\Big).\end{aligned} \tag{8.12.1}$$

So on W' we found the sheaf we are looking for. Finally, Corollary 5.5 allows us to replace $\hat{Y}_1$ by some modification, and to assume that condition (d) in 8.7 holds for $(0, \kappa)$. $\square$

By 8.11(a) one can construct $\widetilde{\mathcal{M}}_{\hat{Z}}$, $\widetilde{\mathcal{M}}_{Z}$ and $\widetilde{\mathcal{M}}_{\hat{X}}$ such that this tuple forms a κ-saturated extension of $\mathcal{L}_0$. Perhaps some of the sheaves $\mathcal{B}^{(\nu,\mu)}_{\bullet}$ or the sheaves $\mathcal{B}_{\bullet}$, depending on $\mathcal{E}_{\hat{Y}}$ in 8.2 are no longer invertible. If so, for $\widetilde{\mathcal{M}}_{\bullet}$ and for the given set I we have to perform again the alterations needed to get the invertible sheaves in 8.4. Lemma 8.11(d) allows us to do so, without losing the κ-saturatedness. So one is allowed to modify condition (vi) in 8.7, keeping all the other ones:

Conclusion and Notation 8.13 (saturated polarizations). We consider an invertible sheaf $\mathcal{L}$ on X, with $\mathcal{L}_0 = \mathcal{L}|_{X_0}$ relatively ample over Y_0, and we start again with a finite set I of tuples (ν, μ) of natural numbers. We choose $\eta_0 > 0$ and $\gamma_0 > 0$ such that the evaluation maps

$$f_0^* f_{0*}\omega^{\eta_0}_{X_0/Y_0} \to \omega^{\eta_0}_{X_0/Y_0} \quad \text{and} \quad f_0^* f_{0*}\mathcal{L}_0^{\gamma_0} \to \mathcal{L}_0^{\gamma_0}$$

are surjective.

We fix some subset $\tilde{I}$ of I consisting of tuples (β, α) with α divisible by γ_0 and with β divisible by η_0. We also fix a positive number κ with $(0, \kappa) \in \tilde{I}$.

Then we can find $\hat{Y}$ and the diagram (4.6.1) (or in (8.0.1)) such that conditions (i)–(v) in 8.5 hold true and conditions (vii)–(x) in 8.7 with $\mathcal{M}_{\bullet}$ given by:

(vi) There exists a tuple of κ-saturated extensions $(\mathcal{M}_{\hat{Z}}, \mathcal{M}_{Z}, \mathcal{M}_{\hat{X}})$ of $\mathcal{L}$.

Note that by Lemma 8.11(d) the "Allowed Constructions" in 8.8 remain allowed, i.e. they respect condition (vi) in 8.13.

COROLLARY 8.14. *The conditions in* 8.13 *imply that for all* $\varepsilon \geq 0$ *the direct images*

$$\hat{g}_*\mathcal{B}^{(0,\kappa)}_{\hat{Z}}, \quad \hat{g}_*\mathcal{M}^{\kappa}_{\hat{Z}} \quad \textit{and} \quad \hat{g}_*\Big(\mathcal{M}^{\kappa}_{\hat{Z}} \otimes \mathcal{O}_{\hat{Z}}\big(\varepsilon \cdot \Pi^{(\eta_0)}_{\hat{Z}}\big)\Big)$$

coincide, and that they are locally free and compatible with base change for morphisms $\varrho : T \to \hat{Y}$ *with* $\varrho^{-1}(\hat{Y}_g)$ *dense in* T.

Proof. By definition of "saturated" and by the choice of $\mathcal{B}^{(0,\kappa)}_{\hat{Z}}$,

$$\hat{g}_*\mathcal{B}^{(0,\kappa)}_{\hat{Z}} = \hat{g}_*\mathcal{M}^{\kappa}_{\hat{Z}} = \hat{g}_*\Big(\mathcal{M}^{\kappa}_{\hat{Z}} \otimes \mathcal{O}_{\hat{Z}}\big(\varepsilon \cdot \Pi^{(\eta_0)}_{\hat{Z}}\big)\Big).$$

Since we assumed that $(0,\kappa)\in I$ the direct image $\hat{g}_*\mathcal{M}^{\kappa}_{\hat{Z}}$ is compatible with base change for alterations. By Addendum 8.3 the same holds true for $\hat{g}_*\mathcal{B}^{(0,\kappa)}_{\hat{Z}}$ and by 8.9 for $\hat{g}_*(\mathcal{M}^{\kappa}_{\hat{Z}}\otimes\mathcal{O}_{\hat{Z}}(\varepsilon\cdot\Pi^{(\eta_0)}_{\hat{Z}}))$. So 8.14 follows from Lemma 5.1(ii). $\square$

Thus, for $\kappa=1$ we could choose $\mathcal{M}_{\hat{Z}}$ to be equal to $\mathcal{B}^{(0,1)}_{\hat{Z}}$, but we will allow other choices. Anyway, it is easy to see that the direct image sheaves are independent of the choices.

9. **The definition of certain multiplier ideals**

The alterations, sheaves and divisors as described in the Conclusions and Notation 8.6, 8.7 or 8.13 depend on the choice of certain numbers and data. Each time we add some numbers, we have to reemploy the constructions of Section 8. In order not to run into an infinite circle of constructions we have to give a complete list of data at some point, and this is done in the first part of this section.

However, we still have to extend the base change property stated in 8.7(ix) to certain multiplier ideals

$$\hat{g}_*\big(\omega^{\nu}_{\hat{Z}/\hat{Y}}\otimes\mathcal{M}^{\mu}_{\hat{Z}}\otimes\mathcal{J}(-e\cdot D)\big),$$

using Theorem 6.5. As in the proof of Variant 2.4 the multiplier ideals we want to consider depend on the tautological map Ξ and on a large number of integers. So in this section we will include those maps in our bookkeeping. In order to get the local freeness and the compatibility with base change for certain morphisms, we will use again the left-hand side of the diagram (4.6.1). Then the compatibility conditions for the sheaves $\mathcal{M}_\bullet$ will allow us, as in Lemma 5.7, to pass to the right-hand side.

Conventions 9.1. Consider for a smooth fiber F a finite tuple Ξ of determinants and their natural inclusion in the tensor products, i.e., $\Xi=(\Xi_1,\ldots,\Xi_s)$ and

$$\Xi_i:\bigwedge^{r_i}H^0\big(F,\omega_F^{\eta_i}\otimes\mathcal{L}_0^{\gamma_i}|_F\big)\longrightarrow\bigotimes^{r_i}H^0\big(F,\omega_F^{\eta_i}\otimes\mathcal{L}_0^{\gamma_i}|_F\big),$$

where $r_i=\dim(H^0(F,\omega_F^{\eta_i}\otimes\mathcal{L}_0^{\gamma_i}|_F))$. Then for any r, divisible by $r_1,\ldots,r_s$ and for each i one obtains a map

$$\Big(\bigwedge^{r_i}H^0\big(F,\omega_F^{\eta_i}\otimes\mathcal{L}_0^{\gamma_i}|_F\big)\Big)^{\otimes\frac{r}{r_i}}\longrightarrow\bigotimes^{r}H^0\big(F,\omega_F^{\eta_i}\otimes\mathcal{L}_0^{\gamma_i}|_F\big)$$

and finally, for $\gamma=\gamma_1+\cdots+\gamma_s$ and for $\eta=\eta_1+\cdots+\eta_s$ one has the product

$$\bigotimes_{i=1}^{s}\Big(\bigwedge^{r_i}H^0\big(F,\omega_F^{\eta_i}\otimes\mathcal{L}_0^{\gamma_i}|_F\big)\Big)^{\otimes\frac{r}{r_i}}\xrightarrow{\Xi^{(r)}}\bigotimes^{r}H^0\big(F,\omega_F^{\eta}\otimes\mathcal{L}_0^{\gamma}|_F\big).$$

We will later require certain divisibilities. For example, we will need that the integers η_0 and γ_0 in 8.5 or 8.7 divide η and γ. This can be achieved by replacing Ξ by s' copies $(\Xi, \ldots, \Xi)$ for a suitable s'.

Note that those conventions carry over to the smooth part of our families, provided the direct image sheaves $f_{0*}(\omega_{X_0/Y_0}^{\eta_i} \otimes \mathcal{L}_0^{\gamma_i})$ are all locally free and compatible with base change. This holds by 5.3 for $\eta_i > 0$. For $\eta_i = 0$ we listed this in 8.7 as an additional condition.

Next we have to explain how to choose the finite sets $\widetilde{I}$ and I in 8.5 or 8.7.

Set-up 9.2. *The canonically polarized case.* Here we start by choosing an integer $\eta_0 > 0$ such that the evaluation map $f_0^* f_{0*}\omega_{X_0/Y_0}^{\eta_0} \to \omega_{X_0/Y_0}^{\eta_0}$ is surjective and we choose $\ell > 0$, divisible by η_0. We will assume in 9.1 that $\gamma_i = 0$ for all i, and that η is divisible by ℓ, hence by η_0. We choose $\widetilde{I} = \{(\eta_0, 0), (\eta, 0)\}$. The set $I \subset \mathbb{N} \times \{0\}$ should contain $\widetilde{I}$, the tuples $(\eta_i, 0)$ for $i = 1, \ldots, s$ and for some $\beta \geq 1$ the tuple $(\beta + \frac{\eta}{\ell}, 0)$. For compatibility of the notation we write $\alpha = \kappa = 0$ and b will denote any positive integer. We choose $\widehat{Y}$ and the different sheaves and divisors according to 8.5.

The polarized case. If $\mathcal{L}_0$ is f_0-ample we start with integers $\eta_0 > 0$ and $\gamma_0 > 0$ such that the evaluation maps

$$f_0^* f_{0*}\omega_{X_0/Y_0}^{\eta_0} \to \omega_{X_0/Y_0}^{\eta_0} \quad \text{and} \quad f_0^* f_{0*}\mathcal{L}_0^{\gamma_0} \to \mathcal{L}_0^{\gamma_0}$$

are both surjective. In addition we will require that for $N \geq \gamma_0$ and for all fibers F of f_0 the sheaves $\mathcal{L}_0^N|_F$ have no higher cohomology. We choose $\ell > 0$, divisible by η_0 and γ_0. In 9.1, replacing Ξ by $(\Xi, \ldots, \Xi)$, and correspondingly s by some multiple, one may assume that ℓ divides γ and η. Fix in addition some tuple (β, α) of natural numbers with $\beta \geq 1$ (or a finite set of such tuples), and some positive integer b, with $b \cdot (\beta - 1, \alpha) \in \eta_0 \cdot \mathbb{N} \times \gamma_0 \cdot \mathbb{N}$. The finite set of tuples $\widetilde{I}$ should contain $\{(\eta_0, 0),\ (0, \gamma_0),\ (\eta, \gamma)\}$, and I should contain $\widetilde{I}$,

$$\left(\beta + \frac{\eta}{\ell}, \alpha + \frac{\gamma}{\ell}\right) \quad \text{and} \quad (\eta_i, \gamma_i) \quad \text{for} \quad i = 1, \ldots, s.$$

For compatibility reasons we choose $\kappa = 0$ in this case, and we choose $\widehat{Y}$ and the different sheaves and divisors according to 8.7.

The saturated polarized case. Everything is as in the polarized case, except that we also choose some positive multiple κ of γ_0 and assume that $(0, \kappa) \in I'$, and apply 8.13 instead of 8.7. In all three cases we fix a natural number e with

$$e \geq \frac{e(\omega_F^{\eta} \otimes \mathcal{L}_0^{\gamma}|_F)}{\ell} \tag{9.2.1}$$

for all fibers F of f_0, where e denotes the threshold introduced in 2.1. The sheaves $\mathcal{F}_{\widehat{Y}}^{(\eta_i, \gamma_i)}$ are locally free. Replacing $\widehat{Y}$ by a nonsingular alteration one finds an invertible sheaf $\mathcal{V}$ on $\widehat{Y}$ with

$$\bigotimes_{i=1}^{s} \det\big(\mathcal{F}_{\widehat{Y}}^{(\eta_i,\gamma_i)}\big)^{\frac{r}{r_i}} = \bigotimes_{i=1}^{s} \det\Big(\hat{g}_*\big(\omega_{\widehat{Z}/\widehat{Y}}^{\eta_i} \otimes \mathcal{M}_{\widehat{Z}}^{\gamma_i}\big)\Big)^{\frac{r}{r_i}} = \mathcal{V}^{r\cdot e\cdot \ell}.$$

Note that all assumptions remains true when we replace r by some multiple or $\widehat{Y}$ by an alteration.

We are not yet done. We will need another auxiliary sheaf.

Assumptions and Notation 9.3. Consider for a locally free sheaf $\mathcal{E}_{\widehat{Y}}$ a morphism $\mathcal{E}_{\widehat{Y}} \to \mathcal{F}_{\widehat{Y}}^{(\beta_0,\alpha_0)}$, where

$$\beta_0 = b\cdot(\beta-1)\cdot e\cdot\ell + \eta\cdot b\cdot(e-1) \quad\text{and}\quad \alpha_0 = b\cdot\alpha\cdot e\cdot\ell + \gamma\cdot b\cdot(e-1).$$

For b sufficiently large, the evaluation map

$$\hat{f}^*\mathcal{E}_{\widehat{Y}} \longrightarrow \omega_{\widehat{X}/\widehat{Y}}^{\beta_0} \otimes \mathcal{M}_{\widehat{X}}^{\alpha_0} \tag{9.3.1}$$

is surjective over $\widehat{X}_0$. We will choose

$$(\beta_0,\alpha_0) = (b\cdot(\beta-1)\cdot e\cdot\ell + \eta\cdot b\cdot(e-1), b\cdot\alpha\cdot e\cdot\ell + \gamma\cdot b\cdot(e-1)) \in \widetilde{I}, \tag{9.3.2}$$

where of course $\alpha_0 = 0$ in the canonically polarized case.

Lemma 8.2 allows us to assume (replacing $\widehat{Y}$ by an alteration) that the image of the evaluation map (9.3.1) is an invertible sheaf $\mathcal{B}_{\widehat{X}}$, and that the image of $\hat{g}^*\mathcal{E}_{\widehat{Y}} \longrightarrow \omega_{\widehat{Z}/\widehat{Y}}^{\beta_0} \otimes \mathcal{M}_{\widehat{Z}}^{\alpha_0}$ is an invertible sheaf $\mathcal{B}_{\widehat{Z}}$. We again write $\Sigma_{\widehat{Z}}$ for the effective divisor with $\mathcal{B}_{\widehat{Z}} = \omega_{\widehat{Z}/\widehat{Y}}^{\beta_0} \otimes \mathcal{M}_{\widehat{Z}}^{\alpha_0} \otimes \mathcal{O}_{\widehat{Z}}(-\Sigma_{\widehat{Z}})$.

Variant 9.4. In the application we have in mind $\mathcal{E}_{\widehat{Y}}$ will be a subsheaf of

$$\mathcal{F}_{\widehat{Y}}^{(\beta_1,\alpha_1)} \otimes \cdots \otimes \mathcal{F}_{\widehat{Y}}^{(\beta_s,\alpha_s)},$$

with cokernel supported in $\widehat{Y}\setminus\widehat{Y}_0$. Here we have to assume that for all $\iota \in \{1,\ldots,s\}$ the evaluation map for $\omega_{\widehat{X}/\widehat{Y}}^{\beta_\iota} \otimes \mathcal{M}_{\widehat{X}}^{\alpha_\iota}$ is surjective over $\widehat{X}_0$.

The morphism $\mathcal{E}_{\widehat{Y}} \to \mathcal{F}_{\widehat{Y}}^{(\beta_0,\alpha_0)}$ will be induced by the multiplication map

$$\mathcal{F}_{\widehat{Y}}^{(\beta_1,\alpha_1)} \otimes \cdots \otimes \mathcal{F}_{\widehat{Y}}^{(\beta_s,\alpha_s)} \xrightarrow{\mathrm{m}} \mathcal{F}_{\widehat{Y}}^{(\beta_0,\alpha_0)}.$$

Of course one needs that $\beta_1+\cdots+\beta_s = \beta_0$ and $\alpha_1+\cdots+\alpha_s = \alpha_0$. In this case one can replace the condition (9.3.2) by

$$(\beta_1,\alpha_1),\ldots,(\beta_s,\alpha_s) \in \widetilde{I}. \tag{9.4.1}$$

Finally remark that here $\mathcal{B}_{\widehat{Z}}$ is contained in the tensor product of the sheaves $\mathcal{B}_{\widehat{Z}}^{(\beta_\iota,\alpha_\iota)}$ and this inclusion is an isomorphism on $\widehat{Z}_0$.

We need a long list of different sheaves and divisors on certain products.

Notation 9.5. (*saturated*) *polarized case.* Let $\hat{g} : \widehat{Z} \to \widehat{Y}$ be the mild morphism constructed in 8.5, 8.7 and 8.13 using the data given in 9.1–9.4 (or by abuse of notation, its pullback under a morphism from a curve to $\widehat{Y}$, if it is mild). Consider the r-fold product

$$\hat{g}^r : \widehat{Z}^r = \widehat{Z} \times_{\widehat{Y}} \cdots \times_{\widehat{Y}} \widehat{Z} \to \widehat{Y}, \quad \text{and} \quad \mathcal{M}_{\widehat{Z}^r} = \mathrm{pr}_1^* \mathcal{M}_{\widehat{Z}} \otimes \cdots \otimes \mathrm{pr}_r^* \mathcal{M}_{\widehat{Z}}.$$

For $(\nu, \mu) \in I$, one obtains by flat base change

$$\hat{g}^r_* \big(\omega^\nu_{\widehat{Z}^r/\widehat{Y}} \otimes \mathcal{M}^\mu_{\widehat{Z}^r} \big) = \bigotimes^r \hat{g}_* \big(\omega^\nu_{\widehat{Z}/\widehat{Y}} \otimes \mathcal{M}^\mu_{\widehat{Z}} \big) = \bigotimes^r \mathcal{F}^{(\nu,\mu)}_{\widehat{Y}}. \tag{9.5.1}$$

For $(\nu, \mu) = (\eta, \gamma)$ the equality (9.5.1) implies that the image of the evaluation map

$$\hat{g}^{r*} \hat{g}^r_* \omega^\eta_{\widehat{Z}^r/\widehat{Y}} \otimes \mathcal{M}^\gamma_{\widehat{Z}^r} \longrightarrow \omega^\eta_{\widehat{Z}^r/\widehat{Y}} \otimes \mathcal{M}^\gamma_{\widehat{Z}^r}$$

is the invertible sheaf $\mathcal{B}^{(\eta,\gamma)}_{\widehat{Z}^r} := \mathrm{pr}_1^* \mathcal{B}^{(\eta,\gamma)}_{\widehat{Z}} \otimes \cdots \otimes \mathrm{pr}_r^* \mathcal{B}^{(\eta,\gamma)}_{\widehat{Z}}$. So the definition of $\mathcal{B}^{(\eta,\gamma)}_{\widehat{Z}^r}$ is compatible with the one in 8.7, and $\mathcal{B}^{(\eta,\gamma)}_{\widehat{Z}^r}$ can be written as

$$\omega^\eta_{\widehat{Z}^r/\widehat{Y}} \otimes \mathcal{M}^\gamma_{\widehat{Z}^r} \otimes \mathcal{O}_{\widehat{Z}^r}\big(-\Sigma^{(\eta,\gamma)}_{\widehat{Z}^r} \big) \quad \text{for} \quad \Sigma^{(\eta,\gamma)}_{\widehat{Z}^r} = \sum_{i=1}^r \mathrm{pr}_i^* \Sigma^{(\eta,\gamma)}_{\widehat{Z}}.$$

Since $\hat{g}^r_* (\omega^\eta_{\widehat{Z}^r/\widehat{Y}} \otimes \mathcal{M}^\gamma_{\widehat{Z}^r}) = \hat{g}^r_* \mathcal{B}^{(\eta,\gamma)}_{\widehat{Z}^r}$, one has an inclusion

$$\mathcal{V}^{r \cdot e \cdot \ell} = \bigotimes_{i=1}^s \det \big(\hat{g}_* \mathcal{M}^{\gamma_i}_{\widehat{Z}} \otimes \varpi^{\eta_i}_{\widehat{Z}} \big)^{\otimes \frac{r}{r_i}} \xrightarrow{\Xi^{(r)}} \bigotimes^r \hat{g}_* \big(\omega^\eta_{\widehat{Z}/\widehat{Y}} \otimes \mathcal{M}^\gamma_{\widehat{Z}} \big) = \hat{g}^r_* \mathcal{B}^{(\eta,\gamma)}_{\widehat{Z}^r}$$

which splits locally, hence a section of $\mathcal{B}^{(\eta,\gamma)}_{\widehat{Z}^r} \otimes \hat{g}^{r*} \mathcal{V}^{-r \cdot e \cdot \ell}$ whose zero divisor $\Gamma_{\widehat{Z}^r}$ does not contain any fiber (but perhaps components of fibers).

In 9.3 one can apply (9.5.1) to see that the invertible sheaf

$$\mathcal{B}_{\widehat{Z}^r} = \mathrm{pr}_1^* \mathcal{B}_{\widehat{Z}} \otimes \cdots \otimes \mathrm{pr}_r^* \mathcal{B}_{\widehat{Z}}$$

is again the image of the evaluation map $\hat{g}^{r*} \mathcal{E}^{\otimes r}_{\widehat{Y}} \to \omega^{\beta_0}_{\widehat{Z}^r/\widehat{Y}} \otimes \mathcal{M}^{\alpha_0}_{\widehat{Z}^r}$.

In Variant 9.4 the same holds true for the sheaves $\mathcal{B}^{(\beta_\iota,\alpha_\iota)}_{\widehat{Z}^r}$, hence for their tensor product and for the image $\mathcal{B}_{\widehat{Z}^r}$ of $\hat{g}^{r*} \mathcal{E}^{\otimes r}_{\widehat{Y}}$. In both cases one finds

$$\mathcal{B}_{\widehat{Z}^r} = \omega^{\beta_0}_{\widehat{Z}^r/\widehat{Y}} \otimes \mathcal{M}^{\alpha_0}_{\widehat{Z}^r} \otimes \mathcal{O}_{\widehat{Z}^r}(-\Sigma_{\widehat{Z}^r}), \quad \text{for} \quad \Sigma_{\widehat{Z}^r} = \sum_{i=1}^r \mathrm{pr}_i^* \Sigma_{\widehat{Z}}.$$

To shorten the expressions, we put

$$\Delta_{\widehat{Z}^r} = b \cdot \big(\Gamma_{\widehat{Z}^r} + \Sigma^{(\eta,\gamma)}_{\widehat{Z}^r} \big) + \Sigma_{\widehat{Z}^r}, \quad N = b \cdot e \cdot \ell$$

$$\text{and} \quad \mathcal{G}^{(\Xi^{(r)}, \mathcal{E}; \beta + \frac{\eta}{\ell}, \alpha + \frac{\gamma}{\ell})}_{\widehat{Y}} = \hat{g}^r_* \bigg(\omega^{\beta + \frac{\eta}{\ell}}_{\widehat{Z}^r/\widehat{Y}} \otimes \mathcal{M}^{\alpha + \frac{\gamma}{\ell}}_{\widehat{Z}^r} \otimes \mathcal{J}\Big(-\frac{1}{N} \cdot \Delta_{\widehat{Z}^r} \Big) \bigg).$$

We will usually write $\mathcal{G}_{\widehat{Y}}^{(\beta+\frac{\eta}{\ell},\alpha+\frac{\gamma}{\ell})}$ instead of $\mathcal{G}_{\widehat{Y}}^{(\Xi^{(r)},\mathcal{E};\beta+\frac{\eta}{\ell},\alpha+\frac{\gamma}{\ell})}$, keeping however in mind that this sheaf depends on the choice of r, of the tautological maps Ξ_i and on $\mathcal{E}$.

Canonically polarized case. We will use the same notation. This is a bit queer, but it allows us to handle both cases in the same way. So in this case $\alpha = \gamma = 0$ and $\mathcal{M}_\bullet = \mathcal{O}_\bullet$.

