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# On a class of $\mathrm{II}_1$ factors with at most one Cartan subalgebra

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*Dedicated to Alain Connes on his* 60th *birthday.*


## Abstract

We prove that the normalizer of any diffuse amenable subalgebra of a free group factor $L(\mathbb{F}_r)$ generates an amenable von Neumann subalgebra. Moreover, any $\mathrm{II}_1$ factor of the form $Q\overline{\otimes}L(\mathbb{F}_r)$, with $Q$ an arbitrary subfactor of a tensor product of free group factors, has no Cartan subalgebras. We also prove that if a free ergodic measure-preserving action of a free group $\mathbb{F}_r$, $2 \leq r \leq \infty$, on a probability space $(X, \mu)$ is profinite then the group measure space factor $L^\infty(X) \rtimes \mathbb{F}_r$ has unique Cartan subalgebra, up to unitary conjugacy.


## 1. **Introduction**

A celebrated theorem of Connes ([Con76]) shows that all amenable $\mathrm{II}_1$ factors are isomorphic to the approximately finite-dimensional (AFD) $\mathrm{II}_1$ factor $R$ of Murray and von Neumann ([MvN43]). In particular, all $\mathrm{II}_1$ group factors $L(\Gamma)$ associated with ICC (infinite conjugacy class) amenable groups $\Gamma$, and all group measure space $\mathrm{II}_1$ factors $L^\infty(X) \rtimes \Gamma$ arising from free ergodic measure-preserving (m.p.) actions of countable amenable groups $\Gamma$ on a probability space $\Gamma \curvearrowright X$, are isomorphic to $R$. Moreover, by [CFW81], any decomposition of $R$ as a group measure space algebra is unique, i.e. if $R = L^\infty(X_i) \rtimes \Gamma_i$, for some free ergodic measure-preserving actions $\Gamma_i \curvearrowright X_i$, $i = 1, 2$, then there exists an automorphism of $R$ taking $L^\infty(X_1)$ onto $L^\infty(X_2)$. In fact, any two Cartan subalgebras of $R$ are conjugate by an automorphism of $R$.

Recall in this respect that a *Cartan subalgebra* $A$ in a $\mathrm{II}_1$ factor $M$ is a maximal abelian $^*$-subalgebra $A \subset M$ with normalizer $\mathcal{N}_M(A) = \{u \in \mathcal{U}(A) \mid uAu^* = A\}$ generating $M$ ([Dix54], [FM77a], [FM77b]). Its presence amounts to realizing $M$ as a generalized, twisted-version of the group measure space construction, corresponding to the equivalence relation induced by the orbits of some ergodic m.p.

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action of a countable group, $\Gamma \curvearrowright X$, and a 2-cocycle, with $A = L^\infty(X)$. Decomposing factors this way is important, especially if one can show uniqueness of their Cartan subalgebras, because then the classification of the factors reduces to the classification of the corresponding actions $\Gamma \curvearrowright X$ up to orbit equivalence ([FM77a], [FM77b]). But beyond the amenable case, very little is known about uniqueness, or possible nonexistence, of Cartan subalgebras in group factors, or other factors that are *a priori* constructed in different ways than as group measure space algebras.

We investigate in this paper Cartan decomposition properties for a class of nonamenable $\mathrm{II}_1$ factors that are in some sense "closest to being amenable". Thus, we consider factors $M$ which satisfy the *complete metric approximation property* (c.m.a.p.) of Haagerup ([Haa79]), which requires existence of normal, finite rank, completely bounded (cb) maps $\phi_n : M \to M$, such that $\|\phi_n\|_{\mathrm{cb}} \le 1$ and $\lim \|\phi_n(x) - x\|_2 = 0$, for all $x \in M$, where $\|\cdot\|_2$ denotes the Hilbert norm given by the trace of $M$ (note that if $\phi_n$ could in addition be taken unital, $M$ would follow amenable). This is the same as saying that the Cowling-Haagerup constant $\Lambda_{\mathrm{cb}}(M)$ equals 1 (see [CH89]). The prototype nonamenable c.m.a.p. factors are the free group factors $L(\mathbb{F}_r)$, $2 \le r \le \infty$ ([Haa79]). Like amenability, the c.m.a.p. passes to subfactors and is well-behaved to inductive limits and tensor products.

We in fact restrict our attention to c.m.a.p. factors of the form $M = Q \rtimes \mathbb{F}_r$, and to subfactors $N$ of such $M$. The aim is to locate all (or prove possible absence of) diffuse AFD subalgebras $P \subset N$ whose normalizer $\mathcal{N}_N(P)$ generates $N$. Our general result along these lines shows:

THEOREM. *Let $\mathbb{F}_r \curvearrowright Q$ be an action of a free group on a finite von Neumann algebra. Assume $M = Q \rtimes \mathbb{F}_r$ has the complete metric approximation property. If $P \subset M$ is a diffuse amenable subalgebra and $N$ denotes the von Neumann algebra generated by its normalizer $\mathcal{N}_M(P)$, then either $N$ is amenable relative to $Q$ inside $M$, or $P$ can be embedded into $Q$ inside $M$.*

The amenability property of a von Neumann subalgebra $N \subset M$ relative to another von Neumann subalgebra $Q \subset M$ is rather self-explanatory: it requires existence of a norm-one projection from the basic construction algebra of the inclusion $Q \subset M$ onto $N$ (see Definition 2.2). The "embeddability of a subalgebra $P \subset M$ into another subalgebra $Q \subset M$ inside an ambient factor $M$" is in the sense of [Pop06c] (see Definition 2.6 below), and roughly means that $P$ can be conjugated into $Q$ via a unitary element of $M$.

We mention three applications of the theorem, each corresponding to a particular choice of $\mathbb{F}_r \curvearrowright Q$ and solving well-known problems. Thus, taking $Q = \mathbb{C}$, we get:

COROLLARY 1. *The normalizer of any diffuse amenable subalgebra $P$ of a free group factor $L(\mathbb{F}_r)$ generates an amenable* (*thus* AFD *by* [Con76]) *von Neumann algebra.*

If we take $Q$ to be an arbitrary finite factor with $\Lambda_{\mathrm{cb}}(Q) = 1$ and let $\mathbb{F}_r$ act trivially on it, then $M = Q \mathbin{\overline{\otimes}} L(\mathbb{F}_r)$, $\Lambda_{\mathrm{cb}}(M) = 1$ and the theorem implies:

COROLLARY 2. *If $Q$ is a* $\mathrm{II}_1$ *factor with the complete metric approximation property then $Q \mathbin{\overline{\otimes}} L(\mathbb{F}_r)$ does not have Cartan subalgebras. Moreover, if $N \subset Q \mathbin{\overline{\otimes}} L(\mathbb{F}_r)$ is a subfactor of finite index* [Jon83], *then $N$ does not have Cartan subalgebras either.*

This shows in particular that any factor of the form $L(\mathbb{F}_r) \mathbin{\overline{\otimes}} R$, $L(\mathbb{F}_{r_1}) \mathbin{\overline{\otimes}} L(\mathbb{F}_{r_2}) \mathbin{\overline{\otimes}} \cdots$, and more generally any subfactor of finite index of such a factor, has no Cartan decomposition. Besides $Q = R, L(\mathbb{F}_r)$, other examples of factors with $\Lambda_{\mathrm{cb}}(Q) = 1$ are the group factors $L(\Gamma)$ corresponding to ICC discrete subgroups $\Gamma$ of $\mathrm{SO}(1,n)$ and $\mathrm{SU}(1,n)$ ([DCH85], [Cow83]), as well as any subfactor of a tensor product of such factors. None of the factors covered by Corollary 2 were known until now not to have Cartan decomposition.

Finally, if we take $\mathbb{F}_r \curvearrowright X$ to be a *profinite* m.p. action on a probability measure space $(X, \mu)$, i.e. an action with the property that $L^\infty(X)$ is a limit of an increasing sequence of $\mathbb{F}_r$-invariant finite-dimensional subalgebras $Q_n \subset L^\infty(X)$, then $M = L^\infty(X) \rtimes \mathbb{F}_r$ is an increasing limit of the algebras $Q_n \rtimes \mathbb{F}_r$, each one of which is an amplification of $L(\mathbb{F}_r)$. Since c.m.a.p. behaves well to amplifications and inductive limits, it follows that $M$ has c.m.a.p., so by applying the theorem and (A.1 in [Pop06a]) we get:

COROLLARY 3. *If $\mathbb{F}_r \curvearrowright X$ is a free ergodic measure-preserving profinite action, then $L^\infty(X)$ is the unique Cartan subalgebra of the* $\mathrm{II}_1$*-factor $L^\infty(X) \rtimes \mathbb{F}_r$, up to unitary conjugacy.*

The above corollary produces the first examples of nonamenable $\mathrm{II}_1$ factors with *all* Cartan subalgebras unitary conjugate. Indeed, the "unique Cartan decomposition" results in [Pop06a], [Pop06c], [IPP08] only showed conjugacy of Cartan subalgebras satisfying certain properties. This was still enough for differentiating factors of the form $L^\infty(\mathbb{T}^2) \rtimes \mathbb{F}_r$ and calculating their fundamental group in [Pop06a], by using [Gab02]. Similarly here, when combined with Gaboriau's results, Corollary 3 shows that any factor $L^\infty(X) \rtimes \mathbb{F}_r$, $2 \le r < \infty$, arising from a free ergodic profinite action $\mathbb{F}_r \curvearrowright X$, has trivial fundamental group. Also, if $\mathbb{F}_s \curvearrowright X$ is another such action, with $r < s \le \infty$, then $L^\infty(X) \rtimes \mathbb{F}_r \not\simeq L^\infty(Y) \rtimes \mathbb{F}_s$. It can be shown that the factors considered in [Pop06a], [Pop06c], [IPP08] cannot even be embedded into the factors arising from profinite actions of free groups. Note that the uniqueness of the Cartan subalgebras of the AFD factor $R$ is up to conjugacy by automorphisms ([CFW81]), but not up to unitary conjugacy, i.e. up to conjugacy by inner automorphisms. Indeed, by [FM77a], [FM77b] there exist uncountably many nonunitary conjugate Cartan subalgebras in $R$. Finally, note that Connes and Jones constructed examples of $\mathrm{II}_1$ factors $M$ with two Cartan subalgebras that are not conjugate by automorphisms of $M$ ([CJ82]).

Corollary 1 strengthens two well-known in-decomposability properties of free group factors: Voiculescu's result in [Voi96], showing that $L(\mathbb{F}_r)$ has no Cartan subalgebras, which in fact exhibited the first examples of factors with no Cartan decomposition, and the first named author's result in [Oza04a], showing that the commutant in $L(\mathbb{F}_r)$ of any diffuse subalgebra must be amenable ($L(\mathbb{F}_r)$ are *solid*), which itself strengthened the in-decomposability of $L(\mathbb{F}_r)$ into tensor product of $\mathrm{II}_1$ factors (*primeness* of free group factors) in [Ge98].

One should point out that Connes already constructed in [Con75] a factor $N$ that does not admit a "classic" group measure space decomposition $L^\infty(X) \rtimes \Gamma$. His factor $N$ is defined as the fixed point algebra of an appropriate finite group of automorphisms of $M = R \mathbin{\bar{\otimes}} L(\mathbb{F}_r)$. But it was left open whether $N$ cannot be obtained as a generalized group measure space factor either, i.e. whether it does not have Cartan decomposition. Corollary 2 shows that indeed it does not.

The proof of the theorem follows a "deformation/rigidity" strategy, being inspired by arguments in [Pop07] and [Pop06a]. A key role is played by a property of group actions $\Gamma \curvearrowright P$ called *weak compactness*, requiring $L^2(P)$ to be a limit of finite dimensional subspaces that are almost invariant to both the left multiplication by elements in $P$ and to the $\Gamma$-action, in the Hilbert-Schmidt norm. In case $P = L^\infty(X)$, this property is weaker than profiniteness and compactness, and it is an orbit equivalence invariant. The first step towards proving the theorem is to show that if a $\mathrm{II}_1$ factor $M$ has c.m.a.p. then given any AFD subalgebra $P \subset M$ the action implemented on $P$ by its normalizer, $\mathcal{N}_M(P) \curvearrowright P$, is weakly compact (see Theorem 3.5). Note that this implies wreath product factors $M = B^\Gamma \rtimes \Gamma$, with $\Gamma$ nonamenable and $B \neq \mathbb{C}$, can never have the c.m.a.p. In particular, $\Lambda_{\mathrm{cb}}(H \wr \Gamma) > 1$, for all $H \neq 1$, a fact that was open until now.

To explain the rest of the argument, assume for simplicity $M = L(\mathbb{F}_r)$. Let $P \subset M$ be AFD diffuse, $N = \mathcal{N}_M(P)''$. Taking

$$\eta \in HS(L^2(M)) \simeq L^2(M) \mathbin{\bar{\otimes}} L^2(\bar{M})$$

to be Følner-type elements, as given by the weak compactness of $\mathcal{N}_M(P) \curvearrowright P$, and $\alpha_t$ the "malleable deformation" of $L(\mathbb{F}_r) * L(\mathbb{F}_r)$ in [Pop06b], [Pop07], it follows that for $t$ small the elements $(\alpha_t \otimes 1)(\eta) \in L^2(M * M) \mathbin{\bar{\otimes}} L^2(\bar{M})$ are still "almost invariant," in the above sense. We finally use this to prove that $L^2(N)$ is weakly contained in a multiple of the *coarse* bimodule $L^2(M) \mathbin{\bar{\otimes}} L^2(\bar{M})$, thus showing $N$ is AFD by the characterizations of amenability in [Con76]. All this is the subject of Theorem 4.1 in the text.

We recall in Section 2 of the paper a number of known results needed in the proofs, for the reader's convenience. This includes a discussion of relative amenability (§2.2), intertwining lemmas for subalgebras (§2.3) and several facts on the complete metric approximation property (§2.4). We mention that in the last section of the paper we prove that for each $2 \leq r \leq \infty$ there exist uncountably many non orbit equivalent profinite actions $\mathbb{F}_r \curvearrowright X$, which by Corollary 3 provide uncountably many nonisomorphic factors $L^\infty(X) \rtimes \mathbb{F}_r$ as well (see Corollary 5.5).

## 2. **Preliminaries**

2.1. *Finite von Neumann algebras.* We fix conventions for (semi-)finite von Neumann algebras, but before that we note that the symbol "Lim" will be used for a state on $\ell^\infty(\mathbb{N})$, or more generally on $\ell^\infty(I)$ with $I$ directed, which extends the ordinary limit, and that the abbreviation "u.c.p." stands for "unital completely positive." We say a map is *normal* if it is ultraweakly continuous. Whenever a *finite* von Neumann algebra $M$ is being considered, it comes equipped with a distinguished faithful normal tracial state, denoted by $\tau$. Any group action on a finite von Neumann algebra is assumed to preserve the tracial state $\tau$. If $M = L(\Gamma)$ is a group von Neumann algebra, then the tracial state $\tau$ is given by $\tau(x) = \langle x\delta_1, \delta_1\rangle$ for $x \in L(\Gamma)$. Any von Neumann subalgebra $P \subset M$ is assumed to contain the unit of $M$ and inherits the tracial state $\tau$ from $M$. The unique $\tau$-preserving conditional expectation from $M$ onto $P$ is denoted by $E_P$. We denote by $\mathscr{Z}(M)$ the center of $M$; by $\mathscr{U}(M)$ the group of unitary elements in $M$; and by

$$\mathscr{N}_M(P) = \{u \in \mathscr{U}(M) : (\operatorname{Ad} u)(P) = P\}$$

the normalizing group of $P$ in $M$, where $(\operatorname{Ad} u)(x) = uxu^*$. A maximal abelian von Neumann subalgebra $A \subset M$ satisfying $\mathscr{N}_M(A)'' = M$ is called a *Cartan subalgebra*. We note that if $\Gamma \curvearrowright X$ is an ergodic essentially-free probability-measure-preserving action, then $A = L^\infty(X)$ is a Cartan subalgebra in the crossed product $L^\infty(X) \rtimes \Gamma$. (See [FM77a], [FM77b].)

We refer the reader to Section IX.2 of [Tak03] for the details of the following facts on noncommutative $L^p$-spaces. Let $\mathscr{N}$ be a semi-finite von Neumann algebra with a faithful normal semi-finite trace Tr. For $1 \le p < \infty$, we define the $L^p$-norm on $\mathscr{N}$ by $\|x\|_p = \mathrm{Tr}(|x|^p)^{1/p}$. By completing $\{x \in \mathscr{N} : \|x\|_p < \infty\}$ with respect to the $L^p$-norm, we obtain a Banach space $L^p(\mathscr{N})$. We only need $L^1(\mathscr{N})$, $L^2(\mathscr{N})$ and $L^\infty(\mathscr{N}) = \mathscr{N}$. The trace Tr extends to a contractive linear functional on $L^1(\mathscr{N})$. We occasionally write $\hat{x}$ for $x \in \mathscr{N}$ when viewed as an element in $L^2(\mathscr{N})$. For any $1 \le p, q, r \le \infty$ with $1/p + 1/q = 1/r$, there is a natural product map

$$L^p(\mathscr{N}) \times L^q(\mathscr{N}) \ni (x, y) \mapsto xy \in L^r(\mathscr{N})$$

which satisfies $\|xy\|_r \le \|x\|_p\|y\|_q$ for any $x$ and $y$. The Banach space $L^1(\mathscr{N})$ is identified with the predual of $\mathscr{N}$ under the duality $L^1(\mathscr{N}) \times \mathscr{N} \ni (\zeta, x) \mapsto \mathrm{Tr}(\zeta x) \in \mathbb{C}$. The Banach space $L^2(\mathscr{N})$ is identified with the GNS-Hilbert space of $(\mathscr{N}, \mathrm{Tr})$. Elements in $L^p(\mathscr{N})$ can be regarded as closed operators on $L^2(\mathscr{N})$ which are affiliated with $\mathscr{N}$ and hence in addition to the above-mentioned product, there are well-defined notion of positivity, square root, etc. We will use many times the generalized Powers-Størmer inequality (Theorem XI.1.2 in [Tak03]):

$$\|\eta - \zeta\|_2^2 \le \|\eta^2 - \zeta^2\|_1 \le \|\eta + \zeta\|_2 \|\eta - \zeta\|_2 \tag{2.1}$$

for every $\eta, \zeta \in L^2(\mathscr{N})_+$. The Hilbert space $L^2(\mathscr{N})$ is an $\mathscr{N}$-bimodule such that $\langle x\xi y, \eta\rangle = \mathrm{Tr}(x\xi y\eta^*)$ for $\xi, \eta \in L^2(\mathscr{N})$ and $x, y \in \mathscr{N}$. We recall that this gives the canonical identification between the commutant $\mathscr{N}'$ of $\mathscr{N}$ in $\mathbb{B}(L^2(\mathscr{N}))$ and the opposite von Neumann algebra $\mathcal{N}^{\mathrm{op}} = \{x^{\mathrm{op}} : x \in \mathcal{N}\}$ of $\mathcal{N}$. Moreover, the opposite von Neumann algebra $\mathcal{N}^{\mathrm{op}}$ is $*$-isomorphic to the complex conjugate von Neumann algebra $\overline{\mathcal{N}} = \{\bar{x} : x \in \mathcal{N}\}$ of $\mathcal{N}$ under the $*$-isomorphism $x^{\mathrm{op}} \mapsto \bar{x}^*$.

