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KAM for the nonlinear Schrödinger
equation

By L. Hakan Eliasson and Sergei B. Kuksin

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Abstract

We consider the d-dimensional nonlinear Schrödinger equation under periodic boundary conditions:

$$-i\dot{u} = -\Delta u + V(x) * u + \varepsilon \frac{\partial F}{\partial \bar{u}}(x, u, \bar{u}), \quad u = u(t, x), \ x \in \mathbb{T}^d$$

where $V(x) = \sum \widehat{V}(a) e^{i\langle a, x\rangle}$ is an analytic function with $\widehat{V}$ real, and F is a real analytic function in $\Re u$, $\Im u$ and x. (This equation is a popular model for the 'real' NLS equation, where instead of the convolution term $V * u$ we have the potential term Vu.) For $\varepsilon = 0$ the equation is linear and has time–quasi-periodic solutions

$$u(t, x) = \sum_{a \in \mathscr{A}} \hat{u}(a) e^{i(|a|^2 + \widehat{V}(a))t} e^{i\langle a, x\rangle}, \quad |\hat{u}(a)| > 0,$$

where $\mathscr{A}$ is any finite subset of $\mathbb{Z}^d$. We shall treat $\omega_a = |a|^2 + \widehat{V}(a)$, $a \in \mathscr{A}$, as free parameters in some domain $U \subset \mathbb{R}^{\mathscr{A}}$.

This is a Hamiltonian system in infinite degrees of freedom, degenerate but with external parameters, and we shall describe a KAM-theory which, under general conditions, will have the following consequence:

If $|\varepsilon|$ is sufficiently small, then there is a large subset U' of U such that for all $\omega \in U'$ the solution u persists as a time–quasi-periodic solution which has all Lyapounov exponents equal to zero and whose linearized equation is reducible to constant coefficients.

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1. **Introduction**

We consider the d-dimensional nonlinear Schrödinger equation

$$(*) \qquad -i\dot{u} = -\Delta u + V(x) * u + \varepsilon \frac{\partial F}{\partial \bar{u}}(x, u, \bar{u}), \quad u = u(t, x)$$

under the periodic boundary condition $x \in \mathbb{T}^d$. The convolution potential $V :$ $\mathbb{T}^d \to \mathbb{C}$ must have real Fourier coefficients $\hat{V}(a)$, $a \in \mathbb{Z}^d$, and we shall suppose it is analytic. F is a real analytic function in $\Re u$, $\Im u$ and x.

The nonlinear Schrödinger as an ∞-dimensional Hamiltonian system. If we write

$$\begin{cases} u(x) = \sum_{a \in \mathbb{Z}^d} u_a e^{i \langle a, x \rangle} \\ \overline{u(x)} = \sum_{a \in \mathbb{Z}^d} v_a e^{i \langle -a, x \rangle} \end{cases} \quad (v_a = \bar{u}_a)$$

and let

$$\zeta_a = \begin{pmatrix} \xi_a \\ \eta_a \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}}(u_a + v_a) \\ \frac{-i}{\sqrt{2}}(u_a - v_a) \end{pmatrix},$$

then, in the symplectic space

$$\{(\xi_a, \eta_a) : a \in \mathbb{Z}^d\} = \mathbb{C}^{\mathbb{Z}^d} \times \mathbb{C}^{\mathbb{Z}^d}, \quad \sum_{a \in \mathbb{Z}^d} d\xi_a \wedge d\eta_a,$$

the equation becomes a real Hamiltonian system with an integrable part

$$\frac{1}{2} \sum_{a \in \mathbb{Z}^d} (|a|^2 + \hat{V}(a))(\xi_a^2 + \eta_a^2)$$

plus a perturbation.

Let $\mathscr{A}$ be a finite subset of $\mathbb{Z}^d$ and fix

$$0 < p_a, \quad a \in \mathscr{A}.$$

The $(\#\mathscr{A})$-dimensional torus

$$\frac{1}{2}(\xi_a^2 + \eta_a^2) = p_a, \quad a \in \mathscr{A},$$

$$\xi_a = \eta_a = 0, \quad a \in \mathscr{L} = \mathbb{Z}^d \setminus \mathscr{A}$$

is invariant for the Hamiltonian flow when $\varepsilon = 0$. Near this torus we introduce action-angle variables (φ_a, r_a), $a \in \mathscr{A}$,

$$\xi_a = \sqrt{2(p_a + r_a)} \cos(\varphi_a), \quad \eta_a = \sqrt{2(p_a + r_a)} \sin(\varphi_a).$$

The integrable Hamiltonian now becomes (modulo a constant)

$$h = \sum_{a \in \mathscr{A}} \omega_a r_a + \frac{1}{2} \sum_{a \in \mathscr{L}} \Omega_a (\xi_a^2 + \eta_a^2),$$

where

$$\omega_a = |a|^2 + \hat{V}(a), \quad a \in \mathscr{A}$$

are the *basic frequencies*, and

$$\Omega_a = |a|^2 + \hat{V}(a), \quad a \in \mathscr{L}$$

are the *normal frequencies* (of the invariant torus). The perturbation $\varepsilon f(\xi, \eta, \varphi, r)$ will be a function of all variables (under the assumption, of course, that the torus lies in the domain of F).

This is a standard form for the perturbation theory of lower-dimensional (isotropic) tori with one exception: it is strongly degenerate. We therefore need external parameters to control the basic frequencies and the simplest choice is to let the basic frequencies (i.e., the potential itself) be our free parameters.

The parameters will belong to a set $U \subset \{\omega \in \mathbb{R}^{\mathscr{A}} : |\omega| \leq C\}$. The normal frequencies will be assumed to verify

$$|\Omega_a| \geq C' > 0 \qquad \forall\, a \in \mathscr{L},$$

$$|\Omega_a + \Omega_b| \geq C' \quad \forall\, a, b \in \mathscr{L},$$

$$|\Omega_a - \Omega_b| \geq C' \quad \forall\, a, b \in \mathscr{L}, |a| \neq |b|.$$

This will be fulfilled, for example, if $\mathscr{A}$ is sufficiently large, or if V is small and $\mathscr{A} \not\ni 0$.

We define the complex domain

$$\mathbb{O}^0(\sigma, \rho, \mu) = \begin{cases} \|\zeta\|_0 = \sqrt{\sum_{a \in \mathscr{L}}(|\xi_a|^2 + |\eta_a|^2)\langle a \rangle^{2m_*}} < \sigma \\ |\Im\varphi| < \rho \\ |r| < \mu, \end{cases}$$

$\langle a \rangle = \max(|a|, 1)$. We assume $m_* > \frac{d}{2}$ because in this space $h + \varepsilon f$ is analytic and the Hamiltonian equations have a well-defined local flow.

By $\langle\,,\,\rangle$ we denote the usual paring $\langle \zeta, \zeta' \rangle = \sum \xi_a \xi'_a + \eta_a \eta'_a$.

THEOREM A. *Under the above assumptions, for ε sufficiently small there exist a subset $U' \subset U$, which is large in the sense that* $\mathrm{Leb}\,(U \setminus U') \leq \mathrm{cte}.\varepsilon^{\exp}$, *and for each $\omega \in U'$, a real analytic symplectic diffeomorphism Φ*

$$\mathbb{O}^0\Big(\frac{\sigma}{2}, \frac{\rho}{2}, \frac{\mu}{2}\Big) \to \mathbb{O}^0(\sigma, \rho, \mu)$$

and a vector ω' such that $(h_{\omega'} + \varepsilon f) \circ \Phi$ equals (modulo a constant)

$$\langle \omega, r \rangle + \frac{1}{2}\langle \xi, Q_1 \xi \rangle + \langle \xi, Q_2 \eta \rangle + \frac{1}{2}\langle \eta, Q_1 \eta \rangle + \varepsilon f',$$

where

$$f' \in \mathbb{O}(|r|^2, |r|\,\|\zeta\|_0, \|\zeta\|_0^3)$$

and $Q = Q_1 + iQ_2$ is a Hermitian and block-diagonal matrix with finite-dimensional blocks.

Moreover $\Phi = (\Phi_\zeta, \Phi_\varphi, \Phi_r)$ verifies, for all $(\zeta, \varphi, r) \in \mathbb{O}^0(\frac{\sigma}{2}, \frac{\rho}{2}, \frac{\mu}{2})$,

$$\big\|\Phi_\zeta - \zeta\big\|_0 + \big|\Phi_\varphi - \varphi\big| + |\Phi_r - r| \leq \beta\varepsilon,$$

and the mapping $\omega \mapsto \omega'(\omega)$ verifies

$$\big|\omega' - \mathrm{id}\big|_{\mathscr{C}^1(U')} \leq \beta\varepsilon.$$

β is a constant that depends on the dimensions d, $\#\mathscr{A}$, on m_*, on the constants C, C' and on V and F.

The consequences of the theorem are well-known. The dynamics of the Hamiltonian vector field of $h_{\omega'} + \varepsilon f$ on $\Phi(\{0\} \times \mathbb{T}^d \times \{0\})$ is the same as that of

$$\langle \omega, r \rangle + \frac{1}{2} \langle \xi, Q_1 \xi \rangle + \langle \xi, Q_2 \eta \rangle + \frac{1}{2} \langle \eta, Q_1 \eta \rangle.$$

The torus $\{\zeta = r = 0\}$ is invariant, since the Hamiltonian system on it is

$$\begin{cases} \dot{\zeta} = 0 \\ \dot{\varphi} = \omega \\ \dot{r} = 0, \end{cases}$$

and the flow on the torus is linear $t \mapsto \varphi + t\omega$.

Moreover, the linearized equation on this torus becomes

$$\begin{cases} \frac{d}{dt}\widehat{\zeta} = J \begin{pmatrix} Q_1(\omega) & Q_2(\omega) \\ {}^t Q_2(\omega) & Q_1(\omega) \end{pmatrix} \widehat{\zeta} + Ja(\varphi + t\omega, \omega)\hat{r} \\ \frac{d}{dt}\widehat{\varphi} = \langle a(\varphi + t\omega, \omega), \widehat{\zeta} \rangle + b(\varphi + t\omega, \omega)\hat{r} \\ \frac{d}{dt}\hat{r} = 0, \end{cases}$$

where $a = \varepsilon \partial_r \partial_\zeta f'$ and $b = \varepsilon \partial_r^2 f'$. Since $Q_1 + iQ_2$ is Hermitian and block diagonal the eigenvalues of the ζ-linear part are purely imaginary $\pm i\,\Omega'_a$, $a \in \mathcal{L}$.

The linearized equation is reducible to constant coefficients if the imaginary part Ω'_a of the eigenvalues are nonresonant with respect to ω, something which can be assumed if we restrict the set U' arbitrarily little. Then the $\widehat{\zeta}$-component (and of course also the $\hat{r}$-component) will have only quasi-periodic (in particular bounded) solutions. The $\widehat{\varphi}$-component may have a linear growth in t, the growth factor (the "twist") being linear in $\hat{r}$.

Reducibility. Reducibility is not only an important outcome of KAM but also an essential ingredient in the proof. It simplifies the iteration since it makes it possible to reduce all approximate linear equations to constant coefficients. But it does not come for free. It requires a lower bound on small divisors of the form

$$(**) \qquad \left| \langle k, \omega \rangle + \Omega'_a - \Omega'_b \right|, \quad k \in \mathbb{Z}^{\mathcal{A}}, \, a, b \in \mathcal{L}.$$

The basic frequencies ω will be kept fixed during the iteration – that is what the parameters are there for – but the normal frequencies will vary. Indeed $\Omega'_a(\omega)$ and $\Omega'_b(\omega)$ are perturbations of Ω_a and Ω_b which are not known *a priori* but are determined by the approximation process.[1]

These are a lot of conditions for a few parameters ω. It is usually possible to make a (scale dependent) restriction of $(**)$ to $|k|, \, |a - b| \leq \Delta = \Delta_\varepsilon$ which improves the situation a bit. Indeed, in one space-dimension ($d = 1$) it improves

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[1] A lower bound on $(**)$, often known as the second Melnikov condition, is strictly speaking not necessary for reducibility. It is necessary, however, for reducibility with a reducing transformation close to the identity.

a lot, and $(**)$ reduces to only finitely many conditions. Not so however when $d \geq 2$, in which case the number of conditions in $(**)$ remains infinite.

To cope with this problem we shall exploit the Töplitz-Lipschitz property which allows for a sort of compactification of the dimensions and reduces the infinitely many conditions $(**)$ to finitely many. These can then be controlled by an appropriate choice of ω.

The Töplitz-Lipschitz property. The Töplitz-Lipschitz property is defined for infinite-dimensional matrices with exponential decay. We say that a matrix

$$A : \mathscr{L} \times \mathscr{L} \to \mathbb{C}$$

is Töplitz at ∞ if, for all $a, b, c \in \mathbb{Z}^d$, the limit

$$\lim_{t \to \infty} A_{a+tc}^{b+tc} \, \exists \quad =: \; A_a^b(c).$$

The Töplitz limit $A(c)$ is a new matrix which is c-invariant: $A_{a+c}^{b+c}(c) = A_a^b(c)$. So it is a simpler object because it is "more constant".

The approach to the Töplitz limit in direction c is controlled by a Lipschitz-condition. This control does not take place everywhere, but on a certain subset

$$D_\Lambda(c) \subset \mathscr{L} \times \mathscr{L}$$

— the Lipschitz domain. Λ is a parameter which, together with $|c|$, determines the size of the domain.

The Töplitz-Lipschitz property permits us to verify certain bounds of the matrix-coefficients or functions of these, like determinants of sub-matrices, in the Töplitz limit and then recover these bounds for the matrix restricted to the Lipschitz domain.

The matrices we shall consider will not be scalar-valued but $\mathrm{gl}(2, \mathbb{C})$-valued:

$$A : \mathscr{L} \times \mathscr{L} \to \mathrm{gl}(2, \mathbb{C})$$

and we shall define a Töplitz-Lipschitz property for such matrices also. These matrices constitute an algebra: one can multiply them and solve linear differential equations. A function f is said to have the Töplitz-Lipschitz property if its Hessian (with respect to ζ) is Töplitz-Lipschitz. If this is the case, as it is for the perturbation f of the nonlinear Schrödinger, then this is also true for the linear part of our KAM–transformations and for the transformed Hamiltonian. This will permit us to formulate an inductive statement which, as usual in KAM, gives Theorem A.

Some references. For finite-dimensional Hamiltonian systems the first proof of persistence of stable (i.e., vanishing of all Lyapounov exponents) lower-dimensional invariant tori was obtained in [Eli85], [Eli88] and there are now many works on this subjects. There are also many works on reducibility (see for example [Kri99], [Eli01]) and the situation in finite dimension is now pretty well understood in the perturbative setting. Not so, however, in infinite dimension.

If $d = 1$ and the space-variable x belongs to a finite segment supplemented by Dirichlet or Neumann boundary conditions, this result was obtained in [Kuk88] (also see [Kuk93], [Pös96]). The case of periodic boundary conditions was treated in [Bou96], using another multi-scale scheme, suggested by Fröhlich-Spencer in their work on the Anderson localization [FS83]. This approach, often referred to as the Craig-Wayne scheme, is different from KAM. It avoids the, sometimes, cumbersome condition (∗∗) but to a high cost: the approximate linear equations are not of constant coefficients. Moreover, it gives persistence of the invariant tori but no reducibility and no information on the linear stability. A KAM-theorem for periodic boundary conditions has recently been proved in [GY05] (with a perturbation F independent of x) and the perturbation theory for quasi-periodic solutions of one-dimensional Hamiltonian PDE is now sufficiently well developed (see for example [Kuk93], [Cra00], [Kuk00]).

The study of the corresponding problems for $d \geq 2$ is at its early stage. Developing further the scheme, suggested by Fröhlich-Spencer, Bourgain proved persistence for the case $d = 2$ [Bou98]. More recently, the new techniques developed by him and collaborators in their work on the linear problem has allowed him to prove persistence in any dimension d [Bou04]. (In this work he also treats the nonlinear wave equation.)

Description of the paper. The paper is divided into three parts. The first part deals with linear algebra of Töplitz-Lipschitz matrices and the analysis of functions with the Töplitz-Lipschitz property. In Section 2 we introduce Töplitz-Lipschitz matrices and prove a product formula. This part is treated in greater generality in [EK08]. In Section 3 we analyze functions with the Töplitz-Lipschitz property.

The second part deals with the bounds on the small divisors (∗∗) which occur in the solution of the homological equation. In Section 4 we analyze the block decomposition of the lattice $\mathbb{Z}^d$ and in Section 5 we study the small divisors. In Section 6 we solve the homological equations. This part is independent of the first part except for basic definitions and properties given in Sections 2.3 and 2.4.

The third part treats KAM-theory with the Töplitz-Lipschitz property and contains a general KAM-theorem, Theorem 7.1. This theorem is applied to the nonlinear Schrödinger to give Theorem 7.2 of which the theorem above is a variant.

Notation. $\langle \, , \, \rangle$ is the standard scalar product in $\mathbb{R}^d$. $\| \ \|$ is an operator-norm or l^2-norm. $| \ |$ will in general denote a supremum norm, with a notable exception: for a lattice vector $a \in \mathbb{Z}^d$ we use $|a|$ for the l^2-norm.

$\mathscr{A}$ is a finite subset of $\mathbb{Z}^d$ and $\mathscr{L}$ is the complement of a finite subset of $\mathbb{Z}^d$. For the nonlinear Schrödinger equation, $\mathscr{L}$ will be the complement of $\mathscr{A}$, but this not assumed in general.

A matrix on $\mathscr{L}$ is just a mapping $A : \mathscr{L} \times \mathscr{L} \to \mathbb{C}$ or $gl(2, \mathbb{C})$. Its components will be denoted A_a^b.

The dimension d will be fixed and m_* will be a fixed constant $> \frac{d}{2}$.

$\lesssim$ means $\leq$ modulo a multiplicative constant that only, unless otherwise specified, depends on d, m_* and $\#\mathcal{A}$.

The points in the lattice $\mathbb{Z}^d$ will be denoted $a, b, c, \dots$. Also d will sometimes be used, without confusion, we hope.

For a vector $c \in \mathbb{Z}^d$, $c^\perp$ will denote the $\perp$ complement of c in $\mathbb{Z}^d$ or in $\mathbb{R}^d$, depending on the context. If $c \neq 0$, for any $a \in \mathbb{Z}^d$ we let $a_c \in (a + \mathbb{R}c) \cap \mathbb{Z}^d$ be the lattice point b on the line $a + \mathbb{R}c$ with smallest norm, i.e., that minimizes $|\langle b, c \rangle|$; if there are two such b's we choose the one with $\langle b, c \rangle \geq 0$. It is the "$\perp$ projection of a to $c^\perp$".

Greek letter $\alpha, \beta, \dots$ will mostly be used for bounds. Exceptions are φ which will denote an element in the torus – an angle – and ω, Ω.

For two subsets X and Y of a metric space,

$$\mathrm{dist}(X, Y) = \inf_{x \in X, y \in Y} d(x, y).$$

(This is not a metric.) X_ε is the ε-neighborhood of X; i.e., $\{y : \mathrm{dist}(y, X) < \varepsilon\}$. Let $B_\varepsilon(x)$ be the ball $\{y : d(x, y) < \varepsilon\}$. Then X_ε is the union, over $x \in X$, of all $B_\varepsilon(x)$.

If X and Y are subsets of $\mathbb{R}^d$ or $\mathbb{Z}^d$ we let

$$X - Y = \{x - y : x \in X, \ y \in Y\}$$

— not to be confused with the set theoretical difference $X \setminus Y$.

Part I. The Töplitz-Lipschitz property

In this part we consider $\mathcal{L} \subset \mathbb{Z}^d$ and matrices $A : \mathcal{L} \times \mathcal{L} \to \mathrm{gl}(2, \mathbb{C})$. We define: the sup-norms $|\cdot|_\gamma$; the notion of being Töplitz at ∞; the Lipschitz-domains $D_\Delta^\pm(c)$; the Lipschitz- norm $\langle \cdot \rangle_{\Lambda, \gamma}$ and the notion of being Töplitz-Lipschitz. (For a more general exposition see [EK08].) We define the Töplitz-Lipschitz property for functions and the norms $[\cdot]_{\Lambda, \gamma, \sigma}$.

2. Töplitz-Lipschitz matrices

2.1. *Spaces and matrices.* We denote by $l_\gamma^2(\mathcal{L}, \mathbb{C}^2)$, $\gamma \geq 0$, the following weighted l_2-spaces:

$$l_\gamma^2(\mathcal{L}, \mathbb{C}^2) = \{\zeta = (\xi, \eta) \in \mathbb{C}^{\mathcal{L}} \times \mathbb{C}^{\mathcal{L}} : \|\zeta\|_\gamma < \infty\},$$

where

$$\|\zeta\|_\gamma^2 = \sum_{a \in \mathcal{L}} (|\xi_a|^2 + |\eta_a|^2) e^{2\gamma|a|} \langle a \rangle^{2m_*}, \quad \langle a \rangle = \max(|a|, 1).$$

We provide $l_\gamma^2(\mathcal{L}, \mathbb{C}^2)$ with the symplectic form

$$\sum_{a \in \mathcal{L}} d\xi_a \wedge d\eta_a.$$

Using the pairing

$$\langle \zeta, \zeta' \rangle = \sum_{a \in \mathscr{L}} (\xi_a \xi'_a + \eta_a \eta'_a)$$

we can write the symplectic form as $\langle \cdot, J \cdot \rangle$ where $J : l^2_\gamma(\mathscr{L}, \mathbb{C}^2) \to l^2_\gamma(\mathscr{L}, \mathbb{C}^2)$ is the standard involution, given by the component-wise application of the matrix

$$J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

We consider the space $gl(2, \mathbb{C})$ of all complex 2×2-matrices provided with the scalar product $\mathrm{Tr}(^t \bar{A} B)$, and consider the orthogonal projection

$$\pi : gl(2, \mathbb{C}) \to M, \quad M = \mathbb{C}I + \mathbb{C}J.$$

It is easy to verify that

$$\begin{cases} M \times M, M^\perp \times M^\perp \subset M \\ M \times M^\perp, M^\perp \times M \subset M^\perp \end{cases}$$

and

$$\begin{cases} \pi(AB) = \pi A \pi B + (I - \pi)A(I - \pi)B \\ (I - \pi)(AB) = (I - \pi)A\pi B + \pi A(I - \pi)B. \end{cases}$$

If $A = (A_i^j)_{i,j=1}^2$ $B = (B_i^j)_{i,j=1}^2$ we define

$$[A] = (|A_i^j|)_{i,j=1}^2,$$

and

$$A \leq B \iff |A_i^j| \leq B_i^j, \quad \forall i, j.$$

Since any Euclidean space E is naturally isomorphic to its dual E^*, the canonical relations

$$E \otimes E \simeq E^* \otimes E^* \simeq \mathrm{Hom}(E, E^*) \simeq \mathrm{Hom}(E, E)$$

permits the identification of the tensor product $\zeta \otimes \zeta'$ with a 2×2-matrix

$$(\zeta \otimes \zeta')_i^j = \zeta_i \zeta'_j.$$

2.2. *Matrices with exponential decay.* Consider now an infinite-dimensional $gl(2, \mathbb{C})$-valued matrix

$$A : \mathscr{L} \times \mathscr{L} \to gl(2, \mathbb{C}), \quad (a, b) \mapsto A_a^b.$$

We define matrix multiplication through

$$(AB)_a^b = \sum_d A_a^d B_d^b,$$

and, for any subset $\mathscr{D}$ of $\mathscr{L} \times \mathscr{L}$, the semi-norms

$$|A|_{\mathscr{D}} = \sup_{(a,b) \in \mathscr{D}} \|A_a^b\|$$

(here $\| \ \|$ is the operator-norm).

We define πA through $(\pi A)_a^b = \pi A_a^b$, for all a, b. Clearly we have

$$\pi(A + B) = \pi A + \pi B$$

(1)
$$\pi(AB) = \pi A \pi B + (I - \pi)A(I - \pi)B$$

$$(I - \pi)(AB) = (I - \pi)A\pi B + \pi A(I - \pi)B.$$

We define

$$A \leq B \iff A_a^b \leq B_a^b, \forall a, b,$$

and

$$(\mathscr{E}_\gamma^\pm A)_a^b = [A_a^b]e^{\gamma|a \mp b|}, \forall a, b.$$

All operators $\mathscr{E}_\gamma^\pm$ commute and we have

$$\begin{cases} \mathscr{E}_\gamma^x(A + B) \leq \mathscr{E}_\gamma^x A + \mathscr{E}_\gamma^x B, & x \in \{+, -\} \\ \mathscr{E}_\gamma^{xy}(AB) \leq (\mathscr{E}_\gamma^x A)(\mathscr{E}_\gamma^y B), & x, y \in \{+, -\}. \end{cases} \quad {}_2$$

We define the norm

$$|A|_\gamma = \max(|\mathscr{E}_\gamma^+ \pi A_a^b|_{\mathscr{L} \times \mathscr{L}}, |\mathscr{E}_\gamma^- (1 - \pi) A_a^b|_{\mathscr{L} \times \mathscr{L}}).$$

We have, by *Young's inequality* (see [Fol76]), that

(2)
$$\|A\zeta\|_{\gamma'} \lesssim \left(\frac{1}{\gamma - \gamma'}\right)^{d + m_*} |A|_\gamma \|\zeta\|_{\gamma'}, \quad \forall \gamma' < \gamma.$$

(Take for example $A = \pi A$ and apply Young's inequality to the matrix $\widetilde{A}$ defined by

$$\widetilde{A}_a^b = e^{\gamma'|a|} \langle a \rangle^{m_*} A_a^b \langle b \rangle^{-m_*} e^{-\gamma'|b|}.)$$

Thus, if $|A|_\gamma < \infty$, then A defines a bounded operator on any $l_{\gamma'}^2(\mathscr{L}, \mathbb{C}^2)$, $\gamma' < \gamma$.

Truncations. Let

$$(\mathscr{T}_\Delta^\pm)A_a^b = \begin{cases} A_a^b & \text{if } |a \mp b| \leq \Delta \\ 0 & \text{if not}, \end{cases}$$

and

$$\mathscr{T}_\Delta A = \mathscr{T}_\Delta^+ \pi A + \mathscr{T}_\Delta^- (I - \pi)A.$$

It is clear that

(3)
$$|\mathscr{T}_\Delta A|_\gamma \leq |A|_\gamma \quad \text{and} \quad |A - \mathscr{T}_\Delta A|_{\gamma'} \leq e^{-\Delta(\gamma - \gamma')} |A|_\gamma.$$

Tensor products. For any two elements $\zeta, \zeta' \in l_\gamma^2(\mathscr{L}, \mathbb{C}^2)$, their tensor product $\zeta \otimes \zeta'$ is a matrix on $\mathscr{L} \times \mathscr{L}$, and it is easy to verify that

(4)
$$\left|\zeta \otimes \zeta'\right|_\gamma \lesssim \|\zeta\|_\gamma \|\zeta'\|_\gamma.$$

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[2]We use the sign convention that $xy = +$ whenever x and y are equal and $xy = -$ whenever they are different.

Multiplication. We have

$$(5) \qquad |AB|_{\gamma'} + |BA|_{\gamma'} \lesssim \left(\frac{1}{\gamma - \gamma'}\right)^d |A|_\gamma \, |B|_{\gamma'} , \quad \forall \gamma' < \gamma.$$

Linear differential equation. Consider the linear system

$$\begin{cases} X' = A(t)X \\ X(0) = I. \end{cases}$$

It follows from (5) that the series

$$I + \sum_{n=1}^\infty \int_0^{t_0} \int_0^{t_1} \dots \int_0^{t_{n-1}} A(t_1)A(t_2)\dots A(t_n)dt_n \dots dt_2 dt_1,$$

as well as its derivative with respect to t_0, converges to a solution which verifies, for $\gamma' < \gamma$,

$$(6) \qquad |X(t) - I|_{\gamma'} \lesssim (\gamma - \gamma')^d \left(\exp\left(\text{cte.} \left(\frac{1}{\gamma - \gamma'}\right)^d |t| \alpha(t)\right) - 1 \right),$$

where $\alpha(t) = \sup_{0 \le |s| \le |t|} |A(s)|_\gamma$.

2.3. *Töplitz-Lipschitz matrices ($d = 2$).* A matrix $A : \mathscr{L} \times \mathscr{L} \to \text{gl}(2, \mathbb{C})$ is said to be *Töplitz at ∞* if, for all a, b, c, the two limits

$$\lim_{t \to +\infty} A_{a+tc}^{b \pm tc} \; \exists \; = A_a^b(\pm, c).$$

It is easy to verify that if $|A|_\gamma < \infty$ and $|B|_\gamma < \infty$, then

$$(\pi A)(-, c) = (I - \pi)A(+, c) = 0$$

and

$$
\begin{aligned}
(7) \quad \pi(AB)(+, c) &= \pi A(+, c)\pi B(+, c) \\
&\quad + (I - \pi)A(-, c)(I - \pi)B(-, -c), \\
(I - \pi)(AB)(-, c) &= (I - \pi)A(-, c)\pi B(+, -c) \\
&\quad + \pi A(+, c)(I - \pi)B(-, c).
\end{aligned}
$$

In the rest of this section we assume that $c \neq 0$. We define

$$(\mathscr{M}_c A)_a^b = \left(\max\left(\frac{|a|}{|c|}, \frac{|b|}{|c|}\right) + 1 \right)[A_a^b], \quad \forall a, b.$$

The operators $\mathscr{M}_c$ and $\mathscr{E}_\gamma^\pm$ all commute and $\mathscr{M}_c(AB) \le (\mathscr{M}_c A)(\mathscr{M}_c B)$.