LEMMA 9.6. *Under the assumptions made in* 9.1–9.4 *one may choose* $\widehat{Y}$ *and* $\widehat{Z}$ *in* 8.5, 8.7 *or* 8.13 *and an open dense subscheme* $\widehat{Y}_g \subset \widehat{Y}_0$ *such that in addition to conditions* (i)–(v) *in* 8.5 *or to* (i)–(x) *in* 8.7 *or* 8.13 *on has*:

(xi) *The multiplier ideal sheaves* $\mathcal{J}\big(-\frac{1}{b\cdot e\cdot\ell}\cdot\Delta_{\widehat{Z}^r}\big)$ *are compatible with pullback, base change and products with respect to* $\widehat{Y}_g$, *as defined in* 6.4. *In particular they are flat over* $\widehat{Y}$ *and the direct image sheaves* $\mathcal{G}_{\widehat{Y}}^{(\beta+\frac{\eta}{\ell},\alpha+\frac{\gamma}{\ell})}$ *are compatible with pullback for morphisms* $\varrho : T \to \widehat{Y}$ *where* ϱ *is either dominant and* T *a normal variety with at most rational Gorenstein singularities*, *or where* T *is a nonsingular curve and* $\varrho^{-1}(\widehat{Y}_g)$ *dense in* T. *Moreover for* $r' > 0$

$$\mathcal{G}_{\widehat{Y}}^{(\Xi^{(r)},\mathcal{E};\beta+\frac{\eta}{\ell},\alpha+\frac{\gamma}{\ell})^{\otimes r'}} = \mathcal{G}_{\widehat{Y}}^{(\Xi^{(r\cdot r')},\mathcal{E};\beta+\frac{\eta}{\ell},\alpha+\frac{\gamma}{\ell})}.$$

Proof. Choose $\mathcal{N} = \omega_{\widehat{Z}^r/\widehat{Y}}^{\beta-1+\frac{\eta}{\ell}} \otimes \mathcal{M}_{\widehat{Z}^r}^{\alpha+\frac{\gamma}{\ell}}$. Then $\mathcal{N}^N \otimes \mathcal{O}_{\widehat{Z}^r}(-\Delta_{\widehat{Z}^r})$ is equal to

$$\Big[\omega_{\widehat{Z}^r/\widehat{Y}}^{\beta_0} \otimes \mathcal{M}_{\widehat{Z}^r}^{\alpha_0} \otimes \mathcal{O}_{\widehat{Z}^r}\big(-\Sigma_{\widehat{Z}^r}\big)\Big] \otimes \Big[\omega_{\widehat{Z}^r/\widehat{Y}}^{\eta\cdot b} \otimes \mathcal{M}_{\widehat{Z}^r}^{\gamma\cdot b} \otimes \mathcal{O}_{\widehat{Z}^r}\Big(-b\Big(\Sigma_{\widehat{Z}^r}^{(\eta,\gamma)} + \Gamma_{\widehat{Z}^r}\Big)\Big)\Big],$$

where the first factor is the image of $\hat{g}^{r*}\mathcal{E}_{\widehat{Y}}^{\otimes r}$ whereas the second one is the b-th power of $\mathcal{B}_{\widehat{Z}^r}^{(\eta,\gamma)} \otimes \mathcal{O}_{\widehat{Z}^r}(-\Gamma_{\widehat{Z}^r}) = \hat{g}^{r*}\mathcal{V}^{r\cdot e\cdot\ell}$. So we obtain:

CLAIM 9.7. *For* $\mathcal{N}$, *for* $\Delta = \Delta_{\widehat{Z}^r}$, *and for* $\mathcal{E} = \mathcal{E}_{\widehat{Y}}^{\otimes r} \otimes \mathcal{V}^{b\cdot r\cdot e\cdot\ell}$ *the assumptions made in* 6.1 *hold true* (*for* Z *replaced by* $\widehat{Z}^r$).

Thus, we are allowed to apply Theorem 6.5. Dropping the index $_1$, assume that $\widehat{Y} = \widehat{Y}_1$, hence that $\mathcal{J}\big(-\frac{1}{N}\cdot\Delta_{\widehat{Z}^r}\big)$ is compatible with pullback, base change and products with respect to $\widehat{Y}_g$.

For $\mathcal{A} = \mathcal{O}_{\widehat{Z}^r}$ in Definition 6.4 the properties (i) and (ii) give the compatibility with pullback under ϱ, and by flat base change also the compatibility with products. □

Before proving an analog of Lemma 5.7 for the sheaves $\mathcal{G}_{\widehat{Y}}^{(\beta+\frac{\eta}{\ell},\alpha+\frac{\gamma}{\ell})}$ we have to extend the definition of the sheaves and divisors to desingularizations of compactifications of $\widehat{X}_0^r \to \widehat{Y}_0$ (or again of the pullback of this morphism to a curve, meeting $\widehat{Y}_0$).

Notation 9.8. Consider the r-fold product $\hat{f}^r : \widehat{X}^r = \widehat{X} \times_{\widehat{Y}} \cdots \times_{\widehat{Y}} \widehat{X} \to \widehat{Y}$. The morphism $\rho' : X^{(r)} \to \widehat{X}^r$ is obtained by desingularizing the main component of $\widehat{X}^r$. By 4.2 the morphism $\hat{g}^r : \widehat{Z}^r \to \widehat{Y}$ in 9.5 and 9.6 is again mild, hence it is a mild model of the induced morphism $f^{(r)} : X^{(r)} \to \widehat{Y}$. Let us write

$$\mathcal{M}_{X^{(r)}} = \rho'^*\big(\mathrm{pr}_1^*\mathcal{M}_{\widehat{X}} \otimes \cdots \otimes \mathrm{pr}_r^*\mathcal{M}_{\widehat{X}}\big).$$

Recall that for ν divisible by η_0 and for μ divisible by γ_0 the evaluation map

$$\hat{f}_0^{r*}\hat{f}^r_{0*}\big(\omega^\nu_{\widehat{X}_0^r/\widehat{Y}_0} \otimes \mathcal{M}^\mu_{\widehat{X}_0^r}\big) \longrightarrow \omega^\nu_{\widehat{X}_0^r/\widehat{Y}_0} \otimes \mathcal{M}^\mu_{\widehat{X}_0^r}$$

is surjective, where again the index $_0$ refers to the preimages of Y_0 or for sheaves to their restriction. Consider a nonsingular modification $\hat{\delta}^r : Z^{(r)} \to \widehat{Z}^r$ which allows a morphism $\delta^{(r)} : Z^{(r)} \to X^{(r)}$, and which dominates the main component of $Z \times_{\widehat{Y}} \cdots \times_{\widehat{Y}} Z$. Defining $\mathcal{M}_{Z^{(r)}}$ as the pullback of $\mathrm{pr}_1^*\mathcal{M}_Z \otimes \cdots \otimes \mathrm{pr}_r^*\mathcal{M}_Z$, one has $\hat{\delta}^{r*}\mathcal{M}_{\widehat{Z}^r} \subset \mathcal{M}_{Z^{(r)}}$ and $\delta^{(r)*}\mathcal{M}_{X^{(r)}} \subset \mathcal{M}_{Z^{(r)}}$.

LEMMA 9.9. *The sheaves* $\mathcal{M}_{Z^{(r)}}$, $\mathcal{M}_{\widehat{Z}^r}$ *and* $\mathcal{M}_{X^{(r)}}$ *satisfy again the assumptions asked for in* 5.4.

Proof. Since $\widehat{Z}^r$ is normal the assumption $\hat{\delta}_*\mathcal{M}_Z = \mathcal{M}_{\widehat{Z}}$ in 5.4 implies that

$$\hat{\delta}^r_*\mathcal{M}_{Z^{(r)}} = \mathcal{M}_{\widehat{Z}^r}.$$

For $\mathcal{M}_{X^{(r)}}$ note first, that $\delta^*\mathcal{M}_{\widehat{X}} \otimes \mathcal{O}_Z(F) = \mathcal{M}_Z$, for some δ-exceptional effective divisor F. Consider the diagram

$$\begin{tikzcd} Z^r \times_{\widehat{X}^r} X^{(r)} \arrow[r, "\theta"] \arrow[d, "p_1"'] & X^{(r)} \arrow[d, "\rho"] \\ Z^r \arrow[r, "\delta^r"] & X^r. \end{tikzcd}$$

Then $\delta^{r*}(\mathrm{pr}_1^*\mathcal{M}_{\widehat{X}} \otimes \cdots \otimes \mathrm{pr}_r^*\mathcal{M}_{\widehat{X}})$ is a subsheaf of $\mathrm{pr}_1^*\mathcal{M}_Z \otimes \cdots \otimes \mathrm{pr}_r^*\mathcal{M}_Z$ and both coincide outside of a divisor F' with $\mathrm{codim}(\hat{\delta}^r(F')) \geq 2$. So the same holds true for the subsheaf

$$p_1^*\delta^{r*}(\mathrm{pr}_1^*\mathcal{M}_{\widehat{X}} \otimes \cdots \otimes \mathrm{pr}_r^*\mathcal{M}_{\widehat{X}}) = \theta^*\mathcal{M}_{X^{(r)}}$$

of $p_1^*(\mathrm{pr}_1^*\mathcal{M}_Z \otimes \cdots \otimes \mathrm{pr}_r^*\mathcal{M}_Z)$. The statement is independent of the desingularization. Hence we may assume that $Z^{(r)}$ dominates the main component of $Z^r \times_{\widehat{X}^r} X^{(r)}$. Now, $\delta^{(r)*}\mathcal{M}_{X^{(r)}} \otimes \mathcal{O}_{Z^{(r)}}(F'') = \mathcal{M}_{Z^{(r)}}$ for some effective $\delta^{(r)}$ exceptional divisor F''. $\square$

Lemma 9.9 allows us to apply Lemma 5.7 and

$$f^{(r)}_*\big(\omega^\nu_{X^{(r)}/\widehat{Y}} \otimes \mathcal{M}^\mu_{X^{(r)}}\big) = \hat{g}^r_*\big(\omega^\nu_{\widehat{Z}^r/\widehat{Y}} \otimes \mathcal{M}^\mu_{\widehat{Z}^r}\big). \tag{9.9.1}$$

For $(\nu, \mu) \in I$ one can use flat base change and the projection formula to identify the right-hand side as

$$\bigotimes^r \hat{g}_*\big(\omega^\nu_{\widehat{Z}/\widehat{Y}} \otimes \mathcal{M}^\mu_{\widehat{Z}}\big).$$

Using 5.7 again, one finds

$$f_*^{(r)}\big(\omega^{\nu}_{X^{(r)}/\hat{Y}} \otimes \mathcal{M}^{\mu}_{X^{(r)}}\big) = \bigotimes^{r} \hat{f}_*\big(\omega^{\nu}_{\hat{X}/\hat{Y}} \otimes \mathcal{M}^{\mu}_{\hat{X}}\big).$$

In particular those sheaves are locally free and compatible with base change for morphisms $\varrho : T \to \widehat{Y}$ with $\varrho^{-1}(\widehat{Y}_g)$ dense in T.

Next we move to the right-hand side of the diagram (4.6.1) and redefine all the sheaves and divisors from 9.5 with $\widehat{Z}^r$ replaced by $\widehat{X}^{(r)}$ or $Z^{(r)}$. As in 9.5 we will give the definitions in the polarized case. For the canonically polarized case the last lines of 9.5 apply.

Notation 9.10. As at the end of the proof of Lemma 8.2, blowing up $X^{(r)}$ with centers outside $\widehat{X}^r_0$, one may assume that the image of

$$f^{(r)*} f_*^{(r)}\big(\omega^{\eta}_{X^{(r)}/\hat{Y}} \otimes \mathcal{M}^{\gamma}_{X^{(r)}}\big) \longrightarrow \omega^{\eta}_{X^{(r)}/\hat{Y}} \otimes \mathcal{M}^{\gamma}_{X^{(r)}}$$

is invertible and we denote it by $\mathcal{B}^{(\eta,\gamma)}_{X^{(r)}}$. The effective divisor $\Sigma^{(\eta,\gamma)}_{X^{(r)}}$ is chosen such that

$$\mathcal{B}^{(\eta,\gamma)}_{X^{(r)}} \otimes \mathcal{O}_{X^{(r)}}\big(\Sigma^{(\eta,\gamma)}_{X^{(r)}}\big) = \omega^{\eta}_{X^{(r)}/\hat{Y}} \otimes \mathcal{M}^{\gamma}_{X^{(r)}},$$

hence is supported outside of $X_0^{(r)}$. If the condition (9.3.2) holds, we can apply 9.8 for the tuple

$$(\beta_0, \alpha_0) = (b\cdot(\beta-1)\cdot e\cdot \ell + \eta\cdot b\cdot(e-1), b\cdot\alpha\cdot e\cdot\ell + \gamma\cdot b\cdot(e-1))$$

and obtain an inclusion $\mathcal{E}^r_{\hat{Y}} \to f_*^{(r)}(\omega^{\beta_0}_{X^{(r)}/\hat{Y}} \otimes \mathcal{M}^{\alpha_0}_{X^{(r)}})$. The image of $f^{(r)*}\mathcal{E}^r_{\hat{Y}}$ under the evaluation map will be denoted by $\mathcal{B}_{X^{(r)}}$.

In Variant 9.4, i.e., if (9.4.1) holds, one applies (9.9.1) for the tuples $(\beta_\iota, \alpha_\iota)$. So one has morphisms

$$\bigotimes^{r} \big(\hat{g}_*\mathcal{B}^{(\beta_\iota,\alpha_\iota)}_{\hat{Z}}\big) \longrightarrow f_*^{(r)}\big(\omega^{\beta_\iota}_{X^{(r)}/\hat{Y}} \otimes \mathcal{M}^{\alpha_\iota}_{X^{(r)}}\big).$$

The image of $f^{(r)*}(\hat{g}_*\mathcal{B}^{(\beta_\iota,\alpha_\iota)}_{\hat{Z}})^{\otimes r}$ is an invertible sheaf $\mathcal{B}^{(\beta_\iota,\alpha_\iota)}_{X^{(r)}}$, and the image of $f^{(r)*}\bigotimes_{\iota=1}^{s}(\hat{g}_*\mathcal{B}^{(\beta_\iota,\alpha_\iota)}_{\hat{Z}})^{\otimes r}$ under the product map is

$$\bigotimes_{\iota=1}^{s} \mathcal{B}^{(\beta_\iota,\alpha_\iota)}_{X^{(r)}} \subset \omega^{\beta_0}_{X^{(r)}/\hat{Y}} \otimes \mathcal{M}^{\alpha_0}_{X^{(r)}}.$$

So the image of $f^{(r)*}\mathcal{E}_{\hat{Y}}$ is a subsheaf $\mathcal{B}_{X^{(r)}}$.

In both cases $\mathcal{B}_{X^{(r)}}$ is isomorphic to $\omega^{\beta_0}_{X^{(r)}/\hat{Y}} \otimes \mathcal{M}^{\alpha_0}_{X^{(r)}}$ on $\widehat{X}^r_0 = f^{(r)^{-1}}(\widehat{Y}_0)$. Blowing up $X^{(r)}$ we find a divisor $\Sigma_{X^{(r)}}$ with

$$\omega^{\beta_0}_{X^{(r)}/\hat{Y}} \otimes \mathcal{M}^{\alpha_0}_{X^{(r)}} = \mathcal{B}_{X^{(r)}} \otimes \mathcal{O}_{X^{(r)}}(\Sigma_{X^{(r)}}).$$

Finally equation (9.9.1) implies that

$$f_*^{(r)}\mathcal{B}_{X^{(r)}}^{(\eta,\gamma)} = f_*^{(r)}\big(\omega_{X^{(r)}/\hat{Y}}^{\eta} \otimes \mathcal{M}_{X^{(r)}}^{\gamma}\big) = \hat{g}_*^r\big(\omega_{\hat{Z}^r/\hat{Y}}^{\eta} \otimes \mathcal{M}_{\hat{Z}^r}^{\gamma}\big).$$

Hence $\Xi^{(r)}: \mathcal{V}^{r\cdot e\cdot \ell} \to \hat{g}_*^r(\omega_{\hat{Z}^r/\hat{Y}}^{\eta} \otimes \mathcal{M}_{\hat{Z}^r}^{\gamma})$ induces a section of $\mathcal{B}_{X^{(r)}}^{(\eta,\gamma)} \otimes f^{(r)*}\mathcal{V}^{-r\cdot e\cdot\ell}$ whose zero divisor will be denoted by $\Gamma_{X^{(r)}}$. Again, we write

$$\Delta_{X^{(r)}} = b\cdot(\Gamma_{X^{(r)}} + \Sigma_{X^{(r)}}^{(\eta,\gamma)}) + \Sigma_{X^{(r)}},$$

and recall that $X_0^{(r)} = \hat{X}_0^r$, $\Sigma_{X_0^{(r)}}^{(\beta_0,\alpha_0)} = \Sigma_{X_0^{(r)}}^{(\eta,\gamma)} = 0$ and $\hat{\delta}^{r*}\Gamma_{\hat{Z}^r} = \delta^{(r)*}\Gamma_{X^{(r)}}$.

LEMMA 9.11. *The sheaf* $\mathcal{G}_{\hat{Y}}^{(\beta+\frac{\eta}{\ell},\alpha+\frac{\gamma}{\ell})}$ *in* 9.6 *is equal to*

$$f_*^{(r)}\Big(\omega_{X^{(r)}/\hat{Y}}^{\beta+\frac{\eta}{\ell}} \otimes \mathcal{M}_{X^{(r)}}^{\alpha+\frac{\gamma}{\ell}} \otimes \mathcal{J}\Big(-\frac{1}{N}\cdot\Delta_{X^{(r)}}\Big)\Big).$$

On $\hat{X}_0^r = f^{(r)^{-1}}(\hat{Y}_0)$ *one has*

$$\mathcal{J}\Big(-\frac{1}{N}\cdot\Delta_{X^{(r)}}\Big)|_{\hat{X}_0^r} = \mathcal{J}\Big(-\frac{1}{e\cdot\ell}\cdot\Gamma_{\hat{X}_0^r}\Big) = \mathcal{O}_{\hat{X}_0^r},$$

and the inclusion $\mathcal{G}_{\hat{Y}}^{(\beta+\frac{\eta}{\ell},\alpha+\frac{\gamma}{\ell})} \to \bigotimes^r \mathcal{F}_{\hat{Y}}^{(\beta+\frac{\eta}{\ell},\alpha+\frac{\gamma}{\ell})}$ *is an isomorphism on* $\hat{Y}_0$.

Proof. We keep the notation from 9.8 and assume in addition that the pullbacks of $\Delta_{\hat{Z}^r}$ and of $\Delta_{X^{(r)}}$ to $Z^{(r)}$ are normal crossing divisors.

Since $\hat{\delta}_*^r\omega_{Z^{(r)}/\hat{Y}} = \omega_{\hat{Z}^r/\hat{Y}}$ and $\delta_*^{(r)}\omega_{Z^{(r)}/\hat{Y}} = \omega_{X^{(r)}/\hat{Y}}$, and since by Lemma 9.9 the same holds for the sheaves $\mathcal{M}_\bullet$ one can find, for all (ν,μ), effective $\hat{\delta}^r$-exceptional divisors $E_{Z^{(r)}/\hat{Z}^r}$ and $F_{Z^{(r)}/\hat{Z}^r}$ and $\delta^{(r)}$-exceptional divisors $E_{Z^{(r)}/X^{(r)}}$ and $F_{Z^{(r)}/X^{(r)}}$ with

$$\begin{aligned}\omega_{Z^{(r)}/\hat{Y}}^{\nu} \otimes \mathcal{M}_{Z^{(r)}}^{\mu} &= \hat{\delta}^{r*}\big(\omega_{\hat{Z}^r/\hat{Y}}^{\nu} \otimes \mathcal{M}_{\hat{Z}^r}^{\mu}\big) \otimes \mathcal{O}_{Z^{(r)}}(\nu\cdot E_{Z^{(r)}/\hat{Z}^r} + \mu\cdot F_{Z^{(r)}/\hat{Z}^r}) \\ &= \delta^{(r)*}\big(\omega_{X^{(r)}/\hat{Y}}^{\nu} \otimes \mathcal{M}_{X^{(r)}}^{\mu}\big) \otimes \mathcal{O}_{Z^{(r)}}(\nu\cdot E_{Z^{(r)}/X^{(r)}} + \mu\cdot F_{Z^{(r)}/X^{(r)}}).\end{aligned}$$

By Lemma 8.2 one has $\hat{\delta}^{r*}\mathcal{B}_{\hat{Z}^r}^{(\eta,\gamma)} = \delta^{(r)*}\mathcal{B}_{X^{(r)}}^{(\eta,\gamma)}$ and $\hat{\delta}^{r*}\mathcal{B}_{\hat{Z}^r} = \delta^{(r)*}\mathcal{B}_{X^{(r)}}$. This implies that

$$\hat{\delta}^{r*}\Sigma_{\hat{Z}^r}^{(\eta,\gamma)} + \eta\cdot E_{Z^{(r)}/\hat{Z}^r} + \gamma\cdot F_{Z^{(r)}/\hat{Z}^r} = \delta^{(r)*}\Sigma_{X^{(r)}}^{(\eta,\gamma)} + \eta\cdot E_{Z^{(r)}/X^{(r)}} + \gamma\cdot F_{Z^{(r)}/X^{(r)}},$$

and that

$$\begin{aligned}&\hat{\delta}^{r*}\Sigma_{\hat{Z}^r} + \beta_0\cdot E_{Z^{(r)}/\hat{Z}^r} + \alpha_0\cdot F_{Z^{(r)}/\hat{Z}^r} \\ &\qquad = \delta^{(r)*}\Sigma_{X^{(r)}} + \beta_0\cdot E_{Z^{(r)}/X^{(r)}} + \alpha_0\cdot F_{Z^{(r)}/X^{(r)}}.\end{aligned}$$

Moreover $\hat{\delta}^{r*}\Gamma_{\widehat{Z}^r} = \delta^{(r)*}\Gamma_{X^{(r)}}$, and putting everything together one finds

$$\hat{\delta}^{r*}\Delta_{\widehat{Z}^r} + (b\cdot(\beta-1)\cdot e\cdot\ell+\eta\cdot b\cdot e)\cdot E_{Z^{(r)}/\widehat{Z}^r} + (b\cdot\alpha\cdot e\cdot\ell+\gamma\cdot b\cdot e)\cdot F_{Z^{(r)}/\widehat{Z}^r} =$$
$$\delta^{(r)*}\Delta_{X^{(r)}} + (b\cdot(\beta-1)\cdot e\cdot\ell+\eta\cdot b\cdot e)\cdot E_{Z^{(r)}/X^{(r)}} + (b\cdot\alpha\cdot e\cdot\ell+\gamma\cdot b\cdot e)\cdot F_{Z^{(r)}/X^{(r)}}$$

and

$$\hat{\delta}^{r*}(\omega_{\widehat{Z}^r/\widehat{Y}}^{\beta+\frac{\eta}{\ell}-1} \otimes \mathcal{M}_{\widehat{Z}^r}^{\alpha+\frac{\gamma}{\ell}}) \otimes \mathcal{O}_{Z^{(r)}}\Big(-\Big[\frac{1}{N}\cdot\hat{\delta}^{r*}\Delta_{\widehat{Z}^r}\Big]\Big)$$
$$= \delta^{(r)*}\Big(\omega_{X^{(r)}/\widehat{Y}}^{\beta+\frac{\eta}{\ell}-1} \otimes \mathcal{M}_{X^{(r)}}^{\alpha+\frac{\gamma}{\ell}}\Big) \otimes \mathcal{O}_{Z^{(r)}}\Big(-\Big[\frac{1}{N}\cdot\delta^{(r)*}\Delta_{X^{(r)}}\Big]\Big).$$

By the definition of multiplier ideals this implies

$$\mathcal{G}_{\widehat{Y}}^{(\beta+\frac{\eta}{\ell},\alpha+\frac{\gamma}{\ell})} = \hat{g}^r_*\hat{\delta}^r_*\Big(\omega_{Z^{(r)}/\widehat{Y}}\otimes\hat{\delta}^{r*}\Big(\omega_{\widehat{Z}^r/\widehat{Y}}^{\beta+\frac{\eta}{\ell}-1}\otimes\mathcal{M}_{\widehat{Z}^r}^{\alpha+\frac{\gamma}{\ell}}\Big)\otimes\mathcal{O}_{Z^{(r)}}\Big(-\Big[\frac{1}{N}\cdot\hat{\delta}^{r*}\Delta_{\widehat{Z}^r}\Big]\Big)\Big)$$
$$= f^{(r)}_*\delta^{(r)}_*\Big(\omega_{Z^{(r)}/\widehat{Y}}\otimes\delta^{(r)*}\Big(\omega_{X^{(r)}/\widehat{Y}}^{\beta+\frac{\eta}{\ell}-1}\otimes\mathcal{M}_{X^{(r)}}^{\alpha+\frac{\gamma}{\ell}}\Big)\otimes\mathcal{O}_{Z^{(r)}}\Big(-\Big[\frac{1}{N}\cdot\delta^{(r)*}\Delta_{X^{(r)}}\Big]\Big)\Big)$$
$$= f^{(r)}_*\Big(\omega_{X^{(r)}/\widehat{Y}}^{\beta+\frac{\eta}{\ell}}\otimes\mathcal{M}_{X^{(r)}}^{\alpha+\frac{\gamma}{\ell}}\otimes\mathcal{J}\Big(-\frac{1}{N}\cdot\Delta_{X^{(r)}}\Big)\Big)$$

as claimed in 9.11. In particular one has a natural inclusion

$$\mathcal{G}_{\widehat{Y}}^{(\beta+\frac{\eta}{\ell},\alpha+\frac{\gamma}{\ell})} \to f^{(r)}_*\Big(\omega_{X^{(r)}/\widehat{Y}}^{\beta+\frac{\eta}{\ell}}\otimes\mathcal{M}_{X^{(r)}}^{\alpha+\frac{\gamma}{\ell}}\Big) = \bigotimes^r \mathcal{F}_{\widehat{Y}}^{(\beta+\frac{\eta}{\ell},\alpha+\frac{\gamma}{\ell})},$$

induced by $\mathcal{J}\big(-\frac{1}{N}\cdot\Delta_{X^{(r)}}\big)\subset\mathcal{O}_{X^{(r)}}$. It remains to show that the latter is an isomorphism over $X_0^{(r)} = \widehat{X}_0^r$.