Whenever $\mathcal{N}_0 \subset \mathcal{N}$ is a von Neumann subalgebra such that the restriction of $\mathrm{Tr}$ to $\mathcal{N}_0$ is still semi-finite, we identify $L^p(\mathcal{N}_0)$ with the corresponding subspace of $L^p(\mathcal{N})$. Anticipating a later use, we consider the tensor product von Neumann algebra $(\mathcal{N} \mathbin{\overline{\otimes}} M, \mathrm{Tr} \otimes \tau)$ of a semi-finite von Neumann algebra $(\mathcal{N}, \mathrm{Tr})$ and a finite von Neumann algebra $(M, \tau)$. Then, $\mathcal{N} \cong \mathcal{N} \mathbin{\overline{\otimes}} \mathbb{C}1 \subset \mathcal{N} \mathbin{\overline{\otimes}} M$ and the restriction of $\mathrm{Tr} \otimes \tau$ to $\mathcal{N}$ is $\mathrm{Tr}$. Moreover, the conditional expectation $\mathrm{id} \otimes \tau : \mathcal{N} \mathbin{\overline{\otimes}} M \to \mathcal{N}$ extends to a contraction from $L^1(\mathcal{N} \mathbin{\overline{\otimes}} M) \to L^1(\mathcal{N})$.

Let $Q \subset M$ be finite von Neumann algebras. Then, the conditional expectation $E_Q$ can be viewed as the orthogonal projection $e_Q$ from $L^2(M)$ onto $L^2(Q) \subset L^2(M)$. It satisfies $e_Q x e_Q = E_Q(x) e_Q$ for every $x \in M$. The *basic construction* $\langle M, e_Q \rangle$ is the von Neumann subalgebra of $\mathbb{B}(L^2(M))$ generated by $M$ and $e_Q$. We note that $\langle M, e_Q \rangle$ coincides with the commutant of the right $Q$-action in $\mathbb{B}(L^2(M))$. The linear span of $\{x e_Q y : x, y \in M\}$ is an ultraweakly dense $*$-subalgebra in $\langle M, e_Q \rangle$ and the basic construction $\langle M, e_Q \rangle$ comes together with the faithful normal semi-finite trace $\mathrm{Tr}$ such that $\mathrm{Tr}(x e_Q y) = \tau(xy)$. See Section 1.3 in [Pop06a] for more information on the basic construction.

2.2. *Relative amenability.* We adapt here Connes' characterization of amenable von Neumann algebras to the relative situation. Recall that for von Neumann algebras $N \subset \mathcal{N}$, a state $\varphi$ on $\mathcal{N}$ is said to be *$N$-central* if $\varphi \circ \mathrm{Ad}(u) = \varphi$ for any $u \in \mathcal{U}(N)$, or equivalently if $\varphi(ax) = \varphi(xa)$ for all $a \in N$ and $x \in \mathcal{N}$.

THEOREM 2.1. *Let $Q, N \subset M$ be finite von Neumann algebras. Then, the following are equivalent*:

(1) *There exists a $N$-central state $\varphi$ on $\langle M, e_Q \rangle$ such that $\varphi|_M = \tau$.*

(2) *There exists a $N$-central state $\varphi$ on $\langle M, e_Q \rangle$ such that $\varphi$ is normal on $M$ and faithful on $\mathcal{Z}(N' \cap M)$.*

(3) *There exists a conditional expectation $\Phi$ from $\langle M, e_Q \rangle$ onto $N$ such that $\Phi|_M = E_N$.*

(4) *There exists a net $(\xi_n)$ in $L^2\langle M, e_Q \rangle$ such that* $\lim_n \langle x\xi_n, \xi_n \rangle = \tau(x)$ *for every $x \in M$ and that* $\lim \|[u, \xi_n]\|_2 = 0$ *for every $u \in N$.*

*Definition* 2.2. Let $Q, N \subset M$ be finite von Neumann algebras. We say $N$ is *amenable relative to $Q$ inside $M$*, denoted by $N \lessdot_M Q$, if any of the conditions in Theorem 2.1 holds. We say $Q$ is *co-amenable in $M$* if $M \lessdot_M Q$ (cf. [Pop86], [AD95]).

*Proof of Theorem* 2.1. The proof follows a standard recipe of the theory (cf. [Con76], [Haa85], [Pop86]). The implication (1) $\Rightarrow$ (2) is obvious. To prove the converse, assume condition (2). Then, there exists $b \in L^1(M)_+$ such that

$\varphi(x) = \tau(bx)$ for $x \in M$. Since $\varphi$ is $N$-central, one has $ubu^* = b$ for all $u \in \mathcal{U}(N)$, i.e. $b \in L^1(N' \cap M)$. We consider the directed set $I$ of finite subsets of $\mathcal{U}(N' \cap M)$. For each element $i = \{u_1, \ldots, u_n\} \in I$ and $m \in \mathbb{N}$, we define $b_i = n^{-1} \sum u_k b u_k^* \in L^1(N' \cap M)_+$, $c_{i,m} = \chi_{(1/m,\infty)}(b_i) b_i^{-1/2} \in N' \cap M$ and

$$\psi_{i,m}(x) = \frac{1}{n} \sum_{k=1}^{n} \varphi(u_k^* c_{i,m} x c_{i,m} u_k)$$

for $x \in \langle M, e_Q \rangle$. Since $c_{i,m} u_k \in N' \cap M$, the positive linear functionals $\psi_{i,m}$ are still $N$-central and $\psi_{i,m}(x) = \tau(\chi_{(1/m,\infty)}(b_i)x)$ for $x \in M$. We note that

$$\lim_i \lim_m \chi_{(1/m,\infty)}(b_i) = \lim_i s(b_i) = \lim_i \bigvee s(u_k b u_k^*) = z,$$

where $s(\cdot)$ means the support projection and $z$ is the central support projection of $b$ in $N' \cap M$. Since $\varphi(z^\perp) = \tau(bz^\perp) = 0$ and $\varphi$ is faithful on $\mathcal{Z}(N' \cap M)$, one has $z = 1$. Hence, the state $\psi = \mathrm{Lim}_i \, \mathrm{Lim}_m \, \psi_{i,m}$ on $\langle M, e_Q \rangle$ is $N$-central and satisfies $\psi|_M = \tau$. This proves (1).

We prove (1) $\Rightarrow$ (4): Let a $N$-central state $\varphi$ on $\langle M, e_Q \rangle$ be given such that $\varphi|_M = \tau$. Take a net $(\zeta_n)$ of positive norm-one elements in $L^1\langle M, e_Q \rangle$ such that $\mathrm{Tr}(\zeta_n \, \cdot)$ converges to $\varphi$ pointwise. Then, for every $x \in \langle M, e_Q \rangle$ and $u \in \mathcal{U}(N)$, one has

$$\lim_n \mathrm{Tr}((\zeta_n - \mathrm{Ad}(u)\zeta_n)x) = \varphi(x) - \varphi(\mathrm{Ad}(u^*)(x)) = 0$$

by assumption. It follows that for every $u \in \mathcal{U}(N)$, the net $\zeta_n - \mathrm{Ad}(u)(\zeta_n)$ in $L^1\langle M, e_Q \rangle$ converges to zero in the weak-topology. By the Hahn-Banach separation theorem, one may assume, by passing to convex combinations, that it converges to zero in norm. Thus, $\|[u, \zeta_n]\|_1 \to 0$ for every $u \in \mathcal{U}(N)$. By (2.1), if we define $\xi_n = \zeta_n^{1/2} \in L^2\langle M, e_Q \rangle$, then one has $\|[u, \xi_n]\|_2 \to 0$ for every $u \in \mathcal{U}(N)$. Moreover, for any $x \in M$,

$$\lim_n \langle x\xi_n, \xi_n \rangle = \lim_n \mathrm{Tr}(\zeta_n x) = \varphi(x) = \tau(x).$$

We prove (4) $\Rightarrow$ (3): For each $x \in \langle M, e_Q \rangle$, denote $\varphi(x) = \mathrm{Lim}_n \langle x\xi_n, \xi_n \rangle$. Note that $\varphi$ is an $N$-central sate on $\langle M, e_Q \rangle$ with $\varphi_M = \tau$. Since

$$|\varphi(bcyz)| = |\varphi(cyzb)| \le \varphi(cyy^*c^*)^{1/2} \varphi(b^*z^*zb)^{1/2} \le \|b\|_2 \|c\|_2 \|y\| \|z\|$$

for every $b, c \in N$ and $y, z \in \langle M, e_Q \rangle$, one has $|\varphi(ax)| \le \|a\|_1 \|x\|$ for every $a \in N$ and $x \in \langle M, e_Q \rangle$. Hence, for every $x \in \langle M, e_Q \rangle$, we may define $\Phi(x) \in N = L^1(N)^*$ by the duality $\tau(a\Phi(x)) = \varphi(ax)$ for all $a \in N$. It is clear that $\Phi$ is a conditional expectation onto $N$ such that $\Phi|_M = E_N$.

We prove (3) $\Rightarrow$ (1): If there is a conditional expectation $\Phi$ from $\langle M, e_Q \rangle$ onto $N$ such that $\Phi|_M = E_N$, then $\varphi = \tau \circ \Phi$ is an $N$-central state such that $\varphi|_M = \tau$. $\square$

Let $N_0 \subset M$ be a von Neumann subalgebra whose unit $e$ does not coincide with the unit of $M$. We say $N_0$ is amenable relative to $Q$ inside $M$, denoted by

$N_0 \prec_M Q$, if $N_0 + \mathbb{C}(1-e) \prec_M Q$. We observe that $N_0 \prec_M Q$ if and only if there exists an $N_0$-central state $\varphi$ on $e\langle M, e_Q\rangle e$ such that $\varphi(exe) = \tau(exe)/\tau(e)$ for $x \in M$.

COROLLARY 2.3. *Let $Q_1, \dots, Q_k, N \subset M$ be finite von Neumann algebras and $\mathcal{G} \subset \mathcal{U}(N)$ be a subgroup such that $\mathcal{G}'' = N$. Assume that for every nonzero projection $p \in \mathcal{Z}(N' \cap M)$, there exists a net $(\xi_n)$ of vectors in a multiple of $\bigoplus_{j=1}^k L^2\langle M, e_{Q_j}\rangle$ such that*:

(1) $\limsup \|x\xi_n\|_2 \le \|x\|_2$ *for all* $x \in M$;

(2) $\liminf \|p\xi_n\|_2 > 0$; *and*

(3) $\lim \|[u, \xi_n]\|_2 = 0$ *for every* $u \in \mathcal{G}$.

*Then, there exist projections $p_1, \dots, p_k \in \mathcal{Z}(N' \cap M)$ such that $\sum_{j=1}^k p_j = 1$ and $Np_j \prec_M Q_j$ for every $j$.*

*Proof.* We observe that if there exists an increasing net $(e_i)_i$ of projections in $\mathcal{Z}(N' \cap M)$ such that $Ne_i \prec_M Q$ for all $i$, then $Ne \prec_M Q$ for $e = \sup e_i$. Hence, by Zorn's lemma, there is a maximal $k$-tuple $(p_1, \dots, p_k)$ of projections in $\mathcal{Z}(N' \cap M)$ such that $\sum_j p_j \le 1$ and $Np_j \prec_M Q_j$ for every $j$. We prove that $\sum_j p_j = 1$. Suppose by contradiction that $p = 1 - \sum_j p_j \ne 0$, and take a net $(\xi_n)$ as in the statement of the corollary. We may assume that all $\xi_n$'s are in a multiple of $L^2\langle M, e_{Q_j}\rangle$ for some fixed $j \in \{1, \dots, k\}$. We define a state $\psi$ on $\langle M, e_{Q_j}\rangle$ by

$$\psi(x) = \operatorname*{Lim}_n \|p\xi_n\|_2^{-2} \langle xp\xi_n, p\xi_n\rangle$$

for $x \in \langle M, e_{Q_j}\rangle$. It is not hard to see that $\psi(p) = 1$, $\psi \circ \mathrm{Ad}(u) = \psi$ for every $u \in \mathcal{G}$ and $\psi(x^*x) \le (\liminf \|p\xi_n\|)^{-2} \|xp\|_2^2$ for every $x \in M$. It follows that $\psi|_M$ is normal and $\psi$ is $N$-central. Let $q$ be the minimal projection in $\mathcal{Z}(N' \cap M)$ such that $\psi(q) = 1$. We finish the proof by showing $Nr \prec_M Q_j$ for $r = p_j + q$ (which gives the desired contradiction to maximality). Since $Np_j \prec_M Q_j$, there is an $Np_j$-central state $\varphi$ on $p_j\langle M, e_{Q_j}\rangle p_j$ such that $\varphi(p_jxp_j) = \tau(p_jxp_j)/\tau(p_j)$ for $x \in M$. We fix a state extension $\tilde{\tau}$ of $\tau$ on $\langle M, e_{Q_j}\rangle$ and define a state $\widetilde{\varphi}$ on $\langle M, e_{Q_j}\rangle$ by

$$\widetilde{\varphi}(x) = \tau(p_j)\varphi(p_jxp_j) + \tau(q)\psi(qxq) + \tilde{\tau}((1-r)x(1-r))$$

for $x \in \langle M, e_{Q_j}\rangle$. The state $\widetilde{\varphi}$ is $(Nr + \mathbb{C}(1-r))$-central, normal on $M$ and faithful on $\mathcal{Z}((Nr + \mathbb{C}(1-r))' \cap M) = \mathcal{Z}(N' \cap M)r + \mathcal{Z}(M)(1-r)$. Hence Theorem 2.1 implies $Nr \prec_M Q_j$. □

Compare the following result with [Pop86] and [AD95].

PROPOSITION 2.4. *Let $P, Q, N \subset M$ be finite von Neumann algebras. Then, the following are true*:

(1) *Suppose that $M = Q \rtimes \Gamma$ is the crossed product of $Q$ by a group $\Gamma$. Then, $L(\Gamma) \prec_M Q$ if and only if $\Gamma$ is amenable.*

(2) *Suppose that* $Q$ *is* AFD. *Then*, $P \prec_M Q$ *if and only if* $P$ *is* AFD.

(3) *If* $N \prec_M P$ *and* $P \prec_M Q$, *then* $N \prec_M Q$.

*Proof.* Denote by $\lambda_g$ the unitary element in $M$ which implements the action of $g \in \Gamma$. Since $e_Q\lambda(g)e_Q = 0$ for $g \in \Gamma \setminus \{1\}$, the projections $\{\lambda_g e_Q \lambda_g^* : g \in \Gamma\}$ are mutually orthogonal and generate an isomorphic copy of $\ell^\infty(\Gamma)$ in $\langle M, e_Q\rangle$. Hence, if there exists an $L(\Gamma)$-central state on $\langle M, e_Q\rangle$, then its restriction to $\ell^\infty(\Gamma)$ becomes a $\Gamma$-invariant mean. This proves the "only if" part of assertion (1). The "if" part is trivial. The assertion (2) easily follows from the fact that $\langle M, e_Q\rangle$ is injective if (and only if) $Q$ is AFD ([Con76]).

Let us finally prove (3). Fix a conditional expectation $\Phi$ from $\langle M, e_Q\rangle$ onto $P$ such that $\Phi|_M = E_P$. For $\xi = \sum_{i=1}^m a_i \otimes b_i \in M \otimes M$, we denote

$$\|\xi\|_2 = \|\sum_{i=1}^m a_i e_P b_i\|_{L^2\langle M, e_P\rangle} = \Big(\sum_{i,j} \tau(b_i^* E_P(a_i^* a_j) b_j)\Big)^{1/2}.$$

For $\xi = \sum_{i=1}^m a_i \otimes b_i$ and $\eta = \sum_{j=1}^n c_j \otimes d_j$ in $M \otimes M$, we define a linear functional $\varphi_{\eta,\xi}$ on $\langle M, e_Q\rangle$ by

$$\varphi_{\eta,\xi}(x) = \sum_{i,j} \tau(b_i^* \Phi(a_i^* x c_j) d_j).$$

We claim that $\|\varphi_{\eta,\xi}\| \leq \|\eta\|_2 \|\xi\|_2$. Indeed, if $\Phi(x) = V^*\pi(x)V$ is a Stinespring dilation, then one has

$$\varphi_{\eta,\xi}(x) = \langle \pi(x) \sum_j \pi(c_j) V d_j \hat{1}_P, \sum_i \pi(a_i) V b_i \hat{1}_P\rangle$$

and $\|\sum_i \pi(a_i) V b_i \hat{1}_P\| = \|\xi\|_2$ and likewise for $\eta$. It follows that $\varphi_{\eta,\xi}$ is defined for $\xi, \eta \in L^2\langle M, e_P\rangle$ in such a way that $\|\varphi_{\eta,\xi}\| \leq \|\eta\|_2\|\xi\|_2$. Now take a net of unit vectors $(\xi_n)$ in $L^2\langle M, e_P\rangle$ satisfying condition (4) in Theorem 2.1, and let $\varphi = \operatorname{Lim} \varphi_{\xi_n,\xi_n}$ be the state on $\langle M, e_Q\rangle$. Then, one has

$$\varphi \circ \operatorname{Ad}(u) = \operatorname{Lim}_n \varphi_{\operatorname{Ad}(u)(\xi_n), \operatorname{Ad}(u)(\xi_n)} = \operatorname{Lim}_n \varphi_{\xi_n,\xi_n} = \varphi$$

for all $u \in \mathcal{U}(N)$ and

$$\varphi(x) = \operatorname{Lim}_n \langle x\xi_n, \xi_n\rangle_{L^2\langle N, e_P\rangle} = \tau(x)$$

for all $x \in M$. This proves that $N \prec_M Q$. $\square$

2.3. *Intertwining subalgebras inside* $\mathrm{II}_1$ *factors.* We extract from [Pop06a], [Pop06c] some results which are needed later. The following are Theorem A.1 in [Pop06a] and its corollary (also, a particular case of 2.1 in [Pop06c]).

THEOREM 2.5. *Let* $N$ *be a finite von Neumann algebra and* $P, Q \subset N$ *be von Neumann subalgebras. Then, the following are equivalent*:

(1) *There exists a nonzero projection* $e \in \langle N, e_Q\rangle$ *with* $\operatorname{Tr}(e) < \infty$ *such that the ultraweakly closed convex hull of* $\{w^* e w : w \in \mathcal{U}(P)\}$ *does not contain* 0.

(2) *There exist nonzero projections* $p \in P$ *and* $q \in Q$, *a normal* $*$-*homomorphism* $\theta\colon pPp \to qQq$ *and a nonzero partial isometry* $v \in N$ *such that*

$$\text{for all } x \in pPp, \quad xv = v\theta(x)$$

*and* $v^*v \in \theta(pPp)' \cap qNq$, $vv^* \in p(P' \cap N)p$.