Lipschitz domains. For a nonnegative constant Λ, let $D_\Lambda^+(c) \subset \mathscr{L} \times \mathscr{L}$ be the set of all (a, b) such that there exist $a', b' \in \mathbb{Z}^d$ and $t \ge 0$ such that

$$\begin{cases} |a = a' + tc| \ge \Lambda(|a'| + |c|)\,|c| \\ |b = b' + tc| \ge \Lambda(|b'| + |c|)\,|c| \end{cases} \quad \text{and} \quad \frac{|a|}{|c|}, \; \frac{|b|}{|c|} \ge 2\Lambda^2.$$

We give here some elementary properties of the Lipschitz domains. They will be studied further in Section 4.

LEMMA 2.1. *Let $t \geq 0$.*

(i) *For $\Lambda \geq 1$, we have $t \geq \Lambda |c| \geq \Lambda$ if $|a = a' + tc| \geq \Lambda(|a'| + |c|) |c|$.*

(ii) *For $\Lambda > 1$,*

$$\begin{cases} |a'| \leq \frac{t}{\Lambda-1} - |c| \ \ \text{if } |a = a' + tc| \geq \Lambda(|a'| + |c|) |c| \,, \\ |a'| \geq \frac{t}{\Lambda+1} - |c| \ \ \text{if not.} \end{cases}$$

(iii) *For $\Lambda > 1$,*

$$\left| \frac{|a|}{|c|} - t \right| \leq \frac{t}{\Lambda - 1} \ \text{and} \ \left| \frac{\langle a, c \rangle}{|c|^2} - t \right| \leq \frac{t}{\Lambda - 1},$$

if $|a = a' + tc| \geq \Lambda(|a'| + |c|) |c|$.

(iv) *For $\Omega \geq (\Lambda + 1)(|a - b| + 1)$, we have $|b = b' + tc| \geq \Lambda(|b'| + |c|)|c|$ with $b' = a' + b - a$, if $|a = a' + tc| \geq \Omega(|a'| + |c|) |c|$.*

Proof. This is a direct computation. □

COROLLARY 2.2. *Let $\Lambda \geq 3$.*

(i) $(a, b) \in D_\Lambda^+(c) \Longrightarrow \dfrac{|a|}{|c|} \approx \dfrac{|b|}{|c|} \approx \dfrac{\langle a, c \rangle}{|c|^2} \approx \dfrac{\langle b, c \rangle}{|c|^2} \gtrsim \Lambda |c| \,.$

(ii) $(a, b) \in D_\Lambda^+(c) \Longrightarrow (a + tc, b + tc) \in D_\Lambda^+(c) \quad \forall t \geq 0.$

(iii) $(a, b) \in D_\Lambda^+(c) \Longrightarrow (\tilde{a}, \tilde{b}) \in D_\Omega^+(c),$ *where*

$$\Omega = \Lambda - \max(|\tilde{a} - a|, \ |\tilde{b} - b|) - 2.$$

(iv) $(a, b) \in D_{\Lambda+3}^+(c), \ (a, d) \notin D_\Lambda^+(c) \Longrightarrow |a - d| \,, \ |b - d| \gtrsim \dfrac{1}{\Lambda^2} \dfrac{|a|}{|c|}.$

Proof. (i) follows from Lemma 2.1, (i) and (iii), if we just observe that

$$t \approx t + \frac{t}{\Lambda - 1} \approx t - \frac{t}{\Lambda - 1}.$$

In order to see (ii) we write $a = a' + sc$, $s \geq 0$, with $|a| \geq \Lambda(|a'| + |c|) |c|$. Then

$$|a + tc|^2 = |a|^2 + t^2|c|^2 + 2t \langle a, c \rangle |a|^2 + t^2|c|^2 + 2ts|c|^2 + 2t \langle a', c \rangle.$$

By Lemma 2.1(ii)

$$2ts|c|^2 + 2t \langle a', c \rangle \geq 2ts \left(1 - \frac{1}{\Lambda-1} \right) |c|^2 \geq 0.$$

Hence

$$|a + tc|^2 \geq |a|^2 + t^2|c|^2 \geq |a|^2 \geq \Lambda(|a'| + |c|) |c| \,.$$

Moreover, for all $t \geq 0$

$$\frac{|a + tc|}{|c|} \geq \frac{|a|}{|c|} \geq 2\Lambda^2.$$

The same argument applies to b.

To see (iii), let $\Delta = \max(|\tilde{a} - a|, |\tilde{b} - b|) + 2$ and write $a = a' + tc$ with $|a| \geq \Lambda(|a'| + |c|)|c|$. Then $\tilde{a} = a' + \tilde{a} - a + tc$, and if $|\tilde{a}| < \Omega(|a' + \tilde{a} - a| + |c|)|c|$ then by Lemma 2.1(ii),

$$|\tilde{a} - a| \geq \frac{t\Delta}{(\Omega + 1)(\Lambda - 1)}.$$

This implies that $t \leq (\Omega + 1)(\Lambda - 1)$ and, hence, $\frac{|a|}{|c|} < 2\Lambda^2$ which is impossible. Therefore $|\tilde{a}| \geq \Omega(|a' + \tilde{a} - a| + |c|)|c|$. Moreover

$$\frac{|\tilde{a}|}{|c|} \geq \frac{|a|}{|c|} - \frac{\Delta}{|c|} \geq 2\Lambda^2 - \Delta \geq 2\Omega^2.$$

The same argument applies to b.

To see (iv), assume that $\frac{|d|}{|c|} < 2\Lambda^2$. As $\frac{|b|}{|c|} \geq 2(\Lambda + 3)^2$ it follows that

$$\frac{|b - d|}{|c|} \geq 12\Lambda.$$

So $|b - d| \geq \Lambda^{-2} \frac{|a|}{|c|}$, unless $\frac{|a|}{|c|} \geq 12\Lambda^3|c|$. In this case, due to Lemma 2.1(iii), $\frac{|b|}{|c|} \geq \frac{\Lambda+1}{\Lambda+3} \frac{|a|}{|b|} \geq 12\Lambda^2$. So we must have

$$\frac{|d|}{|c|} \leq 2\Lambda^2 \leq \frac{1}{6} \frac{|b|}{|c|}$$

which implies that

$$\frac{|b - d|}{|c|} \geq \frac{5}{6} \frac{|b|}{|c|} \geq \frac{1}{\Lambda^2} \frac{|a|}{|c|}.$$

Therefore we can assume that $\frac{|d|}{|c|} \geq 2\Lambda^2$. Since $(a, b) \in D_{\Lambda+3}(c)$, then $b = b' + tc$, where $|b| \geq (\Lambda + 3)(|b'| + |c|)|c|$. Let us write d as $d = b + (d - b) = d' + tc$, $d' = b' + (d - b)$. Since $(a, d) \notin D_\Lambda^+(c)$ while $(a, b) \in D_{\Lambda+3}^+(c) \subset D_\Lambda^+(c)$ and $\frac{|d|}{|c|} \geq 2\Lambda^2$, then $|d| < \Lambda(|d'| + |c|) + |c|$. Applying Lemma 2.1(ii) we get that

$$|b'| \leq \frac{t}{\Lambda + 2} - |c|, \quad |d'| \geq \frac{t}{\Lambda + 1} - |c|.$$

Hence, $|b - d| = |d' - b'| \geq \frac{t}{\Lambda+1} - \frac{t}{\Lambda+2} \gtrsim \frac{t}{\Lambda^2} \gtrsim \frac{1}{\Lambda^2} \frac{|a|}{|c|}$, where we used Lemma 2.1(iii).

Now the required estimate for $|b - d|$ is established. Similar arguments apply to $|a - d|$. □

Lipschitz constants and norms. Define the Lipschitz constants

$$\text{Lip}_{\Lambda,\gamma}^x A = \sup_c |\mathscr{E}_\gamma^x \mathscr{M}_c (A - A(x, c))|_{D_\Lambda^x(c)}, \quad x \in \{+, -\},$$

(see the notation of §§2.2, 2.3) and the Lipschitz norm

$$\langle A \rangle_{\Lambda,\gamma} = \max(\operatorname{Lip}^+_{\Lambda,\gamma} \pi A, \operatorname{Lip}^-_{\Lambda,\gamma}(1-\pi)A) + |A|_\gamma .$$

Here we have defined $(a,b) \in D^-_\Lambda(c) \Longleftrightarrow (a,-b) \in D^+_\Lambda(c)$. The matrix A is *Töplitz-Lipschitz* if it is Töplitz at ∞ and $\langle A \rangle_{\Lambda,\gamma} < \infty$ for some Λ, γ.

Truncations. It is easy to see that

$$(8) \qquad \langle \mathcal{T}_\Delta A \rangle_{\Lambda,\gamma} \le \langle A \rangle_{\Lambda,\gamma}, \qquad \langle A - \mathcal{T}_\Delta A \rangle_{\Lambda,\gamma'} \le e^{-\Delta(\gamma-\gamma')} \langle A \rangle_{\Lambda,\gamma}.$$

Tensor products. We can now verify that

$$(9) \qquad \langle \zeta \otimes \zeta' \rangle_{\Lambda,\gamma} \lesssim \|\zeta\|_\gamma \|\zeta'\|_\gamma .$$

Multiplications and differential equations are more delicate and we shall need the following proposition.

PROPOSITION 2.3. *For all $x, y \in \{+, -\}$, all $\gamma' < \gamma$ and any $c \neq 0$*

(i) $$\left| \mathcal{E}^{xy}_{\gamma'} \mathcal{M}_c(AB) \right|_{D^{xy}_{\Lambda+3}(c)} \lesssim \left(\frac{1}{\gamma-\gamma'} \right)^d \left| \mathcal{E}^x_{\gamma_1} \mathcal{M}_c(A) \right|_{D^x_\Lambda(c)} \left| \mathcal{E}^y_{\gamma_2} B \right|_{\mathcal{L}\times\mathcal{L}}$$
$$+ \Lambda^2 \left(\frac{1}{\gamma-\gamma'} \right)^{d+1} \left| \mathcal{E}^x_{\gamma_1} A \right|_{\mathcal{L}\times\mathcal{L}} \left| \mathcal{E}^y_{\gamma_2} B \right|_{\mathcal{L}\times\mathcal{L}},$$

where one of γ_1, γ_2 is $= \gamma$ and the other one is $= \gamma'$. The same bound holds for BA.

(ii) $$\left| \mathcal{E}^{xyz}_{\gamma'} \mathcal{M}_c(ABC) \right|_{D^{xyz}_{\Lambda+6}(c)}$$
$$\lesssim \left(\frac{1}{\gamma-\gamma'} \right)^{2d} \left| \mathcal{E}^x_{\gamma_1} A \right|_{\mathcal{L}\times\mathcal{L}} \left| \mathcal{E}^y_{\gamma_2} \mathcal{M}_c(B) \right|_{D^y_\Lambda(c)} \left| \mathcal{E}^z_{\gamma_3} C \right|_{\mathcal{L}\times\mathcal{L}}$$
$$+ \Lambda^2 \left(\frac{1}{\gamma-\gamma'} \right)^{2d+1} \left| \mathcal{E}^x_{\gamma_1} A \right|_{\mathcal{L}\times\mathcal{L}} \left| \mathcal{E}^y_{\gamma_2} B \right|_{\mathcal{L}\times\mathcal{L}} \left| \mathcal{E}^z_{\gamma_3} C \right|_{\mathcal{L}\times\mathcal{L}},$$

where two of $\gamma_1, \gamma_2, \gamma_3$ are $= \gamma$ and the third one is $= \gamma'$. The same bound holds if we permute the factors A, B and C.

Proof. To prove (i), let first $x = y = +$. We shall only prove the estimate for AB – the estimate for BA being the same. Notice that for $(a,b) \in D^+_{\Lambda+3}(c)$ we have, by Corollary 2.2(i), that

$$M_c(a,b) = \max \left(\frac{|a|}{|c|}, \frac{|b|}{|c|} \right) + 1 \approx \frac{|a|}{|c|} + 1.$$

Now, for $(a,b) \in D^+_{\Lambda+3}(c)$ we have

$$(\mathcal{E}^+_{\gamma'} \mathcal{M}_c(AB))^b_a \le \sum_d M_c(a,b)[A^d_a][B^b_d] e^{\gamma'|a-b|}$$

$$= \sum_{(a,d)\in D^+_\Lambda(c)} \ldots + \sum_{(a,d)\notin D^+_\Lambda(c)} \ldots = (\text{I}) + (\text{II}).$$

In the domain of (I) we have, by Corollary 2.2(i), that

$$M_c(a,b) \approx \frac{|a|}{|c|} + 1 \approx M_c(a,d),$$

so that

$$(I) \lesssim \left|\mathscr{E}_{\gamma_1}^+ \mathscr{M}_c A\right|_{D_\Lambda^+(c)} \left|\mathscr{E}_{\gamma_2}^+ B\right|_{\mathscr{L}\times\mathscr{L}} \sum_d e^{-(\gamma_1-\gamma')|a-d|-(\gamma_2-\gamma')|d-b|}.$$

Since one of $\gamma_1 - \gamma'$ and $\gamma_2 - \gamma'$ is $\gamma - \gamma'$ the sum is $\lesssim (\frac{1}{\gamma-\gamma'})^d$.

In the domain of (II) we have, by Corollary 2.2(iv), that

$$|a-d|,\ |b-d| \gtrsim \frac{1}{\Lambda^2}\frac{|a|}{|c|},$$

so (II) is

$$\lesssim \left|\mathscr{E}_{\gamma_1}^+ A\right|_{\mathscr{L}\times\mathscr{L}} \left|\mathscr{E}_{\gamma_2}^+ B\right|_{\mathscr{L}\times\mathscr{L}}$$
$$\times \sum_{|a-d|,|d-b|\gtrsim\frac{1}{\Lambda^2}\frac{|a|}{|c|}} \left(\frac{|a|}{|c|}+1\right) e^{-(\gamma_1-\gamma')|a-d|-(\gamma_2-\gamma')|d-b|}.$$

Since one of $\gamma_1 - \gamma'$ and $\gamma_2 - \gamma'$ is $\gamma - \gamma'$ the sum is $\lesssim \Lambda^2 (\frac{1}{\gamma-\gamma'})^{d+1}$. The three other cases of (i) are treated in the same way.

To prove (ii), let first $x = y = z = +$. Notice that for $(a,b) \in D_{\Lambda+6}^+(c)$ we have, by Corollary 2.2(i), that

$$M_c(a,b) = \max\left(\frac{|a|}{|c|},\frac{|b|}{|c|}\right) + 1 \approx \frac{|a|}{|c|} + 1.$$

Now

$$(\mathscr{E}_{\gamma'}^+ \mathscr{M}_c(ABC))_a^b \leq \sum_{d,e} M_c(a,b)[A_a^d][B_d^e][C_e^b]e^{\gamma'|a-b|} \leq \sum_{|d|\geq|e|} \cdots + \sum_{|e|\geq|d|} \cdots .$$

We shall only consider the first of these sums – the second one being analogous. We decompose this sum as

$$\sum_{\substack{(a,d)\in D_{\Lambda+3}^+(c)\\(d,e)\in D_\Lambda^+(c)}} \cdots + \sum_{\substack{(a,d)\in D_{\Lambda+3}^+(c)\\(d,e)\notin D_\Lambda^+(c)}} \cdots + \sum_{(a,d)\notin D_{\Lambda+3}^+(c)} \cdots = (I) + (II) + (III).$$

In the domain of (I) we have, by Corollary 2.2(i), that $M_c(d,e) \approx M_c(a,b)$, so (I) is

$$\lesssim \left|\mathscr{E}_{\gamma_1}^+ A\right|_{\mathscr{L}\times\mathscr{L}} \left|\mathscr{E}_{\gamma_2}^+ \mathscr{M}_c B\right|_{D_\Lambda^+(c)} \left|\mathscr{E}_{\gamma_3}^+ C\right|_{\mathscr{L}\times\mathscr{L}}$$
$$\times \sum_{d,e} e^{-(\gamma_1-\gamma')|a-d|-(\gamma_2-\gamma')|d-e|-(\gamma_3-\gamma')|e-b|}.$$

Since two of $\gamma_1 - \gamma'$, $\gamma_2 - \gamma'$ and $\gamma_3 - \gamma'$ are $\gamma - \gamma'$ the sum is $\lesssim (\frac{1}{\gamma-\gamma'})^{2d}$.

By Corollary 2.2(iv) we have, in the domain of (II),

$$|a-d|, \ |d-e| \gtrsim \frac{1}{\Lambda^2} \frac{|a|}{|c|}.$$

and, in the domain of (III),

$$|a-d|, \ |d-b| \gtrsim \frac{1}{\Lambda^2} \frac{|a|}{|c|}.$$

Hence in both these domains we have

$$s(d,e) = \max(|a-d|, |d-e|, |e-b|) \gtrsim \frac{1}{\Lambda^2} \frac{|a|}{|c|},$$

so (II) + (III) is

$$\lesssim \big|\mathscr{E}^+_{\gamma_1} A\big|_{\mathscr{L}\times\mathscr{L}} \big|\mathscr{E}^+_{\gamma_2} B\big|_{\mathscr{L}\times\mathscr{L}} \big|\mathscr{E}^+_{\gamma_3} C\big|_{\mathscr{L}\times\mathscr{L}}$$
$$\times \sum_{s(d,e)\gtrsim \frac{1}{\Lambda^2}\frac{|a|}{|c|}} \Big(\frac{|a|}{|c|}+1\Big) e^{-(\gamma_1-\gamma')|a-d|-(\gamma_2-\gamma')|d-e|-(\gamma_3-\gamma')|e-b|}.$$

Since two of $\gamma_1 - \gamma'$, $\gamma_2 - \gamma'$ and $\gamma_3 - \gamma'$ are $\gamma - \gamma'$ the sum is $\lesssim \Lambda^2 (\frac{1}{\gamma-\gamma'})^{2d+1}$.

The seven other cases of (ii) are treated in the same way, as well as the case when the factors A, B and C are permuted. $\qquad\square$

We give a more compact and slightly weaker formulation of this result.

COROLLARY 2.4. *For all $x, y \in \{+, -\}$, all $\gamma' < \gamma$ and any $c \neq 0$*

(i)
$$\Big|\mathscr{E}^{xy}_{\gamma'} \mathcal{M}_c(AB)\Big|_{D^{xy}_{\Lambda+3}(c)} \lesssim \Lambda^2 \Big(\frac{1}{\gamma-\gamma'}\Big)^{d+1} \Big[\big|\mathscr{E}^x_{\gamma_1} A\big|_{\mathscr{L}\times\mathscr{L}}$$
$$+ \big|\mathscr{E}^x_{\gamma_1} \mathcal{M}_c(A)\big|_{D^x_\Lambda(c)} \Big] \big|\mathscr{E}^y_{\gamma_2} B\big|_{\mathscr{L}\times\mathscr{L}},$$

where one of γ_1, γ_2 is $= \gamma$ and the other one is $= \gamma'$. The same bound holds for BA.

(ii)
$$\Big|\mathscr{E}^{xyz}_{\gamma'} \mathcal{M}_c(ABC)\Big|_{D^{xyz}_{\Lambda+6}(c)} \lesssim \Lambda^2 \Big(\frac{1}{\gamma-\gamma'}\Big)^{2d+1} \big|\mathscr{E}^x_{\gamma_1} A\big|_{\mathscr{L}\times\mathscr{L}} \big[\big|\mathscr{E}^y_{\gamma_2} \mathcal{M}_c(B)\big|_{D^y_\Lambda(c)}$$
$$+ \big|\mathscr{E}^y_{\gamma_2} B\big|_{\mathscr{L}\times\mathscr{L}}\big] \big|\mathscr{E}^z_{\gamma_3} C\big|_{\mathscr{L}\times\mathscr{L}},$$

where two of $\gamma_1, \gamma_2, \gamma_3$ are $= \gamma$ and the third one is $= \gamma'$. The same bound holds when the factors A, B and C are permuted.

Multiplication. Using relations (1) and (7) we obtain from Corollary 2.4(i) that a product of two Töplitz-Lipschitz matrices is again Töplitz-Lipschitz and for all $\gamma' < \gamma$

(10) $$\langle AB\rangle_{\Lambda+3,\gamma'} \lesssim \Lambda^2 \Big(\frac{1}{\gamma-\gamma'}\Big)^{d+1} \big[\langle A\rangle_{\Lambda,\gamma_1} |B|_{\gamma_2} + |A|_{\gamma_1} \langle B\rangle_{\Lambda,\gamma_2}\big],$$

where one of γ_1, γ_2 is $= \gamma$ and the other one is $= \gamma'$.

This formula cannot be iterated without consecutive loss of the Lipschitz domain. However Corollary 2.4(ii) together with (5) gives for all $\gamma' < \gamma$

$$(11) \quad \langle A_1 \cdots A_n \rangle_{\Lambda+6,\gamma'}$$
$$\leq (\text{cte.})^n \Lambda^2 \Big(\frac{1}{\gamma-\gamma'}\Big)^{(n-1)d+1} \Big[\sum_{1 \leq k \leq n} \prod_{\substack{1 \leq j \leq n \\ j \neq k}} |A_j|_{\gamma_j} \langle A_k \rangle_{\Lambda,\gamma_k} \Big],$$

where all $\gamma_1, \ldots, \gamma_n$ are $= \gamma$ except one which is $= \gamma'$.

Linear differential equation. Consider the linear system

$$\begin{cases} \frac{d}{dt} X = A(t)X \\ X(0) = I, \end{cases}$$

where $A(t)$ is Töplitz-Lipschitz with exponential decay. The solution verifies

$$X(t_0) = I + \sum_{n=1}^{\infty} \int_0^{t_0} \int_0^{t_1} \ldots \int_0^{t_{n-1}} A(t_1)A(t_2)\ldots A(t_n)dt_n \ldots dt_2 dt_1.$$

Using (11) we get for $\gamma' < \gamma$

$$(12) \qquad \langle X(t) - I \rangle_{\Lambda+6,\gamma'}$$
$$\lesssim \Lambda^2 \Big(\frac{1}{\gamma-\gamma'}\Big)|t| \exp\Big(\text{cte.}\Big(\frac{1}{\gamma-\gamma'}\Big)^d |t|\alpha(t)\Big) \sup_{|s| \leq |t|} \langle A(s) \rangle_{\Lambda,\gamma},$$

where $\alpha(t) = \sup_{0 \leq |s| \leq |t|} |A(s)|_\gamma$.

2.4. *Töplitz-Lipschitz matrices* ($d \geq 2$). Let $A : \mathscr{L} \times \mathscr{L} \to \text{gl}(2, \mathbb{C})$ be a matrix. We say that A is 1-Töplitz if all Töplitz limits $A(\pm, c)$ exist, and we define, inductively, that A is n-Töplitz if all Töplitz limits $A(\pm, c)$ are $(n-1)$-Töplitz. We say that A is *Töplitz* if it is $(d-1)$-Töplitz.

In Section 2.3 we have defined $\langle A \rangle_{\Lambda,\gamma}$ which we shall now denote by $^1\langle A \rangle_{\Lambda,\gamma}$. We define, inductively,

$$^n\langle A \rangle_{\Lambda,\gamma} = \sup_{c \in \mathbb{Z}^d} \Big({}^{n-1}\langle A(+,c) \rangle_{\Lambda,\gamma}, {}^{n-1}\langle A(-,c) \rangle_{\Lambda,\gamma} \Big)$$

($c = 0$ is allowed and $A(\pm, 0) = A$) and we denote $\langle A \rangle_{\Lambda,\gamma} = {}^{d-1}\langle A \rangle_{\Lambda,\gamma}$. The matrix A is *Töplitz-Lipschitz* if it is Töplitz at ∞ and $\langle A \rangle_{\Lambda,\gamma} < \infty$ for some Λ, γ.

Proposition 2.3, Corollary 2.4 and (9)–(12) remain valid with this norm in any dimension d.

3. Functions with Töplitz-Lipschitz property

3.1. *Töplitz-Lipschitz property.* Let $\mathbb{O}^\gamma(\sigma)$ be the set of vectors in the complex space $l_\gamma^2(\mathscr{L}, \mathbb{C}^2)$ of norm less than σ, i.e.,

$$\mathbb{O}^\gamma(\sigma) = \{\zeta \in \mathbb{C}^{\mathscr{L}} \times \mathbb{C}^{\mathscr{L}} : \|\zeta\|_\gamma < \sigma\}.$$

Our functions $f : \mathbb{O}^0(\sigma) \to \mathbb{C}$ will be defined and real analytic on the domain $\mathbb{O}^0(\sigma)$.[3]

Its first differential

$$l_0^2(\mathscr{L}, \mathbb{C}^2) \ni \hat{\zeta} \mapsto \langle \hat{\zeta}, \partial_\zeta f(\zeta) \rangle$$

defines a unique vector $\partial_\zeta f(\zeta)$ (the gradient with respect to the paring $\langle \, , \, \rangle$), and its second differential

$$l_0^2(\mathscr{L}, \mathbb{C}^2) \ni \hat{\zeta} \mapsto \langle \hat{\zeta}, \partial_\zeta^2 f(\zeta) \hat{\zeta} \rangle$$

defines a unique symmetric matrix $\partial_\zeta^2 f(\zeta) : \mathscr{L} \times \mathscr{L} \to \mathrm{gl}(2, \mathbb{C})$ (the Hessian with respect to the paring $\langle \, , \, \rangle$). A matrix $A : \mathscr{L} \times \mathscr{L} \to \mathrm{gl}(2, \mathbb{C})$ is symmetric if $^t A_a^b = A_b^a$.

We say that f is *Töplitz at* ∞ if the matrix $\partial_\zeta^2 f(\zeta)$ is Töplitz at ∞ for all $\zeta \in \mathbb{O}^0(\sigma)$. We define the norm $[f]_{\Lambda, \gamma, \sigma}$ to be the smallest C such that

$$\begin{cases} |f(\zeta)| \le C & \forall \zeta \in \mathbb{O}^0(\sigma), \\ \|\partial_\zeta f(\zeta)\|_{\gamma'} \le \frac{1}{\sigma} C & \forall \zeta \in \mathbb{O}^{\gamma'}(\sigma), \ \forall \gamma' \le \gamma, \\ \langle \partial_\zeta^2 f(\zeta) \rangle_{\Lambda, \gamma'} \le \frac{1}{\sigma^2} C & \forall \zeta \in \mathbb{O}^{\gamma'}(\sigma), \ \forall \gamma' \le \gamma. \end{cases}$$

PROPOSITION 3.1.   (i) $[fg]_{\Lambda, \gamma, \sigma} \lesssim [f]_{\Lambda, \gamma, \sigma} [g]_{\Lambda, \gamma, \sigma}$.

(ii) *If* $g(\zeta) = \langle c, \partial_\zeta f(\zeta) \rangle$, *then* $[g]_{\Lambda, \gamma, \sigma'} \lesssim \frac{1}{\sigma - \sigma'} \|c\|_\gamma [f]_{\Lambda, \gamma, \sigma}$ *for* $\sigma' < \sigma$.

(iii) *If* $g(\zeta) = \langle C\zeta, \partial_\zeta f(\zeta) \rangle$, *then, for* $\sigma' < \sigma$ *and* $\gamma' < \gamma$,

$$[g]_{\Lambda + 3, \gamma', \sigma'} \lesssim \left(\left(1 + \frac{\sigma'}{\sigma - \sigma'}\right) \left(\frac{1}{\gamma - \gamma'}\right)^{d + m_*} |C|_\gamma \right.$$
$$\left. + \Lambda^2 \left(\frac{1}{\gamma - \gamma'}\right)^{d + 1} \langle C \rangle_{\Lambda, \gamma} \right) [f]_{\Lambda, \gamma, \sigma}.$$

Proof. We have

$$fg(\zeta) = f(\zeta) g(\zeta),$$
$$\partial_\zeta fg(\zeta) = f(\zeta) \partial_\zeta g(\zeta) + \partial_\zeta f(\zeta) g(\zeta),$$
$$\partial_\zeta^2 fg(\zeta) = f(\zeta) \partial_\zeta^2 g(\zeta) + \partial_\zeta^2 f(\zeta) g(\zeta) + 2(\partial_\zeta f(\zeta) \otimes \partial_\zeta g(\zeta)).$$

(i) now follows from (9).

For $\zeta \in \mathbb{O}^0(\sigma')$ we have

$$|g(\zeta)| \le \|c\|_0 \|\partial_\zeta f(\zeta)\|_0 \le \|c\|_0 \frac{1}{\sigma} \alpha,$$

where $\alpha = [f]_{\Lambda, \gamma, \sigma}$.

---

[3] The space $l_\gamma^2(\mathscr{L}, \mathbb{C}^2)$ is the complexification of the space $l_\gamma^2(\mathscr{L}, \mathbb{R})$ of real sequences. Now, "real analytic" means that $l_\gamma^2(\mathscr{L}, \mathbb{C}^2)$ is a holomorphic function which is real on $\mathbb{O}^0(\sigma) \cap l_\gamma^2(\mathscr{L}, \mathbb{R})$.