Since $\Sigma_{X^{(r)}}|_{X_0^{(r)}} = \Sigma_{X^{(r)}}^{(\eta,\gamma)}|_{X_0^{(r)}} = 0$,

$$\mathcal{G}_{\widehat{Y}}^{(\beta+\frac{\eta}{\ell},\alpha+\frac{\gamma}{\ell})}|_{\widehat{Y}_0} = f^{(r)}_{0*}\Big(\omega_{X_0^{(r)}/\widehat{Y}_0}^{\beta+\frac{\eta}{\ell}}\otimes\mathcal{M}_{X_0^{(r)}}^{\alpha+\frac{\gamma}{\ell}}\otimes\mathcal{J}\Big(-\frac{1}{e\cdot\ell}\cdot\Gamma_{X_0^{(r)}}\Big)\Big).$$

By definition, $X_0^{(r)} = \widehat{X}_0^r$ and by [Vie95, Prop. 5.19],

$$e(\Gamma_{\widehat{X}_0^r}) \le \mathrm{Max}\{e(\omega_{F^r}^\eta\otimes\mathcal{M}_{X^{(r)}}^\gamma|_{F^r});\ F \text{ a fiber of } \hat{f}_0\}.$$

By [Vie95, Cor. 5.21] the right-hand side is equal to

$$\mathrm{Max}\{e(\omega_F^\eta\otimes\mathcal{M}_{\widehat{X}}^\gamma|_F);\ F \text{ a fiber of } \hat{f}_0\}.$$

So the choice of e in (9.2.1) implies that $\mathcal{J}\big(-\frac{1}{e\cdot\ell}\cdot\Gamma_{\widehat{X}_0^r}\big) = \mathcal{O}_{\widehat{X}_0^r}$. □

Remark 9.12. Replacing e by some larger number one can force the multiplier ideal $\mathcal{J}\big(-\frac{1}{b\cdot e\cdot\ell}\cdot\Delta_{\widehat{Z}^r}\big)$ to be equal to $\mathcal{O}_{\widehat{Z}^r}$ and

$$\mathcal{G}_{\widehat{Y}}^{(\beta+\frac{\eta}{\ell},\alpha+\frac{\gamma}{\ell})} \hookrightarrow \bigotimes^r \mathcal{F}_{\widehat{Y}}^{(\beta+\frac{\eta}{\ell},\alpha+\frac{\gamma}{\ell})}$$

in 9.11 to be an isomorphism on $\widehat{Y}$. However, changing e one loses the compatibility of the multiplier ideals with pullbacks and, as remarked before in 6.12, one cannot expect the same e to work over the alterations needed to enforce this condition.

10. **Mild reduction over curves**

The sheaves $\mathcal{F}_{\widehat{Y}}^{(\nu,\mu)}$ and $\mathcal{G}^{(\beta+\frac{\eta}{\ell},\alpha+\frac{\gamma}{\ell})}=\mathcal{G}_{\widehat{Y}}^{(\Xi^{(r)},\mathcal{E};\beta+\frac{\eta}{\ell},\alpha+\frac{\gamma}{\ell})}$ are only compatible with base change for dominant morphisms, and for morphisms from curves whose image meets a certain open subscheme $\widehat{Y}_g$ of $\widehat{Y}_0$. We will extend the latter in Proposition 10.5 to morphisms whose image meets $\widehat{Y}_0$. We need in addition that the pullback family over C has a semistable or mild model, as will be defined in this section.

First we consider the sheaves $\mathcal{F}_{\widehat{Y}}^{(\nu,\mu)}$. The necessary changes for $\mathcal{G}_{\widehat{Y}}^{(\beta+\frac{\eta}{\ell},\alpha+\frac{\gamma}{\ell})}$ will be discussed in the next section. For the canonically polarized case the last lines of 9.5 apply. One just has to choose $\mathcal{M}_\bullet=\mathcal{O}_\bullet$ and choose $\alpha=\gamma=\kappa=0$.

We also need the sheaves $\mathcal{M}_\bullet$ to be well defined for the restrictions of our families to curves. This is evidently true for the dualizing sheaves, and for the pullback of the invertible sheaf $\mathcal{L}$ on X. For the saturated extensions of the polarization, we will need some additional arguments. So at some points we will handle the two cases separately.

We keep in this section the setup and the assumptions from 9.1–9.4, and we choose the morphisms in the diagram (4.6.1) according to 8.5, 8.7 or 8.13.

Definition 10.1. Consider a nonsingular curve $\widehat{C}$, an open dense subscheme $\widehat{C}_0$ and a morphism $\chi':\widehat{C}\to Y$ with $\chi'(\widehat{C}_0)\subset Y_0$. We say that $\chi:\widehat{C}\to\widehat{Y}$ has a *mild reduction*, if there exists a commutative diagram

(10.1.1)
$$\begin{tikzcd} \widehat{S} \arrow[r, "\zeta"] \arrow[d, "\hat{h}"'] & X\times_Y\widehat{C} \arrow[dl, "\mathrm{pr}_2"] \\ \widehat{C} & \end{tikzcd}$$

of morphisms of normal projective varieties such that

(i) $\hat{h}$ is mild.

(ii) $\zeta:\widehat{S}\to X\times_Y\widehat{C}$ is a modification of $X\times_Y\widehat{C}$.

The canonically polarized case. We call $\hat{h}:\widehat{S}\to\widehat{C}$ a *mild reduction of* $\chi':\widehat{C}\to Y$.

The polarized case. We call $(\hat{h}:\widehat{S}\to\widehat{C},\mathcal{M}_{\widehat{S}})$ a *mild reduction of* $\chi':\widehat{C}\to Y$ (*for* $\mathcal{L}$), if in addition to (i) and (ii) one has

(iii) $\mathcal{M}_{\widehat{S}}=\zeta^*\mathrm{pr}_1^*\mathcal{L}$.

It is easy to find a mild reduction over $\widehat{C}$ whenever $\widehat{C}\to\chi'(\widehat{C})$ is sufficiently ramified. As in Section 7 one can desingularize $X\times_Y\widehat{C}$ such that all the fibers become normal crossing divisors, and then one can replace $\widehat{C}$ by a larger covering, to get rid of multiple fiber components.

In the saturated case we have to be more careful. We cannot choose $\mathcal{M}_{\widehat{S}}$ as the pullback, since we do not want to require the existence of a morphism from $\widehat{S}$ to $\widehat{X}$.

Definition 10.2. *The saturated case.* We call $(\hat{h} : \widehat{S} \to \widehat{C}, \mathcal{M}_{\widehat{S}})$ in 10.1 a *mild reduction of* $\chi' : \widehat{C} \to Y$ (*for* $\mathcal{L}$ *or for* $\mathcal{L}$ *and* η_0), if in addition to (i) and (ii) in 10.1 one has:

(iii$'$) There exists a Cartier divisor $\Pi_{\widehat{S}}^{(\eta_0)}$ on $\widehat{S}$ with

$$\hat{h}^* \hat{h}_* \omega_{\widehat{S}/\widehat{C}}^{\eta_0} \longrightarrow \varpi_{\widehat{S}}^{(\eta_0)} = \omega_{\widehat{S}/\widehat{C}}^{\eta_0} \otimes \mathcal{O}_{\widehat{S}}(-\Pi_{\widehat{S}}^{(\eta_0)})$$

surjective. Moreover $\mathcal{M}_{\widehat{S}}$ is a κ-saturated extension of $\zeta^* \mathrm{pr}_1^* \mathcal{L}$, i.e. it satisfies the condition required for $\mathcal{M}_{\widehat{Z}}$ in 8.10:

$$\zeta^* \mathrm{pr}_1^* \mathcal{L} \subset \mathcal{M}_{\widehat{S}} \subset \zeta^* \mathrm{pr}_1^* \mathcal{L} \otimes (\mathcal{O}_{\widehat{S}}(* \cdot \Pi_{\widehat{S}}^{(\eta_0)}) \cap \mathcal{O}_{\widehat{S}}(* \hat{h}^{-1}(\widehat{C} \setminus \chi'^{-1}(Y_0))),$$

and $\hat{h}_* \mathcal{M}_{\widehat{S}}^{\kappa} = \hat{h}_* (\mathcal{M}_{\widehat{S}}^{\kappa} \otimes \mathcal{O}_{\widehat{S}}(\varepsilon \cdot \Pi_{\widehat{S}}^{(\eta_0)}))$ for all $\varepsilon \geq 0$.

In all cases, if $(\hat{h} : \widehat{S} \to \widehat{C}, \mathcal{M}_{\widehat{S}})$ is a mild reduction of $\chi' : \widehat{C} \to Y$ for $\mathcal{L}$, we define $\mathcal{F}_{\widehat{C}}^{(\nu,\mu)} = \hat{h}_* (\omega_{\widehat{S}/\widehat{C}}^{\nu} \otimes \mathcal{M}_{\widehat{S}}^{\mu})$. We will need the compatibility of this sheaf with pullback:

LEMMA 10.3. *Let* $(\hat{h} : \widehat{S} \to \widehat{C}, \mathcal{M}_{\widehat{S}})$ *be a mild reduction for* $\chi' : \widehat{C} \to Y$ *and for* $\mathcal{L}$.

(1) *If* $\theta : \widehat{C}_1 \to \widehat{C}$ *is a finite morphism between nonsingular curves, then*

$$(\widehat{S} \times_{\widehat{C}} \widehat{C}_1 \to \widehat{C}_1, \mathrm{pr}_1^* \mathcal{M}_{\widehat{S}})$$

is a mild reduction for $\chi' \circ \theta$.

(2) *In* (1) *base change induces an isomorphism* $\theta^* \mathcal{F}_{\widehat{C}}^{(\nu,\mu)} \xrightarrow{\cong} \mathcal{F}_{\widehat{C}_1}^{(\nu,\mu)}$ (*which we will write again as an equality of sheaves*).

(3) *Let* $\sigma : S \to X \times_Y \widehat{C}$ *be a modification of* $X \times_Y \widehat{C}$ *with* S *nonsingular*, *and* $h = \mathrm{pr}_2 \circ \sigma$. *In the* (*canonically*) *polarized case choose* $\mathcal{M}_S = \sigma^* \mathrm{pr}_1^* \mathcal{L}$. *In the saturated case choose* $\mathcal{M}_S$ *according to Lemma* 8.11(a). *Then*

$$\mathcal{F}_{\widehat{C}}^{(\nu,\mu)} = h_* \big(\omega_{S/\widehat{C}}^{\nu} \otimes \mathcal{M}_S^{\mu} \big).$$

In particular, *the sheaf* $\mathcal{F}_{\widehat{C}}^{(\nu,\mu)}$ *is independent of the mild model.*

Proof. Since $\widehat{S} \times_C C_1 \to C_1$ is again mild, (1) is obvious and (2) follows by flat base change. Now, (3) is a special case of Lemma 5.7, where in the saturated case we use Lemma 8.11(a) for a smooth model dominating both $\widehat{S}$ and S. $\square$

LEMMA 10.4. *In* 8.5 *and* 8.7, *or in* 8.13 *one may choose an open dense subscheme* $Y_g \subset Y_0$ *such that for all morphisms*

$$\chi' : \widehat{C} \xrightarrow{\pi'} \widehat{Y} \xrightarrow{\varphi} Y$$

with $\widehat{C}_g = \chi'^{-1}(Y_g) \neq \varnothing$ *the tuple* $(\widehat{S} := \widehat{Z} \times_{\widehat{Y}} \widehat{C} \to \widehat{C}, \mathcal{M}_{\widehat{S}} := \mathrm{pr}_1^* \mathcal{M}_{\widehat{Z}})$ *is a mild reduction for* χ' *and*

$$\mathcal{F}_{\widehat{C}}^{(\nu,\mu)} = \pi'^* \mathcal{F}_{\widehat{Y}}^{(\nu,\mu)} \quad \textit{for} \quad (\nu, \mu) \in I. \tag{10.4.1}$$

Proof. Choose Y_g, such that $\varphi^{-1}(Y_g)$ is contained in the open set $\widehat{Y}_g$ in 8.5(iv) or 8.7(ix) and such that $\widehat{Z}$ is smooth over $\varphi^{-1}(Y_g)$. Then the definition of a mild morphism in 4.1 implies that $\hat{h} = \mathrm{pr}_2 : \widehat{S} = \widehat{Z} \times_{\widehat{Y}} \widehat{C} \to \widehat{C}$ is mild. In the diagram (4.6.1) in 4.6 we require the existence of a morphism $\hat{\varphi} : \widehat{Z} \to X$ lifting $\varphi : \widehat{Y} \to Y$; hence there is a modification $\varphi' : \widehat{Z} \to X \times_Y \widehat{Y}$. The fibers of $\widehat{Z}$ and $X \times_Y \widehat{Y}$ over $\widehat{Y}_g$ are smooth, and φ' restricts to a modification of those fibers. This implies that the induced morphism $\widehat{Z} \times_{\widehat{Y}} \widehat{C} \to \widehat{X} \times_{\widehat{Y}} \widehat{C}$ is birational. The equality in (10.4.1) follows from 8.7(ix) and from the choice of Y_g.

It remains to verify condition (iii′) in the saturated case, as stated in 10.2. By assumption 8.5(iv) the direct image $\hat{g}_* \omega_{\widehat{Z}/\widehat{Y}}^{\eta_0} = \hat{g}_* \varpi_{\widehat{Z}}^{(\eta_0)}$ is locally free and compatible with base change for π'. Then the evaluation map for $\varpi_{\widehat{S}}^{(\eta_0)} := \mathrm{pr}_1^* \varpi_{\widehat{Z}}^{(\eta_0)}$ is surjective, and the first part of condition (iii′) in 10.2 holds true. The second condition just says that the pullback of $\mathcal{L}$ to $\widehat{S}$ coincides with $\mathcal{M}$ over some open subscheme of $\widehat{C}$. This follows, since the same holds for $\mathcal{M}_{\widehat{Z}}$ over $\widehat{Y}_0$. The last condition follows from Corollary 8.14. □

PROPOSITION 10.5. *Let* $\widehat{C}$ *be an irreducible curve, and let* $\pi' : \widehat{C} \to \widehat{Y}$ *be a morphism. If* $\widehat{C}_0 = \pi'^{-1}(\widehat{Y}_0) \neq \varnothing$ *and if* $\chi' = \varphi \circ \pi'$ *admits a mild reduction* $(\hat{h} : \widehat{S} \to \widehat{C}, \mathcal{M}_{\widehat{S}})$, *then* $\mathcal{F}_{\widehat{C}}^{(\nu,\mu)} = \pi'^* \mathcal{F}_{\widehat{Y}}^{(\nu,\mu)}$ *for* $(\nu, \mu) \in I$.

Proof. Note that one may replace $\widehat{Y}$ in 8.5, 8.7 or 8.13 by any modification, without losing the properties (i)—(x). In particular the sheaves $\mathcal{F}_{\widehat{Y}}^{(\nu,\mu)}$ are compatible with pullback by dominant morphisms for $(\nu, \mu) \in I$. Part (1) of Lemma 10.3 allows us to replace $\widehat{C}$ by any covering. Hence dropping as usual the lower index $_1$ one can assume that $\widehat{Y} = \widehat{Y}_1$ in 7.2 and use the three properties stated there. Let us write $h : S \to \widehat{C}$ for the induced morphism and $\mathcal{M}_S = \mathcal{M}_{\widehat{X}}|_S$.

In the (canonically) polarized case $\mathcal{M}_S$ is the pullback of $\mathcal{L}$ to S. By assumption $\widehat{C} \to Y$ has a mild reduction $(\hat{h} : \widehat{S} \to \widehat{C}, \mathcal{M}_{\widehat{S}})$. By 10.3(3), $\mathcal{F}_{\widehat{C}}^{(\nu,\mu)} = h_*(\omega_{S/\widehat{C}}^{\nu} \otimes \mathcal{M}_S^{\mu})$ and by Lemma 5.8 this is the pullback of $\mathcal{F}_{\widehat{Y}}^{(\nu,\mu)}$.

For the saturated case, we have to argue in a slightly different way. Recall that we defined in 8.13 the invertible sheaves $\mathcal{B}_{\widehat{X}}^{(0,\kappa)}$ and $\varpi_{\widehat{X}}^{(\eta_0)}$ as the images of the evaluation maps

$$\hat{f}^*\hat{f}_*\mathcal{M}^\kappa_{\widehat{X}} \longrightarrow \mathcal{M}^\kappa_{\widehat{X}} \quad\text{and}\quad \hat{f}^*\hat{f}_*\omega^{\eta_0}_{\widehat{X}/\widehat{Y}} \longrightarrow \omega^{\eta_0}_{\widehat{X}/\widehat{Y}}.$$

Lemma 5.8 implies that the direct images $\hat{f}_*\mathcal{M}^\kappa_{\widehat{X}}$ and $\hat{f}_*\omega^{\eta_0}_{\widehat{X}/\widehat{Y}}$ are compatible with pullback. The sheaves

$$\mathcal{B}^{(0,\kappa)}_S = \mathcal{B}^{(0,\kappa)}_{\widehat{X}}\big|_S \quad\text{and}\quad \varpi^{(\eta_0)}_S = \varpi^{(\eta_0)}_{\widehat{X}}\big|_S,$$

are again invertible and the images of the evaluation maps for $\mathcal{M}^\kappa_S$ and $\omega^{\eta_0}_{S/\widehat{C}}$, respectively. The latter implies that the divisor $\Pi^{(\eta_0)}_S$ is the pullback of $\Pi^{(\eta_0)}_{\widehat{X}}$. By the definition of κ-saturated in 8.10 and by Lemma 8.11(c) one knows that

$$\hat{f}_*\mathcal{B}^{(0,\kappa)}_{\widehat{X}} = \hat{f}_*\mathcal{M}^\kappa_{\widehat{X}} = \hat{f}_*\Big(\mathcal{M}^\kappa_{\widehat{X}} \otimes \mathcal{O}_{\widehat{X}}\big(*\cdot\Pi^{(\eta_0)}_{\widehat{X}}\big)\Big) = \hat{f}_*\Big(\rho^*\mathcal{L}^\kappa \otimes \mathcal{O}_{\widehat{X}}\big(*\cdot\Pi^{(\eta_0)}_{\widehat{X}}\big)\Big).$$

Lemma 5.8 implies that the corresponding property holds true for S instead of $\widehat{X}$.

By assumption $\widehat{C} \to Y$ has a mild κ-saturated reduction $(\hat{h} : \widehat{S} \to \widehat{C}, \mathcal{M}_{\widehat{S}})$. Let $\Psi : W \to S$ and $\Psi' : W \to \widehat{S}$ be modifications, with W smooth. By 5.7,

$$\hat{h}_*\varpi^{(\eta_0)}_{\widehat{S}} = \hat{h}_*\omega^{\eta_0}_{\widehat{S}/\widehat{C}} = h_*\omega^{\eta_0}_{S/\widehat{C}} = h_*\varpi^{(\eta_0)}_S;$$

hence, $\Psi'^*\varpi^{(\eta_0)}_{\widehat{S}} = \Psi^*\varpi^{(\eta_0)}_S$. Call this sheaf $\varpi^{(\eta_0)}_W$. The divisor $\Pi^{(\eta_0)}_W$ with

$$\omega^{\eta_0}_{W/\widehat{C}} = \varpi^{(\eta_0)}_W \otimes \mathcal{O}_W\big(-\Pi^{(\eta_0)}_W\big)$$

is of the form $\Psi'^*\Pi^{(\eta_0)}_{\widehat{S}} + \eta_0\cdot E_{W/\widehat{S}} = \Psi'^*\Pi^{(\eta_0)}_S + \eta_0\cdot E_{W/S}$, where $E_{W/\widehat{S}}$ and $E_{W/S}$ are relative canonical divisors. If $\mathcal{L}_\bullet$ denotes the pullback of $\mathcal{L}$, as in 8.9 one finds that for all $\varepsilon \geq 0$

$$\hat{h}_*\Big(\mathcal{L}^\kappa_{\widehat{S}} \otimes \mathcal{O}_{\widehat{S}}\big(\varepsilon\cdot\Pi^{(\eta_0)}_{\widehat{S}}\big)\Big) = h_*\Big(\mathcal{L}^\kappa_S \otimes \mathcal{O}_S\big(\varepsilon\cdot\Pi^{(\eta_0)}_S\big)\Big),$$

and that for some ε_0 and all $\varepsilon \geq \varepsilon_0$, both sheaves are independent of ε. Since for those ε the left-hand side is $\hat{h}_*\mathcal{B}^{(0,\kappa)}_{\widehat{S}}$ and the right-hand side $h_*\mathcal{B}^{(0,\kappa)}_S$, the two sheaves are equal. This implies that $\Psi'^*\mathcal{B}^{(0,\kappa)}_{\widehat{S}} = \Psi^*\mathcal{B}^{(0,\kappa)}_S$.

The divisor $\Sigma^{(0,\kappa)}_{\widehat{S}}$ and $\Sigma^{(0,\kappa)}_S$ have the same support as $\Pi^{(\eta_0)}_{\widehat{S}} \cap \hat{h}^{-1}(\widehat{C}\setminus\widehat{C}_0)$ and $\Pi^{(\eta_0)}_S$, respectively. Define Σ to be the smallest divisor on W, larger than $\Psi'^*\Sigma^{(0,\kappa)}_{\widehat{S}} \cap \hat{h}^{-1}(\widehat{C}\setminus\widehat{C}_0)$ and $\Psi^*\Sigma^{(0,\kappa)}_S$. Adding components of $\Pi^{(\eta_0)}_W$ one finds some $\Sigma^{(0,\kappa)}_W$ such that

$$\Psi'^*\mathcal{B}^{(0,\kappa)}_{\widehat{S}} \otimes \mathcal{O}_W\big(\Sigma^{(0,\kappa)}_W\big)$$

is the κ-th power of an invertible subsheaf $\mathcal{M}_W$ of $\mathcal{L}_W \otimes \mathcal{O}_W(*\cdot\Pi^{(\eta_0)}_W)$. Obviously $\Psi'_*\mathcal{M}_W = \mathcal{M}_{\widehat{S}}$ and $\Psi_*\mathcal{M}_W = \mathcal{M}_S$, hence we are allowed to apply 5.7 and find

$$\hat{h}_*\big(\omega^\nu_{\widehat{S}/C} \otimes \mathcal{M}^\mu_{\widehat{S}}\big) = h_*\big(\omega^\nu_{S/C} \otimes \mathcal{M}^\mu_S\big) = \mathcal{F}^{(\nu,\mu)}_{\widehat{Y}}\big|_{\widehat{C}}. \qquad\square$$

11. A variant for multiplier ideals

Let us return to the set-up in 9.1 and to the assumptions introduced in 9.3 or in Variant 9.4. As in Section 10 we assume that $\mathcal{M}_{\widehat{Z}}$ and $\mathcal{M}_{\widehat{X}}$ are either the structure sheaves, or the pullback of an invertible sheaf $\mathcal{L}$ on X, or κ-saturated extensions of $\mathcal{L}$.

