

# ANNALS OF MATHEMATICS

**On the formation of singularities in the  
critical  $O(3)$   $\sigma$ -model**

By IGOR RODNIANSKI and JACOB STERBENZ



SECOND SERIES, VOL. 172, NO. 1

July, 2010

ANMAAH

# On the formation of singularities in the critical $O(3)$ $\sigma$ -model

By IGOR RODNIANSKI and JACOB STERBENZ

## Abstract

We study the phenomena of energy concentration for the critical  $O(3)$  sigma model, also known as the wave map flow from  $\mathbb{R}^{2+1}$  Minkowski space into the sphere  $\mathbb{S}^2$ . We establish rigorously and constructively existence of a set of smooth initial data resulting in a dynamic finite time formation of singularities. The construction and analysis are done in the context of the  $k$ -equivariant symmetry reduction, and we restrict to maps with homotopy class  $k \geq 4$ . The concentration mechanism we uncover is essentially due to a resonant self-focusing (shrinking) of a corresponding harmonic map. We show that the phenomenon is generic (e.g. in certain Sobolev spaces) in that it persists under small perturbations of initial data, while the resulting blowup is bounded by a log-modified self-similar asymptotic.

## 1. Introduction

One of the simplest nontrivial models of Quantum Field Theory is based on the  $(2 + 1)$ -dimensional Lorentz invariant  $O(3)$  classical  $\sigma$ -model. It is a nonlinear scalar field Lagrangian theory for a map  $\Phi : \mathbb{R}^{2+1} \rightarrow \mathbb{S}^2 \subset \mathbb{R}^3$  with the Lagrangian density:

$$(1) \quad \mathcal{L}[\Phi] = \frac{1}{2} \partial_\alpha \Phi \cdot \partial_\beta \Phi m^{\alpha\beta},$$

where  $m_{\alpha\beta}$  is the Minkowski metric on  $\mathbb{R}^{2+1}$ . Evolution of the nonlinear scalar field  $\Phi$  is described by the Euler-Lagrange equations:

$$(2) \quad \square \Phi = -\Phi (\partial^\alpha \Phi \cdot \partial_\alpha \Phi).$$

Equation (2) belongs to the more general class of “wave-map” problems, in which  $\Phi$  is a map from Minkowski space  $\mathbb{R}^{2+1}$  to a Riemannian manifold  $(\mathcal{M}, g)$ . The map  $\Phi$  is a solution of the Euler-Lagrange equations:

$$(3) \quad D^\alpha \partial_\alpha \Phi = 0,$$

---

This work was conducted when I.R. was visiting the Department of Mathematics, MIT. He was also partially supported by the NSF grant DMS-0406627. J.S. was supported by an NSF postdoctoral fellowship.

corresponding to the Lagrangian density:

$$(4) \quad \mathcal{L}[\Phi] = \frac{1}{2} g_{ij} \partial_\alpha \Phi^i \partial_\beta \Phi^j m^{\alpha\beta}.$$

Here  $\{\Phi^i\}$  denote local coordinates on  $\mathcal{M}$ , which in turn (under the map) depend on the Minkowski variables  $\{x^\alpha\}_{\alpha=0,1,2}$ .  $D$  is the pullback of the Levi-Civita connection to the (trivial) bundle  $\Phi^*(T\mathcal{M})$ . In terms of the local coordinates  $\{\Phi^i\}$  this pull-back connection acting on sections of  $\Phi^*(T\mathcal{M})$  reads:

$$(5) \quad D_\alpha = \partial_\alpha + \bar{\Gamma}_{\alpha j}^k, \quad \bar{\Gamma}_{\alpha j}^k = \Gamma_{ij}^k(\Phi) \partial_\alpha \Phi^i,$$

where  $\Gamma_{ij}^k$  is the Christoffel symbol in the coordinates  $\{\Phi^i\}$ . The wave-map equation (3) then has the intrinsic form:

$$(6) \quad \partial^\alpha \partial_\alpha \Phi^k = -\Gamma_{ij}^k(\Phi) \partial^\alpha \Phi^i \partial_\alpha \Phi^j.$$

The goal of this paper is to establish and rigorously analyze a catastrophic instability in the  $(2 + 1)$ -dimensional  $O(3)$   $\sigma$ -model represented by equation (2). We will exhibit a spontaneous and monotonic self-focusing mechanism responsible for a dynamic formation of singularities for a rather large and stable set of initial data. This will be done through an entirely explicit and constructive<sup>1</sup> description of this phenomena. Our basic result is as follows:

**THEOREM 1.1.** *For every  $0 < \varepsilon \ll 1$  and  $4 \leq k$  there exists a set of smooth initial data  $(\Phi_0, \dot{\Phi}_0) \in (\mathbb{S}^2, T\mathbb{S}^2)$  with energy  $E = 4\pi k + \varepsilon^2$  (i.e. the Dirichlet energy defined below) and a finite time  $T^{**} = T^{**}(\Phi_0, \dot{\Phi}_0)$ , such that the corresponding solution  $\Phi(t, x)$  of problem (2) remains smooth on the interval  $[0, T^{**})$  and develops a singularity at  $T^{**}$ . More specifically, there exists a (smooth) decomposition  $\Phi = \underline{\Phi} + \mathcal{R}$ , such that as  $t \rightarrow T^{**}$  we have that for any large  $0 < M$ , an  $L^\infty$  bound of the form<sup>2</sup>:*

$$(7) \quad \frac{M}{(T^{**} - t)} \leq \sup_{x \in \mathbb{R}^2} |\nabla_x \underline{\Phi}| \leq \frac{\sqrt{|\ln(T^{**} - t)|}}{T^{**} - t},$$

as well as a uniform bound on the energy of the remainder:

$$(8) \quad E[\mathcal{R}] \lesssim \varepsilon^2.$$

Furthermore, sufficiently small equivariant perturbations of  $(\Phi_0, \dot{\Phi}_0)$  also lead to blowup with the bounds (7)–(8).

The problem of a finite time breakdown of solutions of the problem (2) has been a subject of intense study. From a purely analytical perspective, the context

<sup>1</sup>This is in contrast to some of the examples of the focusing nonlinear Schrödinger and wave equations, where a finite time blowup can be shown by nonconstructive arguments (see [13] and [24]). See, however, the work of Martel/Merle [27] on the critical KdV problem for an example of a constructive finite time blowup mechanism.

<sup>2</sup>More precise asymptotic behavior in terms of the energy concentration will be given below, including both upper and lower bounds.

is the global regularity theory for the general wave-map equations (3), where it is suspected that the formulation of singularities is ultimately tied to certain convexity properties of the target manifold  $\mathcal{M}$ .

From a more physical or gauge theoretic perspective and in a specific context of the  $O(3)$  model, the issue of possible singularity development is thought to be connected to the *incompleteness* of a certain moduli space which characterizes the associated static solutions and provides an approximation for the dynamical evolution through the so called *geodesic hypothesis*. Due to its analytic and physical interest, equation (2) has also been a popular subject of numerical and heuristic studies which all universally pointed in the direction of singularity formation.

Before proceeding with a more detailed description of our main result, we believe it is useful to give a more thorough description of these various points of view. This begins with a discussion of the static solutions of (2); that is, harmonic maps into the sphere. Historically, one of the primary motivating factors of interest in the  $O(3)$   $\sigma$ -model was due to the richness of the set of its static solutions. An ingenious procedure of Belavin and Polyakov [1] allows one to find these solutions in a given homotopy class characterized by the topological degree:

$$k = \frac{1}{4\pi} \int_{\mathbb{R}^2} \Phi_* (dA_{\mathbb{S}^2}),$$

as solutions of the *first order* ‘‘Bogomol’nyi equations’’ (see [4]). To realize this, one factors the energy functional:

$$(9) \quad V[\Phi] = \frac{1}{2} \int_{\mathbb{R}^2} \nabla_x \Phi \cdot \nabla_x \Phi \, dx,$$

which is the potential part of the Dirichlet type energy:

$$(10) \quad E[\Phi](t) = \frac{1}{2} \int_{\mathbb{R}^2} (\partial_t \Phi \cdot \partial_t \Phi + \nabla_x \Phi \cdot \nabla_x \Phi) \, dx = T[\Phi](t) + V[\Phi](t),$$

associated with the action of the Lagrangian (1):

$$(11) \quad \int_{\mathbb{R}^{2+1}} \mathcal{L}[\Phi] \, dx \, dt = - \int_{\mathbb{R}} (T[\Phi] - V[\Phi])(t) \, dt.$$

Using the notation  $\varepsilon_{ij}$  for the antisymmetric tensor on two indices, this factorization reads:

$$(12) \quad \begin{aligned} V[\Phi] &= \frac{1}{4} \int_{\mathbb{R}^2} \left[ (\partial_i \Phi \pm \varepsilon_i^j \Phi \times \partial_j \Phi) \cdot (\partial^i \Phi \pm \varepsilon^{ij} \Phi \times \partial_j \Phi) \right] \, dx \\ &\quad \pm \frac{1}{2} \int_{\mathbb{R}^2} \varepsilon^{ij} \Phi \cdot (\partial_i \Phi \times \partial_j \Phi) \, dx \\ &= \frac{1}{4} \int_{\mathbb{R}^2} \left[ (\partial_i \Phi \pm \varepsilon_i^j \Phi \times \partial_j \Phi) \cdot (\partial^i \Phi \pm \varepsilon^{ij} \Phi \times \partial_j \Phi) \right] \, dx \pm 4\pi k, \end{aligned}$$

from which it is more or less immediate that an absolute minimum of the energy functional  $V[\Phi]$  in a given topological sector  $k$  must be a solution of the equation:

$$(13) \quad \partial_i \Phi \pm \varepsilon_i^j \Phi \times \partial_j \Phi = 0.$$

In terms of complex coordinates on  $\mathbb{R}^2$  and  $\mathbb{S}^2$ , the identities (13) are seen to be nothing other than the Cauchy-Riemann and conjugate Cauchy-Riemann equations (this is a general phenomena, see [15]). Therefore, the moduli space  $M_k$  of static energy minimizing solutions of (2) in a homotopy sector  $k$  can be identified with the rational maps (in  $z$  or  $\bar{z}$  using complex variables)  $I : \mathbb{C} \rightarrow \mathbb{C}$  with degree  $k$ . Of particular importance to us will be the  $k$ -equivariant static solutions (of positive polarity) which are defined via the relation  $I(e^{i\theta} z) = e^{i\theta k} I(z)$ . We will label such solutions by  $I^k$  and refer to them as *solitons*.

Having uncovered the structure of the space of static minimizing solutions, one is then led to the “geodesic” ansatz alluded to above for the approximate dynamics of time-dependent solutions (see also [26]). To understand this, the first thing to notice is that minimizers of the variational problem (9) are highly degenerate. Specifically, they are invariant under the full conformal group of linear fractional transformations acting on  $\mathbb{C}$ . If one restricts to  $k$ -equivariant solutions, then most of this symmetry is lost, and the only remaining degree of freedom which fixes the energy (9) is the scaling transformations:

$$(14) \quad I^k(t, x) \rightarrow I_\lambda^k(t, x) = I^k(\lambda t, \lambda x).$$

Based on this, one would expect that the path of least resistance according to the action (11) would be for (symmetric) solutions close to the family of static solutions to “slide” along the moduli space  $M_k$  via the transformations (14). That is, for a fully dynamic solution  $\Phi$  sufficiently close to some  $I^k$ , i.e.  $E[\Phi] = 4\pi k + \varepsilon^2$ , there should be a splitting<sup>3</sup> as follows:

$$(15) \quad \Phi(t, x) = I^k(\lambda(t)x) + \{\text{small error}\},$$

and the goal is to understand the *lower-dimensional* dynamics of the parameter  $\lambda(t)$ . Plugging the ansatz (15) in the action (11) yields the following effective Lagrangian<sup>4</sup> for  $\lambda(t)$ :

$$(16) \quad \underline{\mathcal{L}}[\lambda](t) = \mathcal{C}_k \frac{\dot{\lambda}^2}{\lambda^4}(t) + 4\pi k + \{\text{small error}\},$$

---

<sup>3</sup>This is very similar to what is done in the modulational theory of dispersive solitons, and we will expound on this in much more detail in the sequel.

<sup>4</sup>It must be kept in mind that this calculation is purely heuristic as the original Lagrangian (1) itself is only a *formal* way to derive the equations (2).

where  $\dot{\lambda} = \frac{d\lambda}{dt}$  and where the normalization constant  $\mathcal{C}_k$  is given by the explicit integral (note that this is only finite for  $2 \leq k$ ):

$$\mathcal{C}_k = -\frac{1}{2} \int_{\mathbb{R}^2} \|r \partial_r I^k\|^2 dx.$$

Here  $\|\cdot\|^2$  is the norm on  $\mathbb{R}^3$ . The effective dynamics generated by (16) are now given by the formula:

$$(17) \quad -\frac{d^2}{dt^2}(\lambda^{-1}) = \frac{d}{dt}\left(\frac{\dot{\lambda}}{\lambda^2}\right) = \{\text{small error}\}.$$

If one were to ignore the contribution on the right-hand side of this last equation, then the evolution generated by (16) would imply that dynamically the soliton radius collapses as a linear function of time or equivalently that  $\lambda \sim (T^{**} - t)^{-1}$  for some fixed  $T^{**}$ . In this sense, the moduli space  $M_k$  is said to be incomplete.

