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via the period map

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Abstract

We characterize the image of the period map for cubic fourfolds with at worst simple singularities as the complement of an arrangement of hyperplanes in the period space. It follows then that the geometric invariant theory (GIT) compactification of the moduli space of cubic fourfolds is isomorphic to the Looijenga compactification associated to this arrangement. This paper builds on and is a natural continuation of our previous work on the GIT compactification of the moduli space of cubic fourfolds.

1. Introduction

An indispensable tool for the study of K3 surfaces is the period map. The period map for K3 surfaces is both injective (satisfies the global Torelli theorem) and surjective. It follows that the moduli space of algebraic K3 surfaces of a given degree is the quotient of a 19-dimensional bounded symmetric domain by an arithmetic group. This explicit description of the moduli space has numerous geometric applications, for example to the study of deformations of certain classes of surface singularities (e.g. the work of Pinkham [39], Looijenga [28], and others). The study of deformations of higher-dimensional singularities led us to consider the period map for cubic fourfolds, which is well-known to behave similarly to the period map for K3 surfaces. Specifically, as in the case of K3 surfaces, the period domain $\mathscr{D}$ for cubic fourfolds is a bounded symmetric domain of type IV of the right dimension, and the global Torelli theorem holds (Voisin [48]). In this paper we analyze the remaining open question, the question of surjectivity (or more precisely of characterizing the image) of the period map for cubic fourfolds. A surjectivity type statement is needed in applications, especially the applications to the study of degenerations and singularities of cubic fourfolds.

Our main result is a characterization of the image of the period map for cubic fourfolds, giving a positive answer to a conjecture of Hassett [21, §4.4]. This builds on earlier results of Voisin [48] and Hassett [21], and was inspired by the recent work of Allcock-Carlson-Toledo [1], [2] and Looijenga-Swierstra [31] on the moduli space of cubic threefolds.

THEOREM 1.1. *The image of the period map for cubic fourfolds* $\mathscr{P}_0 : \mathcal{M}_0 \to \mathscr{D}/\Gamma$ *is the complement of the hyperplane arrangement* $\mathscr{H}_\infty \cup \mathscr{H}_\Delta$ (*see Definition 2.13*). *Furthermore, the period map extends to a regular morphism* $\mathscr{P} : \mathcal{M} \to \mathscr{D}/\Gamma$ *over the simple singularities locus with image the complement of the arrangement* $\mathscr{H}_\infty$.

We note that the theorem is analogous to the corresponding statement for degree two K3 surfaces. Specifically, the hyperplane arrangement $\mathscr{H}_\infty \cup \mathscr{H}_\Delta$ is the analogue of the hyperplane arrangement corresponding to the (-2)-curves in the K3 case. As in the case of low degree K3 surfaces, the two components $\mathscr{H}_\Delta$ and $\mathscr{H}_\infty$ are distinguished by certain arithmetic properties and they parametrize two kinds of degenerations of cubic fourfolds. Finally, again completely analogous to the K3 situation (see [29, §8]), a stronger result holds: the compactification $\overline{\mathcal{M}}$ of the moduli of cubic fourfolds obtained by means of geometric invariant theory (GIT) is an explicit birational modification of the Baily-Borel compactification $(\mathscr{D}/\Gamma)^*$ of the period space.

THEOREM 1.2. *The period map for cubic fourfolds induces an isomorphism*

$$\overline{\mathcal{M}} \cong \widetilde{\mathscr{D}/\Gamma},$$

where $\overline{\mathcal{M}}$ *is the* GIT *compactification of moduli space of cubic fourfolds, and* $\widetilde{\mathscr{D}/\Gamma}$ *denotes the Looijenga compactification associated to the arrangement of hyperplanes* $\mathscr{H}_\infty$.

Theorem 1.2 follows immediately from Theorem 1.1 and the general results of Looijenga (especially [29, Th. 7.6]). Thus, we are mainly concerned here with establishing Theorem 1.1. For this we take an incremental approach, following the arguments of Shah [42], [43] on the surjectivity of the period map for low degree K3 surfaces. We start by computing the GIT compactification $\overline{\mathcal{M}}$ of the moduli space of cubic fourfolds. By studying the monodromy around cubic fourfolds with closed orbits, we conclude that the indeterminacy of the period map is a curve χ. Next, we successively blow-up $\overline{\mathcal{M}}$ first in a special point $\omega \in \chi$, and then in the strict transform of χ. A new monodromy analysis for the blow-up, allows us to conclude that Theorem 1.1 holds. Further details on the organization of the paper and the main intermediary results are given below.

The GIT analysis for cubic fourfolds was done in Laza [27] (some partial results were also obtained by Allcock and Yokoyama). We recall the relevant details in Section 2. Basically, what we need from the GIT computation are the following results. First, a cubic fourfold having at worst simple isolated singularities is GIT stable. In particular, we can talk about the moduli space $\mathcal{M}$ of such cubic fourfolds. For monodromy reasons, over $\mathcal{M}$ the period map naturally extends. The second GIT result used is that the boundary of $\mathcal{M}$ in the GIT compactification $\overline{\mathcal{M}}$ is naturally stratified in 3 types, labeled II, III, and IV. This stratification is closely related to the stratification of Shah [43, Th. 2.4]. Essentially, the singularities that occur

for Type II and III fourfolds are the insignificant limit singularities of Shah [42]. Consequently, the Type II and III fourfolds cause no problem for the period map. Finally, the GIT results identify the locus of Type IV fourfolds in $\overline{\mathcal{M}}$ (the indeterminacy locus of the period map) to be a rational curve χ containing a special point ω. In our analysis, the Type IV fourfolds play the same role as the triple conic for plane sextics (see [43]) or the chordal cubic for cubic threefolds (see [2], [31]).

Based on the GIT results mentioned above, the proof of the Theorem 1.1 follows in two main steps. The first step, completed in Section 3, is to prove that we can control the monodromy of 1-parameter degenerations of cubic fourfolds with central fiber not of Type IV (strictly speaking, this is weaker than an extension statement for the period map $\mathscr{P}$ for Type II and III fourfolds, but since it suffices for our purposes, we will only talk about the Type IV locus as the indeterminacy for $\mathscr{P}$). There are two ingredients going in the proof of the previous statement: a reduction to the central fiber X_0, followed by a Hodge theoretical computation for X_0. Specifically, in Section 3.1, we prove that the natural specialization morphism associated to a degeneration is injective on certain pieces of the corresponding mixed Hodge structures. Thus, the question about the monodromy of the family can be reduced to checking some statement about the mixed Hodge structure of the central fiber X_0. This in turn is relatively easy in our situation. Namely, the mixed Hodge structure of a singular cubic fourfold X_0 can be computed by using the projection from a singular point (see §3.2). This reduces the computation to standard facts about degenerations of K3 surfaces. As hinted in the previous paragraph, the essential fact that makes the proofs of Section 3 work is that the singularities of Type II and III fourfolds are double suspensions of special surface singularities, the so-called insignificant limit singularities of Mumford and Shah.

In Section 4, we analyze the degenerations to Type IV fourfolds, completing the second step of our proof. A partial analysis of the degenerations to Type IV fourfolds was done by Hassett [21, §4.4] and Allcock-Carlson-Toledo [2, §5]. We complete the analysis by proceeding as follows. First, we note that the indeterminacy locus of the period map, the curve χ parametrizing the Type IV fourfolds, is the locus where the GIT quotient $\overline{\mathcal{M}}$ has the worst singularities. Specifically, the Type IV fourfolds are characterized among the semi-stable cubic fourfolds by the fact that their stabilizer is not virtually abelian. Therefore, it is natural to consider a partial desingularization $\widetilde{\mathcal{M}}$ of $\overline{\mathcal{M}}$ as constructed by Kirwan [23]. Namely, we let $\widetilde{\mathcal{M}}$ be the blow-up the special point $\omega \in \chi$, followed by the blow-up of the strict transform of χ (see §4.1). The space $\widetilde{\mathcal{M}}$ has only toric singularities and the period map essentially extends over $\widetilde{\mathcal{M}}$. More precisely, we note that the effect of these two blow-ups is to replace the Type IV fourfolds by some fourfolds, that we call of Type I′–III′, having the same type of singularities as the cubic fourfolds of Type I–III. Thus, we can control the monodromy as in Section 3. We then show that the limit period point corresponding to a Type I′ fourfold belongs to $\mathscr{H}_\infty / \Gamma$. This is enough to complete the proof of Theorem 1.1.

Finally, in Section 5, we put everything together and prove the two main theorems 1.1 and 1.2. As a simple application, we note that from Theorem 1.2 and the results of Looijenga [29] we can recover some information about the GIT compactification purely in arithmetic terms. This is discussed in Section 6.

The study of the periods of cubic fourfolds is closely related to the study of the periods of irreducible holomorphic symplectic fourfolds. Namely, the Fano variety of a cubic fourfold X is a symplectic fourfold F deformation equivalent to the resolution of the symmetric square of a K3 surface (see Beauville-Donagi [6]). The periods of X are essentially the periods of F. It follows that the period domain $\mathscr{D}/\Gamma$ can be interpreted also as the period domain of such symplectic fourfolds (with a degree 6 polarization). A surjectivity type statement for irreducible symplectic fourfolds is known (see Huybrechts [22]), but the linear systems on symplectic fourfolds are not sufficiently understood to obtain a characterization of the image of the period map in this way. We note however that, from the perspective of symplectic fourfolds, the complement of the image of the period map, the divisor $\mathscr{H}_\infty/\Gamma$, has a geometric meaning: it is the locus of symmetric squares of degree two K3 surfaces in the period domain. These type of fourfolds are not Fano varieties of cubic fourfolds, but rather of certain quadric bundles associated to degree two K3 surfaces (see §4.2). Thus, going back to cubics, we can interpret the Kirwan blow-up as enlarging the moduli space of cubic fourfolds $\overline{\mathcal{M}}$ to include these quadric bundles. The first blow-up corresponds to general degree two K3 surface, and the second to the special (i.e., unigonal) ones.

An alternative independent proof of our main result was obtained by Eduard Looijenga [30]. While the proof of [30] also uses some of the GIT results, the techniques (based on [32]) to handle the period map for degenerations of cubic fourfolds are different. We would like to thank Eduard Looijenga for informing us about his work.

Notation and conventions. Our notation and terminology are based on Mumford [35] when we refer to GIT, on Arnold et. al. [3] when we refer to singularities, and on Griffiths et. al. [19] when we refer to Hodge Theory. Additionally, we are using freely the notation of Laza [27] (especially [27, §1.2]). In particular, $\mathcal{M}_0$, $\mathcal{M}$, and $\overline{\mathcal{M}}$ denote the moduli space of smooth cubic fourfolds, of cubic fourfolds with simple (A-D-E) singularities, and the GIT compactification respectively.

All hypersurface singularities are considered up to stable equivalence (i.e., up to adding squares of new variables to the defining equation). In particular, it makes sense to say that two hypersurface singularities of different dimension have the same analytic type.

2. Preliminary results

In this section we collect a series of results that we will need in the subsequent sections. Some are well-known general results, others are specific to cubic fourfolds and are based mostly on Voisin [48], Hassett [21], and Laza [27].

2.1. *The* GIT *compactification of the moduli space of cubic fourfolds.* The computation of the GIT compactification $\overline{\mathcal{M}}$ of the moduli space of smooth cubic fourfolds $\mathcal{M}_0$ was carried out in Laza [27]. Here we recall the GIT results that are needed in what follows (for details see [27]).

We start by noting that a cubic fourfold with only simple singularities is stable.

THEOREM 2.1. *A cubic fourfold with at worst simple singularities is* GIT *stable. In particular, it makes sense to talk about the moduli space $\mathcal{M}$ of such fourfolds as an open subset of the* GIT *quotient $\overline{\mathcal{M}}$. The boundary $\overline{\mathcal{M}} \setminus \mathcal{M}$ consists of six closed irreducible components, that we label α–ϕ. The general point of a boundary component corresponds to a semistable cubic fourfold (with closed orbit) with singularities as given in Table* 1.

Dim.	Component	Singularities of the cubics parametrized by the boundary component
1	α	an elliptic normal curve of degree 4 and a rational normal curve of degree 1
3	β	two isolated $\widetilde{E}_8$ singularities
2	γ	one isolated $\widetilde{E}_7$ singularity and an elliptic normal curve of degree 2
1	δ	three isolated $\widetilde{E}_6$ singularities
3	ε	a rational normal curve of degree 4
2	ϕ	an elliptic normal curve of degree 6

Table 1. The boundary of $\mathcal{M}$ in $\overline{\mathcal{M}}$

Explicit equations for the cubic fourfolds with closed orbit parametrized by the boundary components are given in [27]. Here we only need that, with only one exception (the Type IV described below), the singularities occurring for these cubics are the insignificant limit singularities of Mumford and Shah. Specifically, we recall the following list of singularities from Shah [42, Th. 1]:

Definition 2.2. We say that a 2-dimensional hypersurface singularity $(X, 0) \subset (\mathbb{C}^3, 0)$ is of type t_1–t_6 if its defining equation is one of the following where g is a convergent power series in the appropriate variables:

(t_1): the equation of an isolated rational double point, i.e., A_n, D_m, or E_r;

(t_2): $x_2 x_3$, i.e., double line or A_∞;

(t_3): $x_1^2 x_2 + x_3^2$, i.e., ordinary pinch point or D_∞;

(t_4): $x_3^2 + (x_1 + a_1 x_2^2)(x_1 + a_2 x_2^2)(x_1 + a_3 x_2^2) + g(x_1, x_2)$ with $\mathrm{ord}_0(g) > 6$ with respect to the weights 2 and 1 for x_1 and x_2 respectively, and such that at least two of the a_i are distinct;

(t_5): $x_3^2 + f_4(x_1, x_2) + g(x_1, x_2)$ with f_4 a homogeneous polynomial of degree 4 and $\mathrm{ord}_0(g) > 4$, and such that f_4 has no triple root;

(t_6): $f_3(x_1, x_2, x_3) + g(x_1, x_2, x_3)$ with f_3 a homogeneous polynomial of degree 3 and $\mathrm{ord}_0(g) > 4$, and such that f_3 has at worst ordinary double points as singularities.

Remark 2.3. The types t_4–t_6 cover the simple elliptic (denoted $\widetilde{E_r}$), cusp, and degenerate cusp singularities. In fact, the singularities occurring in the list are the 2-dimensional semi-log-canonical hypersurface singularities (cf. [3, Ch. 15] and [25, Th. 4.21]). The relevance of this fact is that for K3 surfaces three different concepts of measuring the complexity of a singularity are almost equivalent: semi-log-canonical, cohomologically insignificant (cf. [42, Ths. 1 and 2] and [12], [46]), and GIT stable (see [34, §3]). This gives a conceptual explanation of the close relation between the GIT and Hodge theoretical constructions of the moduli spaces of low degree K3 surfaces.

We adapt the previous definition for fourfold singularities as follows:

Definition 2.4. We say that a 4-dimensional hypersurface singularity $(X, 0) \subset (\mathbb{C}^5, 0)$ is of type t_1–t_6 if it is a double suspension of a surface singularity of type t_1–t_6 respectively (i.e., the defining equation can be taken of the form $f(x_1, x_2, x_3) + x_4^2 + x_5^2$ with f of type t_1–t_6).

In analogy to the work of Shah [42], [43] on degenerations of low degree K3 surfaces, it is natural to introduce the following stratification in types of the cubic fourfolds:

Definition 2.5. Let X be a semistable cubic fourfold. We say that

i) *X is of Type* I if X has at worst simple singularities (N.B. X is stable);

ii) *X is of Type* II if X is not of Type I and all its singularities are of type A_n, D_m, E_r, A_∞ (type t_2), D_∞ (type t_3), or $\widetilde{E_r}$ for $r = 6, \ldots, 8$ (the generic case of type t_4–t_6);

iii) *X is of Type* III if X is not of Type I or II and all its singularities are of type t_1–t_6;

iv) *X is of Type* IV if X is not of Type I, II, or III.