Consider again a nonsingular curve $\widehat{C}$ and a morphism $\chi' : \widehat{C} \to Y$ whose image meets Y_0, and a mild reduction $(\hat{h} : \widehat{S} \to \widehat{C}, \mathcal{M}_{\widehat{S}})$ for $\mathcal{L}$, as defined in 10.2. In particular one has a morphism $\zeta : \widehat{S} \to X$, and the sheaves

$$\mathcal{F}_{\widehat{C}}^{(\nu,\mu)} = \hat{h}_*\big(\omega_{\widehat{S}/\widehat{C}}^{\nu} \otimes \mathcal{M}_{\widehat{S}}^{\mu}\big)$$

are defined. Lemma 10.3 and Proposition 10.5 imply that $\chi'^*\mathcal{F}_{\widehat{Y}}^{(\nu,\mu)} = \tau^*\mathcal{F}_{\widehat{C}}^{(\nu,\mu)}$, whenever one has a lifting

(11.0.1)
$$\begin{array}{ccc} C' & \xrightarrow{\chi'} & \widehat{Y} \\ {\scriptstyle\tau}\downarrow & & \downarrow{\scriptstyle\varphi} \\ \widehat{C} & \xrightarrow{\chi} & Y \end{array}$$

with C' a nonsingular curve.

We will need that the different invertible sheaves and divisors introduced in 8.7, 8.9 or 8.13, and in Section 9 are defined for the morphism $\hat{h} : \widehat{S} \to \widehat{C}$.

Assumption 11.1. Assume that the assumptions made in 9.1 and 9.3 hold true, and that $\widehat{Y}$, $\widehat{Z}$ and $\widehat{X}$ are chosen according to Lemma 9.6.

(1) $(\hat{h} : \widehat{S} \to \widehat{C}, \mathcal{M}_{\widehat{S}})$ is a mild reduction for $\chi' : \widehat{C} \to Y$ and for $\mathcal{L}$. For η_0 the image $\varpi_{\widehat{S}}^{(\eta_0)}$ of $\hat{h}^*\hat{h}_*\omega_{\widehat{S}/\widehat{C}}^{\eta_0}$ in $\omega_{\widehat{S}/\widehat{C}}^{\eta_0}$ is locally free, and for $(\beta,\alpha) \in \widetilde{I}$ the images $\mathcal{B}_{\widehat{S}}^{(\beta,\alpha)}$ of the evaluation maps of $\omega_{\widehat{S}/\widehat{C}}^{\beta} \otimes \mathcal{M}_{\widehat{S}}^{\alpha}$ are also.

(2) There exists a subsheaf $\mathcal{E}_{\widehat{C}}$ of $\mathcal{F}_{\widehat{C}}^{(\beta_0,\alpha_0)}$, with $\chi'^*\mathcal{E}_{\widehat{Y}} = \tau^*\mathcal{E}_{\widehat{C}}$, for all liftings $\hat{\varphi}$ as in (11.0.1). Moreover the image $\mathcal{B}_{\widehat{S}}$ of the evaluation map
$$\hat{h}^*\mathcal{E}_{\widehat{C}} \longrightarrow \omega_{\widehat{S}/\widehat{C}}^{\beta_0} \otimes \mathcal{M}_{\widehat{S}}^{\alpha_0}$$
is invertible.

Remark 11.2. If in 9.4 one has $\mathcal{E}_{\widehat{Z}} = \bigotimes_{\iota=1}^{s} \hat{g}_*\mathcal{B}_{\widehat{Z}}^{(\beta_\iota,\alpha_\iota)}$, condition (2) in 11.1 follows from the assumption $(\beta_1,\alpha_1),\ldots(\beta_s,\alpha_s) \in \widetilde{I}$ for $\iota = 1,\ldots,s$.

In fact, the latter implies that the pullbacks of the sheaves $\mathcal{F}_{\widehat{S}}^{(\beta_\iota,\alpha_\iota)}$ and $\mathcal{F}_{\widehat{Y}}^{(\beta_\iota,\alpha_\iota)}$ coincide on C', and so does their image under the multiplication map.

If $\mathcal{E}_{\widehat{Z}}$ is smaller, we will need that it is defined on a compactification of Y, in order to enforce the compatibility condition (2) in 11.1.

We will write again $\Pi^{(\eta_0)}_{\widehat{S}}$, $\Sigma^{(\beta,\alpha)}_{\widehat{S}}$ and $\Sigma_{\widehat{S}}$ for the divisors given by the inclusions $\varpi^{(\eta_0)}_{\widehat{S}} \subset \omega^{\eta_0}_{\widehat{S}/\widehat{C}}$, $\mathcal{B}^{(\beta,\alpha)}_{\widehat{S}} \subset \omega^{\beta}_{\widehat{S}/\widehat{C}} \otimes \mathcal{M}^{\alpha}_{\widehat{S}}$ and $\mathcal{B}_{\widehat{S}} \subset \omega^{\beta_0}_{\widehat{S}/\widehat{C}} \otimes \mathcal{M}^{\alpha_0}_{\widehat{S}}$.

As in 9.5 one defines the different products, models, sheaves and divisors, with $\hat{g} : \widehat{Z} \to \widehat{Y}$ replaced by $\hat{h} : \widehat{S} \to \widehat{C}$. In particular we have again the divisor

$$\Delta_{\widehat{S}^r} = b \cdot \big(\Gamma_{\widehat{S}^r} + \Sigma^{(\eta,\gamma)}_{\widehat{S}^r}\big) + \Sigma_{\widehat{S}^r},$$

on the r-fold fiber product $\hat{h}^r : \widehat{S}^r \to \widehat{C}$, and we define

$$\begin{aligned}\mathcal{G}^{(\beta+\frac{\eta}{\ell},\alpha+\frac{\gamma}{\ell})}_{\widehat{C}} &= \hat{h}^r_*\Big(\omega^{\beta+\frac{\eta}{\ell}}_{\widehat{S}^r/\widehat{C}} \otimes \mathcal{M}^{\alpha+\frac{\gamma}{\ell}}_{\widehat{S}^r} \otimes \mathcal{J}\Big(-\frac{1}{N}\cdot\Delta_{\widehat{S}^r}\Big)\Big)\\ &= \hat{h}^r_*\Big(\omega^{\beta+\frac{\eta}{\ell}}_{\widehat{S}^r/\widehat{C}} \otimes \mathcal{M}^{\alpha+\frac{\gamma}{\ell}}_{\widehat{S}^r} \otimes \mathcal{J}\Big(-\frac{1}{e\cdot\ell}\cdot\big(\Gamma_{\widehat{S}^r} + \Sigma^{(\eta,\gamma)}_{\widehat{S}^r}\big) - \frac{1}{N}\cdot\Sigma_{\widehat{S}^r}\Big)\Big),\end{aligned}$$

where $N = b\cdot e\cdot\ell$ and $\Gamma_{\widehat{S}^r}$ is the zero divisor induced by the natural inclusion

$$\bigotimes_{i=1}^{s} \det\big(\hat{h}_*\mathcal{M}^{\gamma_i}_{\widehat{S}} \otimes \varpi^{\eta_i}_{\widehat{S}}\big)^{\otimes\frac{r}{r_i}} \xrightarrow{\Xi^{(r)}} \bigotimes^{r} \hat{h}_*\big(\omega^{\eta}_{\widehat{S}/\widehat{Y}} \otimes \mathcal{M}^{\gamma}_{\widehat{Z}}\big) = \hat{h}^r_*\mathcal{B}^{(\eta,\gamma)}_{\widehat{S}^r}.$$

Again we should have written $\mathcal{G}^{(\Xi^{(r)},\mathcal{E};\beta+\frac{\eta}{\ell},\alpha+\frac{\gamma}{\ell})}_{\widehat{C}}$ since the sheaf depends on $\Xi^{(r)}$ and $\mathcal{E}$, but we hope that the reader will not forget.

LEMMA 11.3. *Let $\theta : \widehat{C}_1 \to \widehat{C}$ be a finite, nonsingular covering, and let*

$$\begin{tikzcd} \widehat{S}^r_1 \arrow[r, "\theta'"] \arrow[d, "\hat{h}^r_1"'] & \widehat{S}^r \arrow[d, "\hat{h}^r"] \\ \widehat{C}_1 \arrow[r, "\theta"] & \widehat{C} \end{tikzcd}$$

be the induced morphism. Then:

(a) *If $\hat{h} : \widehat{S} \to \widehat{C}$ satisfies Assumption* 11.1 *then $\hat{h}_1 : \widehat{S}_1 \to \widehat{C}_1$ satisfies the same assumption.*

(b) $\mathcal{J}\big(-\frac{1}{N}\cdot\Delta_{\widehat{S}^r_1}\big)$ *is a subsheaf of* $\theta'^*\mathcal{J}\big(-\frac{1}{N}\cdot\Delta_{\widehat{S}^r}\big)$.

(c) *There is a natural inclusion*

$$\mathcal{G}^{(\beta+\frac{\eta}{\ell},\alpha+\frac{\gamma}{\ell})}_{\widehat{C}_1} \longrightarrow \theta^*\mathcal{G}^{(\beta+\frac{\eta}{\ell},\alpha+\frac{\gamma}{\ell})}_{\widehat{C}}.$$

Proof. As in the proof of Lemma 8.2, part (a) of 11.3 follows from Lemma 10.3(1) and (2).

For (b), note that $\mathrm{pr}_1 : \widehat{S}^r_1 \to \widehat{S}^r$ is flat; hence $\theta'^*\mathcal{J}\big(-\frac{1}{N}\cdot\Delta_{\widehat{S}^r}\big)$ has no torsion. Consider a desingularization $\tau : S \to \widehat{S}^r$ such that all fibers are normal crossing divisors, and such that $\tau^*\Gamma_{\widehat{S}^r}$ is a relative normal crossing divisor. Then $\tau^*(\Delta_{\widehat{S}^r})$ is a normal crossing divisor, as well.

Let $\tau_1 : S_1 \to \widehat{S}_1^r$ be the normalization of the pullback family,

$$\begin{tikzcd} S_1 \arrow[r, "\sigma"] \arrow[dr, "\tau_1"'] & S \times_{\widehat{S}^r} \widehat{S}_1^r \arrow[r, "\mathrm{pr}_1"] \arrow[d, "\mathrm{pr}_2"'] & S \arrow[d, "\tau"] \\ & \widehat{S}_1^r \arrow[r, "\theta'"] & \widehat{S}^r \end{tikzcd}$$

and $\theta'' = \mathrm{pr}_1 \circ \sigma$ the induced morphisms. By flat base change

$$\theta'^* \tau_* \omega_{S/\widehat{C}}\Big(-\Big[\frac{1}{N}\cdot \tau^*\Delta_{\widehat{S}^r}\Big]\Big) = \mathrm{pr}_{2*}\mathrm{pr}_1^*\Big(\omega_{S/\widehat{C}_1}\Big(-\Big[\frac{1}{N}\cdot \tau^*\Delta_{\widehat{S}^r})\Big]\Big)\Big).$$

Dualizing sheaves become smaller under normalizations, and this sheaf contains

$$\tau_{1*}\omega_{S_1/\widehat{C}_1} \otimes \theta''^* \mathcal{O}_S\Big(-\Big[\frac{1}{N}\cdot \tau^*\Delta_{\widehat{S}^r}\Big]\Big).$$

Since S_1 has at most rational Gorenstein singularities, this sheaf remains the same if we replace S_1 by a desingularization. Hence by abuse of notation we may assume that S_1 is nonsingular, that the fibers of $S_1 \to \widehat{C}_1$ are normal crossing divisors, and that $\theta''^*\tau^*\Gamma_{\widehat{S}^r}$ is a relative normal crossing divisor.

Obviously one has an inclusion

$$\mathcal{O}_{S_1}\Big(-\Big[\frac{1}{N}\cdot \theta''^*\tau^*\Delta_{\widehat{S}^r})\Big]\Big) \subset \theta''^*\mathcal{O}_S\Big(-\Big[\frac{1}{N}\cdot \tau^*\Delta_{\widehat{S}^r}\Big]\Big)$$

and hence

$$\tau_{1*}\omega_{S_1/\widehat{C}_1}\Big(-\Big[\frac{1}{N}\cdot \tau_1^*\Delta_{\widehat{S}_1^r}\Big]\Big) \subset \theta'^*\tau_*\omega_{S/\widehat{C}}\Big(-\Big[\frac{1}{N}\cdot \tau^*\Delta_{\widehat{S}^r}\Big]\Big)$$

as claimed in (b). By flat base change (c) follows from (b). □

Let $\tau : S \to X \times_Y \widehat{C}$ be any desingularization of the main component, and let $h : S \to \widehat{C}$ denote the induced morphism. Recall that we assumed $\chi' : \widehat{C} \to Y$ to have a mild reduction. So we may choose $\mathcal{M}_S$ as the pullback of $\mathcal{L}$ in the (canonically) polarized case or by Lemma 8.11(a) in the saturated case. Blowing up, we may assume that for $(\nu, \mu) \in \widetilde{I}$ the images $\mathcal{B}_S^{(\nu,\mu)}$ of the evaluation maps are invertible, in particular the image $\varpi_S^{(\eta_0)}$ of $h^*h_*\omega^{\eta_0}_{S/\widehat{C}} \to \omega^{\eta_0}_{S/\widehat{C}}$. We write $h^{(r)} : S^{(r)} \to \widehat{C}$ for the family obtained by desingularizing the r-fold product $S^r = S \times_{\widehat{C}} \cdots \times_{\widehat{C}} S$, where again we assume that $\varpi^{(\eta_0)}_{S^{(r)}}$ is invertible.

As above, or in Section 9 one chooses the sheaf $\mathcal{M}_{S^r}$ as the exterior tensor product. $\mathcal{M}_{S^{(r)}}$ will denotes its pullback to $S^{(r)}$. Since $\chi' : \widehat{C} \to Y$ has a mild reduction, 5.5 implies that one has again the inclusions

$$h^{(r)*}\bigotimes_{i=1}^{s} \det\big(\hat{g}_*\mathcal{M}_{\widehat{S}}^{\gamma_i} \otimes \varpi_{\widehat{S}}^{\eta_i}\big)^{\otimes \frac{r}{r_i}} \longrightarrow \mathcal{B}^{(\eta,\gamma)}_{S^{(r)}}$$

with zero locus $\Gamma_{S^{(r)}}$. Writing $S_0^{(r)} = h^{(r)^{-1}}(\chi'^{-1}(Y_0))$ for the smooth part of $h^{(r)}$ one obtains by 9.11:

LEMMA 11.4.

$$\mathcal{G}_{\widehat{C}}^{(\beta+\frac{\eta}{\ell},\alpha+\frac{\gamma}{\ell})} = h_*^{(r)}\Big(\omega_{S^{(r)}/\widehat{C}}^{\beta+\frac{\eta}{\ell}} \otimes \mathcal{M}_{S^{(r)}}^{\alpha+\frac{\gamma}{\ell}} \otimes \mathcal{J}\Big(-\frac{1}{e\cdot\ell}\cdot\big(\Gamma_{S^{(r)}}+\Sigma_{S^{(r)}}^{(\eta,\gamma)}\big)-\frac{1}{N}\cdot\Sigma_{S^{(r)}}\Big)\Big),$$

and

$$\mathcal{J}\Big(-\frac{1}{e\cdot\ell}\cdot\big(\Gamma_{S^{(r)}}+\Sigma_{S^{(r)}}^{(\eta,\gamma)}\big)-\frac{1}{N}\cdot\Sigma_{S^{(r)}}\Big)\Big|_{S_0^{(r)}} = \mathcal{O}_{S_0^{(r)}}.$$

In particular the inclusion

$$\mathcal{G}_{\widehat{C}}^{(\beta+\frac{\eta}{\ell},\alpha+\frac{\gamma}{\ell})} \subset \bigotimes^r \mathcal{F}_{\widehat{C}}^{(\beta+\frac{\eta}{\ell},\alpha+\frac{\gamma}{\ell})}$$

is an isomorphism over $\chi'^{-1}(Y_0)$.

Definition 11.5. The mild reduction $(\hat{h}:\widehat{S}\to\widehat{C},\mathcal{M}_{\widehat{S}})$ is *exhausting* (or *exhausting for* $(\Xi^{(r)},\mathcal{E};\beta+\frac{\eta}{\ell},\alpha+\frac{\gamma}{\ell})$) if the properties (1) and (2) in 11.1 hold true and if:

(3) For all finite coverings of nonsingular curves $\widehat{C}_1\to\widehat{C}$ the inclusion

$$\mathcal{G}_{\widehat{C}_1}^{(\beta+\frac{\eta}{\ell},\alpha+\frac{\gamma}{\ell})} \longrightarrow \theta^*\mathcal{G}_{\widehat{C}}^{(\beta+\frac{\eta}{\ell},\alpha+\frac{\gamma}{\ell})}$$

in 11.3(c) is an isomorphism.

Lemmas 8.2 and 9.6 imply that given $\chi':\widehat{C}\to Y$ one can always find a finite covering $\widehat{C}_1\to\widehat{C}$ and a mild reduction for the induced morphism $\widehat{C}_1\to Y$ which is exhausting. Repeating the argument used to prove 10.4 one obtains in addition:

LEMMA 11.6. *There exists in* 9.6 *an open dense subscheme* $Y_g\subset Y_0$ *such that for all morphisms*

$$\chi':\widehat{C}\xrightarrow{\pi'}\widehat{Y}\xrightarrow{\varphi}Y$$

from a nonsingular curve $\widehat{C}$, *with* $\chi'^{-1}(Y_g)$ *dense*, χ' *admits a mild exhausting reduction* $(\hat{h}:\widehat{S}\to\widehat{C},\mathcal{M}_{\widehat{S}})$. *Moreover*

$$\mathcal{G}_{\widehat{C}}^{(\beta+\frac{\eta}{\ell},\alpha+\frac{\gamma}{\ell})} = \pi'^*\mathcal{G}_{\widehat{Y}}^{(\beta+\frac{\eta}{\ell},\alpha+\frac{\gamma}{\ell})}.$$

PROPOSITION 11.7. *Consider in* 9.6 *a morphism* $\pi':\widehat{C}\to\widehat{Y}$ *from a nonsingular curve* $\widehat{C}$ *with* $\pi'^{-1}(\widehat{Y}_0)\neq\varnothing$.

If $\chi'=\varphi\circ\pi'$ *admits a mild exhausting reduction* $(\hat{h}:\widehat{S}\to\widehat{C},\mathcal{M}_{\widehat{S}})$, *then*

$$\mathcal{G}_{\widehat{C}}^{(\beta+\frac{\eta}{\ell},\alpha+\frac{\gamma}{\ell})} = \pi'^*\mathcal{G}_{\widehat{Y}}^{(\beta+\frac{\eta}{\ell},\alpha+\frac{\gamma}{\ell})}.$$

Proof. By 10.5

$$\mathcal{F}_{\widehat{C}}^{(\beta+\frac{\eta}{\ell},\alpha+\frac{\gamma}{\ell})} = \pi'^*\mathcal{F}_{\widehat{Y}}^{(\beta+\frac{\eta}{\ell},\alpha+\frac{\gamma}{\ell})},$$

and both sheaves remain unchanged if one replaces $\widehat{C}$ by some finite covering or $\widehat{Y}$ by some alteration. The same holds for the subsheaves $\mathcal{G}_{\widehat{C}}^{(\beta+\frac{\eta}{\ell},\alpha+\frac{\gamma}{\ell})}$ and

$\pi'^{*}\mathcal{G}_{\widehat{Y}}^{(\beta+\frac{\eta}{\ell},\alpha+\frac{\gamma}{\ell})}$. Hence they coincide, if and only if they coincide on some C' finite over $\widehat{C}$.

By assumption the multiplier ideal $\mathcal{J}\big(-\frac{1}{N}\cdot\Delta_{\widehat{Z}}\big)$ is compatible with pullback, base change, and products for all alterations. In particular, $\mathcal{J}\big(-\frac{1}{N}\cdot\Delta_{\widehat{Z}}\big)$ is flat over $\widehat{Y}$.

We are allowed to replace $\widehat{Y}$ by an alteration or by an open neighborhood of the image of C, hence by a local alteration for $\widehat{C}$. So by Proposition 7.8 we may assume that π' is an embedding, that $\hat{f}$ is flat and that $S=\hat{f}^{-1}(\widehat{C})$ is nonsingular and semistable over $\widehat{C}$. By abuse of notation, we will allow $\widehat{X}$ to be normal with rational Gorenstein singularities. By Lemma 7.6 this holds for the total space of pullbacks under local alterations for $\widehat{C}$, and for the fiber products. So we will work with the condition that $\hat{f}:\widehat{X}\to\widehat{Y}$ is a weak semistable reduction for $\widehat{C}$, a condition which is compatible with pullbacks and products. In particular S^r is normal with at most rational Gorenstein singularities and $h^r: S^r\to\widehat{C}$ has reduced fibers.

As stated in 8.8 one is allowed to replace the mild family $\hat{g}^r:\widehat{Z}^r\to\widehat{Y}$ by some mild model, dominating the flat part of the weak semistable reduction $\hat{f}^r:\widehat{X}^r\to\widehat{Y}$. Here we might lose the compatibility of $\mathcal{J}\big(-\frac{1}{N}\cdot\Delta_{\widehat{Z}}\big)$ with pullback, base change, and products for all alterations. Theorem 6.5 allows us to repair this defect, by replacing $\widehat{Y}$ by some larger local alteration.

The morphism $\hat{f}$ is smooth over $\widehat{Y}_0$, and $\mathcal{J}\big(-\frac{1}{N}\cdot\Delta_{\widehat{X}_1^r}\big)\big|_{S_1^r}$ is trivial over $\widehat{X}_0$. So we may apply Proposition 7.11. □

12. Uniform mild reduction and the extension theorem

Constructing the locally free sheaves $\mathcal{F}_{\widehat{Y}}^{(\nu,\mu)}$ and $\mathcal{G}_{\widehat{Y}}^{(\beta+\frac{\eta}{\ell},\alpha+\frac{\gamma}{\ell})}$ we used the Weak Semistable Reduction Theorem several times and we have no control on the alteration $\widehat{Y}$ of Y. As already indicated in 5.9 we will show how to use Gabber's Extension Theorem, recalled in 12.6, to obtain those sheaves on a finite covering of a projective compactification of Y_0. Again the latter will be denoted by Y and the covering will be written as $\phi: W\to Y$. In Proposition 12.8 the corresponding result is formulated for the sheaves $\mathcal{F}_{\bullet}^{(\nu,\mu)}$ whereas the extension to the direct images $\mathcal{G}_{\bullet}^{(\beta+\frac{\eta}{\ell},\alpha+\frac{\gamma}{\ell})}$ of multiplier sheaves is given in Variant 12.11. The Variant 12.10 handles the case of the determinants of $\mathcal{F}_{\bullet}^{(\nu,\mu)}$, starting from a covering of a coarse moduli scheme. As an application we will state and prove a generalization of Theorem 1 allowing arbitrary polarizations in Theorem 12.12 and a variant for saturated polarizations in 12.13.

We will need in all those cases that the trace map of $\phi: W\to Y$ splits, i.e., that $\mathcal{O}_Y$ is a direct factor of $\phi_*\mathcal{O}_W$. By [Vie95, Lemma 2.2] each finite surjective morphism $\widetilde{W}\to Y$ of reduced schemes, with $\widetilde{W}$ normal, factors through a finite covering $\phi: W\to Y$ with a splitting trace map and with $\widetilde{W}\to W$ birational. We will give here a different construction, starting with a fixed embedding $Y\hookrightarrow\mathbb{P}^N$, or more generally with any embedding $Y \hookrightarrow \mathbb{P}$ for $\mathbb{P}$ irreducible, normal and projective. The latter will have the advantage of allowing the gluing, needed to show the uniform mild extension over curves, required by the extension theorem.