Let $\zeta \in \mathbb{O}^{\gamma'}(\sigma')$ and $h(z) = \partial_{\xi} f(\zeta + zc)$. Then, h is a holomorphic function (with values in the Hilbert-space $l^2_{\gamma'}(\mathcal{L}, \mathbb{C}^2)$) in the disk $|z| < \frac{\sigma-\sigma'}{\|c\|_{\gamma'}}$ and

$$\|h(z)\|_{\gamma'} \le \frac{1}{\sigma}\alpha.$$

Since $\partial_{\xi} g(\zeta) = \partial_z h(0)$, we get by a Cauchy estimate that

$$\left\|\partial_{\xi} g(\zeta)\right\|_{\gamma'} \le \frac{1}{\sigma'}\left(\frac{\sigma'}{\sigma}\frac{1}{\sigma-\sigma'}\|c\|_{\gamma'}\,\alpha\right).$$

Let $\zeta \in \mathbb{O}^{\gamma'}(\sigma')$ and $k(z) = \partial_{\xi}^2 f(\zeta + zc)$. Now, k is a holomorphic function (with values in the Banach-space of matrices with the norm $\langle \cdot \rangle_{\Lambda,\gamma'}$) in the disk $|z| < \frac{\sigma-\sigma'}{\|c\|_{\gamma'}}$ and $\langle k(z) \rangle_{\Lambda,\gamma'} \le \frac{1}{\sigma^2}\alpha$. Since $\partial_{\xi}^2 g(\zeta) = \partial_{\xi} k(0)$, we get by a Cauchy estimate that

$$\langle \partial_{\xi} g(\zeta) \rangle_{\Lambda,\gamma'} \le \left(\frac{1}{\sigma'}\right)^2\left(\left(\frac{\sigma'}{\sigma}\right)^2\frac{1}{\sigma-\sigma'}\|c\|_{\gamma'}\,\alpha\right).$$

This proves (ii).

To see (iii) we replace c by $C\zeta$ and notice that

$$\partial_{\xi} g(\zeta) = \partial_z h(0) + {}^tC\,\partial_{\xi} f(\zeta)$$

and

$$\partial_{\xi}^2 g(\zeta) = \partial_z k(0) + {}^tC\,\partial_{\xi}^2 f(\zeta) + {}^t\partial_{\xi}^2 f(\zeta)C.$$

Also, $\partial_z h(0)$ and $\partial_z k(0)$ are estimated as above and $\|C\zeta\|_{\gamma'}$ is estimated with Young's inequality (2). The matrix products are estimated by (10). □

3.2. *Truncations.* Let Tf be the Taylor polynomial of order 2 of f at $\zeta = 0$.

PROPOSITION 3.2.   (i) $[Tf]_{\Lambda,\gamma,\sigma} \lesssim [f]_{\Lambda,\gamma,\sigma}$.

(ii) $[f - Tf]_{\Lambda,\gamma,\sigma'} \lesssim (\frac{\sigma'}{\sigma})^3\frac{\sigma}{\sigma-\sigma'}[f]_{\Lambda,\gamma,\sigma}$.

Proof. Let $\zeta \in \mathbb{O}^0(\sigma')$ and let $g(z) = f(z\zeta)$. Then g is a real holomorphic function in the disk of radius $\frac{\sigma}{\sigma'}$ and is bounded by $\alpha = [f]_{\Lambda,\gamma,\sigma}$. Since $Tf(z\zeta) = g(0) + g'(0)z + \frac{1}{2}g''(0)z^2$ we get by a Cauchy estimate that

$$|(f - Tf)'\zeta)| = \left|g(1) - g(0) - g'(0) - \frac{1}{2}g''(0)\right| \le \left(\frac{\sigma'}{\sigma}\right)^3\frac{\sigma}{\sigma-\sigma'}\alpha.$$

Let $\zeta \in \mathbb{O}^{\gamma'}(\sigma')$ and let $h(z) = \partial_{\xi} f(z\zeta)$. Then h is a holomorphic function in the disk of radius $\frac{\sigma}{\sigma'}$ and bounded by $\frac{\alpha}{\sigma}$. Since $\partial_{\xi} Tf(\zeta) = h(0) + h'(0)z$ we get by a Cauchy estimate that

$$\left\|\partial_{\xi}(f - Tf)(\zeta)\right\|_{\gamma'} \le \left(\frac{\sigma'}{\sigma}\right)^2\frac{\sigma}{\sigma-\sigma'}\frac{\alpha}{\sigma}.$$

Let $\zeta \in \mathbb{O}^{\gamma'}(\sigma')$ and let $k(z) = \partial_{\xi}^2 f(z\zeta)$. Then k is a holomorphic function in the disk of radius $\frac{\sigma}{\sigma'}$ and bounded by $\frac{\alpha}{\sigma^2}$. Since $\partial_{\xi}^2 Tf(\zeta) = k(0)$ we get by a

Cauchy estimate that

$$\langle \partial_\zeta^2 (f - Tf)(\zeta) \rangle_{\Lambda,\gamma'} \le \left(\frac{\sigma'}{\sigma}\right) \frac{\sigma}{\sigma - \sigma'} \frac{\alpha}{\sigma^2}.$$

This gives (ii).

The first statement is obtained by taking $\sigma' = \frac{1}{2}\sigma$. Since f is a quadratic polynomial it satisfies the same (modulo a constant) estimate on σ as on $\frac{1}{2}\sigma$. □

3.3. *Poisson brackets.* The Poisson bracket of two functions f and g is defined by $\{f, g\}(\zeta) = \langle \partial_\zeta f(\zeta), J \partial_\zeta g(\zeta) \rangle$.

PROPOSITION 3.3.   (i) *If g is a quadratic polynomial, then*

$$[\{f, g\}]_{\Lambda+3,\gamma',\sigma'} \lesssim \left[\frac{1}{\sigma_1 \sigma_2} + \Lambda^2 \left(\frac{1}{\gamma - \gamma'}\right)^{d+1} \left(\frac{\sigma'}{\sigma_1 \sigma_2}\right)^2 \right] [f]_{\Lambda,\gamma,\sigma_1} [g]_{\Lambda,\gamma,\sigma_2},$$

for $0 < \sigma_1 - \sigma' \approx \sigma_1$, $0 < \sigma_2 - \sigma' \approx \sigma_2$ and $\gamma' < \gamma$.

(ii) *If g is a quadratic polynomial and $f(\zeta) = \langle \zeta, A\zeta \rangle$, then*

$$[\{f, g\}]_{\Lambda+3,\gamma',\sigma'}$$
$$\lesssim \left[\left(\frac{1}{\gamma - \gamma'}\right)^{d+m_*} \frac{1}{\sigma_1^2} + \Lambda^2 \left(\frac{1}{\gamma - \gamma'}\right)^{d+1} \frac{1}{\sigma_1^2} \right] [f]_{\Lambda,\gamma,\sigma_1} [g]_{\Lambda,\gamma,\sigma_2},$$

for $0 < \sigma_1 - \sigma' \approx \sigma_1$, $0 < \sigma_2 - \sigma' \approx \sigma_2$ and $\gamma' < \gamma$.

Proof. We have

$$\partial_\zeta \{f, g\}(\zeta) = \partial_\zeta^2 f(\zeta) J \partial_\zeta g(\zeta) - \partial_\zeta^2 g(\zeta) J \partial_\zeta f(\zeta)$$

and $\partial_\zeta^2 \{f, g\}(\zeta)$ is the symmetrization of the infinite matrix

$$\partial_\zeta^3 f(\zeta) J \partial_\zeta g(\zeta) - \partial_\zeta^3 g(\zeta) J \partial_\zeta f(\zeta) + \partial_\zeta^2 f(\zeta) J \partial_\zeta^2 g(\zeta) + \partial_\zeta^2 f(\zeta) J \partial_\zeta^2 g(\zeta).$$

For $\zeta \in \mathbb{O}^0(\sigma')$ we get, by Cauchy-Schwartz, that

$$|\{f, g\}(\zeta)| \le \left\| \partial_\zeta f(\zeta) \right\|_0 \left\| \partial_\zeta g(\zeta) \right\|_0 \le \left(\frac{\alpha\beta}{\sigma_1 \sigma_2}\right),$$

where $\alpha = [f]_{\Lambda,\gamma,\sigma_1}$ and $\beta = [g]_{\Lambda,\gamma,\sigma_2}$.

For $\zeta \in \mathbb{O}^{\gamma'}(\sigma')$, let $h(z) = \partial_\zeta f(\zeta + zJ\partial_\zeta g(\zeta))$. For $|z| < \frac{\sigma_1 - \sigma'}{\|\partial_\zeta g(\zeta)\|_{\gamma'}}$ we have

$$\|h(z)\|_{\gamma'} \le \frac{\alpha}{\sigma_1}.$$

Since $\partial_z h(0) = \partial_\zeta^2 f(\zeta) J \partial_\zeta g(\zeta)$ and $\sigma_1 - \sigma' \approx \sigma_1$, we get by a Cauchy estimate that

$$\left\| \partial_\zeta^2 f(\zeta) J \partial_\zeta g(\zeta) \right\|_{\gamma'} \lesssim \frac{1}{\sigma_1^2 \sigma_2} \alpha\beta.$$

The same estimate holds with f and g interchanged.

For $\zeta \in \mathbb{O}^{\gamma'}(\sigma')$, let $k(z) = \partial_\zeta^2 f(\zeta + zJ\partial_\zeta g(\zeta))$. By a Cauchy estimate we get as above that $\langle \partial_\zeta^3 f(\zeta) J \partial_\zeta g(\zeta) \rangle_{\Lambda, \gamma'} \lesssim \frac{1}{\sigma_1^3 \sigma_2} \alpha \beta$. The same estimate holds with f and g interchanged.

Finally, for $\zeta \in \mathbb{O}^{\gamma'}(\sigma')$ we get by (10) that

$$\langle \partial_\zeta^2 f(\zeta) J \partial_\zeta^2 g(\zeta) \rangle_{\Lambda+3, \gamma'} \lesssim \Lambda^2 (\gamma - \gamma')^{-d-1} \langle \partial_\zeta^2 f(\zeta) \rangle_{\Lambda, \gamma'} \langle \partial_\zeta^2 g(\zeta) \rangle_{\Lambda, \gamma}.$$

By hypothesis we have $\langle \partial_\zeta^2 g(\zeta) \rangle_{\Lambda, \gamma} \le \frac{\beta}{\sigma_2^2}$ for ζ only in $\mathbb{O}^{\gamma}(\sigma')$. But since g is quadratic, $\partial_\zeta^2 g(\zeta)$ is independent of ζ and, hence, this also holds in the larger domain $\zeta \in \mathbb{O}^{\gamma'}(\sigma')$. The symmetrized matrices satisfy the same estimates, and (i) is established.

The second part follows directly from Proposition 3.1(iii). $\qquad \square$

3.4. *The flow map.* Consider the linear system

$$\dot{\zeta} = J \partial_\zeta f_t(\zeta)$$

where $f_t(\zeta) = \langle \zeta, a_t \rangle + \frac{1}{2} \langle \zeta, A_t \zeta \rangle$, and let $\alpha(t) = \sup_{|s| \le |t|} |A_s|_\gamma$ and $\beta(t) = \sup_{|s| \le |t|} \|a_s\|_{\gamma'}$. Consider the nonlinear system $\dot{z} = g(\zeta, z)$ where $g(\zeta, z)$ is real analytic in $\mathbb{O}^0(\sigma) \times \mathbb{D}(\mu)$. $\mathbb{D}(\mu)$ is the disk of radius μ in $\mathbb{C}$. Let $0 < \mu' < \mu$.

PROPOSITION 3.4. (i) *The flow map of the linear system has the form*

$$\zeta_t : \zeta \mapsto \zeta + b_t + B_t \zeta,$$

and for $\gamma' < \gamma$

$$\|\zeta_t(\zeta) - \zeta\|_{\gamma'} \lesssim \left(\frac{1}{\gamma - \gamma'}\right)^{m*} \left[e^{\text{cte.}(\frac{1}{\gamma-\gamma'})^d |t| \alpha(t)} |t| \beta(t) \right.$$

$$\left. + \left[e^{\text{cte.}(\frac{1}{\gamma-\gamma'})^d |t| \alpha(t)} - 1 \right] \|\zeta\|_{\gamma'} \right]$$

and

$$\langle B_t \rangle_{\Lambda+6, \gamma'} \lesssim \Lambda^2 \left(\frac{1}{\gamma - \gamma'}\right) |t| e^{\text{cte.}(\frac{1}{\gamma-\gamma'})^d |t| \alpha(t)} \sup_{|s| \le |t|} \langle A_s \rangle_{\Lambda, \gamma}.$$

(ii) *For $|z| < \mu'$, the flow of the nonlinear system is defined for $|t| \le \frac{\mu - \mu'}{2\varepsilon}$ and*

$$[z_t(\cdot, z) - z]_{\Lambda, \gamma, \sigma} \lesssim \left(1 + \frac{\mu - \mu'}{\varepsilon} \left(e^{\text{cte.}|t| \frac{1}{\mu-\mu'} \varepsilon} - 1 \right) \right)^2 \varepsilon,$$

where $\varepsilon = \sup_{z \in \mathbb{D}(\mu)} [g(\cdot, z)]_{\Lambda, \gamma, \sigma} \le 1$.

Proof. (i) We have

$$b_t = \sum_{n=1}^{\infty} \int_0^t \dots \int_0^{t_{n-1}} JA_{t_1} \dots JA_{t_{n-1}} Ja_{t_n} dt_n dt_{n-1} \dots dt_1$$

and

$$B_t = \sum_{n=1}^{\infty} \int_0^t \dots \int_0^{t_{n-1}} J A_{t_1} \dots J A_{t_n} dt_n \dots dt_1.$$

By (5),

$$|B_t|_{\gamma'} \lesssim (\gamma - \gamma')^d (\delta(t) - 1), \quad \delta(t) = \exp(\text{cte.}(\gamma - \gamma')^{-d} |t| \alpha(t))$$

and by (2) we have

$$\|B_t \zeta\|_{\gamma'} \lesssim \left(\frac{1}{\gamma - \gamma'}\right)^{m_*} (\delta(t) - 1) \|\zeta\|_{\gamma'}.$$

By (2) and (5),

$$\|b_t\|_{\gamma'} \lesssim \left(\frac{1}{\gamma - \gamma'}\right)^{m_*} \delta(t) |t| \beta(t).$$

By (12) we have

$$\langle B_t \rangle_{\Lambda+6,\gamma'} \lesssim \Lambda^2 (\gamma - \gamma')^{-1} \delta(t) \sup_{|s| \leq |t|} \langle A_s \rangle_{\Lambda,\gamma}.$$

The proof of (ii) is easier. Now, $\partial_\zeta \dot{z}_t = \partial_\zeta g(\dots) + \partial_z g(\dots) \partial_\zeta z_t$ which implies that

$$\partial_\zeta z_t = \int_0^t e^{\int_s^t \partial_z g(\zeta, z_r) dr} \partial_\zeta g(\zeta, z_s) ds.$$

This is easy to estimate. We also have

$$\partial_\zeta^2 \dot{z}_t = \partial_\zeta^2 g(\dots) + \partial_z \partial_\zeta g(\dots) \otimes \partial_\zeta z_t + \partial_z g(\dots) \partial_\zeta^2 z_t$$

which is treated in the same way. $\square$

Remark. This result holds for $z = (z_1, \dots, z_n) \in \mathbb{D}(\mu)^n$ and $g = (g_1, \dots, g_n)$.

Remark. If $|t| \leq 1$ and $\sup_{|s| \leq |t|} |A_s|_\gamma \lesssim (\gamma - \gamma')^d$, then

$$\|\zeta_t(\zeta) - \zeta\|_{\gamma'} \lesssim \left(\frac{1}{\gamma - \gamma'}\right)^{m_*} \sup_{|s| \leq |t|} \|a_s\|_{\gamma'} + \left(\frac{1}{\gamma - \gamma'}\right)^{m_*+d} \sup_{|s| \leq |t|} |A_s|_\gamma \|\zeta\|_{\gamma'}$$

and

$$\langle B_t \rangle_{\Lambda+6,\gamma'} \lesssim \Lambda^2 \left(\frac{1}{\gamma - \gamma'}\right) \sup_{|s| \leq |t|} \langle A_s \rangle_{\Lambda,\gamma}.$$

If $|t| \leq 1$ and $\varepsilon = \sup_{z \in \mathbb{D}(\mu)} [g(\cdot, z)]_{\Lambda,\gamma,\sigma} \lesssim \mu - \mu'$, then $[z_t(\cdot, z) - z]_{\Lambda,\gamma,\sigma} \lesssim \varepsilon$.

3.5. *Compositions.* Let $f(\zeta, z)$ be a real analytic function on $\mathbb{O}^0(\sigma) \times \mathbb{D}(\mu)$ and $\sup_{z \in \mathbb{D}(\mu)} [f(\cdot, z)]_{\Lambda,\gamma,\sigma} < \infty$. Let $0 < \sigma' < \sigma$, $0 < \mu' < \mu$ and $\Phi(\zeta, z) = \zeta + b(z) + B(z)\zeta$ with

$$\|b(z) + B(z)\zeta\|_{\gamma'} < \sigma - \sigma', \quad \forall (\zeta, z) \in \mathbb{O}^{\gamma'}(\sigma') \times \mathbb{D}(\mu')$$

for all $\gamma' \leq \gamma$. This implies that

$$\Phi(\cdot, z) : \mathbb{O}^{\gamma'}(\sigma') \to \mathbb{O}^{\gamma'}(\sigma), \quad \forall \gamma' \leq \gamma, \quad \forall z \in \mathbb{D}(\mu').$$

Let $g(\zeta, z)$ be a real holomorphic function on $\mathbb{O}^0(\sigma') \times \mathbb{D}(\mu')$ such that $|g| \leq \frac{1}{2}(\mu - \mu')$.

PROPOSITION 3.5. *For all $z \in \mathbb{D}(\mu')$ and $\gamma' < \gamma$,*

$$[f(\Phi(\cdot, z), z + g(\cdot, z))]_{\Lambda+6,\gamma',\sigma'}$$
$$\lesssim \max\left(1, \alpha, \Lambda^2\left(\frac{1}{\gamma-\gamma'}\right)\alpha^2\right) \sup_{z\in\mathbb{D}(\mu)} [f(\cdot, z)]_{\Lambda,\gamma,\sigma},$$

where

$$\alpha = \frac{1}{\mu-\mu'} \sup_{z\in\mathbb{D}(\mu)} [g(\cdot, z)]_{\Lambda,\gamma,\sigma'} + \left(\frac{1}{\gamma-\gamma'}\right)^{d+m_*} \sup_{z\in\mathbb{D}(\mu)} \langle B\rangle_{\Lambda,\gamma}.$$

Proof. Let $\varepsilon = \sup_{z\in\mathbb{D}(\mu)}[f(\cdot, z)]_{\Lambda,\gamma,\sigma}$ and $\beta = \sup_{z\in\mathbb{D}(\mu')}[g(\cdot, z)]_{\Lambda,\gamma,\sigma'}$. Let $h(\zeta, z) = f(\Phi(\zeta, z), z + g(\zeta, z))$. Then

$$\partial_\zeta h = \partial_z f(\ldots)\partial_\zeta g + {}^t B \partial_\zeta f(\ldots)$$

and

$$\partial_\zeta^2 h = \partial_z^2 f(\ldots)(\partial_\zeta g \otimes \partial_\zeta g) + \partial_z f(\ldots)\partial_\zeta^2 g$$
$$+ 2\,{}^t B(\partial_\zeta \partial_z f(\ldots) \otimes \partial_\zeta g) + {}^t B \partial_\zeta^2 f(\ldots)B.$$

For $(\zeta, z) \in \mathbb{O}^0(\sigma') \times \mathbb{D}(\mu')$ we get: $|h(\zeta)| \leq \varepsilon$. For $(\zeta, z) \in \mathbb{O}^{\gamma'}(\sigma') \times \mathbb{D}(\mu')$,

$$\left\|\partial_z f(\ldots)\partial_\zeta g\right\|_{\gamma'} |\partial_z f(\ldots)| \left\|\partial_\zeta g\right\|_{\gamma'} \lesssim \left(\frac{1}{\mu-\mu'}\right)\varepsilon\frac{\beta}{\sigma'};$$
$$\left\|{}^t B \partial_\zeta f(\ldots)\right\|_{\gamma'} \lesssim \left(\frac{1}{\gamma-\gamma'}\right)^{d+m_*} |B|_\gamma \frac{\varepsilon}{\sigma}$$

by Young's inequality (2).

For $(\zeta, z) \in \mathbb{O}^{\gamma'}(\sigma') \times \mathbb{D}(\mu')$,

$$\langle\partial_z^2 f(\ldots)\partial_\zeta g \otimes \partial_\zeta g\rangle_{\Lambda,\gamma'} \lesssim \left(\frac{1}{\mu-\mu'}\right)^2\varepsilon\left(\frac{\beta}{\sigma'}\right)^2$$

by (9);

$$\langle\partial_z f(\ldots)\partial_\zeta^2 g\rangle_{\Lambda,\gamma'} \lesssim \left(\frac{1}{\mu-\mu'}\right)\varepsilon\left(\frac{\beta}{(\sigma')^2}\right);$$
$$\langle{}^t B(\partial_\zeta \partial_z f(\ldots) \otimes \partial_\zeta g)\rangle_{\Lambda+3,\gamma'} \lesssim \Lambda^2\left(\frac{1}{\gamma-\gamma'}\right)^{d+1}\langle B\rangle_{\Lambda,\gamma}\left(\frac{1}{\mu-\mu'}\right)\varepsilon\frac{\beta}{\sigma\sigma'}$$

by (9)–(10);

$$\langle{}^t B \partial_\zeta^2 f(\ldots)B\rangle_{\Lambda+6,\gamma'} \lesssim \Lambda^2\left(\frac{1}{\gamma-\gamma'}\right)^{2d+1}\langle B\rangle_{\Lambda,\gamma}^2\frac{\varepsilon}{\sigma^2}$$

by (11). □

Remark. This result holds for $z = (z_1, \ldots, z_n) \in \mathbb{D}(\mu)^n$ and $g = (g_1, \ldots, g_n)$.

Part II. The homological equations

In this part we consider scalar-valued matrices $Q : \mathscr{L} \times \mathscr{L} \to \mathbb{C}$ which we identify with $\mathrm{gl}(2, \mathbb{C})$-valued matrices through the identification $Q_a^b = Q_a^b I$.

We will only consider the Lipschitz domains $D_\Lambda^+(c)$ which we denote by $D_\Lambda(c)$.

We define the block decomposition $\mathscr{E}_\Delta$ together with the blocks $[\cdot]_\Delta$ and the bound d_Δ of the block diameter. We consider parameters $U \subset \mathbb{R}^{\mathscr{A}}$, $\mathscr{A} = \mathbb{Z}^d \setminus \mathscr{L}$, and define the norms $| \cdot |_{\{{\gamma \atop U}\}}$ and $\langle \cdot \rangle_{\{{\Lambda, \gamma \atop U}\}}$.

4. Decomposition of $\mathscr{L}$

In this section $d \geq 2$. For a nonnegative integer Δ we define an *equivalence relation* on $\mathscr{L}$ generated by the pre-equivalence relation

$$a \sim b \iff \begin{cases} |a|^2 = |b|^2 \\ |a - b| \leq \Delta. \end{cases}$$

Let $[a]_\Delta$ denote the equivalence class (*block*) of a, and let $\mathscr{E}_\Delta$ be the set of equivalence classes. It is trivial that each block $[a]_\Delta$ is finite with cardinality $\lesssim |a|^{d-1}$ that depends on a. But there is also a uniform Δ-dependent bound. Indeed, let d_Δ be the supremum of all block diameters. We will see (Proposition 4.1) $d_\Delta \lesssim \Delta^{\frac{(d+1)!}{2}}$.

Δ will be fixed in this section and we will write $[\cdot]$ for $[\cdot]_\Delta$.

4.1. Blocks.
For any $X \subset \mathbb{Z}^d$ we define its *rank* to be the dimension of the smallest affine subspace in $\mathbb{R}^d$ containing X.

PROPOSITION 4.1. *Let $c \in \mathbb{Z}^d$ and $\mathrm{rank}[c] = k$, $k = 1, \ldots, d$. Then the diameter of $[c]$ is $\lesssim \Delta^{\frac{(k+1)!}{2}}$.*

Proof. Let Δ_j, $j \geq 1$ be an increasing sequence of numbers. Assume that for any $1 \leq l \leq k$,

$$(*)_l \qquad\qquad \mathrm{rank}(B_{\Delta_l}(c) \cap [c]) \geq l \quad \forall c \in [c],$$

where $B_r(c)$ is the ball of radius r centered at c. This means that for any $c \in [c]$, there exist linearly independent vectors $a_1, \ldots, a_l$ in $\mathbb{Z}^d$ such that $c + a_j \in [c]$ and $|a_j| \leq \Delta_l$, $1 \leq j \leq l$.

$(*)_l$ implies that the $\perp$ projection $\tilde{c}$ of c onto $\sum \mathbb{R} a_j$ verifies

$$(**) \qquad\qquad |\tilde{c}| \lesssim \begin{cases} \Delta_l & l = 1 \\ \Delta_l^{l+1} & l \geq 2. \end{cases}$$

Proof. In order to see this we observe that, since $|c + a_j|^2 = |c|^2$ for each j, the (row) vector c verifies $cM = -\frac{1}{2}(|a_1|^2 \ldots |a_l|^2)$, where M is the $d \times l$-matrix

whose columns are ${}^t a_1, \dots {}^t a_l$. Now there exists an orthogonal matrix Q such that

$$QM = \begin{pmatrix} B \\ 0 \end{pmatrix},$$

where B is an invertible $l \times l$-matrix. We have $(\det B)^2 = \det({}^t BB) = \det({}^t MM) \geq 1$, and (the absolute values of) the entries of B are bounded by $\lesssim \Delta_l$.

Define x now by

$$\begin{cases} (x_1 \dots x_l) = -\frac{1}{2}(|a_1|^2 \dots |a_l|^2) B^{-1} \\ x_{l+1} = \cdots = x_d = 0, \end{cases}$$

and $y = xQ$. Then $c - y \perp \sum \mathbb{R} a_j$, so that $|\tilde{c}| \leq |y|$. An easy computation gives $|y| = |x| \lesssim \Delta_l^{l+1}$ and $\lesssim \Delta_1$ (if $l = 1$). □

We shall now determine Δ_l so that $(*)_l$ holds. This will be done by induction on l. For $l = 1$, $\Delta_1 = \Delta$ works, so let us assume that $(*)_l$ holds for some $1 \leq l < k$. If $(*)_{l+1}$ does not hold, it is violated for some c. Let us fix this $c \in [c]$, and let X be the real subspace generated by $(B_{\Delta_{l+1}}(c)) \cap [c]) - c$. Now, X has rank $= l$.

For any $b \in [c]$ with $|b - c| \leq \Delta_{l+1} - \Delta_l$ we have

$$B_{\Delta_l}(b) \cap [c] \subset B_{\Delta_{l+1}}(c) \cap [c].$$

By the induction assumption the $\perp$ projection $\tilde{b}$ of b onto X verifies $(**)$.

Take now $b \in [c]$ such that $\Delta_{l+1} - \Delta_l - \Delta \leq |b - c| \leq (\Delta_{l+1} - \Delta_l)$; such a b exists since the rank of $[c]$ is $\geq l + 1$. Since $b - c$ is parallel to X we have

$$\Delta_{l+1} - \Delta_l - \Delta \leq |b - c| = |\tilde{b} - \tilde{c}| \lesssim \begin{cases} \Delta_l & l = 1 \\ \Delta_l^{l+1} & l \geq 2. \end{cases}$$

So if we take $\Delta_{l+1} \approx$ the RHS, then the assumption that $(*)_{l+1}$ does not hold leads to a contradiction. Hence with this choice $(*)_l$ holds for all $l \leq k$.

To conclude we observe now that $[c] \subset c + X$ where X is a subspace of dimension k. Clearly the diameter of $[c]$ is the same as the diameter of its $\perp$ projection onto X, and, by $(**)$, the diameter of the projection is $\leq \Delta_k$. □

We say that $[a]$ and $[b]$ have the same *block-type* if there are $a' \in [a]$ and $b' \in [b]$ such that $[a] - a' = [b] - b'$. It follows from the proposition that there are only finitely many block-types. We say that the block type of $[a]$ is *orthogonal to* c if $[a] - a \perp c$.

Description of blocks when $d = 2, 3$. Outside $\{|a| :\leq d_\Delta \approx \Delta^3\}$, for $d = 2$ we have

- ★ rank$[a] = 1$ if, and only if, $a \in \frac{b}{2} + b^\perp$ for some $0 < |b| \leq \Delta$; then $[a] = \{a, a - b\}$;
- ★ rank$[a] = 0$; then $[a] = \{a\}$.

For $d = 3$, we have outside $\{|a| :\leq d_\Delta \approx \Delta^{12}\}$

* rank$[a] = 2$ if, and only if, $a \in \frac{b}{2} + b^\perp \cap \frac{c}{2} + c^\perp$ for some $0 < |b|, |c| \leq 2\Delta$ linearly independent; then $[a] \supset \{a, a - b, a - c\}$;

* rank$[a] = 1$ if, and only if, $a \in \frac{b}{2} + b^\perp$ for some $0 < |b| \leq \Delta$; then $[a] = \{a, a - b\}$;

* rank$[a] = 0$; then $[a] = \{a\}$.