By definition the singularities of Type I–III fourfolds are quite mild. Consequently one can control the monodromy of 1-parameter degenerations to Type I–III fourfolds. In fact, the type is simply the index of nilpotency of the monodromy. In contrast, the degenerations to Type IV fourfolds are more complicated and should be treated separately. Thus, we will need the following GIT result that identifies the locus of Type IV fourfolds:

THEOREM 2.6. *The semistable cubic fourfolds of Type* IV *with closed orbits are parametrized by a rational curve χ (including the special point ω). Furthermore, the Type* IV *fourfolds are characterized among the semistable cubics with*

closed orbits by the fact that their stabilizer is not virtually abelian; the stabilizer is SL(2) *for a general point on* χ *(resp.* SL(3) *for* ω).

The Type IV fourfolds parametrized by the curve χ are given by the equations:

$$(2.7) \quad \chi: \ g(x_0,\dots,x_5) = \begin{vmatrix} x_0 & x_1 & x_2 \\ x_1 & x_2 & x_3 \\ x_2 & x_3 & x_4 \end{vmatrix} + ax_5(x_0x_4 - 4x_1x_3 + 3x_2^2) + bx_5^3,$$

for $(a : b) \in W\mathbb{P}(1 : 3)$. As a special case, one obtains the determinantal cubic (the case labeled ω):

$$(2.8) \qquad\qquad \omega: \ g(x_0,\dots,x_5) = \begin{vmatrix} x_0 & x_1 & x_2 \\ x_1 & x_5 & x_3 \\ x_2 & x_3 & x_4 \end{vmatrix}.$$

The fourfold of type ω is the secant to the Veronese surface in $\mathbb{P}^5$. In particular, it is singular along the Veronese surface. More generally, referring to [27], we note the following:

PROPOSITION 2.9. *Let X be a semistable cubic fourfold with closed orbit.*

i) *If X is of Type* II, *then the nonsimple singularities of X are of the following types*:
   - *isolated singularities of type $\widetilde{E_r}$;*
   - *a rational normal curve C of singularities; at a general point of C the singularity is of type A_∞; at 4 special points on C the singularities are of type D_∞;*
   - *an elliptic normal curve C of singularities; at every point of C the singularity is of type A_∞.*

   Additionally, the combinations of nonsimple singularities that occur are given in Table 1.

ii) *If X is of Type* III, *then the locus of nonsimple singularities consists of a connected union of rational normal curves. At a general point of a curve of singularities, the singularity is of type A_∞. On each such curve, there exist two special points with singularities of type t_4–t_6. Furthermore, the only case when all the curves of singularities are lines is the case ζ described below.*

iii) *If X is of Type* IV, *then, except the case ω, X is singular along a rational normal curve C of degree 4 with A_2 transversal singularities. If X is the fourfold ω, then X is singular along a Veronese surface in $\mathbb{P}^5$. Furthermore, this is the only case of a semistable cubic fourfold with 2-dimensional singular locus.*

As suggested by the proposition, the most degenerate case of fourfold of Type II or III is the toric fourfold (labeled ζ):

$$(2.10) \qquad \zeta : \quad g(x_0, \ldots, x_5) = x_0 x_4 x_5 + x_1 x_2 x_3.$$

The fourfold of type ζ is singular along nine lines, meeting in triples. We also note that the point ζ in $\overline{\mathcal{M}}$ (corresponding to the orbit of the fourfold given by (2.10)) is the intersection of all the boundary strata α–ϕ.

Remark 2.11. Some of the Type IV fourfolds appeared previously in literature in connection to the period map. Namely, the case ω was considered by Hassett [21, §4.4], and the special case of χ with extra μ_3-automorphisms (i.e., $a = 0$ in (2.7)) by Allcock-Carlson-Toledo [2].

2.2. *The period map for cubic fourfolds.* The period map for cubic fourfolds is defined by sending a smooth cubic fourfold X to its periods:

$$\mathcal{P}_0 : \mathcal{M}_0 \to \mathcal{D}/\Gamma.$$

The space $\mathcal{D}$ is the classifying space of polarized Hodge structures on the middle cohomology of a cubic fourfold, and Γ is the monodromy group.

Notation 2.12. We denote by $\Lambda := \langle 1 \rangle^{\oplus 21} \oplus \langle -1 \rangle^{\oplus 2}$ the abstract lattice isometric to the integral cohomology of a cubic fourfold, by $h \in \Lambda$ the polarization class (the square of the class of a hyperplane section), and by $\Lambda_0 := \langle h \rangle^{\perp} \cong E_8^{\oplus 2} \oplus U^{\oplus 2} \oplus A_2$ the primitive cohomology.

The Hodge numbers of a cubic fourfold are $h^{4,0} = h^{0,4} = 0$, $h^{3,1} = h^{1,3} = 1$, and $h^{2,2} = 21$. Thus, the classifying space of Hodge structures for cubic fourfolds is a 20-dimensional Type IV bounded symmetric domain

$$\mathcal{D} \cong \mathrm{SO}_0(20, 2)/\mathrm{S}\left(\mathrm{O}(20) \times \mathrm{O}(2)\right).$$

We regard $\mathcal{D}$ as a choice of one of the two connected components of the space of lines in $\Lambda_0 \otimes \mathbb{C}$ satisfying the two Riemann-Hodge bilinear relations:

$$\mathcal{D} = \{\omega \in \mathbb{P}(\Lambda_0 \otimes_{\mathbb{Z}} \mathbb{C}) \mid \omega^2 = 0, \ \omega.\bar{\omega} < 0\}_0.$$

(The subscript 0 indicates the choice of a connected component.) By the results of Ebeling and Beauville [5, Th. 2], Γ can be canonically identified to both $\mathrm{O}_h^+(\Lambda)$ (the automorphisms of Λ that preserve the polarization and the orientation of a negative definite 2-plane in Λ) and $\mathrm{O}^*(\Lambda_0)$ (the automorphisms of Λ_0 that act trivially on the discriminant group $A_{\Lambda_0} := (\Lambda_0)^*/\Lambda_0$ and preserve the orientation of a negative definite 2-plane in Λ_0). Note that $\Gamma \cong \mathrm{O}^*(\Lambda_0)$ respects the choice of connected component for $\mathcal{D}$, and thus acts naturally on $\mathcal{D}$. The orbit space $\mathcal{D}/\Gamma$ is a quasi-projective variety with a natural compactification, the Baily-Borel compactification $(\mathcal{D}/\Gamma)^*$.

It is a general fact that the period map for cubic fourfolds is a local isomorphism. In fact, the stronger global Torelli theorem holds in this situation by a result

of Voisin [48]. Since the period map is algebraic, it follows that $\mathscr{P}_0$ is a birational isomorphism between the quasi-projective varieties $\mathcal{M}_0$ and $\mathscr{D}/\Gamma$. For us, it is more convenient to regard $\mathscr{P}_0$ as a birational map between the projective varieties $\overline{\mathcal{M}}$ and $(\mathscr{D}/\Gamma)^*$ that compactify $\mathcal{M}_0$ and $\mathscr{D}/\Gamma$ respectively.

Hassett [21] has studied the period map of cubic fourfolds in relation to the question whether a smooth cubic fourfold is rational or not. A key aspect of his work is the study of the *special cubic fourfolds*, i.e., the cubic fourfolds that acquire additional algebraic classes in $H^4(X)$. (Since the Hodge conjecture holds for cubic fourfolds, this is equivalent to $H^{2,2}(X) \cap H^4(X, \mathbb{Z})$ having rank at least 2.) In particular, one considers the following notion:

Definition 2.13. Fix a lattice $\Lambda = \langle 1 \rangle^{\oplus 21} \oplus \langle -1 \rangle^{\oplus 2}$ and an element $h \in \Lambda$ of square 3 with even orthogonal complement. Let $\Lambda_0 = \langle h \rangle_\Lambda^\perp \cong E_8^{\oplus 2} \oplus U^{\oplus 2} \oplus A_2$, and $\mathscr{D}$ and Γ as above. For a rank-2 lattice $M \hookrightarrow \Lambda$ primitively embedded in Λ with $h \in M$, we define the hyperplane $\mathscr{D}_M$ by

$$\mathscr{D}_M = \{\omega \in \mathscr{D} \mid \omega \perp M\}.$$

We say $\mathscr{D}_M$ is a hyperplane of determinant $d = \det(M)$. Note that indeed $\mathscr{D}_M$ is the restriction of a hyperplane $H_M \subset \mathbb{P}(\Lambda_0 \otimes_\mathbb{Z} \mathbb{C})$ to $\mathscr{D}$. The hyperplanes of a given determinant form an arithmetic arrangement (with respect to Γ) of hyperplanes in the domain $\mathscr{D}$. We denote the arrangements of hyperplanes of determinants 2 and 6 by $\mathscr{H}_\infty$ and $\mathscr{H}_\Delta$, respectively.

As noted below, the hyperplanes $\mathscr{D}_M$ of a given determinant d form a single Γ-orbit (cf. [21, Props. 3.2.2 and 3.2.4]). In particular, $\mathscr{H}_\infty/\Gamma$ and $\mathscr{H}_\Delta/\Gamma$ are irreducible hypersurfaces in $\mathscr{D}/\Gamma$.

LEMMA 2.14. *With notation as in Definition* 2.13, *the following holds*:

i) $\mathscr{D}_M$ *is nonempty if and only if* $\det(M) \equiv 0, 2 \mod 6$;

ii) *If M and M' have the same determinant, then $\mathscr{D}_M$ and $\mathscr{D}_{M'}$ are conjugated by Γ.*

The relevance for us of the arrangements $\mathscr{H}_\infty$ and $\mathscr{H}_\Delta$ comes from the observation of Hassett [21, §4.4] (see also [48, p. 596, Prop. 1]) that the period map for smooth cubic fourfolds misses these two hyperplane arrangements. Specifically, the following holds:

PROPOSITION 2.15. *Let X be a smooth cubic fourfold. Then,* $H_0^4(X, \mathbb{Z}) \cap H^{2,2}(X)$ *does not contain a primitive class δ such that either*

i) $\delta^2 = 2$,

ii) *or $\delta^2 = 6$ and $\delta.x \equiv 0 \mod 3$ for all $x \in H_0^4(X, \mathbb{Z})$.*

The hyperplanes orthogonal to δ are the hyperplanes of determinant 6 and 2 respectively. Consequently, the image of the period map satisfies:

$$\text{Im}(\mathscr{P}_0) \subset (\mathscr{D} \setminus (\mathscr{H}_\infty \cup \mathscr{H}_\Delta))/\Gamma.$$

We recall that a primitive element δ in a lattice L is called a *generalized root* if and only if $\delta^2 \mid 2\delta.x$ for all $x \in L$. For such an element δ, the reflection s_δ, given by $s_\delta(x) = x - 2\frac{(x,\delta)}{(\delta,\delta)}x$, is an element of the orthogonal group $O(L)$. In our situation, both cases occurring in the previous proposition correspond to generalized roots in Λ_0, of norm 2 and 6 respectively. Consequently, we define:

Definition 2.16. We call an element $\delta \in \Lambda_0$ satisfying $\delta^2 = 2$ a *root* of Λ_0, and an element $\delta \in \Lambda_0$ with $\delta^2 = 6$ and $\delta.\Lambda_0 \equiv 0 \mod 3$ a *long root* of Λ_0.

Since a root δ corresponds geometrically to a vanishing cycle, the subgroup of $O(\Lambda_0)$ generated by reflections s_δ in the roots δ of Λ_0 coincides with the monodromy group $\Gamma \cong O^*(\Lambda_0)$. A reflection s_δ in a long root δ acts nontrivially on the discriminant group $A_{\Lambda_0} \cong \mathbb{Z}/3$ by switching the generators. Thus, $s_\delta \in O^+(\Lambda_0)$, but $s_\delta \notin O^*(\Lambda_0) \cong \Gamma$ (N.B. $O^+(\Lambda_0) \cong \{\pm\mathrm{id}\} \times O^*(\Lambda_0) \cong \mathbb{Z}/2 \times O^*(\Lambda_0)$). It follows that the hyperplanes $\mathscr{H}_\Delta$ are precisely the reflection hyperplanes for Γ (the hyperplanes in $\Lambda_0 \otimes_\mathbb{Z} \mathbb{C}$ pointwise fixed by some element of Γ). Similarly, the hyperplanes in $\mathscr{H}_\Delta \cup \mathscr{H}_\infty$ are the reflection hyperplanes for $O^+(\Lambda_0)$. Note however that the period domain is a subset of $\mathbb{P}(\Lambda_0 \otimes_\mathbb{Z} \mathbb{C})$. In particular, we get that the hyperplanes from $\mathscr{H}_\infty$ are anti-invariant hyperplanes in $\Lambda_0 \otimes_\mathbb{Z} \mathbb{C}$, but become (pointwise) invariant in $\mathscr{D}$ with respect to some s_δ, where δ is a long root (N.B. by abuse of notation we use $\mathscr{H}_\infty$ to refer to hyperplanes both in $\Lambda_0 \otimes_\mathbb{Z} \mathbb{C}$ and $\mathscr{D} \subset \mathbb{P}(\Lambda_0 \otimes_\mathbb{Z} \mathbb{C})$).

LEMMA 2.17. *Let $\delta \in \Lambda_0$ be a long root. Then $(-s_\delta) \in O^*(\Lambda_0)$ and, via the isomorphism $O_h^+(\Lambda) \to O^*(\Lambda_0)$, $(-s_\delta)$ acts as follows on Λ:*

i) *It acts trivially on a primitive sublattice $M \cong \begin{pmatrix} 3 & 2 \\ 2 & 2 \end{pmatrix}$ of Λ, with M containing h and δ;*

ii) *It acts as multiplication by -1 on $M_\Lambda^\perp \cong A_1 \oplus E_8^{\oplus 2} \oplus U^{\oplus 2}$.*

Proof. Since $\delta.\Lambda_0 \equiv 0 \mod 3$, it follows that $h + \delta$ is divisible by 3 in Λ (possibly after changing δ to $-\delta$). The sublattice M of Λ generated by h and $\frac{1}{3}(h + \delta)$ obviously satisfies the conditions of the lemma. For reasons that will be apparent later, we chose generators h and $h - \frac{1}{3}(h + \delta)$ for M. $\square$

Remark 2.18. The element $\gamma = -s_\delta \in \Gamma$ (where δ is a long root) is a Picard-Lefschetz type transformation associated to degenerations to the secant variety X_0 of the Veronese surface. Specifically, γ is the monodromy transformation associated to a loop around the exceptional divisor of the Kirwan blow-up of the point ω (corresponding to X_0) in $\overline{\mathscr{M}}$ (see §4). The invariant cohomology under γ (i.e., the lattice M) is the cohomology that comes from X_0. The anti-invariant cohomology is the vanishing cohomology, i.e., the cohomology that it is determined by the direction of the degeneration to X_0. Furthermore, the Hodge structure supported on the vanishing cohomology is essentially the Hodge structure of a degree two

K3 surface (see §4.2, especially Theorem 4.11). Indeed, $M_\Lambda^\perp$ is (up to sign) the lattice associated to the primitive cohomology of a degree two K3 surface.

2.3. *Degenerations of Hodge structures for cubic fourfolds.* One of the main aspects of our work is the study of 1-parameter families of cubic fourfolds from the point of view of variations of Hodge structures. In this section we collect a series of general results of this theory.