LEMMA 12.1. *Let $\Psi : \mathbb{P}' \to \mathbb{P}$ be a finite normal covering and let $Y \subset \mathbb{P}$ be a closed subvariety. Then $\phi : W = \Psi^{-1}(Y) \to Y$ has a splitting trace map.*

Proof. Since $\mathbb{P}'$ is normal, $\mathcal{O}_{\mathbb{P}}$ is a direct factor of $\Psi_*\mathcal{O}_{\mathbb{P}'}$; hence there is a surjection $\Psi_*\mathcal{O}_{\mathbb{P}'} \to \mathcal{O}_Y$. Obviously this factors through $\phi_*\mathcal{O}_W \to \mathcal{O}_Y$. □

Definition 12.2. Let $\widetilde{\phi} : \widetilde{W} \to Y$ be a surjective finite map and $Y \to \mathbb{P}$ a closed embedding with $\mathbb{P}$ irreducible, normal and projective. Then $\Psi : \mathbb{P}' \to \mathbb{P}$ *dominates* $\widetilde{W}$, if $\mathbb{P}'$ is normal and irreducible, if Ψ is a finite covering and if the normalization $V \to \Psi^{-1}(Y) \to Y$ factors as $V \to \widetilde{W} \to Y$.

LEMMA 12.3. *Given $\widetilde{\phi} : \widetilde{W} \to Y$ and $\mathbb{P}$ as in Definition* 12.2, *there exists a finite normal covering $\Psi : \mathbb{P}' \to \mathbb{P}$ dominating $\widetilde{W}$. Moreover one may choose $\Psi : \mathbb{P}' \to \mathbb{P}$ to be a Galois covering.*

Proof. In order to prove Lemma 12.3 we may assume that $\widetilde{W}$ is normal. Let us first assume that Y and $\widetilde{W}$ are both irreducible and write K for the function field of Y. The function field of $\widetilde{W}$ can be written as $K[T]/f$ for a monic irreducible polynomial $f \in K[T]$. For some open subscheme $U \subset \mathbb{P}$ the polynomial f lies in $\mathcal{O}_Y(U \cap Y)[T]$ and lifts to a monic irreducible polynomial $F \in \mathcal{O}_{\mathbb{P}}(U)[T]$. Choose $\mathbb{P}'$ as the normalization of $\mathbb{P}$ in $L[X]/F$, where L denotes the function field of $\mathbb{P}$. The preimage of Y in $\mathbb{P}'$ is birational to $\widetilde{W}$ and since $\widetilde{W} \to Y$ is finite, the normalization V of $\Psi^{-1}(Y)$ dominates $\widetilde{W}$.

Next assume that Y is irreducible, and that $W_1, \ldots, W_s$ are the components of $\widetilde{W}$. We already know how to construct $\Psi_\iota : \mathbb{P}'_\iota \to \mathbb{P}$, dominating $W_\iota \to Y$. We choose $\Psi : \mathbb{P}' \to \mathbb{P}$ as the normalization of $\mathbb{P}_1 \times_{\mathbb{P}} \cdots \times_{\mathbb{P}} \mathbb{P}_s$. Then $\Psi^{-1}(Y)$ is finite over $W_1 \times_Y \cdots \times_Y W_s$, and we obtain the factorization of the normalization. This remains true if we replace $\mathbb{P}'$ by a larger covering, hence we can glue the construction for different components of Y in the same way.

Finally, if $\Psi : \mathbb{P}' \to \mathbb{P}$ is a finite covering, dominating $\widetilde{W}$, the normalization of $\mathbb{P}'$ in the Galois hull of the function field $\mathbb{C}(\mathbb{P}')$ over $\mathbb{C}(\mathbb{P})$ will again dominate $\widetilde{W}$. So we can add the property "Galois" as well. □

LEMMA 12.4. *Let $\Psi : \mathbb{P}' \to \mathbb{P}$ be a finite morphism between normal schemes, let $Y \subset \mathbb{P}$ be a closed subscheme and $Y_0 \subset Y$ an open set. Let $\overline{W}$ be a modification of $W = \Psi^{-1}(Y)$ with centers outside $W_0 = \Psi^{-1}(Y_0)$. Then there exist normal modifications $\widehat{\mathbb{P}} \to \mathbb{P}$ and $\widehat{\mathbb{P}}' \to \mathbb{P}'$ with centers in $Y \setminus Y_0$ and $W \setminus W_0$, respectively, such that the induced rational map $\Psi_1 : \widehat{\mathbb{P}}' \to \widehat{\mathbb{P}}$ is a finite morphism and such that the proper transform of W is $\overline{W}$.*

Proof. It is sufficient to consider irreducible varieties $\mathbb{P}$ and $\mathbb{P}'$. Assume first that $\mathbb{P}'$ is Galois over $\mathbb{P}$, say with Galois group Γ. One can extend the modification $\overline{W} \to W$ to a modification $\mathbb{P}'_{\mathrm{id}} \to \mathbb{P}'$ by blowing up an ideal $\mathcal{J}$, with $\mathcal{O}_{\mathbb{P}'}/\mathcal{J}$ supported in $W \setminus W_0$. Blowing up the conjugates of $\mathcal{J}$ under $\sigma \in \Gamma$ we see that the action of $\sigma \in \Gamma$, given by $\sigma : \mathbb{P}' \to \mathbb{P}'$, extends to a morphism $\sigma : \mathbb{P}'_{\mathrm{id}} \to \mathbb{P}'_{\sigma}$. So Γ acts on the fiber product $\bigtimes_{\mathbb{P}} \mathbb{P}'_{\sigma}$, taken over all $\sigma \in \Gamma$.

Obviously $\bigtimes_{\mathbb{P}} \mathbb{P}'_{\sigma}$ contains an open dense subset of $\mathbb{P}'$, embedded diagonally, and we choose $\widehat{\mathbb{P}}'$ to be the normalization of its closure. The projection to $\mathbb{P}'_{\mathrm{id}}$ induces the morphism $\widehat{\mathbb{P}}' \to \mathbb{P}'$. The group Γ acts on $\widehat{\mathbb{P}}'$, and we can choose the quotient for $\widehat{\mathbb{P}}$.

If $\mathbb{P}'$ is not Galois, we replace it by its normalization $\mathbb{P}''$ in the Galois hull of the function field extension for $\mathbb{P}' \to \mathbb{P}$. So $\mathbb{P}'$ is the quotient of $\mathbb{P}''$ by some subgroup $\Gamma' \subset \Gamma$. Having constructed $\widehat{\mathbb{P}}''$, we choose $\widehat{\mathbb{P}}' = \widehat{\mathbb{P}}''/\Gamma'$. □

Let us recall Gabber's Extension Theorem. We start with the following set-up.

Set-up 12.5. Let $\mathbb{P}$ be a normal projective scheme, $\widetilde{Y} \subset \mathbb{P}$ a closed reduced subscheme, and let $\widetilde{Y}_0 \subset \widetilde{Y}$ be open and dense. Let $\Psi : \mathbb{P}' \to \mathbb{P}$ be a finite covering, with $\mathbb{P}'$ normal, and write $W = \Psi^{-1}(\widetilde{Y})$, $W_0 = \Psi^{-1}(\widetilde{Y}_0)$, $\widetilde{\phi} = \Psi|_W$ and $\widetilde{\phi}_0 = \Psi|_{W_0}$. Consider a modification $\xi_0 : \widehat{Y}_0 \to W_0$ with $\widehat{Y}_0$ nonsingular, and a projective manifold $\widehat{Y}$ containing $\widehat{Y}_0$ as an open dense subscheme.

Let $\mathcal{C}_{\widetilde{Y}_0}$ and $\mathcal{C}_{\widehat{Y}}$ be locally free sheaves on $\widetilde{Y}_0$ and $\widehat{Y}$ respectively, such that for $\mathcal{C}_{W_0} = \widetilde{\phi}_0^* \mathcal{C}_{\widetilde{Y}_0}$ one has:

(i) $\xi_0^* \mathcal{C}_{W_0} = \mathcal{C}_{\widehat{Y}}|_{\widehat{Y}_0}$.

(ii) For each morphism $\pi : C \to \mathbb{P}'$ from a nonsingular projective curve C with $C_0 = \pi^{-1}(W_0)$ dense in C, the sheaf $(\pi|_{C_0})^* \mathcal{C}_{W_0}$ extends to a locally free sheaf $\mathcal{C}_C$ such that:

  (a) If $\pi' : \widehat{C} \to \mathbb{P}'$ factors through $\iota : \widehat{C} \to C$ then $\mathcal{C}_{\widehat{C}} = \iota^* \mathcal{C}_C$.

  (b) If $\pi : C \to \mathbb{P}'$ lifts to a morphism $\chi : C \to \widehat{Y}$ then $\mathcal{C}_C = \chi^* \mathcal{C}_{\widehat{Y}}$.

THEOREM 12.6. *In* 12.5, *blowing up* $\mathbb{P}$ *with centers not meeting* $\widetilde{Y}_0$ *and replacing* $\mathbb{P}'$ *by the normalization of* $\mathbb{P}$ *in its function field, one finds an extension of* $\mathcal{C}_{W_0}$ *to a locally free sheaf* $\mathcal{C}_W$ *on* $W = \Psi^{-1}(\widetilde{Y})$ *such that for all commutative diagrams*

$$\begin{tikzcd} \widehat{Y}_0 \arrow[r, "\subset"] \arrow[d, "\xi_0"'] & \widehat{Y} & \Lambda \arrow[l, "\psi"'] \arrow[dl, "\rho"] \\ W_0 \arrow[r, "\subset"] & W & \end{tikzcd}$$

with ψ *either a dominant morphism, or a morphism from a nonsingular curve* Λ *with* $\psi^{-1}(W_0) \neq \varnothing$, *one has* $\psi^* \mathcal{C}_W = \rho^* \mathcal{C}_{\widehat{Y}}$.

Proof. This is more or less what is shown in [Vie95, Th. 5.1]. There we constructed a compactification $\overline{W}$ of W_0 and the sheaf $\mathcal{C}_{\overline{W}}$. Of course, we may replace $\overline{W}$ by a modification of W, and by Lemma 12.4 we can embed $\overline{W}$ in a modification of $\mathbb{P}'$, finite over a modification of $\mathbb{P}$. □

Applying Theorem 12.6 to prove Theorem 1 we will start with $\widetilde{Y}_0 = Y_0$ and with any compactification $\widetilde{Y}$. Theorem 12.6 will force us to choose for Y a modification of $\widetilde{Y}$ in the diagram (4.6.1). This one is a closed subscheme of $\mathbb{P}$. If we have to reapply Theorem 12.6, as will happen in the proof of Theorem 2(iv), we may have to replace Y by some modification. Since it is obtained by blowing up $\mathbb{P}$ we can choose W as the preimage in the corresponding normal modification of $\mathbb{P}'$.

The statement of 12.6 is compatible with further blowing ups of $\widehat{Y}$. So by abuse of notation, we may assume that there is a morphism $\widehat{Y} \to \widetilde{Y}$, as required in the diagram (4.6.1) in case $\widetilde{Y} = Y$. We will denote the induced morphisms by

(12.6.1)
$$\begin{tikzcd}
\widehat{Y}_0 \arrow[r, "\iota'"] \arrow[d, "\xi_0"'] & \widehat{Y} \arrow[d, "\xi"'] & \\
W_0 \arrow[r, "\iota"] \arrow[d, "\widetilde{\phi}_0"'] & W \arrow[r, "\subset"] \arrow[d, "\widetilde{\phi}"'] & \mathbb{P}' \arrow[d, "\Psi"] \\
\widetilde{Y}_0 \arrow[r, "\subset"] & \widetilde{Y} \arrow[r, "\subset"] & \mathbb{P}
\end{tikzcd}$$

and $\varphi = \widetilde{\phi} \circ \xi$. If $\widetilde{Y}_0 = Y_0$ we will drop all the $\widetilde{\ }$.

Addendum 12.7. (1) If we consider a finite set of sheaves $\mathcal{C}_\bullet$, we can choose the same compactification W for all of them. Assume for example that $\mathcal{C}_\bullet$ and $\mathcal{C}'_\bullet$ are two systems of locally free sheaves satisfying conditions (i) and (ii) in 12.6. Then one may choose W such that both locally free sheaves, $\mathcal{C}_W$ and $\mathcal{C}'_W$ exist, as well as the morphism ξ in (12.6.1).

(2) Let $\mathcal{R}$ be the sheaf of $\mathcal{O}_W$ algebras $\mathcal{R} = \xi_*\mathcal{O}_{\widehat{Y}} \cap \iota_*\mathcal{O}_{W_0}$. The scheme $\mathrm{Spec}(\mathcal{R})$ is finite and birational over W, the inclusion ι lifts to an open embedding of W_0 in $\mathrm{Spec}(\mathcal{R})$. Lemma 12.4 allows us to replace W by this covering, hence we may assume that $\xi_*\mathcal{O}_{\widehat{Y}} \cap \iota_*\mathcal{O}_{W_0} = \mathcal{O}_W$.

(3) If in (1) one has morphisms $\iota : \mathcal{C}'_{\widetilde{Y}_0} \to \mathcal{C}_{\widetilde{Y}_0}$ and $\iota' : \mathcal{C}'_{\widehat{Y}} \to \mathcal{C}_{\widehat{Y}}$, compatible with the pullback in 12.6(i), one has a natural map

$$\mathcal{C}'_W \longrightarrow \xi_*\xi^*\mathcal{C}'_W = \xi_*\mathcal{C}'_{\widehat{Y}} \xrightarrow{\iota'} \xi_*\mathcal{C}_{\widehat{Y}} = \mathcal{C}_W \otimes \xi_*\mathcal{O}_{\widehat{Y}}.$$

So $\mathcal{C}'_W$ maps to $\mathcal{C}_W \otimes \mathcal{R}$, for the coherent sheaf $\mathcal{R}$ considered in (2). Replacing W by $\mathrm{Spec}(\mathcal{R})$ one obtains $\iota'' : \mathcal{C}'_W \to \mathcal{C}_W$, and this morphism is compatible with all further pullbacks.

Let us state three slightly different applications of Theorem 12.6 which will be proved together after Variant 12.11.

PROPOSITION 12.8. *One may choose* Y, $\widehat{Y}$ *and* $\widehat{Z}$ *in* 8.5 *and* 8.7 (*or* 8.13 *in the saturated case*) *such that in addition to conditions* (*i*)–(*x*) *one has a diagram* (12.6.1) *with* $\widetilde{Y} = Y$ *and* $\widetilde{\phi} = \phi$ *such that*:

I. *Ψ is a finite covering, $\mathbb{P}$ and $\mathbb{P}'$ are normal and projective, $\Psi^{-1}(Y) = W$, and ξ is birational.*

II. *Let C be a smooth curve and $\chi : C \to Y$ a morphism. Assume that χ factors through $C \xrightarrow{\pi} W \xrightarrow{\phi} Y$, and that $C_0 = \chi^{-1}(Y_0)$ is dense in C. Then χ admits a mild reduction.*

III. *For $(\nu,\mu) \in I$ there exists a locally free sheaf $\mathcal{F}_W^{(\nu,\mu)}$ on W with $\xi^*\mathcal{F}_W^{(\nu,\mu)} = \mathcal{F}_{\hat{Y}}^{(\nu,\mu)}$, and such that $\mathcal{F}_W^{(\nu,\mu)}|_{W_0} = \phi_0^* f_*(\omega_{X/Y}^{\nu} \otimes \mathcal{L}^{\mu})$.*

IV. *For all curves considered in II one has $\pi^*\mathcal{F}_W^{(\nu,\mu)} = \mathcal{F}_C^{(\nu,\mu)}$.*

Assume for a moment, that a coarse moduli scheme M_h exists for families of polarized manifolds with Hilbert polynomial h, and that the family $f_0 : X_0 \to Y_0$ lies in $\mathfrak{M}_h(Y_0)$ for the corresponding moduli functor. Assume the induced morphism $Y_0 \to M_h$ is finite. Then we want to factor $\hat{Y}_0 \to \tilde{Y}_0 = M_h$ through some W_0, birational to $\hat{Y}_0$, and finite over M_h with a splitting trace map. In general the different direct image sheaves do not descend to the moduli schemes, just their determinants. So we only can expect that certain "natural" invertible sheaves descend to the compactification of the moduli scheme. In the canonically polarized case, those sheaves will be of the form $\det(\mathcal{F}_{\hat{Y}}^{(\nu)})$. If one allows arbitrary polarizations, one has to choose certain rigidifications.

Definition 12.9. Let ι and ι' be integers. We call the sheaf

$$\det\left(\mathcal{F}_\bullet^{(\nu,\mu)}\right)^{\iota} \otimes \det\left(\mathcal{F}_\bullet^{(\nu',\mu')}\right)^{\iota'}$$

a *rigidified determinant sheaf*, if $\iota \cdot \mu \cdot \mathrm{rk}(\mathcal{F}_\bullet^{(\nu,\mu)}) + \iota' \cdot \mu' \cdot \mathrm{rk}(\mathcal{F}_\bullet^{(\nu',\mu')}) = 0$.

Recall that for the moduli problem of polarized manifolds one does not distinguish between families

$$(f_0 : X_0 \to Y_0, \mathcal{L}) \quad \text{and} \quad (f_0 : X_0 \to Y_0, \mathcal{L} \otimes f_0^*\mathcal{N}),$$

where $\mathcal{N}$ is an invertible sheaf on $\hat{Y}$. Definition 12.9 is made in such a way, that the rigidified determinant sheaves are invariant under this relation. It follows from the construction of moduli schemes that some power of a rigidified determinant sheaf descends to M_h (see [Vie95, Prop. 7.9], for example).

VARIANT 12.10. *Assume that Y_0 is normal and that the family $f_0 : X_0 \to Y_0$ (or $(f_0 : X_0 \to Y_0, \mathcal{L}_0)$) induces a finite morphism $Y_0 \to M_h$. Then one can find for a compactification Y of Y_0, the schemes $\hat{Y}$ and $\hat{Z}$ in* 8.5 *and* 8.7 (*or* 8.13 *in the saturated case*), *such that in addition to conditions* (*i*)–(*x*) *one has for $\tilde{Y}_0 = M_h$ the diagram* (12.6.1) *and*:

I. *Ψ is a finite covering, $\mathbb{P}$ and $\mathbb{P}'$ are normal and projective, $\Psi^{-1}(\overline{M}_h) = W$, and ξ is birational.*

II. *Let C be a smooth curve and $\chi : C \to \overline{M_h}$ a morphism. Assume that χ factors through $C \xrightarrow{\pi} W \xrightarrow{\widetilde{\phi}} \overline{M}_h$, and that $C_0 = \chi^{-1}(M_h)$ is dense in C. Then the induced morphism $C \to Y$ admits a mild reduction.*

III. *For (ν, μ), $(\nu', \mu') \in I$ and ι, $\iota' \in \mathbb{Z}$ let $\det(\mathcal{F}_\bullet^{(\nu,\mu)})^\iota \otimes \det(\mathcal{F}_\bullet^{(\nu',\mu')})^{\iota'}$ be a rigidified determinant. Then there exists some $p \gg 1$ and an invertible sheaf $\mathcal{C}_{\overline{M}_h}$ on $\overline{M}_h$ with*

$$\mathcal{C}_{\hat{Y}} := \Big(\det \big(\mathcal{F}_{\hat{Y}}^{(\nu,\mu)}\big)^\iota \otimes \det \big(\mathcal{F}_{\hat{Y}}^{(\nu',\mu')}\big)^{\iota'} \Big)^p = \xi^* \widetilde{\phi}^* \mathcal{C}_{\overline{M}_h}.$$

IV. *Under the assumption made in* III, *for all curves as in* II

$$\mathcal{C}_C := \Big(\det \big(\mathcal{F}_C^{(\nu,\mu)}\big)^\iota \otimes \det \big(\mathcal{F}_C^{(\nu',\mu')}\big)^{\iota'} \Big)^p = \pi^* \widetilde{\phi}^* \mathcal{C}_{\overline{M}_h}.$$

VARIANT 12.11. *Assume again that $\widetilde{Y} = Y$ and that the assumptions made in* 8.5 *and* 8.7 (*or* 8.13 *in the saturated case*) *hold true*, *as well as those made in* 9.1.

Assume there exist for $(\nu, \mu) \in I$ locally free sheaves $\mathcal{F}_Y^{(\nu,\mu)}$ on Y whose pullback to $\hat{Y}$ coincides with $\mathcal{F}_{\hat{Y}}^{(\nu,\mu)}$ and whose restriction to Y_0 is $f_*(\omega_{X/Y}^\nu \otimes \mathcal{L}^\mu)$. Assume moreover, *that there are a locally free sheaf $\mathcal{E}_Y$ and a morphism $\mathcal{E}_Y \to \mathcal{F}_Y^{(\beta_0,\alpha_0)}$ satisfying Assumptions* 9.3 *or* 9.4.

Then, *replacing Y by a modification with center in $Y \setminus Y_0$*, *one can find $\hat{Y}$ and $\hat{Z}$ such that* 8.5, 8.7 *and* 9.6 *hold*, *and such that one has a diagram* (12.6.1) *with*:

I. *Ψ is a finite covering*, *$\mathbb{P}$ and $\mathbb{P}'$ are normal and projective*, *$\Psi^{-1}(Y) = W$*, *and ξ is birational.*

II. *Let C be a smooth curve and $\chi : C \to Y$ a morphism. Assume that χ factors through $C \xrightarrow{\pi} W \xrightarrow{\phi} Y$*, *and that $C_0 = \chi^{-1}(Y_0)$ is dense in C. Then χ admits a mild exhausting reduction for $(\Xi^{(r)}, \mathcal{E}; \beta + \frac{\eta}{\ell}, \alpha + \frac{\gamma}{\ell})$.*

III. *There exists a locally free sheaf $\mathcal{G}_W^{(\beta+\frac{\eta}{\ell},\alpha+\frac{\gamma}{\ell})}$ on W whose pullback to $\hat{Y}$ is the sheaf $\mathcal{G}_{\hat{Y}}^{(\beta+\frac{\eta}{\ell},\alpha+\frac{\gamma}{\ell})}$*, *defined in* 9.5. *One has an inclusion*

$$\mathcal{G}_W^{(\beta+\frac{\eta}{\ell},\alpha+\frac{\gamma}{\ell})} \xrightarrow{\subset} \bigotimes^r \mathcal{F}_W^{(\beta+\frac{\eta}{\ell},\alpha+\frac{\gamma}{\ell})},$$

and over W_0 both sheaves are isomorphic.

Proof of 12.8, 12.10 *and* 12.11. The verification of the properties I and II in each of the cases goes along the same line.

In 12.8 and 12.11 one starts with $Y = \widetilde{Y}$, where $X_0 \to Y_0$ extends to a flat morphism $f : X \to Y$, as required in Step I of 4.4 or in Variant 4.8. We choose an embedding $Y \to \mathbb{P} = \mathbb{P}^M$.

In 12.10 we start with an embedding $M_h \to \mathbb{P} = \mathbb{P}^M$. In order to be able to use induction on certain strata, we will allow $\widetilde{Y}_0$ to be a subscheme of M_h, and we choose $\widetilde{Y}$ as the closure of $\widetilde{Y}$ in $\mathbb{P}$. Correspondingly we will replace Y_0 by the preimage of $\widetilde{Y}$, and Y will be a compactification Y_0 such that $\tau_0 : Y_0 \to \widetilde{Y}_0$ extends to a morphism $\tau : Y \to \widetilde{Y}$, and such that $f_0 : X_0 \to Y_0$ extends to a flat projective morphism $f : X \to Y$.