While the above scenario is appealing for its simple geometric and physical motivation, it has been rigorously known for some time that it cannot be quite correct. This is due to the fundamental regularity results of Shatah and Tahvildar-Zadeh [30]–[32] (see also [9], [10]) which rule out the existence of purely (i.e. linear) self-similar collapse:

**THEOREM 1.2** (Regularity theory for symmetric wave-maps). *Let  $\Phi$  be an equivariant solution to equation (2) with smooth Cauchy data. There exists an  $\varepsilon > 0$  with the following property: let  $T^*$  be any time such that this solution is  $C^\infty$  for all times  $0 \leq t < T^*$  and such that the following condition holds:*

- *For any  $0 \leq t < T^*$  the energy content  $E_{\mathcal{B}(t)}[\Phi(t)]$  inside the ball  $\mathcal{B}(t)$  centered at the origin  $r = 0$  is such that  $\lim_{t \rightarrow T^*} E_{\mathcal{B}(t)}[\Phi(t)] \leq \varepsilon$  whenever  $|\mathcal{B}(t)| = (T^* - t) \cdot o(1)$ .*

*Then the wave-map  $\Phi$  extends past  $T^*$  as a  $C^\infty$  solution. That is, if the energy  $E[\Phi(t)]$  concentrates at most at a (linear) self-similar rate up to time  $T^*$ , then the solution cannot break down at time  $T^*$ .*

This theorem shows that the error terms on the right-hand side of (17) *cannot* be ignored and that any complete theory of how  $\lambda(t)$  should evolve must take them into account. In fact, the above result leaves the question of breakdown in finite time for equation (2) open to a much wider range of possibilities because while it gives a necessary lower bound on any possible blowup rate for  $\lambda(t)$ , it does not give *any* upper bound in case collapse might occur.

At this point we should further mention that the general wave-map equations (3) have also been studied intensely from an analytic perspective. For the static case of (3), we point out the references [16] and Chapter 8 of [18] and the references therein. In the case of dynamic solutions and the Cauchy problem, the only general understanding of the equations (3) that is yet available is for the local theory (see [19]) and the small data scale invariant (global) results of [36] (in the Besov case) and [35], [21], and [37] for the case of small energy. Large data global regularity

has been conjectured in the case of a hyperbolic space  $\mathbb{H}^2$  target, while singularity formation has been expected for the  $O(3)$   $\sigma$ -model for some time. We explain this more in a moment. We point out to the reader that this is in stark contrast to what is known for the parabolic analog of (3) (i.e. the harmonic map heat flow), where the global regularity theory at all energy levels is much better understood (see [12], [7], [8]).

However, in the case where the dynamic solutions of (3) possess a large amount of symmetry, there has been considerable progress toward our understanding of the general Cauchy problem in the case of arbitrarily large initial data. The global regularity question was first handled in the work of Shatah/Tahvildar-Zadeh (see [30]–[32]) and Christodoulou/Tahvildar-Zadeh (see [9], [10]), where the context is spherical symmetry or more generally  $k$ -equivariance. There is also the important and closely related work of Struwe (see [33], [34]), where breakdown is studied in the general (symmetric) case including maps into  $\mathbb{S}^2$ . From all these works, it is known that if the target manifold  $\mathcal{M}$  is “geodesically convex,” then symmetric solutions to (3) cannot break down in finite time. Furthermore, this behavior has been shown to be stable under small rough perturbations in the recent work of [22]. Finally, it is known in general (i.e. without geodesic convexity) that if a symmetric solution to (3) does break down in finite time, then the singularity formation must be tied to the existence of a static solution to (3) and, in fact, will rescale to a nontrivial harmonic map in the limit.

While the works mentioned above furnish a great deal of understanding, they also leave completely open the issue of whether or not singularities do in fact form in the specific case of dynamic solutions to equation (2). The most convincing evidence to date that breakdown does occur in finite time is the analytic work of R. Côte [11] on strong asymptotic instability in the energy space and the many numerical studies that have been performed (see for example [3], [17], [23], [25], and [29]). We mention here that the work [25] suggests a universal log-modified self-similar behavior similar to (7).

In this work we show that singularities will form in finite time for the critical  $O(3)$   $\sigma$ -model in such a way that the stable dynamics is bounded by a log-modified self-similar collapse that is not so far<sup>5</sup> from what is predicted by (17). One of the major points of this paper is to uncover the precise analytic mechanism which is responsible for this upper bound.

Before closing this subsection, let us make several remarks. The first is that the  $O(3)$   $\sigma$ -model also enjoys many analogies with other more complicated field theories such as the  $(4 + 1)$  Yang-Mills and the  $(3 + 1)$  Yang-Mills-Higgs equations. For this reason, it has been an important testing ground for ideas concerning the structural behavior of these more complicated models. We would like mention here the work of Bizon, Ovchinnikov, and Sigal for the case of Yang-Mills instantons [2],

---

<sup>5</sup>In terms of power law type behavior.

which proposes a collapse scenario for the critical  $(4 + 1)$ -dimensional Yang-Mills equations similar to what we deal with here. The reader will see that some of our methods are inspired by certain calculations performed in that paper.

Secondly, existence of finite time blowup solutions had been known for some time in the case of a super-critical higher-dimensional wave map problem with Minkowski space  $\mathbb{R}^{n+1}$  with  $n > 2$  as a base and a rotationally symmetric Riemannian manifold  $\mathcal{M}$  as a target manifold. The construction of blowup solutions is based on the existence of  $k$ -equivariant *self-similar* solutions of finite energy for the higher-dimensional wave map problem. Such solutions have been exhibited in the work of Shatah (see [31]) for the  $\mathbb{R}^{3+1} \rightarrow \mathbb{S}^3$  problem. This was later extended to other target manifolds in [32] and higher dimensions  $n \geq 4$  in [6]. In the latter work it was also shown that for  $n \geq 7$  self-similar blowup

can occur even in the case when the target manifold is negatively curved.

Thirdly, an interesting issue that we would like to draw to the readers attention is that the  $k$ -equivariant heat flow corresponding to the instance of (2) we study here is known to be *globally regular* (it is expected that the corresponding Schrödinger flow is also globally regular). That is, for the equivariant maps into the sphere  $\mathbb{S}^2$  with the homotopy index  $k > 1$ , the harmonic map heat flow *does not* break down in finite time [14]. The reason why finite time breakdown can occur in the wave flow analog of this problem is essentially due to the second order nature of the equations. See Remark 1.7 below for more thorough discussion.

Lastly, we point out that our work is essentially independent of previous techniques used for wave-maps, although we find it extremely useful to keep in mind that the self-similar blowup is *a priori* ruled out by Theorem 1.2. However, we do refer to that result for the statement of “small energy implies regularity,” which underlies much of what we do in the sequel. We again stress that the fundamental structure we rely on in this paper is the “quasi-integrable” and “super-symmetric” aspects of the *static* (elliptic) case of equation (2). Specifically, the fact that such solutions may be constructed by solving the first order Bogomol’nyi equations, as opposed to the full second order Euler-Lagrange equations. These aspects enter prominently into our analysis of the *time-dependent* problem.

In the remainder of this section, we will give a detailed discussion of the symmetry reduction we use in this work, as well as two separate statements of our main theorem.

1.1. *Symmetric reduction of the problem and the statement of the main theorem.* As we have already mentioned, we will restrict our study of the system (2) by enforcing some fairly rigid<sup>6</sup> symmetry and “size” assumptions. As seen in the introduction, the class of solutions one has access to under these restrictions

---

<sup>6</sup>It would be extremely interesting to remove these in some way. For example, by either studying large initial deviations from an equivariant soliton or by studying small nonequivariant perturbations of an equivariant soliton.



already exhibits some interesting and striking phenomena and, in many ways, is still quite far from being understood. We now give an alternative derivation of the symmetry assumption we use here. This corresponds to solutions behaving rigidly with respect to rotations on the base manifold  $\mathbb{R}^{2+1}$ . That is, we require that along some fixed time-line  $(t, 0) \in \mathbb{R}^{2+1}$ , a rotation of  $2\pi$  corresponds to a rotation of  $2k\pi$  on the sphere  $\mathbb{S}^2$  about some fixed axis. This type of symmetry dictates a more or less canonical set of coordinates on the target, which is simply polar coordinates centered about the axis of rotation. We write this in the usual way in terms of two angles:

$$(18) \quad \{\Phi^1, \Phi^2\} = (\phi, \theta), \quad g = ds^2 = d\phi^2 + \sin^2(\phi)d\theta^2,$$

where we restrict  $0 \leq \phi \leq \pi$  and  $0 \leq \theta < 2\pi$ , with  $\phi = \pi, 0$  the respective north and south poles of the rotation axis.

With this choice of coordinates, our symmetry assumption boils down to the simple relation  $\theta \equiv k\Theta$  where  $(t, r, \Theta)$  are polar coordinate on the base  $\mathbb{R}^{2+1}$  and we mod with respect to  $2\pi$ . In this case, the only remaining degree of freedom is given by the quantity  $\phi$  which can only depend on the variables  $(t, r)$ . Because all of the Christoffel symbols  $\Gamma_{ij}^\phi$  vanish except in the case  $i = j = \theta$ , where we have by a simple calculation  $\Gamma_{\theta\theta}^\phi = -\frac{1}{2} \frac{d}{d\phi}(\sin^2 \phi)$ , the general system (6) reduces to the single equation:

$$(19) \quad -\partial_t^2 \phi + \left( \partial_r^2 + \frac{1}{r} \partial_r \right) \phi = k^2 \frac{\sin(2\phi)}{2r^2}, \quad k \in \mathbb{N}^+,$$

where we implicitly enforce the boundary conditions  $\phi(0) = 0$  and  $\phi(\infty) = \pi$ .

Before we continue, it is useful for us to record here the formula for the Lagrangian density (4) under this  $k$ -equivariant symmetry reduction and in terms of the local coordinates on the sphere  $(\phi, \theta)$ :

$$(20) \quad \mathcal{L}[\phi] = \frac{1}{2} \left[ -(\partial_t \phi)^2 + (\partial_r \phi)^2 + \frac{k^2}{2r^2} (1 - \cos(2\phi)) \right].$$

In this notation the conserved energy (10) becomes:

$$(21) \quad E[\phi] = \pi \int_{\mathbb{R}^+} \left[ (\partial_t \phi)^2 + (\partial_r \phi)^2 + \frac{k^2}{r^2} \sin^2(\phi) \right] r dr.$$

The statement of our main theorem is now the following:

**THEOREM 1.3** (Finite time energy concentration for wave-maps). *Consider the full wave-map (6) with  $\mathbb{S}^2$  target under the equivariant restriction to (19). Then for any integer  $4 \leq k$  and for any sufficiently small constant  $0 < c_0 \ll 1$  with the property that for any  $\varepsilon \leq c_0^2$ , we can find a set of smooth (in the sense of the full map on  $\mathbb{R}^{2+1}$ ) Cauchy data:*

$$\phi(0) = \phi_0^\varepsilon, \quad \partial_t \phi(0) = \dot{\phi}_0^\varepsilon.$$

with energy size  $E[\phi^\varepsilon] = 4\pi k + \varepsilon^2$  such that this solution collapses at a finite time  $T^{**}$ . More specifically, this solution collapses at a rate bounded by a “log-modified self-similar” dynamic in the sense that there exists a universal time independent profile  $\underline{\phi}^k$ , and a real parameter  $0 < \lambda(t)$ , such that:

$$(22) \quad E \left[ \phi(t, r) - \underline{\phi}^k(\lambda r) \right] \lesssim \varepsilon^2$$

and such that for any  $0 < M$  and times sufficiently close to  $T^{**}$ , one has the bound:

$$(23) \quad \frac{M}{(T^{**} - t)} \leq \lambda(t) \leq c_0^{\frac{1}{4}} \frac{\sqrt{|\ln(T^{**} - t)|}}{(T^{**} - t)}.$$

Finally, this type of blowup is stable within the class of initial data in the sense that there exists a weighted Sobolev space  $H^{s,m}$  (see (30) for a definition), such that the  $c_0\varepsilon$  ball about  $(\phi_0^\varepsilon, \dot{\phi}_0^\varepsilon)$  in  $H^{s,m}$  also leads to collapse with the same universal profile  $\underline{\phi}^k$  and the same bound (23).

1.2. *The family of static solutions and a modulational version of Theorem 1.3.*

As we have mentioned previously, it is well known from the work of Struwe (see again [33]) that any blowup of the form described in Theorem 1.3 must in fact be a “bubbling off” of a static solution to equation (19). That is, after rescaling the solution  $\phi(t)$  as described in Theorem 1.3, the resulting profile should be a solution to (19). In the sequel, we will actually take the converse approach and give an explicit construction of such bubbling off solutions. This will be done in a way which is generally consistent with the decomposition (15) of the introduction. Our method also naturally shows that this process is reached from a generic (in the symmetric sense) set of initial data and that it enjoys a certain universality which is embodied by the blowup rate (23).