Notation 2.19. *A 1-parameter degeneration* $f : \mathcal{X} \to \Delta$ is a proper analytic map, with generic fiber a smooth projective variety (where $\mathcal{X}$ is an analytic variety and Δ is the unit disk). We denote by X_0 the special fiber, by X_t the generic fiber, and by $X_\infty := \mathcal{X}^* \times_{\Delta^*} \mathfrak{h}$ the canonical fiber, where $\mathcal{X}^*$ is the restriction of $\mathcal{X}$ to the punctured disk Δ^* and $\mathfrak{h} \to \Delta^*$ is the universal cover of Δ^* ($\mathfrak{h}$ is the upper half plane). We also say that f is *a 1-parameter smoothing of X_0.*

2.3.1. *Limit mixed Hodge structures.* Schmid and Steenbrink associate a canonical limit mixed Hodge structure on $\mathrm{H}_{\lim}^n := \mathrm{H}^n(X_\infty, \mathbb{Q})$ to a 1-parameter degeneration. To fix the notation and terminology we recall a few general facts (for a survey see [33]). The monodromy T of the family over the punctured disk is quasi-unipotent. After a ramified base change of type $t \to t^r$ one can further assume that T is actually unipotent. We will assume this in what follows. We fix n to be the dimension of the fiber.

Let $N = \log T$ be the logarithm of the monodromy. It is a nilpotent endomorphism acting on the cohomology $\mathrm{H}_{\lim}^n$ with index of nilpotence ν (i.e., $N^\nu = 0$, but $N^{\nu-1} \neq 0$). On the vector space $\mathrm{H}_{\lim}^n$ there are two filtrations: a decreasing limit Hodge filtration F^p and an increasing weight filtration W_k induced by N (see [33, pp. 106, 107]). Together, the two filtrations define a mixed Hodge structure on $\mathrm{H}_{\lim}^n$. We are interested in determining the index of nilpotency ν of a 1-parameter degeneration. For this, we note the relation:

$$\nu = \max\{k \mid \mathrm{Gr}_{n-k}^W \mathrm{H}_{\lim}^n \neq 0\} + 1.$$

We then note that the possibilities for the Hodge numbers

$$h^{p,q} := \dim \mathrm{Gr}_F^p \mathrm{Gr}_{p+q}^W (\mathrm{H}_{\lim}^n)$$

are quite restricted. Namely, $\mathrm{H}_{\lim}^n$ is isomorphic as a vector space to $\mathrm{H}^n(X_t)$ and the following holds:

  i) $\mathrm{Gr}_k^W (\mathrm{H}_{\lim}^n)$ carries a pure Hodge structure (induced by the filtration F^p) of weight k;

  ii) $N^k : \mathrm{Gr}_{n+k}^W (\mathrm{H}_{\lim}^n) \to \mathrm{Gr}_{n-k}^W (\mathrm{H}_{\lim}^n)$ is an isomorphism of Hodge structures of type $(-k, -k)$;

 iii) $\dim_\mathbb{C} F^p \mathrm{H}_{\lim}^n = \dim_\mathbb{C} F^p \mathrm{H}^n(X_t)$ (see [38, Cor. 11.25]).

2.3.2. *The case of cubic fourfolds.* We specialize the above discussion to the case of degeneration of cubic fourfolds. First, since the Hodge structure on the middle cohomology of a smooth cubic fourfold is of level 2 (i.e., $H^{p,q} = 0$ if $|p - q| > 2$), it follows that the index of nilpotence of N is at most 3. Similarly to the case of K3 surfaces, we then define 3 types of degenerations.

Definition 2.20. Let $f : \mathscr{X} \to \Delta$ be a 1-parameter degeneration of cubic four-folds. Assume that the monodromy T is unipotent. We say that f is a *Type* I (II, *or* III) *degeneration* if the index of nilpotence ν of $N = \log T$ is 1 (2, or 3 respectively).

A simple observation is that, for cubic fourfolds, to determine the type of the degenerations it suffices to know the nonvanishing of a graded piece of $H^4_{\lim}$.

LEMMA 2.21. *Let $f : \mathscr{X} \to \Delta$ be a 1-parameter degeneration of cubic four-folds. Then f is a Type II degeneration if and only if $\mathrm{Gr}^W_3 H^4_{\lim} \neq 0$. In this case $\mathrm{Gr}^W_3 H^4_{\lim}$ is a Tate twist of the Hodge structure of an elliptic curve E, i.e., $\mathrm{Gr}^W_3 H^4_{\lim} \cong H^1(E)(-1)$. Similarly, f is a Type III degeneration if and only if $\mathrm{Gr}^W_2 H^4_{\lim} \neq 0$, in which case $\mathrm{Gr}^W_2 H^4_{\lim}$ is a trivial 1-dimensional Hodge structure of weight 2.*

Proof. Since X_t is a smooth cubic fourfold, we have $\dim_{\mathbb{C}} F^4 H^4(X_t) = 0$ and $\dim_{\mathbb{C}} F^3 H^4(X_t) = 1$. Thus, the only possibly nonzero Hodge numbers in weight at most 3 are $h^{2,1}$, $h^{1,2}$, and $h^{1,1}$. Additionally, they satisfy (by item iii) above) $h^{3,1} + h^{2,1} + h^{1,1} = 1$. The claim follows.   □

We note that the type of a degeneration is closely related to the Baily-Borel compactification $(\mathscr{D}/\Gamma)^*$ as follows. A 1-parameter degeneration of cubic four-folds induces a period map $g : \Delta^* \to \mathscr{D}/\Gamma$, which always extends to an analytic map $\Delta \to (\mathscr{D}/\Gamma)^*$. For Type I degenerations the limit point $\lim_{z \to 0} g(z)$ belongs to the interior $\mathscr{D}/\Gamma$. In contrast, the limit point of a Type II (Type III) degeneration belongs to a Type II (Type III respectively) boundary component (see §6.1).

2.3.3. *The specialization morphism.* The formalism of vanishing cycles ([10, Exp. XIII]) relates the limit mixed Hodge structure of a smoothing to the mixed Hodge structure of the central fiber via the specialization morphism:

$$\mathrm{sp}_n : H^n(X_0) \to H^n_{\lim}.$$

We recall the basic construction as needed in our situation.

First, from the specialization diagram:

(2.22)
$$
\begin{array}{ccccc}
X_0 & \xrightarrow{\ i\ } & \mathscr{X} & \xleftarrow{\quad\pi\quad} & X_\infty \\
& & \downarrow{\scriptstyle f} & & \downarrow \\
& & \Delta & \longleftarrow \Delta^* \longleftarrow & \mathfrak{h}
\end{array}
$$

we define *the functor of nearby cycles* $\psi_f : D_c^b(\mathscr{X}) \to D_c^b(X_0)$ (where $D_c^b(.)$ denotes the bounded derived category of constructible complexes) by

$$\psi_f \mathscr{F}^\bullet := i^* R\pi_* \pi^* \mathscr{F}^\bullet$$

(see [11, Def. 4.2.1]). There exists a natural comparison map $i^* \mathscr{F}^\bullet \xrightarrow{c} \psi_f \mathscr{F}^\bullet$, and the *specialization morphism* sp_n is defined to be the cohomology map associated to c. The *functor of vanishing cycles* ϕ_f is the cone over the morphism c. By definition there exists a distinguished triangle:

(2.23)

$$
\begin{array}{ccc}
i^*\mathscr{F}^\bullet & \xrightarrow{\quad c \quad} & \psi_f \mathscr{F}^\bullet \\
& \underset{[1]}{\searrow} \quad \swarrow {\scriptstyle \mathrm{can}} & \\
& \phi_f \mathscr{F}^\bullet &
\end{array}
$$

in derived category $D_c^b(X_0)$ (see [11, §4.2]).

    We are interested in the situation when $\mathscr{F}^\bullet$ is the constant sheaf $\underline{\mathbb{C}}_{\mathscr{X}}$. Taking the hypercohomology associated to (2.23), we obtain a long exact sequence relating the cohomology of the central fiber with the cohomology of the canonical fiber X_∞:

(2.24) $$\cdots \to \mathrm{H}^n(X_0) \xrightarrow{\mathrm{sp}_n} \mathrm{H}^n_{\lim} \to \mathbb{H}^n(\phi_f \underline{\mathbb{C}}_{\mathscr{X}}) \to \mathrm{H}^{n+1}(X_0) \to \cdots .$$

Furthermore, *the vanishing cohomology* $\mathbb{H}^n(\phi_f \underline{\mathbb{C}}_{\mathscr{X}})$ can be endowed with a natural mixed Hodge structure making (2.24) an exact sequence of mixed Hodge structures. Since a morphism of mixed Hodge structures is strict with respect to both the weight and Hodge filtration, sp_n maps $\mathrm{H}^{p,q}(X_0)$ to $\mathrm{H}^{p,q}_{\lim}$ (where $\mathrm{H}^{p,q} = \mathrm{Gr}_F^p \mathrm{Gr}_{p+q}^W \mathrm{H}$ for a mixed Hodge structure H). In particular, the statement sp_n *is an isomorphism on the* (p,q) *components* is well-defined.

    In our situation, X_t is either a cubic fourfold or a K3 surface, and n represents the dimension of X_t. In particular, there is no odd cohomology. Thus, we have:

(2.25) $$\mathrm{H}^{n-1}(X_t) = \mathrm{H}^{n+1}(X_t) = 0.$$

It follows that (2.24) reduces to a five-term exact sequence:

(2.26) $$0 \to \mathbb{H}^{n-1}(\phi_f \underline{\mathbb{C}}_{\mathscr{X}}) \to \mathrm{H}^n(X_0) \xrightarrow{\mathrm{sp}_n} \mathrm{H}^n_{\lim} \to \mathbb{H}^n(\phi_f \underline{\mathbb{C}}_{\mathscr{X}}) \to \mathrm{H}^{n+1}(X_0) \to 0.$$

    To compute the vanishing cohomology we note the following spectral sequence:

(2.27) $$E_2^{p,q} = \mathrm{H}^p(X_0, \mathscr{H}^q(\phi_f \underline{\mathbb{C}}_{\mathscr{X}})) \Longrightarrow \mathbb{H}^{p+q}(\phi_f \underline{\mathbb{C}}_{\mathscr{X}}).$$

Furthermore, the stalk of the cohomology sheaf $\mathscr{H}^q$ is the reduced cohomology of the Milnor fiber:

(2.28) $$\mathscr{H}^q(\phi_f \underline{\mathbb{C}}_{\mathscr{X}})_x = \widetilde{\mathrm{H}}^q(F_x; \mathbb{C}),$$

where, as usual, F_x is the intersection of the generic nearby fiber with a small open ball centered at x. In particular, $\mathcal{H}^q(\phi_f \underline{\mathbb{C}}_{\mathcal{X}})$ are supported on the singular locus of X_0 ([11, Prop. 4.2.8]). The total space $\mathcal{X}$ of the degeneration might be singular, but in our situation (degeneration of hypersurfaces) it has at worst local complete intersection singularities. Therefore, the range for which there exists nonvanishing cohomology for the Milnor fiber is the same as in the smooth case ([11, Prop. 6.1.2]).

In particular, in the case where the special fiber X_0 has at worst isolated singularities, one obtains that the specialization morphism sp_n is injective ([12, Prop. 2.7], [11, 6.2.2, 6.2.4]) and an exact sequence of mixed Hodge structures:

$$(2.29) \qquad 0 \to \mathrm{H}^n(X_0) \xrightarrow{\mathrm{sp}_n} \mathrm{H}^n_{\lim} \to \oplus_{x_i \in \mathrm{Sing}(X_0)} \mathrm{H}^n(X_i) \to \mathrm{H}^{n+1}(X_0) \to 0$$

where $\mathrm{H}^n(X_i)$ is the cohomology of the Milnor fiber at x_i endowed with the mixed Hodge structure of Steenbrink [45]. Similarly, if the central fiber has 1-dimensional locus the first part of the exact sequence (2.26) reads:

$$(2.30) \qquad 0 \to \mathrm{H}^0(X_0, \mathcal{H}^{n-1}(\phi_f \underline{\mathbb{C}}_{\mathcal{X}})) \to \mathrm{H}^n(X_0) \xrightarrow{\mathrm{sp}_n} \mathrm{H}^n_{\lim} \to \cdots .$$

3. The monodromy around semistable cubic fourfolds

The main result of this section is the control of the monodromy for degenerations of cubic fourfolds which are not of Type IV.

THEOREM 3.1. *Let $f : \mathcal{X} \to \Delta$ be a 1-parameter smoothing of a semi-stable cubic fourfold X_0 with closed orbit. The following holds*:

  i) *If X_0 has Type I, then f is a Type I degeneration*;

 ii) *If X_0 has Type II, then f is a Type II degeneration*;

iii) *If X_0 has Type III, then f is a Type III degeneration.*

(*see Definitions 2.5 and 2.20*).

Proof. By Lemma 2.21, the monodromy of the family is determined by the nonvanishing of $\mathrm{H}^{p,1}_{\lim}$ for $p = 3$, 2, or 1. By Proposition 3.3, the nonvanishing of $\mathrm{H}^{p,1}(X_0)$ implies the nonvanishing of $\mathrm{H}^{p,1}_{\lim}$. The claim now follows from the computation, done in Proposition 3.6, of the mixed Hodge structure of the central fiber. $\square$

Due to the finiteness of the monodromy for Type I fourfolds a stronger result holds. Namely, the period map extends over the locus $\mathcal{M}$ of such fourfolds.

PROPOSITION 3.2. *The period map for a cubic fourfolds $\mathcal{P}_0 : \mathcal{M}_0 \to \mathcal{D}/\Gamma$ extends to a morphism $\mathcal{P} : \mathcal{M} \to \mathcal{D}/\Gamma$ over the simple singularity locus $\mathcal{M} \subset \overline{\mathcal{M}}$. The image of $\mathcal{M} \setminus \mathcal{M}_0$ under the extended period map $\mathcal{P}$ is contained in $\mathcal{H}_\Delta/\Gamma$.*

Proof. Let $o \in \mathcal{M} \setminus \mathcal{M}_0$ correspond to a cubic fourfold X_0 with simple isolated singularities. The statement is analytically local at o, and stable by finite

base changes. Since $\mathcal{M}$ is a geometric quotient, after shrinking and a possible finite cover, we can assume that a neighborhood of o in $\mathcal{M}$ is a 20-dimensional ball S. We can further assume that there exists a family of cubic fourfolds $\mathcal{X} \to S$ with at worst simple isolated singularities and fiber X_0 over o. Let $o \in \Sigma$ be discriminant hypersurface. Over $S \setminus \Sigma$ the family $\mathcal{X}$ gives a variation of Hodge structures defining the period map $\mathcal{P}_0$. By the removable singularity theorem [19, p. 41], the extension statement is equivalent to the monodromy representation $\pi_1(S \setminus \Sigma, t) \to \mathrm{Aut}(\mathrm{H}^4(X_t, \mathbb{Z}))$ (for $t \in S \setminus \Sigma$) having finite image Γ_0 (N.B. Γ_0 is the local monodromy group around X_0).