*Definition* 2.6. Let $P, Q \subset N$ be finite von Neumann algebras. Following [Pop06c], we say that $P$ *embeds into* $Q$ *inside* $N$, and write $P \preceq_N Q$, if any of the conditions in Theorem 2.5 holds.

Let $\phi$ be a $\tau$-preserving u.c.p. map on $N$. Then, $\phi$ extends to a contraction $T_\phi$ on $L^2(N)$ by $T_\phi(\hat{x}) = \widehat{\phi(x)}$. Suppose that $\phi|_Q = \mathrm{id}_Q$. Then, $\phi$ automatically satisfies $\phi(axb) = a\phi(x)b$ for any $a, b \in Q$ and $x \in N$. It follows that $T_\phi \in \mathbb{B}(L^2(N))$ commutes with the right action of $Q$, i.e., $T_\phi \in \langle N, e_Q\rangle$. We say $\phi$ is *compact over* $Q$ if $T_\phi$ belongs to the "compact ideal" of $\langle N, e_Q\rangle$ (see §1.3.2 in [Pop06a]). If $\phi$ is compact over $Q$, then for any $\varepsilon > 0$, the spectral projection $e = \chi_{[\varepsilon,1]}(T_\phi^* T_\phi) \in \langle N, e_Q\rangle$ has finite $\mathrm{Tr}(e)$ and

$$\langle w^* e w \hat{1}, \hat{1}\rangle_{L^2(N)} \geq \langle T_\phi^* T_\phi \widehat{w}, \widehat{w}\rangle_{L^2(N)} - \varepsilon = \|\phi(w)\|_2^2 - \varepsilon$$

for all $w \in \mathcal{U}(P)$. These observations imply the following corollary [Pop06a].

COROLLARY 2.7. *Let* $P, Q \subset N$ *be finite von Neumann algebras. Suppose that* $\phi$ *is a* $\tau$-*preserving* u.c.p. *map on* $N$ *such that* $\phi|_Q = \mathrm{id}_Q$ *and* $\phi$ *is compact over* $Q$. *If* $\inf\{\|\phi(w)\|_2 : w \in \mathcal{U}(P)\} > 0$, *then* $P \preceq_N Q$.

Finally, recall that A.1 in [Pop06a] shows the following:

LEMMA 2.8. *Let* $A$ *and* $B$ *be maximal abelian* $^*$-*subalgebras of a type* $\mathrm{II}_1$-*factor* $N$. *If* $A \preceq_N B$, *then there exists a nonzero partial isometry* $v \in N$ *such that* $v^*v \in A$, $vv^* \in B$ *and* $vAv^* = Bvv^*$. *If, moreover,* $\mathcal{N}_N(A)''$, $\mathcal{N}_N(B)''$ *are factors* (*i.e.* $A$, $B$ *are semiregular* [Dix54]), *then* $v$ *can be taken a unitary element.*

2.4. *The complete metric approximation property.* Let $\Gamma$ be a discrete group. For a function $f$ on $\Gamma$, we write $m_f$ for the multiplier on $\mathbb{C}\Gamma \subset L(\Gamma)$ defined by $m_f(g) = fg$ for $g \in \mathbb{C}\Gamma$. We simply write $\|f\|_{\mathrm{cb}}$ for $\|m_f\|_{\mathrm{cb}}$ and call it the Herz-Schur norm. If $\|f\|_{\mathrm{cb}}$ is finite and $f(1) = 1$, then $m_f$ extends to a $\tau$-preserving normal unital map on $L(\Gamma)$. We refer the reader to Sections 5 and 6 in [Pis01] for an account of Herz-Schur multipliers.

*Definition* 2.9. A discrete group $\Gamma$ is *weakly amenable* if there exist a constant $C \geq 1$ and a net $(f_n)$ of finitely supported functions on $\Gamma$ such that $\limsup \|f_n\|_{\mathrm{cb}} \leq C$ and $f_n \to 1$ pointwise. The Cowling-Haagerup constant $\Lambda_{\mathrm{cb}}(\Gamma)$ of $\Gamma$ is defined as the infimum of the constant $C$ for which a net $(f_n)$ as above exists.

We say a von Neumann algebra $M$ has the (*weak*$^*$) *completely bounded approximation property* if there exist a constant $C \geq 1$ and a net $(\phi_n)$ of normal finite-rank maps on $M$ such that $\limsup \|\phi_n\|_{\mathrm{cb}} \leq C$ and $\|x - \phi_n(x)\|_2 \to 0$ for every $x \in M$. The Cowling-Haagerup constant $\Lambda_{\mathrm{cb}}(M)$ of $M$ is defined as the infimum of the constant $C$ for which a net $(\phi_n)$ as above exists. Also, we say that $M$ has the (*weak**) *complete metric approximation property* (c.m.a.p.) if $\Lambda_{\rm cb}(M) = 1$. Note that, by Connes' theorem [Con76], amenability trivially implies c.m.a.p.

By routine perturbation arguments, one may arrange $\phi_n$'s in the above definition to be unital and trace-preserving when $M$ is finite. We are interested here in the case $\Lambda_{\rm cb}(M) = 1$, i.e. when $M$ has the complete metric approximation property. We summarize below some known results in this direction. For part (7), recall that an action of a group $\Gamma$ on a finite von Neumann algebra $P$ is *profinite* if there exists an increasing sequence of $\Gamma$-invariant finite-dimensional von Neumann subalgebras $P_n \subset P$ that generate $P$. Note that this implies $P$ is AFD. If $P = L^\infty(X)$ is abelian and $\Gamma \curvearrowright P$ comes from a m.p. action $\Gamma \curvearrowright X$, then the profiniteness of $\Gamma \curvearrowright P$ amounts to the existence of a sequence of $\Gamma$-invariant finite partitions of $X$ that generate the $\sigma$-algebra of measurable subsets of $X$.

THEOREM 2.10. (1) $\Lambda_{\rm cb}(L(\Gamma)) = \Lambda_{\rm cb}(\Gamma)$ *for any* $\Gamma$.

(2) *If* $\Gamma$ *is a discrete subgroup of* $\mathrm{SO}(1,n)$ *or of* $\mathrm{SU}(1,n)$, *then* $\Lambda_{\rm cb}(\Gamma) = 1$.

(3) *If* $\Gamma$ *acts properly on a finite-dimensional* CAT(0) *cubical complex*, *then* $\Lambda_{\rm cb}(\Gamma) = 1$.

(4) *If* $\Lambda_{\rm cb}(\Gamma_i) = 1$ *for* $i = 1, 2$, *then* $\Lambda_{\rm cb}(\Gamma_1 \times \Gamma_2) = 1$ *and* $\Lambda_{\rm cb}(\Gamma_1 * \Gamma_2) = 1$.

(5) *If* $N \subset M$ *are finite von Neumann algebras*, *then* $\Lambda_{\rm cb}(N) \leq \Lambda_{\rm cb}(M)$. *Moreover*, *if* $N, M$ *are factors and* $[M : N] < \infty$, *then* $\Lambda_{\rm cb}(M) = \Lambda_{\rm cb}(N)$ *and* $\Lambda_{\rm cb}(M^t) = \Lambda_{\rm cb}(M)$, *for all* $t > 0$.

(6) *Let* $M$ *be a finite von Neumann algebra and* $(M_n)$ *be an increasing net of von Neumann subalgebras of* $M$ *such that* $M = (\bigcup M_n)''$. *Then*, $\Lambda_{\rm cb}(M) = \sup \Lambda_{\rm cb}(M_n)$.

(7) *If* $P$ *is a finite von Neumann algebra and* $\Gamma \curvearrowright P$ *is a profinite action*, *then* $\Lambda_{\rm cb}(P \rtimes \Gamma) = \Lambda_{\rm cb}(\Gamma)$.

The assertions (1), (2), (3) and (4) are respectively due to [CH89], [DCH85], [Cow83], [GH07] and [RX06]. The rest are trivial. We will see in Corollary 3.3 that property (7) generalizes to compact actions of groups $\Gamma$, and even to actions of $\Gamma$ that are "weakly compact", in the sense of Definition 3.1.

We prove in this paper a general property about normal amenable subgroups of groups with $\Lambda_{\rm cb}$-constant equal to 1. While this property is a consequence of Theorem 3.5 (via (3) $\Leftrightarrow$ (4) in Proposition 3.2), we give here a direct proof in group-theoretic framework. To this end, note that if $\Lambda \triangleleft \Gamma$ is a normal subgroup then the semi-direct product group $\Lambda \rtimes \Gamma$ acts on $\Lambda$ by $(a, g)b = agbg^{-1}$, for $(a, g) \in \Lambda \rtimes \Gamma$ and $b \in \Lambda$.

PROPOSITION 2.11. *Suppose that* $\Gamma$ *has an infinite normal amenable subgroup* $\Lambda \triangleleft \Gamma$ *and that* $\Lambda_{\rm cb}(\Gamma) = 1$. *Then there exists a* $\Lambda \rtimes \Gamma$-*invariant mean on* $\ell^\infty(\Lambda)$ (*i.e.*, $\Gamma$ *is co-amenable in* $\Lambda \rtimes \Gamma$). *In particular*, $\Gamma$ *is inner-amenable.* (*See* §5 *for the definition of inner-amenability.*)

*Proof.* Let $f_n$ be a net of finitely supported functions such that $\sup \|f_n\|_{\mathrm{cb}} = 1$ and $f_n \to 1$ pointwise. By the Bożejko-Fendler theorem (Theorem 6.4 in [Pis01]), there are Hilbert space vectors $\xi_n(a)$ and $\eta_n(b)$ of norm at most one such that $f_n(ab^{-1}) = \langle \eta_n(b), \xi_n(a)\rangle$ for all $a, b \in \Gamma$. Then, for every $g \in \Gamma$, one has

$$\begin{aligned}\lim_n \sup_{a\in\Gamma} \|\xi_n(ga) - \xi_n(a)\|^2 &\le \lim_n \sup_{a\in\Gamma} 2\big(\|\xi_n(ga) - \eta_n(a)\|^2 + \|\eta_n(a) - \xi_n(a)\|^2\big)\\ &\le \lim_n 2(2 - 2\Re f_n(g) + 2 - 2\Re f_n(1)) = 0,\end{aligned}$$

and similarly $\lim_n \sup_{b\in\Gamma} \|\eta_n(gb) - \eta_n(b)\| = 0$ for every $g \in \Gamma$. It follows that

$$\lim_n \|f_n - f_n^g\|_{\mathrm{cb}} = 0$$

for every $g \in \Gamma$, where $f_n^g \in \mathbb{C}\Gamma$ is defined by $f_n^g(a) = f_n(gag^{-1})$. Now since $\Lambda \lhd \Gamma$ is amenable, the trivial representation $\tau_0 \colon C^*_{\mathrm{red}}(\Lambda) \to \mathbb{C}$ is continuous. We define a linear functional $\omega_n$ on $C^*_{\mathrm{red}}(\Lambda)$ by $\omega_n = \tau_0 \circ m_{f_n}|_{C^*_{\mathrm{red}}(\Lambda)}$. Since $f_n$ is finitely supported, $\omega_n$ is ultraweakly continuous on $L(\Lambda)$. We note that $\lim \omega_n(\lambda(a)) = 1$ for all $a \in \Lambda$ and

$$\lim_n \|\omega_n - \omega_n \circ \mathrm{Ad}(g)\| \le \lim_n \|f_n - f_n^g\|_{\mathrm{cb}} = 0$$

for all $g \in \Gamma$. Since $\|\omega_n\| \le 1$ and $\lim \omega_n(1) = 1$, we have $\lim \|\omega_n - |\omega_n|\| = 0$. We view $|\omega_n|$ as an element in $L^1(L(\Lambda))$ (which is $L^1(\widehat{\Lambda})$ if $\Lambda$ is abelian) and consider $\zeta_n = |\omega_n|^{1/2} \in L^2(L(\Lambda)) = \ell^2(\Lambda)$. Then, the net $(\zeta_n)$ satisfies $\lim_n \langle \lambda(a)\zeta_n, \zeta_n\rangle = 1$ for all $a \in \Lambda$ and $\lim_n \|\zeta_n - \mathrm{Ad}(g)(\zeta_n)\|_2 = 0$ for all $g \in \Gamma$ by (2.1). Therefore, the state $\omega$ on $\ell^\infty(\Lambda) \subset \mathbb{B}(\ell^2(\Lambda))$ defined by

$$\omega(x) = \operatorname*{Lim}_n \langle x\zeta_n, \zeta_n\rangle = \operatorname*{Lim}_n \sum_{a\in\Lambda} x(a)\zeta_n(a)^2$$

is $\Lambda \rtimes \Gamma$-invariant. Since $\Lambda$ is infinite, the $\Lambda$-invariant mean $\omega$ is singular, i.e, $\zeta_n \to 0$ weakly. This implies inner-amenability of $\Gamma$. $\square$

Recall that the wreath product $H \wr \Gamma_0$ of a group $H$ by a group $\Gamma_0$ is defined as the semi-direct product $(\bigoplus_{\Gamma_0} H) \rtimes \Gamma_0$ of $\bigoplus_{\Gamma_0} H$ by the shift action $\Gamma_0 \curvearrowright \bigoplus_{\Gamma_0} H$.

COROLLARY 2.12. *If* $\Gamma_0$ *is nonamenable and* $H \ne \{1\}$, *then* $\Lambda_{\mathrm{cb}}(H \wr \Gamma_0) > 1$, *i.e.* $L(H \wr \Gamma_0)$ *does not have* c.m.a.p. *Also*, *if* $\Gamma$ *is a nonamenable group having a nontrivial normal amenable subgroup* $\Lambda$ *such that the centralizer* $\mathfrak{Z}(a) = \{g \in \Gamma : ga = ag\}$ *of any nonneutral element* $a \in \Lambda$ *is amenable*, *then* $\Lambda_{\mathrm{cb}}(\Gamma) > 1$.

*Proof.* Suppose that $\Gamma_0$ is nonamenable and $\Lambda_{\mathrm{cb}}(H \wr \Gamma_0) = 1$. Passing to a subgroup if necessary, we may assume that $H$ is cyclic. Thus $\Lambda = \bigoplus_{\Gamma_0} H$ is a nontrivial normal amenable subgroup of $\Gamma = H \wr \Gamma_0$ such that the centralizer of any nonneutral element of $\Lambda$ is amenable (finite). It is thus sufficient to prove the second part of the statement. We consider $\Lambda$ as a set on which $\Gamma$ acts by conjugation. Then, $\Lambda \setminus \{1\} = \bigsqcup_{a\in X} \Gamma/\mathfrak{Z}(a)$ as a $\Gamma$-set, where $X$ is a system of representatives of $\Gamma$-orbits of $\Lambda \setminus \{1\}$. We observe that there is a $\Gamma$-equivariant u.c.p. map from $\ell^\infty(\Gamma)$ into $\ell^\infty(\Gamma/\mathfrak{Z}(a))$, which is given by a fixed right $\mathfrak{Z}(a)$-invariant mean applied to each coset $g\mathcal{Z}(a) \subset \Gamma$. Hence, there is a $\Gamma$-equivariant u.c.p. map from $\ell^\infty(\Gamma)$ into $\ell^\infty(\Lambda \setminus \{1\})$. Since $\Gamma$ is nonamenable, there is no $\Gamma$-invariant mean on $\Lambda \setminus \{1\}$. Hence, any $\Gamma$-invariant mean on $\Lambda$ has to be concentrated on $\{1\}$. Such mean cannot be $\Lambda$-invariant. This is in contradiction with Proposition 2.11. $\square$

*Remark* 2.13. Let $\Gamma = (\mathbb{Z}/2\mathbb{Z}) \wr \mathbb{F}_2$. Since $\Lambda_{\mathrm{cb}}$ is multiplicative ([CH89]) and satisfies $\Lambda_{\mathrm{cb}}(\Gamma) > 1$ (by 2.12), the direct product $\bigoplus \Gamma$ of infinitely many copies of $\Gamma$ is not weakly amenable, i.e. $\Lambda_{\mathrm{cb}}(\bigoplus \Gamma) = \infty$. It is plausible that $\Gamma$ itself is not weakly amenable. De Cornulier-Stalder-Valette ([dCSV08]) recently proved the surprising result that, despite satisfying $\Lambda_{\mathrm{cb}}(\Gamma) > 1$, the group $\Gamma$ (and hence $\bigoplus \Gamma$) has Haagerup's compact approximation property [Haa79]. Taken together, these results falsify one implication of the so-called Cowling's conjecture, which asserts that Haagerup's compact approximation property for a group $\Gamma$ is equivalent to the condition $\Lambda_{\mathrm{cb}}(\Gamma) = 1$. There are still no known examples of groups $\Gamma$ which satisfy $\Lambda_{\mathrm{cb}}(\Gamma) = 1$ but fail Haagerup's compact approximation property.

## 3. **Weakly compact actions**

We introduce in this section a new property for group actions, weaker than compactness (thus weaker than profiniteness as well) and closely related to the complete metric approximation property of the corresponding crossed product algebras. The main result of this section will show that if a $\mathrm{II}_1$ factor $M$ has the c.m.a.p. then, given any maximal abelian subalgebra $A$ of $M$, the action on $A$ of its normalizer, $\mathcal{N}_M(A) \curvearrowright A$, is weakly compact. Also, if a group $\Gamma$ satisfies $\Lambda_{\mathrm{cb}}(\Gamma) = 1$ and $\Gamma \curvearrowright X$ is weakly compact, then $M = L^\infty(X) \rtimes \Gamma$ has c.m.a.p.

*Definition* 3.1. Let $\sigma$ be an action of a group $\Gamma$ on a finite von Neumann algebra $P$. Recall that $\sigma$ is called *compact* if $\sigma(\Gamma) \subset \mathrm{Aut}(P)$ is pre-compact in the point-ultraweak topology. We call the action $\sigma$ *weakly compact* if there exists a net $(\eta_n)$ of unit vectors in $L^2(P \mathbin{\bar\otimes} \bar{P})_+$ such that:

(1) $\|\eta_n - (v \otimes \bar{v})\eta_n\|_2 \to 0$ for every $v \in \mathcal{U}(P)$.

(2) $\|\eta_n - (\sigma_g \otimes \bar{\sigma}_g)(\eta_n)\|_2 \to 0$ for every $g \in \Gamma$.

(3) $\langle (x \otimes 1)\eta_n, \eta_n\rangle = \tau(x) = \langle \eta_n, (1 \otimes \bar{x})\eta_n\rangle$ for every $x \in P$ and every $n$.

Here, we consider the action $\sigma$ on $P$ as the corresponding unitary representation on $L^2(P)$. By the proof of Proposition 3.2, condition (3) can be replaced with a formally weaker condition

(3′) $\langle (x \otimes 1)\eta_n, \eta_n\rangle \to \tau(x)$ for every $x \in P$.