4.2. Neighborhood at ∞.

PROPOSITION 4.2. *For any* $|a| \gtrsim \Lambda^{2d-1}$, *there exists* $c \in \mathbb{Z}^d$, $0 < |c| \lesssim \Lambda^{d-1}$, *such that*

$$|a| \geq \Lambda(|a_c| + |c|)\,|c|, \quad \langle a, c \rangle \geq 0.$$

(a_c *is the lattice element on* $a + \mathbb{R}c$ *closest to the origin.*)

Proof. For all $K \gtrsim 1$ there is a $c \in \mathbb{Z}^d \cap \{|x| \leq K\}$ such that

$$\delta = \text{dist}(c, \mathbb{R}a) \leq C_1 \left(\frac{1}{K}\right)^{\frac{1}{d-1}}$$

where C_1 only depends on d.

To see this we consider the segment $\Gamma = [0, \frac{K}{|a|}a]$ in $\mathbb{R}^d$ and a tubular neighborhood Γ_ε of radius ε:

$$\text{vol}(\Gamma_\varepsilon) \approx K\varepsilon^{d-1}.$$

The projection of $\mathbb{R}^d$ onto $\mathbb{T}^d$ is locally injective and locally volume-preserving. If $\varepsilon \gtrsim (\frac{1}{K})^{\frac{1}{d-1}}$, then the projection of Γ_ε cannot be injective (for volume reasons), so there are two different points $x, x' \in \Gamma_\varepsilon$ such that $x - x' = c \in \mathbb{Z}^d \setminus 0$. Then $|a_c| \lesssim \frac{|a|}{|c|}\delta$. Now,

$$\Lambda(|a_c| + |c|)\,|c| \leq \Lambda K^2 + C_2 \frac{\Lambda}{K^{\frac{1}{d-1}}}\,|a|.$$

If we choose $K = (2C_2\Lambda)^{d-1}$, then this is $\leq |a|$. $\square$

COROLLARY 4.3. *For any* $\Lambda, N > 1$, *the subset*

$$\{|a| + |b| \gtrsim \Lambda^{2d-1}\} \cap \{|a - b| \leq N\} \subset \mathbb{Z}^d \times \mathbb{Z}^d$$

is contained in

$$\bigcup_{0 < |c| \lesssim \Lambda^{d-1}} D_\Omega(c)$$

for any $\Omega \leq \frac{\Lambda}{N+1} - 1$.

Proof. Let $|a| \gtrsim \Lambda^{2d-1}$. Then there exists $0 < |c| \lesssim \Lambda^{d-1}$ such that $|a| \geq \Lambda(|a_c| + |c|)\,|c|$. Clearly (because $d \geq 2$)

$$\frac{|a|}{|c|} \geq 2\Lambda^2 \geq 2\Omega^2.$$

If we write $a = a_c + tc$ then $b = a_c + b - a + tc$. According to Lemma 2.1(iv)

$$|b| \geq \Omega(|a_c + b - a| + |c|)|c|,$$

and moreover

$$\frac{|b|}{|c|} \geq \frac{|a|}{|c|} - N \geq 2\Lambda^2 - N \geq 2\Omega^2. \qquad \square$$

Remark. This corollary is essential. It says that any neighborhood

$$\{(a,b) : |a - b| \leq N\} \subset \mathbb{Z}^d \times \mathbb{Z}^d$$

of the diagonal, outside some *finite* set, is covered by *finitely* many Lipschitz domains.

4.3. *Lines* $(a + \mathbb{R}c) \cap \mathbb{Z}^d$.

PROPOSITION 4.4.   (i) *If* $[a + tc] = [b + tc]$ *for all* $t \gg 1$, *then* $[a + tc] = [b + tc]$ *for all* t;

(ii) $[a + tc] - (a + tc)$ *is constant and* $\perp$ *to* c *for all* t *such that*

$$|a + tc| \geq d_\Delta^2(|a_c| + |c|)\,|c|.$$

Proof. To prove (i) we observe that $|a + tc| = |b + tc|$ for all $t \gg 1$, which clearly implies that $|a + tc| = |b + tc|$ for all t. If $|a - b| \leq \Delta$ then this implies that $[a + tc] = [b + tc]$ for all t. Otherwise, for all $t \gg 1$ there is a $d_t \notin \{a, b\}$ such that $[d_t + tc] = [a + tc]$. Since the diameter of each block is $\leq d_\Delta$, it follows that $|d_t - a| \leq d_\Delta$. Since there are infinitely many t's and only finitely many d_t's, there is some d such that $d = d_t$ for at least three different t's. Then $|d + tc| = |a + tc|$ for all t.

If now $|a - d| \leq \Delta$ and $|d - b| \leq \Delta$, then $[a + tc] = [b + tc]$ for all t. Otherwise, for all $t \gg 1$ there is an $e_t \notin \{a, b, d\}$ such that $[e_t + tc] = [a + tc]$, and the statement follows by a finite induction.

To prove (ii) it is enough to consider $a = a_c$. Let $b \in [a + tc] - (a + tc)$ for some $t = t_0$, such that $|a + tc| \geq d_\Delta^2(|a_c| + |c|)\,|c|$. Then $|a + tc + b|^2 = |a + tc|^2$; i.e., $2t\langle b, c\rangle + 2\langle b, a\rangle + |b|^2 = 0$.

If $\langle b, c\rangle \neq 0$, then

$$|a + tc| \leq |a| + |t\langle b, c\rangle|\,|c| \leq |a| + \left(|\langle b, a\rangle| + \frac{1}{2}|b|^2\right)|c|$$

which is less than

$$\left((d_\Delta + 1)\,|a| + \frac{1}{2}d_\Delta^2\right)|c|.$$

But this is impossible under the assumption on $a + tc$. Therefore $\langle b, c\rangle = 0$, i.e., $[a + tc] - (a + tc) \perp$ to c.

Moreover it follows that $|a + tc + b| = |a + tc|$ for all t. If $|b| \leq \Delta$ it follows that $[a + b + tc] = [a + tc]$ for all t. If not, there is a sequence of points $0 = b_1$,

$b_2, \ldots, b_k = b$ in $[a + t_0 c] - (a + t_0 c)$ such that $\left| b_{j+1} - b_j \right| \leq \Delta$ for all j. By a finite induction it follows that $[a + b + tc] = [a + tc]$ for all t. Hence

$$[a + tc] = (t - t_0)c + [a + t_0 c]$$

for all $t \geq t_0$. $\qquad\qquad\square$

More on Töplitz-Lipschitz matrices. For a matrix $Q : \mathscr{L} \times \mathscr{L} \to \mathbb{C}$ we denote by $Q(tc)$ the matrix whose components are

$$Q_a^b(tc) =: Q(tc)_a^b = Q_{a+tc}^{b+tc}.\,^{4}$$

Clearly for any subset I, J of $\mathscr{L}$

$$Q_I^J(tc) =: Q(tc)_I^J = Q_{I+tc}^{J+tc}$$

in an obvious sense.

COROLLARY 4.5. *Let* $\Lambda \geq d_\Delta^2$. *If* $(a, b) \in D_\Lambda(c)$, *then*

$$Q_{[a]_\Delta}^{[b]_\Delta}(tc) = Q_{[a+tc]_\Delta}^{[b+tc]_\Delta}$$

for all $t \geq 0$. *In particular, if* Q *is Töplitz at* ∞, *then*

$$\lim_{t \to \infty} \left\| Q_{[a]_\Delta}^{[b]_\Delta}(tc) - Q_{[a]_\Delta}^{[b]_\Delta}(\infty c) \right\| = 0.$$

Proof. This follows immediately from Proposition 4.4(ii). $\qquad\square$

5. **Small divisor estimates**

Let $\omega \in U \subset \mathbb{R}^{\mathscr{A}}$ be a set contained in

(13) $$\{|\omega| \leq C_1\}, \qquad C_1 \geq 1.$$

If $A : \mathscr{L} \times \mathscr{L} \to \mathrm{gl}(2, \mathbb{C})$ depends on the parameters $\omega \in U$ we define

$$|A|_{\{^\gamma_U\}} = \sup_{\omega \in U} (|A(\omega)|_\gamma, |\partial_\omega A(\omega)|_\gamma),$$

where the derivative should be understood in the sense of Whitney.[5] If the matrices $A(\omega)$ and $\partial_\omega A(\omega)$ are Töplitz at ∞ for all $\omega \in U$, then we can define

$$\langle A \rangle_{\{^{\Lambda,\gamma}_U\}} = \sup_{\omega \in U} (\langle A(\omega) \rangle_{\Lambda,\gamma}, \langle \partial_\omega A(\omega) \rangle_{\Lambda,\gamma}).$$

(This Lipschitz norm is defined in §§2.3, 2.4.) When $\gamma = 0$ we denote these norms by $|A|_U$ and $\langle A \rangle_{\{^\Lambda_U\}}$.

It is clear that if $\langle A \rangle_{\{^{\Lambda,\gamma}_U\}}$ is finite, then the convergence to the Töplitz limit is uniform in ω both for A and $\partial_\omega A$.

---

[4]Notice the abuse of notation. In order to avoid confusion we shall sometimes denote the Töplitz limit in the direction c by $Q(\infty c)$.

[5]This implies that $\langle A \rangle_{\{^\gamma_U\}}$ bounds a $\mathscr{C}^1$-extension of $A(\omega)$ to a ball containing U.

5.1. *Normal form matrices.* A matrix $A : \mathscr{L} \times \mathscr{L} \to \mathrm{gl}(2, \mathbb{C})$ is on *normal form* – denoted $\mathscr{NF}_\Delta$ – if

(i) A is real valued;

(ii) A is symmetric, i.e., $A_b^a = {}^t(A_a^b)$;

(iii) $\pi A = A$ (π is defined in §2.1);

(iv) A is *block-diagonal* over $\mathscr{E}_\Delta$, i.e., $A_a^b = 0$ for all $[a]_\Delta \neq [b]_\Delta$.

For a normal form matrix A the quadratic form $\frac{1}{2}\langle \zeta, A\zeta \rangle$ takes the form

$$\frac{1}{2}\langle \xi, A_1\xi \rangle + \langle \xi, A_2\eta \rangle + \frac{1}{2}\langle \eta, A_1\eta \rangle$$

where $A_1 + iA_2$ is a Hermitian (scalar-valued) matrix.

Let

$$w = \begin{pmatrix} u_a \\ v_a \end{pmatrix} = C^{-1} \begin{pmatrix} \xi_a \\ \eta_a \end{pmatrix}, \quad C = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{-i}{\sqrt{2}} & \frac{i}{\sqrt{2}} \end{pmatrix}$$

and define ${}^tCAC : \mathscr{L} \times \mathscr{L} \to \mathrm{gl}(2, \mathbb{C})$ through $({}^tCAC)_a^b = {}^tCA_a^bC$.

Then A is on normal form if, and only if,

$$\frac{1}{2}\langle w, {}^tCACw \rangle = \frac{1}{2}\langle u, Qv \rangle,$$

where $Q : \mathscr{L} \times \mathscr{L} \to \mathbb{C}$ is

(i) Hermitian, i.e., $Q_b^a = \overline{Q_a^b}$,

(ii) block-diagonal over $\mathscr{E}_\Delta$.

We say that a scalar-valued matrix Q with this property is on *normal form*, denoted $\mathscr{NF}_\Delta$.

Remark. Notice that a scalar valued normal form matrix Q will in general not become a $\mathrm{gl}(2, \mathbb{R})$-valued normal form matrix through the identification $Q_a^b = Q_a^b I$, because the identification with ${}^tCAC$ is different. However, the Töplitz properties are the same and the two Lipschitz norms (obtained by these two different identifications) are equivalent.

We denote for any subset I of $\mathscr{L}$

$$Q_I = Q_I^I = Q|_{I \times I}.$$

5.2. *Small divisor estimates.* Let $\Omega = \Omega(\omega) : \mathscr{L} \times \mathscr{L} \to \mathbb{R}$ be a real scalar valued diagonal matrix with diagonal elements $\Omega_a(\omega)$, $\omega \in U$. Consider the conditions

(14) $$\begin{cases} \left| \partial_\omega^\nu(\Omega_a(\omega) - |a|^2) \right| \leq C_2 e^{-C_3|a|}, \quad C_3 > 0 \\ (a, \omega) \in \mathscr{L} \times U, \quad \nu = 0, 1, \end{cases}$$

and

$$(15) \quad \begin{cases} \langle \partial_\omega(\langle k, \omega \rangle + \Omega_a(\omega)), \frac{k}{|k|} \rangle \geq C_4 > 0, \\ \langle \partial_\omega(\langle k, \omega \rangle + \Omega_a(\omega) + \Omega_b(\omega)), \frac{k}{|k|} \rangle \geq C_4, \\ \langle \partial_\omega(\langle k, \omega \rangle + \Omega_a(\omega) - \Omega_b(\omega)), \frac{k}{|k|} \rangle \geq C_4, \\ a, b \in \mathcal{L}, \ k \in \mathbb{Z}^{\mathscr{A}} \setminus 0, \ \omega \in U. \end{cases}$$

Let $H = H(\omega) : \mathcal{L} \times \mathcal{L} \to \mathbb{C}$ and consider

$$(16) \qquad \qquad \|\partial_\omega H(\omega)\| \leq \frac{C_4}{4}, \quad \omega \in U.$$

(Here $\| \ \|$ is the operator norm.)

Let us first formulate and prove the easy case.

PROPOSITION 5.1. *Let $\Delta' > 1$ and $1 > \kappa > 0$. Assume that U verifies (13), that Ω is real diagonal and verifies (14) and (15), and that H verifies (16). Assume also that $H(\omega)$ is $\mathcal{NF}_\Delta$ for all $\omega \in U$.*

Then there exists a closed set $U' \subset U$,

$$\text{Leb}(U \setminus U') \leq \text{cte.} \max(\Delta', d_\Delta^2)^{2d + \#\mathscr{A} - 1} (C_1 + \sup_U \|H(\omega)\|)^d \kappa C_1^{\#\mathscr{A}-1}$$

such that for all $\omega \in U'$, all $0 < |k| \leq \Delta'$ and for all

$$(17) \qquad \qquad [a]_\Delta, [b]_\Delta$$

we have

$$(18) \qquad \qquad |\langle k, \omega \rangle| \geq \kappa,$$

$$(19) \qquad |\langle k, \omega \rangle + \alpha(\omega)| \geq \kappa \quad \forall \, \alpha(\omega) \in \sigma((\Omega + H)(\omega)_{[a]_\Delta})$$

and

$$(20) \quad |\langle k, \omega \rangle + \alpha(\omega) + \beta(\omega)| \geq \kappa \quad \forall \begin{cases} \alpha(\omega) \in \sigma((\Omega + H)(\omega)_{[a]_\Delta}) \\ \beta(\omega) \in \sigma((\Omega + H)(\omega)_{[b]_\Delta}). \end{cases}$$

Moreover the κ-neighborhood of $U' \subset U$ satisfies the same estimate. The constant cte. *depends on the dimensions d and $\#\mathscr{A}$ and on C_2, C_4.*

Proof. It is enough to prove the statement for $\Delta' \geq d_\Delta^2$. Let us prove the estimate (20), the other two being the same, but easier. Let $C_5 = \sup_U \|H(\omega)\|$. Since $|k| \leq \Delta'$, $|\langle k, \omega \rangle| \lesssim C_1 \Delta'$.[6] If the block I intersects $\{|c| \gtrsim \sqrt{C_1 \Delta' + C_5}\}$, then any eigenvalue α of $(\Omega + H)(\omega)_I$ verifies $\alpha \gtrsim C_1 \Delta'$. Hence $|\langle k, \omega \rangle + \alpha + \beta| \gtrsim 1$.

So it suffices to consider a pair of eigenvalues $\alpha \in \sigma((\Omega + H)(\omega)_I)$ and $\beta \in \sigma((\Omega + H)(\omega)_J)$ with blocks $I, J \subset \{|c| \lesssim \sqrt{C_1 \Delta' + C_5}\}$. These are at most $\lesssim (C_1 \Delta' + C_5)^d$ many possibilities.

Now, $(\langle k, \omega \rangle + \alpha + \beta)$ is an eigenvalue of the Hermitian operator

$$\mathcal{H}(\omega) : X \mapsto \langle k, \omega \rangle X + (\Omega + H)(\omega)_I X + (\Omega + H)(\omega)_J X$$

_____

[6]In this proof $\lesssim$ depends on d, $\#\mathscr{A}$ and on C_2, C_4.

which extends $\mathscr{C}^1$ to a ball around U in $\{|\omega| < C_1\}$. Assumptions (15) and (16), via Proposition 9.3 (Appendix), now imply that the inverse of $\mathscr{H}(\omega)$ is bounded from above by $\frac{1}{\kappa}$ (this gives a lower bound for its eigenvalues) outside a set of Lebesgue measure $\lesssim d_\Delta^d \frac{\kappa}{|k|} C_1^{\#\mathscr{A}-1}$.

Summing now over all these blocks I, J and all $|k| \leq \Delta'$ gives the result. $\square$

We now turn to the main problem.

PROPOSITION 5.2. *Let $\Delta' > 1$ and $0 < \kappa \lesssim 1$. Assume that U verifies* (13), *that Ω is real diagonal and verifies* (14) *and* (15), *and that H verifies* (16). *Assume also that $H(\omega)$ and $\partial_\omega H(\omega)$ are Töplitz at ∞ and $\mathscr{NF}_\Delta$ for all $\omega \in U$.*

Then there exists a subset $U' \subset U$,

$$\operatorname{Leb}(U \setminus U') \leq \text{cte.} \max(\Delta', d_\Delta^2, \Lambda)^{\exp + \#\mathscr{A}-1}\left(C_1 + \langle H\rangle_{\substack{\{\Lambda\} \\ \{U\}}}\right)^d \kappa^{(\frac{1}{d+1})^d} C_1^{\#\mathscr{A}-1},$$

such that, for all $\omega \in U'$, $0 < |k| \leq \Delta'$ and all

$$(21) \qquad \operatorname{dist}([a]_\Delta, [b]_\Delta) \leq \Delta'$$

we have

$$(22) \qquad |\langle k, \omega\rangle + \alpha(\omega) - \beta(\omega)| \geq \kappa \qquad \forall \begin{cases} \alpha(\omega) \in \sigma((\Omega + H)(\omega)_{[a]_\Delta}) \\ \beta(\omega) \in \sigma((\Omega + H)(\omega)_{[b]_\Delta}). \end{cases}$$

Moreover the κ-neighborhood of $U \setminus U'$ satisfies the same estimate.

The exponent exp *depends only on d. The constant* cte. *depends on the dimensions d and $\#\mathscr{A}$ and on C_2, C_3, C_4.*

Proof. The proof goes in the following way: first we prove an estimate in a large finite part of $\mathscr{L}$ (this requires parameter restriction); then we assume an estimate "at ∞" of $\mathscr{L}$ and we prove, using the Lipschitz property, that this estimate propagates from "∞" down to the finite part (this requires no parameter restriction); in a third step we have to prove the assumption at ∞. This will be done by a finite induction on the "Töplitz-invariance" of H. Notice that it is enough to prove the statement for $\Delta' \geq \max(\Lambda, d_\Delta^2)$ and let $[\,]$ denote $[\,]_\Delta$. Let $N \approx (\Delta')^2$.

1. *Finite part.* For the finite part, let us suppose a belongs to

$$(23) \qquad \{a \in \mathscr{L} : |a| \lesssim (C_1 + \frac{1}{\kappa_1} d_\Delta^d \langle H\rangle_{\substack{\{\Lambda\} \\ \{U\}}}) N^{2d-1}\}, \ ^{7}$$

where $\kappa_1 = \kappa^{\frac{1}{d+1}}$. There are finitely many possibilities and $(22)_\kappa$ is fulfilled, for all $[a]$ satisfying (23), all $[b]$ with $|a - b| \lesssim \Delta'$ and all $0 < |k| \leq \Delta'$, outside a set of Lebesgue measure

$$(24) \qquad \lesssim d_\Delta^d (C_1 + d_\Delta^d \langle H\rangle_{\substack{\{\Lambda\} \\ \{U\}}})^d N^{d(2d-1)}(\Delta')^{d+\#\mathscr{A}-1} \frac{\kappa}{\kappa_1^d} C_1^{\#\mathscr{A}-1}.$$

(This is the same argument as in Proposition 5.1.)

---

[7] In this proof $\lesssim$ depends on d, $\#\mathscr{A}$ and on C_2, C_3, C_4.

We now get rid of the diagonal terms $\hat{V}(a,\omega) = \Omega_a(\omega) - |a|^2$ which, by (14), are $\leq C_2 e^{-|a|C_3}$. We include them into H. Since they are diagonal, H will remain on normal form. Due to the exponential decay of $\hat{V}$, H and $\partial_\omega H$ will remain Töplitz at ∞. The Lipschitz norm gets worse but this is innocent in view of the estimates. Also the estimate of $\partial_\omega H(\omega)$ gets larger, but if a is outside (23) then condition (16) remains true with a slightly larger bound, say

$$\|\partial_\omega H(\omega)\| \leq \frac{3C_4}{8}, \quad \omega \in U.$$

So from now on, a is outside (23) and $\Omega_a = |a|^2$.

2. *Condition at* ∞. For each vector $c \in \mathbb{Z}^d$ such that $0 < |c| \lesssim N^{d-1}$, we suppose that the Töplitz limit $H(c,\omega)$ verifies $(22)_{\kappa_1}$ for (21) and for

(25) $$([a] - [b]) \perp c.$$

It will become clear in the next part why we only need $(22)_{\kappa_1}$ and (21) under the supplementary restriction (25).

3. *Propagation of the condition at* ∞. We must now prove that for $|b - a| \lesssim \Delta'$ and an $a \in \mathscr{L}$ outside (23), $(22)_\kappa$ is fulfilled.

By Corollary 4.3 we get

$$(a,b) \in \bigcup_{0 < |c| \lesssim N^{d-1}} D_{N'}(c), \quad N' \approx \frac{N}{\Delta'}.$$

Fix now $0 < |c| \lesssim N^{d-1}$ and $(a,b) \in D_{N'}(c)$.

By Corollary 2.2(iii)

$$[a] \times [a], \ [a] \times [b], \ [b] \times [b] \subset D_{N'}(c) \subset D_\Delta(c)$$

if we replace N' by $N' - d_\Delta - 2$ — still denoted N'.

By Proposition 4.4(ii) (notice that $N' \geq d_\Delta^2$), $[a + tc] = [a] + tc$ and $[b + tc] = [b] + tc$ for $t \geq 0$ and $[a] - a, \ [b] - b \perp c$. It follows (Corollary 4.5) that

$$\lim_{t \to \infty} H(\omega)_{[a+tc]} = H(c,\omega)_{[a]} \quad \text{and} \quad \lim_{t \to \infty} H(\omega)_{[b+tc]} = H(c,\omega)_{[b]}.$$

The matrices $\Omega_{[a+tc]}$ and $\Omega_{[b+tc]}$ do not have limits as $t \to \infty$. However, for any $(\#[a] \times \#[b])$-matrix X,

$$\Omega_{[a+tc]} X - X \Omega_{[b+tc]} = \Omega_{[a]} X - X \Omega_{[b]} + 2t \langle a - b, c \rangle X$$

for $t \geq 0$, and we must discuss two different cases according to if $\langle c, b - a \rangle = 0$ or not.

Consider for $t \geq 0$ a pair of continuous eigenvalues

$$\begin{cases} \alpha_t \in \sigma((\Omega + H(\omega))_{[a+tc]}) \\ \beta_t \in \sigma((\Omega + H(\omega))_{[b+tc]}). \end{cases}$$

Case I: $\langle c, b - a \rangle = 0$. Here

$$(\Omega + H(\omega))_{[a+tc]}X - X(\Omega + H(\omega))_{[b+tc]}$$

equals

$$(|a|^2 + H(\omega))_{[a+tc]}X - X(|b|^2 + H(\omega))_{[b+tc]}.$$

The linear and quadratic terms in t cancel!

By continuity of eigenvalues, $\lim_{t\to\infty}(\alpha_t - \beta_t) = (\alpha_\infty - \beta_\infty)$, where

$$\begin{cases} \alpha_\infty \in \sigma((|a|^2 + H(c,\omega))_{[a]}) \\ \beta_\infty \in \sigma((|b|^2 + H(c,\omega))_{[b]}). \end{cases}$$

Since $[a]$ and $[b]$ verify (25), our assumption on $H(c,\omega)$ implies that $(\alpha_\infty - \beta_\infty)$ verifies $(22)_{\kappa_1}$.

For any two $a, a' \in [a]$ we have $|a| = |a'|$. Hence

$$\left\| H(\omega)_{[a]} - H(c,\omega)_{[a]} \right\| \frac{|a|}{|c|} \lesssim d_\Delta^d \langle H \rangle_{\{\Lambda\}\atop\{U\}},$$

and the same for $[b]$. Recalling that a and, hence, b violate (23) this implies

$$\left\| H(\omega)_{[d]} - H(c,\omega)_{[d]} \right\| \leq \frac{\kappa_1}{4}, \quad d = a, b.$$

By Lipschitz-dependence of eigenvalues (of Hermitian operators) on parameters, this implies that

$$|(\alpha_0 - \beta_0) - (\alpha_\infty - \beta_\infty)| \leq \frac{\kappa_1}{2}$$

and we are done.

Case II: $\langle c, b - a \rangle \neq 0$. We write $a = a_c + \tau c$. Since $|a| \geq N'(|a_c| + |c|)|c|$, it follows that $|a_c| \leq \frac{1}{N'}\frac{|a|}{|c|}$.

Now, $\alpha_0 - \beta_0$ differs from $|a|^2 - |b|^2$ by at most $2\|H(\omega)\| \lesssim d_\Delta^d \langle H \rangle_{\{\Lambda\}\atop\{U\}}$, and

$$|a|^2 - |b|^2 = -2\tau\langle c, b - a \rangle - 2\langle a_c, b - a \rangle - |b - a|^2.$$

Since $|\langle c, b - a \rangle| \geq 1$ it follows that

$$\tau \lesssim |\alpha_0 - \beta_0| + |a_c|\Delta' + (\Delta')^2 + d_\Delta^d \langle H \rangle_{\{\Lambda\}\atop\{U\}}.$$

If now $|\alpha_0 - \beta_0| \lesssim C_1\Delta'$ then $|a| \leq |a_c| + |\tau c|$ is

$$\leq \text{cte.}(|a_c|\Delta'|c| + C_1(\Delta')^2|c| + d_\Delta^d \langle H \rangle_{\{\Lambda\}\atop\{U\}}|c|)$$

$$\leq \frac{1}{2}|a| + \text{cte.}(C_1(\Delta')^2|c| + d_\Delta^d \langle H \rangle_{\{\Lambda\}\atop\{U\}}|c|).$$

Since a violates (23) this is impossible. Therefore $|\alpha_0 - \beta_0| \gtrsim C_1\Delta'$ and $(22)_\kappa$ holds.

Hence, we have proved that $(22)_\kappa$ holds for any

$$\begin{cases} a \in (23)_{\kappa_1} \\ (a,b) \in (21) \end{cases} \cup \begin{cases} (a,b) \in \bigcup_{0 < |c| \lesssim N^{d-1}} D_{N'}(c) \\ (a,b) \in (21) \end{cases}$$

under the condition at ∞. Therefore $(22)_\kappa$ holds for any $(a,b) \in (21)$.

4. *Proof of condition at ∞ — induction.* Let c_1 be a primitive vector in $0 < |c_1| \lesssim N^{d-1}$, and let G be the Töplitz limit $H(c_1)$. Then G verifies (16), $G(\omega)$ and $\partial_\omega G(\omega)$ are Töplitz at ∞ and

$$\langle G \rangle_{\{\Lambda\} \atop \{U\}} \leq \langle H \rangle_{\{\Lambda\} \atop \{U\}}.$$

Clearly $G(\omega)$ is Hermitian and, by Proposition 4.4(i), $G(\omega)$ and $\partial_\omega G(\omega)$ are block diagonal over $\mathscr{E}_\Delta$, i.e., $G(\omega)$ and $\partial_\omega G(\omega)$ are $\mathscr{NF}_\Delta$. Moreover G is Töplitz in the direction c_1, $G_{a+tc_1}^{b+tc_1} = G_a^b$, for all a, b, tc_1. Now, $\Omega_a = |a|^2$ for all a, so Ω verifies (14) and (15) .

We want to prove that G verifies $(22)_{\kappa_1}$ for all $(a,b) \in (21) + (25)_{c_1}$, i.e., for all $|a - b| \lesssim \Delta'$ and $([a] - [b]) \perp c_1$. Since G is Töplitz in the direction c_1 it is enough to show this for

$$(26) \qquad\qquad \left| \text{proj}_{\text{Lin}(c_1)} a \right| \lesssim N^{d-1}.$$

To prove this we repeat the previous arguments.

Finite part. In the set $(23)_{\kappa_2}$, $\kappa_2 = \kappa_1^{\frac{1}{d+1}}$, there are only finitely many possibilities and $(22)_{\kappa_1}$ will be fulfilled outside a set of ω of Lebesgue measure $(24)_{\frac{\kappa_1}{\kappa_2^d}}$.

A second condition at ∞. For each vector $c \in \mathbb{Z}^d$ such that $0 < |c| \lesssim N^{d-1}$ with c and c_1 being linearly independent, we suppose that the Töplitz limit $G(c, \omega)$ verifies $(22)_{\kappa_2}$ for all $(a,b) \in (21) + (25)_{c_1} + (25)_c$, i.e., for all $|a - b| \lesssim \Delta'$ and $([a] - [b]) \perp c_1, c$.