In all cases our starting point are morphisms

$$Y \xrightarrow{\tau} \widetilde{Y} \xrightarrow{\subset} \mathbb{P} \quad \text{and} \quad f : X \longrightarrow Y,$$

and $f_0 : X_0 \to Y_0$ is the smooth part of f. So in 12.8 and 12.10 the data $\widetilde{I}$ and I allow us (by 8.5, 8.7 or 8.13) to choose a diagram as in (4.6.1):

$$\begin{tikzcd}
 & X \arrow[dr, "f"'] & \widehat{Z} \arrow[l, "\hat{\varphi}"'] \arrow[dr, "\hat{g}"'] & Z \arrow[l, "\hat{\delta}"'] \arrow[r, "\delta"] \arrow[d, "g"] & \widehat{X} \arrow[r, "\rho"] \arrow[dl, "\hat{f}"] & X \arrow[dl, "f"] & \\
\mathbb{P} & \widetilde{Y} \arrow[l, "\subset"'] & Y \arrow[l, "\tau"'] & \widehat{Y} \arrow[l, "\varphi"] \arrow[r, "\varphi"'] & Y \arrow[r, "\tau"] & \widetilde{Y} \arrow[r, "\subset"] & \mathbb{P}.
\end{tikzcd}$$

In Variant 12.11 we use Lemma 9.6 to get the same diagram, starting with the data collected in 9.1–9.4. Recall that all those conditions are compatible with pullback under alterations of $\widehat{Y}$.

Consider the Stein factorization $\widetilde{\eta} : \widetilde{V} \to \widetilde{Y}$ of $\varphi : \widehat{Y} \to \widetilde{Y}$. By 12.3 we can find an irreducible normal covering $\Psi : \mathbb{P}' \to \mathbb{P}$ dominating $\widetilde{V} \to \widetilde{Y}$. So the normalization of $W := \Psi^{-1}(\widetilde{Y})$ dominates $\widetilde{V}$. The compatibility of our constructions with further pullback, allows us to assume that $\widehat{Y}$ is a nonsingular modification of V, and we obtain all the morphisms in (12.6.1), except that they are not yet coming by an application of the extension theorem. In the course of the verification of II we will have to replace $\mathbb{P}'$ by finite coverings, and by some modification with center in $W \setminus W_0$. The Lemma 12.4 allows us to replace Y by a modification with center in $Y \setminus Y_0$, and to keep the conditions in I.

By Lemma 10.4 there exists an open dense subscheme $Y_g \subset Y_0$ such that $\chi : C \to Y$ admits a mild reduction if $\chi^{-1}(Y_g) \neq \varnothing$ and if χ lifts to a morphism $C \to \widehat{Y}$. In 12.11 we assumed that the sheaf $\mathcal{E}_{\widehat{Y}}$ is the pullback of a sheaf on Y. So as remarked in 10.2 this allows us to apply 11.6, and the same holds for mild exhausting reductions.

Replacing Y_g by some open dense subscheme, we may assume in addition that:

(1) In 12.10 one has $Y_g = \tau^{-1}(\widetilde{Y}_g)$ for some open dense subscheme $\widetilde{Y}_g$ of $\widetilde{Y}$.

(2) $W_g = \widetilde{\phi}^{-1}(\widetilde{Y}_g)$ is normal and the restriction of ξ to $\widehat{Y}_g = \xi^{-1}(W_g)$ is an isomorphism $\widehat{Y}_g \to W_g$.

(2) implies that a morphism $\pi : C \to W$ from a nonsingular projective curve C whose image meets W_g lifts to a morphism $C \to \widehat{Y}$. So the conditions II in 12.8, 12.10 or 12.11 hold for morphisms $\pi : C \to W$ with $\pi^{-1}(W_g)$ dense in C.

The open set W_g will be the large stratum, and next we will construct a similar open subset of the complement. Let us write $\widetilde{Y}_b$ for the closure of $\widetilde{Y}_{0b} = \widetilde{Y}_0 \setminus \widetilde{Y}_g$ in $\widetilde{Y}$. Correspondingly Y_b will be equal to $\widetilde{Y}_b$ in 12.8 and 12.11, and equal to $\tau^{-1}(\widetilde{Y}_b)$ in 12.10.

The dimension of $\widetilde{Y}_b$ is strictly smaller than $\dim(\widetilde{Y})$. By induction on the dimension we assume that we have found a nonsingular alteration $\widehat{Y}_b \to Y_b$ and the covering $\Psi_b : \mathbb{P}'_b \to \mathbb{P}$, satisfying conditions (i)–(v) in 8.5 and (vi)–(x) in 8.7 (or 8.13) and the assumptions made in 9.1 and 9.3 or 9.4, such that the conditions II hold for $\widetilde{Y}_b$ instead of $\widetilde{Y}$.

We choose $\mathbb{P}'_1$ to be one of the irreducible components of the normalization of $\mathbb{P}' \times_{\mathbb{P}} \mathbb{P}'^{(b)}$. Writing $\Psi_1 : \mathbb{P}'_1 \to \mathbb{P}$ for the induced map, we choose $\widehat{Y}_1$ to be a desingularization of $W_1 = \Psi_1^{-1}(\widetilde{Y})$, which maps to $\widehat{Y}$. So all the conditions needed in 8.5, 8.7, 8.13 and stated in 9.1, 9.3 and 9.4 remain true.

Let $\chi : C \to \widetilde{Y}$ be a morphism with $\chi^{-1}(\widetilde{Y}_0) \neq \varnothing$, and factoring through W_1. If $\chi^{-1}(\widetilde{Y}_g) \neq \varnothing$, we are done. Otherwise $\chi(C_0)$ is contained in $\widetilde{Y}_b$. By the choice of $\mathbb{P}'_1$, the morphism χ factors through $\mathbb{P}'_b$, hence $C \to Y$ allows again a mild (exhausting) reduction.

So in each of the three cases considered, we found a nonsingular alteration satisfying I and II. Dropping as usual the lower index $_1$ we will use the notation from the diagram (12.6.1).

Conditions III and IV will follow from the Extension Theorem, so again we will have to modify all the morphisms in (12.6.1). In order to apply it, we have to define the sheaves $\mathcal{C}_C$ in the Set-up 12.5 and to verify properties (i) and (ii) stated there. This will be done in each case separately.

Let us start with 12.8. Recall that by 8.5 and 8.7 on $\widetilde{Y}_0 = Y_0$ the sheaves $\mathcal{C}_{Y_0} = f_{0*}(\omega^{\nu}_{X_0/Y_0} \otimes \mathcal{L}_0^{\mu})$ are locally free and compatible with base change for $(\nu, \mu) \in I$. Correspondingly we choose $\mathcal{C}_{\widehat{Y}} = \mathcal{F}^{(\nu,\mu)}_{\widehat{Y}}$, and $\mathcal{C}_C = \mathcal{F}^{(\nu,\mu)}_C$, as defined in 10.2. Then (i) is obviously true, and (ii) follows from II, by Proposition 10.5.

The same argument works for 12.10. However here we have to choose for $\mathcal{C}_{Y_0}$ the rigidified determinant

$$\det\Big(f_{0*}\big(\omega^{\nu}_{X_0/Y_0} \otimes \mathcal{L}_0^{\mu}\big)\Big)^{\iota} \otimes \det\Big(f_{0*}\big(\omega^{\nu'}_{X_0/Y_0} \otimes \mathcal{L}_0^{\mu'}\big)^{\iota'}\Big)^{p}.$$

As mentioned already, by [Vie95, Prop. 7.9] for p sufficiently large, this sheaf is the pullback of an invertible sheaf $\mathcal{C}_{M_h}$. Then for $\mathcal{C}_{\widehat{Y}}$ and $\mathcal{C}_C$, as defined in 12.10 III and IV, property (i) follows from the compatibility of $\mathcal{C}_{Y_0}$ with pullback, and (ii) follows again from II, by Proposition 10.5. So the Extension Theorem 12.6 gives the existence of the sheaf $\mathcal{C}_W$. It remains to show, that $\mathcal{C}_W$, or some tensor power of $\mathcal{C}_W$ descends to $\overline{M}_h$.

To this aim, we can replace $\mathbb{P}'$ by a finite covering, and assume that $\mathbb{C}(\mathbb{P}')$ is Galois over $\mathbb{C}(\mathbb{P})$. So the Galois group Γ acts on W and the quotient is $\overline{M}_h$. For $\sigma \in \Gamma$ one has $\sigma^*\mathcal{C}_W = \mathcal{C}_W$. In fact, this holds true on the open dense subscheme

W_0, and on every curve mapping to W and meeting W_0. Replacing p by some multiple, one finds the sheaf $\mathcal{C}_{\overline{M}_h}$.

In 12.11 we start with

$$\mathcal{C}_{Y_0} = \bigotimes^r f_{0*}\Big(\omega_{X_0/Y_0}^{\beta+\frac{\eta}{\ell}} \otimes \mathcal{L}_0^{\alpha+\frac{\gamma}{\ell}}\Big)$$

and with $\mathcal{C}_{\widehat{Y}} = \mathcal{G}_{\widehat{Y}}^{(\beta+\frac{\eta}{\ell},\alpha+\frac{\gamma}{\ell})}$. Again, those sheaves are compatible with pullback, and (i) follows from Lemma 9.11. Since $\mathcal{E}_{\widehat{Y}}$ is the pullback of a sheaf on Y, we are allowed to use the constructions in Section 11. We choose for $\mathcal{C}_C$ the sheaf $\mathcal{G}_C^{(\beta+\frac{\eta}{\ell},\alpha+\frac{\gamma}{\ell})}$, defined just before Lemma 11.3. The condition (ii) in the set-up 12.5 follows again from II and from 11.7. So the Extension Theorem gives the existence of the locally free sheaf $\mathcal{G}_W^{(\beta+\frac{\eta}{\ell},\alpha+\frac{\gamma}{\ell})}$ and as remarked in 12.7(5) we can assume that it is a subsheaf of $\bigotimes^r \mathcal{F}_W^{(\beta+\frac{\eta}{\ell},\alpha+\frac{\gamma}{\ell})}$. By 9.11 the pullback of both to $\widehat{Y}_0$ are equal, hence their restrictions to W_0 as well. $\square$

Let us formulate what we obtained up to now for the sheaves $\mathcal{F}_\bullet^{(\nu,\mu)}$.

THEOREM 12.12. *Let $f : X \to Y$ be a flat projective morphism of quasi-projective reduced schemes, and let $\mathcal{L}$ be an invertible sheaf on X. Let $Y_0 \subset Y$ be a dense open set, with $f_0 : X_0 = f^{-1}(Y_0) \to Y_0$ smooth. Assume that ω_{X_0/Y_0} and $\mathcal{L}_0 = \mathcal{L}|_{X_0}$ are both f_0 semiample.*

Let I be a finite set of tuples (ν, μ) of natural numbers. Assume that for all $(0, \mu') \in I$ the direct image $f_{0*}\mathcal{L}_0^{\mu'}$ is locally free and compatible with arbitrary base change. Then, replacing Y by a modification with centers in $Y \setminus Y_0$, there exists a finite covering $\phi : W \to Y$ with a splitting trace map and for $(\nu, \mu) \in I$ a locally free sheaf $\mathcal{F}_W^{(\nu,\mu)}$ on W with:

(i) *For $W_0 = \phi^{-1}(Y_0)$ and $\phi_0 = \phi|_{W_0}$ one has $\phi_0^* f_{0*}(\omega_{X_0/Y_0}^\nu \otimes \mathcal{L}_0^\mu) = \mathcal{F}_W^{(\nu,\mu)}|_{W_0}$.*

(ii) *Let $\theta : T \to W$ be a morphism from a nonsingular variety T. Assume that either $T \to W$ is dominant or that T is a curve and $T_0 = \theta^{-1}(W_0)$ dense in T. For some $r \geq 1$ let $X_T^{(r)}$ be a desingularization of*

$$X_T^r = (X \times_Y \cdots \times_Y X) \times_Y T.$$

Let $\hat{\varphi}_T : X_T^{(r)} \to X_T^r$ and $f_T^{(r)} : X_T^{(r)} \to T$ be the induced morphisms and

$$\mathcal{M} = \hat{\varphi}_T^*(\mathrm{pr}_1^*\mathcal{L} \otimes \cdots \otimes \mathrm{pr}_r^*\mathcal{L}).$$

Then $f_{T*}^{(r)}(\omega_{X_T^{(r)}/T}^\nu \otimes \mathcal{M}^\mu) = \bigotimes^r \theta^*\mathcal{F}_W^{(\nu,\mu)}$.

For $\mu = 0$ one obtains, in particular, parts (i) and (ii) of Theorem 1, and we have seen in Section 2 that those two conditions imply (iii) in Theorem 2, saying that the sheaf $\mathcal{F}_W^{(\nu,0)} = \mathcal{F}_W^{(\nu)}$ is nef. So it remains to prove the "weak stability" condition (iv). This will be done in Section 13. Let us formulate first a variant of the last theorem allowing saturated extensions of polarizations.

VARIANT 12.13. *In* 12.12 *fix some* η_0 *such that the evaluation map for* $\omega^{\eta_0}_{X_0/Y_0}$ *is surjective, and some* $\kappa > 0$, *with* $(\eta_0, 0),\ (0, \kappa) \in I$. *Then there exists a finite covering* $\phi : W \to Y$ *with a splitting trace map, and for* $(\nu, \mu) \in I$ *a locally free sheaf* $\mathcal{F}^{(\nu,\mu)}_W$ *on* W *with property* (i) *and*

(ii) *Let* $\theta : T \to W$ *be a morphism from a nonsingular variety* T. *Assume that either* $T \to W$ *is dominant or that* T *is a curve and* $T_0 = \theta^{-1}(W_0)$ *is dense in* T. *For some* $r \geq 1$ *let* $X^{(r)}_T$ *be a desingularization of* $(X \times_Y \cdots \times_Y X) \times_Y T$. *Let* $\hat{\varphi}_T : X^{(r)}_T \to X^r$ *and* $f^{(r)}_T : X^{(r)}_T \to T$ *be the induced morphisms. Assume that* $X^{(r)}_T$ *is chosen such that the image of the evaluation map for* $\omega^{\eta_0}_{X^{(r)}_T/T}$ *is invertible, hence equal to* $\omega^{\eta_0}_{X^{(r)}_T/T} \otimes \mathcal{O}_{X^{(r)}_T}(\Pi_{X^{(r)}_T})$ *for an effective Cartier divisor* $\Pi_{X^{(r)}_T}$. *Then for* $\mathcal{M} = \hat{\varphi}^*_T(\mathrm{pr}^*_1\mathcal{L} \otimes \cdots \otimes \mathrm{pr}^*_r\mathcal{L})$ *one has*

$$f^{(r)}_{T*}\Big(\omega^{\nu}_{X^{(r)}_T/T} \otimes \mathcal{M}^{\mu} \otimes \mathcal{O}_{X^{(r)}_T}\Big(* \cdot \Pi_{X^{(r)}_T}\Big)\Big) = \bigotimes^{r} \theta^* \mathcal{F}^{(\nu,\mu)}_W.$$

Proof of 12.12 *and* 12.13. Start with $\widehat{Y}$, $\widehat{Z}$ and $\widehat{X}$ according to 8.5 and 8.7 (or 8.13 in 12.13). Choose the compactification Y, and W using Proposition 12.8.

So there are locally free sheaves $\mathcal{F}^{(\nu)}_W$ (or $\mathcal{F}^{(\nu,\mu)}_W$), whose pullbacks under ξ are the sheaves $\mathcal{F}^{(\nu)}_{\widehat{Y}}$ (or $\mathcal{F}^{(\nu,\mu)}_{\widehat{Y}}$). It remains to verify condition (ii) in all cases.

Recall that $\widehat{X} \to \widehat{Y}$ has a mild model $\widehat{Z} \to \widehat{Y}$; hence $X^{(r)}_T \to \widehat{Y}$ has $\widehat{Z}^r \to \widehat{Y}$ as a mild model. If T dominates $\widehat{Y}$ property (ii) in 12.12 follows for $r = 1$ from 8.5 and 8.7, and for $r > 1$ by flat base change. In 12.13 the same argument works for a κ saturated extension $\mathcal{M}_{X^{(r)}_T}$, and one finds that

$$f^{(r)}_{T*}\Big(\omega^{\nu}_{X^{(r)}_T/T} \otimes \mathcal{M}^{\mu}_{X^{(r)}_T}\Big) = \bigotimes^{r} \theta^* \mathcal{F}^{(\nu,\mu)}_W.$$

In general there is some nonsingular modification $\theta' : T' \to T$ such that (ii) holds on T'. The sheaf $f^{(r)}_{T*}(\omega^{\nu}_{X^{(r)}_T/T} \otimes \mathcal{M}^{\mu})$ is independent of the desingularization $X^{(r)}_T$, and we may assume that $f^{(r)}_T$ factors through $h : X^{(r)}_T \to T'$. Then

$$h_*\Big(\omega^{\nu}_{X^{(r)}_T/T} \otimes \mathcal{M}^{\mu}\Big) = \bigotimes^{r} \theta'^* \theta^* \mathcal{F}^{(\nu,\mu)}_W \otimes \omega^{\nu}_{T'/T},$$

and the projection formula implies that

$$f^{(r)}_{T*}\Big(\omega^{\nu}_{X^{(r)}_T/T} \otimes \mathcal{M}^{\mu}\Big) = \bigotimes^{r} \theta^* \mathcal{F}^{(\nu,\mu)}_W \otimes \theta'_* \omega^{\nu}_{T'/T} = \bigotimes^{r} \theta^* \mathcal{F}^{(\nu,\mu)}_W,$$

as claimed in 12.12. In the situation considered in 12.13 the same equality holds with $\mathcal{M}$ replaced by the κ saturated extension $\mathcal{M}_{X^{(r)}_T}$. However both differ by some positive multiple of $\Pi_{X^{(r)}_T}$ and

$$f^{(r)}_{T*}\Big(\omega^{\nu}_{X^{(r)}_T/T}\otimes\mathcal{M}^{\mu}_{X^{(r)}_T}\Big)=f^{(r)}_{T*}\Big(\omega^{\nu}_{X^{(r)}_T/T}\otimes\mathcal{M}^{\mu}_{X^{(r)}_T}\otimes\mathcal{O}_{X^{(r)}_T}\big(*\cdot\Pi_{X^{(r)}_T}\big)\Big)$$
$$=f^{(r)}_{T*}\Big(\omega^{\nu}_{X^{(r)}_T/T}\otimes\mathcal{M}^{\mu}\otimes\mathcal{O}_{X^{(r)}_T}\big(*\cdot\Pi_{X^{(r)}_T}\big)\Big).$$

If T is a curve, then by Proposition 12.8 II we know that $T\to W\to Y$ admits a mild reduction and, by part IV, the pullback of $\mathcal{F}^{(\nu,\mu)}_W$ is the sheaf $\mathcal{F}^{(\nu,\mu)}_C$ defined in Section 10. So it is equal to $h_*(\omega^{\nu}_{S/T}\otimes\mathcal{M}^{\mu}_S)$ for a mild model $h:S\to T$ of the pullback family.

The r-fold fiber product $h^r:S^r\to T$ is again mild, and for the exterior tensor product $\mathcal{M}_{S^r}$ one has by flat base change $h^r_*(\omega^{\nu}_{S^r/T}\otimes\mathcal{M}^{\mu}_{S^r})=\bigotimes^r\theta^*\mathcal{F}^{(\nu,\mu)}_W$. So property (ii) in 12.12 or 12.13 for T a curve follows from 5.7. $\square$

13. **Positivity of direct images II**

Now we are all set to finish the proof of Theorem 2(iv), allowing W_0 to be singular, contrary to Variant 2.4. Lemma 1.9 allows us to replace W by a larger covering, and we will do so several times. We will also formulate and prove a generalization to the polarized case, which in particular will imply Lemma 3.2.

As explained in 2.5 we will construct to this aim a locally free subsheaf $\mathcal{G}$ of $\mathcal{F}^{(\nu,\mu)}_W$, isomorphic to $\mathcal{F}^{(\nu,\mu)}_W$ over W_0, whose pullback to $\widehat{Y}$ remains nef after tensoring with a "negative" invertible sheaf. The sheaf $\mathcal{G}$ will depend on the data defined in Section 9.

Set-up 13.1. We will specify in each case the tautological maps in 9.1, and the sheaf $\mathcal{E}_{\widehat{Y}}$ according to 9.3 or 9.4. We choose the sets $\widetilde{I}$ and I as in Set-up 9.2 and enlarge them such that 9.3 or 9.4 applies. By abuse of notation we will assume that the alteration $\widehat{Y}$ and W are chosen according to Theorem 12.12 and Variant 12.13, and moreover we will assume that Variant 12.11 applies, i.e., that the locally free subsheaf $\mathcal{G}^{(\beta+\frac{\eta}{\ell},\alpha+\frac{\gamma}{\ell})}_W$ of $\mathcal{F}^{(\beta+\frac{\eta}{\ell},\alpha+\frac{\gamma}{\ell})}_W$ exists.

In all situations the locally free sheaf $\mathcal{E}_{\widehat{Y}}$ in 9.3 or 9.4 will be the pullback of a locally free sheaf $\mathcal{E}_W$, and the invertible sheaf $\mathcal{V}$ in 9.1 will be the pullback of an invertible sheaf $\mathcal{V}_W$; hence the $r\cdot e\cdot\ell$-th root out of $\det(\mathcal{F}^{(\eta_1,\gamma_1)}_W)^{\frac{r}{r_1}}\otimes\cdots\otimes\det(\mathcal{F}^{(\eta_s,\gamma_s)}_W)^{\frac{r}{r_s}}$ will exist on W.

Note that all those conditions can be realized, after blowing up Y with centers in $Y\setminus Y_0$ for some finite covering $W\to Y$ with a splitting trace map. The conclusion stated in the sequel remain true over any model, where the different sheaves are defined on W, locally free and compatible with pullback.

PROPOSITION 13.2. *In* 13.1 *one has*:

(a) *If* $\mathcal{E}_W$ *is nef, the sheaf* $\mathcal{G}^{(\beta+\frac{\eta}{\ell},\alpha+\frac{\gamma}{\ell})}_W\otimes\mathcal{V}^{-r}_W$ *is nef and the sheaf* $\mathcal{F}^{(\beta+\frac{\eta}{\ell},\alpha+\frac{\gamma}{\ell})}_W\otimes\mathcal{V}^{-1}_W$ *is weakly positive over* W_0.

(b) *If for some invertible sheaf* $\mathcal{H}$ *on* W *the sheaf* $\mathcal{E}_W \otimes \mathcal{H}^{b\cdot e\cdot \ell}$ *is nef, the sheaf* $\mathcal{G}_W^{(\beta+\frac{\eta}{\ell},\alpha+\frac{\nu}{\ell})} \otimes \mathcal{H}^r \otimes \mathcal{V}_W^{-r}$ *is nef and the sheaf* $\mathcal{F}_W^{(\beta+\frac{\eta}{\ell},\alpha+\frac{\nu}{\ell})} \otimes \mathcal{H} \otimes \mathcal{V}_W^{-1}$ *is weakly positive over* W_0.