To get things started, we derive the formula for the energy minimizer of the (full) action (20). This is just a recalculation of lines (12) in the current notation. Completing the square in the spatial terms in the energy (21), we can write it as:

$$(24) \quad \begin{aligned} E[\phi] &= \pi \int_{\mathbb{R}^+} \left[ (\partial_t \phi)^2 + \left( \partial_r \phi - \frac{k}{r} \sin(\phi) \right)^2 \right] r dr + 2\pi \int_0^\infty k \sin(\phi) \partial_r \phi \, dr, \\ &= \pi \int_{\mathbb{R}^+} \left[ (\partial_t \phi)^2 + \left( \partial_r \phi - \frac{k}{r} \sin(\phi) \right)^2 \right] r dr + 4k\pi. \end{aligned}$$

Thus, one has the universal lower bound  $4k\pi \leq E[\phi]$ , which can be reached if we can find a function  $I^k$  with the property that  $I^k(0) = 0$  and  $I^k(\infty) = \pi$  and which satisfies the following equations:

$$(25) \quad \begin{aligned} \partial_t I^k &= 0, \\ r \partial_r I^k &= k \sin(I^k). \end{aligned}$$

We shall refer to the solution  $I^k$  as the harmonic map *soliton*. A direct calculation reveals that the function  $I^k$  is given by the explicit formula:

$$(26) \quad I^k(r) = 2 \tan^{-1}(r^k).$$

We also denote:

$$(27) \quad I(r) := I^k(r), \quad J(r) := r \partial_r I(r).$$

Note that since the equations (25) are homogeneous, the general solution  $I_\lambda$  is only defined up to a rescaling, as we have already mentioned on line (14) above. Now, in terms of these objects we can state the following more technical and precise version of Theorem 1.3, which is what we shall actually prove in the sequel:

**THEOREM 1.4 (Modulational version of the main theorem).** *Consider the reduced wave-map (19) with  $4 \leq k$ . Suppose we are given a pair of sufficiently small positive constants  $\varepsilon, c_0$  with  $\varepsilon \leq c_0^2$  and an initial data set of the form<sup>7</sup>:*

$$(28) \quad \phi(0) = I + u_0, \quad \partial_t \phi(0) = \frac{\varepsilon}{\pi} \|J\|_{L^2(rdr)}^{-2} \cdot J + g_0,$$

where:

$$\int_{\mathbb{R}^+} u_0 \cdot J \, r dr = 0,$$

and obeys the smallness condition:

$$(29) \quad \|(u_0, g_0)\|_{H^{2,1}}^2 \leq c_0^2 \varepsilon^2,$$

where we have set:

$$(30) \quad \|(u_0, g_0)\|_{H^{2,1}}^2 = \sum_{i=0}^1 \int_{\mathbb{R}^+} (1+r^2)^{1-i} \left[ (\partial_r^i g_0)^2 + \frac{(g_0)^2}{r^2} + (\partial_r^{i+1} u_0)^2 + \frac{(\partial_r^i u_0)^2}{r^2} \right] r dr.$$

Then we have that the following is true: there exists a continuous time-dependent parameter  $\lambda(t)$  with  $\lambda(0) = 1$ , such that the solution  $\phi$  to (19) with initial data (28) splits into the sum:

$$(31) \quad \phi(t, r) = I(\lambda(t)r) + u(t, r),$$

where the “remainder” term  $u$  satisfies the bounds:

$$(32) \quad \int_{\mathbb{R}^+} \left[ (\partial_t u)^2 + (\partial_r u)^2 + \frac{u^2}{r^2} \right] r dr \lesssim \varepsilon^2,$$

for all times the solution exists. Furthermore, there exists a finite time  $T^{**}$  such that  $\lim_{t \rightarrow T^{**}} \lambda(t) \rightarrow \infty$ . Finally, this parameter obeys the following bounds for

---

<sup>7</sup>Note that our choice of initial data already requires  $2 \leq k$  as for  $k = 1$  we have that  $\|J\|_{L^2(rdr)} = \infty$ .

times  $t$  sufficiently close to  $T^{**}$ :

$$(33) \quad \frac{M}{(T^{**} - t)} \leq \lambda(t) \leq c_0^{\frac{1}{4}} \frac{\sqrt{|\ln(T^{**} - t)|}}{(T^{**} - t)}.$$

*Remark 1.5.* The lower bound in the blowup rate (33) follows easily from the orbital stability bound (32) and Theorem 1.2. Therefore, in the sequel we shall concentrate on establishing blowup with the *upper* bound on line (33). The reader should note, however, that the presence of the extra small constant  $c_0^{\frac{1}{4}}$ , which may go to zero with  $\varepsilon$ , indicates that the true blowup rate is even closer to self-similar than the  $\sqrt{|\ln(T^{**} - t)|}$  correction. We will return to this delicate issue in a later work.

*Remark 1.6.* The extra decay provided by the bounds (29) is not essential to what we do here and the result also holds in the space  $H^{2,0}$ . It is assumed here as a convenience that will simplify the exposition. However, the extra regularity afforded to us in the norm (30) will be used in a crucial way. We also remark that the norm (30) is consistent with smoothness of the derivative of the initial data when considered as map from  $\mathbb{R}^2$  into the pullback bundle  $\cup_{x \in \mathbb{R}^{2+1}} \Phi_x^*(T\mathbb{S}^2)$ . This is a consequence of some simple calculations involving the frame Christoffel symbols (5).

*Remark 1.7.* As we have already mentioned, the blowup mechanism we exhibit here is ignited by a spectral phenomenon. The choice of initial data (28) guarantees that the time derivative of the wave-map  $\dot{\phi}_0$  has a “strong” projection onto the “ground state”  $J(r)$  of equation (19) linearized around the soliton  $I(r)$ . This is precisely the coefficient in a Riccati equation for the scaling parameter  $\lambda(t)$  (see formula (63) below). The Riccati equation generates the first self-similar epoch of collapse which lasts on the time interval of size  $\sim \varepsilon^{-1}$  and, as the projection of the time derivative  $\partial_t \phi$  on the “ground state”  $J_\lambda$  of the modulated soliton  $I_\lambda$  goes to zero, is eventually replaced by a more violent accelerated regime leading to the blowup. For this initial phenomenon to take place, it is crucial that the linearized ground state  $J$  is an  $L^2(rdr)$  function and that the projection of the time derivative of  $\phi$  on the ground state  $J$  is initially nontrivial. That is, one of the main things which makes our analysis possible is that the first order<sup>8</sup> field quantities  $(\phi - I_\lambda, \partial_t \phi)$  cannot be *both* orthogonal to the eigenfunction of (19) linearized around  $I(\lambda r)$  (unless one restricts the initial data to a co-dimension one submanifold).

In this regard there are some interesting open questions connected with the value of the homotopy index. For  $k = 2$  the linearized ground state is still in  $L^2$ , so it is likely that an adaptation of our methods is possible. This is important because it is this case which is most closely related to Yang-Mills (see the next

<sup>8</sup>That is, in this notation the second order wave (19) can be written as a first order system.

remark). For the unit homotopy class,  $k = 1$ , the situation appears to be more complicated. In this case the linearized ground state just misses  $L^2$  by a log. The major open problem here seems to be whether there is complete instability of the kind stated in Theorem 1.3 or if small enough perturbations (in some space) are asymptotically stable, with blowup occurring as some kind of “critical phenomena” depending delicately on the size of the perturbation. Another interesting thing is that in the case of  $k = 1$ , there are some numerical simulations which seem to indicate that the blowup, while taking place, occurs at an algebraically *different* rate from (33) (see again [3]). On the other hand, there are further numerical and heuristic results (see [25]) which suggest the validity of the log-modified behavior even in this case ( $k = 1$ ). We believe that both the  $k = 1, 2$  cases of (a possible analog of) Theorem 1.3 deserve further serious investigation in terms of numerics, heuristics, and theory.

*Remark 1.8.* Another important issue is that in the case of the critical Yang-Mills, heuristic arguments as well as numerical evidence point to blowup with a modified self-similar asymptotic of the same form as (33) (see again [2]). In fact, the spherically symmetric reduction of the  $(4 + 1)$ -dimensional Yang-Mills equations is very closely connected with the homotopy  $k = 2$  case for the  $O(3)$   $\sigma$ -model. This strongly suggests that the methods we develop here will transfer to the case of the Yang-Mills model as well, and this will be the subject of a forthcoming work of the authors.

1.3. *A few more calculations.* Before continuing on, we list here some simple formulas involving the unit solitons  $I^k$  which will be of particular importance to us in the sequel:

$$(34) \quad r \partial_r I^k = k \sin(I^k) = k \frac{2r^k}{1 + r^{2k}}, \quad \cos(I^k) = \frac{1 - r^{2k}}{1 + r^{2k}},$$

$$(35) \quad \sin(2I^k) = 4 \frac{r^k - r^{3k}}{(1 + r^{2k})^2}, \quad \cos(2I^k) = \frac{1 - 6r^{2k} + r^{4k}}{(1 + r^{2k})^2}.$$

Also, in the sequel we will refer to any specific instance of  $I^k$  as simply  $I$ , and we remind the reader that we are assuming  $4 \leq k$ .

1.4. *Vanishing of the wave-map.* We end this section by recording and proving a simple geometric lemma which will be of central importance to us throughout the sequel. We will show that a wave-map  $\Phi$  together with its derivative vanish at the origin  $r = 0$  when computed in the pair of local coordinates (18) and  $(r, \Theta)$ .

LEMMA 1.9 (Admissibility condition for the wave-map  $\Phi$ ). *Let  $\Phi$  be a smooth  $k$ -equivariant function from the plane  $\mathbb{R}^2$  into the sphere  $\mathbb{S}^2$ . Then if  $2 \leq k$ ,*

$$(36) \quad |\partial_r \phi| \leq C_\phi r, \quad 0 \leq r \leq 1.$$

*Remark 1.10.* Note that the condition  $2 \leq k$  for estimate (36) is crucial, as the formula (26) shows for unit homotopy class instanton  $I^1$ .

*Proof of estimate (36).* Our first step is to establish that  $\partial_r \phi$  is continuous and vanishes at  $r = 0$ .

First of all, notice that along any fixed radial line  $\Theta$  is constant and the vector-field  $\partial_r$  is a continuous section of  $T\mathbb{R}^2$ . The same is true of the field  $\frac{1}{r}\partial_\Theta$ . Furthermore, one has that:

$$\lim_{\substack{\Theta=0 \\ r \rightarrow 0}} \partial_r = - \lim_{\substack{\Theta=\frac{\pi}{2} \\ r \rightarrow 0}} \frac{1}{r} \partial_\Theta.$$

Therefore, by continuity we must have that:

$$\lim_{\substack{\Theta=0 \\ r \rightarrow 0}} \|\partial\Phi(\partial_r)\|^2 = \lim_{\substack{\Theta=\frac{\pi}{2} \\ r \rightarrow 0}} \left\| \partial\Phi\left(\frac{1}{r}\partial_\Theta\right) \right\|^2,$$

where  $\|\cdot\|^2$  refers to the metric (18). Computing both sides of this last equation, we see that not only is  $\partial_r \phi$  continuous (and hence bounded) on the interval  $[0, 1]$ , but that we also have:

$$(37) \quad \lim_{r \rightarrow 0} |\partial_r \phi| = \lim_{r \rightarrow 0} \frac{k |\sin(\phi)|}{r}.$$

Using now the fact that  $\phi(0) = 0$  to write  $\phi(r) = \int_0^r \partial_y \phi(y) dy$ , upon substitution of this integral into the right-hand side of (37), we see from the fundamental theorem of calculus and the condition  $2 \leq k$ , that we must in fact have  $\partial_r \phi(0) = 0$ .

It remains to show that  $\partial_r \phi$  vanishes uniformly (with nonuniform constant) in  $r$ . To do this, we compute the Dirichlet energy:

$$e(\phi) = g_{ij} \left[ \partial_r \Phi^i \partial_r \Phi^j + \frac{1}{r^2} \partial_\Theta \Phi^i \partial_\Theta \Phi^j \right] = |\partial_r \phi|^2 + \frac{k^2}{r^2} |\sin(\phi)|^2.$$

This is a  $C^\infty$  function on  $\mathbb{R}^2$  which depends on the radial variable only. Furthermore, we have that  $e(\phi)(0) = 0$ . Thus, it is necessary that:

$$|e(\phi)| \leq C r^2, \quad 0 \leq r \leq 1,$$

for some constant that depends on  $\phi$ . In particular, we have the bound (36).  $\square$

## 2. Some notational conventions and an overview

In this section, we will first list some standard notational conventions that will be useful throughout the sequel. We then give a quick technical overview of the main result.

2.1. *Some notation.* Throughout this paper, we shall employ the standard notation  $A \lesssim B$  to mean  $A \leq CB$  for two quantities  $A$  and  $B$ , where  $C$  is a fixed constant. There is no uniformity in this notation for *separate* instances. That is, separate occurrences of  $\lesssim$  on the same page will not necessarily imply  $C$  is the

same for each. Another less standard notation which will be of great use is the following:

*Notation 2.1.* For any pair of nonnegative integers  $0 \leq m, n$  we will denote by  $F^{m,n}$  any  $C^\infty(\mathbb{R}^+)$  function which satisfies the following bounds:

$$|(r\partial_r)^i F^{m,n}| \lesssim C_i \frac{r^m}{(1+r)^{m+n}}.$$

We will also use a shorthand notation for the case  $m = 0$ . Here we shall set  $F^n = F^{0,n}$ , so that we have:

$$|(r\partial_r)^i F^n| \lesssim \frac{C_i}{(1+r)^n}.$$

We also denote the  $\lambda$  rescaling of these functions by  $F_\lambda^{m,n}(r) = F^{m,n}(\lambda r)$  and similarly for  $F_\lambda^n$ . Finally, let us remark that different instances of  $F_\lambda^{m,n}$  on any line, or between lines, can mean separate functions.

This notation will occur so frequently in the sequel that it will be useful for us to record here several instances which involve either time or space derivatives or multiplication by powers of  $r$ . Collectively these are the following, where we assume  $j \in \mathbb{Z}$  is such that  $-m \leq j$  in the first identity and  $1 \leq m$  in the third:

$$(38) \quad r^j F_\lambda^{m,n} = \lambda^{-j} F_\lambda^{m+j,n-j}, \quad \partial_t F_\lambda^{m,n} = \dot{\lambda} \lambda^{-1} F_\lambda^{m,n}, \quad \partial_r F_\lambda^{m,n} = \lambda F_\lambda^{m-1,n+1}.$$

All of these are immediate from Notation 2.1 above.