Weak compactness is manifestly weaker than profiniteness, which is why in an initial version of this paper we called it *weak profiniteness*. We are very grateful to Adrian Ioana, who pointed out to us that the condition is even weaker than compactness (cf. (2) $\Rightarrow$ (3) below) and suggested a change in terminology.

PROPOSITION 3.2. *Let $\sigma$ be an action of a group $\Gamma$ on a finite von Neumann algebra $P$ and consider the following conditions*:

(1) *The action $\sigma$ is profinite.*

(2) *The action $\sigma$ is compact and the von Neumann algebra $P$ is* AFD.

(3) *The action $\sigma$ is weakly compact.*

(4) *There exists a state $\varphi$ on $\mathbb{B}(L^2(P))$ such that $\varphi|_P = \tau$ and $\varphi \circ \operatorname{Ad} u = \varphi$ for all $u \in \mathcal{U}(P) \cup \sigma(\Gamma)$.*

(5) *The von Neumann algebra $L(\Gamma)$ is co-amenable in $P \rtimes \Gamma$.*

*Then*, *one has* (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Leftrightarrow$ (4) $\Leftrightarrow$ (5).

(Note that, by a result of Høegh-Krohn-Landstad-Størmer ([HKLS81]), if in the above statement we restrict our attention to ergodic actions $\Gamma \curvearrowright P$, then the condition that $P$ is AFD in part (2) follows automatically from the assumption $\Gamma \curvearrowright P$ compact. We observe that weak compactness also implies that $P$ is AFD by Connes' theorem ([Con76]).)

*Proof.* We have (1) $\Rightarrow$ (2), by the definitions. We prove (2) $\Rightarrow$ (4). Since $P$ is AFD, there is a net $\Phi_n$ of normal u.c.p. maps from $\mathbb{B}(L^2(P))$ into $P$ such that $\tau \circ (\Phi_n|_P) = \tau$ and $\|a - \Phi_n(a)\|_2 \to 0$ for all $a \in P$. Let $G$ be the SOT-closure of $\sigma(\Gamma)$ in the unitary group on $L^2(P)$. By assumption, $G$ is a compact group and has a normalized Haar measure $m$. We define a state $\varphi_n$ on $\mathbb{B}(L^2(P))$ by

$$\varphi_n(x) = \int_G \tau \circ \Phi_n(gxg^{-1})\, dm(g).$$

It is clear that $\varphi_n$ is $\operatorname{Ad}(\Gamma)$-invariant and $\varphi_n|_P = \tau$. We will prove that the net $\varphi_n$ is approximately $P$-central. Let $\Phi_n(x) = V^*\pi(x)V$ be a Stinespring dilation. Then, for $x \in \mathbb{B}(L^2(P))$ and $a \in P$, one has

$$\begin{aligned}
\|\Phi_n(xa) - \Phi_n(x)\Phi_n(a)\|_2 &= \|V^*\pi(x)(1 - VV^*)\pi(a)V\hat{1}\|_{L^2(P)} \\
&\le \|x\| \|(1 - VV^*)^{1/2}\pi(a)V\hat{1}\|_{L^2(P)} \\
&= \|x\| \tau(\Phi_n(a^*a) - \Phi_n(a^*)\Phi_n(a))^{1/2} \\
&\le 2\|x\| \|a\|^{1/2} \|a - \Phi_n(a)\|_2^{1/2}.
\end{aligned}$$

It follows that for every $x \in \mathbb{B}(L^2(P))$ and $a \in P$, one has

$$|\varphi_n(xa) - \varphi_n(ax)| \le 4\|x\| \|a\|^{1/2} \sup_{g \in G} \|gag^{-1} - \Phi_n(gag^{-1})\|_2^{1/2},$$

which converge to zero since $\{gag^{-1} : g \in G\}$ is compact in $L^2(P)$ and $\Phi_n$'s are contractive on $L^2(P)$. Hence $\varphi_n$ is approximately $P$-central and $\varphi = \operatorname{Lim}_n \varphi_n$ satisfies the requirement.

We prove (3) $\Leftrightarrow$ (4). Take a net $\eta_n$ satisfying conditions (1), (2) and (3′) of Definition 3.1. We define a state $\varphi$ on $\mathbb{B}(L^2(P))$ by $\varphi = \operatorname{Lim}_n \varphi_n$ with $\varphi_n(x) =$

$\langle (x \otimes 1)\eta_n, \eta_n\rangle$. Then, for any $u \in \mathcal{U}(P) \cup \sigma(\Gamma)$, one has

$$\varphi(u^* x u) = \operatorname*{Lim}_n \langle (x \otimes 1)(u \otimes \bar{u})\eta_n, (u \otimes \bar{u})\eta_n\rangle = \varphi(x)$$

by conditions (1) and (2) of Definition 3.1. That $\varphi|_P = \tau$ follows from (3′). Conversely, suppose now that $\varphi$ is given. We recall that $\mathbb{B}(L^2(P))$ is canonically identified with the dual Banach space of the space $S_1(L^2(P))$ of trace class operators. Take a net of positive elements $T_n \in S_1(L^2(P))$ with $\mathrm{Tr}(T_n) = 1$ such that $\mathrm{Tr}(T_n x) \to \varphi(x)$ for every $x \in \mathbb{B}(L^2(P))$. Let $b_n \in L^1(P)_+$ be such that $\mathrm{Tr}(T_n a) = \tau(b_n a)$ for $a \in P$. Since $\mathrm{Tr}(T_n a) \to \varphi(a) = \tau(a)$ for $a \in P$, the net $(b_n)$ converges to 1 weakly in $L^1(P)$. Thus, by the Hahn-Banach separation theorem, one may assume, by passing to a convex combinations, that $\|b_n - 1\|_1 \to 0$. By a routine perturbation argument, we may further assume that $b_n = 1$. For the reader's convenience we give an argument for this. Let $h(t) = \max\{1, t\}$ and $k(t) = \max\{1 - t, 0\}$ be functions on $[0, \infty)$, and let $c_n = h(b_n)^{-1}$. We note that $0 \le c_n \le 1$ and $b_n c_n + k(b_n) = 1$. We define $T_n' = c_n^{1/2} T_n c_n^{1/2} + k(b_n)^{1/2} P_0 k(b_n)^{1/2}$, where $P_0$ is the orthogonal projection onto $\mathbb{C}\hat{1}$. Then, one has

$$\begin{aligned} \|T_n - T_n'\|_1 &\le 2\|T_n^{1/2} - c_n^{1/2} T_n^{1/2}\|_2 + \|k(b_n)\|_1 \\ &= 2\tau(b_n(1 - c_n^{1/2})^2)^{1/2} + \|k(b_n)\|_1 \\ &\le 2\tau(b_n(1 - c_n))^{1/2} + \|k(b_n)\|_1 \\ &\le 2\|b_n - 1\|_1^{1/2} + \|1 - b_n\|_1 \to 0. \end{aligned}$$

Hence, by replacing $T_n$ with $T_n'$, we may assume that $\mathrm{Tr}(T_n a) = \tau(a)$ for $a \in P$. Since for every $x \in \mathbb{B}(L^2(P))$ and $u \in \mathcal{U}(P) \cup \sigma(\Gamma)$, one has

$$\mathrm{Tr}((T_n - \mathrm{Ad}(u)T_n)x) \to \varphi(x) - \varphi(\mathrm{Ad}(u^*)(x)) = 0,$$

by applying the Hahn-Banach separation theorem again, one may furthermore assume that $\|T_n - \mathrm{Ad}(u)(T_n)\|_{S_1} \to 0$ for every $u \in \mathcal{U}(P) \cup \sigma(\Gamma)$. Then by (2.1), the Hilbert-Schmidt operators $T_n^{1/2}$ satisfy $\|T_n^{1/2} - \mathrm{Ad}(u)(T_n^{1/2})\|_{S_2} \to 0$ for every $u \in \mathcal{U}(P) \cup \sigma(\Gamma)$. Now, if we use the standard identification between $S_2(L^2(P))$ and $L^2(P \mathbin{\bar{\otimes}} \bar{P})$ given by

$$S_2(L^2(P)) \ni \sum_k \langle \cdot\,, \eta_k\rangle \xi_k \mapsto \sum_k \xi_k \otimes \bar{\eta}_k \in L^2(P \mathbin{\bar{\otimes}} \bar{P})$$

and view $T_n^{1/2}$ as an element $\zeta_n \in L^2(P \mathbin{\bar{\otimes}} \bar{P})$, then we have $\langle (a \otimes 1)\zeta_n, \zeta_n\rangle = \tau(a) = \langle \zeta_n, (1 \otimes \bar{a})\zeta_n\rangle$ and $\|\zeta_n - (u \otimes \bar{u})\zeta_n\|_2 \to 0$ for every $u \in \mathcal{U}(P) \cup \sigma(\Gamma)$. Therefore, the net of $\eta_n = (\zeta_n \zeta_n^*)^{1/2} \in L^2(P \mathbin{\bar{\otimes}} \bar{P})_+$ verifies the conditions of weak compactness.

Finally, we prove (4) $\Leftrightarrow$ (5). We consider $P \rtimes \Gamma$ as the von Neumann subalgebra of $\mathbb{B}(L^2(P) \mathbin{\bar{\otimes}} \ell^2(\Gamma))$ generated by $P \mathbin{\bar{\otimes}} \mathbb{C}1$ and $(\sigma \otimes \lambda)(\Gamma)$. This gives an identification between $L^2(P \rtimes \Gamma)$ and $L^2(P) \mathbin{\bar{\otimes}} \ell^2(\Gamma)$. Moreover, the basic construction $\langle P \rtimes \Gamma, e_{L(\Gamma)}\rangle$ becomes $\mathbb{B}(L^2(P)) \mathbin{\bar{\otimes}} L(\Gamma)$, since it is the commutant of the right $L(\Gamma)$-action (which is given by $(1 \otimes \rho)(\Gamma)$). Now suppose that $\varphi$ is given as in condition (4). Then, $\widetilde{\varphi} = \varphi \otimes \tau$ on $\mathbb{B}(L^2(P)) \mathbin{\bar{\otimes}} L(\Gamma)$ is $\mathrm{Ad}(\mathcal{U}(P \mathbin{\bar{\otimes}} \mathbb{C}1) \cup (\sigma \otimes \lambda)(\Gamma))$-invariant and $\widetilde{\varphi}|_{P \rtimes \Gamma} = \tau$. This implies that $L(\Gamma)$ is co-amenable in $P \rtimes \Gamma$. Conversely, if $\widetilde{\varphi}$ is a $(P \rtimes \Gamma)$-central state such that $\widetilde{\varphi}|_{P \rtimes \Gamma} = \tau$, then the restriction $\varphi$ of $\widetilde{\varphi}$ to $\mathbb{B}(L^2(P))$ satisfies condition (4). $\square$

Note that by part (7) in Theorem 2.10, if $\Lambda_{\mathrm{cb}}(\Gamma) = 1$ and $\Gamma \curvearrowright P$ is a profinite action then $\Lambda_{\mathrm{cb}}(P \rtimes \Gamma) = 1$. More generally we have the following. (Compare this with [Jol07].)

COROLLARY 3.3. *Let $\Gamma$ be weakly amenable and $\Gamma \curvearrowright P$ be a weakly compact action on an* AFD *von Neumann algebra. Then, $P \rtimes \Gamma$ has the completely bounded approximation property and $\Lambda_{\mathrm{cb}}(P \rtimes \Gamma) = \Lambda_{\mathrm{cb}}(\Gamma)$.*

*Proof.* By Proposition 3.2, $L(\Gamma)$ is co-amenable in $P \rtimes \Gamma$. Hence, Theorem 4.9 of [AD95] implies that $\Lambda_{\mathrm{cb}}(P \rtimes \Gamma) = \Lambda_{\mathrm{cb}}(L(\Gamma)) = \Lambda_{\mathrm{cb}}(\Gamma)$. $\square$

PROPOSITION 3.4. *Let $P \subset M$ be an inclusion of finite von Neumann algebras such that $P' \cap M \subset P$. Assume the normalizer $\mathcal{N}_M(P)$ contains a subgroup $\mathcal{G}$ such that its action on $P$ is weakly compact and $(P \cup \mathcal{G})'' = \mathcal{N}_M(P)''$. Then the action of $\mathcal{N}_M(P)$ on $P$ is weakly compact. Moreover, if $\mathcal{N}_M(P) \curvearrowright P$ is weakly compact and $p \in \mathcal{P}(P)$ then $\mathcal{N}_{pMp}(pPp) \curvearrowright pPp$ is weakly compact.*

*Proof.* We may clearly assume $\mathcal{N}_M(P)'' = M$. Denote by $\sigma$ the action of $\mathcal{N}_M(P)$ on $P$. If $u \in \mathcal{N}_M(P)$, then by the conditions $P' \cap M = \mathcal{Z}(P)$ and $(P \cup \mathcal{G})'' = M$ it follows that there exists a partition $\{p_i\}_i \subset \mathcal{Z}(P)$ and unitary elements $v_i \in P$ such that $v = \Sigma_i p_i v_i u_i$ for some $u_i \in \mathcal{G}$ (see e.g. [Dye59]). Then $\sigma_v(x) = vxv^* = \Sigma_i p_i \sigma_{v_i u_i}(x)$. Let now $\eta_n \in L^2(P \mathbin{\bar{\otimes}} \bar{P})_+$ satisfy the conditions in Definition 3.1 for the action $\sigma_{|\mathcal{G}}$. By 3.1(1) we have $\|\Sigma_i (p_i \otimes \bar{p}_i)\eta_n - \eta_n\|_2 \to 0$, and thus $\|(p_i \otimes \bar{p}_j)\eta_n\|_2 \to 0$, for all $i \neq j$. Since $q_i = \sigma_{u_i^*}(p_i)$ are mutually orthogonal as well, this also implies that for $i \neq j$ we have

$$\begin{aligned}\|(p_i \otimes \bar{p}_j)&(\sigma_{v_i u_i} \otimes \bar{\sigma}_{v_j u_j})(\eta_n)\|_2 \\ &= \|(\sigma_{v_i u_i} \otimes \bar{\sigma}_{v_j u_j})((q_i \otimes \bar{q}_j)\eta_n)\|_2 = \|(q_i \otimes \bar{q}_j)\eta_n\|_2 \to 0.\end{aligned}$$

Also, since $w_i = u_i^* v_i u_i \in \mathcal{U}(P)$, we have $\|\sigma_{w_i} \otimes \bar{\sigma}_{w_i})(\eta_n) - \eta_n\|_2 \to 0$. Combining with condition 3.2(2) on the action $\mathcal{G} \curvearrowright P$, one gets

$$\|(p_i \otimes \bar{p}_i)(\eta_n - (\sigma_{v_i u_i} \otimes \bar{\sigma}_{v_i u_i})(\eta_n))\|_2 \to 0.$$

By Pythagoras' theorem, and using that $\sum_{i,j} \|p_i \otimes \bar{p}_j\|_2^2 = 1$, all this entails

$$\begin{aligned}\|\eta_n - (\sigma_v \otimes \bar{\sigma}_v)(\eta_n)\|_2^2 &= \Sigma_{i,j} \|(p_i \otimes \bar{p}_j)\eta_n - (p_i \otimes \bar{p}_j)(\sigma_v \otimes \bar{\sigma}_v)(\eta_n)\|_2^2 \\ &= \Sigma_{i,j} \|(p_i \otimes \bar{p}_j)\eta_n - (p_i \otimes \bar{p}_j)(\sigma_{v_i u_i} \otimes \bar{\sigma}_{v_j u_j})(\eta_n)\|_2^2 \to 0,\end{aligned}$$

showing that $\mathcal{N}_M(P) \curvearrowright P$ satisfies 3.1.(2), thus being weakly compact.

To see that weak compactness behaves well to reduction by projections, note that any $v \in \mathcal{N}_{pMp}(pPp)$ extends to a unitary in $\mathcal{N}_M(P)$. Thus, if $\varphi$ satisfies 3.2(4) for $\mathcal{N}_M(P) \curvearrowright P$ then $\varphi^p = \varphi(p \cdot p)$ clearly satisfies the same condition for $\mathcal{N}_{pMp}(pPp) \curvearrowright pPp$. $\square$

The above result shows in particular that if a measure-preserving action of a countable group $\Gamma$ on a probability space $(X, \mu)$ is weakly compact (i.e., $\Gamma \curvearrowright L^\infty(X)$ weakly compact), then the action of its associated full group $[\Gamma]$, as defined in [Dye59], is weakly compact. Thus, weak compactness is an orbit equivalence invariant for group actions, unlike profiniteness and compactness which are of course not. In fact, Proposition 3.4 shows that weak compactness is even invariant to stable orbit equivalence (also called *measure equivalence*).

An embedding of finite von Neumann algebras $P \subset M$ is called *weakly compact* if the action $\mathcal{N}_M(P) \curvearrowright P$ is weakly compact. The next result shows that the complete metric approximation property of a factor $M$ imposes the weak compactness of all embeddings into $M$ of AFD (in particular abelian) von Neumann algebras.

THEOREM 3.5. *Let $M$ be a finite von Neumann algebra with the* c.m.a.p., *i.e.* $\Lambda_{\mathrm{cb}}(M) = 1$. *Then any embedding of an* AFD *von Neumann algebra $P \subset M$ is weakly compact, i.e., $\mathcal{N}_M(P) \curvearrowright P$ is weakly compact, for all $P \subset M$* AFD *subalgebra.*

For the proof, we need the following consequence of Connes' theorem [Con76]. This is well-known, but we include a proof for the reader's convenience.

LEMMA 3.6. *Let $M$ be a finite von Neumann algebra, $P \subset M$ be an* AFD *von Neumann subalgebra and $u \in \mathcal{N}_M(P)$. Then, the von Neumann algebra $Q$ generated by $P$ and $u$ is* AFD.

*Proof.* Since $P$ is injective, the $\tau$-preserving conditional expectation $E_P$ from $M$ onto $P$ extends to a u.c.p. map $\widetilde{E}_P$ from $\mathbb{B}(L^2(M))$ onto $P$. We note that $\widetilde{E}_P$ is a conditional expectation: $\widetilde{E}_P(axb) = a\widetilde{E}_P(x)b$ for every $a, b \in P$ and $x \in \mathbb{B}(L^2(M))$. We define a state $\sigma$ on $\mathbb{B}(L^2(M))$ by

$$\sigma(x) = \operatorname*{Lim}_n \frac{1}{n} \sum_{k=0}^{n-1} \tau\big(\widetilde{E}_P\big(u^k x u^{-k}\big)\big).$$

It is not hard to check that $\sigma|_M = \tau$, $\sigma \circ \operatorname{Ad} u = \sigma$ and $\sigma \circ \operatorname{Ad} v = \sigma$ for every $v \in \mathcal{U}(P)$. It follows that $\sigma$ is a $Q$-central state with $\sigma|_Q = \tau$. By Connes' theorem, this implies that $Q$ is AFD. $\square$

*Proof of Theorem* 3.5. First we note the following general fact: Let $\omega$ be a state on a C$^*$-algebra $N$ and $u \in \mathcal{U}(N)$. We define $\omega_u(x) = \omega(xu^*)$ for $x \in N$. Then, one has

$$\max\{\|\omega - \omega_u\|, \|\omega - \omega \circ \operatorname{Ad}(u)\|\} \le 2\sqrt{2|1 - \omega(u)|}. \tag{3.1}$$

Indeed, one has $\|\xi_\omega - u^*\xi_\omega\|^2 = 2(1-\Re\omega(u)) \le 2|1-\omega(u)|$, where $\xi_\omega$ is the GNS-vector for $\omega$.