Proof. Since $\mathcal{H} = \mathcal{O}_W$ in (a) we will handle both cases at once. By 12.11 one has an inclusion

$$\mathcal{G}_W^{(\beta+\frac{\eta}{\ell},\alpha+\frac{\nu}{\ell})} \subset \bigotimes^r \mathcal{F}_W^{(\beta+\frac{\eta}{\ell},\alpha+\frac{\nu}{\ell})}$$

and both sheaves are isomorphic on W_0. Hence using the equivalence of (1) and (2) in Lemma 1.6, it is sufficient to verify that the first sheaf, tensorized by $\mathcal{H}^r \otimes \mathcal{V}_W^{-r}$ is nef. By Lemma 1.3 this follows if for $\widehat{\mathcal{H}} = \xi^*\mathcal{H}$ the sheaf

$$\mathcal{G}_{\widehat{Y}}^{(\beta+\frac{\eta}{\ell},\alpha+\frac{\nu}{\ell})} \otimes \widehat{\mathcal{H}}^r \otimes \mathcal{V}^{-r}$$

is nef. We work with the mild model and use the notation from Claim 9.7. There we verified that the sheaf $\mathcal{N}^N \otimes \hat{g}^{r*}\mathcal{V}^{-N\cdot r} \otimes \mathcal{O}_{\hat{Z}^r}(-\Delta_{\hat{Z}^r})$ is the image of $\hat{g}^{r*}\mathcal{E}_{\widehat{Y}}^{\otimes r}$, for $N = b\cdot e\cdot \ell$. So Lemma 2.3 implies for $\mathcal{N}$ replaced by $\mathcal{N} \otimes \hat{g}^{r*}\widehat{\mathcal{H}}^r$ that for a very ample sheaf $\mathcal{A}$ on $\widehat{Y}$ the sheaf

$$\begin{aligned}\omega_{\widehat{Y}} \otimes \mathcal{A}^{m+2} &\otimes \mathcal{G}_W^{(\beta+\frac{\eta}{\ell},\alpha+\frac{\nu}{\ell})} \otimes \mathcal{H}^r \otimes \mathcal{V}_W^{-r} \\ &= \omega_{\widehat{Y}} \otimes \mathcal{A}^{m+2} \otimes \hat{g}^r_*\Big(\omega_{\hat{Z}^r/\widehat{Y}} \otimes \mathcal{N} \otimes \mathcal{J}\Big(-\frac{1}{N}\cdot \Delta_{\hat{Z}^r}\Big)\Big) \otimes \widehat{\mathcal{H}}^r \otimes \mathcal{V}^{-r}\end{aligned}$$

is globally generated. This remains true for $r\cdot r'$ instead of r. Since

$$\mathcal{G}_W^{(\Xi^{(r\cdot r')},\mathcal{E};\beta+\frac{\eta}{\ell},\alpha+\frac{\nu}{\ell})} = \bigotimes^{r'} \mathcal{G}_W^{(\Xi^{(r)},\mathcal{E};\beta+\frac{\eta}{\ell},\alpha+\frac{\nu}{\ell})} = \bigotimes^{r'} \mathcal{G}_W^{(\beta+\frac{\eta}{\ell},\alpha+\frac{\nu}{\ell})}$$

Proposition 13.2 follows from the equivalence of (1) and (3) in Lemma 1.6. □

Proof of Theorem 2. Note that we already obtained Theorem 1 in Theorem 12.12, and we keep the choice of $\phi : W \to Y$ made there. The part (iii) of Theorem 2 has been verified in Section 2.

For $r_\iota = \dim(H^0(F,\omega_F^{\eta_\iota}))$ choose $\Xi = (\Xi_1,\ldots,\Xi_s)$ in Proposition 13.4 as the tuple of tautological maps

$$\Xi_\iota : \bigwedge^{r_\iota} H^0\big(F,\omega_F^{\eta_\iota}\big) \longrightarrow \bigotimes^{r_\iota} H^0\big(F,\omega_F^{\eta_\iota}\big).$$

For some η_0 the evaluation map for $\omega_{X_0/Y_0}^{\eta_0}$ is surjective. Replacing Ξ by $\xi,\ldots,\xi$ we may assume that η_0 divides $\eta = \eta_1 + \cdots + \eta_s$. We choose $\ell = \eta$, for r we choose some positive common multiple of $r_1,\ldots,r_s$, for e any integer larger that $\frac{1}{\ell}e(\omega_F^{\eta})$, and for b we choose any natural number with $b\cdot(\nu-2)$ divisible by η_0. We choose $\widetilde{I}$ and I such that the numerical conditions in 9.1 hold true.

Thus, $\beta = \nu - 1$, and $\beta_0 = b\cdot\beta\cdot e\cdot\ell + \eta\cdot b\cdot(e-1)$. As in 9.3 we assume that $\beta_0 \in \widetilde{I}$ and for $\mathcal{E}_{\widehat{Y}}$ we choose $\mathcal{F}_{\widehat{Y}}^{(\beta_0)}$; hence $\mathcal{E}_W = \mathcal{F}_W^{(\beta_0)}$ in 13.1.

Lemma 9.6 and Proposition 12.8 allow us to replace W by some larger covering with a splitting trace map, and to assume that the conditions in Set-up 13.1 hold. Doing so we are allowed to apply Proposition 13.2(a) and we obtain the weak positivity of

$$\Big(\bigotimes^{\alpha} \mathcal{F}_W^{(\nu)}\Big) \otimes \bigotimes_{\iota=1}^{s} \det\big(\mathcal{F}_W^{(\eta_\iota)}\big)^{-\frac{r}{r_\iota}}$$

over W_0 for some $\alpha > 0$. We know by part (iii) of Theorem 2 that the sheaves $\det(\mathcal{F}_W^{(\eta_i)})$ are all nef. Hence we can enlarge the a_ι and assume that $a_\iota \cdot r_\iota$ is independent of ι, hence that $\bigotimes_{\iota=1}^{s} \det(\mathcal{F}_W^{(\eta_\iota)})^{\frac{r}{r_\iota}}$ is ample with respect to W_0. □

Remark 13.3. If one wants to avoid using s copies of Ξ, one can also argue in the following way: $\Xi^{(r)}$ defines an embedding of some linear combination $\mathcal{N}$ of the sheaves $\det(\mathcal{F}_W^{(\eta_\iota)})$ in the sheaf $\bigotimes \mathcal{F}_W^{(\eta)}$, and it is easy to see that this inclusion locally splits. By part (iii) of Theorem 1 the second sheaf is nef. Then the quotient is locally free and nef, hence the determinant of $\bigotimes \mathcal{F}_W^{(\eta)}$ must be ample. So we can replace, in the assumptions of Theorem 2(iv), s by 1 and η_i by some large number η. In particular, we may assume that the evaluation map for $\omega_{X_0/Y_0}^{\eta_0}$ is surjective.

Proof of Corollary 3. As already stated in the introduction, (a) follows from Theorem 1 and 2. Moreover in order to prove (b), Lemma 1.7(4) allows us to apply Theorem 1, and to replace $\mathcal{F}_{Y_0}^{(\bullet)}$ by $\mathcal{F}_{W_0}^{(\bullet)}$. In fact, we will choose $\tilde{I}$ and I as in the proof of Theorem 2, given above, and we will choose W and $\mathcal{E}_W$ as we did there. There is however a subtle point: Even if $\bigotimes \det(\mathcal{F}_{W_0}^{(\eta_i)})^{a_i}$ is ample and even if $\bigotimes \det(\mathcal{F}_W^{(\eta_i)})^{a_i}$ is nef, the latter does not have to be ample with respect to W_0.

We will not refer to Y anymore so we may blow up the boundary $W \setminus W_0$. Now choosing the a_i large enough, we may assume that for some divisor B supported in $W \setminus W_0$ the sheaf

$$\mathcal{O}_W(B) \otimes \bigotimes \det\big(\mathcal{F}_W^{(\eta_i)}\big)^{a_i}$$

is semiample and ample with respect to W_0. Moreover, replacing W by a finite covering with a splitting trace map, we can assume that the multiplicities of B are as divisible as needed. So applying 13.2(b) instead of (a) one finds a divisor B', still supported in $W \setminus W_0$ with

$$\Big(\bigotimes^{\alpha} \mathcal{F}_W^{(\nu)}\Big) \otimes \mathcal{O}_W(-B') \otimes \bigotimes_{\iota=1}^{s} \det\big(\mathcal{F}_W^{(\eta_\iota)}\big)^{-\frac{r}{r_\iota}}$$

nef and with $\mathcal{O}_W(B') \otimes \bigotimes_{\iota=1}^{s} \det(\mathcal{F}_W^{(\eta_\iota)})^{\frac{r}{r_\iota}}$ ample with respect to W_0, which implies part (b) of Corollary 3. □

Next we will show analogs of Theorem 2 for the sheaves $\mathcal{F}_W^{(\nu,\mu)}$. As in the proof of Theorem 2 we will rely on Proposition 13.2; however it will be a bit more complicated to choose the right data to start with.

PROPOSITION 13.4. *Assume in Theorem* 12.12 *or Variant* 12.13 *that for some* $\kappa > 0$ *with* $(0,\kappa) \in \widetilde{I}$ *one has* $\det(\mathcal{F}_W^{(0,\kappa)}) = \mathcal{O}_W$. *In* 12.13 *assume in addition, that the sheaves* $\mathcal{M}_\bullet$ *are* κ*-saturated.*

Choose some $\eta_0 > 0$ *such that the evaluation map for* $\omega_{X_0/Y_0}^{\eta_0}$ *is surjective, and let* ϵ *be a positive multiple of* η_0, *with* $\epsilon \geq e(\mathcal{L}^{\kappa\cdot\eta_0}|_F)$ *for all fibers* F *of* $f_0 : X_0 \to Y_0$.

(i) *Assume that* $(\epsilon\cdot\nu, \kappa\cdot\nu)$ *and* $(\eta_0, 0)$ *are in* I. *Then the sheaf* $\mathcal{F}_W^{(\epsilon\cdot\nu,\kappa\cdot\nu)}$ *is weakly positive over* W_0.

(ii) *Assume that for some* $\nu' > 0$, *divisible by* η_0 *and* ν

$$((\epsilon+1)\cdot\nu, \kappa\cdot\nu),\ (\epsilon\cdot\nu,\kappa\cdot\nu),\ ((\epsilon+1)\cdot\nu', \kappa\cdot\nu'),\ (\eta_0, 0) \in I.$$

Then for some positive integer c *the sheaf*

$$S^c\big(\mathcal{F}_W^{((\epsilon+1)\cdot\nu,\kappa\cdot\nu)}\big) \otimes \det\big(\mathcal{F}_W^{((\epsilon+1)\cdot\nu',\kappa\cdot\nu')}\big)^{-1}$$

is weakly positive over W_0.

Proof. For simplicity we will replace $\mathcal{L}$ by $\mathcal{L}^\kappa$ and assume that $\kappa = 1$. Choose an ample invertible sheaf $\mathcal{H}$ on W and define

$$\rho = \mathrm{Min}\big\{\mu > 0;\ \mathcal{F}_W^{(\epsilon\cdot\nu,\nu)} \otimes \mathcal{H}^{\epsilon\cdot\nu\cdot\mu-1} \text{ is weakly positive over } W_0\big\}.$$

CLAIM 13.5. *The sheaf* $\mathcal{F}_W^{(\epsilon\cdot\nu,\nu)} \otimes \mathcal{H}^a$ *is weakly positive over* W_0 *for* $a = \nu\cdot\rho\cdot(\epsilon - \frac{\ell}{\nu})$.

Part (i) follows directly from 13.5. In fact, by the choice of ρ

$$\nu\cdot\rho\cdot\Big(\epsilon - \frac{\ell}{\nu}\Big) > \epsilon\cdot\nu\cdot(\rho-1), \quad \text{or} \quad \rho < \frac{\epsilon\cdot\nu}{\ell}.$$

Then $\mathcal{F}_W^{(\epsilon\cdot\nu,\nu)} \otimes \mathcal{H}^{\frac{\epsilon^2\cdot\nu^2}{\ell}}$ is weakly positive over W_0. The exponent $\frac{\epsilon^2\cdot\nu^2}{\ell}$ is independent of W and $\mathcal{H}$. So the same holds true for any ample invertible sheaf $\mathcal{H}'$ on any finite covering W' of W, and the weak positivity of $\mathcal{F}_W^{(\epsilon\cdot\nu,\nu)}$ over W_0 follows from Lemma 1.6.

Proof of Claim 13.5. In the proof we will blow up W with centers in $W \setminus W_0$, and so we will not use the ampleness of $\mathcal{H}$, just the condition that $\mathcal{F}_W^{(\epsilon\cdot\nu,\nu)} \otimes \mathcal{H}^{\epsilon\cdot\nu\cdot\rho}$ is ample with respect to W_0.

For $r' = \mathrm{rk}(\mathcal{F}_W^{(0,1)})$ one has the natural locally splitting inclusion

$$\mathcal{O}_W = \det\big(\mathcal{F}_W^{(0,1)}\big) \longrightarrow \bigotimes^{r'} \mathcal{F}_W^{(0,1)},$$

whose pullback to $\widehat{Y}$ is $\Xi_1 : \mathcal{O}_{\widehat{Y}} = \det(\hat{g}_*\mathcal{M}_{\widehat{Z}}) \longrightarrow \bigotimes^{r'} \hat{g}_*\mathcal{M}_{\widehat{Z}}$.

Choose in 9.1 $\ell = \eta_0$ and for Ξ the tuple consisting of ℓ copies of Ξ_1. Then

$$\gamma_1 = \cdots = \gamma_\ell = 1,\quad \gamma = \ell \quad \text{and} \quad \eta_1 = \cdots = \eta_\ell = \eta = 0.$$

By assumption $\ell \cdot e = \epsilon \geq e(\mathcal{L}^{\gamma}|_F)$, as required in 9.1. We choose $\beta = \epsilon \cdot \nu = e \cdot \ell \cdot \nu$ and $\alpha = \nu - 1$, and for b' any positive integer satisfying $b' \cdot (\beta - 1, \alpha) \in \ell \cdot \mathbb{N} \times \mathbb{N}$. We may assume that ν and $\ell = \eta_0$ divide b'.

By the choice of ρ the sheaf

$$S^{b'\cdot\epsilon - \frac{b'}{\nu}\cdot\ell}\big(\mathcal{F}_W^{(\epsilon\cdot\nu,\nu)}\big) \otimes S^{\frac{b'}{\ell}\cdot\epsilon\cdot(\ell-1)}\big(\mathcal{F}_W^{(\eta_0)}\big) \otimes \mathcal{H}^{\epsilon\cdot\nu\cdot\rho\cdot(b'\cdot\epsilon - \frac{b'}{\nu}\cdot\ell)}$$

is ample with respect to W_0. We can find some $d \gg 1$, a very ample sheaf $\mathcal{A}$ on W and a morphism

$$\bigoplus \mathcal{A} \longrightarrow S^d\Big(S^{b'\cdot\epsilon - \frac{b'}{\nu}\cdot\ell}\big(\mathcal{F}_W^{(\epsilon\cdot\nu,\nu)}\big) \otimes S^{\frac{b'}{\ell}\cdot\epsilon\cdot(\ell-1)}\big(\mathcal{F}_W^{(\eta_0)}\big) \otimes \mathcal{H}^{\epsilon\cdot\nu\cdot\rho\cdot(b'\cdot\epsilon - \frac{b'}{\nu}\cdot\ell)}\Big)$$

surjective over W_0. Blowing up W with centers in $W \setminus W_0$ we can assume that the image of this map is locally free, hence nef. We write this image as $\mathcal{E}_W \otimes \mathcal{H}^{\epsilon\cdot d\cdot b'\cdot a}$, and its pullback to $\hat{Y}$ as $\mathcal{E}_{\hat{Y}} \otimes \tau^*\mathcal{H}^{\epsilon\cdot d\cdot b'\cdot a}$. Let us choose $b = d \cdot b'$. Multiplication of sections gives a map to $\mathcal{F}_{\hat{Y}}^{(\beta_0,\alpha_0)} \otimes \tau^*\mathcal{H}^{\epsilon\cdot b\cdot a}$ for

$$\beta_0 = b\cdot\epsilon^2\cdot\nu - b\cdot\ell\cdot\epsilon + b\cdot\epsilon\cdot(\ell-1) \quad \text{and} \quad \alpha_0 = b\cdot\epsilon\cdot\nu - b\cdot\ell.$$

Since $\epsilon = e\cdot\ell$, $\beta = \epsilon\cdot\nu$ and $\alpha = \nu - 1$ one has

$$\beta_0 = b\cdot(\beta-1)\cdot e\cdot\ell \quad \text{and} \quad \alpha_0 = b\cdot\alpha\cdot e\cdot\ell + \ell\cdot b\cdot(e-1).$$

Since $\eta = 0$ and $\gamma = \ell$ this is just what we required in 9.1, and for a suitable choice of I the assumptions in 9.1 and 9.4 hold true.

Since the sheaf $\mathcal{E}_{\hat{Y}}$ is the pullback of a locally free sheaf $\mathcal{E}_W$ on W we can use 12.11 for W instead of Y, and obtain $\hat{Y}_1 \to W_1$ and a finite covering $\tau : W_1 \to W$ with a splitting trace map, such that the sheaf $\mathcal{G}_{W_1}^{(\beta+\frac{\eta}{\ell},\alpha+\frac{\gamma}{\ell})} = \mathcal{G}_{W_1}^{(\epsilon\cdot\nu,\nu)}$ exists on W_1. The conditions in Set-up 13.1 hold on W_1, and for $\mathcal{H}_1 = \tau^*\mathcal{H}$ the sheaf $\mathcal{E}_{W_1} \otimes \mathcal{H}_1^{e\cdot\ell\cdot b\cdot a}$ is globally generated, hence nef. Proposition 13.2(b) implies that $\mathcal{F}_{W_1}^{(\epsilon\cdot\nu,\nu)} \otimes \mathcal{H}_1^a$ is weakly positive over $\tau^{-1}(W_0)$. By Lemma 1.6 the sheaf $\mathcal{F}_W^{(\epsilon\cdot\nu,\nu)} \otimes \mathcal{H}^a$ is weakly positive over W_0. □

So we finished the proof of part one and we can use in (ii) that the sheaf $\mathcal{F}_W^{(\epsilon\cdot\nu,\nu)}$ is weakly positive over W_0. In particular in the first part we can choose $\rho = 1$ and $\mathcal{F}_W^{(\epsilon\cdot\nu,\nu)} \otimes \mathcal{H}^{\epsilon\cdot\nu}$ is ample with respect to W_0. In the proof of Claim 13.5 we obtain a bit more.

Addendum 13.6. Under the assumptions made in 13.4, there exists a projective morphism $\tau : W_1 \to W$ such that its restriction $\tau^{-1}(W_0) \to W_0$ is finite with a splitting trace map, and there exists an inclusion

$$\mathcal{G}_{W_1} = \mathcal{G}_{W_1}^{(\epsilon\cdot\nu,\nu)} \subset \bigotimes^{\mathrm{rk}(\mathcal{F}_W^{(0,\kappa)})} \mathcal{F}_{W_1}^{(\epsilon\cdot\nu,\nu)},$$

surjective over $\tau^{-1}(W_0)$ with $\mathcal{G} \otimes \tau^*(\mathcal{H})^{\nu\cdot(\epsilon-1)\cdot\mathrm{rk}(\mathcal{F}_W^{(0,\kappa)})}$ nef.

Replacing W by W_1 we will assume that the subsheaf $\mathcal{G}_W$ of $\bigotimes^{r'} \mathcal{F}_W^{(\epsilon\cdot\nu,\nu)}$ exists on W, for $r' = \mathrm{rk}(\mathcal{F}_W^{(0,\kappa)})$. We will use 13.2 a second time, so we have to choose again data as in Section 9. For $r = \mathrm{rk}(\mathcal{F}_W^{((\epsilon+1)\cdot\nu',\nu')})$, we start with the tautological morphism

$$\Xi : \det\big(\mathcal{F}_W^{((\epsilon+1)\cdot\nu',\nu')}\big)^{r'} \longrightarrow \bigotimes^{r\cdot r'} \mathcal{F}_W^{((\epsilon+1)\cdot\nu',\nu')}.$$

So $\eta = \eta_1 = (\epsilon+1)\cdot\nu'$ and $\ell = \gamma = \gamma_1 = \nu'$. Necessarily one needs $\beta = (\epsilon+1)\cdot(\nu-1)$ and $\alpha = \nu - 1$. For e we choose a natural number with $\ell \cdot e \geq e(\omega_F^{(\epsilon+1)\cdot\nu'} \otimes \mathcal{L}^{\nu'})$, for all fibers F of f_0. For b we choose any positive integer with

$$b \cdot (\beta - 1, \alpha) \in \eta_0 \cdot \mathbb{N} \times \mathbb{N},$$

such that $r' \cdot \epsilon \cdot \nu$ divides $\alpha_0 = b \cdot (\nu - 1) \cdot e \cdot \ell + \gamma \cdot b \cdot (e - 1)$. Comparing the different constants one finds

$$\begin{aligned} \beta_0 &= b \cdot ((\epsilon + 1) \cdot (\nu - 1) - 1) \cdot e \cdot \ell + \eta \cdot b \cdot (e - 1) \\ &= b \cdot \epsilon \cdot (\nu - 1) \cdot e \cdot \ell + \epsilon \cdot \ell \cdot b \cdot (e - 1) + b \cdot \ell \cdot ((\nu - 1) \cdot e - 1) \\ &= \epsilon \cdot \alpha_0 + b \cdot \ell \cdot ((\nu - 1) \cdot e - 1). \end{aligned}$$

We choose

$$\begin{aligned} \mathcal{E}_W &= \Big(\bigotimes^{r} \mathcal{G}_W^{\frac{\alpha_0}{\nu}} \Big) \otimes \bigotimes^{r\cdot r'} \mathcal{F}_W^{(\eta_0)}\big(\tfrac{b\cdot\ell\cdot((\nu-1)\cdot e-1)}{\eta_0}\big) \\ &\subset \Big(\bigotimes^{r\cdot r'} \mathcal{F}_W^{(\epsilon\cdot\nu,\nu)\,\frac{\alpha_0}{\nu}} \Big) \otimes \bigotimes^{r\cdot r'} \mathcal{F}_W^{(\eta_0)}\big(\tfrac{b\cdot\ell\cdot((\nu-1)\cdot e-1)}{\eta_0}\big) \end{aligned}$$

and $\mathcal{E}_{\hat{Y}}$ will denote its pullback to $\hat{Y}$. The $r \cdot r'$-tensor product of the multiplication map gives

$$\mathcal{E}_{\hat{Y}} \longrightarrow \bigotimes^{r\cdot r'} \mathcal{F}_{\hat{Y}}^{(\beta_0,\alpha_0)}.$$

Since $\mathcal{F}_W^{(\eta_0)}$ is nef, the choice of $\mathcal{G}$ in 13.6 implies that $\mathcal{E}_W \otimes \mathcal{H}^{\alpha_0\cdot(\epsilon-1)\cdot r'}$ is nef. Replacing W by a larger covering, we may also assume that $\det(\mathcal{F}_W^{((\epsilon+1)\cdot\nu',\nu')})$ is the $r \cdot e \cdot \ell$-th power of an invertible sheaf $\mathcal{V}_W$, and that $\mathcal{H}^{\alpha_0\cdot(\epsilon-1)}$ is the $b \cdot e \cdot \ell$-th power of an invertible sheaf.

So all the conditions made in 13.1 hold, and we can apply Proposition 13.2. One obtains the weak positivity over W_0 of

$$\mathcal{F}_W^{(\epsilon\cdot\nu,\nu)} \otimes \mathcal{H}^{\frac{\alpha_0\cdot(\epsilon-1)}{b\cdot e\cdot\ell}} \otimes \mathcal{V}_W^{-1}.$$

The exponent $\frac{\alpha_0\cdot(\epsilon-1)}{b\cdot e\cdot\ell}$ is independent of W and of the ample invertible sheaf $\mathcal{H}$. So Lemma 1.6 implies that $\mathcal{F}_W^{(\epsilon\cdot\nu,\nu)} \otimes \mathcal{V}_W^{-1}$ is already weakly positive over W_0, hence

$$S^{r\cdot e\cdot\ell}\big(\mathcal{F}_W^{(\epsilon\cdot\nu,\nu)}\big) \otimes \mathcal{V}_W^{-r\cdot e\cdot\ell} = S^{r\cdot e\cdot\ell}\big(\mathcal{F}_W^{(\epsilon\cdot\nu,\nu)}\big) \otimes \det\big(\mathcal{F}_W^{((\epsilon+1)\cdot\nu',\nu')}\big)^{-1}$$

as well, as claimed in Proposition 13.4(ii). □

Proof of Lemma 3.2. We start with the models of our morphisms, constructed in Theorem 12.12 or its Variant 12.13, with $\epsilon+1$ replaced by ϵ. We may add the condition $\det(\mathcal{F}_W^{(\kappa)})=\mathcal{O}_W$. In fact, replacing W by a larger covering with a splitting trace map, [Vie95, Lemma 2.1] allows one to assume that $\det(\mathcal{F}_W^{(\kappa)})=\mathcal{W}^{\mathrm{rk}(\mathcal{F}_W^{(\kappa)})}$ for an invertible sheaf $\mathcal{W}$. Then one can replace the polarization on $\hat{X}\to\hat{Y}$ and on $X_0\times_{Y_0}W_0\to W_0$ by $\mathcal{M}\otimes\hat{f}^*\mathcal{W}^{-1}$ and $\mathrm{pr}_1^*\mathcal{L}_0\otimes\mathrm{pr}_2^*\mathcal{W}^{-1}$.