Also, in the sequel we will in general use the  $\lambda$ -subscript notation to denote the  $\lambda$  rescaling of a given function. For example,  $I_\lambda(r) = I(\lambda r)$ .

*2.2. An overview.* As is perhaps already clear at this point, our method for establishing Theorem 1.4 is to control directly a certain modulational equation for the time-dependent scaling parameter  $\lambda(t)$  and to show that this evolves according to a blowup ODE. Thus, in this sense our work is closely related in spirit to the modulational stability approach originally pioneered by M. Weinstein (see [38]) and later sharpened by Buslaev and Perelman (see [5]) to study solitons dynamics of the focusing nonlinear Schrödinger equation. The major difference, however, is that we are actually trying to show that there is an extremely strong asymptotic *instability*. This, of course, introduces a serious problem when one tries to control the nonlinear (19) linearized around the modulated soliton  $I_\lambda$ . Nonetheless, we begin by using the decomposition  $\phi = I_\lambda + u$  from line (31) and then linearizing (19) around  $I_\lambda$ :

$$(39) \quad \partial_t^2 u + H_\lambda u = -\ddot{I}_\lambda + \mathcal{N}(u),$$

where the Hamiltonian is given by  $H_\lambda = -\partial_r^2 - r^{-1}\partial_r + Q_\lambda(r)$  and the nonlinear term  $\mathcal{N}(u)$  containing quadratic and higher order terms in  $u$  (also containing a factor of  $r^{-2}$ ).

Our first task, dealt with in Section 4, is to prove orbital stability of the modulated soliton  $I_\lambda$  under the condition that the “radiation” part of the solution (i.e.  $u$ ) is orthogonal to the function  $J_\lambda = r \partial_r I_\lambda$ , which is the unique eigenfunction of the Hamiltonian  $H_\lambda$ . The latter is a consequence of the fact that  $I_\lambda$  realizes an absolute minimum of the energy (21) associated with the full nonlinear problem (19). The orthogonality condition provides us with an ODE for the scaling parameter  $\lambda(t)$ , which is coupled to the radiation term  $u$ :

$$(40) \quad \dot{\lambda} \left( \langle J, J \rangle - \langle u(\lambda^{-1}r), r \partial_r J \rangle \right) = \left( \lambda \langle \phi_t, J_\lambda \rangle \right) \lambda^2,$$

while the orbital stability statement will give us a very weak control of the remainder  $u$ :

$$(41) \quad \int_{\mathbb{R}^+} \left[ (\partial_t \phi)^2 + (\partial_r u)^2 + \frac{k^2}{r^2} u^2 \right] r dr \lesssim \varepsilon^2.$$

Due to the coupling between the scaling parameter  $\lambda(t)$  and the radiation  $u$ , to control  $\lambda(t)$  to the extent that we can show  $\lambda(t) \rightarrow \infty$  in finite time requires much better control on the radiation term  $u$ . The usual procedure for dealing with this is to scale out the modulational parameter  $\lambda(t)$  at each fixed time, so *time independent* spectral methods can be used to control the linearized equation. This procedure works well if one can prove that there is a *slow* limit of the parameter  $\lambda(t)$ , but it obviously causes a catastrophe if  $\lambda(t)$  grows rapidly. In this case, a truly nonlinear approach is needed.

The reason why standard nonlinear estimates, for example the kind used to prove orbital (Lyapunov) stability (i.e. (41) above), are not sufficient to reach the blowup time  $\sim \varepsilon^{-1}$  is basically due to the fact that their application to the ODE (40) is not truly scale invariant. That is, the use of *fixed time* estimates which result from orbital stability analysis causes a loss (of scaling) when one integrates over time. Such integrations seem unavoidable when analyzing (40). To overcome this problem requires uncovering a nonlinear dispersion phenomenon<sup>9</sup> in equation (39) for the radiation term  $u$ . That such a dispersive process indeed takes place is in some sense the miracle of equation (39). More specifically, as the soliton  $I_\lambda$  collapses it actually repels the excess radiation away from the origin. This evacuation process only causes the soliton to collapse at a faster rate, and it is what is ultimately responsible for the acceleration of self-similar behavior governed by the left-hand side of the blowup rate (33).

One of the most interesting issues in this paper is the mechanism by which this “repulsive” behavior of the linearized equations manifests itself mathematically. This is where the “quasi-integrable” system aspect of the static version of (19) comes in. As we have already discussed in the introduction, static solutions to

---

<sup>9</sup>A nonlinear dispersion phenomenon (of a different nature) has been observed and used by Merle and Raphael in their work [28] on the blowup analysis for the critical focusing nonlinear Schrödinger equation.



(19) are generated by the *first order* Bogomol'nyi (25). When one linearizes (19) around these static solutions, the corresponding Hamiltonian  $H_\lambda$  splits as a product of two first order operators which are adjoints of each other. That is:

$$H_\lambda = A_\lambda^* A_\lambda,$$

where  $A_\lambda$  is the linearization of (25). The reason why this splitting is so useful is that the Hamiltonian  $H_\lambda$  also possesses its super-symmetric companion:

$$\tilde{H}_\lambda = A_\lambda A_\lambda^*,$$

with  $A_\lambda^*$  and  $A_\lambda$  being the analogs of the creation and annihilation operators, and  $H_\lambda$  and  $\tilde{H}_\lambda$  related to each other according to the remarkable intertwining relation:

$$(42) \quad A_\lambda H_\lambda = \tilde{H}_\lambda A_\lambda.$$

Such a splitting elucidates the nonnegativity of  $H_\lambda$  and identifies the function  $J_\lambda$ , which is the kernel of  $A_\lambda$ , as the ground (vacuum) state of  $H_\lambda$ . In addition, the intertwining property (42) allows us to simply conjugate the problem (39) to one whose linear part involves the more manageable Hamiltonian  $\tilde{H}_\lambda$ :

$$(43) \quad \partial_t^2(A_\lambda u) + \tilde{H}_\lambda(A_\lambda u) = -A_\lambda(\ddot{I}_\lambda) + A_\lambda \mathcal{N}(u) + [\partial_t^2, A_\lambda]u.$$

The Hamiltonian  $\tilde{H}_\lambda$ , which is obtained from  $H_\lambda$  by the process of “removing” its ground state, is of the explicit form  $\tilde{H}_\lambda = -\partial_r^2 - r^{-1}\partial_r + V_\lambda(r)$  and involves a *space-time repulsive* time-dependent potential. This means that for problem (43) one may proceed via purely physical space methods, and there is no difficulty in handling extremely violent growth of the scaling parameter  $\lambda(t)$ . What we can do is to establish quite strong (i.e. scale invariant) integrated and fixed time energy estimates (i.e. so called Morawetz type estimates), while keeping precise track of the influence of the source terms on the right-hand side of (43) involving the scaling parameter  $\lambda$ . To undo the conjugation procedure embodied in (43), we only need to use the fact that  $u$  is orthogonal to the kernel of  $A_\lambda$ , because through a little elementary functional analysis, this allows one to turn our Morawetz estimates into ones involving only the term  $u$  (as opposed to  $A_\lambda u$ ). Once these estimates are established, it is possible to show, through a somewhat lengthy calculation, that after a long self-similar epoch where  $C_0\dot{\lambda} \sim \varepsilon_0\lambda^2$ , the modulation ODE for  $\lambda(t)$  enters another monotonic regime where it takes the final form:

$$C_0\dot{\lambda}(t) \sim \varepsilon_0\lambda^2(t) - \lambda^2(t) \int_0^t \mathcal{O}\left(\frac{\dot{\lambda}^4}{\lambda^7}(s)\right) ds.$$

It is this ODE which leads to the blowup

and also gives the tight *upper* bounds in (33). The reader should compare this last formula to the blowup ODE for modulated Yang-Mills instantons derived in [2] through heuristic arguments.

We now turn to the details of all of this. As is common in this type of work, many of our assumptions will be bootstrapped. We shall follow the outline:

- In Section 3 we discuss the Hamiltonians  $H_\lambda$  and  $\tilde{H}_\lambda$ .
- In Section 4 we derive the basic ODE for the scaling parameter  $\lambda(t)$  and prove the orbital stability statement.
- In Section 5, assuming certain estimates on the “radiation” part of the solution, we obtain a much refined *closed form* (i.e. without explicit dependence on the  $u$ ) of the modulation ODE, establish its monotonic and algebraic properties, and prove blowup along with an explicit rate bound.
- In Section 6, assuming the monotonic properties of the scaling parameter  $\lambda(t)$ , we prove the integrated space-time and fixed time bounds for the radiation by making use of the conjugated Hamiltonian  $\tilde{H}_\lambda$ .
- In Appendix A we give some further explicit computations needed in the analysis of the blowup ODE.
- In Appendix B we establish some general coercive properties for the class of first order operators related to  $A_\lambda$ .

### 3. The linearized equations and a basic spectral calculation

Our purpose here is to derive and record certain calculations involving the linearization of equation (19) around a time-dependent modulation of the soliton  $I(\lambda(t)r)$ . That is, we decompose the full solution as:

$$(44) \quad \phi(t, r) = I(\lambda(t)r) + u(t, r).$$

This yields the following set of formulas for (19) linearized around  $I_\lambda$ :

$$(45) \quad \partial_t^2 u + H_\lambda u = -\ddot{I}_\lambda + \mathcal{N}(u),$$

where we have set:

$$\mathcal{N}(u) = \frac{k^2 \sin(2I_\lambda)}{2r^2} \cdot (1 - \cos(2u)) + \frac{k^2 \cos(2I_\lambda)}{r^2} \cdot \left(u - \frac{1}{2} \sin(2u)\right).$$

Here  $H_\lambda$  is the linearized Hamiltonian:

$$(46) \quad H_\lambda = -\partial_r^2 - \frac{1}{r} \partial_r + \frac{k^2}{r^2} \cos(2I_\lambda) = A_\lambda^* A_\lambda,$$

where the first order operators  $A, A^*$  are given by:

$$(47) \quad A_\lambda = -\partial_r + \frac{k}{r} \cos(I_\lambda), \quad A_\lambda^* = \partial_r + \frac{1}{r} + \frac{k}{r} \cos(I_\lambda).$$

The spectrum of  $H_\lambda$ , defined as a self-adjoint operator on  $L^2(rdr)$  and obtained by taking the closure of  $C_0^\infty(\mathbb{R}^+)$  in the graph norm of  $H_\lambda$ , is easily computed via the factorization (46) or via the knowledge that the ground state  $I_\lambda$  of the (static

form of the) (19) is unique. There is a unique eigenfunction, which has zero energy, and it is given by the formula:

$$(48) \quad J_\lambda = r \partial_r I_\lambda = k \sin(I_\lambda).$$

In particular,  $J_\lambda$  solves the first order ‘‘linearized Bogomol’nyi equation’’:

$$(49) \quad A_\lambda J_\lambda = 0.$$

The absolutely continuous spectrum fills the half-line  $[0, \infty)$ .

Of primary importance for use here will be the conjugate operator formed by  $A_\lambda A_\lambda^*$ , the super-symmetric companion  $\tilde{H}_\lambda$  of  $H_\lambda$

$$(50) \quad \begin{aligned} \tilde{H}_\lambda = A_\lambda A_\lambda^* &= -\partial_r^2 - \frac{1}{r} \partial_r + \frac{k^2 + 1}{r^2} + \frac{2k}{r^2} \cos(I_\lambda) \\ &= -\partial_r^2 - \frac{1}{r} \partial_r + V_\lambda(r). \end{aligned}$$

Recall that the Hamiltonians  $H_\lambda$  and  $\tilde{H}_\lambda$  are related via an intertwining relation (42). As opposed to  $H_\lambda$ , the spectrum of  $\tilde{H}_\lambda$  has only an absolutely continuous component filling  $[0, \infty)$ . What is more important is that while in our application the Hamiltonian  $\tilde{H}_\lambda$  will be *time-dependent*, it has a remarkable structure which allows one to prove strong local energy decay estimates even if the parameter  $\lambda(t)$  grows in an unconstrained fashion. The needed properties follow from entirely elementary calculations and are as follows:

$$(51) \quad V_\lambda \geq \frac{(k-1)^2}{r^2}, \quad (\text{Positive}),$$

$$-\partial_r V_\lambda = \frac{2(k^2+1)}{r^3} + \frac{4k}{r^3} \cos(I_\lambda) + \frac{2k^2}{r^3} \sin^2(I_\lambda)$$

$$(52) \quad \geq \frac{2(k-1)^2}{r^3}, \quad (\text{Space-Repulsive}),$$

$$(53) \quad -\partial_t V_\lambda = \frac{\dot{\lambda}}{\lambda} \cdot \frac{2k^2}{r^2} \sin^2(I_\lambda), \quad (\text{Time-Repulsive}).$$

We conclude this section by refining the decomposition (44). This will be extremely important for us in the sequel, and it will ultimately lead us to the precise asymptotic (33). What we will need to do is further decompose the radiation term as:

$$u = w_0 + w,$$

where the ‘‘leading term’’  $w_0$  is meant solely to eliminate the term  $\ddot{I}_\lambda$  on the right-hand side of (45), while at the same retaining the orthogonality relation:

$$(54) \quad \langle w_0, J_\lambda \rangle = 0.$$

The equation we use to generate  $w_0$  is (the reason for this will become apparent in the sequel):

$$(55) \quad A_\lambda(H_\lambda w_0) = \tilde{H}_\lambda(A_\lambda w_0) = -A_\lambda(\ddot{I}_\lambda).$$