The fourfold X_0 is GIT stable; thus it has finite stabilizer. For cubic fourfolds we have:

$$(*) \qquad\qquad\qquad n(d-2) - 2 = d,$$

where $n - 1 = 4$ is the dimension and $d = 3$ the degree. It follows that the conditions (i.e., finite stabilizer and $(*)$) from du Plessis-Wall [13, Cor. 1.6] and [14] are satisfied. Thus, the family $\mathcal{X}$ gives a simultaneous versal deformation of the singularities of X_0. It follows that the local monodromy group Γ_0 is the product of the monodromy groups associated to the singularities of X_0. Since the singularities of X_0 are of type A-D-E and the dimension is even, these monodromy groups are (finite) Weyl groups of type A-D-E. Thus, as needed, Γ_0 is finite. Furthermore, the period point corresponding to X_0 is left invariant by the reflections in the vanishing cycles. Thus, it belongs to $\mathcal{H}_\Delta / \Gamma$. $\qquad\square$

3.1. *Reduction to the central fiber.*

PROPOSITION 3.3. *Let X_0 be a semistable cubic fourfold with closed orbit of Type* I–III. *Let $\mathcal{X} \to \Delta$ be any 1-parameter smoothing of X_0, and consider the associated specialization morphism* $\mathrm{sp}_4 : \mathrm{H}^4(X_0) \to \mathrm{H}^4_{\lim}$. *If X_0 has isolated (nonisolated) singularities then* sp_4 *induces an isomorphism (resp. injection) on the (p, q) components of corresponding mixed Hodge structures for all p and q with $p + q \leq 4$ and $(p, q) \neq (2, 2)$.*

Proof. We divide the proof in two cases: either X_0 has isolated singularities, or not.

Case 1 (*Isolated singularities*). Assume that X_0 has only isolated singularities which are suspensions of the types listed in Definition 2.2. From the exact sequence (2.29):

$$0 \to \mathrm{H}^4(X_0) \xrightarrow{\mathrm{sp}_4} \mathrm{H}^4_{\lim} \to \oplus_{x_i \in \mathrm{Sing}(X_0)} \mathrm{H}^4(X_i) \cdots$$

and the strictness of the morphisms of mixed Hodge structures, we see that it is enough to prove the following claim:

($*$) *If* $\operatorname{Gr}_F^p \operatorname{Gr}_{p+q}^W \operatorname{H}^4(X_i) \neq 0$ *then* $(p, q) \in \{(2, 2), (3, 2), (2, 3), (3, 3)\}$, *where* $\operatorname{H}^4(X_i)$ *is the vanishing coholomogy of a smoothing of a simple, simple elliptic, or cusp singularity.*

We note the following facts about the mixed Hodge structure on the vanishing cohomology:

i) (*the local nature*) The mixed Hodge structure on $\operatorname{H}^4(X_i)$ depends only on the germ of the smoothing $(\mathscr{X}, x_i)$ (cf. [47, p. 560]);

ii) (*semicontinuity*) $\dim \operatorname{Gr}_F^p \operatorname{H}^4(X_i)$ is independent of the smoothing (cf. [47, Cor. 2.6]).

From the semicontinuity property, it follows immediately that if ($*$) holds for a smoothing, then it holds for any smoothing. We therefore check ($*$) for the standard Milnor fibrations of the singularities A_n, D_m, E_r, $\widetilde{E_r}$, and $T_{p,q,r}$ respectively. For these singularities the computation of the mixed Hodge structure on the Milnor fiber is well-known (see [26, II.8]). In fact, ($*$) is equivalent to the statement that the spectrum of those singularities (in dimension 4) is included in the interval $[1, 2]$ (see also Remark 3.5). This settles the case of isolated singularities.

Case 2 (*Nonisolated singularities*). Let C be the 1-dimensional singular locus of X_0, and C_i its irreducible components. According to Proposition 2.9, the singularities along C are of type A_∞ at all but a finite number of points. At these special points the singularities are of type D_∞ or degenerate cusps (nonisolated singularities of type t_4–t_6; see Definition 2.2). We denote by $\mathring{C}$ (and $\mathring{C}_i$) the non-special locus. We note two simple facts:

i) $\mathring{C} = \cup_i \mathring{C}_i$ is disconnected (i.e., any point of intersection is special);

ii) $\pi_1(\mathring{C}_i)$ is nontrivial. (In fact, either C_i is elliptic and $\mathring{C}_i = C_i$, or C_i is rational and there are either 2 or 4 special points depending on the type of the degeneration.)

We recall the exact sequence (2.30):

$$(3.4) \qquad 0 \to \operatorname{H}^0(X_0, \mathscr{H}^3(\phi_f \underline{\mathbb{C}}_{\mathscr{X}})) \to \operatorname{H}^4(X_0) \xrightarrow{\operatorname{sp}_4} \operatorname{H}^4_{\lim} \to \cdots$$

and that

$$\mathscr{H}^3(\phi_f \underline{\mathbb{C}}_{\mathscr{X}})_x \cong \operatorname{H}^3(F_x, \mathbb{C})$$

where F_x is the Milnor fiber at x. In particular, the sheaf of vanishing cycles is supported on the 1-dimensional singular locus. The injectivity statement is equivalent to saying that the mixed Hodge structure on the vanishing cohomology satisfies $\operatorname{Gr}_F^p \operatorname{H}^0(X_0, \mathscr{H}^3(\phi_f \underline{\mathbb{C}}_{\mathscr{X}})) = 0$ for $p < 2$.

The analysis of the vanishing cycles in the case of 1-dimensional singular locus is relatively well understood (see [44], [26, II.8.10]). Specifically, we stratify $C = \Sigma_1 \cup \Sigma_0$ such that the vanishing cycles form a local system over Σ_1. In our

situation, $\Sigma_1 = \cup_i \mathring{C}_i$, and Σ_0 are the special points. Since the transversal singularity is A_1, we obtain a 1-dimensional local system over Σ_1. Since the vertical monodromy, given by the natural action of $\pi_1(\mathring{C}_i)$ is nontrivial on each component (see [44, Ch. 3]), there are no nonzero sections of the local system over Σ_1. Thus, the sections of $\mathcal{H}^3(\phi_f \underline{\mathbb{C}}_{\mathscr{X}})$ are supported on the special points. In the type II case we only have special points of type D_∞, but then $H^3(F_x) = 0$ ([44, p. 5]). Thus, $H^0(X_0, \mathcal{H}^3(\phi_f \underline{\mathbb{C}}_{\mathscr{X}})) = 0$ for the Type II case. For the Type III case, we note that the situation is local around the special points. Via the generalized Thom-Sebastiani theorem [40], [20] the computation of the vanishing cohomology is reduced to the surface case, but since the singularities are restricted to the list given in Definition 2.2, we obtain $\mathrm{Gr}_F^p H^0(X_0, \mathcal{H}^3(\phi_f \underline{\mathbb{C}}_{\mathscr{X}})) = 0$ for $p < 2$ (see [42, Th. 2]). $\qquad\square$

Remark 3.5. We note that the analogue of $(*)$ in the surface case is equivalent to the condition that spectrum is included in $[0, 1]$ (N.B. the spectrum is translated by $\frac{1}{2}$ for each suspension). This in turn is equivalent to the singularities being log canonical. In conclusion, the isolated singularities for which Proposition 3.3 holds are precisely the simple, simple-elliptic, and cusp singularities (see also Remark 2.3, [42, Th. 1.2], and [12, Th. 4.13]).

3.2. *The Mixed Hodge Structure of Type* I–III *fourfolds.* In this section we discuss the mixed Hodge structure on the middle cohomology of a singular cubic fourfold. The basic observation is that a singular cubic fourfold X_0 is rational: X_0 is birational to $\mathbb{P}^4$ via the projection from any singular point p. This allows us to reduce the computation of the mixed Hodge structure on $H^4(X_0)$ to a computation for degenerations of K3 surfaces. We obtain the following result, concluding the proof of Theorem 3.1.

PROPOSITION 3.6. *Let X_0 be a semistable cubic fourfold with closed orbit. Then the following hold*:

  i) *If X_0 is of Type* I, *then $H^4(X_0)$ carries a pure Hodge structure of weight* 4 *with $H^{3,1}(X_0) \neq 0$;*

 ii) *If X_0 is of Type* II, *then $\mathrm{Gr}_3^W H^4(X_0) \neq 0$;*

iii) *If X_0 is of Type* III, *then $\mathrm{Gr}_2^W H^4(X_0) \neq 0$.*

Proof. If X_0 is smooth there is nothing to prove. Assume that X_0 is singular. Let $p \in \mathrm{Sing}(X_0)$ (suitably chosen) and $\pi_p : X_0 \dashrightarrow \mathbb{P}^4$ the projection map. In Section 3.2.1 (especially Corollary 3.12), we relate the Hodge structure of X_0 to the Hodge structure of the surface S_p, the base locus of π_p^{-1}. The proposition then follows from the analysis of the mixed Hodge structure on $H^2(S_p)$. But since the surface S_p is a degeneration of K3 surfaces with insignificant limit singularities, the computation is standard. The case by case analysis is done in Lemmas 3.14, 3.16, and 3.17 for Type I–III respectively. $\qquad\square$

3.2.1. *Singular cubic fourfolds and degree* 6 K3 *surfaces.* Let X_0 be a singular cubic fourfold (irreducible and reduced). We choose a singular point $p \in X_0$ and assume additionally:

(∗) $\operatorname{corank}_p(X_0) \leq 3$ *and no line contained in* $\operatorname{Sing}(X_0)$ *passes through* p.

The linear projection π_p with center p gives a birational isomorphism between X_0 and $\mathbb{P}^4$. The birational map π_p can be resolved by blowing-up the point p. The result is the following diagram:

(3.7)
$$
\begin{array}{ccc}
Q_p \lhook\joinrel\longrightarrow & \widetilde{X_0} & \longleftarrow\!\!\rsh E_p \\
\end{array}
$$

$$
\begin{array}{ccccc}
& Q_p \lhook\!\!\longrightarrow \widetilde{X_0} \longleftarrow\!\!\rsh E_p & \\
& \swarrow \quad {}^f\!\swarrow \qquad \searrow{}^g \qquad \searrow & \\
p \lhook\!\!\longrightarrow X_0 & \dashrightarrow^{\ \pi_p} & \mathbb{P}^4 \longleftarrow\!\!\rsh S_p
\end{array}
$$

where Q_p is the projectivized tangent cone at p, $S_p \subset \mathbb{P}^4$ is the surface parametrizing the lines of X_0 through p, and $E_p = g^{-1}(S_p)$ is the exceptional divisor of g. The following facts about the surface S_p are well-known (see especially O'Grady [37, §5.4]).

PROPOSITION 3.8. *With notation as above, assume that* $p \in X_0$ *is a singular point satisfying* (∗). *Then the following hold*:

i) S_p *is the complete intersection of a quadric and cubic in* $\mathbb{P}^4$;

ii) S_p *is reduced* (*but possibly reducible*);

iii) S_p *has only hypersurface singularities*;

iv) *The singularities of* S_p *are in one-to-one correspondence, including the type, with the singularities of* $\widetilde{X_0}$.

Furthermore, the morphism g of (3.7) *is the blow-up of* $\mathbb{P}^4$ *along* S_p. *In particular,* E_p *is a* $\mathbb{P}^1$-*bundle over* S_p.

Proof. The first statement is well-known (e.g. [37, Rem. 5.11]). Namely, let X_0 be given by:

$$X_0: \ (x_0 Q(x_1, \ldots, x_5) + F(x_1, \ldots, x_5) = 0),$$

with Q and F nonvanishing homogeneous polynomials of degree 2 and 3 respectively. Then, the surface S_p is the complete intersection of Q and F. The two assumptions from (∗) imply that Q is reduced and that Q and F are not simultaneously singular. In particular, since at any point of S_p we can choose either Q or F as a local coordinate, we obtain iii). The relation between the singularities of S_p and those of X_0 and $\widetilde{X_0}$ was analyzed in O'Grady [37, Prop. 5.15] and Wall [49, §I.2]. In particular, it is not hard to see that, under the assumption (∗), the singularities of $\widetilde{X_0}$ are double suspensions of the singularities of S_p (see [49, p. 7]), i.e., the singularities of $\widetilde{X_0}$ and S_p have the same analytic type.

The statement about g is [37, Prop. 5.14]. Finally, the exceptional divisor E_p is the projectivization of the normal bundle $\mathcal{N}_{S_P/\mathbb{P}^4}$. Since S_p is a complete intersection, $\mathcal{N}_{S_P/\mathbb{P}^4}$ is a rank-2 vector bundle over S_p (see [17, App. B §6–7]) and the claim follows. $\qquad\square$

Since both f and g are explicit blow-ups, we are able to compute the Hodge structures of X_0 from that of S_p. Specifically, we recall that for a proper birational modification there exists a Mayer-Vietoris sequence relating the cohomologies of the spaces involved. Given a diagram

$$
\begin{array}{ccc}
E & \longrightarrow & \widetilde{X} \\
\downarrow & & \downarrow \\
D & \longrightarrow & X
\end{array}
$$

with X and $\widetilde{X}$ projective varieties, $f : \widetilde{X} \to X$ a projective birational morphism, D the discriminant of f and $E = \pi^{-1}(D)$, there exists a long exact sequence of mixed Hodge structures

(3.9)
$$\cdots \to \mathrm{H}^{n-1}(E) \to \mathrm{H}^n(X) \to \mathrm{H}^n(\widetilde{X}) \oplus \mathrm{H}^n(D) \to \mathrm{H}^n(E) \to \mathrm{H}^{n+1}(X) \to \cdots ;$$

(see [38, Cor. 5.37]). Applying this exact sequence to the discriminant square associated to the morphism f (see (3.7)), we relate the cohomology of X_0 with that of the blow-up $\widetilde{X_0}$:

LEMMA 3.10. *Let X_0 be a singular cubic fourfold (irreducible, reduced), and p a singular point satisfying* (∗). *Then, the pullback $f^* : \mathrm{H}^4(X_0) \to \mathrm{H}^4(X_0)$ induces a (p,q)-isomorphism for all $(p,q) \neq (2,2)$.*

Proof. The morphism f is the blow-up of the singular point p. Thus, the relevant part of the sequence (3.9) reads:

$$\cdots \to \mathrm{H}^3(Q_p) \to \mathrm{H}^4(X_0) \to \mathrm{H}^4(\widetilde{X_0}) \to \mathrm{H}^4(Q_p) \to \cdots .$$

Since for the reduced quadric Q_p we have $\mathrm{H}^3(Q_p) = 0$ and $\mathrm{H}^4(Q_p)$ is of pure weight 4 and type $(2,2)$, the claim follows. $\qquad\square$

Similarly, we relate the cohomology of the $\widetilde{X_0}$ with that of the surface S_p.

LEMMA 3.11. *With notation and assumptions as above,*

$$\dim \mathrm{Gr}_F^1 \mathrm{Gr}_{n+2}^W \mathrm{H}^4(\widetilde{X_0}) = \dim \mathrm{Gr}_F^0 \mathrm{Gr}_n^W \mathrm{H}^2(S_p)$$

for $n = 0, 1, 2$.

Proof. We consider the exact sequence (3.9) for the birational morphism g of (3.7). We obtain:

$$\cdots \to \mathrm{H}^4(\mathbb{P}^4) \to \mathrm{H}^4(\widetilde{X_0}) \oplus \mathrm{H}^4(S_p) \to \mathrm{H}^4(E_p) \to \mathrm{H}^5(\mathbb{P}^4) = 0.$$

Both $\mathrm{H}^4(\mathbb{P}^4) \cong \mathbb{C}$ and $\mathrm{H}^4(S_p)$ carry a pure weight 4 Hodge structure of type $(2,2)$ (for S_p this follows from [38, Th. 6.32]). Thus, the restriction map $\mathrm{H}^4(\widetilde{X_0}) \to$

$H^4(E_p)$ is a (p,q)-isomorphism for all $(p,q) \neq (2,2)$. The claim now follows from the fact that E_p is a projective $\mathbb{P}^1$-bundle over S_p (cf. Proposition 3.8). $\square$

In conclusion, we obtain:

COROLLARY 3.12. *Let* X_0 *be an irreducible, reduced cubic fourfold, and* $p \in X_0$ *a singular point satisfying* (∗). *Then* $\mathrm{Gr}_F^0 \mathrm{Gr}_{n+2}^W H^4(X_0) = 0$ *and*

$$(3.13) \qquad \dim \mathrm{Gr}_F^1 \mathrm{Gr}_{n+2}^W H^4(X_0) = \dim \mathrm{Gr}_F^0 \mathrm{Gr}_n^W H^2(S_p)$$

for $n = 0, 1, 2$.