Let $(\phi_n)$ be a net of normal finite rank maps on $M$ such that $\limsup \|\phi_n\|_{\mathrm{cb}} \le 1$ and $\|x-\phi_n(x)\|_2 \to 0$ for all $x \in M$. We observe that the net $(\tau\circ\phi_n)$ converges to $\tau$ weakly in $M_*$. Hence by the Hahn-Banach separation theorem, one may assume, by passing to convex combinations, that $\|\tau - \tau\circ\phi_n\| \to 0$. Let $\mu$ be the $*$-representation of the algebraic tensor product $M \otimes \bar{M}$ on $L^2(M)$ defined by

$$\mu\Big(\sum_k a_k \otimes \bar{b}_k\Big)\xi = \sum_k a_k \xi b_k^*.$$

We define a linear functional $\mu_n$ on $M \otimes \bar{M}$ by

$$\mu_n\Big(\sum_k a_k \otimes \bar{b}_k\Big) = \Big\langle \mu\Big(\sum_k \phi_n(a_k)\otimes \bar{b}_k\Big)\hat{1}, \hat{1}\Big\rangle_{L^2(M)} = \tau\Big(\sum_k \phi_n(a_k) b_k^*\Big).$$

Since $\phi_n$ is normal and of finite rank, $\mu_n$ extends to a normal linear functional on $M \mathbin{\bar{\otimes}} \bar{M}$, which is still denoted by $\mu_n$. For an AFD von Neumann subalgebra $Q \subset M$, we denote by $\mu_n^Q$ the restriction of $\mu_n$ to $Q \mathbin{\bar{\otimes}} \bar{Q}$. Since $Q$ is AFD, the $*$-representation $\mu$ is continuous with respect to the spatial tensor norm on $Q \otimes \bar{Q}$ and hence $\|\mu_n^Q\| \le \|\phi_n\|_{\mathrm{cb}}$. We denote $\omega_n^Q = \|\mu_n^Q\|^{-1}|\mu_n^Q|$. Since $\limsup \|\mu_n^Q\| \le 1$ and $\lim \mu_n^Q(1\otimes 1) = 1$, the inequality (3.1), applied to $\omega_n^Q$, implies that

$$\limsup_n \|\mu_n^Q - \omega_n^Q\| = 0. \tag{3.2}$$

Now, consider the case $Q = P$. Since $\mu_n^P(v \otimes \bar{v}) = \tau(\phi_n(v)v^*) \to 1$ for any $v \in \mathcal{U}(P)$, one has

$$\limsup_n \|\omega_n^P - (\omega_n^P)_{v\otimes\bar{v}}\| = 0 \tag{3.3}$$

by (3.1) and (3.2). Now, let $u \in \mathcal{N}_M(P)$ and consider the case $Q = \langle P, u\rangle$, which is AFD by Lemma 3.6. Since $\mu_n^{\langle P,u\rangle}(u\otimes\bar{u}) = \tau(\phi_n(u)u^*) \to 1$, one has

$$\limsup_n \|\mu_n^{\langle P,u\rangle} - \mu_n^{\langle P,u\rangle}\circ \mathrm{Ad}(u\otimes\bar{u})\| = 0 \tag{3.4}$$

by (3.1) and (3.2). But since $(\mu_n^{\langle P,u\rangle}\circ\mathrm{Ad}(u\otimes\bar{u}))|_{P\bar{\otimes}\bar{P}} = \mu_n^P \circ \mathrm{Ad}(u\otimes\bar{u})$, one has

$$\limsup_n \|\omega_n^P - \omega_n^P \circ \mathrm{Ad}(u\otimes\bar{u})\| = 0 \tag{3.5}$$

by (3.2) and (3.4). Now, we view $\omega_n^P$ as an $\zeta_n$ element in $L^1(P \mathbin{\bar{\otimes}} \bar{P})_+$ and let $\eta_n = \zeta_n^{1/2}$. By (2.1), the net $\eta_n$ satisfies all the required conditions. □

## 4. **Main results**

We prove in this section the main results of the paper. They will all follow from the following stronger version of the theorem stated in the introduction:

THEOREM 4.1. *Let* $\Gamma = \mathbb{F}_{r(1)} \times \cdots \times \mathbb{F}_{r(k)}$ *be a direct product of finitely many free groups of* rank $2 \leq r(j) \leq \infty$ *and denote by* $\Gamma_j$ *the kernel of the projection from* $\Gamma$ *onto* $\mathbb{F}_{r(j)}$. *Let* $M = Q \rtimes \Gamma$ *be the crossed product of a finite von Neumann algebra* $Q$ *by* $\Gamma$ (*action need not be ergodic nor free*). *Let* $P \subset M$ *be such that* $P \not\prec_M Q$. *Let* $\mathcal{G} \subset \mathcal{N}_M(P)$ *be a subgroup which acts weakly compactly on* $P$ *by conjugation*, *and denote* $N = \mathcal{G}''$. *Then there exist projections* $p_1, \ldots, p_k \in \mathcal{Z}(N' \cap M)$ *with* $\sum_{j=1}^{k} p_j = 1$ *such that* $Np_j \prec_M Q \rtimes \Gamma_j$ *for every* $j$.

From the above result, we will easily deduce several (in)decomposability properties for certain factors constructed out of free groups and their profinite actions. Note that Corollaries 4.2 and 4.3 below are just Corollaries 1 and 2 in the introduction, while Corollary 4.5 is a generalization of Corollary 3 therein.

COROLLARY 4.2. *If* $P \subset L(\mathbb{F}_r)^t$ *is a diffuse* AFD *von Neumann subalgebra of the amplification by some* $t > 0$ *of a free group factor* $L(\mathbb{F}_r)$, $2 \leq r \leq \infty$, *then* $\mathcal{N}_{L(\mathbb{F}_r)^t}(P)''$ *is* AFD.

Note that the above corollary generalizes the (in)-decomposability results for free group factors in [Oza04a] and [Voi96]. Indeed, Voiculescu's celebrated result in [Voi96], showing that the normalizer of any amenable diffuse subalgebra $P \subset L(\mathbb{F}_r)$ cannot generate all $L(\mathbb{F}_r)$, follows from Corollary 4.2 because $L(\mathbb{F}_r)$ is nonAFD by [MvN43]. Also, since any unitary element commuting with a subalgebra $P \subset L(\mathbb{F}_r)$ lies in the normalizer of $P$, Corollary 4.2 shows in particular that the commutant of any diffuse AFD subalgebra $P \subset L(\mathbb{F}_r)$ is amenable, i.e. $L(\mathbb{F}_r)$ is *solid* in the sense of [Oza04a], which amounts to the free group case of a result in [Oza04a]. Note however that the (in)-decomposability results in [Voi96] and [Oza04a] cover much larger classes of factors, e.g. all free products of diffuse von Neumann algebras in [Voi96] (for absence of Cartan subalgebras) and all $\mathrm{II}_1$ factors arising from word-hyperbolic groups in [Oza04a] (for solidity).

Calling *strongly solid* (or *s-solid*) the factors satisfying the property that the normalizer of any diffuse amenable subalgebra generates an amenable von Neumann algebra, it would be interesting at this point to produce examples of $\mathrm{II}_1$ factors that are s-solid, have both c.m.a.p. and Haagerup property, yet are not isomorphic to an amplification of a free group factor (i.e., to an interpolated free group factor [Dyk94], [Rădu94]).

COROLLARY 4.3. *If* $Q$ *is a type* $\mathrm{II}_1$*-factor with* c.m.a.p., *then* $Q \mathbin{\overline{\otimes}} L(\mathbb{F}_r)$ *does not have Cartan subalgebras. Moreover*, *if* $N \subset Q \mathbin{\overline{\otimes}} L(\mathbb{F}_r)$ *is a subfactor of finite index*, *then* $N$ *does not have Cartan subalgebras either.*

This corollary shows in particular that if $Q$ is an arbitrary subfactor of a tensor product of free group factors, then $Q \mathbin{\overline{\otimes}} L(\mathbb{F}_r)$ (or any of its finite index subfactors) has no Cartan subalgebras. When applied to $Q = R$, this shows that the subfactor $N \subset R \mathbin{\overline{\otimes}} L(\mathbb{F}_r)$ with $N \not\simeq N^{\mathrm{op}}$ constructed in [Con75], as the fixed point algebra of an appropriate free action of a finite group on $R \mathbin{\overline{\otimes}} L(\mathbb{F}_r)$ (which thus has finite index in $R \mathbin{\overline{\otimes}} L(\mathbb{F}_r)$), does not have Cartan subalgebras.

Another class of factors without Cartan subalgebras is provided by part (2) of the next corollary.

COROLLARY 4.4. *Let* $\Gamma = \mathbb{F}_{r(1)} \times \cdots \times \mathbb{F}_{r(k)}$, *as in Theorem* 4.1 , *and* $\Gamma \curvearrowright X$ *an ergodic probability-measure-preserving action. Then* $M = L^\infty(X) \rtimes \Gamma$ *is a* $\mathrm{II}_1$ *factor and for each* $t > 0$ *we have*:

(1) *Assume* $M^t$ *has a maximal abelian* $^*$*-subalgebra* $A$ *such that* $\mathcal{N}_{M^t}(A) \curvearrowright A$ *is weakly compact and* $N = \mathcal{N}_{M^t}(A)''$ *is a subfactor of finite index in* $M^t$. *Then* $\Gamma \curvearrowright X$ *is necessarily a free action*, $L^\infty(X)$ *is Cartan in* $M$ *and there exists a unitary element* $u \in M^t$ *such that* $uAu^* = L^\infty(X)^t$.

(2) *Assume* $\Gamma \curvearrowright X$ *is profinite* (*or merely compact*). *Then* $M$ *has a Cartan subalgebra if and only if* $\Gamma \curvearrowright X$ *is free.*

(3) *Assume* $\Gamma = \mathbb{F}_r$. *If* $M^t$ *has a weakly compact maximal abelian* $^*$*-subalgebra* $A$ *whose normalizer generates a von Neumann algebra without amenable direct summand*, *then* $\Gamma \curvearrowright X$ *follows free and* $A$ *is unitary conjugate to* $L^\infty(X)^t$.

Note that one can view part (1) of the above corollary as a *strong rigidity result*, in the spirit of results in ([Pop06a], [Pop06c], [IPP08]). Indeed, by taking $A = L^\infty(Y)$ to be Cartan in $M^t$, it follows that any isomorphism between group measure space $\mathrm{II}_1$ factors $\theta : (L^\infty(X) \rtimes \Gamma)^t \simeq L^\infty(Y) \rtimes \Lambda$, with the "source" $\Gamma$ a direct product of finitely many free groups and the "target" $\Lambda$ arbitrary but the action $\Lambda \curvearrowright Y$ weakly compact (e.g. profinite, or compact), is implemented by a stable orbit equivalence of the free ergodic actions $\Gamma \curvearrowright X$, $\Lambda \curvearrowright Y$, up to perturbation by an inner automorphism and by an automorphism coming from a 1-cocycle of the target action.

COROLLARY 4.5. *Let* $\Gamma = \mathbb{F}_{r(1)} \times \cdots \times \mathbb{F}_{r(k)}$ (*as in Theorem* 4.1, *Corollary* 4.4) *and* $\Gamma \curvearrowright X$ *a free ergodic profinite* (*or merely compact*) *action. Then*, $L^\infty(X)$ *is the unique Cartan subalgebra of the* $\mathrm{II}_1$*-factor* $L^\infty(X) \rtimes \Gamma$, *up to unitary conjugacy. Moreover*, *if* $\mathcal{FP}$ *denotes the class of all* $\mathrm{II}_1$ *factors that can be embedded as subfactors of finite index in an amplification of some* $L^\infty(X) \rtimes \Gamma$, *with* $\Gamma \curvearrowright X$ *free ergodic compact action and* $\Gamma$ *as above*, *then any* $M \in \mathcal{FP}$ *has unique Cartan subalgebra*, *up to unitary conjugacy. The class* $\mathcal{FP}$ *is closed under amplifications*, *tensor product and finite index extension/restriction. Also*, *if* $M \in \mathcal{FP}$ *and* $N \subset M$ *is an irreducible subfactor of finite index*, *then* $[M : N]$ *is an integer.*

The above corollary implies that any isomorphism between factors $M \in \mathcal{FP}$ comes from an isomorphism of the orbit equivalence relations $\mathcal{R}_M$ associated with their unique Cartan decomposition. Hence, like in the case of the $\mathcal{HT}$-factors in [Pop06a], invariants of equivalence relations, such as Gaboriau's cost and $L^2$-Betti numbers ([Gab02]), are isomorphism invariants of $\mathrm{II}_1$ factors in $\mathcal{FP}$. The subfactor theory within the class $\mathcal{FP}$ is particularly interesting: By Corollary 4.5 and its proof (see Proposition 4.12), and Section 7 in [Pop06a], any irreducible inclusion of finite index $N \subset M$ in this class has a canonical decomposition $N \subset Q \subset P \subset M$, with

$P \subset M$ coming from a subequivalence relation of $\mathcal{R}_M$, $N \subset Q$ from a quotient of $\mathcal{R}_Q$ and $Q \subset P$ from an irreducible $\mathcal{U}(n)$-valued 1-cocycle for $\mathcal{R}_Q$.

Note that all factors in the class $\mathcal{FP}$ have $\Lambda_{\mathrm{cb}}$-constant equal to 1 by Theorem 2.10 and have Haagerup's compact approximation property by [Haa79]. The sub-class of $\mathrm{II}_1$ factors $L^\infty(X) \rtimes \mathbb{F}_r \in \mathcal{FP}$, arising from free ergodic profinite probability-measure-preserving actions of free groups $\mathbb{F}_r \curvearrowright X$, is of particular interest, as they are inductive limits of (amplifications of) free group factors. We call such a factor $L^\infty(X) \rtimes \mathbb{F}_r$ an *approximate free group factor* of rank $r$. By Corollary 4.5, more than being in the class $\mathcal{FP}$, such a factor has the property that any maximal abelian *-subalgebra with normalizer generating a von Neumann algebra with no amenable summand is unitary conjugate to $L^\infty(X)$. When combined with [Gab02], we see that approximate free group factors of different rank are not isomorphic and that for $r < \infty$ they have trivial fundamental group. Also, they are prime by [Oza06], in fact by Theorem 4.1 the normalizer (in particular the commutant) of any AFD $\mathrm{II}_1$ subalgebra of such a factor must generate an AFD von Neumann algebra. We will construct uncountably many approximate free group factors in Section 5 and comment more on this class in Remark 5.6.

For the proof of Theorem 4.1, recall from [Pop06b], [Pop07] the construction of 1-parameter automorphisms $\alpha_t$ ("malleable deformation") of $L(\mathbb{F}_r * \widetilde{\mathbb{F}}_r)$. Let $\widetilde{\mathbb{F}}_r$ be a copy of $\mathbb{F}_r$ and $a_1, a_2, \dots$ (resp. $b_1, b_2, \dots$) be the standard generators of $\mathbb{F}_r$ (resp. $\widetilde{\mathbb{F}}_r$) viewed as unitary elements in $L(\mathbb{F}_r * \widetilde{\mathbb{F}}_r)$. Let $h_s = (\pi\sqrt{-1})^{-1} \log b_s$, where log is the principal branch of the complex logarithm so that $h_s$ is a self-adjoint element with spectrum contained in $[-1, 1]$. For simplicity, we write $b_s^t$ ($s = 1, 2, \dots$ and $t \in \mathbb{R}$) for the unitary element $\exp(t\pi\sqrt{-1}h_s)$. The $*$-automorphism $\alpha_t$ is defined by $\alpha_t(a_s) = b_s^t a_s$ and $\alpha_t(b_s) = b_s$.

In this paper, we adapt this construction to $\Gamma = \mathbb{F}_{r(1)} \times \cdots \times \mathbb{F}_{r(k)}$ acting on $Q$ and $M = Q \rtimes \Gamma$. We extend the action $\Gamma \curvearrowright Q$ to that of

$$\widetilde{\Gamma} = (\mathbb{F}_{r(1)} * \widetilde{\mathbb{F}}_{r(1)}) \times \cdots \times (\mathbb{F}_{r(k)} * \widetilde{\mathbb{F}}_{r(k)}),$$

where $\widetilde{\mathbb{F}}_{r(j)}$'s act trivially on $Q$. We denote by $a_{j,1}, a_{j,2}, \dots$ (resp. $b_{j,1}, b_{j,2}, \dots$) the standard generators of $\mathbb{F}_{r(j)}$ (resp. $\widetilde{\mathbb{F}}_{r(j)}$) We redefine the $*$-homomorphism

$$\alpha_t \colon M \to \widetilde{M} = Q \rtimes \widetilde{\Gamma}$$

by $\alpha_t(x) = x$ for $x \in Q$ and $\alpha_t(a_{j,s}) = b_{j,s}^t a_{j,s}$ for each $1 \le j \le k$ and $s$. (We can define $\alpha_t$ on $\widetilde{M}$, but we do not need it.)

Let

$$\gamma(t) = \tau(b_{j,s}^t) = \frac{1}{2}\int_{-1}^{1} \exp(t\pi\sqrt{-1}h)\, dh = \frac{\sin(t\pi)}{t\pi} = \gamma(-t)$$

and $\phi_{j,\gamma(t)} \colon L(\mathbb{F}_{r(j)}) \to L(\mathbb{F}_{r(j)})$ be the Haagerup multiplier ([Haa79]) associated with the positive type function $g \mapsto \gamma(t)^{|g|}$ on $\mathbb{F}_{r(j)}$. We may extend

$$\phi_{\gamma(t)} = \phi_{1,\gamma(t)} \otimes \cdots \otimes \phi_{k,\gamma(t)}$$

to $M$ by defining $\phi_{\gamma(t)}(x\lambda(g)) = x\phi_{\gamma(t)}(\lambda(g))$ for $x \in Q$ and $\lambda(g) \in L(\Gamma)$. We relate $\alpha_t$ and $\phi_{\gamma(t)}$ as follows (cf. [Pet09]).