To further the computation, we use the fact that  $A_\lambda J_\lambda = 0$  to write:

$$A_\lambda(\ddot{I}_\lambda) = \frac{\dot{\lambda}}{\lambda} A_\lambda \partial_t (r \partial_r I_\lambda) = \frac{\dot{\lambda}^2}{\lambda^2} A_\lambda r \partial_r (J_\lambda) = \frac{\dot{\lambda}^2}{\lambda^2} A_\lambda (J + r \partial_r J)_\lambda.$$

Therefore, by peeling off the common factor of  $A_\lambda$  from both sides of (55), it suffices to solve:

$$H_\lambda w_0 = -\frac{\dot{\lambda}^2}{\lambda^2} (J + r \partial_r J)_\lambda.$$

We now use the ansatz  $w_0 = \dot{\lambda}^2 \lambda^{-4} [K(\lambda r) - \gamma J_\lambda]$ , where  $\gamma$  is a normalization constant which will be chosen as to satisfy (54). Eliminating common factors and rescaling the spatial variable, we find that:

$$H_1(K) = -(J + r \partial_r J),$$

from which a direct computation shows that  $K(r) = \frac{r^2}{4} J(r)$  is the desired solution. Therefore, we have that:

$$(56) \quad w_0(t, r) = \frac{\dot{\lambda}^2}{\lambda^4} (t) \cdot (a J_\lambda(r) + b (r^2 J)_\lambda),$$

where the coefficients are given by the explicit formulas:

$$(57) \quad a = -\frac{1}{4} \langle J, (r^2 J) \rangle \cdot \|J\|_{L^2(rdr)}^{-2}, \quad b = \frac{1}{4}.$$

Before ending this section, let us translate the abstract function notation on line (38) into some specific bounds which will be used many times in the sequel. These are a consequence of simple explicit formulas and the restriction  $4 \leq k$ :

$$(58a) \quad w_0 = \dot{\lambda}^2 \lambda^{-4} (F_\lambda^{4,4} + F_\lambda^{6,2}), \quad \partial_t(w_0) = \ddot{\lambda} \dot{\lambda} \lambda^{-4} (F_\lambda^{4,4} + F_\lambda^{6,2}) + \dot{\lambda}^3 \lambda^{-5} (F_\lambda^{4,4} + F_\lambda^{6,2}),$$

$$(58b) \quad |J_\lambda| \lesssim F_\lambda^{4,4}, \quad |\partial_t J_\lambda| \lesssim \dot{\lambda} \lambda^{-1} F_\lambda^{4,4}, \\ |\partial_t^2 J_\lambda| \lesssim (|\ddot{\lambda}| \lambda^{-1} + \dot{\lambda}^2 \lambda^{-2}) F_\lambda^{4,4},$$

$$(58c) \quad \partial_t(A_\lambda) = \dot{\lambda} F_\lambda^{7,9}, \quad |\partial_t^2(A_\lambda)| \lesssim (|\ddot{\lambda}| + \dot{\lambda}^2 \lambda^{-1}) F_\lambda^{7,9}, \\ A_\lambda(F_\lambda^{m,n}) = \lambda F_\lambda^{m-1, n+1}.$$

We note here that we are assuming  $1 \leq m$  for the last identity on line (58c).

#### 4. Orbital stability

We now begin with the first step in our proof of Theorem 1.4. This is to show that one can make a rough decomposition of the full wave-map  $\phi$  into a bulk piece

which is a rescaled soliton, plus a small remainder which we can estimate in a certain energy space. Of particular importance to us will be that we can construct this decomposition in such a way as to retain a certain orthogonality between the bulk piece and the small “radiation term.” Of course, this is precisely the modulational approach to orbital stability first pioneered by M. Weinstein in his study of the nonlinear Schrödinger equation (see [38]). What we intend to prove is the following:

LEMMA 4.1 (Orbital stability with orthogonal decomposition). *Suppose that  $\phi$  is a solution to the problem (19), and suppose that initially the Cauchy data for  $\phi$  decomposes as:*

$$(59) \quad \phi(0) = I_{\lambda_0} + u_0, \quad \partial_t \phi(0) = \dot{\phi}_0.$$

*Assume that the energy satisfies  $E[\phi] = 4\pi k + \varepsilon^2$ , with  $\varepsilon$  chosen small enough. Then as long as the solution  $\phi$  exists, there is a time-dependent parameter  $0 < \lambda(t) < \infty$ , with  $\lambda(0) = \lambda_0$  and the property that the following conditions hold for all times of existence provided that they hold initially:*

$$(60) \quad E_0[u] = \frac{1}{2} \int_{\mathbb{R}^+} \left[ (\partial_t \phi)^2 + (\partial_r u)^2 + \frac{k^2}{r^2} u^2 \right] r dr \lesssim \varepsilon^2,$$

$$(61) \quad 0 = \langle u(t), J_{\lambda(t)} \rangle.$$

*Here  $u$  is defined by the relation  $u = \phi - I_\lambda$ . In addition we have that:*

$$(62) \quad \left| \frac{\dot{\lambda}}{\lambda^2} \right| \lesssim \varepsilon.$$

*Finally, we remark that (conversely) the full wave map  $\Phi$  is  $C^\infty$  up to any time  $T$  as long as  $\lambda(t) < \infty$  and (60) holds true for any  $t \in [0, T]$ .*

*Proof of Lemma 4.1.* The proof essentially reduces to defining an appropriate equation for the evolution of  $\lambda(t)$ , basic existence and uniqueness theory of ODEs, followed by the coercive estimate (162) proved in Appendix B. We note here that the last remark of Lemma 4.1 follows from the local “small energy implies regularity” statement for symmetric wave-maps and is, for instance, contained in Theorem 1.2.

We will use here the following observation: notice that if  $\lambda \rightarrow \infty$  or  $\lambda \rightarrow 0$  on some time interval such that (60) holds, then the full wave-map  $\Phi$  must break down also on that time interval because its energy concentrates at  $r = 0$  or  $r = \infty$  respectively (in fact, by finite speed of propagation it cannot happen that  $\lambda \rightarrow 0$  in finite time, so any “blowup” of this type must occur at  $t = \infty$ ).

Now, for a strictly positive real valued function of time  $\lambda(t)$ , we define the equation:

$$(63) \quad \dot{\lambda} \left( 2\langle I, J \rangle + \langle \phi(\lambda^{-1}r), r \partial_r J \rangle \right) = -\langle \phi_t, J_\lambda \rangle \lambda^3.$$

Notice that as long as the wave-map  $\phi$  exists and is smooth, this equation is of the form  $\alpha(\lambda, t)\dot{\lambda} = \beta(\lambda, t)$  for two  $C^1$  functions  $(\lambda, t)$  of the variables  $\alpha, \beta$ . We now construct  $\lambda(t)$  from (63) via a simple bootstrapping procedure. Our goal is to provide a strict lower bound for  $|\alpha|$  so that we may simply apply the usual existence theory to (63) which then produces  $\lambda(t)$ .

First of all, notice that a simple calculation involving the identity:

$$\langle I, r\partial_r J \rangle = -2\langle I, J \rangle - \langle J, J \rangle$$

shows that the bound (60) initially implies that:

$$|\alpha(\lambda_0, 0) + \langle J, J \rangle| \leq C\varepsilon,$$

for some, possibly large, constant  $C$  which we choose in a moment. Therefore, by continuity there exists a small time  $T^*$  such that the solution  $\phi$  exists and is regular on  $[0, T^*]$ , and a solution  $\lambda(t)$  to equation (63) exists, is contained in  $(0, \infty)$ , and obeys the bounds:

$$(64) \quad |\alpha(\lambda(t), t) + \langle J, J \rangle| \leq 2C\varepsilon.$$

The heart of the matter is now the following: we will show that if  $T^*$  is *any* time such that the above holds (i.e. existence for  $\phi$  and  $\lambda$  and the bound (64)), then we must also necessarily have the conditions (61) and (60), *as well as* the following improvement of (64):

$$(65) \quad |\alpha(\lambda(t), t) + \langle J, J \rangle| \leq C\varepsilon.$$

By continuing this process, we will have shown that on any time interval  $[0, T^*]$  such that the solution  $\phi$  exists and is regular, there is a continuous solution of (63)  $\lambda(t) \in (0, \infty)$  such that  $\lambda(0) = \lambda_0$  and (60)–(62) holds (assuming, of course, that these conditions hold initially).

We now show that existence and (64) imply (60) and (61), and that these two together imply the improved bound (65). Everything rests on the orthogonality condition (61). We define  $u = \phi - I_\lambda$  and use a few simple integration by parts to compute that:

$$\begin{aligned} \frac{d}{dt} \langle u, J_\lambda \rangle &= \langle u_t, J_\lambda \rangle + \frac{\dot{\lambda}}{\lambda} \langle u, (r\partial_r J)_\lambda \rangle \\ &= \frac{\dot{\lambda}}{\lambda^3} \left( 2\langle I, J \rangle + \langle \phi(\lambda^{-1}r), r\partial_r J \rangle \right) + \langle \phi_t J_\lambda \rangle \\ &= 0, \end{aligned}$$

where the last line follows from the assumption that  $\lambda$  solves (63). Therefore, since (61) holds initially, it holds on  $[0, T^*]$ .

It remains to show (60) and that this bound implies (65). In fact, given the form of  $\alpha$ , this latter implication is immediate from (60) and the Cauchy-Schwartz inequality, where the sufficiently large constant  $C$  is chosen according to the implicit constant appearing on line (60).

Thus, we have reduced things to proving that, assuming that orthogonality condition (61) holds, (60) also holds with a fixed implicit constant which *does not* depend on the size of  $\lambda$  or  $C$ . This is where the condition  $E[\phi] - E[I_\lambda] = \varepsilon^2$  enters. Notice that this equality follows from the conservation of energy and our assumption  $E[\phi] = 4\pi k + \varepsilon^2$ . Computing the energy difference and using line (24), we have that:

$$\begin{aligned} \varepsilon^2 &= \pi \int_{\mathbb{R}^+} \left[ (\partial_t \phi)^2 + \left( \partial_r \phi - \frac{k}{r} \sin(\phi) \right)^2 \right] r dr \\ &= \pi \int_{\mathbb{R}^+} \left[ (\partial_t \phi)^2 + \left( \partial_r u - \frac{k}{r} \cos(I_\lambda) u \right)^2 \right] r dr - \pi \mathcal{R}_\lambda(u), \end{aligned}$$

where we are defining higher order integrated nonlinearity:

$$\mathcal{R}_\lambda(u) = \int_{\mathbb{R}^+} \left( \partial_r \phi - \frac{k}{r} \sin(\phi) \right)^2 r dr - \int_{\mathbb{R}^+} \left( \partial_r u - \frac{k}{r} \cos(I_\lambda) u \right)^2 r dr.$$

Now, using the first coercive bound (162) of Appendix B we have the estimate:

$$(66) \quad E_0[u] \lesssim \varepsilon^2 + |\mathcal{R}_\lambda(u)|.$$

Finally, using the simple algebraic formula for the difference of squares, using equation (25), and writing the first few terms in the Taylor series for  $\sin(I_\lambda + u)$ , we easily have the nonlinear bound:

$$\begin{aligned} (67) \quad |\mathcal{R}_\lambda(u)| &\lesssim \int_{\mathbb{R}^+} \left( |\partial_r u| + \frac{|u|}{r} \right) \cdot \left( \frac{u^2}{r} \right) r dr \\ &\lesssim (E_0[u])^{\frac{3}{2}}, \end{aligned}$$

where the last line follows from Cauchy-Schwartz and the Poincaré type estimate:

$$(68) \quad |u(r)|^2 \leq 2 \left( \int_{\mathbb{R}^+} (\partial_r u)^2 r dr \right)^{\frac{1}{2}} \cdot \left( \int_{\mathbb{R}^+} \frac{u^2}{r^2} r dr \right)^{\frac{1}{2}}.$$

The bounds (66)–(67) taken together show that we may conclude (60) for  $\varepsilon$  small enough and some universal implicit constant (e.g. this can be shown through another continuity argument). Having established (60) and (65) we can easily show from (63) that:

$$\left| \frac{\dot{\lambda}}{\lambda^2} \right| \lesssim \varepsilon.$$

This concludes our demonstration of Lemma 4.1. □

### 5. The effective evolution and the main blowup argument

We now begin in earnest the proof of the main Theorem 1.4. This centers around computing a more effective form of the modulation ODE (63). This will be followed by an ODE analysis giving the desired blowup together with its asymptotic profile. The main technical result of this section is the following:

PROPOSITION 5.1 (Refined structure for the modulation equation (63)). *Consider the scaling parameter  $\lambda(t)$  which is defined through Lemma 4.1 and equation (63). Suppose that the initial data (59) is given according to Theorem 1.4. Then on a time interval where  $\lambda(t) \in (0, \infty)$  and  $t \in [0, \varepsilon^{-4}]$ , the scaling parameter  $\lambda(t)$  satisfies a first order ODE:*

$$(69) \quad (C_0 - \varepsilon_1(t))\dot{\lambda}(t) = \varepsilon_0\lambda^2(t) - \lambda^2(t) \int_0^t \mathfrak{E}(s) ds.$$

Here  $C_0 = \langle J, J \rangle$ , and  $\varepsilon_0, \varepsilon_1(t)$  and  $\mathfrak{E}(t)$  obey the conditions ( $\varepsilon_0$  is fixed):

$$(70) \quad \left| \varepsilon_0 - \frac{\varepsilon}{\pi} \right| \lesssim c_0\varepsilon,$$

$$(71) \quad |\varepsilon_1(t)| \lesssim \varepsilon,$$

$$(72) \quad |\mathfrak{E}(t)| \lesssim c_0\varepsilon^2 + c_0^{\frac{1}{2}} \sup_{0 \leq s \leq t} \dot{\lambda}^4 \lambda^{-7}(s).$$

Here  $\varepsilon, c_0$  are the small constants from line (29). In addition, one has the following “structure bounds” for the acceleration of  $\lambda(t)$ :

$$(73) \quad \left| \ddot{\lambda}(t) - 2\frac{\dot{\lambda}^2}{\lambda}(t) \right| \leq Cc_0^{\frac{1}{2}}\frac{\dot{\lambda}^2}{\lambda}(t) + C \left( c_0\varepsilon^2 + \sup_{0 \leq s \leq t} \frac{\dot{\lambda}^4}{\lambda^7}(s) \right) \lambda^2.$$

We remark that the constant  $C$  is universal, i.e. independent of  $c_0$  and  $\varepsilon$ .