3.2.2. *The Type* I *case.* The surface S_p associated to a singular Type I fourfold X_0 has at worst du Val singularities. Thus, S_p is a K3 surface. It follows that the Hodge structure on $H^4(X_0)$ is pure.

LEMMA 3.14. *Let* X_0 *be a cubic fourfold with at worst simple isolated singularities. Then* $H^4(X_0)$ *carries a pure Hodge structure of weight 4. Furthermore,* $H^{3,1}(X_0) \neq 0$.

Proof. Assume that X_0 is singular, and choose any singular point $p \in X_0$. According to Proposition 3.8, the singularities of S_p are in bijective correspondence to the singularities of $\widetilde{X_0}$ and of the same analytic type. The singularities of $\widetilde{X_0}$ are of two kinds: either they come from $X_0 \setminus \{p\}$, or they lie over p. In either case they are simple singularities. It follows that S_p is a $(2,3)$-complete intersection in $\mathbb{P}^4$ with at worst du Val singularities. It is well-known that the minimal desingularization of S_p is a K3 surface and that the Hodge structure on $H^2(S_p)$ is pure. The lemma follows (cf. Corollary 3.12). $\square$

Remark 3.15. An essentially equivalent statement is O'Grady [37, Prop. 5.9, 5.28]. We remark that the proof of [37, Prop. 5.9] (see [37, §5.4.4]) actually works under the assumption that S_p is a surface with at worst du Val singularities. This gives another proof to the lemma.

3.2.3. *The Type* II *and* III *case.* In the case of Type II and III fourfolds, we can reduce the computation of the mixed Hodge structure on $H^4(X_0)$ to a computation for degenerations of degree 6 K3 surfaces. This reduction is made possible by the correspondence of singularities given by Proposition 3.8 and the fact that the singularities that occur are the insignificant singularities of Shah [42].

LEMMA 3.16. *Let* X_0 *be a Type* II *cubic fourfold with closed orbit. Then we can chose a singular point* $p \in X_0$ *satisfying* (∗) *such that the surface* S_p *associated to the projection from* p *is a Type* II *degeneration of* K3 *surfaces; i.e.,* S_p *has insignificant limit singularities and* $\mathrm{Gr}_1^W H^2(S_p) \neq 0$. *Thus,* $\mathrm{Gr}_3^W H^4(X_0) \neq 0$.

Proof. All singularities of a Type II fourfold X_0 have corank at most 3. Also, in each of the cases α–ϕ we can find a singular point p not lying on a singular line (see Table 1). Thus, we can choose a point p satisfying (∗). We can further assume that p is either of type $\widetilde{E}_r$ or A_∞. It is immediate to check that the nonsimple

singularities of the fourfold $\widetilde{X_0}$ (obtained by blowing-up p) are of type $\widetilde{E_r}$, A_∞, or D_∞. By Proposition 3.8, the same holds for S_p. In particular, we conclude that S_p has only insignificant limit singularities and is a degeneration of (degree 6) K3 surfaces. From Shah [42, Th. 2] it follows that $\dim \operatorname{Gr}_F^0 \operatorname{H}^2(S_p) = 1$. Since by construction S_p has at least one non-du Val singularity, the Hodge structure on S_p is not pure. Thus, $\operatorname{Gr}_k^W \operatorname{H}^2(S_p) \neq 0$ for either $k = 1$ (Type II) or $k = 0$ (Type III). It is known that if all the singularities are of type $\widetilde{E_r}$, A_∞, or D_∞, the degeneration is of Type II (see [43, Th. 3.2]), concluding the proof of the lemma.

For concreteness, we illustrate the case δ. In this case, the cubic fourfold X_0 has 3 singularities of type $\widetilde{E_6}$. We choose one of them as the projection center p. The resulting surface S_p is the union of two surfaces S_1 and S_2. Both S_1 and S_2 are cones over the same elliptic curve $C = S_1 \cap S_2$. By the Mayer-Vietoris sequence, we have

$$\cdots \to \operatorname{H}^1(S_1) \oplus \operatorname{H}^1(S_2) \to \operatorname{H}^1(C) \to \operatorname{H}^2(S_p) \to \operatorname{H}^2(S_1) \oplus \operatorname{H}^2(S_2) \to \cdots .$$

The resolution of S_i (for $i = 1, 2$) is a ruled surface $\widetilde{S_i}$ over C. Moreover, the exceptional divisor of $\widetilde{S_i} \to S_i$ is isomorphic to C. It then follows that $\operatorname{H}^1(S_i) = 0$ and $\operatorname{H}^2(S_i)$ is 1-dimensional, carrying a pure Hodge structure of type $(1, 1)$. In conclusion, we have $\operatorname{Gr}_1^W \operatorname{H}^2(S_p) \cong \operatorname{H}^1(C)$ and the claim follows. $\qquad\square$

Similarly, for Type III fourfolds we obtain:

LEMMA 3.17. *Let X_0 be a Type* III *cubic fourfold with closed orbit. Then* $\operatorname{Gr}_2^W \operatorname{H}^4(X_0) \neq 0$.

Proof. By Proposition 2.9 (ii), except the case ζ, we can find a singular point $p \in X_0$ not lying on a line of singularities. We can further assume that p is of type A_∞. As in Lemma 3.16, S_p has insignificant limit singularities. Since some of the singularities of S_p are degenerate cusps, S_p is a Type III degeneration of K3 surface. It remains to consider the case ζ. Assume X_0 is given by equation (2.10), and let $p = (1 : 0 : \cdots : 0)$. Items iii) and iv) of Proposition 3.8 do not hold for S_p (p lies on a line of singularities). In particular, since some of the singularities of S_p are not hypersurface singularities, we cannot apply the results of Shah [42] to get information about $\operatorname{H}^2(S_p)$. On the other hand, since the singularities of S_p are complete intersections, all the statements of Proposition 3.8, except (iii) and (iv), are still valid. It follows then that the relation (3.13) between the Hodge structures of X_0 and S_p also holds. Finally, we find the Hodge structure on $\operatorname{H}^2(S_p)$ by a direct computation as follows. The surface S_p is the complete intersection $(x_4 x_5 = 0,\ x_1 x_2 x_3 = 0)$ in $\mathbb{P}^4$ (with homogeneous coordinates $(x_1 : \cdots : x_5)$). Let S_1 and S_2 be the restrictions of S_p to the hyperplanes $(x_4 = 0)$ and $(x_5 = 0)$ respectively. The surface S_p is obtained by gluing S_1 and S_2 along the curve C given by $(x_1 x_2 x_3 = 0)$ in the plane $(x_4 = x_5 = 0)$. Thus, by Mayer-Vietoris, we have

$$\cdots \to \operatorname{H}^1(S_1) \oplus \operatorname{H}^1(S_2) \to \operatorname{H}^1(C) \to \operatorname{H}^2(S_p) \to \cdots .$$

Since S_1 and S_2 are normal crossing varieties, it follows easily that $H^1(S_i) = 0$ for $i = 1, 2$ (e.g. [33, p. 111]). Also, since C is a cycle of rational curves, $\mathrm{Gr}_0^W H^1(C)$ is 1-dimensional. Thus, $\mathrm{Gr}_0^W H^2(S_p) \neq 0$ and the lemma follows. $\qquad\square$

Remark 3.18. It is possible to compute the mixed Hodge structure of a Type II fourfold X_0 by means of a resolution of the singularities of X_0. This approach identifies the Hodge structure on $\mathrm{Gr}_3^W H^4(X_0)$ with the Hodge structure of an elliptic curve C naturally associated to the singularities of X_0. For example, in the simplest case, the case ϕ, X_0 is singular along an elliptic normal curve C. The singularities of X_0 are resolved by the blow-up of C. The resulting exceptional divisor E is a quadric bundle over C. Since the intermediate Jacobian of E is (up to isogeny) the Jacobian of C, we obtain an identification between $\mathrm{Gr}_3^W H^4(X_0)$ and $H^1(C)(-1)$. The situation is similar in the case when X_0 is singular along a rational normal curve with 4 special points on it (see Proposition 2.9 (i)). Finally, in the case of $\widetilde{E}_r$ singularities, a weighted blow-up of the $\widetilde{E}_r$ singularity produces a fourfold singular along an elliptic curve, bringing us to the situation discussed above.

4. **Degenerations to Type IV fourfolds**

Due to the presence of bad singularities for Type IV fourfolds, the period map has singularities along the curve $\chi \subset \overline{\mathcal{M}}$. In this section, we note that a certain blow-up $\widetilde{\mathcal{M}}$ of $\overline{\mathcal{M}}$ along χ essentially resolves these singularities. We say essentially because our result (see Theorem 4.11) is weaker than an extension statement. However, *a posteriori*, one can indeed conclude that the blow-up $\widetilde{\mathcal{M}}$ resolves the period map (see Remark 6.2).

4.1. *The Kirwan desingularization along χ.* We start by recalling the setup of Kirwan [23]. Let G be a reductive group, acting on a smooth projective variety P. For R a reductive subgroup of G we denote by Z_R^{ss} the subset of semistable points stabilized by R. Then, to each conjugacy class of reductive subgroups with $Z_R^{\mathrm{ss}} \neq \varnothing$ there is associated a natural blow-up of the GIT quotient $P /\!\!/ G$. The center of the blow-up is the closed subset $(G \cdot Z_R^{\mathrm{ss}}) /\!\!/ G$. The resulting space is again a GIT quotient, but with simpler singularities. By performing this procedure a finite number of times starting with the highest dimensional subgroup R, one obtains a space with only quotient singularities.

In our situation $\overline{\mathcal{M}} = \mathbb{P}(\mathrm{Sym}^3 W) /\!\!/ G$, where $G = \mathrm{SL}(6)$ and W is the standard G-representation. The conjugacy classes of subgroups R are easily determined from the results of [27]. Specifically, the minimal subgroups are $R_\alpha, \ldots, R_\delta$ corresponding to the stabilizer of a generic point in the boundary strata α–δ (see [27, §4]). These boundary strata further specialize to the strata τ, χ, ζ, and ω. The resulting subgroups (except when specified otherwise, these subgroups are tori of the indicated dimension) and the corresponding inclusions are given in Figure 1 (for a similar situation see [24, p. 66]).
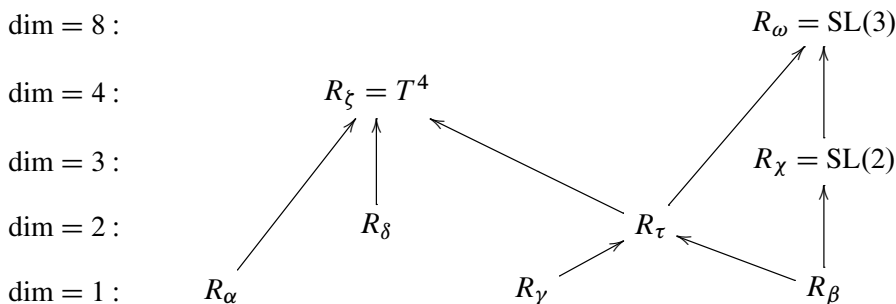
$$\dim = 8: \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad R_\omega = \mathrm{SL}(3)$$

$$\dim = 4: \qquad\qquad R_\zeta = T^4$$

$$\dim = 3: \qquad\qquad\qquad\qquad\qquad\qquad\qquad R_\chi = \mathrm{SL}(2)$$

$$\dim = 2: \qquad\qquad R_\delta \qquad\qquad\qquad R_\tau$$

$$\dim = 1: \qquad R_\alpha \qquad\qquad\qquad R_\gamma \qquad\qquad R_\beta$$

Figure 1. The adjacencies of connected reductive subgroups with $Z_R^{\mathrm{ss}} \neq \varnothing$

Here, we consider only the first two steps of the Kirwan partial desingularization of $\overline{\mathcal{M}}$. Namely, we blow up $\overline{\mathcal{M}}$ with respect to R_ω, followed by the blow-up with respect to R_χ. We denote the resulting variety $\widetilde{\mathcal{M}}$, and by $\widehat{\mathcal{M}}$ the intermediary blow-up. Thus, $\widehat{\mathcal{M}}$ is the blow-up of $\overline{\mathcal{M}}$ in the point ω, and $\widetilde{\mathcal{M}}$ is the blow-up of $\widehat{\mathcal{M}}$ along the strict transform of χ.

4.1.1. *The blow-up of the point ω.* The moduli of cubic fourfolds is obtained as the quotient of the action of $G = \mathrm{SL}(6)$ on

$$P := \mathbb{P}(H^0(\mathbb{P}^5, \mathcal{O}_{\mathbb{P}^5}(3))) = \mathbb{P}(\mathrm{Sym}^3 W).$$

Let $\omega \in \mathcal{M}$ be the special boundary point. The preimage of ω in P contains a unique closed orbit, say the orbit of $x \in P^{\mathrm{ss}}$.

The local structure (in the étale topology) of the quotient $\overline{\mathcal{M}}$ is described by Luna's slice theorem [35, App. D]. Specifically, there exists a G_x-invariant slice W to the orbit $G \cdot x$. The slice W can be taken to be a smooth, affine, locally closed subvariety of P such that $U = G \cdot W$ is open in P. We then have the following commutative diagram with Cartesian squares:

$$
\begin{array}{ccccccc}
G \times_{G_x} \mathcal{N}_x & \xleftarrow{\text{étale}} & G \times_{G_x} W & \xrightarrow{\text{étale}} & U & \subset & P \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\mathcal{N}_x /\!\!/ G_x & \xleftarrow{\text{étale}} & W /\!\!/ G_x & \xrightarrow{\text{étale}} & U /\!\!/ G & \subset & P /\!\!/ G \cong \overline{\mathcal{M}},
\end{array}
$$

where G_x denotes the stabilizer of x, and $\mathcal{N}_x$ the fiber of the normal bundle of the orbit $G \cdot x$. The Kirwan desingularization is compatible with Luna's slice. In particular, the exceptional divisor of the blow-up of $\overline{\mathcal{M}}$ at the point corresponding to $G \cdot x$ is the quotient $\mathbb{P}(\mathcal{N}_x) /\!\!/ G_x$ ([23, Rem. 6.4]).