LEMMA 4.6. *One has $E_M \circ \alpha_t = \phi_{\gamma(t)}$.*

*Proof.* Since $E_M(x\lambda(g)) = xE_{L(\Gamma)}(\lambda(g))$ for $x \in Q$ and $\lambda(g) \in L(\widetilde{\Gamma})$, one has $E_M \circ \alpha_t(x\lambda(g)) = xE_{L(\Gamma)}(\alpha_t(\lambda(g)))$ for $x \in Q$ and $\lambda(g) \in L(\Gamma)$. Hence it suffices to show $E_{L(\Gamma)} \circ \alpha_t = \phi_{\gamma(t)}$ on $L(\Gamma)$. Since all $E_{L(\Gamma)}$, $\alpha_t$ and $\phi_{\gamma(t)}$ split as tensor products, we may assume that $k = 1$. Since $a_1, \ldots, b_1, \ldots$ are mutually free, it is not hard to check

$$(E_{L(\mathbb{F}_r)} \circ \alpha_t)(a_{i_1}^{\pm 1} \cdots a_{i_n}^{\pm 1}) = \gamma(t)^n a_{i_1}^{\pm 1} \cdots a_{i_n}^{\pm 1} = \phi_{\gamma(t)}(a_{i_1}^{\pm 1} \cdots a_{i_n}^{\pm 1})$$

for every reduced word $a_{i_1}^{\pm 1} \cdots a_{i_n}^{\pm 1}$ in $\mathbb{F}_r$. $\square$

In particular, the u.c.p. map $E_M \circ \alpha_t$ on $M$ is compact over $Q$ provided that $r(j) < \infty$ for every $j$. In case of $r(j) = \infty$, we need a little modification: we replace the defining equation $\alpha_t(a_{j,s}) = b_{j,s}^t a_{j,s}$ with $\alpha_t(a_{j,s}) = b_{j,s}^{st} a_{j,s}$. Then, the u.c.p. map $E_M \circ \alpha_t$ is compact over $Q$ and $\alpha_t \to \mathrm{id}_M$ as $t \to 0$.

Let $\Gamma_j$ be the kernel of the projection from $\Gamma$ onto $\mathbb{F}_{r(j)}$ and $Q_j = Q \rtimes \Gamma_j \subset M$. We consider the basic construction $\langle M, e_{Q_j} \rangle$ of $(Q_j \subset M)$. Then, $L^2\langle M, e_{Q_j} \rangle$ is naturally an $M$-bimodule.

LEMMA 4.7. *Let $Q_j \subset M \subset \widetilde{M}$ be as above. Then, $L^2(\widetilde{M}) \ominus L^2(M)$ is isomorphic as an $M$-bimodule to a submodule of a multiple of $\bigoplus_{j=1}^k L^2\langle M, e_{Q_j} \rangle$.*

*Proof.* Let $\widetilde{\Gamma}_j$ be the kernel of the projection from $\widetilde{\Gamma}$ onto $\mathbb{F}_{r(j)} * \widetilde{\mathbb{F}}_{r(j)}$. By permuting the position appropriately, we consider that $\widetilde{\Gamma}_j \times \mathbb{F}_{r(j)} \subset \widetilde{\Gamma}$ and $\bigcap \widetilde{\Gamma}_j \times \mathbb{F}_{r(j)} = \Gamma$. Let $\widetilde{Q}_j = Q \rtimes \widetilde{\Gamma}_j$ and $\widetilde{M}_j = Q \rtimes (\widetilde{\Gamma}_j \times \mathbb{F}_{r(j)})$. Since $L^2(M) = \bigcap_{j=1}^k L^2(\widetilde{M}_j)$, it suffices to show $L^2(\widetilde{M}) \ominus L^2(\widetilde{M}_j)$ is isomorphic as an $M$-bimodule to a multiple of $L^2\langle M, e_{Q_j} \rangle$.

We observe that

$$L^2(\widetilde{M}) \ominus L^2(\widetilde{M}_j) = \bigoplus_d \Big[ \widetilde{Q}_j \lambda(\mathbb{F}_{r(j)} d\, \mathbb{F}_{r(j)}) \Big],$$

where the square bracket means the $L^2$-closure and the direct sum runs all over $d \in \mathbb{F}_{r(j)} * \widetilde{\mathbb{F}}_{r(j)}$ whose initial and final letters in the reduced form come from $\widetilde{\mathbb{F}}_{r(j)}$. Let $\pi_j : \mathbb{F}_{r(j)} * \widetilde{\mathbb{F}}_{r(j)} \to \mathbb{F}_{r(j)}$ be the projection sending $\widetilde{\mathbb{F}}_{r(j)}$ to $\{1\}$. It is not difficult to see that

$$x\lambda(gdh) \mapsto x\lambda(g) e_{Q_j} \lambda(\pi_j(d)h)$$

extends to an $M$-bimodule isometry from $\Big[ \widetilde{Q}_j \lambda(\mathbb{F}_{r(j)} d\, \mathbb{F}_{r(j)}) \Big]$ onto $L^2\langle M, e_{Q_j} \rangle$. $\square$

We summarize the above two lemmas as follows.

PROPOSITION 4.8. *Let $Q \subset Q_j \subset M$ be as above. Then, there are a finite von Neumann algebra $\widetilde{M} \supset M$ and trace-preserving $*$-homomorphisms $\alpha_t\colon M \to \widetilde{M}$ such that:*

(1) $\lim_{t\to 0} \|\alpha_t(x) - x\|_2 \to 0$ *for every* $x \in M$;

(2) $E_M \circ \alpha_t$ *is compact over* $Q$ *for every* $t > 0$; *and*

(3) $L^2(\widetilde{M}) \ominus L^2(M)$ *is isomorphic as an $M$-bimodule to a submodule of a multiple of* $\bigoplus_{j=1}^k L^2\langle M, e_{Q_j}\rangle$.

We complete the proof of Theorem 4.1 in this abstract setting.

THEOREM 4.9. *Let $Q \subset Q_j \subset M$ be as in Proposition 4.8. Let $P \subset M$ be such that $P \not\preceq_M Q$. Let $\mathcal{G} \subset \mathcal{N}_M(P)$ be subgroup which acts weakly compactly on $P$ by conjugation, and $N = \mathcal{G}''$. Then there exist projection $p_1, \dots, p_k \in \mathcal{Z}(N' \cap M)$ with $\sum_{j=1}^k p_j = 1$ such that $Np_j \lessdot_M Q_j$ for every $j$.*

*Proof.* We may assume that $\mathcal{U}(P) \subset \mathcal{G}$. We use Corollary 2.3 to conclude the relative amenability. Let a nonzero projection $p$ in $\mathcal{Z}(N' \cap M)$, a finite subset $F \subset \mathcal{G}$ and $\varepsilon > 0$ be given arbitrary. It suffices to find $\xi \in \bigoplus \bigoplus_{j=1}^k L^2\langle M, e_{Q_j}\rangle$ such that $\|x\xi\|_2 \le \|x\|_2$ for all $x \in M$, $\|p\xi\|_2 \ge \|p\|_2/8$ and $\|[\xi, u]\|_2^2 < \varepsilon$ for every $u \in F$.

Let $\delta = \|p\|_2/8$. We choose and fix $t > 0$ such that $\alpha = \alpha_t$ satisfies $\|p - \alpha(p)\|_2 < \delta$ and $\|u - \alpha(u)\|_2 < \varepsilon/6$ for every $u \in F$. We still denote by $\alpha$ when it is viewed as an isometry from $L^2(M)$ into $L^2(\widetilde{M})$. Let $(\eta_n)$ be the net of unit vectors in $L^2(P \mathbin{\bar{\otimes}} \bar{P})_+$ as in Definition 3.1 and denote

$$\tilde{\eta}_n = (\alpha \otimes 1)(\eta_n) \in L^2(\widetilde{M}) \mathbin{\bar{\otimes}} L^2(\bar{M}).$$

We note that

$$\|(x \otimes 1)\tilde{\eta}_n\|_2^2 = \tau(\alpha^{-1}(E_{\alpha(M)}(x^*x))) = \|x\|_2^2 \tag{4.1}$$

for every $x \in \widetilde{M}$. In particular, one has

$$\|[u \otimes \bar{u}, \tilde{\eta}_n]\|_2 \le \|[u \otimes \bar{u}, \eta_n]\|_2 + 2\|u - \alpha(u)\|_2 < \varepsilon/2 \tag{4.2}$$

for every $u \in F$ and large enough $n \in \mathbb{N}$. We denote $\zeta_n = (e_M \otimes 1)(\tilde{\eta}_n)$ and $\zeta_n^\perp = \tilde{\eta}_n - \zeta_n$. Noticing that $L^2(M) \mathbin{\bar{\otimes}} L^2(\bar{M})$ is an $M \mathbin{\bar{\otimes}} \bar{M}$-bimodule, it follows from (4.2) that

$$\|[u \otimes \bar{u}, \zeta_n]\|_2^2 + \|[u \otimes \bar{u}, \zeta_n^\perp]\|_2^2 = \|[u \otimes \bar{u}, \tilde{\eta}_n]\|_2^2 < (\varepsilon/2)^2 \tag{4.3}$$

for every $u \in F$ and large enough $n \in \mathbb{N}$. We claim that

$$\operatorname*{Lim}_n \|(p \otimes 1)\zeta_n^\perp\|_2 > \delta. \tag{4.4}$$

Suppose this is not the case. Then, for any $v \in \mathcal{U}(P)$, one has

$$\begin{aligned}
&\operatorname*{Lim}_n \|(p\otimes 1)\tilde{\eta}_n - (e_M\alpha(v)p\otimes\bar{v})\zeta_n\|_2 \\
&\quad\le \operatorname*{Lim}_n \|(p\otimes 1)\tilde{\eta}_n - (e_M\alpha(v)p\otimes\bar{v})\tilde{\eta}_n\|_2 + \operatorname*{Lim}_n \|(p\otimes 1)\zeta_n^{\perp}\|_2 \\
&\quad\le \operatorname*{Lim}_n \|(p\otimes 1)\tilde{\eta}_n - (e_M p\otimes 1)(\alpha(v)\otimes\bar{v})\tilde{\eta}_n\|_2 + \|[\alpha(v),p]\|_2 + \delta \\
&\quad\le \operatorname*{Lim}_n \|(p\otimes 1)\zeta_n^{\perp}\|_2 + \operatorname*{Lim}_n \|\tilde{\eta}_n - (\alpha(v)\otimes\bar{v})\tilde{\eta}_n\|_2 + 2\|p-\alpha(p)\|_2 + \delta \\
&\quad\le 4\delta
\end{aligned}$$

since $pe_M = e_M p$. It follows that

$$\begin{aligned}
\|(E_M\circ\alpha)(v)p\|_2 &= \operatorname*{Lim}_n \|\big((E_M\circ\alpha)(v)p\otimes\bar{v}\big)\tilde{\eta}_n\| \\
&\ge \operatorname*{Lim}_n \|(e_M\otimes 1)\big((E_M\circ\alpha)(v)p\otimes\bar{v}\big)\tilde{\eta}_n\| \\
&= \operatorname*{Lim}_n \|\big(e_M\alpha(v)p\otimes\bar{v}\big)\zeta_n\| \\
&\ge \|p\|_2 - 4\delta > 0
\end{aligned}\tag{4.5}$$

for all $v \in \mathcal{U}(P)$. (One has $\|(E_M\circ\alpha)(vp)\|_2 \ge \|p\|_2 - 6\delta$ as well.) Since $E_M\circ\alpha$ is compact over $Q$, this implies $P \preceq_M Q$ by Corollary 2.7, contradicting the assumption. Thus by (4.3) and (4.4), there exists $n\in\mathbb{N}$ such that $\zeta = \zeta_n^{\perp} \in (L^2(\widetilde{M})\ominus L^2(M))\mathbin{\bar{\otimes}} L^2(\bar{M})$ satisfies $|[u\otimes\bar{u},\zeta]\|_2 < \varepsilon/2$ for every $u\in F$ and $\|(p\otimes 1)\zeta\|_2 \ge \delta$. We note that for all $x\in M$, equation (4.1) implies

$$\|(x\otimes 1)\zeta\|_2^2 = \|(e_M^{\perp}\otimes 1)(x\otimes 1)\tilde{\eta}_n\|_2^2 \le \|(x\otimes 1)\tilde{\eta}_n\|_2^2 = \|x\|_2^2. \tag{4.6}$$

By Proposition 4.8, we may view $\zeta$ as a vector $(\zeta_i)$ in $\bigoplus_i L^2\langle M, e_{Q_{j(i)}}\rangle \mathbin{\bar{\otimes}} L^2(\bar{M})$. We consider $\zeta_i\zeta_i^* \in L^1(\langle M, e_{Q_{j(i)}}\rangle \mathbin{\bar{\otimes}} \bar{M})$ and define $\xi_i = ((\mathrm{id}\otimes\tau)(\zeta_i\zeta_i^*))^{1/2}$ and then $\xi = (\xi_i) \in \bigoplus_i L^2\langle M, e_{Q_{j(i)}}\rangle$. Then, the inequality (4.6) implies

$$\|x\xi\|_2^2 = \sum_i \tau(x^*x(\mathrm{id}\otimes\tau)(\zeta_i\zeta_i^*)) = \|(x\otimes 1)\zeta\|_2^2 \le \|x\|_2^2,$$

and for all $x\in M$. In particular,

$$\|p\xi\|_2 = \|(p\otimes 1)\zeta\|_2 \ge \delta.$$

Finally, by (2.1), one has

$$\begin{aligned}
\|[\xi,u]\|_2^2 &= \sum_i \|\xi_i - (\operatorname{Ad} u)(\xi_i)\|_2^2 && \le \sum_i \|\xi_i^2 - (\operatorname{Ad} u)(\xi_i^2)\|_1 \\
&\le \sum_i \|\zeta_i\zeta_i^* - \operatorname{Ad}(u\otimes\bar{u})(\zeta_i\zeta_i^*)\|_1 && \le \sum_i 2\|\zeta_i\|_2\|[u\otimes\bar{u},\zeta_i]\|_2 \\
&\le 2\|\zeta\|_2\|[u\otimes\bar{u},\zeta]\|_2 && < \varepsilon
\end{aligned}$$

for every $u\in F$. $\square$

Before proving the corollaries to Theorem 4.1, we mention one more result in the spirit of Theorem 4.1. Its proof is similar to the above, but requires more involved technique from [IPP08].

THEOREM 4.10. *Let $M = M_1 * M_2$ be the free product of finite von Neumann algebras and $P \subset M$ be a von Neumann subalgebra such that $P \not\preceq_M M_i$ for $i = 1, 2$. If the action of $\mathcal{G} \subset \mathcal{N}_M(P)$ on $P$ is weakly compact, then $\mathcal{G}''$ is* AFD.

*Proof.* We follow the proof of Theorem 4.1, but use instead the deformation $\alpha_t$ given in Lemma 2.2.2 in [IPP08]. Let a nonzero projection $p \in \mathcal{Z}(\mathcal{G}' \cap M)$, a finite subset $F \subset \mathcal{G}$ and $\varepsilon > 0$ be given arbitrary. Since $P \not\preceq_M M_i$ for $i = 1, 2$, one has

$$\lim_{t \to 0} \inf\{\|(E_M \circ \alpha_t)(vp)\|_2 : v \in \mathcal{U}(P)\} < (999/1000)\|p\|_2$$

by Proposition 3.4 and Theorem 4.3 in [IPP08]. (N.B. This is because Proposition 3.4 is the only part where the rigidity assumption in Theorem 4.3 of that paper is being used.) Hence, if we choose $\delta > 0$ small enough and $t > 0$ accordingly, then one obtains as in the proof of Theorem 4.1 that

$$\operatorname*{Lim}_n \|(p \otimes 1)\zeta_n^{\perp}\|_2 \geq \delta$$

for $\zeta_n^{\perp} = ((1 - e_M) \otimes 1)\tilde{\eta}_n \in L^2(\widetilde{M} \ominus M) \mathbin{\bar{\otimes}} L^2(\bar{M})$. Since $L^2(\widetilde{M} \ominus M)$ is a multiple of $L^2(M \mathbin{\bar{\otimes}} M)$ as an $M$-bimodule, one obtains $\xi \in \bigoplus L^2(M \mathbin{\bar{\otimes}} M)$ such that $\|x\xi\|_2 = \|\xi x\|_2 \leq \|x\|_2$ for all $x \in M$, $\|p\xi\|_2 \geq \delta$ and $\|[u, \xi]\|_2 < \varepsilon$ for every $u \in F$. This proves that $\mathcal{G}''$ is AFD. □

*Proof of Corollary* 4.2. This is a trivial consequence of Theorems 3.5 and Theorem 4.1 . □

*Proof of Corollary* 4.3. Suppose there is a Cartan subalgebra $A \subset M$ where $M \subset N = Q \mathbin{\bar{\otimes}} L(\mathbb{F}_r)$ is a subfactor of finite index. Since $\mathbb{F}_r$ is nonamenable, $N$ is not amenable relative to $Q$, so by Proposition 2.4, $M$ is not amenable relative to $Q$ inside $N$. Hence, by Theorems 3.5 and 4.1, one has $A \preceq_N Q$. By Theorem 2.5, this implies there exist projections $p \in A' \cap N$, $q \in Q$, an abelian von Neumann subalgebra $A_0 \subset qQq$ and a nonzero partial isometry $v \in N$ such that $p_0 = vv^* \in p(A' \cap N)p$, $q_0 = v^*v \in A_0' \cap qNq$ and $v^*(Ap_0)v = A_0q_0$. Since $Q = L(\mathbb{F}_r)' \cap N$, by "shrinking" $q$ if necessary we may clearly assume $q = \bigvee\{uq_0u^* : u \in \mathcal{U}(L(\mathbb{F}_r))\}$. Since $L(\mathbb{F}_r)q$ is contained in $(A_0q)' \cap qNq$, this implies $q_0$ has central support 1 in the von Neumann algebra $(A_0q)' \cap qNq$. But $(A_0q_0)' \cap q_0Nq_0 = v^*(A' \cap N)v$ by spatiality and since $M \subset N$ has finite index, $A \subset A' \cap N$ has finite index as well (in the sense of [PP86]) so $A' \cap N$ is type I, implying $(A_0q_0)' \cap q_0Nq_0$ type I, and thus $(A_0q)' \cap qNq$ type I as well. But $L(\mathbb{F}_r) \simeq L(\mathbb{F}_r)q \subset (A_0q)' \cap qNq$, contradiction. □

For the proof of Corollary 4.4, we will need the following general observation.

LEMMA 4.11. *Let* $\Gamma$ *be an* ICC *group and* $\Gamma \curvearrowright X$ *an ergodic measure-preserving action. Let* $M = L^\infty(X) \rtimes \Gamma$. *Then* $M$ *is a factor. Moreover*, *the following conditions are equivalent*:

(1) $\Gamma \curvearrowright X$ *is free.*

(2) $L^\infty(X)$ *is maximal abelian* (*thus Cartan*) *in* $M$.

(3) *There is a maximal abelian* $^*$*-subalgebra* $A \subset M$ *such that* $A \preceq_M L^\infty(X)$.

*Proof.* The first part is well-known, its proof being identical to the Murray-von Neumann classical argument in [MvN43], showing that if a group $\Gamma$ is ICC then its group von Neumann algebra $L(\Gamma)$ is a factor.