So we may assume that the assumptions in 13.2 and 13.4 hold and Lemma 3.2 follows from Proposition 13.4, by Lemma 1.6. □

14. **The Proof of Theorems** 5 **and** 6

In the construction of the compactifications $\overline{M}_h$ and the sheaf $\lambda_\nu^{(p)}$ we will use the negativity of the kernels of the multiplication map, stated and proved in [Vie95, Th. 4.33]. Unfortunately there we did not keep track of what happens along the boundary, so we have to indicate the necessary modifications of the statements and proofs.

THEOREM 14.1. *Let W be a reduced projective scheme, let $W_0\subset W$ be open and dense, let $\mathcal{P}$ and $\mathcal{Q}$ be locally free sheaves on W. For a morphism* $\mathrm{m}:S^\mu(\mathcal{P})\to\mathcal{Q}$, *surjective over W_0, assume that the kernel of* m *has maximal variation in all points $w\in W_0$.*

If $\mathcal{P}$ is weakly positive over W_0 then for $b\gg a\gg 0$ the sheaf $\det(\mathcal{Q})^a\otimes\det(\mathcal{P})^b$ *is ample with respect to W_0.*

We will not recall the definition of "maximal variation" given in [Vie95, Def. 4.32]. Instead we will just explain this notion in the special situation where the theorem will be applied.

Example 14.2. Assume that over W_0 there exists a flat family $f_0:X_0\to W_0$ and an f_0-ample invertible sheaf $\mathcal{L}_0$ on X_0. Assume that $\mathcal{L}$ is fiberwise very ample, and without higher cohomology. So for all fibers F one has an embedding

$$F\hookrightarrow\mathbb{P}=\mathbb{P}(H^0(F,\mathcal{L}_0|_F).$$

Choose $\beta\gg 1$ such that the homogeneous ideal of F is generated in degree β, for all fibers. Assume that $\mathcal{P}|_{W_0}=S^\beta(f_{0*}\mathcal{L}_0)$, that $\mathcal{Q}|_{W_0}=f_{0*}\mathcal{L}_0^\beta$ and that m is the multiplication map. Then the kernel of m has maximal variation in all points $w\in W_0$ if and only if for each fiber F the set

$$\left\{w'\in W_0;\ \text{for } F'=f_0^{-1}(w') \text{ there is an isomorphism } (F,\mathcal{L}_0|_F)\cong(F',\mathcal{L}_0|_{F'})\right\}$$

is finite. Moreover this condition is compatible with base change under finite morphisms.

Sketch of the proof of 14.1. We will just recall the main steps of the proof of [Vie95, Th. 4.33], to convince the reader that one controls the sections along the boundary, and explain where the condition "maximal variation" enters the scene.

Writing $r = \mathrm{rk}(\mathcal{P})$ we consider the projective bundle $\mathbb{P} = \mathbb{P}(\bigoplus^r \mathcal{P}^\vee)$ with $\pi : \mathbb{P} \to W$. On $\mathbb{P}$ one has the "universal basis"

$$\underline{s} : \bigoplus^r \mathcal{O}_{\mathbb{P}}(-1) \longrightarrow \pi^*\mathcal{P},$$

and $\underline{s}$ is an isomorphism outside of an effective divisor Δ on $\mathbb{P}$ with

$$\mathcal{O}_{\mathbb{P}}(\Delta) = \mathcal{O}_{\mathbb{P}}(r) \otimes \pi^* \det(\mathcal{P}).$$

The universal basis is induced by the tautological map $\bigoplus^r \pi^*\mathcal{P}^\vee \to \mathcal{O}_{\mathbb{P}}(1)$. The latter gives a surjection

$$\bigoplus^r \pi^*\Big(\bigwedge^{r-1} \mathcal{P}\Big) \cong \bigoplus^r \pi^*(\mathcal{P}^\vee \otimes \det(\mathcal{P})) \longrightarrow \mathcal{O}_{\mathbb{P}}(1) \otimes \pi^* \det(\mathcal{P}).$$

Hence $\mathcal{O}_{\mathbb{P}}(1) \otimes \pi^* \det(\mathcal{P}) = \mathcal{O}_{\mathbb{P}}(r-1) \otimes \mathcal{O}_{\mathbb{P}}(\Delta)$ is weakly positive over $\pi^{-1}(W_0)$.

The sheaf $\mathcal{B}$ denotes the image of the composite

$$S^\mu\Big(\bigoplus^r \mathcal{O}_{\mathbb{P}}(-1)\Big) = \mathcal{O}_{\mathbb{P}}(-\mu) \otimes S^\mu\Big(\bigoplus^r \mathcal{O}_{\mathbb{P}}\Big) \xrightarrow{S^\mu(\underline{s})} S^\mu(\mathcal{P}) \xrightarrow{\pi^*(\mathfrak{m})} \pi^*\mathcal{Q}.$$

Remark that $\mathcal{B} \to \mathcal{Q}$ is an isomorphism outside $\Delta \cup \pi^{-1}(W \setminus W_0)$. So there is a modification $\tau : \mathbb{P}' \to \mathbb{P}$ with center in this set, such that $\mathcal{B}' = \mathcal{B}/_{\text{torsion}}$ is locally free. When $\mathcal{O}_{\mathbb{P}'}(-\eta)$ is the pullback of $\mathcal{O}_{\mathbb{P}}(-\eta)$, the surjection

$$S^\mu\Big(\bigoplus^r \mathcal{O}_{\mathbb{P}'}\Big) \longrightarrow \mathcal{B}' \otimes \mathcal{O}_{\mathbb{P}'}(\mu)$$

defines a morphism to a Grassmann variety $\rho' : \mathbb{P}' \to \mathbb{G}r$.

The condition on the "maximal variation" is used here. One needs the fact that ρ' is quasi-finite on $(\pi \circ \tau)^{-1}(W_0) \setminus \tau^{-1}\Delta$. In the situation considered in Example 14.2 this is obviously true. The kernel of $\mathfrak{m}$ determines the fiber F as a subscheme of $\mathbb{P}(H^0(F, \mathcal{L}_0|_F))$. So by assumption there are only finitely many $\mathbb{P}\mathrm{Gl}(r-1, \mathbb{C})$ orbits, hence fibers of $\pi|_{\mathbb{P}\setminus\Delta}$, whose images in $\mathbb{G}r$ can meet. And obviously ρ' is injective on those fibers.

The Plücker embedding gives an ample invertible sheaf on $\mathbb{G}r$, and its pullback to $\mathbb{P}'$ is $\det(\mathcal{B}') \otimes \mathcal{O}_{\mathbb{P}'}(\gamma)$ with $\gamma = \mu \cdot \mathrm{rk}(\mathcal{Q})$. So this sheaf is ample with respect to $(\pi \circ \tau)^{-1}(W_0) \setminus \tau^{-1}\Delta$.

Next, blowing up $\mathbb{P}'$ a bit more, one can also assume that for some $\nu > 0$ and for some divisor E, supported in $\tau^{-1}(\Delta)$ the sheaf

$$\det(\tau^*\pi^*\mathcal{Q})^\nu \otimes \mathcal{O}_{\mathbb{P}'}(\gamma \cdot \nu) \otimes \mathcal{O}_{\mathbb{P}'}(-E)$$

is ample with respect to $(\pi \circ \tau)^{-1}(W_0)$. Also the pullback of a weakly positive sheaf

$$\tau^*\pi^* \det(\mathcal{P})^{r-1} \otimes \mathcal{O}_{\mathbb{P}'}(\tau^*\Delta)$$

is weakly positive over $(\pi \circ \tau)^{-1}(W_0)$.

Using the equality $\mathcal{O}_{\mathbb{P}'}(r) = \tau^*\pi^* \det(\mathcal{P})^{-1} \otimes \mathcal{O}_{\mathbb{P}'}(\tau^*\Delta')$, one finds that for all $\eta > 0$ the sheaf

$$\begin{aligned}\tau^*\pi^*(\det(\mathcal{Q})^{\nu\cdot\gamma} \otimes \det(\mathcal{P})^{\eta\cdot r-\eta}) \otimes \mathcal{O}_{\mathbb{P}'}(\nu\cdot r\cdot\gamma) \otimes \mathcal{O}_{\mathbb{P}'}(-r\cdot E + \eta\cdot\tau^*\Delta)\\ = \tau^*\pi^*(\det(\mathcal{Q})^{\nu\cdot\gamma} \otimes \det(\mathcal{P})^{\eta\cdot r-\eta-\nu\cdot\gamma}) \otimes \mathcal{O}_{\mathbb{P}'}(-r\cdot E + (\eta+\nu\cdot\gamma)\cdot\tau^*\Delta)\end{aligned}$$

is still ample with respect to $(\pi\circ\tau)^{-1}(W_0)$. For η sufficiently large the correction divisor $-r\cdot E + (\eta+\nu\cdot\gamma)\cdot\tau^*\Delta$ will be effective. So we found some effective divisor Δ'', supported in $\tau^{-1}(\Delta)$ and $a, b > 0$ such that

$$\tau^*\pi^*(\det(\mathcal{Q})^a \otimes \det(\mathcal{P})^b) \otimes \mathcal{O}_{\mathbb{P}'}(\Delta'')$$

is ample with respect to $(\pi\circ\tau)^{-1}(W_0)$.

Next, by [Vie95, Lemma 4.29], for all $c > 0$ one has a natural splitting

$$\mathcal{O}_W \longrightarrow (\pi\circ\tau)_*\mathcal{O}_{\mathbb{P}'}(c\cdot\Delta'') \longrightarrow \mathcal{O}_W, \tag{14.2.1}$$

compatible with pullbacks. As in [Vie95, Prop. 4.30] this implies that “ampleness with respect to $(\pi\circ\tau)^{-1}W_0$ descends from $\mathbb{P}'$ to W:

Let $\mathcal{N} = \det(\mathcal{Q})^a \otimes \det(\mathcal{P})^b$. Consider two points w and w' in W_0 and $T = w\cup w'$. Let $\mathbb{P}'_T$ be the proper transform of $\pi^{-1}(T)$ in $\mathbb{P}'$. The splitting (14.2.1) gives a commutative diagram

$$\begin{tikzcd} H^0(\mathbb{P}', \tau^*\pi^*\mathcal{N}^\nu \otimes \mathcal{O}_{\mathbb{P}'}(\nu\cdot\Delta'')) \arrow[r] \arrow[d, "\chi'"'] & H^0(W, \mathcal{N}^\nu) \arrow[d, "\chi"] \\ H^0(\mathbb{P}'_T, \tau^*\pi^*(\mathcal{N}^\nu \otimes \mathcal{O}_{\mathbb{P}'}(\nu\cdot\Delta''))|_{\mathbb{P}'_T}) \arrow[r] & H^0(T, \mathcal{N}^\nu|_T) \end{tikzcd}$$

with surjective horizontal maps. For some $\nu \geq \nu(w, w')$ the map χ' and hence χ will be surjective. For those ν the sheaf $\mathcal{N}^\nu$ is generated in a neighborhood of w' by global sections t, with $t(w) = 0$. By Noetherian induction one finds some $\nu_0 > 0$ such that, for $\nu \geq \nu_0$, the sheaf $\mathcal{N}^\nu$ is generated by global sections $t_1, \ldots, t_r$, on $W_0 \setminus \{w\}$ with $t_1(w) = \cdots = t_r(w) = 0$, and moreover there is a global section t_0 with $t_0(w) \neq 0$. For the subspace V_ν of $H^0(W, \mathcal{N}^\nu)$, generated by $t_0, \ldots, t_r$, the morphism $g_\nu : W \to \mathbb{P}(V_\nu)$ is quasi-finite in a neighborhood of $g_\nu^{-1}(g_\nu(w))$. In fact, $g_\nu^{-1}(g_\nu(w)) \cap W_0$ is equal to w.

Again by Noetherian induction one finds some ν_1 and for $\nu \geq \nu_1$ some subspace V_ν such that the restriction of g_ν to W_0 is quasi-finite. Then $g_\nu^*\mathcal{O}_{\mathbb{P}(V_\nu)}(1) = \mathcal{N}^\nu$ is ample with respect to W_0. □

We keep the notation introduced in Section 3. We also use the terminology of [Vie95, 7.4] for the relevant cases, as follows. For Theorem 5 we consider

Case CP: the moduli functor $\mathfrak{M}_h$ of canonically polarized manifolds.

As shown in Lemma 3.3(2), for Theorem 6 it is sufficient to consider

Case PO: the moduli functor $\mathfrak{M}_h^{(\nu)}$ of minimal manifolds F with $\omega_F^\nu = \mathcal{O}_F$, and with a very ample polarization $\mathcal{L}_F$ without higher cohomology.

As in the construction of M_h or $M_h^{(\nu)}$ in Section 3 we will by abuse of notation consider M_h and $M_h^{(\nu)}$ with their reduced structure.

In general M_h is not a fine moduli space; hence there is no universal family. However Seshadri's theorem on the elimination of finite isotropies, recalled in [Vie95, Th. 3.49], provides us with a finite normal covering $\phi_0 : Y_0 \to M_h$ which factors over the moduli stack, i.e., which is induced by a family $f_0 : X_0 \to Y_0$ (or by $(f_0 : X_0 \to Y_0, \mathcal{L}_0)$). So we are in the situation considered in Variant 12.10, and for each rigidified determinant sheaf, as defined in Definition 12.9, we can find $\overline{M}_h$ and $\phi : W \to \overline{M}_h$ such that $\mathcal{C}_{\overline{M}_h}$ exists. Recall that its pullback is the p-th tensor power of the given rigidified determinant.

We apply 12.10 to $\det(\mathcal{F}_{\bullet}^{(\nu)})$ and we obtain a morphism $\phi : Y \to \overline{M}_h$. The corresponding sheaf $\mathcal{C}_{\overline{M}_h}$ is just the sheaf $\lambda_\nu^{(p)}$ in Theorem 5 (or $\lambda_\nu^{(p)}$ in Theorem 6). So in order to prove both theorems, it remains to show:

($\star$) The sheaf $\lambda_\nu^{(p)}$ is nef and ample with respect to M_h.

To do so, Lemma 1.9 allows us to replace $\overline{M}_h$ by any finite covering, for example by the normalization of W or by a modification Y of the latter with centers outside the preimage of M_h.

The preimage of M_h in Y maps to Y_0, and we may assume that both are equal. So we are exactly in the situation considered in Section 4. Replacing Y by some alteration, finite over Y_0, we can assume that the mild morphism $\hat{Z} \to \hat{Y}$ in Proposition 4.5 exists over a desingularization $\varphi : \hat{Y} \to Y$ of Y, hence over all the morphisms in the diagram 4.2. Moreover we can assume that the locally free sheaf $\mathcal{F}_{\hat{Y}}^{(\nu)}$ (invertible for $\mathfrak{M}_h^{(\nu)}$) in Theorem 1 exists and that it is the pullback of a locally free sheaf $\mathcal{F}_Y^{(\nu)}$ on Y. So ($\star$) and hence Theorems 5 and 6 follow from:

CLAIM 14.3. *The locally free sheaf $\mathcal{F}_Y^{(\nu)}$ is nef and ample with respect to Y_0.*

Proof of 14.3 *in Case* CP. In addition to fixing ν let us fix an η_0 such that for all $F \in \mathfrak{M}_h(\mathrm{Spec}(\mathbb{C}))$ the sheaf $\omega_F^{\eta_0}$ is very ample. Choose $\eta_1 = \beta \cdot \eta_0$ such that the multiplication map

$$\mathrm{m} : S^\beta(H^0(F, \omega_F^{\eta_0})) \longrightarrow H^0(F, \omega_F^{\eta_1})$$

is surjective and such that its kernel generates the homogeneous ideal, defining $F \subset \mathbb{P}(H^0(F, \omega_F^{\eta_0}))$. By Theorem 1 the sheaves $\mathcal{F}_W^{(\eta_0)}$, $\mathcal{F}_W^{(\eta_1)}$ and $\mathcal{F}_W^{(\nu)}$ exist on some alteration of Y, finite over Y_0. So we can replace Y by the normalization of this alteration, and assume that they exist on Y itself. The multiplication of sections defines a morphism $S^\beta(\mathcal{F}_{\hat{Y}}^{(\eta_0)}) \to \mathcal{F}_{\hat{Y}}^{(\eta_1)}$; hence as in Addendum 12.7(5) this is the pullback of $\mathrm{m} : S^\beta(\mathcal{F}_Y^{(\eta_0)}) \to \mathcal{F}_Y^{(\eta_1)}$.

Both sheaves are locally free and by Theorem 5(iii) they are nef. The kernel of m is of maximal variation, as explained in Example 14.2. By Example 14.2(iii) one finds that for some positive integers the sheaf $\det(\mathcal{F}_Y^{(\eta_1)})^a \otimes \det(\mathcal{F}_Y^{(\eta_0)})^b$ is ample with respect to Y_0 and by part (iv) the same holds for $\mathcal{F}_Y^{(\nu)}$. □

Proof of 14.3 *in Case* PO. The proof of Theorem 6 is similar. We choose a positive integer β, divisible by $s = h(1)$ such that the multiplication map

$$\mathrm{m} : S^{\beta}(H^0(F, \mathcal{L}|_F)) \longrightarrow H^0(F, \mathcal{L}^{\beta}|_F)$$

is surjective for all $F \in \mathfrak{M}_h^{(\beta)}(\mathbb{C})$, and such that its kernel defines the homogeneous ideal of the image of F in $\mathbb{P}(H^0(F, \mathcal{L}|_F))$. We choose a natural number ϵ divisible by β with $\epsilon > e(\mathcal{L}^{\beta}|_F)$.

Since we are allowed to replace Y by some finite covering, we can apply 8.13, Proposition 12.8 and [Vie95, Lemma 2.1] and assume:

(1) The sheaves $(\mathcal{M}_{\hat{Z}}, \mathcal{M}_Z, \mathcal{M}_{\hat{X}})$ are β-saturated.

(2) The invertible sheaf $\lambda = \mathcal{F}_Y^{(\nu)}$, and the locally free sheaves $\mathcal{F}_Y^{(0,1)}$ and $\mathcal{F}_Y^{(0,\beta)}$ exist on Y.

(3) For $s = \mathrm{rk}(\mathcal{F}_Y^{(0,1)})$ the sheaf $\det(\mathcal{F}_Y^{(0,1)})$ is the s-th tensor power of an invertible sheaf $\mathcal{N}$.

Replacing $(\mathcal{M}_{\hat{Z}}, \mathcal{M}_Z, \mathcal{M}_{\hat{X}})$ and $\mathcal{F}_Y^{(\beta,\mu)}$ by

$$(\mathcal{M}_{\hat{Z}} \otimes \hat{g}^*\varphi^*\mathcal{N}^{-1},\ \mathcal{M}_Z \otimes g^*\varphi^*\mathcal{N}^{-1},\ \mathcal{M}_{\hat{X}} \otimes \hat{f}^*\varphi^*\mathcal{N}^{-1}) \quad \text{and} \quad \mathcal{F}_Y^{(\beta,\mu)} \otimes \mathcal{N}^{-\mu}$$

we can add:

(4) $\det(\mathcal{F}_Y^{(0,1)}) = \mathcal{O}_Y$ and hence $\det(\mathcal{F}_{\hat{Y}}^{(0,1)}) = \mathcal{O}_{\hat{Y}}$.

CLAIM 14.4. *Assumptions* (1)–(4) *imply for all ϵ' divisible by ν that*:

(5) $\mathcal{F}_Y^{(\epsilon'\cdot\beta,\beta)} = \lambda^{\frac{\epsilon'\cdot\beta}{\nu}} \otimes \mathcal{F}_Y^{(0,\beta)}$.

(6) $\det\big(\mathcal{F}_Y^{(\epsilon'\cdot\beta,\beta)}\big) = \lambda^{\frac{\epsilon'\cdot\beta\cdot r}{\nu}} \otimes \det\big(\mathcal{F}_Y^{(0,\beta)}\big)$ *for* $r = \mathrm{rk}\big(\mathcal{F}_Y^{(0,\beta)}\big)$.

(7) $\mathcal{F}_Y^{(\epsilon',1)} = \lambda^{\frac{\epsilon'}{\nu}} \otimes \mathcal{F}_Y^{(0,1)}$.

(8) $\det\big(\mathcal{F}_Y^{(\epsilon',1)}\big) = \lambda^{\frac{s\cdot\epsilon'}{\nu}}$ *for* $s = \mathrm{rk}\big(\mathcal{F}_{\hat{Y}}^{(0,1)}\big)$.

Proof. It is sufficient to verify these four equations on $\hat{Y}$. Let $\Pi_{\hat{X}}^{(\nu)}$ be the divisor with

$$\hat{f}^*\mathcal{F}_{\hat{Y}}^{(\nu)} = \hat{f}^*\hat{f}_*\omega_{\hat{X}/\hat{Y}}^{\nu} = \omega_{\hat{X}/\hat{Y}}^{\nu} \otimes \mathcal{O}_{\hat{X}}\big(-\Pi_{\hat{X}}^{(\nu)}\big).$$

By Lemma 8.11(c)

(14.4.1)

$$\hat{f}_*(\omega_{\hat{X}/\hat{Y}}^{\epsilon'\cdot\beta} \otimes \mathcal{M}_{\hat{X}}^{\beta}) = \lambda^{\frac{\epsilon'\cdot\beta}{\nu}} \otimes \hat{f}_*\Big(\mathcal{M}_{\hat{X}}^{\beta} \otimes \mathcal{O}_{\hat{X}}\Big(\frac{\epsilon'\cdot\beta}{\nu}\cdot\Pi_{\hat{X}}^{(\beta)}\Big)\Big) = \lambda^{\frac{\epsilon'\cdot\beta}{\nu}} \otimes \hat{f}_*(\mathcal{M}_{\hat{X}}^{\beta}).$$

So (5) holds true, and (6) as well. For (7) we apply Lemma 8.11(e) saying that the sheaves $(\mathcal{M}_{\hat{Z}}, \mathcal{M}_Z, \mathcal{M}_{\hat{X}})$ are also 1-saturated. Then the equality (14.4.1) holds for β replaced by 1. Since $\det(\hat{f}_*\mathcal{M}_{\hat{X}}) = \mathcal{O}_{\hat{Y}})$ one obtains (8). □

Note that Claim 14.4 implies in particular, that the sheaves $\mathcal{F}_Y^{(\epsilon'\cdot\beta,\beta)}$, and $\mathcal{F}_Y^{(\epsilon',1)}$ automatically exist, with all the properties asked for in 12.8.

By Proposition 13.4 we may assume that the sheaves $\mathcal{F}_Y^{(\epsilon,1)}$ and $\mathcal{F}_Y^{(\epsilon\cdot\beta,\beta)}$ are both weakly positive over Y_0. Since Y is normal, the multiplication of sections on $\widehat{Y}$ is the pullback of a morphism $m : S^\beta(\mathcal{F}_Y^{(\epsilon,1)}) \to \mathcal{F}_Y^{(\epsilon\cdot\beta,\beta)}$. It is surjective over Y_0 with kernel of maximal variation, as explained in Example 14.2. By Theorem 14.1, for some positive integers a and b the sheaf

$$\det(\mathcal{F}_Y^{(\epsilon,1)})^a \otimes \det\left(\mathcal{F}_Y^{(\epsilon\cdot\beta,\beta)}\right)^b = \lambda^{\frac{a\cdot s\cdot\epsilon+b\cdot\epsilon\cdot\beta\cdot r}{\nu}} \otimes \det\left(\mathcal{F}_Y^{(0,\beta)}\right)^b \tag{14.4.2}$$

is ample with respect to Y_0. Since $\mathcal{F}_Y^{(\epsilon,1)}$ is nef, we can replace a by a larger integer, and assume that $a\cdot s$ is divisible by $b\cdot\beta\cdot r$. So for $\epsilon' = \epsilon\cdot(\frac{a\cdot s}{b\cdot\beta\cdot r}+1)$ the sheaf in (14.4.2) is of the form $\det(\mathcal{F}_Y^{(\epsilon'\cdot\beta,\beta)})^b$ and 13.4(ii) implies that $\mathcal{F}_Y^{(\epsilon',1)}$ is ample with respect to Y_0, hence $\mathcal{F}_Y^{(\nu)}$ as well. □

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TABLE OF CONTENTS