We recall that the point ω corresponds to the secant variety X of the Veronese surface S in $\mathbb{P}^5$. It follows that $G_x^0 \cong \mathrm{SL}(3)$, acting in the standard way on the singular locus S of X. The Veronese surface $S \subset \mathbb{P}^5$ is the image of $\mathbb{P}^2$ embedded

by $\mathbb{O}_{\mathbb{P}^2}(2)$ in $\mathbb{P}^5$. Thus, there is a natural identification $W \cong \mathrm{Sym}^2 V$ of SL(3)-representations such that $S \subset \mathbb{P}(W)$ is left invariant (where V is the standard SL(3) representation). We obtain the decomposition in irreducible summands

$$(4.1) \qquad \mathrm{Sym}^3 W \cong \mathrm{Sym}^3 \mathrm{Sym}^2 V \cong \mathrm{Sym}^6 V \oplus \Gamma_{2,2} \oplus \mathbb{C}$$

(cf. [18, (13.15)]). It then follows easily that the natural representation of SL(3) on the normal slice $\mathcal{N}_x$ is isomorphic to $\mathrm{Sym}^6 V$. In conclusion, from Luna's slice theorem and this computation we have obtained the following precise description of the blow-up of the point ω:

COROLLARY 4.2. *The point $\omega \in \overline{M}$ has an étale (or analytic) neighborhood isomorphic to the affine cone over the moduli space of plane sextic curves. The Kirwan blow-up of $\overline{M}$ at the point ω is isomorphic when restricted to this neighborhood to the natural blow-up of the vertex of the cone. In particular the exceptional divisor is isomorphic to the GIT quotient for plane sextics.*

4.1.2. *The blow-up of the curve χ.* We repeat the computation from the previous section for the strict transform of the curve χ. We recall that the singular locus of a fourfold giving a minimal orbit of type χ is a rational normal curve of degree 4. It follows that (at least generically) $G_x \cong \mathrm{SL}(2)$, for $x \in P$ with closed orbit and mapping to χ. As in the case ω, we obtain:

LEMMA 4.3. *Let $x \in P$ be a semistable point with closed orbit, mapping to χ. Then the natural representation of $G_x^0 \cong \mathrm{SL}(2)$ on the normal $\mathcal{N}_x$ is isomorphic to*

$$(4.4) \qquad\qquad \mathrm{Sym}^{12} V \oplus \mathrm{Sym}^8 \oplus \mathbb{C},$$

where V denotes the standard SL(2) *representation.*

Proof. A fourfold of type χ is singular along a rational normal curve of degree 4 contained in a hyperplane of $\mathbb{P}^5$. It follows (as in §4.1.1) that we can naturally identify W to the SL(2) representation $\mathrm{Sym}^4 V \oplus \mathbb{C}$. We then have

$$
\begin{aligned}
(4.5) \quad \mathrm{Sym}^3 W &\cong \mathrm{Sym}^3(\mathrm{Sym}^4 V \oplus \mathbb{C}) \\
&\cong \mathrm{Sym}^3(\mathrm{Sym}^4 V) \oplus \mathrm{Sym}^2(\mathrm{Sym}^4 V) \oplus \mathrm{Sym}^4 V \oplus \mathbb{C} \\
&\cong \mathrm{Sym}^{12} V \oplus (\mathrm{Sym}^8 V)^{\oplus 2} \oplus \mathrm{Sym}^6 V \oplus (\mathrm{Sym}^4 V)^{\oplus 3} \oplus \mathbb{C}^{\oplus 3}.
\end{aligned}
$$

The lemma now follows from the identification of the summands in the normal sequence:

$$0 \to \mathscr{T}_{G \cdot x, x} \to \mathscr{T}_{P, x} \to \mathcal{N}_x \to 0$$

(e.g. the representation on $\mathscr{T}_{G \cdot x, x}$ is computed by using $G \cdot x \cong G/G_x$). $\qquad \square$

To interpret geometrically the result of the previous lemma, we recall the situation of plane sextics studied by Shah [43]. Namely, to resolve the rational map from the GIT quotient of plane sextics to the period domain of degree two K3 surfaces, one needs to blow-up the point corresponding to the triple conic. The

stabilizer is again SL(2), and the representation on the normal slice is

$$(4.6) \qquad\qquad \mathrm{Sym}^{12}V \oplus \mathrm{Sym}^8 V.$$

The two summands of (4.6) correspond to Φ and Ξ respectively in Shah's notation ([43, pp. 498–500]). In conclusion, by comparing (4.4) and (4.6), we can say that the blow-up of χ is locally the product of the affine line with the exceptional divisor $\mathfrak{N}$ obtained by Shah [43, §5]. The divisor $\mathfrak{N}$ corresponds to elliptic degree two K3 surfaces. These special degree two K3 surfaces are not double covers of $\mathbb{P}^2$, but instead are double covers of the Hirzebruch surface $\mathbb{F}_4$ embedded in $\mathbb{P}^5$ as the cone Σ_4^0 over a rational normal curve of degree 4.

Remark 4.7. It is well-known that for a degree two K3 surface (S, H) the polarization $2H$ is base point free and defines a morphism to $\mathbb{P}^5$. In the generic case the image of S is the Veronese surface, and in the special (unigonal) case the image is Σ_4^0. In both cases the branch curve is cut by a cubic fourfold. Thus, from our point of view the two cases behave similarly.

4.2. *Modular interpretation of the blow-up $\widetilde{\mathcal{M}} \to \overline{\mathcal{M}}$.* In Sections 4.1.1 and 4.1.2 we have noted the natural occurrence of the moduli space of degree two K3 surfaces in the blow-up of $\overline{\mathcal{M}}$ along χ. The statement there is purely representation theoretic. We now interpret geometrically this observation and relate it to the Hodge theory of the degenerations to Type IV fourfolds. Most of the discussion here is about the case ω. Since the case χ is quite similar (see §4.1.2, especially Remark 4.7), there is no difficulty in adapting the arguments to the case χ. In particular we note that the semi-stable reduction in this case is essentially contained in Allcock-Carlson-Toledo [2, §5].

To start, we recall the computation of the semi-stable reduction for a general pencil $\mathcal{X} \to \Delta$ of cubic fourfolds degenerating to the secant variety X_0 of Veronese surface V (cf. Hassett [21, §4.4]). The total space $\mathcal{X}$ is singular along the degree 12 curve $C = X_t \cap V$ (a plane sextic). After a base change of order 2, and the blow-up of the family $\mathcal{X}$ along the Veronese surface V, we obtain a semi-stable model $\mathcal{X}' \to \Delta$. The new central fiber X_0' is the union of two smooth components $\overline{X}_0$ and E, intersecting transversely in a $\mathbb{P}^1$-bundle E_0 over $\mathbb{P}^2$. The fourfold $\overline{X}_0 \cong \mathrm{Hilb}^2(\mathbb{P}^2)$ is the natural resolution of $X_0 \cong \mathrm{Sym}^2(\mathbb{P}^2)$. The other component E of the central fiber is a quadric bundle over $V \cong \mathbb{P}^2$ with discriminant the curve C. As it is standard, to E one associates the degree two K3 surface S obtained as the double cover of $\mathbb{P}^2$ branched along C. (Intrinsically $S \xrightarrow{2:1} \mathbb{P}^2$ parametrizes the rulings of the fibers of $E \to \mathbb{P}^2$.) The Hodge structure of E (and that of X_0') is essentially the Hodge structure of S. Specifically, in the generic case, the Hodge structure on the transcendental sublattice of $\mathrm{H}^4(E, \mathbb{Z})$ is a Tate twist of the Hodge structure on the primitive cohomology of S. It then follows that $\mathrm{H}_0^2(S)(-1)$ embeds in the limit Hodge structure $\mathrm{H}_{\mathrm{lim}}^4$. This implies that $\mathrm{H}_{\mathrm{lim}}^4$ is pure and the corresponding period point belongs to the divisor $\mathscr{H}_\infty/\Gamma$. By

interpreting the pencil $\mathcal{X} \to \Delta$ as a generic arc meeting the exceptional divisor (corresponding to ω) of the blow-up $\widehat{\mathcal{M}} \to \overline{\mathcal{M}}$, we conclude that the monodromy around this exceptional divisor is of order 2 and corresponds to a reflection in a long root (see Remark 2.18).

The above computation still remains valid for any arc transversally intersecting the exceptional divisor of the blow-up of ω. (The computations should be done in étale slice modeling the local structure of $\overline{\mathcal{M}}$ near ω.) Thus, we can interpret this exceptional divisor as parametrizing fourfolds of type $X_0' = \overline{X}_0 \cup_{E_0} E$, where E is a (possibly singular) quadric bundle over $\mathbb{P}^2$ (i.e., $E \to \mathbb{P}^2$ is an algebraic fiber space with generic fiber a smooth quadric in $\mathbb{P}^3$). This identification is rather tautological. Namely, by Section 4.1.1, the points of the exceptional divisor of $\widetilde{\mathcal{M}} \to \overline{\mathcal{M}}$ naturally correspond to isomorphism classes of plane sextics. The following lemma shows that to a plane sextic we can associate a quadric bundle E in a compatible way with our identification.

LEMMA 4.8. *The quadric bundles E occurring in the semistable reduction of degenerations to the secant to Veronese are given by sections $q \in H$, where*

$$H := H^0 \left(\mathbb{P}^2, \operatorname{Sym}^2(\mathcal{N}^*_{V/\mathbb{P}^5} \oplus \mathcal{O}_{\mathbb{P}^2}) \otimes \mathcal{O}_{\mathbb{P}^2}(6) \right).$$

Furthermore, with respect to the natural $G = \operatorname{SL}(3)$ action on H, we have the decomposition:
(4.9)
$$H \cong H^0 \left(\mathbb{P}^2, \operatorname{Sym}^2(\mathcal{N}^*_{V/\mathbb{P}^5}) \otimes \mathcal{O}_{\mathbb{P}^2}(6) \right) \oplus H^0 \left(\mathbb{P}^2, \mathcal{N}_{V/\mathbb{P}^5} \right) \oplus H^0 \left(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(6) \right).$$

In particular, q can be taken of type $q = q_0 + f$ with

$$q_0 \in H^0 \left(\mathbb{P}^2, \operatorname{Sym}^2(\mathcal{N}^*_{V/\mathbb{P}^5}) \otimes \mathcal{O}_{\mathbb{P}^2}(6) \right)$$

a tautological G-invariant section and $f \in H^0 \left(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(6) \right)$ the equation of the discriminant curve C.

Proof. As the semi-stable reduction is obtained by blowing up the Veronese surface, we have $E \subset \mathbb{P}(\mathcal{N}^*_{V/\mathbb{P}^5} \oplus \mathcal{O}_V)$ (see also [21, Lemma 4.4.3]). The quadric bundle E corresponds to a section $q \in H^0 \left(\mathbb{P}^2, \operatorname{Sym}^2(\mathcal{N}^*_{V/\mathbb{P}^5} \oplus \mathcal{O}_V) \otimes \mathcal{O}_{\mathbb{P}^2}(\alpha) \right)$ for some $\alpha \in \mathbb{Z}$ (see [4, Prop. 1.2]). Then, the discriminant curve is given by the induced section

$$\Delta \in H^0 \left(\mathbb{P}^2, \operatorname{Sym}^2 \left(\det \left(\mathcal{N}^*_{V/\mathbb{P}^5} \oplus \mathcal{O}_V \right) \right) \otimes \mathcal{O}_{\mathbb{P}^2}(4\alpha) \right).$$

The following basic facts about the normal (and conormal) bundle of the Veronese surface V in $\mathbb{P}^5$ are well-known (e.g. [15, Lemma 2.7]):

i) $\mathcal{N}^*_{V/\mathbb{P}^5} \cong \operatorname{Sym}^2(\Omega^1_{\mathbb{P}^2})$;

ii) $\mathcal{N}^*_{V/\mathbb{P}^5}(2) \cong \mathcal{N}_{V/\mathbb{P}^5}(-1)$ (the twist is taken with respect to $\mathcal{O}_{\mathbb{P}^5}(1)$);

iii) The isomorphism of ii) induces a G-invariant section q_0 via the natural inclusion:

$$q_0 \in \mathrm{H}^0\left(\mathrm{Sym}^2(\mathcal{N}_{V/\mathbb{P}^5}(-1)) \otimes \mathcal{O}_V(-1)\right)$$
$$\hookrightarrow \mathrm{Hom}\left(\mathcal{N}^*_{V/\mathbb{P}^5}(1), \mathcal{N}_{V/\mathbb{P}^5}(-1) \otimes \mathcal{O}_V(-1)\right).$$

Using ii), we rewrite iii) as $q_0 \in \mathrm{H}^0(\mathrm{Sym}^2(\mathcal{N}^*_{V/\mathbb{P}^5}) \otimes \mathcal{O}_{\mathbb{P}^2}(6))$. The lemma now follows from [21, Lemma 4.4.3] by noting the surjection

$$\mathrm{H}^0(\mathbb{P}^5, \mathcal{O}_{\mathbb{P}^5}(3)) \twoheadrightarrow \mathrm{H}^0(V, \mathcal{O}_V(3)) \cong \mathrm{H}^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(6)). \qquad \square$$

The previous lemma associates a quadric bundle over $\mathbb{P}^2$ to a plane sextic, in a canonical way. Since we also are considering singular sextics, we need to understand the possible singularities of the resulting quadric bundle.

LEMMA 4.10. *Let E be a quadric bundle as in Lemma* 4.8. *Then the singularities of E are in one-to-one correspondence*, *including the analytic type*, *with the singularities of the discriminant curve C.*

Proof. We choose $U \subset \mathbb{P}^2$ an open affine over which $\mathcal{N}_{V/\mathbb{P}^5}$ is trivialized. Locally over U, $E_0 \subset \mathbb{P}^2 \times U \to U$ is given by $x_1^2 + x_0 x_2$ and $E \subset \mathbb{P}^3 \times U \to U$ by $f(x, y)t^2 + x_1^2 + x_0 x_2$, where (x, y) are affine coordinates on U, f is the equation of the discriminant curve, and $(x_0 : x_1 : x_2 : t)$ are appropriate homogeneous coordinates on $\mathbb{P}^3$. Since E_0 is the smooth hyperplane section ($t = 0$) of E, we can assume $t = 1$. Clearly, the singularities of E correspond to the singularities of C. $\qquad \square$

In the case of arcs $\tau : \Delta \to \widehat{\mathcal{M}}$ meeting nontransversally the exceptional divisor of the blow-up of the point $\omega \in \overline{\mathcal{M}}$, one might need several blow-ups to obtain the quadric bundle E. More precisely, starting with a 1-parameter degeneration $\mathscr{X} \to \Delta$ to ω, repeating several times the computation from the transversal case, we arrive at a partial semi-stable reduction $\mathscr{X}' \to \Delta$ such that one of the components of the central fiber is a quadric bundle E attached to the other components along a smooth divisor E_0, and such that the condition of semi-stable family for $\mathscr{X}'$ holds away from E. As before, the quadric bundle E is determined by the point where the arc meets the exceptional divisor. By standard arguments (e.g. [43, Prop. 2.1]), we can further assume that discriminant curve C associated to E is a semi-stable plane sextic with closed orbit (see Shah [43, Th. 2.4]). Then, depending on the singularities of E (or equivalently C) we call the central fiber X'_0 of $\mathscr{X}'$ a fourfold of Type I′–IV′, where the types are defined by:

(I′) The cases when E has at worst simple singularities;
(II′) The cases when E has at worst $\widetilde{E_r}$, A_∞ or D_∞ singularities (corresponding to the Type II in Shah [43, Th. 2.4]);
(III′) The cases when E has at worst degenerate cusp singularities (Type III in [43]);

(IV′) The cases corresponding to the intersection of the exceptional divisor with the strict transform of the curve χ (i.e., the discriminant sextic is the triple conic).

We conclude that any degenerations of cubic fourfolds to the secant variety of the Veronese surface can be filled in with a fourfold of type I′–IV′ (as defined above). Since the interesting cohomology of X_0' comes from E, which in turn comes from the cohomology of a degeneration of K3 surfaces with insignificant singularities, we can control the monodromy for degenerations with central fiber of Type I′–III′ as in Section 3 (i.e., the index of nilpotency is the numerical label of the type). By considering the blow-up of $\widetilde{M} \to \widehat{M}$ of the strict transform of χ in $\overline{M}$, we can eliminate the remaining cases of Type IV (corresponding to $\chi \setminus \omega$) and the Type IV′, introducing instead fourfolds analogous to the Types I′–III′ above (this is essentially Shah [43, Th. 4.3]). In conclusion, using the blow-up $\widetilde{M} \to \overline{M}$, any 1-parameter family of smooth cubic fourfolds degenerating to a Type IV fourfold can be filled in with a fourfold of Type I′–III′, for which we can control the monodromy. In particular, for the Type II′ and III′ cases the limit period belongs to the boundary of $(\mathscr{D}/\Gamma)^*$. On the other hand, as already noted above in the generic case, the limit period for a Type I′ fourfold belongs to $\mathscr{H}_\infty/\Gamma$.