The equivalence of (1) and (2) is a classical result of Murray and von Neumann, and (2) $\Rightarrow$ (3) is trivial. To prove (3) $\Rightarrow$ (2), denote $B = L^\infty(X)$ and let $A \subset M$ be maximal abelian satisfying $A \preceq_M B$. Then there exists a nonzero partial isometry $v \in M$, projections $p \in A = A' \cap M$, $q \in B$ and a unital isomorphism $\theta$ of $Ap$ onto a unital subalgebra $B_0$ of $Bq$ such that $va = \theta(a)v$, for all $a \in Ap$. Denoting $q' = vv^* \in B_0' \cap qMq$, it follows that $q'(B_0' \cap qMq)q' = (B_0q')' \cap q'Mq'$. Since by spatiality $B_0q' = vAv^*$ is maximal abelian, this implies $q'(B_0' \cap qMq)q' = vAv^*$. Thus, $B_0' \cap qMq$ has a type I direct summand. Since $(Bq)' \cap qMq$ is a subalgebra of $B_0' \cap pMp$, it follows that $B' \cap M$ has a type I summand. Since $\Gamma$ acts ergodically on $\mathscr{Z}(B' \cap M) \supset B$ (or else $M$ would not be a factor), the algebra $B' \cap M$ is homogeneous of type $\mathrm{I}_n$, for some $n < \infty$.

Note at this point that since all maximal abelian subalgebras of the type I summand of $B_0' \cap qMq$ containing $q'$ are unitary conjugate (cf. [Kad84]), we may assume that $q'$ is in a maximal abelian algebra containing $Bq$. Thus, if $\mathscr{Z}$ denotes the center of $B' \cap M$, then $\mathscr{Z}q' \subset q'(B_0' \cap qMq)q' = B_0q' \subset Bq'$, showing that $\mathscr{Z}q' = Bq'$. Since $B, \mathscr{Z}$ are $\Gamma$-invariant with the corresponding $\Gamma$-actions ergodic, it follows that there exists a partition of 1 with projections of equal trace $p_1, \dots, p_m \in \mathscr{Z}$ such that $\mathscr{Z} = \Sigma_i Bp_i$ and $E_B(p_i) = m^{-1}1$, for all $i$. Since $B' \cap M = \mathscr{Z}' \cap M$ has an orthonormal basis over $\mathscr{Z}$ with $n^2$ unitary elements, this shows that $B' \cap M$ has a finite unitary orthonormal basis over $B$. But if $x \in (B' \cap M) \setminus B$, and $x = \Sigma_g a_g u_g$ is its Fourier series, with $a_g \neq 0$ for some $g \neq e$, then $p_g u_g \in B' \cap M$, where $p_g$ denotes the support projection of $a_g$. Now, since $\Gamma$ is ICC there exist infinitely many $h_n \in \Gamma$ such that $g_n = h_n g h_n^{-1}$ are distinct. This shows that all $\sigma_{h_n}(p_g)u_{g_n} \subset B' \cap M$ are mutually orthogonal relative to $B$. By [PP86], this contradicts the finiteness of the index of $B \subset B' \cap M$. Thus, we must have $B' \cap M = B$, showing that $\Gamma \curvearrowright X$ is free and $B = L^\infty(X)$ is maximal abelian, hence Cartan. $\square$

*Proof of Corollary* 4.4. The factoriality of $M$ was shown in Lemma 4.11 above.

To prove part (1), note that $\mathscr{N}_{M^t}(A) \curvearrowright A$ weakly compact implies $\mathscr{N}_M(A^{1/t}) \curvearrowright A^{1/t}$ weakly compact, where $A^{1/t} \subset M$ is the semiregular maximal abelian

$^*$-subalgebra obtained by amplifying $A \subset M^t$ by $1/t$ (see Proposition 3.4 and the comments following its proof). This shows that it is sufficient to prove the case $t = 1$. Let $\Gamma_j$ be as in Theorem 4.1 . If $N = \mathcal{N}_M(A)'' \prec_M L^\infty(X) \rtimes \Gamma_j$ for some $j$, then by $[M : N] < \infty$ it follows that $M \prec_M L^\infty(X) \rtimes \Gamma_j$ as well. But this implies $\mathbb{F}_{r(j)}$ amenable, a contradiction. Thus, by Theorem 4.1 we have $A \preceq L^\infty(X)$ and the statement follows from Lemma 4.11.

Part (2) follows trivially from part (1), since $\Gamma \curvearrowright X$ compact implies $M$ has c.m.a.p., by Proposition 3.2.

An obvious maximality argument shows that in order to prove (3) it is sufficient to show: (3′) for all $p \in \mathcal{P}(A)$, $p \neq 0$, $\exists\ v \in M^t$, nonzero partial isometry, such that $v^*v \in Ap$, $vAv^* \subset L^\infty(X)^t$. By amplifying $A \subset M^t$ by suitable integers, we see that in order to prove (3′) for arbitrary $t > 0$, it is sufficient to prove it for $t = 1$. Since $N \prec_M L^\infty(X)$ would imply $N$ amenable, by Theorem 4.1 we must have $A \preceq L^\infty(X)$. Then Lemma 4.11 implies $L^\infty(X)$ maximal abelian in $M$ and Lemma 2.8 applies to get (3′), thus (3) as well. □

The proof of Corollary 4.5 will follow readily from the next general "principle".

PROPOSITION 4.12. *Assume a* $\mathrm{II}_1$ *factor* $M$ *has the property*:

(a) $\exists\ A \subset M$ *Cartan and any maximal abelian* $^*$*-subalgebra* $A_0 \subset M$ *with* $\mathcal{N}_M(A_0)''$ *a subfactor of finite index in* $M$ *is unitary conjugate to* $A$.

*Then any amplification and finite index extension/restriction of* $M$ *satisfies* (a) *as well. Moreover*, *if* $M$ *satisfies* (a) *and* $N \subset M$ *is an irreducible subfactor of finite index*, *then* $[M : N]$ *is an integer.*

*Proof.* For the proof, we call an abelian von Neumann subalgebra $B$ of a $\mathrm{II}_1$ factor $P$ *virtually Cartan* if it is maximal abelian and $Q = \mathcal{N}_P(B)''$ has finite-dimensional center with $[qPq : Qq] < \infty$ for any atom $q \in \mathcal{Z}(Q)$. We first prove that if $P \subset N$ is an inclusion of factors with finite index and $B \subset P$ is virtually Cartan in $P$ then any maximal abelian $^*$-subalgebra $A$ of $B' \cap N$ is virtually Cartan in $N$.

To see this, note that, by commuting squares, the index of $B \subset B' \cap N$ (in the sense of [PP86]) is majorized by $[N : P] < \infty$, implying that $B' \cap N$ is a direct sum of finitely many homogeneous type $\mathrm{I}_{n_i}$ von Neumann algebras $B_i$, with $1 \leq n_1 < n_2 < \cdots < n_k < \infty$. Since any two maximal abelian $^*$-subalgebras of a finite type I von Neumann algebra are unitary conjugate and $\mathcal{N}_P(B)$ leaves $B' \cap N$ globally invariant, it follows that given any $u \in \mathcal{N}_P(B)$, there exists $v(u) \in \mathcal{U}(B' \cap N)$ such that $v(u)uAu^*v(u)^* = A$. Moreover, $A$ is Cartan in $B' \cap N$, i.e. $\mathcal{N}_{B'\cap N}(A)'' = B' \cap N$. This shows in particular that the von Neumann algebra generated by $\mathcal{N}_N(A)$ contains $B' \cap N$ and $v(u)u$, and thus it contains $u$, i.e. $\mathcal{N}_P(B) \subset \mathcal{N}_N(A)''$. Thus, the [PP86]-index of $\mathcal{N}_N(A)''$ in $N$ is majorized by the index of $P$ in $N$, and is thus finite. Since $N$ is a factor, this implies $Q = \mathcal{N}_N(A)''$ has finite-dimensional center and $[qNq : Qq] < \infty$ for any atom in its center, i.e. $A$ is virtually Cartan in $N$.

Now notice that since any unitary conjugacy of subalgebras $A, A_0 \subset M$ as in (a) can be "amplified" to a unitary conjugacy of $A^t$, $A_0^t$ in $M^t$, property (a) is stable to amplifications. This also shows that (a) holds true for a factor $M$ if and only if $M$ satisfies:

(b) *$\exists A \subset M$ Cartan and any virtually Cartan subalgebra $A_0$ of $M$ is unitary conjugate to $A$.*

Since if a subfactor $N \subset M$ satisfies $[M : N] < \infty$ then $\langle M, e_N \rangle$ is an amplification of $N$ (see e.g. [PP86]), it follows that in order to finish the proof of the statement it is sufficient to prove that if $M$ satisfies (b) and $N \subset M$ is a subfactor with finite index, then $N$ satisfies (b).

Let $A \subset M$ be a Cartan subalgebra of $M$. Let $P \subset N$ be such that $N \subset M$ is the basic construction of $P \subset N$ (cf. [Jon83]). Thus $P$ is isomorphic to an amplification of $M$ and so it has a Cartan subalgebra $A_2 \subset P$. By the first part of the statement any maximal abelian subalgebra $A_1$ of $A_2' \cap N$ is virtually Cartan in $N$. Applying again the first part, any maximal abelian $A_0$ of $A_1' \cap M$ is virtually Cartan in $M$, so it is unitary conjugate to $A$. Thus, $A_0 \subset M$ follows Cartan. Thus, $L^2(M) = \oplus u_n L^2(A_0)$, for some partial isometries $u_n \in M$ normalizing $A_0$. Since $A_0$ is a finitely generated $A_1$-module, it follows that each $u_n L^2(A_0)$ is finitely generated both as left and as right $A_1$ module, i.e. there exist finitely many $\xi_i, \xi_j' \in u_n L^2(A_0)$ such that $\Sigma_i \xi_i A_1$ and $\Sigma A_1 \xi_i'$ are dense in $u_n L^2(A_0)$. Thus, if we denote by $\mathcal{H}_n$ the closure of the range of the projection of $u_n L^2(A_0)$ onto $L^2(N)$ and by $\eta_i, \eta_j'$ the projection of $\xi_i, \xi_j'$ onto $L^2(N)$, then $\mathcal{H}_n$ is a Hilbert $A_1$-bimodule generated as left Hilbert $A_1$-module by $\eta_i \in L^2(N)$ and as a right Hilbert $A_1$-module by $\eta_j' \in L^2(N)$. Moreover, since $\vee_n u_n L^2(A_0) = L^2(M)$, we have $\vee_n \mathcal{H}_n = L^2(N)$. Thus, by Section 1.4 in [Pop06a], $A_1$ is Cartan in $N$.

Note that the above argument shows that $N$ has Cartan subalgebra, but also that any virtually Cartan subalgebra of $N$ is in fact Cartan. If now $B_1 \subset N$ is another Cartan subalgebra of $N$, then let $B_0$ be a maximal abelian subalgebra of $B_1' \cap M$. By the first part of the proof $B_0$ is virtually Cartan, so by (b) there exists $v \in \mathcal{U}(M)$ such that $vA_0v^* = B_0$. Thus, if we let $v_n = vu_n$ then $L^2(M) = \oplus_n v_n L^2(A_0) = \oplus_n L^2(B_0)v_n$. Since $A_0$ (resp. $B_0$) is a finitely generated $A_1$ (resp. $B_1$) module, there exist $\xi_i, \xi_j' \in v_n L^2(A_0) = L^2(B_0)v_n$ such that $\Sigma_i \xi_i A_1$ is dense in $v_n L^2(A_0)$ and $\Sigma_j B_1 \xi_j'$ is dense in $L^2(B_0)v_n$. But then exactly the same argument as above shows that $L^2(N)$ is spanned by Hilbert $B_1 - A_1$ bimodules $\mathcal{H}_n$ which are finitely generated both as right $A_1$ Hilbert modules and as left Hilbert $B_1$ modules. By Section 1.4 in [Pop06a], it follows that $A_1, B_1$ are unitary conjugate.

Finally, to see that for irreducible inclusions of factors $N \subset M$ satisfying $(a)$ the index $[M : N]$ is an integer, when finite, let $N \subset Q \subset P \subset M$ be the canonical intermediate subfactors constructed in 7.1 of [Pop06a]. Then $Q, P$ satisfy $(a)$ as well and by 7.1 in [Pop06a] the Cartan subalgebra of $P$ is maximal abelian and Cartan in $M$. Thus, as in the proof of 7.2.3° in [Pop06a], we have $[Q : N]$, $[P : Q]$, $[M : P] \in \mathbb{N}$, implying that $[M : N] \in \mathbb{N}$. □

*Proof of Corollary* 4.5.. Let $M = L^\infty(X) \rtimes \Gamma$ and assume $A \subset M$ is a Cartan subalgebra. By Proposition 3.2 and Corollary 3.3, $M$ follows c.m.a.p. Thus, Theorem 3.5 applies to show that $\mathcal{N}_M(A) \curvearrowright A$ is weakly compact. Since $\mathbb{F}_{r(j)}$ are all nonamenable, $M = \mathcal{N}_M(A)''$ cannot be amenable relative to $L^\infty(X) \rtimes \Gamma_j$ (with $\Gamma_j$ as defined in Theorem 4.1 ), for all $j$. Hence, Theorem 4.1 implies $A \preceq_M L^\infty(X)$. Then Lemma 2.8 shows there is $u \in \mathcal{U}(M)$ such that $uAu^* = L^\infty(X)$, proving the first part of the statement. The rest is a consequence of Proposition 4.12. □

## 5. **Uncountably many approximate free group factors**

In this section we prove that there are uncountably many approximate free group factors of any rank $2 \leq n \leq \infty$. We do this by using a "separability argument," in the spirit of [Pop86], [JP95], [Oza04b]. The proof is independent of the previous sections. The result shows in particular the existence of uncountably many orbit inequivalent profinite actions of $\mathbb{F}_n$. The fact that $\mathbb{F}_n$ has uncountably many orbit inequivalent actions was first shown in [GP05]. A concrete family of orbit inequivalent actions of $\mathbb{F}_n$ was recently obtained in [Ioa09]. Note that the actions $\mathbb{F}_n \curvearrowright X$ in [GP05] and [Ioa09] are not orbit equivalent to profinite actions (because they have quotients that are free and have relative property (T) in the sense of [Pop06a]).

*Definition* 5.1. We say a unitary representation $(\pi, \mathcal{H})$ of $\Gamma$ *has* (*resp. essential*) *spectral gap* if there is a finite subset $F$ of $\Gamma$ and $\varepsilon > 0$ such that the self-adjoint operator

$$\frac{1}{2|F|} \sum_{g \in F} (\pi(g) + \pi(g^{-1}))$$

has (resp. essential) spectrum contained in $[-1, 1-\varepsilon]$. We say such $(F, \varepsilon)$ *witnesses* (*resp. essential*) *spectral gap* of $(\pi, \mathcal{H})$.

It is well-known that $(\pi, \mathcal{H})$ has spectral gap if and only if it does not contain approximate invariant vectors.

*Definition* 5.2. Let $\Gamma$ be a group. We say $\Gamma$ is *inner-amenable* ([Eff75]) if the conjugation action of $\Gamma$ on $\ell^2(\Gamma \setminus \{1\})$ does not have spectral gap.

Let $\{\Gamma_n\}$ be a family of finite index (normal) subgroups of $\Gamma$. We say $\Gamma$ has the property $(\tau)$ with respect to $\{\Gamma_n\}$ if the unitary $\Gamma$-representation on

$$\bigoplus_n \ell^2(\Gamma/\Gamma_n)^o$$

has spectral gap, where $\ell^2(\Gamma/\Gamma_n)^o = \ell^2(\Gamma/\Gamma_n) \ominus \mathbb{C}1_{\Gamma/\Gamma_n}$.

Let $I$ be a family of decreasing sequences

$$i = \big(\Gamma = \Gamma_0^{(i)} \geq \Gamma_1^{(i)} \geq \Gamma_2^{(i)} \geq \cdots\big)$$

of finite index normal subgroups of $\Gamma$ such that $\bigcap \Gamma_n^{(i)} = \{1\}$. We allow the possibility that $\Gamma_n^{(i)} = \Gamma_{n+1}^{(i)}$. We say the family $I$ is *admissible* if $\Gamma$ has the property $(\tau)$ with respect to $\{\Gamma_m^{(i)} \cap \Gamma_n^{(j)} : i, j \in I,\ m, n \in \mathbb{N}\}$ and

$$\sup\{[\Gamma : \Gamma_m^{(i)}\Gamma_n^{(j)}] : m, n \in \mathbb{N}\} < \infty$$

for any $i, j \in I$ with $i \neq j$.

LEMMA 5.3. *Let $\Gamma \leq \mathrm{SL}(d, \mathbb{Z})$ with $d \geq 2$ be a finite index subgroup and*

$$\Gamma_n = \Gamma \cap \ker\big(\mathrm{SL}(d, \mathbb{Z}) \to \mathrm{SL}(d, \mathbb{Z}/n\mathbb{Z})\big).$$

*Let $I$ be a family of infinite subsets of prime numbers such that $|i \cap j| < \infty$ for any $i, j \in I$ with $i \neq j$. (We note that there exists such an uncountable family $I$.) Associate each $i = \{p_1 < p_2 < \cdots\} \in I$ with the decreasing sequence of finite index normal subgroups $\Gamma_n^{(i)} = \Gamma_{i(n)}$ where $i(n) = p_1 \cdots p_n$. Then, the family $I$ is admissible.*

*Proof.* First, we note that $\Gamma_m \cap \Gamma_n = \Gamma_{\gcd(m,n)}$. By the celebrated results of Kazhdan for $d \geq 3$ (see [BdlHV08]) and Selberg for $d = 2$ (see [Lub94]) the group $\Gamma$ has the property $(\tau)$ with respect to the family $\{\Gamma_n : n \in \mathbb{N}\}$. We observe that the index $[\Gamma : \Gamma_m^{(i)}\Gamma_n^{(j)}]$ is the cardinality of $\Gamma$-orbits of $(\Gamma/\Gamma_m^{(i)}) \times (\Gamma/\Gamma_n^{(j)})$. Since

$$\mathrm{SL}(d, \mathbb{Z}/p_1 \cdots p_l\mathbb{Z}) = \prod_{k=1}^{l} \mathrm{SL}(d, \mathbb{Z}/p_k\mathbb{Z})$$

for any mutually distinct primes $p_1, \ldots, p_l$, one has a group isomorphism

$$\mathrm{SL}(d, \mathbb{Z}/i(m)\mathbb{Z}) \times \mathrm{SL}(d, \mathbb{Z}/j(n)\mathbb{Z}) \cong \mathrm{SL}(d, \mathbb{Z}/k\mathbb{Z}) \times \mathrm{SL}(d, \mathbb{Z}/l\mathbb{Z}),$$

where $k = \gcd(i(m), j(n))$ and $l = i(m)j(n)/\gcd(i(m), j(n))$. Since

$$(\Gamma/\Gamma_m^{(i)}) \times (\Gamma/\Gamma_n^{(j)}) \subset \mathrm{SL}(d, \mathbb{Z}/i(m)\mathbb{Z}) \times \mathrm{SL}(d, \mathbb{Z}/j(n)\mathbb{Z})$$

as a $\Gamma$-set, one has

$$[\Gamma : \Gamma_m^{(i)}\Gamma_n^{(j)}] \leq |\mathrm{SL}(d, \mathbb{Z}/k\mathbb{Z})|\,[\mathrm{SL}(d, \mathbb{Z}/l\mathbb{Z}) : \Gamma/\Gamma_l].$$

Therefore, the condition $\sup\{[\Gamma : \Gamma_m^{(i)}\Gamma_n^{(j)}] : m, n \in \mathbb{N}\} < \infty$ follows from the fact that $|i \cap j| < \infty$. □

For example, we can take $\Gamma \leq \mathrm{SL}(2, \mathbb{Z})$ to be $\langle \left(\begin{smallmatrix} 1 & 2 \\ 0 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & 0 \\ 2 & 1 \end{smallmatrix}\right) \rangle \cong \mathbb{F}_2$. By [Sha99], one may relax the assumption that "$\Gamma \leq \mathrm{SL}(d, \mathbb{Z})$ has finite index" to "$\Gamma \leq \mathrm{SL}(d, \mathbb{Z})$ is co-amenable," so that one can take $\Gamma$ to be isomorphic to $\mathbb{F}_\infty$.