*Remark 5.2.* The reason the explicit constant  $C$  appears in estimate (73) is merely a notational convenience. The bound (73) will be one of our main bootstrapping assumptions in the sequel.

The remainder of this section is divided into two parts. First, we will show that the identities and estimates (69)–(73) imply that the parameter  $\lambda(t)$  goes to infinity at some time  $T^{**} \in [0, \varepsilon^{-4}]$  as long as  $\varepsilon$  is chosen small enough. We then establish the bounds (33) on the asymptotic rate of  $\lambda(t)$  as  $t \rightarrow T^{**}$ . Finally, in the last subsection we state the “Main Estimate” (a certain fixed time energy estimate) of our paper, and we use it to derive all of the assumptions (69)–(73). This main technical estimate will be the subject of the final section of the paper.

5.1. *Proof of the blowup and the universal bound for  $\lambda(t)$ .* Using equation (69), as well as the assumptions (70)–(73), we now show that  $\lambda$  must blowup in finite time. The basic idea is the following: without the contribution of the integral on the right-hand side of (69), the desired blowup would occur in finite time  $\sim C_0\varepsilon_0^{-1}$  in a self-similar Riccati fashion. Therefore, the only problem is that one must guarantee the integral term (which adds a negative<sup>10</sup> contribution, creating a delay effect) does not interfere to the extent that  $\dot{\lambda}$  is driven to zero too quickly before the blowup can occur. The fact that *a priori* (i.e. again by Theorem 1.2) at any supposed blowup time one *must* have that  $\dot{\lambda}\lambda^{-2} \rightarrow 0$  indicates a very

<sup>10</sup> That the contribution is overall a negative one follows from Theorem 1.2.



delicate balancing between the two main terms on right-hand side of (69). This constitutes one of the main technical difficulties in this paper and why many of the estimates which appear in the sequel are so involved. To establish the needed control, we will first show that the initial self-similar behavior, approximated by the ODE  $C_0 \dot{\lambda} = \varepsilon_0 \lambda^2$ , forces  $\lambda(t)$  into a different monotonic regime, where in particular, the terms  $\dot{\lambda}^4 \lambda^{-7}$  dominate the error estimate for  $\mathcal{E}$  (i.e. (72)). At that point, blowup is assured. Our last task is then to analyze the balance of the integral and  $\varepsilon_0$  terms on the right-hand side of (69) and to derive the precise blowup bounds (33) from this. Along the way, we will show that all of this can be accomplished before the time interval  $[0, \varepsilon^{-4}]$  expires, so that we still have access to all of the structure included in Proposition 5.1.

We now proceed with the details outlined above. The first main thing is to control the size of the interval where one cannot guarantee good bounds on  $\mathcal{E}(s)$ . The key is to show that  $\dot{\lambda}^4 \lambda^{-7}$  becomes monotonic soon enough and with enough force to cover the constant  $c_0 \varepsilon^2$  which is lost on line (73). Luckily it is not hard to show that these two things happen at essentially the same time. Computing the time derivative and then using the identity (73) we have:

$$(74) \quad \frac{d}{dt} \left( \frac{\dot{\lambda}^4}{\lambda^7} \right) = \frac{\dot{\lambda}^5}{\lambda^8} + O \left( c_0^{\frac{1}{2}} \frac{\dot{\lambda}^2}{\lambda^3} + c_0 \varepsilon^2 + \sup_{0 \leq s \leq t} \frac{\dot{\lambda}^4}{\lambda^7}(s) \right) \frac{\dot{\lambda}^3}{\lambda^5}.$$

We now let  $T_0$  be the first time such that:

$$(75) \quad \dot{\lambda}^4 \lambda^{-7}(T_0) = C c_0^{\frac{1}{2}} \varepsilon^2,$$

for some large constant  $C$  which is larger than twice the implicit constants in (72). First, we argue that such a time must occur.

If such a time does not occur, then if  $c_0$  is chosen small enough (we remind the reader that this is done by simply choosing initial data according to (29) and does not affect the size of constants in estimates like (72)), one sees immediately from (72) and (69) that the following bound holds for all times:

$$(76) \quad \frac{\dot{\lambda}}{\lambda^2}(t) \geq \frac{1}{2C_0} \varepsilon_0 - \int_0^t C c_0 \varepsilon_0^2 ds,$$

where  $C$  is fixed and independent of  $c_0, \varepsilon$ . A simple argument, which we leave to the reader, shows that if  $\lambda(0) = 1$ , then this last inequality implies that  $\lambda \rightarrow \infty$  at some finite time  $T^{**} \leq 4C_0 \varepsilon_0^{-1}$  and that in addition:

$$\frac{\dot{\lambda}}{\lambda^2}(t) \geq \frac{\varepsilon_0}{4C_0},$$

for all  $t \in [0, T^{**})$ . In that case, however, we would also have that:

$$\frac{\dot{\lambda}^4}{\lambda^7}(t) = \left( \frac{\dot{\lambda}}{\lambda^2}(t) \right)^4 \lambda(t) \geq \left( \frac{\varepsilon_0}{4C_0} \right)^4 \lambda(t) \rightarrow \infty \quad \text{as } t \rightarrow T^{**},$$

which shows that in fact the time  $T_0$  defined above must occur. What is more, a simple analysis of the previous argument shows that this time must also satisfy the

conditions:

$$(77) \quad T_0 < 4C_0\varepsilon_0^{-1}, \quad \dot{\lambda}\lambda^{-2}(T_0) \geq \frac{1}{4C_0}\varepsilon_0.$$

Notice it is also clear that for all times  $t \in [0, T_0]$ , we have  $\dot{\lambda}(t) > 0$ .

Now, by a direct application of the orbital stability bound (62), we have that for any time where the solution exists, there is the bound:

$$\frac{\dot{\lambda}^4}{\lambda^7}(t) \lesssim \varepsilon^2 \frac{\dot{\lambda}^2}{\lambda^3}(t).$$

Applying this on the right-hand side of (74), we see that for all  $t \geq T_0$  and as long as  $\dot{\lambda}^4\lambda^{-7}(t) \geq Cc_0\varepsilon^2$ , we have that:

$$\frac{d}{dt} \left( \frac{\dot{\lambda}^4}{\lambda^7} \right) (t) \geq (1 - Cc_0^{\frac{1}{2}}) \frac{\dot{\lambda}^5}{\lambda^8}(t) > 0.$$

By bootstrapping this argument, we see that  $\dot{\lambda}^4\lambda^{-7}$  is monotonically increasing for all times  $t \geq T_0$ .

Before continuing, with the proof of blowup, we pause for a moment to upgrade the bound (75). This will be used in a crucial way in the sequel (see the proof of Proposition 6.3 in §6). We claim that there exists a time  $T^* = T_0 + O(1)$  such that the following improvement of (75) is valid:

$$(78) \quad \dot{\lambda}^4\lambda^{-7}(T^*) = c_0\varepsilon.$$

Again by contradiction, if this were not the case by (69), the definition (75) of  $T_0$ , and the time bounds (77), we would have a bound of the form:

$$\frac{\dot{\lambda}}{\lambda^2}(t) \geq \frac{1}{4C_0}\varepsilon_0 - \int_{T_0}^t Cc_0\varepsilon_0 ds \geq \frac{1}{8C_0}\varepsilon_0,$$

for times  $T^* = T_0 + O(1)$  as long as  $c_0 \ll 1$ . Integrating this and again applying (75) as well as the orbital stability bounds (62) to the term  $\lambda^{-1}(T_0)$ , we arrive at the inequality:

$$(t - T_0)\varepsilon_0 \lesssim \frac{1}{\lambda(T_0)} - \frac{1}{\lambda(t)} \lesssim c_0^{-1}\varepsilon^2.$$

Using the condition that  $\varepsilon \leq c_0^2$ , we see that such a bound must expire in  $t - T_0 = O(1)$  time. Finally, notice that by using the time bounds (77), the definition of  $T_0$  (75), the definition of  $T^*$  (78), as well as the relation  $T^* = T_0 + O(1)$ , we may integrate the quantity  $\dot{\lambda}^4\lambda^7$  over  $[0, T^*]$  to obtain:

$$(79) \quad \int_0^{T^*} \frac{\dot{\lambda}^4}{\lambda^7}(s) ds \leq \varepsilon.$$

We now return to the main thread of the blowup argument. So far we have achieved the following. There exists a time  $T_0$  such that for  $T_0 \leq t$  we have:

- $\dot{\lambda}^4\lambda^{-7}(t)$  is a monotonically increasing function.

- $\dot{\lambda}^4 \lambda^{-7}(t) \geq C c_0^{\frac{1}{2}} \varepsilon^2$ , where  $C$  is at least twice the implicit constant in the bound (72).
- The time  $T_0$  is associated with the bounds (77).

We are now at the point where blowup with the rate bounds (33) is assured. From the above conditions, we have that for all times  $t \geq T_0$ :

$$(80) \quad \mathcal{E} \lesssim c_0^{\frac{1}{2}} \dot{\lambda}^4 \lambda^{-7}.$$

Writing:

$$\gamma(t) = -\varepsilon_0 + \int_0^t \mathcal{E}(s) ds,$$

we have that  $\dot{\lambda} \lambda^{-2} \sim -\gamma$ . Differentiating  $\gamma$  and using the bound (80) we see that:

$$\dot{\gamma} \lesssim c_0^{\frac{1}{2}} \frac{\dot{\lambda}^4}{\lambda^7} \sim -\gamma^3 c_0^{\frac{1}{2}} \frac{\dot{\lambda}}{\lambda}.$$

By dividing through by  $-\gamma^3$  and integrating both sides of this last inequality over the interval  $[T_0, t_1]$ , we arrive at the estimate (we may assume that  $0 \leq -\gamma$  throughout this argument as will become apparent on the next line):

$$\begin{aligned} \frac{1}{\gamma^2(t_1)} &\lesssim \frac{1}{\gamma^2(T_0)} + c_0^{\frac{1}{2}} \ln(\lambda(t_1)) - c_0^{\frac{1}{2}} \ln(\lambda(T_0)) \\ &\lesssim \varepsilon^{-2} + c_0^{\frac{1}{2}} \ln(\lambda(t_1)), \end{aligned}$$

where to obtain the last inequality, we have used the second bound on line (77) as well as the fact that  $1 \leq \lambda(T_0)$ . We now recast this last expression in terms of  $\dot{\lambda}$ :

$$(81) \quad \frac{1}{\sqrt{\varepsilon^{-2} + c_0^{\frac{1}{2}} \ln(\lambda(t_1))}} \lesssim \frac{\dot{\lambda}}{\lambda^2}(t_1).$$

Integrating this last line over time intervals past  $[0, T_0]$ , we see that within  $O(\varepsilon^{-1})$  time we must have  $\varepsilon^{-2} \leq c_0^{\frac{1}{2}} \ln(\lambda(t))$ . Therefore, we are assured of the bound:

$$\lambda^{2-\delta}(t) \lesssim \frac{\lambda^2(t)}{\sqrt{c_0^{\frac{1}{2}} \ln(\lambda(t))}} \lesssim \dot{\lambda}(t), \quad 0 < \delta \ll 1,$$

inside of some interval  $[0, C\varepsilon^{-1}]$ , for a uniform constant  $C$ . This is a Riccati type inequality which easily implies that  $\lambda(t) \rightarrow \infty$  in  $O(1)$  time starting with  $1 \leq \lambda$  at the first time where it holds. Let us call the blowup time  $T^{**}$ .

Finally, we need to recover the rate bounds (33). By Remark 1.5 we need only establish the upper bound. From (81) we have the inequality:

$$c_0^{-\frac{1}{4}} \lesssim \sqrt{\ln(\lambda(t))} \frac{\dot{\lambda}}{\lambda^2}(t),$$

for times  $t$  sufficiently close to  $T^{**}$ . Making the substitution  $\lambda(t) = e^{s^2}$  and integrating from  $t$  to the blowup time, we have the estimate:

$$\begin{aligned} c_0^{-\frac{1}{4}}(T^{**} - t) &\lesssim 2 \int_{\sqrt{\ln(\lambda(t))}}^{\infty} s^2 e^{-s^2} ds \\ &= \frac{\sqrt{\ln(\lambda(t))}}{\lambda(t)} + \int_{\sqrt{\ln(\lambda(t))}}^{\infty} e^{-s^2} ds \\ &= \frac{\sqrt{\ln(\lambda(t))}}{\lambda(t)} + \frac{O(1)}{\lambda(t)\sqrt{\ln(\lambda(t))}} \\ &\lesssim \frac{\sqrt{\ln(\lambda(t))}}{\lambda(t)}, \end{aligned}$$

where the last line above follows from the well known asymptotics of the error function. The above identity easily implies that as  $t \rightarrow T^{**}$ , one has the bound:

$$\lambda(t) \lesssim c_0^{\frac{1}{4}} \frac{\sqrt{|\ln(T^{**} - t)|}}{(T^{**} - t)}.$$

This establishes the upper bound in (33).