THEOREM 4.11. *Let $f : \mathscr{X} \to \Delta$ be a 1-parameter degeneration with finite monodromy of cubic fourfolds such that the central fiber is of Type* IV. *Then, the period point corresponding to the limit Hodge structure* $H^4_{\lim}$ *(which is automatically pure) belongs to* $\mathscr{H}_\infty/\Gamma$.

Proof. For simplicity, we restrict the discussion to the case ω; the computation in the case χ is similar (in fact, ω is the degenerate case of χ; one obtains an additional class in the case χ). Assume that the central fiber X_0 of the degeneration is the secant variety of the Veronese surface $S \hookrightarrow \mathbb{P}^5$. We have $X_0 \cong \mathrm{Sym}^2(\mathbb{P}^2)$. Since X_0 is the quotient of $\mathbb{P}^2 \times \mathbb{P}^2$ by the finite group μ_2, the cohomology (over $\mathbb{Q}$) of X_0 is the μ_2-invariant cohomology of $\mathbb{P}^2 \times \mathbb{P}^2$. It follows that $H^4(X_0)$ is two-dimensional carrying a pure Hodge structure of type $(2, 2)$. There are two natural generators h and e for $H^4(X_0)$: the square of the hyperplane section and the class corresponding to the diagonal in $\mathbb{P}^2 \times \mathbb{P}^2$. Via the natural double cover $\mathbb{P}^2 \times \mathbb{P}^2 \to X_0$, one computes the intersection matrix of h and e to be $\begin{pmatrix} 3 & 2 \\ 2 & 2 \end{pmatrix}$ (since X_0 is the quotient of a smooth variety by a finite group, the intersection makes sense). By the discussion of Section 2.2 (especially Lemma 2.17), the limit period point belongs to $\mathscr{H}_\infty/\Gamma$ provided that the algebraic classes h and e are preserved in the limit $H^4_{\lim}$.

Let $\mathscr{X} \to \Delta$ be a degeneration of cubic fourfolds to X_0, the secant of the Veronese surface S. Assume that $\mathscr{X}' \to \Delta$ is a semi-stable reduction of the degeneration. We can write the new central fiber as $X_0' = \overline{X}_0 \cup E$, where $\overline{X}_0$ is the component dominating X_0 (thus a resolution of X_0) and E the union of the other

components in the central fiber X_0'. Let $E_0 = \overline{X}_0 \cap E$. We get a diagram:

(4.12)
$$
\begin{array}{ccc}
E \longleftarrow E_0 \longhookrightarrow \overline{X}_0 \\
\searrow \quad \downarrow \qquad \downarrow \\
S \longrightarrow X_0.
\end{array}
$$

The limit Hodge structure H^4_{lim} is computed by the Clemens-Schmid exact sequence for the semi-stable family $\mathscr{X}' \to \Delta$. Specifically, under the assumption of finite monodromy, we have:

(4.13) $\qquad 0 \to H^2_{\mathrm{lim}} \to \mathrm{Gr}^W_{-6} H_6(X_0') \xrightarrow{\alpha} \mathrm{Gr}^W_4 H^4(X_0') \to H^4_{\mathrm{lim}} \to 0$

([33, p. 109]). The relevant part of the cohomology of X_0' is computed by the exact sequence

(4.14) $\qquad 0 \to \mathrm{Gr}^W_4 H^4(X_0') \to H^4(\overline{X}_0) \oplus \mathrm{Gr}^W_4 H^4(E) \to \mathrm{Gr}^W_4 H^4(E_0).$

Combining (4.14) with the exact sequence (3.9) for the resolution $\overline{X}_0 \to X_0$ (see diagram (4.12)), we get a natural injective map $H^4(X_0) \hookrightarrow \mathrm{Gr}^W_4 H^4(X_0')$. Since the morphism $\alpha : H_6(X_0') \to H^4(X_0')$ is obtained as the composition of the Poincaré duality map $H_6(X_0') \cong H_6(\mathscr{X}') \to H^4(\mathscr{X}', \partial\mathscr{X}')$ with the restriction map $H^4(\mathscr{X}', \partial\mathscr{X}') \to H^4(\mathscr{X}') \cong H^4(X_0')$ ([33, p. 108]), no class $s_0 \in H^4(X_0')$ that can be extended to a nonzero class $s_t \in H^4(X_t')$ belongs to the image of α. The classes coming from the embedding $H^4(X_0) \hookrightarrow H^4(X_0')$ extend to the nearby X_t' fibers for the simple reason that $H^4(X_0)$ is generated by the polarization class and a class supported on the smooth locus of X_0. Thus, the image of $H^4(X_0)$ in H^4_{lim} is two-dimensional and the claim follows. $\qquad\qquad\square$

5. **Proof of the main results**

5.1. *Proof of Theorem* 1.1. The period map $\mathscr{P}_0 : \mathcal{M}_0 \to \mathscr{D}/\Gamma$ is a birational morphism of quasi-projective varieties with the image contained in $(\mathscr{D} \setminus (\mathscr{H}_\Delta \cup \mathscr{H}_\infty))/\Gamma$. By Proposition 3.2, $\mathscr{P}_0$ extends to a regular morphism $\mathscr{P} : \mathcal{M} \to \mathscr{D}/\Gamma$ over the simple singularities locus $\mathcal{M} \subset \widetilde{\mathcal{M}}$. Additionally, the image of the discriminant is included in $\mathscr{H}_\Delta$. Now, Theorem 1.1 follows provided the properness of the extended period map $\mathscr{P} : \mathcal{M} \to (\mathscr{D} \setminus \mathscr{H}_\infty)/\Gamma$.

By the valuative criterion, it suffices to consider 1-parameter families $\mathscr{X}^* \to \Delta^*$ of smooth cubic fourfolds over the punctured disk Δ^* with the limit period point in $(\mathscr{D} \setminus \mathscr{H}_\infty)/\Gamma$. The properness means that $\mathscr{X}^* \to \Delta^*$ can be filled in (after a possible finite base change) to a family $\mathscr{X} \to \Delta$ of cubic fourfolds with at worst simple singularities. By using the GIT quotient $\overline{\mathcal{M}}$, we can extend $\mathscr{X}^* \to \Delta^*$ to a family $\mathscr{X} \to \Delta$ of cubic fourfolds such that the central fiber X_0 is a semi-stable cubic fourfold with closed orbit (see [43, Prop. 2.1]). If X_0 is of Type I (see §2.1), we are done. Thus, assume X_0 is of Type II–IV. Since the limit period point for

$\mathscr{X}^* \to \Delta^*$ belongs to the period domain, it is known that the monodromy of the family $\mathscr{X}^*$ is finite (see [8, Th. 13.4.5]). By Theorem 3.1, we conclude that X_0 cannot be of Type II or III. Similarly, the Type IV case is excluded by Theorem 4.11 (i.e., the limit period belongs to $\mathscr{H}_\infty/\Gamma$ in this case). □

Proof of Theorem 1.2. The statement follows directly from Theorem 1.1 and the general results of Looijenga [29, Th. 7.6]. In fact, the situation is formally the same as that of low degree K3 surfaces. This case was analyzed in detail by Looijenga [29, §8] (especially [29, Th. 8.6]). A sketch of the argument is given below.

Let $G = \mathrm{SL}(6)$, and $P = \mathbb{P}\mathrm{Sym}^3 W$ be the linear system of cubics in $\mathbb{P}^5$. On P we consider the natural polarization $\eta = \mathcal{O}_P(n)$ (for some n to be chosen later), and define $U \subset P$ to be the G-invariant open subset parametrizing cubic fourfolds with at worst simple isolated singularities. The GIT results of [27] imply in particular that:

i) $P \setminus U$ has high codimension (at least 2) in P;

ii) The points of U are stable with respect to the action of G on P ([27, Th. 1.1]).

From the general theory, we note that the GIT quotient $\overline{\mathcal{M}}$ is isomorphic to the Proj of the ring of G-invariant sections (i.e., $\mathrm{Proj}(\oplus_{k\geq 0} H^0(P, \eta^{\otimes k})^G)$) and that some power of η descends to an ample line bundle on $\overline{\mathcal{M}}$.

On the $\mathscr{D}/\Gamma$ side, there exists a natural automorphic line bundle $\mathbb{L}$ on the domain $\mathscr{D}$, which descends to an ample line bundle $\mathscr{L}$ on $\mathscr{D}/\Gamma$ (by the Baily-Borel theory). Let $\mathscr{D}^0 = \mathscr{D} \setminus \mathscr{H}_\infty$ and $X^0 = \mathscr{D}^0/\Gamma$. The content of Theorem 1.1 is that the period map induces an isomorphism between the quasi-projective varieties $\mathcal{M} = U/G$ and $X^0 = \mathscr{D}^0/\Gamma$. A basic observation of Looijenga is that, for hypersurfaces, one has a stronger statement: The isomorphism given by the period map is an isomorphism of polarized varieties, i.e., the period map gives an identification of $(U, \eta_{|U})/G$ with $(X^0, \mathscr{L}_{|X^0})$. Namely, given the identification between $\mathcal{M}$ and X^0, the identification of line bundles follows from Griffiths' residue theory: The period point corresponding to a cubic fourfold is determined by the 1-dimensional subspace $H^{3,1}$ of the middle cohomology. This in turn is generated by the residue of a rational form on $\mathbb{P}^5$, which depends only on the equation of the cubic. (In particular, since this rational form depends quadratically on the equation of the cubic, the right choice of n in the definition of η is 2.)

We conclude that the assumptions (in particular (i) and (ii) above) of [29, Th. 7.6] are satisfied. Thus, we obtain an isomorphism of graded algebras:

$$\oplus_{k\geq 0} H^0(P, \eta^{\otimes k})^G \cong \oplus_{k\in\mathbb{Z}} H^0(\mathscr{D}^0, \mathcal{O}(\mathbb{L})^{\otimes k})^\Gamma.$$

Taking Proj of both sides, one obtains an isomorphism between the GIT compactification $\overline{\mathcal{M}}$ and the Looijenga compactification associated to $\mathscr{H}_\infty$. (For a geometric description of this compactification see the next section, especially diagram (6.1).) Thus, Theorem 1.2 holds. □

6. **Arithmetic properties of the arrangement of hyperplanes $\mathcal{H}_\infty$**

We close by noting several arithmetic properties of the hyperplane arrangement $\mathcal{H}_\infty$, which give (*a posteriori*) some information about the GIT compactification $\overline{\mathcal{M}}$. Namely, we recall that as a consequence of Theorem 1.2 and of the general results of Looijenga [29] the birational map between the GIT compactification $\overline{\mathcal{M}}$ and the Baily-Borel compactification $(\mathcal{D}/\Gamma)^*$ fits into a diagram:

$$(6.1) \qquad \begin{array}{ccc} \widetilde{\mathcal{M}} & \longrightarrow & \widehat{\mathcal{D}/\Gamma} \\ \downarrow & & \downarrow \\ \overline{\mathcal{M}} & \dashrightarrow & (\mathcal{D}/\Gamma)^*. \end{array}$$

The space $\widehat{\mathcal{D}/\Gamma}$ is a small modification of $(\mathcal{D}/\Gamma)^*$ such that the Weil divisor defined by $\mathcal{H}_\infty$ becomes $\mathbb{Q}$-Cartier. Then $\widetilde{\mathcal{M}}$ is obtained from $\widehat{\mathcal{D}/\Gamma}$ by blowing up the strict transform of the arrangement $\mathcal{H}_\infty$ in the usual way (starting with the highest codimension self-intersection stratum). Finally, the morphism $\widetilde{\mathcal{M}} \to \overline{\mathcal{M}}$ contracts the hyperplane arrangement in the opposite direction. The net effect of this birational modification is to flip a codimension k intersection of hyperplanes in $(\mathcal{D}/\Gamma)^*$ to a $(k-1)$-dimensional stratum in $\overline{\mathcal{M}}$. Since all the steps involved in passing from $(\mathcal{D}/\Gamma)^*$ to $\overline{\mathcal{M}}$ are explicit and of relatively simple arithmetic nature, we can recover some of the GIT results obtained in [27]. See the discussion from [31, Rem. 3.2] for the similar situation of cubic threefolds. For instance, the dimensions of the boundary strata from Table 1 can be obtained by understanding how the arrangement of hyperplanes $\mathcal{H}_\infty$ meets the boundary of $(\mathcal{D}/\Gamma)^*$.

Remark 6.2. An unpublished argument of Looijenga (based on the universal property of the blow-up of the arrangement $\mathcal{H}_\infty$; see [29, Prop. 7.2]) shows that an immediate consequence of Theorem 1.2 is that the space $\widetilde{\mathcal{M}}$ of diagram (6.1) is isomorphic to the Kirwan blow-up considered in Section 4. This explains our choice of notation. However, in what follows, we do not need to assume this.

6.1. *The Baily-Borel Compactification of $\mathcal{D}/\Gamma$*. The Baily-Borel theory compactifies the locally symmetric space $\mathcal{D}/\Gamma$ to the normal projective variety $(\mathcal{D}/\Gamma)^*$. The compactification is done by adding points, *Type* III *boundary components*, and curves, *Type* II *boundary components*. The Type II and III boundary components correspond 1-to-1 to the classes (modulo Γ) of the isotropic sublattices of Λ_0 of rank 2 and 1 respectively. Furthermore, the incidences between the boundary components correspond to the inclusions of the isotropic lattices. In our situation, we obtain:

PROPOSITION 6.3. *The period domain $\mathcal{D}/\Gamma$ for cubic fourfolds is compactified by adding six type* II *boundary components meeting in a single point, the unique type* III *component. The type* II *boundary components are distinguished*

by the root sublattice R contained in $E^\perp/E$, where E is the rank-2 isotropic sublattices associated to the component. The six possibilities for R are $E_8^{\oplus 2} \oplus A_2$, $D_{16} \oplus A_2$, $E_7 \oplus D_{10}$, A_{17}, $E_6^{\oplus 3}$ and $A_{11} \oplus D_7$.

Proof. We first note that up to the action of Γ there exists only one class of isotropic rank-1 sublattices in Λ_0. Since Λ_0 contains two hyperbolic planes, the claim follows from general lattice theory (e.g. [41, Lemma 4.1.2]). Thus, there exists a unique type III component.

For the classification of rank-2 isotropic sublattices in Λ_0 we apply the approach of Scattone [41]. Namely, let E be an isotropic rank-2 lattice. A standard invariant of E is the isometry class of the lattice $N := E^\perp/E$. The lattice N is positive definite of rank 18 and in the same genus as $E_8^{\oplus 2} \oplus A_2$. By gluing N and E_6 along the discriminant groups, we obtain an even unimodular lattice L of rank 24. Conversely, any lattice N in the genus of $E_8^{\oplus 2} \oplus A_2$ can be realized as the orthogonal complement of an embedding of E_6 into an even unimodular lattice L of rank 24 (see [41, Prop. 6.1.1]). The possibilities for L were classified by Niemeier. There are 24 isometry classes of even unimodular rank-24 lattices, which are distinguished by the sublattice spanned by the roots. It is clear that E_6 can be embedded only into those unimodular lattices L for which the sublattice spanned by the roots contains an E_r summand (for $r = 6, 7, 8$). We get six possibilities for L corresponding to the root lattices: $E_6^{\oplus 4}$, $E_6 \oplus A_{11} \oplus D_7$, $E_7^{\oplus 2} \oplus D_{10}$, $E_8^{\oplus 3}$ and $E_8 \oplus D_{16}$ (see [9, §18.4]). Since E_6 embeds into an E_r summand uniquely up to isometries, we obtain only six possibilities for N. We label (unambiguously) the six cases by the root sublattice contained in N, i.e., $E_8^{\oplus 2} \oplus A_2$, $D_{16} \oplus A_2$, $E_7 \oplus D_{10}$, A_{17}, $E_6^{\oplus 3}$ and $A_{11} \oplus D_7$ respectively.