Propagation of condition at ∞. The same argument as before shows that $(22)_{\kappa_1}$ holds for any

$$\begin{cases} a \in (23)_{\kappa_2} \\ (a,b) \in (21) \end{cases} \cup \begin{cases} (a,b) \in \bigcup_{0 < |c| \lesssim N^{d-1} \atop c \nparallel c_1} D_{N'}(c) \\ (a,b) \in (21) + (25)_{c_1} \end{cases}$$

under the condition at ∞.

Since a verifies (26), it follows that $a \in (23)_{\kappa_2}$ or $(a,b) \notin D_{N'}(c_1)$. Indeed, if $(a,b) \in D_{N'}(c_1)$, then (Corollary 2.2(i))

$$|a| \approx \frac{|\langle a, c_1 \rangle|}{|c_1|} \lesssim N^{d-1}$$

which implies that $a \in (23)_{\kappa_2}$. Therefore $(22)_{\kappa_1}$ holds for any $(a,b) \in (21) + (25)_{c_1}$.

5. *The last inductive step.* Suppose we have a matrix G verifying (16) and such that $G(\omega)$ and $\partial_\omega G(\omega)$ are Töplitz at ∞ and $\mathcal{N}\mathcal{F}_\Delta$ and $\langle G \rangle_{\{\Lambda\} \atop \{U\}} \leq \langle H \rangle_{\{\Lambda\} \atop \{U\}}$. Suppose also that there are primitive and linearly independent vectors $c_1, \ldots, c_{d-1}$ of norm $\lesssim N^{d-1}$, such that G is Töplitz in these directions, i.e.,

$$G_{a+tc_j}^{b+tc_j} = G_a^b, \quad \forall a, b, tc_j, \quad j = 1, \ldots, d-1.$$

We want to prove that G verifies $(22)_{\kappa_{d-1}}$, $\kappa_{d-1} = \kappa_{d-2}^{\frac{1}{d+1}}$, for all $(a, b) \in (21) + (25)_{c_1} + \cdots + (25)_{c_{d-1}}$. Since G is Töplitz in the directions $c_1, \ldots, c_{d-1}$ it suffices to prove this for $a \in (26)_{c_1,\ldots,c_{d-1}}$; i.e., $\left| \text{proj}_{\text{Lin}(c_1,\ldots,c_{d-1})} a \right| \lesssim N^{d-1}$.

If $a \in (23)_{\kappa_d}$, $\kappa_d = \kappa_{d-1}^{\frac{1}{d+1}}$, then $(22)_{\kappa_{d-1}}$ will be fulfilled outside a set of ω of Lebesgue measure $(24)_{\kappa_{d-1}/\kappa_d^d}$. By assumptions $(25)_{c_1} + \cdots + (25)_{c_{d-1}}$, $[a]$ and $[b]$ are contained in one and the same affine line, so $\#[a], \#[b] \leq 2$. If now $a \notin (23)_{\kappa_d}$, then $|a| \gtrsim \left| \text{proj}_{\text{Lin}(c_1,\ldots,c_{d-1})} a \right|$, and the same for b. Therefore $\#[a] = \#[b] = 1$ and

$$|a + b| \gtrsim (C_1 + \sup_U \|G(\omega)\|) N^{2d-1}.$$

It follows that

$$\left| |a|^2 - |b|^2 \right| \gtrsim (C_1 + \sup_U \|G(\omega)\|) N^{2d-1},$$

unless $[a] = [b] = \{a\}$. In the first case we are done because $|\langle k, \omega \rangle| \lesssim C_1 \Delta'$ and in the second case condition $(22)_{\kappa_{d-1}}$ reduces to $|\langle k, \omega \rangle| > \kappa$.

This completes the proof of the last inductive step and, hence, of the proposition. $\qquad \square$

6. **The homological equations**

6.1. *A first equation.* For $k \in \mathbb{Z}^n$ consider the equation

$$(27) \qquad i \langle k, \omega \rangle S + i (\Omega(\omega) + H(\omega)) S = F(\omega),$$

where $F(\omega)$ and $\partial_\omega F(\omega)$ are elements in $l_\gamma^2(\mathcal{L}, \mathbb{C}) = \{\xi = (\xi_a)_{a \in \mathcal{L}} : \|\xi\|_\gamma < \infty\}$,

$$\|\xi\|_\gamma = \sqrt{\sum_{a \in \mathcal{L}} |\xi_a|^2 e^{2\gamma |a|} \langle a \rangle^{2m_*}}$$

$(\langle a \rangle = \max(1, |a|))$. Denote

$$\|F\|_{\{\gamma\} \atop \{U\}} = \sup_{\omega \in U} (\|F(\omega)\|_\gamma, \|\partial_\omega F(\omega)\|_\gamma).$$

Let $U' \subset U$ be a set such that for all $\omega \in U_\kappa'$ the small divisor condition (19) holds for all a, i.e.,

$$|\langle k, \omega \rangle + \alpha(\omega)| \geq \kappa, \quad \forall \, \alpha(\omega) \in \sigma((\Omega + H)(\omega)).$$

PROPOSITION 6.1. *Let $0 < \kappa < 1$. Assume that Ω is real diagonal and verifies* (14) *and that H verifies* (16). *Assume also that $H(\omega)$ and $\partial_\omega H(\omega)$ are $\mathcal{NF}_\Delta$ for all $\omega \in U$.*

Then equation (27) *has for all $\omega \in U'$ a unique solution $S(\omega)$ such that*

$$\|S\|_{\{^\gamma_{U'}\}} \leq \text{cte.} \frac{1}{\kappa^2} d_\Delta^{2m_*} e^{2\gamma d_\Delta} (1 + |k|) \|F\|_{\{^\gamma_{U'}\}}.$$

The constant cte. *only depends on d, $\#\mathcal{A}$, m_* and C_2, C_3, C_4.*

Proof. This is a standard result. Equation (27) has a unique solution verifying

$$\|S(\omega)\|_\gamma \lesssim \frac{1}{\kappa} d_\Delta^{m_*} e^{\gamma d_\Delta} \|F(\omega)\|_\gamma.$$

The factor $d_\Delta^{m_*} e^{\gamma d_\Delta}$ comes in because the block-diagonal character of $\Omega(\omega) + H(\omega)$ interferes with the polynomial and exponential decay.

If we differentiate equation (27) with respect to ω we get

$$i\langle k, \omega\rangle \partial_\omega S + i(\Omega(\omega) + H(\omega))\partial_\omega S$$
$$= \partial_\omega F(\omega) - i(\partial_\omega \langle k, \omega\rangle)S - i\partial_\omega(\Omega(\omega) + H(\omega))S.$$

If we apply the same estimate to this equation we get the result on U'.

In order to extend S from U' to a ball we take a $\mathscr{C}^1$ cut-off function χ which is 1 on U' and 0 outside U'_κ. We now first solve the equation on U'_κ as above to get a solution $\widetilde{S}$ and then we define $S = \chi \widetilde{S}$.   □

6.2. *Truncations.* For a matrix $Q : \mathscr{L} \times \mathscr{L} \to \mathbb{C}$ consider three truncations

$$\mathscr{T}_{\Delta'} Q = Q \text{ restricted to } \{(a,b) : |a - b| \leq \Delta'\},$$
$$\mathscr{P}_c Q = Q \text{ restricted to } \{(a,b) : (a - b) \perp c\},$$
$$\mathscr{D}_{\Delta'} Q = Q \text{ restricted to } \{(a,b) : |a - b| \leq \Delta' \text{ and } |a| = |b|\}.$$

These truncations all commute. Moreover,

LEMMA 6.2.   (i)

$$\begin{cases} |\mathscr{T}_{\Delta'} Q|_{\{^\gamma_U\}} \leq |Q|_{\{^\gamma_U\}}, \\ \langle \mathscr{T}_{\Delta'} Q\rangle_{\{^{\Lambda,\gamma}_U\}} \leq \langle Q\rangle_{\{^{\Lambda,\gamma}_U\}} \end{cases}$$

and $(\mathscr{T}_{\Delta'} Q)(c) = \mathscr{T}_{\Delta'}(Q(c))$ for all c.

(ii) *The same result holds for $\mathscr{P}_c$.*

(iii) *For any $\Lambda \geq (d_{\Delta'})^2$,*

$$\begin{cases} |\mathscr{D}_{\Delta'} Q|_{\{^\gamma_U\}} \leq |Q|_{\{^\gamma_U\}}, \\ \langle \mathscr{D}_{\Delta'} Q\rangle_{\{^{\Lambda,\gamma}_U\}} \leq \langle Q\rangle_{\{^{\Lambda,\gamma}_U\}}. \end{cases}$$

Moreover $(\mathscr{P}_c \mathscr{D}_{\Delta'} Q)(c) = (\mathscr{P}_c \mathscr{D}_{\Delta'})(Q(c))$ for all c.

Proof. (i) and (ii) are obvious. Let us consider (iii). We have that $(\mathscr{D}_{\Delta'}Q)_a^b(c)$ is $= Q_a^b(c)$ if $|a - b| \leq \Delta'$, $|a| = |b|$, $(a-b) \perp c$, and is $= 0$ otherwise. This gives immediately the last statement.

If $|a - b| \leq \Delta'$, then $|a| = |b| \implies [a]_{\Delta'} = [b]_{\Delta'}$. Hence, if $(a, b) \in D_\Lambda(c)$ and $|a - b| \leq \Delta'$, then $|a| = |b| \implies (a - b) \perp c$. From this we derive that $(\mathscr{D}_{\Delta'}Q)_a^b - (\mathscr{D}_{\Delta'}Q)_a^b(c)$ is $= Q_a^b - Q_a^b(c)$ or $= 0$. $\qquad \square$

6.3. *A second equation, $k \neq 0$.* For $k \in \mathbb{Z}^n \setminus \{0\}$ consider the equation

$$(28) \qquad i\langle k, \omega \rangle S + i[\Omega(\omega) + H(\omega), S] = \mathscr{T}_{\Delta'} F(\omega)$$

where $F(\omega) : \mathscr{L} \times \mathscr{L} \to \mathbb{C}$ and $\partial_\omega F(\omega)$ are Töplitz at ∞.

Let $U' \subset U$ be a set such that for all $\omega \in U'_\kappa$ the small divisor condition $(21)_{\Delta'+2d_\Delta} + (22)$ holds; i.e.,

$$|\langle k, \omega \rangle + \alpha(\omega) - \beta(\omega)| \geq \kappa \quad \forall \begin{cases} \alpha(\omega) \in \sigma((\Omega + H)(\omega)_{[a]_\Delta}) \\ \beta(\omega) \in \sigma((\Omega + H)(\omega)_{[b]_\Delta}) \end{cases}$$

for $\mathrm{dist}([a]_\Delta, [b]_\Delta) \leq \Delta' + 2d_\Delta$.

PROPOSITION 6.3. *Let $\Delta' > 1$ and $0 < \kappa < 1$. Assume that U verifies (13), that Ω is real diagonal and verifies (14), and that H verifies (16). Assume also that $H(\omega)$ and $\partial_\omega H(\omega)$ are Töplitz at ∞ and $\mathscr{NF}_\Delta$ for all $\omega \in U$.*

Then equation (28), under the condition that $S = \mathscr{T}_{\Delta'+2d_\Delta} S$, have for all $\omega \in U'$, a unique solution $S(\omega)$ verifying

(i) $|S|_{\{\gamma \atop U'\}} \leq$ cte. $\frac{1}{\kappa^2} d_\Delta^{2d} e^{2\gamma d_\Delta} (1 + |k|) |F|_{\{\gamma \atop U'\}}$;

(ii) *$S(\omega)$ and $\partial_\omega S(\omega)$ are Töplitz at ∞ and the Töplitz limits verify*

$$\begin{cases} i\langle k, \omega \rangle S + i[\Omega(\omega) + H(c, \omega), S] = \mathscr{T}_{\Delta'} \mathscr{P}_c F(c, \omega) \\ S = \mathscr{T}_{\Delta'+2d_\Delta} S; \end{cases}$$

(iii) *For any $\Lambda' \gtrsim \max(\Lambda, d_\Delta^2, \Delta', \sup_U \|H(\omega)\|)$,*

$$\langle S \rangle_{\{\Lambda'+d_\Delta+2, \gamma \atop U'\}} \leq \text{cte.} \frac{1}{\kappa^3} d_\Delta^{2d} e^{2\gamma d_\Delta} (1 + |k| + \langle H \rangle_{\{\Lambda \atop U'\}}) \langle F \rangle_{\{\Lambda', \gamma \atop U'\}}.$$

The constant cte. *only depends on the dimensions d and $\#\mathscr{A}$ and on C_1, C_2, C_3, C_4.*

Proof. We first eliminate the diagonal terms $\widehat{V}(a, \omega) = \Omega_a(\omega) - |a|^2$ which by (14) are $\lesssim C_2 e^{-|a|C_3}$.[8] We include them into H; in view of the estimates of the proposition this is innocent. Let us also notice that it is enough to prove the statement for $\Lambda \geq d_\Delta^2$. We first assume that $F = \mathscr{T}_{\Delta'} F$.

So from now on we assume $\Omega_a = |a|^2$ and $\Lambda \geq d_\Delta^2$. We shall denote the blocks $[\;]_\Delta$ by $[\;]$.

---

[8]In this proof $\lesssim$ depends on $d, \#\mathscr{A}$ and on C_1, C_2, C_3, C_4.

We first block decompose equation (28) over $\mathscr{E}_\Delta$ taking into account the truncation of S and the small divisor condition which becomes

(29)
$$
\begin{cases}
L_k(\omega) S_{[a]}^{[b]} = F_{[a]}^{[b]}(\omega) & \text{if dist}([a], [b]) \le \Delta' \\
S_{[a]}^{[b]} = 0 & \text{if not,}
\end{cases}
$$

where

$$
L_k(\omega) S_{[a]}^{[b]} = i \langle k, \omega \rangle S_{[a]}^{[b]} + i (\Omega + H(\omega))_{[a]} S_{[a]}^{[b]} - i S_{[a]}^{[b]} (\Omega + H(\omega))_{[b]}.
$$

Since $\Omega + H$ is Hermitian, under the small divisor condition, equation (29) has a unique solution which is $\mathscr{C}^1$ in ω and verifies

$$
\left| S_a^b \right| \le \left\| S_{[a]}^{[b]} \right\| \le \frac{1}{\kappa} \left\| F_{[a]}^{[b]} \right\|.
$$

(Here, $\| \ \|$ is the operator norm.) Hence

(30)
$$
|S|_\gamma \le \frac{1}{\kappa} d_\Delta^d e^{2\gamma d_\Delta} |F|_\gamma.
$$

The factor d_Δ^d comes from the two different matrix norms used here, and the exponential factor occurs because the block character of $\Omega + H$ interferes with the exponential decay.

In order to estimate the derivatives in ω we just differentiate (29) with respect to ω:

(31) $(i \langle k, \omega \rangle + i (\Omega + H(\omega))_{[a]}) \partial_\omega S_{[a]}^{[b]} - i \partial_\omega S_{[a]}^{[b]} (\Omega + H(\omega))_{[b]}$

$$
= \partial_\omega F_{[a]}^{[b]}(\omega) - i \left(\partial_\omega \langle k, \omega \rangle + \partial_\omega H(\omega)_{[a]} \right) S_{[a]}^{[b]} - S_{[a]}^{[b]} \partial_\omega H(\omega)_{[b]}.
$$

If $G_{[a]}^{[b]}$ is the matrix on RHS, then

$$
\left\| G_{[a]}^{[b]} \right\| \le \left\| \partial_\omega F_{[a]}^{[b]} \right\| + (|k| + \left\| \partial_\omega H_{[a]} \right\| + \left\| \partial_\omega H_{[b]} \right\|) \left\| S_{[a]}^{[b]} \right\|
$$

and $\partial_\omega S_{[a]}^{[b]}$ is now estimated like $S_{[a]}^{[b]}$.

We repeat the same thing on U_κ' and then extend S from U' to be 0 outside U_κ' by a $\mathscr{C}^1$ cut-off. This gives (i).

Töplitz at ∞. Let Q be a matrix on $\mathscr{L}$ and denote by $Q(tc)$ the matrix whose elements are $Q_a^b(tc) = Q_{a+tc}^{b+tc}.$[9]

By Proposition 4.4(ii), for $(a, b) \in D_{\Lambda'}(c)$ (notice that $\Lambda' \ge d_\Delta^2$), $[a + tc] = [a] + tc$ and $[b + tc] = [b] + tc$ for $t \ge 0$ and $[a] - a, [b] - b \perp c$. It follows that

(32) $i \langle k, \omega \rangle S_{[a]}^{[b]}(tc) + i (\Omega + H)_{[a]}(tc) S_{[a]}^{[b]}(tc)$

$$
- i S_{[a]}^{[b]}(tc) (\Omega + H)_{[b]}(tc) = F_{[a]}^{[b]}(tc)
$$

for all $t \ge 0$.

---

[9] In order to avoid confusion we shall denote the Töplitz limit in the direction c by $Q(\infty c)$.

Moreover $H_{[a]}(tc)$, $H_{[b]}(tc)$ and $F_{[a]}^{[b]}(tc)$ have limits as $t \to \infty$ (Corollary 4.5). $\Omega_{[a]}(tc)$ and $\Omega_{[b]}(tc)$ do not have limits, and we must analyze two different cases according to whether $\langle c, a - b \rangle = 0$ or not.

Case I: $\langle c, a - b \rangle = 0$. We have that $\Omega_{[a]}(tc)X - X\Omega_{[b]}(tc)$ (for any (#$[a]$ × #$[b]$)-matrix X) equals $|a|^2 X - X |b|^2$. The linear and quadratic terms in t cancel! Therefore equation (32) has a limit as $t \to \infty$:

$$i \langle k, \omega \rangle X + i(\Omega_{[a]} + H_{[a]}(\infty c))X - iX(\Omega_{[b]} + H_{[b]}(\infty c)) = F_{[a]}^{[b]}(\infty c).$$

Since eigenvalues are continuous in parameters we have

$$|\langle k, \omega \rangle + \alpha - \beta| \geq \kappa \quad \forall \begin{cases} \alpha \in \sigma(|a|^2 + H_{[a]}(\infty c)) \\ \beta \in \sigma(|b|^2 + H_{[b]}(\infty c)). \end{cases}$$

Therefore the limit equation has a unique solution X which is $\mathscr{C}^1$ in ω and verifies

$$\|X\| \leq \frac{1}{\kappa} \left\| F_{[a]}^{[b]}(\infty c) \right\|.$$

Since $S_{[a]}^{[b]}(tc)$ is bounded, it follows from uniqueness that $S_{[a]}^{[b]}(tc) \to S_{[a]}^{[b]}(\infty c) = X$ as $t \to \infty$.

Case II: $\langle c, a - b \rangle \neq 0$. We have $\Omega_{[a]}(tc)X - X\Omega_{[b]}(tc)$ equals

$$(2t \langle a, c \rangle + |a|^2)X - X(2t \langle b, c \rangle |b|^2);$$

only the quadratic terms in t cancel! Dividing (32) by t and letting $t \to \infty$, the limit equation becomes $2\langle c, a - b \rangle X = 0$. It has the unique solution $X = 0$. For the same reason as in the previous case we have that $S_{[a]}^{[b]}(tc) \to S_{[a]}^{[b]}(\infty c) = 0$ as $t \to \infty$.

We have thus shown that, for any c, the solution S has a Töplitz limit $S(\infty c)$ which verifies, for $(a, b) \in D_{\Lambda'}(c)$,

(33) $$\begin{cases} L_k(\omega)S_{[a]}^{[b]} = F_{[a]}^{[b]}(\infty c, \omega) & \text{if dist}([a], [b]) \leq \Delta' \text{ and } (a - b) \perp c \\ S_{[a]}^{[b]} = 0 & \text{if not,} \end{cases}$$

where

$$L_k(\omega)S_{[a]}^{[b]} = i \langle k, \omega \rangle S_{[a]}^{[b]} + i(\Omega + H(\infty c, \omega))_{[a]} S_{[a]}^{[b]} - i S_{[a]}^{[b]}(\Omega + H(\infty c, \omega))_{[b]}.$$

Since $S(\infty c)$ is invariant under c-translations, this implies that $S(\infty c)$ verifies the equation in (ii). Moreover $|S(\infty c)|_\gamma \leq \frac{1}{\kappa} d_\Delta^d e^{2\gamma d_\Delta} |F(\infty c)|_\gamma$.

Estimate of Lipschitz norm. Consider the "derivative" ∂_c:

$$\partial_c Q_{[a]}^{[b]}(tc) = (Q_{[a]}^{[b]}(tc) - Q_{[a]}^{[b]}(\infty c)) \max \left(\frac{|a|}{|c|}, \frac{|b|}{|c|} \right).$$

(Notice that the definition does not depend on the choice of representatives a and b in $[a]$ and $[b]$ respectively.) We shall "differentiate" equation (32) and estimate the solution of the "differentiated" equation over $[a] \times [b] \subset D_{\Lambda'}(c)$ which is $\subset D_\Lambda(c)$

because $\Lambda' \geq \Lambda$. By Corollary 2.2(iii) this will provide us with an estimate of the Lipschitz constant $\mathrm{Lip}^+_{\Lambda'+d_\Delta+2,\gamma}$.

So we take $[a] \times [b] \subset D_{\Lambda'}(c)$. Since S is 0 at distances $\gtrsim \Delta' + d_\Delta$ from the diagonal we only need to treat $|a - b| \lesssim \Delta' + d_\Delta$. Again we must consider two cases.

Case I: $\langle c, a - b \rangle = 0$. Subtracting equation (33) for $S^{[b]}_{[a]}(\infty c)$ from equation (29) for $S^{[b]}_{[a]}$ and multiplying by $\max\left(\frac{|a|}{|c|}, \frac{|b|}{|c|}\right)$ gives

$$i \langle k, \omega \rangle \partial_c S^{[b]}_{[a]} + i (\Omega + H)_{[a]} \partial_c S^{[b]}_{[a]} - \partial_c S^{[b]}_{[a]}(\Omega + H)_{[b]}$$
$$= \partial_c F^{[b]}_{[a]} - \partial_c H_{[a]} S^{[b]}_{[a]}(\infty c) + S^{[b]}_{[a]}(\infty c) \partial_c H_{[b]}.$$

Now we get as for equation (29) that

$$\left\| \partial_c S^{[b]}_{[a]} \right\| \leq \frac{1}{\kappa} \left(\left\| \partial_c F^{[b]}_{[a]} \right\| + \left(\left\| \partial_c H_{[a]} \right\| + \left\| \partial_c H_{[b]} \right\| \right) \left\| S^{[b]}_{[a]}(\infty c) \right\| \right).$$

Case II: $\langle c, a - b \rangle \neq 0$. Then

$$\left| |a|^2 - |b|^2 \right| \approx \frac{|a|}{|c|} |\langle c, a - b \rangle| \approx \frac{|b|}{|c|} |\langle c, a - b \rangle| \gtrsim \Lambda'.$$

Indeed $|a|^2 - |b|^2$ can be written

$$\left| a' + \tau c \right|^2 - \left| b' + \tau c \right|^2 \left| a' \right|^2 - \left| b' \right|^2 + 2\tau \langle c, a - b \rangle,$$

and (recalling Lemma 2.1(ii))

$$\left| \left| a' \right|^2 - \left| b' \right|^2 \right| \leq |a - b| \left(|a'| + |b'| \right) \leq \mathrm{cte.} (\Delta' + d_\Delta) \frac{\tau}{\Lambda'}$$

and this is $\leq \frac{1}{2}\tau$, since $\Lambda' \geq 2\mathrm{cte.}(\Delta' + d_\Delta)$. Moreover (Lemma 2.1(i) and (iii))

$$\frac{|a|}{|c|} \approx \frac{|b|}{|c|} \approx \tau \geq \Lambda'.$$

Since $\Lambda' \gtrsim \|H\|$, assuring that $\|H\|$ is small compared with $|a|^2 - |b|^2|$, we have

$$|\alpha - \beta| \approx 2 |\langle a - b, c \rangle| \geq 2 \quad \forall \begin{cases} \alpha \in \sigma(\frac{1}{\tau}(\Omega + H)_{[a]}) \\ \beta \in \sigma(\frac{1}{\tau}(\Omega + H)_{[b]}). \end{cases}$$

Since $S^{[b]}_{[a]}(\infty c) = 0$, multiplying (28) by $\frac{1}{\tau} \max\left(\frac{|a|}{|c|}, \frac{|b|}{|c|}\right)$ gives,

$$\frac{i}{\tau} \langle k, \omega \rangle \partial_c S^{[b]}_{[a]} + \frac{i}{\tau}(\Omega + H)_{[a]} \partial_c S^{[b]}_{[a]} - \partial_c S^{[b]}_{[a]} \frac{i}{\tau}(\Omega + H)_{[b]}$$
$$= F^{[b]}_{[a]} \frac{1}{\tau} \max\left(\frac{|a|}{|c|}, \frac{|b|}{|c|}\right) \approx F^{[b]}_{[a]}.$$

Since $\Lambda' \geq C_1 \Delta'$, the absolute value of the eigenvalues of the LHS-operator is ≥ 1 and it follows that $\left\| \partial_c S^{[b]}_{[a]} \right\| \lesssim \left\| F^{[b]}_{[a]} \right\|$.

If $(a, b) \in D_{\Lambda' + d_\Delta + 2}(c)$, then both (a, a) and (b, b) belong to $D_{\Lambda' + d_\Delta + 2}(c)$ and, by Corollary 2.2(iii),

$$[a] \times [b], [a] \times [a], [b] \times [b] \subset D_{\Lambda'}(c) \subset D_\Lambda(c).$$

Therefore

$$\left\| \partial_c H_{[a]} \right\| + \left\| \partial_c H_{[b]} \right\| \leq d_\Delta^d \langle H \rangle_\Lambda.$$

Using this, the estimates (in Cases I and II) for $\left\| \partial_c S_{[a]}^{[b]} \right\|$ and the estimate (30) we obtain

$${}^1 \langle S \rangle_{\Lambda' + d_\Delta + 2, \gamma} \lesssim d_\Delta^{2d} e^{2\gamma d_\Delta} \left(\frac{1}{\kappa} \langle F \rangle_{\Lambda', \gamma} + \frac{1}{\kappa^2} \langle H \rangle_\Lambda |F|_\gamma \right).$$

(This norm is defined in §2.4.) The estimate of $\langle S \rangle_{\Lambda' + d_\Delta + 2, \gamma}$ is obtained by a finite induction using this estimate and the equation (ii) for the Töplitz limits.

Estimate of ω-derivatives. In order to estimate the derivatives in ω we consider the differentiated equation (31). The RHS $G_{[a]}^{[b]}$ verifies

$$(34) \qquad \left\| \partial_c G_{[a]}^{[b]} \right\| \leq \left\| \partial_c \partial_\omega F_{[a]}^{[b]} \right\| + \left(|k| + \left\| \partial_\omega H_{[a]} \right\| + \left\| \partial_\omega H_{[b]} \right\| \right) \left\| S_{[a]}^{[b]} \right\|$$
$$+ \left(\left\| \partial_c \partial_\omega H_{[a]} \right\| + \left\| \partial_c \partial_\omega H_{[b]} \right\| \right) \left\| S_{[a]}^{[b]} \right\|$$

and $\partial_c \partial_\omega S_{[a]}^{[b]}$ is now estimated like $\partial_c S_{[a]}^{[b]}$ but with G instead of F. Combining these estimates now gives the result when $F = \mathcal{T}_{\Lambda'} F$. By Lemma 6.2(i) we get the result for a general F. $\qquad\square$

6.4. *A second equation, $k = 0$.* Consider the equation

$$(35) \qquad\qquad i[\Omega(\omega) + H(\omega), S] = (\mathcal{T}_{\Lambda'} - \mathcal{D}_{\Lambda'}) F(\omega)$$

where $F(\omega) : \mathcal{L} \times \mathcal{L} \to \mathbb{C}$ and $\partial_\omega F(\omega)$ are Töplitz at ∞.

Let $U' \subset U$ be a set such that for all $\omega \in U'_\kappa$ the small divisor condition

$$(36) \qquad \begin{cases} |\alpha(\omega) - \beta(\omega)| \geq \kappa & \forall \begin{cases} \alpha(\omega) \in \sigma((\Omega + H)(\omega)_{[a]_\Delta}) \\ \beta(\omega) \in \sigma((\Omega + H)(\omega)_{[b]_\Delta}) \end{cases} \\ \text{dist}([a]_\Delta, [b]_\Delta) \leq \Delta' + 2d_\Delta \quad \text{and} \quad |a| \neq |b|. \end{cases}$$

holds.