*5.2. Derivation of the main ODE and its structure.* We now derive the ODE (69) as well as all of the accompanying structural assumptions (70)–(73). This will require a certain fixed time energy estimate which is the main technical estimate of the paper and will be proved in the following section. We start by recomputing the modulation (63) using the splitting  $\phi = I_\lambda + u$ . Differentiating the orthogonality relation (61) with respect to time, we have the simple identity:

$$(82) \quad \langle \dot{I}_\lambda, J_\lambda \rangle = \langle \phi_t, J_\lambda \rangle + \langle u, \dot{J}_\lambda \rangle.$$

Differentiating one more time and rearranging things with a little help from the equation  $\phi_{tt} + H_\lambda u = \mathcal{N}(u)$ , we have that:

$$(83) \quad \langle \ddot{I}_\lambda, J_\lambda \rangle = 2\langle \partial_t u, \dot{J}_\lambda \rangle + \langle u, \ddot{J}_\lambda \rangle + \langle \mathcal{N}(u), J_\lambda \rangle$$

$$(84) \quad = 2\partial_t \langle u, \dot{J}_\lambda \rangle - \langle u, \ddot{J}_\lambda \rangle + \langle \mathcal{N}(u), J_\lambda \rangle,$$

where  $\mathcal{N}(u)$  denotes the nonlinearity:

$$\begin{aligned} (85) \quad \mathcal{N}(u) &= \frac{k^2 \sin(2I_\lambda)}{2r^2} \cdot (1 - \cos(2u)) + \frac{k^2 \cos(2I_\lambda)}{r^2} \cdot \left(u - \frac{1}{2} \sin(2u)\right) \\ &= \frac{k^2 \sin(2I_\lambda)}{r^2} \cdot u^2 + \tilde{\mathcal{N}}(u). \end{aligned}$$

Notice that:

$$(86) \quad |\tilde{\mathcal{N}}(u)| \lesssim \frac{|u|^3}{r^2}.$$

Next, a short computation shows that we have the identity (recall the definition of  $J_\lambda$  from line (27)):

$$\langle \ddot{I}_\lambda, J_\lambda \rangle = \left( \frac{\ddot{\lambda}}{\lambda} - 2 \frac{\dot{\lambda}^2}{\lambda^2} \right) \langle J_\lambda, J_\lambda \rangle = C_0 \frac{d}{dt} \left( \frac{\dot{\lambda}}{\lambda^2} \right) \lambda^{-1},$$

where we have set  $\langle J, J \rangle = C_0$ . Therefore, from these last two lines as well as the identities (83)–(84), we have our two main structural equations:

$$(87) \quad \ddot{\lambda} - 2 \frac{\dot{\lambda}^2}{\lambda} = C_0^{-1} (2 \langle \partial_t u, \dot{J}_\lambda \rangle + \langle u, \ddot{J}_\lambda \rangle + \langle \mathcal{N}(u), J_\lambda \rangle) \lambda^3,$$

$$(88) \quad C_0 \frac{d}{dt} \left( \frac{\dot{\lambda}}{\lambda^2} \right) = 2 \partial_t [\langle u, \dot{J}_\lambda \rangle \lambda] - 2 \langle u, \dot{J}_\lambda \rangle \dot{\lambda} - \langle u, \ddot{J}_\lambda \rangle \lambda + \langle \mathcal{N}(u), J_\lambda \rangle \lambda.$$

The first (87) is sufficient for us to prove the bound (73). The second (88) will yield (69) upon integration. Doing this over a time interval  $[0, t]$ , we have the identity (recall that  $\lambda(0) = 1$ ):

$$(89) \quad C_0 \frac{\dot{\lambda}}{\lambda^2}(t) - 2 \langle u(t), \dot{J}_{\lambda(t)} \rangle \lambda(t) \\ = \dot{\lambda}(0) (C_0 - 2 \langle u(0), r \partial_r J \rangle) - \int_0^t \left( C_* \frac{\dot{\lambda}^4}{\lambda^7}(s) + \mathcal{E}(s) \right) ds,$$

where we define the constant  $C_*$  as follows:

$$(90) \quad C_* = -k^2 \left\langle \frac{(aJ_\lambda + b(r^2J)_\lambda)^2}{r^2}, \sin(2I_\lambda) \cdot J_\lambda \right\rangle \\ + \lambda^2 \left\langle (aJ_\lambda + b(r^2J)_\lambda), (r \partial_r J)_\lambda \right\rangle - \lambda^2 \left\langle r \partial_r (aJ_\lambda + b(r^2J)_\lambda), (r \partial_r J)_\lambda \right\rangle.$$

In Appendix A, it will be shown that  $C_* = 0$ . This indicates that the precise rate of blowup in the inequality (33) is quite delicate. We will return to this in a later work.

The error term  $\mathcal{E}(s)$  is given by the expression:

$$(91) \quad \mathcal{E} = 2 \langle w, \dot{J}_\lambda \rangle \dot{\lambda} + \langle w, \ddot{J}_\lambda \rangle \lambda + \left\langle w_0, \left( \ddot{\lambda} - 2 \frac{\dot{\lambda}^2}{\lambda} \right) (r \partial_r J)_\lambda \right\rangle \\ - \left\langle \frac{w \cdot (2w_0 + w)}{r^2}, \sin(2I_\lambda) \cdot J_\lambda \right\rangle \lambda - \langle \tilde{\mathcal{N}}(u), J_\lambda \rangle \lambda.$$

Here the terms  $w_0, w$  refer to the decomposition on line (56) above. We list this here again for the convenience of the reader:

$$(92) \quad u = \frac{\dot{\lambda}^2}{\lambda^4} (aJ_\lambda + b(r^2J)_\lambda) + w = w_0 + w,$$

where the constants  $a, b$  are derived on lines (57). Notice that the constant  $C_*$  defined by (90) arises from the expression (and a few integrations by parts):

$$C_* \dot{\lambda}^4 \lambda^{-7} = 2 \langle w_0, \dot{J}_\lambda \rangle \dot{\lambda} + \langle w_0, \ddot{J}_\lambda \rangle \lambda - \left\langle \frac{k^2 w_0^2}{r^2}, \sin(2I_\lambda) \cdot J_\lambda \right\rangle \lambda - \left\langle w_0, \left( \ddot{\lambda} - 2 \frac{\dot{\lambda}^2}{\lambda} \right) (r \partial_r J)_\lambda \right\rangle.$$

Before commencing with the proof of estimates (72) and (73), let us first derive from (89) the identity (69) and also the conditions (70)–(71). First of all, notice that from the orbital stability bounds (60), we have that:

$$\begin{aligned} |\langle u, \dot{J}_\lambda \rangle \lambda| &\lesssim \frac{\dot{\lambda}}{\lambda} \|r^{-1} u\|_{L^2(rdr)} \cdot \|(r^2 \partial_r J)_\lambda\|_{L^2(rdr)} \\ &\lesssim \varepsilon \frac{\dot{\lambda}}{\lambda^2}. \end{aligned}$$

Therefore, we may define  $\varepsilon_1$  on the left-hand side of line (69) as  $\varepsilon_1 = 2\lambda^3 \dot{\lambda}^{-1} \langle u, \dot{J}_\lambda \rangle$  and we have (71).

Similarly, at the initial time, the identity (82) gives the relation (we are assuming  $\lambda(0) = 1$ ):

$$\dot{\lambda}(0) \langle J, J \rangle = \langle \phi_t(0), J \rangle + \dot{\lambda}(0) \langle u(0), r \partial_r J \rangle.$$

Substituting into this last relation the initial data (28) and using the smallness condition (29), we easily have that:

$$\dot{\lambda}(0) (C_0 - 2 \langle u(0), r dr J \rangle) = \frac{\varepsilon}{\pi} + O(c_0 \varepsilon),$$

which gives the condition (70). Finally, notice that this last line also implies the initial expansion:

$$\dot{\lambda}(0) = \frac{\varepsilon}{\pi \langle J, J \rangle} + O(c_0 \varepsilon).$$

Plugging this into the first term on the right-hand side of formula (87) and using the bounds (29) on our chosen initial data (28) to estimate the remaining terms, we see that (73) holds for the initial time  $t = 0$ .

It remains for us to derive the bounds (72) and (73) from the identities (91) and (87) respectively. This will be done through a bootstrapping process and the use of a special energy estimate for the function  $w$  appearing in those expressions. This brings us to the main technical estimate of our paper which is the following:

**PROPOSITION 5.3 (Main technical estimate).** *Let  $u = w_0 + w$  be the decomposition from line (92). Next, let us assume that estimate (73) holds with constant  $2C$ ; that is:*

$$(93) \quad \left| \ddot{\lambda}(t) - 2 \frac{\dot{\lambda}^2}{\lambda}(t) \right| \leq 2C c_0^{\frac{1}{2}} \frac{\dot{\lambda}^2}{\lambda}(t) + 2C \left( c_0 \varepsilon^2 + \sup_{0 \leq s \leq t} \frac{\dot{\lambda}^4}{\lambda^7}(s) \right) \lambda^2.$$

*Then as long as the parameter  $\lambda(t)$  is monotonically nondecreasing, one has the following fixed time energy type estimate for  $t \in [0, \varepsilon^{-4}]$ :*

(94)

$$\int_{\mathbb{R}^+} \lambda^{-1} \frac{(\lambda r)^\delta}{1+r^\delta} \left[ (LA_\lambda w)^2 + \frac{(A_\lambda w)^2}{r^2} \right] (t) r dr \lesssim c_0^2 \varepsilon^2 + \varepsilon \sup_{0 \leq s \leq t} \frac{\dot{\lambda}^4}{\lambda^7}(s).$$

Here the implicit constant depends on  $C$  from line (93) above, but is independent of  $\varepsilon$  and  $c$  from line (29). Also, the operator  $A_\lambda$  is defined on line (47) above. Lastly,  $L = \partial_t + \partial_r$  is the outgoing null derivative.

*Remark 5.4.* Observe that by a direct application of estimate (163) of Appendix B that the bound on line (94) implies:

$$(95) \quad \int_{\mathbb{R}^+} \lambda^{-1} \frac{(\lambda r)^\delta}{1+r^\delta} \frac{w^2}{r^4} (t) r dr \lesssim c_0^2 \varepsilon^2 + \varepsilon \sup_{0 \leq s \leq t} \frac{\dot{\lambda}^4}{\lambda^7}(s).$$

This will be used many times in the sequel.

We now prove the bounds (72) and (73). This will be done separately and in reverse order. To prove the second estimate (73), it will suffice for us to demonstrate the following set of estimates:

LEMMA 5.5 (Estimates for (73)). *Assuming the bootstrapping estimate (93) and the results of Proposition 5.3, one has the following estimates where the implicit constant depends on line (94):*

$$(96) \quad |\langle \partial_t u, J_\lambda \rangle \lambda^3| \lesssim c_0^{\frac{3}{4}} \frac{\dot{\lambda}^2}{\lambda} + c_0^{\frac{1}{4}} \left( c_0 \varepsilon^2 + \sup_{0 \leq s \leq t} \frac{\dot{\lambda}^4}{\lambda^7}(s) \right) \lambda^2,$$

$$(97) \quad \langle u, \ddot{J}_\lambda \rangle \lambda^3 = \eta_1(t) \ddot{\lambda} + \eta_2(t) \frac{\dot{\lambda}^2}{\lambda},$$

$$(98) \quad |\langle \mathcal{N}(u), J_\lambda \rangle \lambda^3| \lesssim \varepsilon^2 \frac{\dot{\lambda}^2}{\lambda} + \left( c_0^2 \varepsilon^2 + \varepsilon \sup_{0 \leq s \leq t} \frac{\dot{\lambda}^4}{\lambda^7}(s) \right) \lambda^2,$$

where we also have the bounds<sup>11</sup>:

$$(99) \quad |\eta_1| \lesssim \varepsilon, \quad |\eta_2| \lesssim \varepsilon.$$

In particular, for  $\varepsilon^{\frac{1}{2}} \leq c_0$  small enough, we have that (73) holds.

*Proof of estimate (96).* This is the most involved of the above three estimates. To prove this, we begin by isolating the explicit piece involving  $w_0$  from the expansion (92). Thus, our first task is to prove that:

$$(100) \quad |\langle \partial_t w_0, J_\lambda \rangle \lambda^3| \lesssim \varepsilon^2 \frac{\dot{\lambda}^2}{\lambda}.$$

To do this we will employ the bootstrapping assumption (93). We will also use the abstract function notation replacements from lines (58). Doing this, we see from

<sup>11</sup>The implicit constants in (99) depend only on the orbital stability bound (60) and thus are independent of the bootstrap constant  $C$ .

lines (58a) and (58b) that we may write:

$$(101) \quad \langle \partial_t w_0, \dot{J}_\lambda \rangle \lambda^3 = \dot{\lambda}^4 \lambda^{-3} \langle F_\lambda^2, F_\lambda^4 \rangle + \frac{\ddot{\lambda} \dot{\lambda}^2}{\lambda^2} \langle F_\lambda^2, F_\lambda^4 \rangle.$$

Notice that from the bootstrapping assumption (93) and estimate (62), as well as the monotonicity of  $\lambda$ , we have that:

$$\begin{aligned} |\ddot{\lambda}| &\lesssim \dot{\lambda}^2 \lambda^{-1} + 2C \left( c_0 \varepsilon^2 + \sup_{0 \leq s \leq t} \dot{\lambda}^4 \lambda^{-7}(s) \right) \lambda^2 \\ &\lesssim \dot{\lambda}^2 \lambda^{-1} + \varepsilon^2 \lambda^3 \lesssim \varepsilon^2 \lambda^3. \end{aligned}$$

Therefore, plugging this into line (101) and using that  $|\langle F_\lambda^2, F_\lambda^4 \rangle| \lesssim \lambda^{-2}$ , we have the bound (using again (62)):

$$\begin{aligned} |\langle \partial_t w_0, \dot{J}_\lambda \rangle \lambda^3| &\lesssim \dot{\lambda}^4 \lambda^{-5} + \varepsilon^2 \dot{\lambda}^2 \lambda^{-1} \\ &\lesssim \varepsilon^2 \dot{\lambda}^2 \lambda^{-1}. \end{aligned}$$

This establishes (100) and, therefore, (96) for the  $w_0$  portion of  $u$ .