To conclude the proof, we have to show that the classification of the possibilities for N is equivalent to the classification of E. One direction is easy. Given N in the genus of $E_8^2 \oplus A_2$, there exists an isotropic lattice E such that $N \cong E^\perp/E$. Namely, since Λ_0 is unique in its genus (see [36]), we have $\Lambda_0 \cong N \oplus U^2$ and the claim follows. Conversely, E is uniquely determined by N as in Scattone [41, Th. 5.0.2 (1)]. Finally, the classification of the isotropic lattices E given above is (*a priori*) only up to the orthogonal group $O(\Lambda_0)$. Since $O^+(\Lambda_0)/O^*(\Lambda_0) \cong \{\pm \mathrm{id}\}$, we obtain in fact the same classification modulo $\Gamma \cong O^*(\Lambda_0)$. $\qquad\square$

The fact that the number of Type II boundary components is equal to the number of GIT boundary components is explained by the fact that a generic point x_0 in a GIT boundary component α–ϕ corresponds to a Type II fourfold X_0. By our results (especially Theorem 1.2) and the general theory, it follows that the period map extends at x_0 mapping it to a point y_0 on a Type II boundary component of $(\mathscr{D}/\Gamma)^*$. This gives a matching between the GIT boundary components and the Type II Baily-Borel boundary components. To compute the actual matching, we note that the point y_0 encodes the information contained in the graded pieces of the limit mixed Hodge structure H_{lim}^4 of a degeneration to X_0 (see [19, Ch. V]). This

$$\varepsilon \to D_{16} \oplus A_2 \qquad \beta \to E_8^{\oplus 2} \oplus A_2 \qquad \alpha \to A_{11} \oplus D_7$$

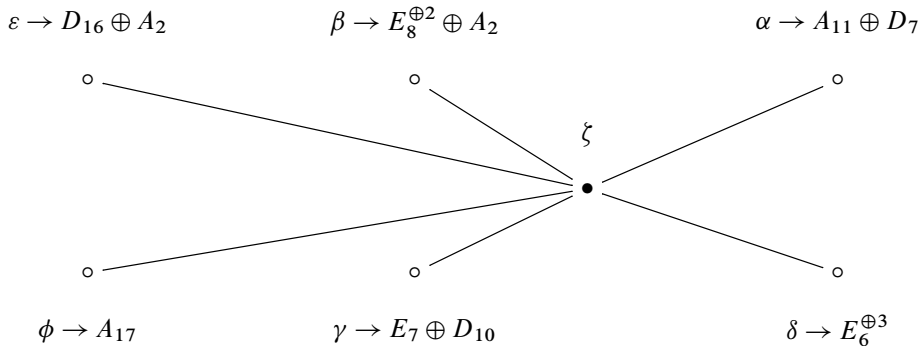$$\phi \to A_{17} \qquad \gamma \to E_7 \oplus D_{10} \qquad \delta \to E_6^{\oplus 3}$$

Figure 2. Matching of GIT and Baily-Borel boundary components

information consists of a discrete part that determines the boundary component and of a continuous part (essentially a j-invariant) coming from the Hodge structure on $\mathrm{Gr}_3^W(\mathrm{H}_{\mathrm{lim}}^4)$. In our situation, both pieces of information can be determined in terms of X_0 (see §3). We obtain a correspondence between the boundary components in the two compactifications as given in Figure 2 (N.B. $\circ$ and $\bullet$ correspond to Type II and III components respectively). We omit the details of the computation. Here, we only give the following heuristic argument. A nonsimple singularity for X_0 produces a lattice of type A-D-E by the following rules:

(1) An $\widetilde{E}_r$ singularity for X_0 gives an E_r lattice;

(2) An elliptic curve E of degree d in the singular locus of X_0 gives an A_{3d-1} lattice (N.B. $3d$ is the number of special points on E, i.e., the points cut on E by a pencil of cubics degenerating to X_0);

(3) A rational curve C of degree d in the singular locus of X_0 gives a D_{3d+4} lattice (N.B. $3d + 4$ special points on C).

The direct sum of the lattices associated to the singularities of X_0 determines the label of the boundary component associated to X_0 (see the discussion for K3 surfaces from [16, Rem. 5.6]).

6.2. *Intersections of hyperplanes from $\mathscr{H}_\infty$.* By the work of Looijenga [29], the structure of the birational map $\overline{\mathcal{M}} \dashrightarrow (\mathscr{D}/\Gamma)^*$ is very explicit provided that we understand how the hyperplanes from the arrangement $\mathscr{H}_\infty$ intersect inside the period domain $\mathscr{D}$. This is quite easily achieved by means of the following observation: *A hyperplane H from $\mathscr{H}_\infty$ is the restriction to $\mathscr{D}$ of a hyperplane $H' \subset \mathscr{D}'$ orthogonal to a root*, where $\mathscr{D}'$ is a certain Type IV domain containing as a subdomain the period domain $\mathscr{D}$. Specifically, we introduce:

Notation 6.4. Let $\Lambda' := \mathrm{II}_{26,2}$ be the unique even unimodular lattice of signature $(26, 2)$. Up to isometries there exists a unique primitive embedding $\Lambda_0 \hookrightarrow \Lambda'$. We fix such an embedding and denote by $M \cong E_6$ the orthogonal complement

of Λ_0 in Λ'. Let $\mathscr{D}'$ be the type IV domains associated to Λ' and $\mathscr{D} \hookrightarrow \mathscr{D}'$ the inclusion induced by the embedding $\Lambda_0 \hookrightarrow \Lambda'$.

With this notation, we have:

LEMMA 6.5. *Let H be a hyperplane from the arrangement $\mathscr{H}_\infty$. Then there exists a root $\delta' \in \Lambda'$ (i.e., $\delta'^2 = 2$) such that*

i) *The sublattice in Λ' spanned by M and δ' is isomorphic to E_7;*

ii) *The restriction to $\mathscr{D}$ of the hyperplane $H'_{\delta'} \subset \mathscr{D}'$ orthogonal to δ' coincides with H.*

Conversely, for any root $\delta' \in \Lambda'$ satisfying condition i)*, the restriction to $\mathscr{D}$ of the associated orthogonal hyperplane $H'_{\delta'}$ is an element of $\mathscr{H}_\infty$.*

Proof. As noted in Section 2.2, given H in $\mathscr{H}_\infty$, there exists a long root $\delta \in \Lambda_0$ orthogonal to H. The condition of long root implies in particular that $\frac{\delta}{3}$ is a generator of the discriminant group A_{Λ_0}. Since M and Λ_0 are glued along the discriminant A_{Λ_0} to give the lattice Λ', it follows easily that the saturation S of the sublattice of Λ' spanned by $M \cong E_6$ and δ is isomorphic to E_7. Clearly, the orthogonal hyperplane $H'_{\delta'}$ to a root $\delta' \in S \setminus M$ restricts on $\mathscr{D}$ to H. Since there is a unique embedding of E_6 into E_7 and the orthogonal complement of E_6 in this embedding is the rank-1 lattice spanned by an element of norm 6, the converse also follows. $\qquad\square$

A similar description also holds for the hyperplanes from $\mathscr{H}_\Delta$. The only difference is to replace E_7 from condition i) by $E_6 \oplus A_1$. In particular, from the work of Borcherds, it follows that $(\mathscr{H}_\infty \cup \mathscr{H}_\Delta)/\Gamma$ is the zero locus of an automorphic form. Thus, $(\mathscr{D} \setminus (\mathscr{H}_\infty \cup \mathscr{H}_\Delta))/\Gamma$ is an affine variety. This also follows from GIT via the isomorphism $(\mathscr{D} \setminus (\mathscr{H}_\infty \cup \mathscr{H}_\Delta))/\Gamma \cong \mathcal{M}_0$.

PROPOSITION 6.6. *There exists an automorphic form Φ on $\mathscr{D}$ such that*
i) *The weight of Φ is 48;*
ii) *The vanishing locus of Φ is $\mathscr{H}_\infty \cup \mathscr{H}_\Delta$;*
iii) *ϕ vanishes of order 1 along $\mathscr{H}_\Delta$ and of order 27 along $\mathscr{H}_\infty$.*
We call Φ the discriminant automorphic form.

Proof. Formally, the situation is the same as that for degree-two K3 surface (see Borcherds et al. [7, Th. 1.2, Ex. 2.1]). Namely, it is well-known that on $\mathscr{D}'$ there exists an automorphic form Φ_{12} vanishing exactly along the hyperplanes $H'_{\delta'}$, where δ' is a root of Λ'. By restricting Φ_{12} to $\mathscr{D}$ and dividing by the product of the linear forms which vanish along the hyperplanes $H'_{\delta'}$ containing $\mathscr{D}$, we obtain an automorphic form Φ on $\mathscr{D}$. The weight of Φ is the weight of Φ_{12} plus the number of hyperplanes $H'_{\delta'}$ containing $\mathscr{D}$ (i.e., half the number of roots of E_6). Clearly, the vanishing locus of Φ is the union of hyperplanes H which are the restrictions to $\mathscr{D}$ of the hyperplanes of type $H'_{\delta'}$, for δ' a root of Λ'. The condition that $H'_{\delta'} \cap \mathscr{D} \neq \varnothing$ implies that the sublattice $\langle M, \delta' \rangle_{\Lambda'}$ (the span of M and δ') is positive definite.

Since $M \cong E_6$, we obtain that $\langle M, \delta' \rangle_{\Lambda'}$ is isomorphic to either $E_6 \oplus A_1$ or E_7. The two cases correspond to $\mathcal{H}_\Delta$ and $\mathcal{H}_\infty$ respectively. The claim follows. $\qquad\square$

Remark 6.7. The advantage of working with Λ' instead of Λ_0 or Λ is that the similarities and differences between the arrangements $\mathcal{H}_\Delta$ and $\mathcal{H}_\infty$ are more apparent. Also, since we are working with an even unimodular lattice, the results of Nikulin [36] are easier to apply.

An immediate consequence of Lemma 6.5 is that a nonempty intersection in $\mathcal{D}$ of hyperplanes from $\mathcal{H}_\infty$ has codimension at most 2 and that all codimension-2 intersections are conjugated by Γ.

LEMMA 6.8. *Assume that $H_1, \ldots, H_k$ (with $k \geq 2$) are distinct hyperplanes from $\mathcal{H}_\infty$ such that $\mathcal{D} \cap H_1 \cap \cdots \cap H_k \neq \varnothing$. Then the locus $\mathcal{D} \cap H_1 \cap \cdots \cap H_k$ has codimension-2 in $\mathcal{D}$. Furthermore, all codimension-2 intersection loci of hyperplanes from $\mathcal{H}_\infty$ are conjugate by Γ.*

Proof. By Lemma 6.5, each hyperplane H_i corresponds to a root $\delta_i' \in \Lambda'$. It follows that the sublattice of Λ' spanned by M and the roots δ_i' is a root lattice R of rank at least 8 (the hyperplanes H_i are distinct). The condition that $\mathcal{D} \cap H_1 \cap \cdots \cap H_k \neq \varnothing$ is equivalent to R being positive definite. Since $\langle M, \delta_i' \rangle_{\Lambda'} \cong E_7$ for each i, it follows that $R = \langle M, \delta_1', \ldots, \delta_k' \rangle_{\Lambda'}$ is an irreducible positive definite root lattice of rank at least 8. Since $M \cong E_6$ embeds in R, we conclude that $R \cong E_8$. The codimension claim follows. Since the embeddings $E_6 \hookrightarrow E_8$, $E_8 \hookrightarrow \Lambda'$, and $E_6 \hookrightarrow \Lambda'$ are unique up to isometries, it is not hard to see that the codimension-2 intersections form a single Γ-orbit. $\qquad\square$

Geometrically, the previous lemma implies that the period map for cubic four-folds is undefined precisely along the curve χ. Moreover, via the diagram (6.1), the curve χ is a flip of the locus in $\mathcal{D}/\Gamma$ corresponding to the codimension-2 intersections from $\mathcal{H}_\infty$ and the point ω is a contraction of the divisor $\mathcal{H}_\infty/\Gamma$.

To understand the effect of the small birational modification $\widehat{\mathcal{M}} \to (\mathcal{D}/\Gamma)^*$, we have to understand the structure of the arrangement $\mathcal{H}_\infty$ near the boundary components. Since a Baily-Borel boundary component corresponds to an isotropic sublattice E of Λ_0, this is essentially equivalent to understanding how the hyperplanes from $\mathcal{H}_\infty$, containing E, intersect. Specifically, one is interested in the following lemma:

LEMMA 6.9. *Let E be a rank-2 isotropic primitive sublattice of Λ_0 and R the root sublattice in $E^\perp/E$ (see Proposition 6.3). Denote by K the restriction of the intersection of hyperplanes from $\mathcal{H}_\infty$ containing E to $E^\perp$, i.e.,*

$$K = \cap_{H \in \mathcal{H}_\infty, \, E \subset H} \, (H \cap E^\perp).$$

Then, the following holds:

  i) (E_6 *type*) *If R is either $E_6^{\oplus 3}$ or $A_{11} \oplus D_7$, then $K = E^\perp$;*

  ii) (E_7 *type*) *If R is either $E_7 \oplus D_{10}$ or A_{17}, then K has codimension-1 in $E^\perp$;*

iii) (E_8 type) If R is either $E_8^{\oplus 2} \oplus A_2$ or $D_{16} \oplus A_2$, then K has codimension-2 in $E^\perp$.

Proof. As before, a hyperplane H from $\mathcal{H}_\infty$ corresponds to a root δ' in Λ' such that $\langle M, \delta' \rangle'_\Lambda \cong E_7$. The condition $E \subset H$ is equivalent to δ' being orthogonal to E. By the proof of Proposition 6.3, it follows that $L = E_{\Lambda'}^\perp / E$ is an even unimodular lattice of rank 24 whose isometry class is uniquely determined by E. Additionally, the projection of $M \subset E_{\Lambda'}^\perp$ in L determines an embedding $E_6 \hookrightarrow L$ (uniquely determined by E). Similarly, δ' projects to a root of L. Now, the claim follows by arguments similar to those of Lemma 6.8 (with L in place of Λ'). $\square$

The role of linear space K associated to a boundary component in the construction of the blow-up $\widehat{\mathcal{M}} \to (\mathcal{D}/\Gamma)^*$ is explained in Looijenga [29, §3]. In particular, the following geometric consequences follow immediately from the general theory and the computations of the lemma:

i) The GIT boundary components α and δ are unaffected by the birational transformation from (6.1). In particular, these strata are 1-dimensional and the period map extends to an isomorphism over these strata.

ii) The strata γ and ϕ are 2-dimensional and the only indeterminacy point for the period map on these strata is the point ω.

iii) The strata β and ε are 3-dimensional, and the indeterminacy locus along these strata is the curve χ (including the special point ω).

Additionally, the open subsets in α–ϕ parametrizing Type II cubic fourfolds (i.e., excluding the surface $\sigma \subset \alpha \cup \cdots \cup \phi$ of [27]) fiber over the Type II boundary components of the Baily-Borel compactification (see the matching of components from Figure 2). Clearly, the dimensions of the fibers are 2 for β and ε, 1 for γ and ϕ, and 0 for α and δ respectively. Each of the Type II boundary components in $(\mathcal{D}/\Gamma)^*$ are quotients of the upper-half space by some arithmetic group. Thus, the fibration mentioned above is obtained geometrically (up to some finite index) by associating a natural j-invariant to a Type II fourfold. For example, the Type II fourfolds parametrized by the boundary component ϕ are singular along an elliptic normal curve of degree 6. The j-invariant mentioned above is the j-invariant of this elliptic curve. A similar situation holds also for the other boundary components: There is a natural j-invariant associated to the singularities of a Type II fourfold (see Proposition 2.9 (i) and Remark 3.18).

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