PROPOSITION 6.4. *Let $\Delta' > 1$ and $0 < \kappa < 1$. Assume that U verifies* (13), *that Ω is real diagonal and verifies* (14), *and that H verifies* (16). *Assume also that $H(\omega)$ and $\partial_\omega H(\omega)$ are Töplitz at ∞ and $\mathcal{NF}_\Delta$ for all $\omega \in U$.*

Then equation (35), *under the condition that $S - \mathcal{T}_{\Lambda' + 2d_\Delta} S = \mathcal{D}_{\Lambda'} S = 0$ have, for all $\omega \in U'$, a unique solution $S(\omega)$ verifying*

(i) $|S|_{\{U'\}}^{\{\gamma\}} \leq \text{cte.} \frac{1}{\kappa^2} d_\Delta^{2d} e^{2\gamma\Delta} |F|_{\{U'\}}^{\{\gamma\}};$

(ii) $S(\omega)$ and $\partial_\omega S(\omega)$ are Töplitz at ∞ and the Töplitz limits verify

$$\begin{cases} i\langle k,\omega\rangle S + i[\Omega(\omega) + H(c,\omega), S] = (\mathcal{T}_{\Delta'} - \mathcal{D}_{\Delta'})\mathcal{P}_c F(c,\omega) \\ S - \mathcal{T}_{\Delta'+2d_\Delta} S = \mathcal{D}_{\Delta'} S = 0; \end{cases}$$

(iii) For any $\Lambda' \gtrsim \max(\Lambda, d_\Delta^2, (d_{\Delta'})^2, \sup_U \|H(\omega)\|)$,

$$\langle S\rangle_{\left\{\substack{\Lambda'+d_\Delta+2,\gamma \\ U'}\right\}} \le \text{cte.}\,\frac{1}{\kappa^3}d_\Delta^{2d}e^{2\gamma\Delta}(1 + \langle H\rangle_{\left\{\substack{\Lambda \\ U'}\right\}})\langle F\rangle_{\left\{\substack{\Lambda',\gamma \\ U'}\right\}}.$$

The constant cte. only depends on the dimensions d and $\#\mathcal{A}$ and on C_1, C_2, C_3, C_4.

Proof. We first assume that $F = (\mathcal{T}_{\Delta'} - \mathcal{D}_{\Delta'})F$. The proof is the same as in Proposition 6.3, with $k = 0$. Notice that the limit equation in (ii) is invariant under c-translations, due to Lemma 6.2(iii).

The proof gives that $\Lambda' \gtrsim \max(\Lambda, d_\Delta^2, \Delta', \sup_U \|H(\omega)\|)$. In order to get the result we need to estimate $(\mathcal{T}_{\Delta'} - \mathcal{D}_{\Delta'})F$ in terms of F. This is done by Lemma 6.2(i) and (iii) and requires a larger Λ'. $\qquad\square$

6.5. *A third equation.* Consider the equation

$$(37) \qquad i\langle k,\omega\rangle S + i(\Omega(\omega) + H(\omega))S + iS\mathcal{I}(\Omega(\omega) + {}^tH(\omega)) = F(\omega)$$

where $F(\omega) : \mathcal{L} \times \mathcal{L} \to \mathbb{C}$ and $\partial_\omega F(\omega)$ are Töplitz at ∞ and $\mathcal{I}Q$ is defined by $(\mathcal{I}Q)_a^b = Q_{-a}^{-b}$. (This equation will be motivated in the proof of Proposition 6.7.)

Let $U' \subset U$ be a set such that for all $\omega \in U'_\kappa$ the small divisor condition (20) holds for all a, b; i.e.,

$$|\langle k,\omega\rangle + \alpha(\omega) + \beta(\omega)| \ge \kappa \quad \forall \begin{cases} \alpha(\omega) \in \sigma((\Omega + H)(\omega)) \\ \beta(\omega) \in \sigma((\Omega + H)(\omega))). \end{cases}$$

PROPOSITION 6.5. *Let $0 < \kappa < 1$. Assume that U verifies* (13), *that Ω is real diagonal and verifies* (14), *and that H verifies* (16). *Assume also that $H(\omega)$ and $\partial_\omega H(\omega)$ are Töplitz at ∞ and $\mathcal{N}\mathcal{F}_\Delta$ for all $\omega \in U$.*

Then equation (37) *has for all $\omega \in U'$ a unique solution $S(\omega)$ verifying*

(i) $|S|_{\left\{\substack{\gamma \\ U'}\right\}} \le \text{cte.}\,\frac{1}{\kappa^2}d_\Delta^{2d}e^{2\gamma\Delta}(1 + |k|)\,|F|_{\left\{\substack{\gamma \\ U'}\right\}};$

(ii) *$S(\omega)$ and $\partial_\omega S(\omega)$ are Töplitz at ∞ and all Töplitz limits $S(c,\omega)$, $c \ne 0$, are $= 0$;*

(iii) *For any $\Lambda' \gtrsim \max(\Lambda, d_\Delta^2, \Delta', \sup_U \|H(\omega)\|)$,*

$$\langle S\rangle_{\left\{\substack{\Lambda'+d_\Delta+2,\gamma \\ U'}\right\}} \le \text{cte.}\,\frac{1}{\kappa^3}d_\Delta^{2d}e^{2\gamma\Delta}\left(1 + |k| + \langle H\rangle_{\left\{\substack{\Lambda \\ U'}\right\}}\right)\langle F\rangle_{\left\{\substack{\Lambda',\gamma \\ U'}\right\}}.$$

The constant cte. only depends on the dimensions d and $\#\mathcal{A}$ and on C_1, C_2, C_3, C_4.

Proof. As before we reduce to $\Omega_a = |a|^2$ and we block decompose the equation over $\mathcal{E}_\Delta$:

$$i\langle k,\omega\rangle S_{[a]}^{[b]} + i(\Omega + H)_{[a]}S_{[a]}^{[b]} + iS_{[a]}^{[b]}(\Omega + {}^tH)_{-[b]}F_{[a]}^{[b]}.$$

We then repeat the proof as for Proposition 6.3. There is a difference in the computation of the Töplitz limits. Equation (32) becomes

$$i\langle k, \omega\rangle S_{[a]}^{[b]}(tc) + i(\Omega + H)_{[a]}(tc) S_{[a]}^{[b]}(tc)$$
$$+ i S_{[a]}^{[b]}(tc)(\Omega + {}^t H)_{[-b]}(-tc) = F_{[a]}^{[b]}(tc)$$

and now $\Omega_{[a]}(tc)X + X\Omega_{[-b]}(-tc)$ equals

$$(t^2 |c|^2 + 2t\langle a, c\rangle + |a|^2)X + X(t^2 |c|^2 + 2t\langle b, c\rangle + |b|^2).$$

The quadratic terms in t do not cancel! Dividing the equation by t^2 and letting $t \to \infty$, the limit equation becomes $2|c|^2 X = 0$, which has the unique solution $X = 0$. Therefore $S_{[a]}^{[b]}(tc) \to S_{[a]}^{[b]}(\infty c) = 0$ as $t \to \infty$; i.e., the Töplitz limits are always 0.

In order to estimate the Lipschitz norm we only need to consider the analogue of Case II (even when $\langle c, a - b\rangle = 0$). We have for $[a] \times [b] \subset D_{\Lambda'}(c)$

$$|a|^2 + |b|^2 \gtrsim \left(\frac{|a|}{|c|}\right)^2 \approx \left(\frac{|b|}{|c|}\right)^2 \gtrsim (\Lambda')^2.$$

To avoid any problems with $\langle k, \omega\rangle$ and H it is sufficient that $(\Lambda')^2$ be $\gtrsim C_1 \Delta'$ and $\gtrsim \|H\|$.   □

6.6. *The homological equations.* Let $\Omega(\omega) : \mathcal{L} \times \mathcal{L} \to \text{gl}(2, \mathbb{C})$ be a real diagonal matrix; i.e.,

$$\Omega_a^b(\omega) = \begin{cases} \Omega_a(\omega)I & a = b \\ 0 & a \neq b. \end{cases}$$

Consider

$$(38) \qquad \begin{cases} |\Omega_a(\omega)| \geq C_5 > 0, \\ |\Omega_a(\omega) + \Omega_b(\omega)| \geq C_5, \\ |\Omega_a(\omega) - \Omega_b(\omega)| \geq C_5, \ |a| \neq |b|, \\ a, b \in \mathcal{L}, \ \omega \in U. \end{cases}$$

Let $H(\omega) : \mathcal{L} \times \mathcal{L} \to \text{gl}(2, \mathbb{C})$ and $\partial_\omega H(\omega)$ be Töplitz at ∞ for all $\omega \in U$ and consider

$$(39) \qquad \begin{cases} \|H(\omega)\| \leq \frac{C_5}{4}, & \omega \in U, \\ \langle H\rangle_{\{\Lambda\}\atop\{U\}} \leq C_6. \end{cases}$$

(Here $\| \ \|$ is the operator norm.)

PROPOSITION 6.6. *Let $\Delta' > 0$ and $0 < \kappa < \frac{C_5}{2}$. Assume that U verifies* (13), *that Ω is real diagonal and verifies* (14), (15), *and* (38), *and that H verifies* (16) *and* (39). *Assume also that $H(\omega)$ and $\partial_\omega H(\omega)$ are $\mathcal{NF}_\Delta$ for all $\omega \in U$. Then there is a subset $U' \subset U$,*

$$\text{Leb}(U \setminus U') \leq \text{cte.} \max(\Delta', d_\Delta^2)^{2d + \#\mathcal{A} - 1} \kappa,$$

such that for all $\omega \in U'$ the following hold:

(i) *For any $0 < |k| \leq \Delta'$, $|\langle k, \omega \rangle| \geq \kappa$.*

(ii) *For any $|k| \leq \Delta'$ and for any vector $F(\omega) \in l_\gamma^2(\mathcal{L}, \mathbb{C}^2)$ there exists a unique vector $S(\omega) \in l_\gamma^2(\mathcal{L}, \mathbb{C}^2)$ such that $i\langle k, \omega \rangle S + J(\Omega + H)S = F$ and satisfying*

$$\|S\|_{\{^\gamma_{U'}\}} \leq \text{cte.} \frac{1}{\kappa^2} \Delta' d_\Delta^{2m_*} e^{2\gamma d_\Delta} \|F\|_{\{^\gamma_{U'}\}} .$$

The constants cte. *only depend on $d, \#\mathcal{A}, m_*$ and on $C_1, \ldots, C_6$.*

Proof. (i) holds outside a set of ω of Lebesgue measure $\lesssim (\Delta')^{\#\mathcal{A}} \kappa$, so that it suffices to consider (ii). Let

$$C = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{-i}{\sqrt{2}} & \frac{i}{\sqrt{2}} \end{pmatrix}$$

and define ${}^t CAC : \mathcal{L} \times \mathcal{L} \to \text{gl}(2, \mathbb{C})$ through $({}^t CAC)_a^b = {}^t C A_a^b C$.

We change to complex coordinates $\tilde{S} = C^{-1}S$ and $\tilde{F} = C^{-1}F$. Then the equation becomes

$$i\langle k, \omega \rangle \tilde{S} - iJ \begin{pmatrix} 0 & \Omega + H \\ \Omega + {}^t H & 0 \end{pmatrix} \tilde{S} = \tilde{F}$$

where $\Omega, H : \mathcal{L} \to \mathbb{C}$ are the scalar-valued normal form matrices associated to Ω, H (see §5.1), Ω is real symmetric and H is Hermitian.

This equation decouples into two equations for (scalar-valued) matrices of type $i\langle k, \omega \rangle R \pm i(\Omega + Q)R = G$, where $Q = H$ or ${}^t H$. By Proposition 6.1 we can solve these equations uniquely for all $\omega \in U'$ such that

$$|\langle k, \omega \rangle + \alpha(\omega)| \geq \kappa \quad \forall \alpha(\omega) \in \sigma((\Omega + H)(\omega)), \ |k| \leq \Delta'.$$

If $k = 0$ this follows from (38) and (39) since $\kappa \leq \frac{C_5}{2}$. If $k \neq 0$ this follows from Proposition 5.1. $\square$

PROPOSITION 6.7. *Let $\Delta' > 0$ and $0 < \kappa < \frac{C_5}{2}$. Assume that U verifies (13), that Ω is real diagonal and verifies (14), (15), and (38), and that H verifies (16) and (39). Assume also that $H(\omega)$ and $\partial_\omega H(\omega)$ are $\mathcal{NF}_\Delta$ for all $\omega \in U$.*

Then there is a subset $U' \subset U$, $\text{Leb}(U - U') \leq \text{cte.} \max(\Lambda, \Delta, \Delta')^{\exp_\kappa(\frac{1}{d+1})^d}$, such that for all $\omega \in U'$ the following hold:

For any $|k| \leq \Delta'$ and for any matrix

$$\begin{cases} F(\omega) : \mathcal{L} \times \mathcal{L} \to \text{gl}(2, \mathbb{C}) \\ F(\omega) \text{ symmetric, i.e., } F_a^b = {}^t F_b^a \\ (\pi F)_a^b = 0 \quad \text{when } |a - b| > \Delta', \end{cases}$$

there exist symmetric matrices $S(\omega)$ and $H'(\omega)$ such that

$$i\langle k, \omega \rangle S + (\Omega + H)JS - SJ(\Omega + H) = F - H'$$

and satisfying, for any $\Lambda' \geq$ cte. $\max(\Lambda, d_\Delta^2, (d_{\Delta'})^2)$,

(i) $\langle S \rangle_{\substack{\{\Lambda'+d_\Delta+2,\gamma\} \\ \{U'\}}} \leq$ cte. $\frac{1}{\kappa^3} \Delta' d_\Delta^{2d} e^{2\gamma d_\Delta} \langle F \rangle_{\substack{\{\Lambda',\gamma\} \\ \{U'\}}}.$

(ii) *For $k \neq 0$ $H'(\omega) = 0$ and for $k = 0$, $H'(\omega)$ and $\partial_\omega H'(\omega)$ are block diagonal over $\mathcal{E}_{\Delta'}$ and*

$$\langle H' \rangle_{\substack{\{\Lambda'+d_\Delta+2\} \\ \{U'\}}} \leq \langle F \rangle_{\substack{\{\Lambda'\} \\ \{U'\}}}.$$

Moreover, if F is real then $H'(\omega)$ and $\partial_\omega H'(\omega)$ are $\mathcal{NF}_{\Delta'}$.

The exponent exp *only depends on d, $\#\mathcal{A}$ and the constants* cte. *also depend on $C_1, \ldots, C_6$.*

Proof. We change to complex coordinates $\widetilde{S} = {}^t C S C$ and $\widetilde{F} = {}^t C F C$. Then the equation becomes

$$\widetilde{F} - \widetilde{H}' = i \langle k, \omega \rangle \widetilde{S} - i \begin{pmatrix} 0 & \Omega + H \\ \Omega + {}^t H & 0 \end{pmatrix} J \widetilde{S} - i \widetilde{S} J \begin{pmatrix} 0 & \Omega + H \\ \Omega + {}^t H & 0 \end{pmatrix}$$

where $\Omega, H : \mathcal{L} \to \mathbb{C}$ are the scalar-valued normal form matrices associated to Ω, H (see §5.1) – Ω is real symmetric and H is Hermitian.

If we write

$$F = \begin{pmatrix} F_1 & F_2 \\ {}^t F_2 & F_3 \end{pmatrix},$$

then

$$\widetilde{F} = \frac{1}{2} \begin{pmatrix} (F_1 - F_3) - i(F_2 + {}^t F_2) & (F_1 + F_3) + i(F_2 - {}^t F_2) \\ (F_1 + F_3) - i(F_2 - {}^t F_2) & (F_1 - F_3) + i(F_2 + {}^t F_2) \end{pmatrix},$$

the diagonal parts coming from $(I - \pi)F$ and the off-diagonal parts from πF.

The equation decouples into four (scalar-valued) matrices of the types

$$i \langle k, \omega \rangle R \pm i((\Omega + Q)R - R(\Omega + Q)) = G - P,$$

for the off-diagonal terms, and

$$i \langle k, \omega \rangle R \pm i((\Omega + Q)R + R(\Omega + {}^t Q)) = G - P,$$

for the diagonal terms. Here $Q = H$ or ${}^t H$.

Let us first consider the off-diagonal equations. By the assumption on F, $\mathcal{T}_{\Delta'} G = G$, G is Töplitz at ∞ and

$$\langle G \rangle_{\substack{\{\Lambda',\gamma\} \\ \{U'\}}} \leq \langle F \rangle_{\substack{\{\Lambda',\gamma\} \\ \{U'\}}}.$$

Moreover, G is Hermitian if F is real.

If $k \neq 0$ we take $P = 0$ and we can solve the equation by Proposition 6.3 for all ω such that

$$|\langle k, \omega \rangle + \alpha(\omega) - \beta(\omega)| \geq \kappa \quad \forall \begin{cases} \alpha(\omega) \in \sigma((\Omega + H)(\omega)_{[a]_\Delta}) \\ \beta(\omega) \in \sigma((\Omega + H)(\omega)_{[b]_\Delta}), \end{cases}$$

for dist$([a]_\Delta, [b]_\Delta) \le \Delta' + 2d_\Delta$. The set of such ω is estimated in Proposition 5.2. The solution is unique if we impose $\mathscr{T}_{\Delta'+2d_\Delta} R - R = 0$.

If $k = 0$ we take $P = \mathscr{D}_{\Delta'} G$ and we can solve the equation by Proposition 6.4 for all ω such that

$$|\alpha(\omega) - \beta(\omega)| \ge \kappa \quad \forall \begin{cases} \alpha(\omega) \in \sigma((\Omega + H)(\omega)_{[a]_\Delta}) \\ \beta(\omega) \in \sigma((\Omega + H)(\omega)_{[b]_\Delta}) \end{cases}$$

for dist$([a]_\Delta, [b]_\Delta) \le \Delta' + 2d_\Delta$ and $|a| \ne |b|$. This condition on ω holds by assumptions (38) and (39) since $\kappa \le \frac{C_5}{2}$. The solution is unique if we impose $\mathscr{T}_{\Delta'+2d_\Delta} R - R = \mathscr{D}_{\Delta'} R = 0$. P is estimated by Lemma 6.2(iii).

To treat the diagonal equations let us consider the operators $(\mathscr{R}G)_a^b = G_a^{-b}$ and $(\mathscr{I}G)_a^b = G_{-a}^{-b}$. Now $\mathscr{R}G$, G coming from $(I - \pi)F$, is Töplitz at ∞ and

$$\langle \mathscr{R}G \rangle_{\{U'\}}^{\{\Delta',\gamma\}} \le \langle F \rangle_{\{U'\}}^{\{\Delta',\gamma\}}.$$

With $T = \mathscr{R}R$ the equation takes the form

$$i\langle k, \omega \rangle T \pm i((\Omega + Q)T + T\mathscr{I}(\Omega + {}^tQ)) = \mathscr{R}G - \mathscr{R}P.$$

We take $\mathscr{R}P = 0$ and then the result follows from Proposition 6.5 under the assumption (20) on ω. This assumption holds for $k = 0$ by (38) and (39) and for $k \ne 0$ on a set U' which is estimated in Proposition 5.1.

By construction H' is symmetric. Moreover, for $k = 0$, $(\pi S)_a^b = 0$ when $|a - b| > \Delta' + 2d_\Delta$ or $[a]_\Delta = [b]_\Delta$; and for $k \ne 0$, $(\pi S)_a^b = 0$ when $|a - b| > \Delta' + 2d_\Delta$. These conditions determine S uniquely and symmetry follows from this. $\square$

Part III. KAM

7. A KAM theorem

7.1. *Statement of the theorem.* Let

$$\mathbb{O}^\gamma(\sigma, \rho, \mu) = \mathbb{O}^\gamma(\sigma) \times \mathbb{T}_\rho^{\mathscr{A}} \times \mathbb{D}(\mu)^{\mathscr{A}}$$

be the set of all ζ, φ, r such that

$$\zeta = (\xi, \eta) \in \mathbb{O}^\gamma(\sigma), \quad |\Im\varphi_a| < \rho, \quad |r_a| < \mu \quad \forall a \in \mathscr{A}.$$

Let

$$h_\omega(\zeta, r) = h(\zeta, r, \omega) = \langle \omega, r \rangle + \frac{1}{2}\langle \zeta, (\Omega(\omega) + H(\omega))\zeta \rangle$$

where $\Omega(\omega)$ is a real diagonal matrix with diagonal elements $\Omega_a(\omega)I$ and $H(\omega)$ and $\partial_\omega H(\omega)$ are Töplitz at ∞ and $\mathscr{NF}_\Delta$ for all $\omega \in U$. We recall (§5.1) that a matrix $H : \mathscr{L} \times \mathscr{L} \to \mathrm{gl}(2, \mathbb{C})$ is $\mathscr{NF}_\Delta$ if it is real, symmetric and can be written

$$H = \begin{pmatrix} Q_1 & Q_2 \\ {}^tQ_2 & Q_1 \end{pmatrix}$$

with $Q = Q_1 + i Q_2$ Hermitian and block-diagonal over the decomposition $\mathscr{E}_\Delta$ of $\mathscr{L}$.

We assume (13)–(15) and (38); i.e.,

$$U \text{ is an open subset of } \{|\omega| < C_1\} \subset \mathbb{R}^{\#\mathscr{A}},$$

$$\begin{cases} \left| \partial_\omega^\nu (\Omega_a(\omega) - |a|^2) \right| \leq C_2 e^{-C_3 |a|}, \quad C_3 > 0, \\ (a, \omega) \in \mathscr{L} \times U, \quad \nu = 0, 1, \end{cases}$$

$$\begin{cases} \langle \partial_\omega(\langle k, \omega \rangle + \Omega_a(\omega)), \frac{k}{|k|} \rangle \geq C_4 > 0, \\ \langle \partial_\omega(\langle k, \omega \rangle + \Omega_a(\omega) + \Omega_b(\omega)), \frac{k}{|k|} \rangle \geq C_4, \\ \langle \partial_\omega(\langle k, \omega \rangle + \Omega_a(\omega) - \Omega_b(\omega)), \frac{k}{|k|} \rangle \geq C_4, \\ a, b \in \mathscr{L}, \ k \in \mathbb{Z}^{\mathscr{A}} \setminus 0, \ \omega \in U, \end{cases}$$

$$\begin{cases} |\Omega_a(\omega)| \geq C_5 > 0, \\ |\Omega_a(\omega) + \Omega_b(\omega)| \geq C_5, \\ |\Omega_a(\omega) - \Omega_b(\omega)| \geq C_5, \ |a| \neq |b|, \\ a, b \in \mathscr{L}, \ \omega \in U. \end{cases}$$

Remark. The conditions on the directional derivative hold trivially for $C_4 = \frac{1}{2}$ if $|\partial_\omega \Omega_a(\omega)| \leq \frac{1}{4}$ for all $(a, \omega) \in \mathscr{L} \times U$.

We also assume (16) and (39); i.e.,

$$\begin{cases} \|\partial_\omega H(\omega)\| \leq \frac{C_4}{4} \\ \|H(\omega)\| \leq \frac{C_5}{4} \\ \langle H \rangle_{\{\Lambda\} \atop \{U\}} \lesssim 1 \end{cases}$$

for some Λ. (Here $\| \ \|$ is the operator norm.)

Remark. For simplicity we assume that $\gamma, \sigma, \rho, \sigma$ are < 1 and that Δ, Λ are ≥ 3.

Let $f : \mathbb{O}^\gamma(\sigma, \rho, \mu) \times U \to \mathbb{C}$ be real analytic in ζ, φ, r and $\mathscr{C}^1$ in $\omega \in U$ and let

$$[f]_{\{\Lambda, \gamma, \sigma\} \atop \{U, \rho, \mu\}} \quad \sup_{\substack{\varphi \in \mathbb{T}_\rho^{\mathscr{A}} \\ r \in \mathbb{D}(\mu)^{\mathscr{A}}}} \ [f(\cdot, \varphi, r, \cdot)]_{\{\Lambda, \gamma, \sigma\} \atop \{U\}}.$$

THEOREM 7.1. *Assume that U verifies (13), that Ω is real diagonal and verifies (14), (15), and (38), that $H(\omega)$ and $\partial_\omega H(\omega)$ are Töplitz at ∞ and $\mathscr{N}\mathscr{F}_\Delta$ for all $\omega \in U$, and that H verifies (16) and (39).*

Then there is a constant Cte. *and an exponent* exp *such that, if*

$$[f]_{\{\Lambda, \gamma, \sigma\} \atop \{U, \rho, \mu\}} = \varepsilon \leq \text{Cte.} \min\left(\gamma, \rho, \frac{1}{\Lambda}, \frac{1}{\Delta}\right)^{\exp} \min(\sigma^2, \mu)^2,$$

there is a $U' \subset U$ with $\mathrm{Leb}(U \setminus U') \le \mathrm{cte}.\varepsilon^{\exp'}$ such that for all $\omega \in U'$ the following hold: *There are a real analytic symplectic diffeomorphism*

$$\Phi : \mathbb{O}^0\left(\frac{\sigma}{2}, \frac{\rho}{2}, \frac{\mu}{2}\right) \to \mathbb{O}^0(\sigma, \rho, \mu)$$

and a vector $\omega' = \omega'(\omega)$ such that $(h_{\omega'} + f) \circ \Phi$ equals (modulo a constant)

$$\langle \omega, r \rangle + \frac{1}{2}\langle \zeta, (\Omega + H')(\omega)\zeta \rangle + f'(\zeta, \varphi, r, \omega)$$

where $\partial_\zeta f' = \partial_r f' = \partial_\zeta^2 f' = 0$ for $\zeta = r = 0$ and

$$H' = \begin{pmatrix} Q_1' & Q_2' \\ {}^t Q_2' & Q_1' \end{pmatrix}$$

with $Q' = Q_1' + i Q_2'$ Hermitian and block diagonal, i.e., $(Q')_a^b = 0$ for all $|a| \ne |b|$.
Moreover, $\Phi = (\Phi_\zeta, \Phi_\varphi, \Phi_r)$ verifies

$$\left\| \Phi_\zeta - \zeta \right\|_0 + \left| \Phi_\varphi - \varphi \right| + |\Phi_r - r| \le \mathrm{cte}.\beta\varepsilon$$

for all $(\zeta, \varphi, r) \in \mathbb{O}^0(\frac{\sigma}{2}, \frac{\rho}{2}, \frac{\mu}{2})$, $\|H'(\omega) - H(\omega)\| \le \mathrm{cte}.\beta\varepsilon$ and the mapping $\omega \mapsto \omega'(\omega)$ verifies $|\omega' - \mathrm{id}|_{\mathscr{C}^1(U')} \le \mathrm{cte}.\frac{\varepsilon}{\mu}$.

The exponents $\exp, \exp'$ only depend on $d, \#\mathscr{A}, m_*$, the constants $\mathrm{Cte}., \mathrm{cte}.$ also depend on $C_1, \dots, C_5$ and $\beta = \beta(\gamma, \sigma, \rho, \mu, \Lambda, \Delta, \omega)$.

Remark 1. Each block-component of $\Omega + H(\omega)$ is of finite dimension but in general there is no uniform bound – they may be of arbitrarily large dimension. Due to this lack of uniformity we lose, in our estimates, all exponential decay in the space modes. However, if there were a uniform bound (as happens in some cases, cf. [GY06]) we would retain some exponential decay.

Remark 2. It follows from the proof that Φ is of the form

$$\begin{cases} \Phi_\zeta(\zeta, \varphi, r) = z(\varphi) + Z(\varphi)\zeta, \\ \Phi_\varphi(\zeta, \varphi, r) = \varphi + a(\varphi), \\ \Phi_r(\zeta, \varphi, r) = r + b(\zeta, \varphi) + c(\varphi)r, \end{cases}$$

where $b(\zeta, \varphi)$ is quadratic in ζ, because Φ is a composition of mappings of this form.

If f does not depend on r, then $a = c = 0$ and $\omega' = \omega$, because Φ is a composition of mappings of this form, and it preserves Hamiltonians of this form.

If $f(\zeta, \varphi) = \frac{1}{2}\langle \zeta, F(\varphi)\zeta \rangle$, then also $z = 0$ and $b(\zeta, \varphi) = \frac{1}{2}\langle \zeta, B(\varphi)\zeta \rangle$, because Φ is a composition of mappings of this form, and it preserves Hamiltonians of this form.

Since the consequences of the theorem are discussed in the introduction, let us instead here discuss a special case. Consider a linear nonautonomous Hamiltonian system with quasiperiodic coefficients

$$\dot{\zeta} = J\big(\Omega + H(\omega) + \varepsilon F(\varphi, \omega)\big)\zeta, \quad \dot{\varphi} = \omega$$

where Ω and $H(\omega)$ are as in Theorem 7.1 and F is symmetric and Töplitz at ∞ and

$$\langle F(\varphi, \cdot) \rangle_{\substack{\{\Lambda, \gamma\} \\ \{U\}}} < \infty$$

for $|\Im\varphi| < \rho$ and for some $\gamma > 0$. Then, by Young's inequality (2),

$$\|F(\varphi, \omega)\zeta\|_{\gamma'} \leq \left(\frac{1}{\gamma - \gamma'}\right)^{d+m_*} |F(\varphi, \omega)|_\gamma \|\zeta\|_{\gamma'} \quad \forall \gamma' < \gamma$$

and

$$|\langle \zeta, F(\varphi, \omega)\zeta \rangle| \leq \left(\frac{1}{\gamma}\right)^{d+m_*} |F(\varphi, \omega)|_\gamma \|\zeta\|_0^2 .$$

Therefore we can apply Theorem 7.1 and Remark 2 to the Hamiltonian

$$h + \varepsilon f = \langle \omega, r \rangle + \frac{1}{2} \langle \zeta, (\Omega + H(\omega) + F(\varphi, \omega))\zeta \rangle.$$

If ε is sufficiently small, it gives a mapping Φ such that

$$(h + \varepsilon f) \circ \Phi(\zeta, \varphi, r) = \langle \omega, r \rangle + \frac{1}{2} \langle \zeta, (\Omega + H'(\omega))\zeta \rangle$$

with

$$\Phi(\zeta, \varphi, r) = \begin{pmatrix} Z(\varphi)\zeta \\ r + \frac{1}{2}\langle \zeta, B(\varphi)\zeta \rangle \\ \varphi \end{pmatrix}.$$

From this form and from the symplectic character of Φ we derive

$$\langle \partial_\varphi Z(\varphi), \omega \rangle = J(\Omega + H + F(\varphi))Z(\varphi) - Z(\varphi)J(\Omega + H').$$

This implies that the mapping $(\zeta, \varphi) \mapsto (w = Z(\varphi)\zeta, \varphi)$ reduces the linear nonautonomous system to the autonomous system $\dot{w} = J(\Omega + H'(\omega))\zeta$, $\dot{\varphi} = \omega$. Notice also that $J(\Omega + H'(\omega))$ is block-diagonal with purely imaginary eigenvalues.