Let $\mathcal{S} = (\Gamma_n)_{n=1}^\infty$ be a decreasing sequence of finite index subgroups of a group $\Gamma$. We write $X_{\mathcal{S}} = \varprojlim \Gamma/\Gamma_n$ for the projective limit of the finite probability space $\Gamma/\Gamma_n$ with uniform measures. We note that $L^\infty(X_{\mathcal{S}}) = (\bigcup \ell^\infty(\Gamma/\Gamma_n))''$, where the inclusion $\iota_n\colon \ell^\infty(\Gamma/\Gamma_n) \hookrightarrow \ell^\infty(\Gamma/\Gamma_{n+1})$ is given by $\iota_n(f)(g\Gamma_{n+1}) = f(g\Gamma_n)$. There is a natural action $\Gamma \curvearrowright L^\infty(X_{\mathcal{S}})$ which is ergodic, measure-preserving and profinite. (Any such action arises in this way.) The action is essentially-free if and only if

$$\text{(5.1)}\quad \text{for all } g \in \Gamma \setminus \{1\} \quad |\{x \in X_{\mathcal{S}} : gx = x\}| = \lim_n \frac{|\{x \in \Gamma/\Gamma_n : gx = x\}|}{|\Gamma/\Gamma_n|} = 0.$$

This condition clearly holds if all $\Gamma_n$ are normal and $\bigcap \Gamma_n = \{1\}$. We denote $A_{\mathcal{S}} = L^\infty(X_{\mathcal{S}})$ and $A_{\mathcal{S},n} = \ell^\infty(\Gamma/\Gamma_n) \subset A_{\mathcal{S}}$. Since

$$L^2(A_{\mathcal{S}}) \cong \mathbb{C}1 \oplus \bigoplus_{n=1}^{\infty} \big(L^2(A_{\mathcal{S},n}) \ominus L^2(A_{\mathcal{S},n-1})\big) \subset \mathbb{C}1 \oplus \bigoplus_{n=1}^{\infty} \ell^2(\Gamma/\Gamma_n)^o$$

as a $\Gamma$-space, the action $\Gamma \curvearrowright A_{\mathcal{S}}$ is strongly ergodic if $\Gamma$ has the property $(\tau)$ with respect to $\mathcal{S}$.

THEOREM 5.4. *Let $\Gamma$ be a countable group which is not inner-amenable, and $I$ be an uncountable admissible family of decreasing sequences of finite index normal subgroups of $\Gamma$. Then, all $M_i = L(X_i) \rtimes \Gamma$ are full factors of type $\mathrm{II}_1$ and the set $\{M_i : i \in I\}$ contains uncountably many isomorphism classes of von Neumann algebras.*

*Proof.* That all $M_i$ are full follows from [Cho82]. Take a finite subset $F$ of $\Gamma$ and $\varepsilon > 0$ such that $(F, \varepsilon)$ witnesses spectral gap for both non-inner-amenability and the property $(\tau)$ with respect to $\{\Gamma_m^{(i)} \cap \Gamma_n^{(j)}\}$. We write $\lambda_i(g)$ for the unitary element in $M_i$ that implements the action of $g \in \Gamma$.

We claim that if $i \neq j$, then $(F, \varepsilon)$ witnesses essential spectral gap of the unitary $\Gamma$-representation $\mathrm{Ad}(\lambda_i \otimes \lambda_j)$ on $L^2(M_i \mathbin{\bar{\otimes}} M_j)$. First, we deal with the $\mathrm{Ad}(\lambda_i \otimes \lambda_j)(\Gamma)$-invariant subspace

$$\text{(5.2)}\qquad L^2(A_i \mathbin{\bar{\otimes}} A_j) \cong \mathbb{C}1 \oplus \bigoplus_{n=1}^{\infty} \big(L^2(A_{i,n} \mathbin{\bar{\otimes}} A_{j,n}) \ominus L^2(A_{i,n-1} \mathbin{\bar{\otimes}} A_{j,n-1})\big).$$

We note that the unitary $\Gamma$-representation on

$$L^2(A_{i,n} \mathbin{\bar{\otimes}} A_{j,n}) \cong \ell^2((\Gamma/\Gamma_n^{(i)}) \times (\Gamma/\Gamma_n^{(j)}))$$

is contained in a multiple of $\ell^2(\Gamma/(\Gamma_n^{(i)} \cap \Gamma_n^{(j)}))$. Hence if we show that the subspace of $\Gamma$-invariant vectors in $L^2(A_i \mathbin{\bar{\otimes}} A_j)$ is finite-dimensional, then we can conclude by the property $(\tau)$ that $(F, \varepsilon)$ witnesses essential spectral gap. Suppose $\xi \in L^2(A_{i,n} \mathbin{\bar{\otimes}} A_{j,n})$ is $\Gamma$-invariant. Since $\Gamma_n^{(i)}$ acts trivially on $L^2(A_{i,n})$, the vector $\xi$ is $\mathrm{Ad}(1 \otimes \lambda_j)(\Gamma_n^{(i)})$-invariant. The same thing is true for $j$. It follows that $\xi$ is in the $\Gamma_n^{(i)}\Gamma_n^{(j)} \times \Gamma_n^{(i)}\Gamma_n^{(j)}$-invariant subspace, whose dimension is $[\Gamma : \Gamma_n^{(i)}\Gamma_n^{(j)}]^2$. Since this number stays bounded as $n$ tends to $\infty$, we are done. Second, we deal with the $\mathrm{Ad}(\lambda_i \otimes \lambda_j)(\Gamma)$-invariant subspace

$$\text{(5.3)}\qquad (L^2(M_i) \ominus L^2(A_i)) \mathbin{\bar{\otimes}} L^2(M_j) \cong \ell^2(\Gamma \setminus \{1\}) \mathbin{\bar{\otimes}} L^2(A_i) \mathbin{\bar{\otimes}} L^2(M_j),$$

where $\Gamma$ acts on the right-hand side Hilbert space (which will be denoted by $\mathcal{H}$) as $\mathrm{Ad}(\lambda(g)\otimes\lambda_i(g)\otimes\lambda_j(g))$. For every vector $\xi\in\mathcal{H}$, we write it as $(\xi_g)_{g\in\Gamma\setminus\{1\}}$ with $\xi_g\in L^2(A_i)\mathbin{\bar\otimes} L^2(M_j)$ and define $|\xi|\in\ell^2(\Gamma\setminus\{1\})$ by $|\xi|(g)=\|\xi_g\|$. It follows that

$$\begin{aligned}\Re\langle\mathrm{Ad}(\lambda(g)\otimes\lambda_i(g)\otimes\lambda_j(g))\xi,\xi\rangle &= \Re\sum_{h\in\Gamma\setminus\{1\}}\langle\mathrm{Ad}(\lambda_i(g)\otimes\lambda_j(g))\xi_h,\xi_{ghg^{-1}}\rangle\\ &\le\sum_{h\in\Gamma\setminus\{1\}}\|\xi_h\|\|\xi_{ghg^{-1}}\|=\langle\mathrm{Ad}\,\lambda(g)|\xi|,|\xi|\rangle\end{aligned}$$

for every $g\in\Gamma$ and $\xi\in\mathcal{H}$. Since $(F,\varepsilon)$ witnesses spectral gap of the conjugation action on $\ell^2(\Gamma\setminus\{1\})$, it also witnesses spectral gap of the $\Gamma$-action on $\mathcal{H}$. Similarly, $(F,\varepsilon)$ witnesses spectral gap of

$$L^2(M_i)\mathbin{\bar\otimes}(L^2(M_j)\ominus L^2(A_j)). \tag{5.4}$$

Since the Hilbert spaces (5.2)–(5.4) cover $L^2(M_i\mathbin{\bar\otimes} M_j)$, we conclude that $(F,\varepsilon)$ witnesses essential spectral gap of the $\Gamma$-action $\mathrm{Ad}(\lambda_i\otimes\lambda_j)$. This argument is inspired by [Cho82].

We claim that for any $i\in I$ and any unitary element $u(g)\in M_i$ with $\|\lambda_i(g)-u(g)\|_2<\varepsilon/4$, the essential spectrum of the self-adjoint operator

$$h_F=\frac{1}{2|F|}\sum_{g\in F}\big(\mathrm{Ad}(\lambda_i(g)\otimes u(g))+\mathrm{Ad}(\lambda_i(g^{-1})\otimes u(g^{-1}))\big)$$

on $L^2(M_i\mathbin{\bar\otimes} M_i)$ intersects with $[1-\varepsilon/2,1]$. We fix $i\in I$ and define for every $n\in\mathbb{N}$ the projection $\chi_n\in M_i\mathbin{\bar\otimes} M_i$ by $\chi_n=\sum e_k\otimes e_k$, where $\{e_k\}$ is the set of nonzero minimal projections in $A_{i,n}\cong\ell^\infty(\Gamma/\Gamma_n^{(i)})$. We normalize $\xi_n=[\Gamma:\Gamma_n^{(i)}]^{1/2}\chi_n$ so that $\|\xi_n\|_2=1$. Then, it is not hard to see

$$\mathrm{Ad}(\lambda_i(g)\otimes\lambda_i(g))\xi_n=\xi_n$$

for all $g\in\Gamma$, and

$$\|(1\otimes a)\xi_n\|_2^2=\|a\|_2^2=\|\xi_n(1\otimes a)\|_2^2$$

for all $a\in M_i$. It follows that

$$\begin{aligned}\langle h_F\xi_n,\xi_n\rangle &= \frac{1}{|F|}\sum_{g\in F}\Re\langle\mathrm{Ad}(\lambda_i(g)\otimes u(g))\xi_n,\xi_n\rangle\\ &\ge\frac{1}{|F|}\sum_{g\in F}(1-2\|\lambda_i(g)-u(g)\|_2)>1-\varepsilon/2.\end{aligned}$$

Since $\xi_n\to 0$ weakly as $n\to\infty$, the claim follows (cf. [Ioa]).

From the above claims, we know that if $i\neq j$, then there is no $*$-isomorphism $\theta$ from $M_i$ onto $M_j$ such that $\|\theta(\lambda_i(g))-\lambda_j(g)\|_2<\varepsilon/4$ for all $g\in F$. Now, if the isomorphism classes of $\{M_i : i\in I\}$ were countable, then there would be $M_0$

and an uncountable subfamily $I_0 \subset I$ such that $M_i \cong M_0$ for all $i \in I_0$. Take an $*$-isomorphism $\theta_i : M_i \to M_0$ for every $i \in I_0$. Since $M_0^F$ is separable in $\| \cdot \|_2$-norm, there has to be $i, j \in I_0$ with $i \neq j$ such that

$$\max_{g \in F} \|\theta_i(\lambda_i(g)) - \theta_j(\lambda_j(g))\|_2 < \varepsilon/4,$$

in contradiction to the above. □

When combined with Lemma 5.3, Theorem 5.4 shows in particular that any arithmetic property (T) group has uncountably many orbit inequivalent free ergodic profinite actions, thus recovering a result in [Ioa]. However, [Ioa] provides a "concrete" family (consequence of a cocycle superrigidity result for profinite actions of Kazhdan groups) rather than an "existence" result, as Theorem 5.4 does. But the consequence of Theorem 5.4 and Lemma 5.3 that is relevant here is the following:

COROLLARY 5.5. *For each* $2 \leq r \leq \infty$, *there exist uncountably many nonisomorphic approximate free group factors of rank* $r$. *In particular*, *there exist uncountably many orbit inequivalent free ergodic profinite actions of* $\mathbb{F}_r$.

*Remark* 5.6. Note that if $2 \leq r \leq \infty$ and $\mathcal{S} = (\Gamma_n)$ is a decreasing sequence of finite index subgroups of the free group $\mathbb{F}_r$ satisfying condition (5.1), then the associated free group factor of rank $r$ is the inductive limit of $A_{\mathcal{S},n} \rtimes \mathbb{F}_r \cong \mathbb{B}(\ell^2(\mathbb{F}_r/\Gamma_n)) \mathbin{\bar{\otimes}} L(\Gamma_n)$, which is isomorphic to $L(\mathbb{F}_{1+(r-1)/[\Gamma:\Gamma_n]})$, by Schreier's and Voiculescu's formulae ([VDN92]). Since $1+(r-1)/[\Gamma:\Gamma_n] \to 1$, this justifies the notation $L(\mathbb{F}_{1^+}^{r;\mathcal{S}})$ for the approximate free group factor $L^\infty(X_{\mathcal{S}}) \rtimes \mathbb{F}_r$. The factors $L(\mathbb{F}_{1^+}^{*})$ can be viewed as complementing the one parameter family of free group factors $L(\mathbb{F}_{1+t}), 0 < t \leq \infty$, in [Dyk94], [Răd94].

As mentioned in Section 4, all $L(\mathbb{F}_{1^+}^{r;\mathcal{S}})$ have Haagerup's compact approximation property (by [Haa79]), the complete metric approximation property (by Theorem 2.10) and unique Cartan subalgebra, up to unitary conjugacy (by Corollary 4.5). Also, by [Oza06], the commutant of any hyperfinite subfactor of $L(\mathbb{F}_{1^+}^{r;\mathcal{S}})$ must be an amenable von Neumann algebra, in particular $L(\mathbb{F}_{1^+}^{r;\mathcal{S}})$ is prime, i.e. it cannot be written as a tensor product of two $\mathrm{II}_1$ factors. By [Pop06a], since the factors $L(\mathbb{F}_{1^+}^{r;\mathcal{S}})$ have Haagerup property they cannot contain factors $M$ which have a diffuse subalgebra with the relative property (T). In particular, the $\mathcal{HT}$-factors considered in [Pop06a]) cannot be embedded into approximate free group factors. Same for the factors arising from Bernoulli actions of "$w$-rigid" groups in [Pop06b].

Corollary 4.5 combined with [Gab02] shows that approximate free group factors of different rank are nonisomorphic, $L(\mathbb{F}_{1^+}^{r;\mathcal{S}}) \not\simeq L(\mathbb{F}_{1^+}^{s;\mathcal{S}})$, for all $2 \leq r \neq s \leq \infty$, and have trivial Murray-von Neumann fundamental group [MvN43] when the rank is finite, $\mathcal{F}(L(\mathbb{F}_{1^+}^{s;\mathcal{S}})) = \{1\}$, for all $2 \leq r < \infty$. (Recall from [MvN43] that if $M$ is a $\mathrm{II}_1$ factor then its fundamental group is defined by $\mathcal{F}(M) = \{t > 0 \mid M^t \simeq M\}$.) The first examples of factors with trivial fundamental group were constructed in [Pop06a], were it is shown that $\mathcal{F}(L^\infty(\mathbb{T}^2) \rtimes \mathbb{F}_r) = \{1\}$, for any finite $r \geq 2$, the action of $\mathbb{F}_r$ on $\mathbb{T}^2$ being inherited from the natural action $\mathrm{SL}(2,\mathbb{Z}) \curvearrowright \mathbb{T}^2 = \widehat{\mathbb{Z}^2}$, for some embedding $\mathbb{F}_r \subset \mathrm{SL}(2,\mathbb{Z})$.

One can show that amplifications of approximate free group factors are related by the formula $L(\mathbb{F}_{1^+}^{r,\mathcal{S}})^t = L(\mathbb{F}_{1^+}^{r',\mathcal{S}'})$, with $r' = t^{-1}(r-1)+1$, whenever $t^{-1}$ is an integer dividing the index of some $[\Gamma : \Gamma_n]$ in the decreasing sequence of groups $\mathcal{S} = (\Gamma_n)$, with $\mathcal{S}'$ appropriately derived from $\mathcal{S}$. It is not clear however if this is still the case for other values of $t$ for which $t^{-1}(r-1)+1$ is still an integer.

Finally, note that $L(\mathbb{F}_{1^+}^{r,\mathcal{S}})$ is non$\Gamma$ if and only if the action $\Gamma \curvearrowright X_{\mathcal{S}}$ has spectral gap. Indeed, since the acting group is $\mathbb{F}_r$, any asymptotically central sequence in $L(\mathbb{F}_{1^+}^{r,\mathcal{S}}) = L^\infty(X_{\mathcal{S}}) \rtimes \mathbb{F}_r$ must lie in $L^\infty(X_{\mathcal{S}})$, so $L(\mathbb{F}_{1^+}^{r,\mathcal{S}})$ is non$\Gamma$ if and only if $\mathbb{F}_r \curvearrowright X_{\mathcal{S}}$ is strongly ergodic, which by [AE] is equivalent to $\mathbb{F}_r \curvearrowright X_{\mathcal{S}}$ having spectral gap. For each $2 \le r \le \infty$, one can easily produce sequences of subgroups $\mathcal{S} = (\Gamma_n)$ such that $\mathbb{F}_r \curvearrowright X_{\mathcal{S}}$ does not have spectral gap, thus giving factors $L(\mathbb{F}_{1^+}^{r,\mathcal{S}})$ with property $\Gamma$. On the other hand, as mentioned before, if $\mathbb{F}_r$ is embedded with finite index in $\mathrm{SL}(2,\mathbb{Z})$ (or merely embedded "co-amenably," see [Sha99]) and $\mathcal{S} = (\Gamma_n)$ is given by congruence subgroups, then $\mathbb{F}_r \curvearrowright X_{\mathcal{S}}$ has spectral gap by Selberg's theorem. Thus, the corresponding approximate free group factors $L(\mathbb{F}_{1^+}^{r,\mathcal{S}})$ are non$\Gamma$. By Corollary 5.5 and its proof, there are uncountably many nonisomorphic such factors $L(\mathbb{F}_{1^+}^{r,\mathcal{S}})$ for each $2 \le r \le \infty$. It is an open problem whether there exist solid factors within this class.

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