7.2. *Application to the Schrödinger equation.* Consider a nonlinear Schrödinger equation

$$(*) \qquad -i\dot{u} = -\Delta u + V(x) * u + \varepsilon \frac{\partial F}{\partial \bar{u}}(x, u, \bar{u}), \quad u = u(t, x), \ x \in \mathbb{T}^d,$$

where $V(x) = \sum \hat{V}(a)e^{i\langle a, x \rangle}$ is an analytic function with $\hat{V}$ real and where F is real analytic in $\Re u, \Im u$ and in $x \in \mathbb{T}^d$.

Let $\mathscr{A} \subset \mathbb{Z}^d$ be a finite set and consider a function

$$u_1(\varphi, x) = \sum_{a \in \mathscr{A}} \sqrt{p_a} e^{i\varphi_a} e^{i\langle a, x \rangle}, \quad p_a > 0,$$

such that $(x, u_1(\varphi, x), \bar{u}_1(\varphi, x))$ belongs to the domain of F for all $(x, \varphi) \in \mathbb{T}^d \times \mathbb{T}^{\mathscr{A}}$. Then $u_1(t, x) = u_1(\varphi + t\omega, x)$ is a solution of $(*)$ for $\varepsilon = 0$.

Let $\mathscr{L}$ be the complement of $\mathscr{A}$ and let

$$\omega = \{\omega_a = |a|^2 + \hat{V}(a) : a \in \mathscr{A}\},$$
$$\Omega = \{\omega_a = |a|^2 + \hat{V}(a) : a \in \mathscr{L}\}.$$

Let V depend $\mathscr{C}^1$ on a parameter $w \in W \subset \mathbb{R}^{\#\mathscr{A}}$ and assume that the mapping

$$W \ni w \mapsto \omega(w) = \{\omega_a = |a|^2 + \hat{V}(a, w); a \in \mathscr{A}\} \subset U$$

is a diffeomorphism whose inverse is bounded in the $\mathscr{C}^1$-norm, i.e.,

(40) $$|\omega^{-1}|_{\mathscr{C}^1} \leq C_6.$$

We also assume conditions analogous to (13)–(15) and (38); i.e.,

$$W \text{ is an open subset of } \{|w| < C_1\} \subset \mathbb{R}^{\#\mathscr{A}},$$

$$\begin{cases} |\partial_w^\nu (\Omega_a(w) - |a|^2)| \leq C_2 e^{-C_3|a|}, \quad C_3 > 0, \\ (a, w) \in \mathscr{L} \times W, \quad \nu = 0, 1, \end{cases}$$

$$\begin{cases} \langle \partial_w(\langle k, \omega(w)\rangle + \Omega_a(w)), v_k \rangle \geq C_4 > 0, \\ \langle \partial_w(\langle k, \omega(w)\rangle + \Omega_a(w) + \Omega_b(w)), v_k \rangle \geq C_4, \\ \langle \partial_w(\langle k, \omega(w)\rangle + \Omega_a(w) - \Omega_b(w)), v_k \rangle \geq C_4, \\ a, b \in \mathscr{L}, \; k \in \mathbb{Z}^{\mathscr{A}} \setminus 0, \; w \in W, \end{cases}$$

where

$$v_k = (D\omega(w))^{-1} \frac{k}{|k|},$$

and

$$\begin{cases} |\Omega_a(w)| \geq C_5 > 0, \\ |\Omega_a(w) + \Omega_b(w)| \geq C_5, \\ |\Omega_a(w) - \Omega_b(w)| \geq C_5, \quad |a| \neq |b|, \\ a, b \in \mathscr{L}, \; \omega \in U. \end{cases}$$

THEOREM 7.2. *For ε sufficiently small, there is a subset $W' \subset W$,*

$$\mathrm{Leb}(W \setminus W') \leq \mathrm{cte}.\varepsilon^{\exp},$$

such that on W' there is a $u(\varphi, x)$, analytic in $\varphi \in \mathbb{T}^d_{\frac{\rho}{2}}$ and of class $\mathscr{C}^{m_* - d}$ in $x \in \mathbb{T}^d$, with

$$\sup_{|\Im\varphi| < \frac{\rho}{2}} \|u(\varphi, \cdot) - u_1(\varphi, \cdot)\|_{H^{m_*}(\mathbb{T}^d)} \leq \beta\varepsilon,$$

and there is a $\omega' : W' \to U$, $|\omega' - \omega|_{\mathbb{C}^1(W')} \leq \beta\varepsilon$, such that

$$u(t, x) = u(\varphi + t\omega'(w), x)$$

is a solution of $(*)$ for any $w \in W'$.

Moreover, the linearized equation

$$-i\dot{v} = \Delta v + V(x) * v + \varepsilon \frac{\partial^2 F}{\partial \bar{u}^2}(x, u(t, x), \bar{u}(t, x))\bar{v}$$

$$+ \varepsilon \frac{\partial^2 F}{\partial u \partial \bar{u}}(x, u(t, x), \bar{u}(t, x))v$$

is reducible to constant coefficients and has only time-quasi-periodic solutions, except for a (#𝒜)-dimensional subspace where solutions may increase at most linearly in t. Also, β is a constant that depends on the dimensions d, #𝒜, m_*, the constants $C_1, \ldots, C_6$ and on w and F.

Proof. We write

$$\begin{cases} u(x) = \sum_{a \in \mathbb{Z}^d} u_a e^{i\langle a, x \rangle}, \\ \overline{u(x)} = \sum_{a \in \mathbb{Z}^d} v_a e^{i\langle -a, x \rangle} \quad (v_a = \bar{u}_a), \end{cases}$$

and let

$$\zeta_a = \begin{pmatrix} \xi_a \\ \eta_a \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}}(u_a + v_a) \\ \frac{-i}{\sqrt{2}}(u_a - v_a) \end{pmatrix}.$$

In the symplectic space

$$\{(\xi_a, \eta_a) : a \in \mathbb{Z}^d\} = \mathbb{R}^{\mathbb{Z}^d} \times \mathbb{R}^{\mathbb{Z}^d}, \quad \sum_{a \in \mathbb{Z}^d} d\xi_a \wedge d\eta_a,$$

the equation becomes a Hamiltonian equation in infinite degrees of freedom. The Hamiltonian function has an integrable part

$$\frac{1}{2} \sum_{a \in \mathbb{Z}^d} (|a|^2 + \hat{V}(a))(\xi_a^2 + \eta_a^2)$$

plus a perturbation.

In a neighborhood of the unperturbed solution $\frac{1}{2}(\xi_a^2 + \eta_a^2) = p_a$, $a \in \mathcal{A}$, we introduce the action angle variables (φ_a, r_a) (notice that each $p_a > 0$ by assumption), defined through the relations

$$\xi_a = \sqrt{2(r_a + p_a)} \cos(\varphi_a), \qquad \eta_a = \sqrt{2(r_a + p_a)} \sin(\varphi_a).$$

The integrable part of the Hamiltonian becomes

$$h(\zeta, r, \omega) = \langle \omega, r \rangle + \frac{1}{2} \sum_{a \in \mathcal{L}} \Omega_a(\omega)(\xi_a^2 + \eta_a^2),$$

while the perturbation

$$\varepsilon f(u, \bar{u}) = \varepsilon \int_{\mathbb{T}^d} F(x, u(x)\bar{u}(x))dx,$$

will be a function of ζ, φ, r. If we write

$$G(x, u_1, \bar{u}_1, u, \bar{u}) = F(x, u_1 + u, \bar{u}_1 + \bar{u})$$

then G is an analytic function in $x, u, \bar{u}$ which depends analytically on φ, r. Then one verifies (see Lemma 1 in [EK08]) that, since $m_* > \frac{d}{2}$, there exist $\gamma, \sigma, \rho, \mu$ such that f is real analytic on $\mathbb{O}^\gamma(\sigma, \rho, \mu)$ and that f has the Töplitz-Lipschitz property:

$$
(41) \qquad [f]_{\substack{\{\Lambda, \gamma, \sigma\} \\ \{U, \rho, \mu\}}} \le C_7
$$

for some constant C_7.

The assumptions of Theorem 7.1 are now fulfilled and gives the result. $\qquad \square$

8. **Proof of theorem**

8.1. *Preliminaries.* Let $f : \mathbb{O}^\gamma(\sigma, \rho, \mu) \times U \to \mathbb{C}$ be real analytic in ζ, φ, r and $\mathscr{C}^1$ in $\omega \in U$ and consider $[f]_{\substack{\{\Lambda, \gamma, \sigma\} \\ \{U, \rho, \mu\}}}$.

Notation. We let

$$
\alpha = \begin{pmatrix} \gamma & \sigma \\ \rho & \mu \end{pmatrix},
$$

and we write this norm as $[f]_{\substack{\{\Lambda \\ U} \alpha\}}$.

Remark. We assume that all $\gamma, \sigma, \rho, \mu$ are < 1, that $0 < \sigma - \sigma' \approx \sigma$, $0 < \mu - \mu' \approx \mu$ and that $\Lambda, \Delta \ge 3$.

Cauchy estimates. It follows by Cauchy estimates that

$$
(42) \qquad [\partial_\varphi f]_{\substack{\{\Lambda \\ U} \alpha'\}} \lesssim \frac{1}{\rho - \rho'} [f]_{\substack{\{\Lambda \\ U} \alpha\}},
$$

$$
[\partial_r f]_{\substack{\{\Lambda \\ U} \alpha'\}} \lesssim \frac{1}{\mu - \mu'} [f]_{\substack{\{\Lambda \\ U} \alpha\}}.
$$

Truncation. We obtain $\mathscr{T}_\Delta f$ from f by: 1) truncating the Taylor expansion in ζ at order 2; 2) truncating the Taylor expansion in r at order 0 for the first and the second order term in ζ and at order 1 for the zero'th order term in ζ; 3) truncating the Fourier modes at order Δ; 4) truncating the space modes of the second order term in ζ at order Δ. Formally $\mathscr{T}_\Delta f$ is

$$
\sum_{|k| \le \Delta} [\hat{f}(0, k, 0, \omega) + \partial_r \hat{f}(0, k, 0, \omega) r + \langle \partial_\zeta \hat{f}(0, k, 0, \omega), \zeta \rangle
$$

$$
+ \frac{1}{2} \langle \zeta, \mathscr{T}_\Delta \partial_\zeta^2 \hat{f}(0, k, 0, \omega) \zeta \rangle] e^{i\langle k, \varphi \rangle}.
$$

We have

$$
(43) \qquad [\mathscr{T}_\Delta f]_{\substack{\{\Lambda \\ U} \alpha\}} \lesssim \Delta^{\#\mathscr{A}} [f]_{\substack{\{\Lambda \\ U} \alpha\}}
$$

and

$$
(44) \qquad [f - \mathscr{T}_\Delta f]_{\substack{\{\Lambda \\ U} \alpha'\}} \lesssim A(\alpha, \alpha', \Delta) [f]_{\substack{\{\Lambda \\ U} \alpha\}},
$$

where $A(\alpha, \alpha', \Delta)$ is

$$\left(\frac{\sigma'}{\sigma}\right)^3 + \left(\frac{\sigma'}{\sigma} + \frac{\mu'}{\mu}\right)\frac{\mu'}{\mu} + \left(\frac{1}{\rho - \rho'}\right)^{\#\mathscr{A}} e^{-\Delta(\rho - \rho')} + e^{-\Delta(\gamma - \gamma')}.$$

This follows from Proposition 3.2, from Cauchy estimates in r and φ, and from formula (8).

Poisson brackets. The Poisson bracket is defined by

$$\{f, g\} = \langle \partial_\zeta f, J \partial_\zeta g \rangle + \partial_\varphi f \, \partial_r g - \partial_r f \, \partial_\varphi g.$$

If g is a quadratic polynomial in ζ, then

(45) $$[\{f, g\}]_{\left\{^{\Lambda + 3}_U \alpha'\right\}} \lesssim B(\gamma - \gamma', \sigma, \rho - \rho', \mu, \Lambda)[f]_{\left\{^\Lambda_U \alpha\right\}}[g]_{\left\{^\Lambda_U \alpha\right\}},$$

where $B = \Lambda^2 \frac{1}{\sigma^2}\left(\frac{1}{\gamma - \gamma'}\right)^{d + m_*} + \frac{1}{\rho - \rho'}\frac{1}{\mu}.$

Also, if f is a quadratic polynomial in ζ and, moreover, independent of φ and of the form $\langle a, r \rangle + \frac{1}{2}\langle \zeta, A\zeta \rangle$, then

(46) $$[\{f, g\}]_{\left\{^{\Lambda + 3}_U \alpha'\right\}} \lesssim B(\bar{\gamma} - \gamma', \sigma_1, \bar{\rho} - \rho', \mu_1, \Lambda)[f]_{\left\{^\Lambda_U \alpha_1\right\}}[g]_{\left\{^\Lambda_U \alpha_2\right\}},$$

$$\alpha_i = \begin{pmatrix} \gamma & \sigma_i \\ \rho & \mu_i \end{pmatrix}, \quad i = 1, 2,$$

and $\bar{\gamma} = \min(\gamma_1, \gamma_2)$, $\bar{\rho} = \min(\rho_1, \rho_2)$.[10]

In both cases, the first term to the right (in the expression for $\{f, g\}$ above) is estimated by Proposition 3.3 and the other two terms by Cauchy estimates.

We shall use both these estimates. Notice that (46) is much better than (45) when σ_2, μ_2 are much smaller than σ_1, μ_1.

Flow maps. Let

$$s = \mathscr{T}_\Delta s = S_0(\varphi, r, \omega) + \langle \zeta, S_1(\varphi, \omega) \rangle + \frac{1}{2}\langle \zeta, S_2(\varphi, \omega)\zeta \rangle.$$

Notice that, since $s = \mathscr{T}_\Delta$, S_0 is of first order in r. Consider the Hamiltonian vector field

$$\frac{d}{dt}\begin{pmatrix} \zeta \\ \varphi \\ r \end{pmatrix} = \begin{pmatrix} J\partial_\zeta s \\ \partial_r s \\ -\partial_\varphi s \end{pmatrix} = \begin{pmatrix} JS_1(\varphi, \omega) + JS_2(\varphi, \omega)\zeta \\ \partial_r S_0(\varphi, 0, \omega) \\ -\partial_\varphi s(\zeta, \varphi, r, \omega) \end{pmatrix}$$

and let

$$\Phi_t = \begin{pmatrix} \zeta_t \\ \varphi_t \\ r_t \end{pmatrix} = \begin{pmatrix} \zeta + b_t(z, \omega) + B_t(z, \omega)\zeta \\ z + g_t(\zeta, z, \omega) \end{pmatrix}$$

be the flow. Here we have denoted φ and r by z.

---

[10] In the expression for B we have assumed that $0 < \sigma_j - \sigma' \approx \sigma$, $0 < \mu_j - \mu' \approx \mu_j$, $j = 1, 2$.

Assume that

(47) $$[s]_{\{\overset{\Lambda}{U}\alpha\}} = \varepsilon \lesssim \min((\rho - \rho')\mu, (\gamma - \gamma')^{d+m_*}\sigma^2).$$

Then for $|t| \leq 1$ we have: $\Phi_t : \mathbb{O}^{\gamma''}(\sigma', \rho', \mu') \to \mathbb{O}^{\gamma''}(\sigma, \rho, \mu)$, for all $\gamma'' \leq \gamma'$;

(48) $$[g_t]_{\{\overset{\Lambda}{U}\alpha'\}} \lesssim \frac{\varepsilon}{\mu} \quad \text{or} \quad \frac{\varepsilon}{\rho - \rho'}$$

depending on if g is a φ-component or an r-component;

(49) $$\|b_t + B_t\zeta\|_{\{\overset{\gamma''}{U,\rho'}\}} \lesssim \left(\left(\frac{1}{\gamma - \gamma'} \right)^{m_*} + \left(\frac{1}{\gamma - \gamma'} \right)^{d+m_*} \frac{1}{\sigma} \|\zeta\|_{\gamma''} \right) \frac{\varepsilon}{\sigma}$$

for all $\gamma'' \leq \gamma'$;

(50) $$\langle B_t \rangle_{\left\{ \overset{\Lambda+6,\gamma'}{U,\rho'} \right\}} \lesssim \Lambda^2 \left(\frac{1}{\gamma - \gamma'} \right) \frac{\varepsilon}{\sigma^2}.$$

Moreover, for $1 \geq \bar{\sigma} \geq \sigma'$ and $1 \geq \bar{\mu} \geq \mu'$, Φ_t has an analytic (because polynomial in ζ and ρ) extension to $\mathbb{O}^{\gamma''}(\bar{\sigma}, \rho', \bar{\mu})$ for all $\gamma'' \leq \gamma'$ and verifies on this set

(51) $$\begin{cases} \|\zeta_t - \zeta\|_{\gamma''} \lesssim (\frac{1}{\gamma - \gamma'})^{d+m_*}(\frac{\bar{\sigma}}{\sigma} + 1)\frac{\varepsilon}{\sigma} \\ |\varphi_t - \varphi| \lesssim \frac{\varepsilon}{\mu} \\ |r_t - r| \lesssim (\frac{1}{\rho - \rho'})(\frac{\bar{\mu}}{\mu} + (\frac{\bar{\sigma}}{\sigma})^2 + 1)\varepsilon. \end{cases}$$

Proof. We have $\varphi_t = \varphi + a_t(\varphi, \omega)$ and since $|\partial_r S_0(\varphi, 0, \omega)| \lesssim \frac{\varepsilon}{\mu}$ for all $\varphi \in \mathbb{T}_\rho^{\mathscr{A}}$, φ_t remains in $\mathbb{T}_\rho^{\mathscr{A}}$ for $|t| \leq 1$ if $\frac{\varepsilon}{\mu} \lesssim (\rho - \rho')$. The ω-derivative verifies

$$\frac{d}{dt}(\partial_\omega \varphi_t) = \partial_\omega \partial_r S_0(\varphi, 0, \omega) + \partial_\varphi \partial_r S_0(\varphi, 0, \omega)(\partial_\omega \varphi_t)$$

and can be solved explicitly by an integral formula. This gives (48) for $z = \varphi$ and the φ-part of (51).

For a fixed ω (49) follows from the first part of Proposition 3.4(i) if $|JS_2|_\gamma \lesssim (\gamma - \gamma')^d$, i.e., if $\varepsilon \lesssim (\gamma - \gamma')^d \sigma^2$. This also gives the ζ-part of (51). In order to get $\|\zeta_t - \zeta\|_{\gamma'} \leq \sigma - \sigma' \approx \sigma$ for $\|\zeta\|_{\gamma'} \leq \sigma$ we need $\varepsilon \lesssim (\gamma - \gamma')^{d+m_*}\sigma^2$. Now, (50) follows from the second part of Proposition 3.4(i). The ω-derivative of ζ_t satisfies

$$\frac{d}{dt}(\partial_\omega \zeta_t) = \partial_\omega JS_1(\varphi, 0, \omega) + \partial_\omega JS_2(\varphi, 0, \omega)\zeta_t + JS_2(\varphi, 0, \omega)(\partial_\omega \zeta_t)$$

which is solved in the same way. We note that $r_t = r + c_t(\zeta, \varphi, \omega) + d_t(\varphi, \omega)r$ and for a fixed ω, (48) follows from Proposition 3.4(ii) if $\varepsilon \lesssim (\rho - \rho')(\mu - \mu') \approx (\rho - \rho')\mu$. The ω-derivative satisfies a similar equation which is solved in the same way. The r-part of (51) follows from these estimates since r_t is linear in r. $\square$

Composition. Consider now the composition $f(\Phi_t, \omega)$. If

(52) $$\varepsilon \lesssim \min((\rho - \rho')\mu, (\gamma - \gamma')^{d+m_*+1}\sigma^2)\sqrt{\gamma - \gamma'}$$

then

$$(53) \qquad [f(\Phi_t, \cdot)]_{\left\{\begin{smallmatrix}\Lambda+18\\U\end{smallmatrix}\alpha'\right\}} \lesssim \Lambda^{14}[f]_{\left\{\begin{smallmatrix}\Lambda\\U\end{smallmatrix}\alpha\right\}}.$$

Proof. Consider first a fixed ω. We have

$$\|\zeta_t(\zeta, z) - \zeta\|_{\gamma'} < \sigma - \sigma' \quad \forall (\zeta, z) \in \mathbb{O}^{\gamma'}(\sigma') \times \mathbb{T}_{\rho'}^{\mathscr{A}} \times \mathbb{D}(\mu')^{\mathscr{A}}$$

by (49) and (52), and we have

$$|g_t(\zeta, z)| < \frac{1}{2}(\mu - \mu') \text{ or } \frac{1}{2}(\rho - \rho') \quad \forall (\zeta, z) \in \mathbb{O}^0(\sigma') \times \mathbb{T}_{\rho'}^{\mathscr{A}} \times \mathbb{D}(\mu')^{\mathscr{A}},$$

depending on whether g is an r-component or a φ-component, by (48) and (52). By Proposition 3.5 we get

$$[f(\Phi_t(\cdot, \omega), \omega)]_{\left\{\begin{smallmatrix}\Lambda+12, \gamma'', \sigma'\\\rho', \mu'\end{smallmatrix}\right\}} \lesssim A \, [f(\cdot, \omega)]_{\left\{\begin{smallmatrix}\Lambda+6, \gamma', \sigma\\\rho, \mu\end{smallmatrix}\right\}},$$

where $A = \max(1, \alpha, \Lambda^2 \frac{1}{\gamma'-\gamma''}\alpha^2)$ and

$$\alpha = \frac{1}{\mu - \mu'}[r_t - r]_{\left\{\begin{smallmatrix}\Lambda+6, \gamma', \sigma'\\\rho', \mu'\end{smallmatrix}\right\}} + \frac{1}{\rho - \rho'}[\varphi_t - \varphi]_{\left\{\begin{smallmatrix}\Lambda+6, \gamma', \sigma'\\\rho', \mu'\end{smallmatrix}\right\}}$$

$$+ \left(\frac{1}{\gamma' - \gamma''}\right)^{d+m_*} \langle B_t \rangle_{\left\{\begin{smallmatrix}\Lambda+6, \gamma'\\\rho'\end{smallmatrix}\right\}}.$$

If we choose $\gamma' - \gamma'' = \gamma - \gamma'$, then (48), (50), and the bound (52) give $A \lesssim \Lambda^6$.

Consider now the dependence on ω. We have

$$\partial_\omega(f(\Phi_t)) = \partial_\omega f(\Phi_t) + \langle \partial_z f(\Phi_t), \partial_\omega g_t \rangle + \langle \partial_\zeta f(\Phi_t), \partial_\omega \zeta_t \rangle.$$

The first term is a composition and we get the same estimate as above but with f replaced by $\partial_\omega f$.

The second term is a finite sum of products, each of which is estimated by Proposition 3.1(i); i.e.,

$$[\langle \partial_z f(\Phi_t, \omega), \partial_\omega g_t \rangle]_{\left\{\begin{smallmatrix}\Lambda+12, \gamma'', \sigma'\\\rho'', \mu''\end{smallmatrix}\right\}}$$

$$\lesssim [\partial_z f(\Phi_t, \omega)]_{\left\{\begin{smallmatrix}\Lambda+12, \gamma'', \sigma'\\\rho'', \mu''\end{smallmatrix}\right\}} [\partial_\omega g_t]_{\left\{\begin{smallmatrix}\Lambda+12, \gamma'', \sigma'\\\rho'', \mu''\end{smallmatrix}\right\}}.$$

The first factor is a composition which is estimated as above: if we take $\rho' - \rho'' = \rho - \rho'$ and $\mu' - \mu'' = \mu - \mu'$, then we get

$$\lesssim \Lambda^6 [\partial_z f(\cdot, \omega)]_{\left\{\begin{smallmatrix}\Lambda+6, \gamma', \sigma\\\rho', \mu'\end{smallmatrix}\right\}} [\partial_\omega g_t]_{\left\{\begin{smallmatrix}\Lambda+12, \gamma'', \sigma'\\\rho', \mu''\end{smallmatrix}\right\}}.$$

Using Cauchy estimates for the first factor and (48) and (50) for the second factor gives

$$\lesssim \Lambda^6 [f(\cdot, \omega)]_{\left\{\begin{smallmatrix}\Lambda+6, \gamma', \sigma\\\rho, \mu\end{smallmatrix}\right\}}.$$

The third term is a composition of the function $\tilde{f} = \langle \partial_\zeta f, (\partial_\omega \zeta_t) \circ \Phi_{-t} \rangle$ with Φ_t. Evaluating $\tilde{f}$ we find that it has the form $\langle \partial_\zeta f, \tilde{b}_t + \tilde{B}_t \zeta \rangle$ where

$$\tilde{b}_t = \partial_\omega b_t(\varphi_{-t}) + \partial_\omega B_t(\varphi_{-t}) b_{-t},$$
$$\tilde{B}_t = \partial_\omega B_t(\varphi_{-t}) + \partial_\omega B_t(\varphi_{-t}) B_{-t}.$$

For $\varphi \in \mathbb{T}^{\mathscr{A}}_{\rho''}$ we get by (48) and (52) that $|\varphi_{-t} - \varphi| \le \rho' - \rho'' = \rho - \rho'$, so that $\tilde{b}_t$ and $\tilde{B}_t$ are defined on $\mathbb{T}^{\mathscr{A}}_{\rho''}$. By (49) and (52), $\|\tilde{b}_t\|_{\gamma'} \le \sigma - \sigma'$, and by (50), (52), and the product formula (10),

$$\langle \tilde{B}_t \rangle_{\left\{ \substack{\Lambda+9,\gamma' \\ \rho''} \right\}} \lesssim \Lambda^6 \left(\frac{1}{\gamma - \gamma'} \right) \frac{\varepsilon}{\sigma^2},$$

and so by Proposition 3.1(ii), (iii) and (52) we obtain

$$[\tilde{f}]_{\left\{ \substack{\Lambda+9,\gamma',\sigma' \\ \rho'',\mu'} \right\}} \lesssim \Lambda^8 [f]_{\left\{ \substack{\Lambda+6,\gamma,\sigma \\ \rho'',\mu'} \right\}}.$$

Finally by the same argument as above we get

$$[\tilde{f}(\Phi_t(\cdot, \omega), \omega)]_{\left\{ \substack{\Lambda+15,\gamma'',\sigma'' \\ \rho''',\mu''} \right\}} \lesssim \Lambda^6 [\tilde{f}(\cdot, \omega)]_{\left\{ \substack{\Lambda+9,\gamma',\sigma' \\ \rho'',\mu'} \right\}},$$

if we choose $\rho'' - \rho''' = \rho' - \rho''$, $\sigma' - \sigma'' = \sigma - \sigma'$ and $\mu' - \mu'' = \mu - \mu'$. This completes the proof. $\qquad \square$

8.2. *A finite induction.* Let $h(\zeta, r, \omega) = \langle \omega, r \rangle + \frac{1}{2} \langle \zeta, (\Omega(\omega) + H(\omega)) \zeta \rangle$ satisfy (13)–(16) and (38)–(39), and let $H(\omega)$ and $\partial_\omega H(\omega)$ be $\mathscr{N}\mathscr{F}_\Delta$. Let $f : \mathbb{O}^\gamma(\sigma, \rho, \mu) \times U \to \mathbb{C}$ be real analytic in ζ, φ, r and $\mathscr{C}^1$ in $\omega \in U$ and consider

$$[f]_{\left\{ \substack{\Lambda \\ U} \alpha \right\}} = \varepsilon, \quad \alpha = \begin{pmatrix} \gamma & \sigma \\ \rho & \mu \end{pmatrix}.$$

Besides the assumption that all constants $\gamma, \sigma, \rho, \mu$ are < 1 and that Δ, Λ are ≥ 3, we shall also assume that $\mu = \sigma^2$ and $d_\Delta \gamma \le 1$. The first assumption is just for convenience, but the second is forced upon us by the occurrence of a factor $e^{d_\Delta \gamma}$ in the estimates of Propositions 6.6 and 6.7 which we must control.

Fix $\rho' < \rho$, $\gamma' < \gamma$ and $0 < \kappa < 1$ and let

$$\Delta' = \left(\log\left(\frac{1}{\varepsilon}\right) \right)^2 \frac{1}{\min(\gamma - \gamma', \rho - \rho')}, \quad n = \left[\log\left(\frac{1}{\varepsilon}\right) \right].$$

Define for $1 \leq j \leq n$,

$$\varepsilon_{j+1} = (\tfrac{\varepsilon}{\sigma^2 \kappa^3}) \varepsilon_j, \qquad \varepsilon_1 = \varepsilon,$$

$$\Lambda_{j+1} = \Lambda_j + d_\Delta + 23, \qquad \Lambda_1 = \text{cte.} \max(\Lambda, d_\Delta^2, (d_{\Delta'})^2), \ ^{11}$$

$$\gamma_j = \gamma - (j-1) \tfrac{\gamma - \gamma'}{n}, \qquad \rho_j = \rho - (j-1) \tfrac{\rho - \rho'}{n},$$

$$\sigma_{j+1} = (\tfrac{\varepsilon}{\sigma^2 \kappa^3})^{\frac{1}{3}} \sigma_j \qquad \sigma_1 = \sigma,$$

$$\mu_{j+1} = (\tfrac{\varepsilon}{\sigma^2 \kappa^3})^{\frac{2}{3}} \mu_j, \qquad \mu_1 = \mu.$$

We have the following proposition.

PROPOSITION 8.1. *Under the above assumptions there exist a constant* Cte. *and an exponent* $\exp_1$ *such that if*

$$\varepsilon \leq \kappa^3 \text{Cte.} \min\left(\gamma - \gamma', \rho - \rho', \frac{1}{\Delta}, \frac{1}{\Lambda}, \frac{1}{\log(\frac{1}{\varepsilon})} \right)^{\exp_1} \min(\sigma^2, \mu),$$

then there is a subset $U' \subset U$, $\text{Leb}(U \setminus U') \leq \text{cte.} \varepsilon^{\exp_2}$, *such that for all* $\omega \in U'$ *the following holds for* $1 \leq j \leq n$: *there is a real analytic symplectic diffeomorphism*

$$\Phi_j : \mathbb{O}^{\gamma''}(\sigma_{j+1}, \rho_{j+1}, \mu_{j+1}) \to \mathbb{O}^{\gamma''}(\sigma_j, \rho_j, \mu_j), \quad \forall \gamma'' \leq \gamma_{j+1},$$

such that

$$(h + h_1 + \ldots + h_{j-1} + f_j) \circ \Phi_j = h + h_1 + \ldots + h_j + f_{j+1}$$

$(f_1 = f)$ *with*

(i)
$$h_j = c_j(\omega) + \langle \chi_j(\omega), r \rangle + \frac{1}{2} \langle \zeta, H_j(\omega) \zeta \rangle,$$

$H_j(\omega)$ *and* $\partial_\omega H_j(\omega)$ *in* $\mathcal{NF}_{\Delta'}$, *and* $[h_j]_{\{U' \, \alpha_j\}}^{\{\Lambda_j} \leq \beta^{j-1} \varepsilon_j.$

(ii) $[f_{j+1}]_{\{U' \, \alpha_{j+1}\}}^{\{\Lambda_{j+1}} \leq \beta^j \varepsilon_{j+1},$

for some

$$\beta \lesssim \text{cte.} \max\left(\frac{1}{\gamma - \gamma'}, \frac{1}{\rho - \rho'}, \Lambda, \Delta, \log\left(\frac{1}{\varepsilon}\right) \right)^{\exp_3}.$$

Moreover, for $1 \geq \bar{\sigma} \geq \sigma_{j+1}$ *and* $1 \geq \bar{\mu} \geq \mu_{j+1}$, $\Phi_j = (\zeta_j, \varphi_j, r_j)$ *has an analytic extension to* $\mathbb{O}^{\gamma''}(\bar{\sigma}, \rho_{r+j}, \bar{\mu})$ *for all* $\gamma'' \leq \gamma_{j+1}$ *and verifies on this set*

$$\begin{cases} \|\zeta_j - \zeta\|_{\gamma''} \lesssim (\frac{1}{\gamma_j - \gamma_{j+1}})^{d+m_*} (\frac{\bar{\sigma}}{\sigma_j} + 1) \beta^{j-1} \frac{\varepsilon_j}{\sigma_j} \\ |\varphi_j - \varphi| \lesssim \beta^{j-1} \frac{\varepsilon_j}{\mu_j} \\ |r_j - r| \lesssim (\frac{1}{\rho_j - \rho_{j+1}})(\frac{\bar{\mu}}{\mu_j} + (\frac{\bar{\sigma}}{\sigma_j})^2 + 1) \beta^{j-1} \varepsilon. \end{cases}$$

---

[11] The constant in the definition of Λ_1 is the one in Proposition 6.7.

The exponents $\exp_1, \exp_2, \exp_3$ *only depend on* $d, \#\mathcal{A}, m_*$ *while the constants* Cte. *and* cte. *also depend on* $C_1, \dots, C_5$.

Proof. We start by solving inductively $\{h, s_j\} = -\mathcal{T}_{\Delta'} f_j + h_j$, where $\mathcal{T}_{\Delta'} f_j$ is the truncation (§8.1) and s_j and h_j are to be found using Propositions 6.6 and 6.7. To see how this works, write

$$s_j = S_0 + \langle \zeta, S_1 \rangle + \tfrac{1}{2} \langle \zeta, S_2 \zeta \rangle,$$
$$\mathcal{T}_{\Delta'} f_j = F_0 + \langle \zeta, F_1 \rangle + \tfrac{1}{2} \langle \zeta, F_2 \zeta \rangle,$$
$$h_j = c_j(\omega) + \langle \chi_j(\omega), r \rangle + \tfrac{1}{2} \langle \zeta, H_j(\omega) \zeta \rangle.$$

The equations written in Fourier modes become

$$-i \langle k, \omega \rangle \hat{S}_0(k) = -\hat{F}_0(k) + \delta_0^k (c_j(\omega) + \langle \chi_j(\omega), r \rangle),$$
$$-i \langle k, \omega \rangle \hat{S}_1(k) + J(\Omega(\omega) + H(\omega)) \hat{S}_1(k) = -\hat{F}_1(k),$$
$$-i \langle k, \omega \rangle \hat{S}_2(k) + (\Omega(\omega) + H(\omega)) J \hat{S}_2(k) - \hat{S}_2(k) J (\Omega(\omega) + H(\omega))$$
$$= -\hat{F}_2(k) + \delta_0^k H_j(\omega).$$

Using Propositions 6.6 and 6.7 these equations can now be solved for ω in a set U_j with $\mathrm{Leb}(U_{j-1} \setminus U_j) \leq \mathrm{cte}.\varepsilon^{\exp}$, $(U_0 = U)$. Indeed with $c_j(\omega) = \hat{F}_0(0)$ and $\chi_j(\omega) = \hat{F}_1(0)$ the first equation follows from Proposition 6.6(i). The second equation follows from Proposition 6.6(ii) and the third from Proposition 6.7. ($H_j(\omega)$ is not the full mean value $\hat{F}_2(0)$ but only the part $\pi \hat{F}_2(0)$.)

This gives, after summing up the (finite) Fourier series,

$$[s_j]_{\left\{ \substack{\Lambda_j + d_\Delta + 2 \\ U_j} \right\}_{\alpha_j}} \leq \mathrm{cte}.(\Delta' \Delta)^{\exp} \frac{1}{\kappa^3} \beta^{j-1} \varepsilon_j = \tilde{\varepsilon}_j,$$

$$[h_j]_{\left\{ \substack{\Lambda_j + d_\Delta + 2 \\ U_j} \right\}_{\alpha_j}} \leq \mathrm{cte}.(\Delta' \Delta)^{\exp} \beta^{j-1} \varepsilon_j.$$

If the solutions s_j and h_j were nonreal (they are not because the construction gives real functions) then their real parts would give real solutions.

In a second step, for $0 \leq t \leq 1$ we estimate

$$f_j - h_j + \{h + h_1 + \dots + h_{j-1} + (1-t)h_j + t f_j, s_j\}$$

which is equal to

$$(f_j - \mathcal{T}_{\Delta'} f_j) + t\{f_j, s_j\} + \{h_1 + \dots + h_{j-1} + (1-t)h_j, s_j\} =: g_1 + g_2 + g_3.$$

According to (44) we have

$$[g_1]_{\left\{ \substack{\Lambda_j + d_\Delta + 2 \\ U_j} \right\}_{\tilde{\alpha}_{j+1}}} \lesssim A(\alpha_j, \tilde{\alpha}_{j+1}, \Delta') \beta^{j-1} \varepsilon_j,$$

where

$$\tilde{\alpha}_{j+1} = \begin{pmatrix} \gamma_j - \frac{\gamma_j - \gamma_{j+1}}{2} & 2\sigma_{j+1} \\ \rho_j - \frac{\rho_j - \rho_{j+1}}{2} & 2\mu_{j+1} \end{pmatrix}.$$

By our choice of constants and the assumption on ε we have

$$A \lesssim \left(\frac{1}{\sigma^2 \kappa^3} + \left(\frac{1}{\rho - \rho'}\right)^{\#\mathscr{A}}\right)\varepsilon \lesssim \frac{1}{\Lambda_j^{14}}\beta \frac{\varepsilon}{\sigma^2 \kappa^3}.$$

According to (45),

$$[g_2]_{\left\{\begin{smallmatrix}\Lambda_j + d_\Delta + 5 \\ U_j\end{smallmatrix}\ \tilde{\alpha}_{j+1}\right\}} \lesssim B_j (\Delta'\Delta)^{\exp}\frac{1}{\kappa^3}\beta^{2j-2}\varepsilon_j^2,$$

where $B_j = B(\gamma_j - \gamma_{j+1}, \sigma_j, \rho_j - \rho_{j+1}, \mu_j, \Lambda_j)$. β takes care of this when $j = 1$ and when $j \geq 2$ we have the factor $\frac{\varepsilon_j}{\varepsilon_1}$ that controls everything, and we get the bound $\lesssim \frac{1}{\Lambda_j^{14}}\beta^j \frac{\varepsilon}{\sigma^2 \kappa^3}\varepsilon_j$.

According to (46),

$$[g_3]_{\left\{\begin{smallmatrix}\Lambda_j + d_\Delta + 5 \\ U_j\end{smallmatrix}\ \tilde{\alpha}_{j+1}\right\}} \lesssim \sum_{1 \leq i \leq n} B_i (\Delta'\Delta)^{\exp}\beta^{i-1}\varepsilon_i \text{cte.}(\Delta'\Delta)^{\exp}\frac{1}{\kappa^3}\beta^{j-1}\varepsilon_j,$$

where $B_i = B(\gamma_j - \gamma_{j+1}, \sigma_i, \rho_j - \rho_{j+1}, \mu_i, \Lambda_j)$. The same argument applies again: β takes care of this when $i = 1$ and when $i \geq 2$ we have the factor $\frac{\varepsilon_i}{\varepsilon_1}$ that controls everything. We get, as before, the bound $\lesssim \frac{1}{\Lambda_j^{14}}\beta^j \frac{\varepsilon}{\sigma^2 \kappa^3}\varepsilon_j$.

In a third step we construct the time-t-map, $|t| \leq 1$, Φ_t of the Hamiltonian vector field $J \partial s_j$. Condition (47),

$$\tilde{\varepsilon}_j \lesssim \min\left((\tilde{\rho}_{j+1} - \rho_{j+1})\tilde{\mu}_{j+1}, (\tilde{\gamma}_{j+1} - \gamma_{j+1})^{d+m_*}\tilde{\sigma}_{j+1}^2\right),$$

is fulfilled for all j by assumption on ε, so that

$$\Phi_t : \mathbb{O}^{\gamma''}(\sigma_{j+1}, \rho_{j+1}, \mu_{j+1}) \to \mathbb{O}^{\gamma''}(\tilde{\sigma}_{j+1}, \tilde{\rho}_{j+1}, \tilde{\mu}_{j+1})$$

for all $\gamma'' < \gamma_{j+1}$, and it will verify conditions (48)–(51) with α, α', Λ replaced by $\tilde{\alpha}_{j+1}, \alpha_{j+1}, \Lambda_j + d_\Delta + 2$. Then the time-1-map Φ_t, $t = 1$, will be our Φ_j and do what we want; this is a well-known relation.

Finally we define

$$f_{j+1} = \int_0^1 (g_1 + g_2 + g_3) \circ \Phi_t \, dt.$$

It only remains to verify the estimate for f_{j+1}. Condition (52),

$$\tilde{\varepsilon}_j \lesssim \min((\tilde{\rho}_{j+1} - \rho_{j+1})\tilde{\mu}_{j+1}, (\tilde{\gamma}_{j+1} - \gamma_{j+1})^{d+m_*+1}\tilde{\sigma}_{j+1}^2)\sqrt{\tilde{\gamma}_{j+1} - \gamma_{j+1}},$$

is fulfilled for all j by assumption on ε, so we get by (53)

$$[f_{j+1}]_{\left\{\begin{smallmatrix}\Lambda_{j+1} \\ U_j\end{smallmatrix}\ \alpha_{j+1}\right\}} \lesssim \Lambda_j^{14}[g]_{\left\{\begin{smallmatrix}\Lambda_j + d_\Delta + 5 \\ U_j\end{smallmatrix}\ \tilde{\alpha}_{j+1}\right\}},$$

and we are done. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \square$

COROLLARY 8.2. *There exist a constant* Cte. *and an exponent* $\exp_1$ *such that, if*

$$\varepsilon \le \text{Cte.} \min\left(\gamma - \gamma', \rho - \rho', \frac{1}{\Delta}, \frac{1}{\Lambda}\right)^{\exp_1} \min(\sigma^2, \mu)^{\frac{1}{1-3\tau}} \quad (\tau = \tfrac{1}{30}), \ ^{12}$$

then there is a subset $U' \subset U$,

$$\text{Leb}(U \setminus U') \le \text{cte.}\varepsilon^{\exp_2},$$

such that for all $\omega \in U'$ *the following hold*: *There is a real analytic symplectic diffeomorphism* $\Phi : \mathbb{O}^{\gamma''}(\sigma', \rho', \mu') \to \mathbb{O}^{\gamma''}(\sigma, \rho, \mu)$ *for all* $\gamma'' \le \gamma'$, *and a vector* ω' *such that* $(h_{\omega'} + f) \circ \Phi = h' + f'$ *with*

(i) $h' = \langle \omega, r \rangle + \frac{1}{2}\langle \zeta, (\Omega(\omega) + H'(\omega))\zeta \rangle$ *(modulo a constant)*, $H'(\omega)$ *and* $\partial_\omega H'(\omega)$ *in* $\mathcal{NF}_{\Delta'}$, *and*

$$\left\| H' - H \right\|_{U'}, \ \langle H' - H \rangle_{\{\Lambda'\} \atop \{U'\}} \le \text{cte.}\frac{\varepsilon}{\sigma^2};$$

(ii) $[f']_{\{\Lambda' \atop \{U'} \ \alpha'\}} \le \varepsilon' \le e^{-\tau(\log(\frac{1}{\varepsilon}))^2}$;

where

$$\Delta' = (\log(\tfrac{1}{\varepsilon}))^2 \frac{1}{\min(\gamma - \gamma', \rho - \rho')},$$
$$\Lambda' = \text{cte.} \max(\Lambda, d_\Delta^2, (d_{\Delta'})^2) + \log(\tfrac{1}{\varepsilon})(d_\Delta + 23),$$
$$\sigma' = (\varepsilon')^{\frac{1}{3}+\tau}\sigma,$$
$$\mu' = (\varepsilon')^{\frac{2}{3}+2\tau}\mu.$$

Moreover, for $1 \ge \bar{\sigma} \ge \sigma'$ *and* $1 \ge \bar{\mu} \ge \mu'$, $\Phi = (\Phi_\zeta, \Phi_\varphi, \Phi_r)$ *has an analytic extension to* $\mathbb{O}^{\gamma''}(\bar{\sigma}, \rho', \bar{\mu})$ *for all* $\gamma'' \le \gamma'$ *and verifies on this set*

$$\begin{cases} \left\| \Phi_\zeta - \zeta \right\|_{\gamma''} \le (\frac{\bar{\sigma}}{\sigma} + 1)\beta\frac{\varepsilon}{\sigma} \\ \left| \Phi_\varphi - \varphi \right| \le \beta\frac{\varepsilon}{\mu} \\ \left| \Phi_r - r \right| \le (\frac{\bar{\mu}}{\mu} + (\frac{\bar{\sigma}}{\sigma})^2 + 1)\beta\varepsilon \end{cases}$$

for some $\beta \le \text{cte.} \max(\frac{1}{\gamma - \gamma'}, \frac{1}{\rho - \rho'}, \Lambda, \Delta, \log(\frac{1}{\varepsilon}))^{\exp_3}$, *and the mapping* $\omega \mapsto \omega'$ *verifies* $|\omega' - \text{id}|_{\mathscr{C}^1(U')} \le \text{cte.}\frac{\varepsilon}{\mu}$.

The exponents $\exp_1, \exp_2, \exp_3$ *only depend on* $d, \#\mathscr{A}, m_*$ *while the constants* Cte. *and* cte. *also depend on* $C_1, \dots, C_5$.

Proof. Take $\kappa^3 = \varepsilon^\tau$. Then $\beta^n \varepsilon_{n+1} = \varepsilon'$, $\sigma_{n+1} \ge (\varepsilon')^{\frac{1}{3}+\tau}\sigma$, $\mu_{n+1} \ge (\varepsilon')^{\frac{2}{3}+2\tau}\mu$, and $\varepsilon' \le e^{-\tau(\log(\frac{1}{\varepsilon}))^2}$ if $\varepsilon^{1-2\tau} \lesssim (\frac{1}{\beta})^{\frac{1+3\tau}{3\tau}}\sigma^2$.

---

[12] The bound on ε in Proposition 8.1 is implicit due to $\log(\frac{1}{\varepsilon}))$ and depends on κ. Here we have an explicit bound, but the price for taking κ to be a fractional power of ε is that the bound must depend on $\max(\sigma^2, \mu)$ to a power larger than 1. The choice of τ is only for convenience.

The result is an immediate consequence of Proposition 8.1 with

$$h'_\omega = \langle \omega + \chi(\omega), r \rangle + \frac{1}{2}\langle \zeta, (\Omega(\omega) + H'(\omega))\zeta \rangle.$$

By Proposition 8.1(ii) we get $|\chi|_{\mathscr{C}^1(U')} \leq \text{cte.}\frac{\varepsilon}{\mu}$. Therefore the image of U' under the mapping $\omega \to \omega + \chi(\omega)$ covers a subset U'' of U of the same complementary Lebesgue measure, and we can replace $\omega + \chi(\omega)$ by ω if we take $\omega' = (\text{Id} + \chi)^{-1}(\omega)$. $\square$

8.3. *The infinite induction.* Let h and f be as in the previous subsection with the same restrictions on the constants $\gamma, \sigma, \rho, \mu, \Delta, \Lambda$.

Choice of constants. We define

$$\varepsilon_{j+1} = e^{-\tau(\log(\frac{1}{\varepsilon_j}))^2} \quad (\tau = \frac{1}{30}), \qquad \varepsilon_1 = \varepsilon$$

$$\gamma_j = (d_{\Delta_j})^{-1}, \qquad\qquad\qquad \gamma_1 = \min(d_\Delta, \gamma)$$

$$\sigma_j = \varepsilon_j^{\frac{1}{3}+\tau}\sigma_{j-1} \; j \geq 2, \qquad\qquad \sigma_1 = \sigma$$

$$\mu_j = \varepsilon_j^{\frac{2}{3}+2\tau}\mu_{j-1} \; j \geq 2, \qquad\qquad \mu_1 = \mu$$

$$\rho_j = (\frac{1}{2} + \frac{1}{2^j})\rho,$$

$$\Delta_{j+1} = (\log(\frac{1}{\varepsilon_j}))^2 \frac{1}{\min(\gamma_j, \rho_j - \rho_{j+1})}, \qquad \Delta_1 = \Delta$$

$$\Lambda_j = \text{cte.}(d_{\Delta_j})^2. \,^{13}$$

With this choice of constants we prove

LEMMA 8.3. *There exist a constant* Cte.$'$ *and an exponent* exp$'$ *such that if*

$$\varepsilon \leq \text{Cte.}' \min\left(\gamma, \rho, \frac{1}{\Delta}, \frac{1}{\Lambda}\right)^{\exp'} \min(\sigma^2, \mu)^{\frac{1}{1-3\tau}},$$

then for all $j \geq 1$,

$$\varepsilon_j \leq \text{Cte.} \min\left(\gamma_j - \gamma_{j+1}, \rho_j - \rho_{j+1}, \frac{1}{\Delta_j}, \frac{1}{\Lambda_j}\right)^{\exp} \min(\sigma_j^2, \mu_j)^{\frac{1}{1-3\tau}}$$

and

$$\sum_{1 \leq i \leq j} (d_{\Delta_i})^2 \varepsilon_i \leq \frac{1}{4} \min(C_4, C_5, 1),$$

where Cte., exp *are those of Corollary* 8.2. *The exponents* exp$'$ *only depend on* $d, \#\mathscr{A}, m_*$ *while the constant* Cte.$'$ *also depend on* $C_1, \ldots, C_5$.

Remark. Notice that Δ_j increases much faster than quadratically at each step — $\Delta_{j+1} \geq \Delta_j^{\frac{(d+1)!}{2}}$ due to its coupling with γ_j. This is the reason why we cannot

---

[13] The constant in the definition of Λ_j is the one in Proposition 6.7.

grant the convergence by a quadratic iteration but need a much faster iteration scheme, as the one provided by Proposition 8.1 and Corollary 8.2.

The proof is an exercise on the theme "superexponential growth beats (almost) everything".

PROPOSITION 8.4. *Under the above assumptions, there exist a constant* Cte. *and an exponent* exp *such that if*

$$\varepsilon \ \leq \ \text{Cte.} \min\left(\gamma - \gamma', \rho - \rho', \frac{1}{\Delta}, \frac{1}{\Lambda}\right)^{\exp} \min(\sigma^2, \mu)^{\frac{1}{1-3\tau}},$$

then there is a subset $U' \subset U$, $\text{Leb}(U \setminus U') \leq \text{cte.}\varepsilon^{\exp'}$, such that for all $\omega \in U'$ the following hold: for all $j \geq 1$ there are an analytic symplectic diffeomorphism

$$\Phi_j : \mathbb{O}^{\gamma''}(\sigma_{j+1}, \rho_{j+1}, \mu_{j+1}) \to \mathbb{O}^{\gamma''}(\sigma_j, \rho_j, \mu_j), \quad \forall \gamma'' \leq \gamma_{j+1},$$

and a vector ω_j such that

$$(h_{j-1} + f_j) \circ \Phi_j = h_j + f_{j+1} \quad (h_0 = h_{\omega_j}, \ f_1 = f)$$

and satisfying:

(i) $h_j = \langle \omega, r \rangle + \frac{1}{2}\langle \zeta, (\Omega(\omega) + H_j(\omega))\zeta \rangle$ *(modulo a constant), $H_j(\omega)$ and $\partial_\omega H_j(\omega)$ in $\mathcal{NF}_{\Delta_{j+1}}$, and*

$$\left\| H_j - H_{j-1} \right\|_{U'}, \ \langle H_j - H_{j-1} \rangle_{\{\Lambda_j\} \atop \{U'\}} \leq \text{cte.} \frac{\varepsilon_j}{\sigma_j^2};$$

(ii) $[f_{j+1}]_{\{\Lambda_{j+1} \ \alpha_{j+1}\} \atop \{U'\}} \leq \varepsilon_{j+1}.$

Moreover, $\Phi_j = (\zeta_j, \varphi_j, r_j)$ has an analytic extension to $\mathbb{O}^0(\frac{\sigma}{2}, \frac{\rho}{2}, \frac{\mu}{2})$ and verifies on this set

$$\begin{cases} \|\zeta_j - \zeta\|_0 \leq (\frac{\sigma}{\sigma_j} + 1)\beta \frac{\varepsilon_j}{\sigma_j} \\ |\varphi_j - \varphi| \leq \beta_j \frac{\varepsilon_j}{\mu_j} \\ |r_j - r| \leq (\frac{\mu}{\mu_j} + (\frac{\sigma}{\sigma_j})^2 + 1)\beta_j \varepsilon_j \end{cases}$$

for some

$$\beta_j \leq \text{cte.} \max\left(\frac{1}{\gamma_j - \gamma_{j+1}}, \frac{1}{\rho_j - \rho_{j+1}}, \Lambda_j, \Delta_j, \log\left(\frac{1}{\varepsilon_j}\right)\right)^{\exp_3},$$

and the mapping $\omega \mapsto \omega_j$ verifies $|\omega_j - \omega_{j-1}|_{\mathscr{C}^1(U')} \leq \text{cte.} \frac{\varepsilon_j}{\mu_j}$.

The exponents $\exp, \exp'$ only depend on $d, \#\mathcal{A}, m_*$ while the constants Cte. and cte. also depend on $C_1, \ldots, C_5$.

Proof. The proof is an immediate consequence of Corollary 8.2 and Lemma 8.3. The first part of the lemma implies that the smallness assumption in the corollary is fulfilled for every $j \geq 1$, and the second part implies that assumptions (16) and (39) hold for every $j \geq 1$. The remaining assumptions are only on Ω. $\quad\square$

Theorem 7.1 now follows from this proposition. Indeed, $\omega_j \to \omega'$ and we have

$$(h_{\omega'} + f) \circ \Phi = \lim_{t \to \infty} (h_{\omega_j} + f) \circ \Phi_1 \circ \cdots \circ \Phi_j = \lim_{t \to \infty} (h_j + f_{j+1}).$$

Since the sequence h_j clearly converges on $\mathbb{O}^0(\frac{\sigma}{2}, \frac{\rho}{2}, \frac{\mu}{2})$, also f_j converges on this set — to a function f'.

Moreover, for $\zeta = r = 0$ and $|\Im \varphi| < \frac{\rho}{2}$ we have, as $j \to \infty$,

$$|f_j|, \ |\partial_r f_j|, \ \|\partial_\xi f_j\|_0 \to 0$$

and, by Young's inequality,

$$\left\| \partial_\xi^2 f_j \hat{\xi} \right\|_0 \lesssim \left(\frac{1}{\gamma_j} \right)^d \left| \partial_\xi^2 f_j \right|_0 \left\| \hat{\xi} \right\|_0 \to 0.$$

Therefore $\partial_\xi f' = \partial_r f' = \partial_\xi^2 f' = 0$ for $\zeta = r = 0$.

9. **Appendix. Some estimates**

LEMMA 9.1. *Let $f : I =]-1, 1[\to \mathbb{R}$ be of class $\mathscr{C}^n$ and $\left| f^{(n)}(t) \right| \geq 1$ for all $t \in I$. Then, for all $\varepsilon > 0$, the Lebesgue measure of $\{t \in I : |f(t)| < \varepsilon\}$ is $\leq \text{cte}.\varepsilon^{\frac{1}{n}}$, where the constant only depends on n.*

Proof. We have $\left| f^{(n)}(t) \right| \geq \varepsilon^{\frac{0}{n}}$ for all $t \in I$. Since

$$f^{(n-1)}(t) - f^{(n-1)}(t_0) = \int_{t_0}^t f^{(n)}(s) ds,$$

we get that $\left| f^{(n-1)}(t) \right| \geq \varepsilon^{\frac{1}{n}}$ for all t outside an interval of length $\leq 2\varepsilon^{\frac{1}{n}}$. By induction we get that $\left| f^{(n-j)}(t) \right| \geq \varepsilon^{\frac{j}{n}}$ for all t outside 2^{j-1} intervals of length $\leq 2\varepsilon^{\frac{1}{n}}$. Now, $j = n$ gives the result. $\square$

Remark. The same is true if $\max_{0 \leq j \leq n} \left| f^{(j)}(t) \right| \geq 1$ for all $t \in I$ and $f \in \mathscr{C}^{n+1}$. In this case the constant will depend on $|f|_{\mathscr{C}^{n+1}}$.

Let $A(t)$ be a real diagonal $N \times N$-matrix with diagonal components a_j which are $\mathscr{C}^1$ on $I =]-1, 1[$ and

$$a_j'(t) \geq 1 \qquad j = 1, \ldots, N, \ \forall t \in I.$$

Let $B(t)$ be a Hermitian $N \times N$-matrix of class $\mathscr{C}^1$ on $I =]-1, 1[$ with

$$\left\| B'(t) \right\| \leq \frac{1}{2} \quad \forall t \in I.$$

LEMMA 9.2. *The Lebesgue measure of the set*

$$\{t \in I : \min_{\lambda(t) \in \sigma(A(t)+B(t))} |\lambda(t)| < \varepsilon\}$$

is $\leq \text{cte}.N\varepsilon$, where the constant is independent of N.

Proof. Assume first that $A(t) + B(t)$ is analytic in t. Then each eigenvalue $\lambda(t)$ and its (normalized) eigenvector $v(t)$ are analytic in t, and

$$\lambda'(t) = \langle v(t), (A'(t) + B'(t))v(t)\rangle$$

(scalar product in $\mathbb{C}^N$). Under the assumptions on A and B, this is $\geq 1 - \frac{1}{2}$. Lemma 9.1 applied to each eigenvalue $\lambda(t)$ gives the result.

If B is nonanalytic we get the same result by analytic approximation. $\square$

PROPOSITION 9.3. $\left\|(A(t) + B(t))^{-1}\right\| \leq \frac{1}{\varepsilon}$ *outside a set of* $t \in I$ *of Lebesgue measure* $\leq \text{cte}.N\varepsilon$.

Proof. There exists a unitary matrix $U(t)$ such that

$$U(t)^*(A(t) + B(t))U(t) \begin{pmatrix} \lambda_1(t) & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \lambda_N(t) \end{pmatrix}.$$

Now

$$\left\|(A(t) + B(t))^{-1}\right\| = \max_{0 \leq j \leq N} \left|\frac{1}{\lambda_j(t)}\right|. \qquad \square$$

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