We shall now prove that:

$$(102) \quad |\langle \partial_t w, \dot{J}_\lambda \rangle \lambda^3| \lesssim c_0^{\frac{3}{4}} \frac{\dot{\lambda}^2}{\lambda} + c_0^{\frac{1}{4}} \left( c_0 \varepsilon^2 + \sup_{0 \leq s \leq t} \frac{\dot{\lambda}^4}{\lambda^7}(s) \right) \lambda^2.$$

The complication in this estimate stems from the fact that the energy estimate (94) does not provide control of the time derivative of  $w$ . In addition, (102) is a fixed time estimate and, therefore, it is not amenable to the procedure of integrating out the time-derivative as was done on the line (88) above. However, we will be able to exploit a structure of the inner product in this expression and convert the  $\partial_t w$  into  $LA_\lambda w$  derivative, which appears in (94). First of all, we write this as:

$$\langle \partial_t w, \dot{J}_\lambda \rangle = \dot{\lambda} \lambda^{-1} \langle \partial_t w, r \partial_r J_\lambda \rangle.$$

We now employ the following identity:

$$A_\lambda^*(rJ_\lambda) = 2J_\lambda + 2r \partial_r J_\lambda + r A_\lambda J_\lambda,$$

along with the conditions  $A_\lambda J_\lambda = 0$  and  $\langle w, J_\lambda \rangle = 0$  to write:

$$\begin{aligned} \langle \partial_t w, r \partial_r J_\lambda \rangle &= \frac{1}{2} \langle \partial_t w, A_\lambda^*(rJ_\lambda) \rangle - \langle \partial_t w, J_\lambda \rangle = \frac{1}{2} \langle \partial_t A_\lambda w, (rJ)_\lambda \rangle \lambda^{-1} \\ &\quad + \frac{1}{2} \langle [A_\lambda, \partial_t] w, (rJ)_\lambda \rangle \lambda^{-1} + \dot{\lambda} \lambda^{-1} \langle w, r \partial_r J_\lambda \rangle. \end{aligned}$$

Therefore, to show estimate (102) we will establish the following bounds:

$$(103) \quad |\langle \partial_t A_\lambda w, (rJ)_\lambda \rangle \dot{\lambda} \lambda| \lesssim c_0^{\frac{3}{4}} \frac{\dot{\lambda}^2}{\lambda} + c_0^{\frac{1}{4}} \left( c_0 \varepsilon^2 + \sup_{0 \leq s \leq t} \frac{\dot{\lambda}^4}{\lambda^7}(s) \right) \lambda^2,$$

$$(104) \quad |\langle [A_\lambda, \partial_t] w, (rJ)_\lambda \rangle \dot{\lambda} \lambda| \lesssim \varepsilon^2 \frac{\dot{\lambda}^2}{\lambda} + \left( c_0^2 \varepsilon^2 + \varepsilon \sup_{0 \leq s \leq t} \frac{\dot{\lambda}^4}{\lambda^7}(s) \right) \lambda^2,$$



$$(105) \quad |\langle w, r \partial_r J_\lambda \rangle \dot{\lambda}^2 \lambda| \lesssim \varepsilon^2 \frac{\dot{\lambda}^2}{\lambda} + \left( c_0^2 \varepsilon^2 + \varepsilon \sup_{0 \leq s \leq t} \frac{\dot{\lambda}^4}{\lambda^7}(s) \right) \lambda^2.$$

We prove these three estimates in order. By the triangle inequality we have that:

$$(106) \quad |\langle \partial_t A_\lambda w, (rJ)_\lambda \rangle| \lesssim |\langle LA_\lambda w, (rJ)_\lambda \rangle| + |\langle \partial_r A_\lambda w, (rJ)_\lambda \rangle|.$$

To estimate the left-hand side of (103) involving the first term in the last, sum we write:

$$(107) \quad \begin{aligned} & |\langle LA_\lambda w, (rJ)_\lambda \rangle \dot{\lambda} \lambda| \\ & \lesssim \dot{\lambda} \lambda \| (\lambda r)^{\frac{\delta}{2}} (1+r^\delta)^{-\frac{1}{2}} LA_\lambda w \|_{L^2(rdr)} \| (1+r^\delta)^{\frac{1}{2}} (r^{1-\frac{\delta}{2}} F^4)_\lambda \|_{L^2(rdr)} \\ & \lesssim \dot{\lambda} \lambda^{-\frac{1}{2}} \| \lambda^{-\frac{1}{2}} (\lambda r)^{\frac{\delta}{2}} (1+r^\delta)^{-\frac{1}{2}} LA_\lambda w \|_{L^2(rdr)} \lambda \\ & \lesssim c_0^{\frac{3}{4}} \dot{\lambda}^2 \lambda^{-1} + c_0^{-\frac{3}{4}} \| \lambda^{-\frac{1}{2}} (\lambda r)^{\frac{\delta}{2}} (1+r^\delta)^{-\frac{1}{2}} LA_\lambda w \|_{L^2(rdr)}^2 \lambda^2 \\ & \lesssim c_0^{\frac{3}{4}} \dot{\lambda}^2 \lambda^{-1} + c_0^{\frac{1}{4}} (c_0 \varepsilon^2 + \sup_{0 \leq s \leq t} \dot{\lambda}^4 \lambda^{-7}) \lambda^2. \end{aligned}$$

In the last line above we have used the assumption  $\varepsilon \leq c_0^2$ .

To conclude estimate (103), it remains to bound the second term on the right-hand side of (106). To do this we see that a simple calculation involving the notation on lines (38) and (58b) above allows us to write  $(\partial_r + r^{-1})(rJ)_\lambda = \lambda F_\lambda^4$ . This leads us to the estimate:

$$(108) \quad \begin{aligned} & |\langle \partial_r A_\lambda w, (rJ)_\lambda \rangle \dot{\lambda} \lambda| = |\langle A_\lambda w, F_\lambda^4 \rangle \dot{\lambda} \lambda^2| \\ & \lesssim \dot{\lambda} \lambda \| (\lambda r)^{\frac{\delta}{2}} (1+r^\delta)^{-\frac{1}{2}} r^{-1} A_\lambda w \|_{L^2(rdr)} \| (1+r^\delta)^{\frac{1}{2}} (r^{1-\frac{\delta}{2}} F^4)_\lambda \|_{L^2(rdr)} \\ & \lesssim c_0^{\frac{3}{4}} \dot{\lambda}^2 \lambda^{-1} + c_0^{-\frac{3}{4}} \| \lambda^{-\frac{1}{2}} (\lambda r)^{\frac{\delta}{2}} (1+r^\delta)^{-\frac{1}{2}} r^{-1} A_\lambda w \|_{L^2(rdr)}^2 \lambda^2 \\ & \lesssim c_0^{\frac{3}{4}} \dot{\lambda}^2 \lambda^{-1} + c_0^{\frac{1}{4}} (c_0 \varepsilon^2 + \sup_{0 \leq s \leq t} \dot{\lambda}^4 \lambda^{-7}) \lambda^2. \end{aligned}$$

This completes our proof of estimate (103).

To finish the proof of (96) we need to establish the estimates (104)–(105) above. As we shall see, the proof of (105), with a minor exception, follows almost verbatim from the estimates we will use for (104). Therefore, we now concentrate on (104). A simple computation shows that we may write the commutator as  $[A_\lambda, \partial_t] = -\partial_t(A_\lambda)$ , which from line (58c) is a multiplication operator given by a

function of the form  $\dot{\lambda} F_\lambda^9$ . Thus, we compute that:

$$\begin{aligned}
& |\langle [A_\lambda, \partial_t]w, (rJ)_\lambda \rangle \dot{\lambda} \lambda| \lesssim |\langle w, F_\lambda^{12} \rangle \dot{\lambda}^2 \lambda| \\
& \lesssim \dot{\lambda}^2 \lambda^{-\frac{1}{2}} \|\lambda^{-\frac{1}{2}} (\lambda r)^{\frac{\delta}{2}} (1+r)^{-\frac{\delta}{2}} r^{-2} w\|_{L^2(rdr)} \|(1+r)^{\frac{\delta}{2}} (r^{2-\frac{\delta}{2}} F^{12})_\lambda\|_{L^2(rdr)} \\
& \lesssim \dot{\lambda}^2 \lambda^{-\frac{3}{2}} \cdot (c_0^2 \varepsilon^2 + \varepsilon \sup_{0 \leq s \leq t} \dot{\lambda}^4 \lambda^{-7}(s))^{\frac{1}{2}} \\
& \lesssim \dot{\lambda}^4 \lambda^{-5} + (c_0^2 \varepsilon^2 + \varepsilon \sup_{0 \leq s \leq t} \dot{\lambda}^4 \lambda^{-7}(s)) \lambda^2 \\
& \lesssim \varepsilon^2 \dot{\lambda}^2 \lambda^{-1} + (c_0^2 \varepsilon^2 + \varepsilon \sup_{0 \leq s \leq t} \dot{\lambda}^4 \lambda^{-7}(s)) \lambda^2,
\end{aligned}$$

where we used that  $\dot{\lambda} \lambda^{-2} \lesssim \varepsilon$  and (95).

The proof of the bound (105) is very similar to what was done above. To set things up in terms of the previous steps, we simply use the notation on line (58b) and the Cauchy-Schwartz inequality to write:

$$\begin{aligned}
& |\langle w, r \partial_r J_\lambda \rangle \dot{\lambda}^2 \lambda| \\
& \lesssim \dot{\lambda}^2 \lambda^{-\frac{1}{2}} \|\lambda^{-\frac{1}{2}} (\lambda r)^{\frac{\delta}{2}} (1+r)^{-\frac{\delta}{2}} r^{-2} w\|_{L^2(rdr)} \|(1+r)^{\frac{\delta}{2}} (r^{2-\frac{\delta}{2}} F^4)_\lambda\|_{L^2(rdr)}
\end{aligned}$$

The steps are now identical to what was done in the previous due to the bound:

$$(109) \quad \|(1+r)^{\frac{\delta}{2}} (r^{2-\frac{\delta}{2}} F^4)_\lambda\|_{L^2(rdr)} \lesssim \lambda^{-1}.$$

This concludes our proof of estimate (96).  $\square$

*Proof of the identity (97) and the bounds (99).* This follows from the orbital stability bound (60) and the Cauchy-Schwartz inequality. Specifically, a short calculation shows that:

$$\langle u, \ddot{J}_\lambda \rangle \lambda^3 = \langle u, (r \partial_r J)_\lambda \rangle \ddot{\lambda} \lambda^2 - \langle u, (r \partial_r J)_\lambda \rangle \dot{\lambda}^2 \lambda + \langle u, ((r \partial_r)^2 J)_\lambda \rangle \dot{\lambda}^2 \lambda.$$

We leave the details of application of the Cauchy-Schwartz and the estimate (60) to the reader.  $\square$

*Proof of estimate (98).* By invoking the explicit formula (85) and the decomposition (92), it suffices to show the two estimates:

$$(110) \quad \left\langle \frac{(w_0)^2}{r^2}, J_\lambda \right\rangle \lambda^3 \lesssim \varepsilon^2 \frac{\dot{\lambda}^2}{\lambda},$$

$$(111) \quad \left\langle \frac{w^2}{r^2}, J_\lambda \right\rangle \lambda^3 \lesssim \left( c_0^2 \varepsilon^2 + \varepsilon \sup_{0 \leq s \leq t} \frac{\dot{\lambda}^4}{\lambda^7}(s) \right) \lambda^2.$$

The proof of (110) is a simple and direct calculation involving definition (56) and estimate (62). We leave the details to the reader.

The proof of the second estimate (111) follows almost immediately from (95). To see this, we compute that:

$$\begin{aligned} \left\langle \frac{w^2}{r^2}, J_\lambda \right\rangle \lambda^3 &\lesssim \lambda^2 \left\| \lambda^{-\frac{1}{2}} (\lambda r)^{\frac{\delta}{2}} (1+r)^{-\frac{\delta}{2}} r^{-2} w \right\|_{L^2(rdr)}^2 \\ &\quad \cdot \left\| (1+r)^{\frac{\delta}{2}} (r^{2-\frac{\delta}{2}} F^4)_\lambda \right\|_{L^\infty} \\ &\lesssim (c_0^2 \varepsilon^2 + \varepsilon \sup_{0 \leq s \leq t} \dot{\lambda}^4 \lambda^{-7}(s)) \lambda^2. \end{aligned}$$

This completes our proof of estimates (98).  $\square$

Having now completed our proof of estimate (96)–(98), our last task in the section is to establish the structure estimate (72) for the function  $\mathcal{E}$  defined on line (91). To do this it clearly suffices to add together the following set of estimates for the individual terms on the right-hand side of (91):

LEMMA 5.6 (Estimates for (72)). *Assuming the bootstrapping estimate (93) and the results of Proposition 5.3, one has the following estimates where the implicit constant depends on line (94):*

$$(112) \quad \left| \langle w, \dot{J}_\lambda \rangle \dot{\lambda} \right| \lesssim c_0 \varepsilon^2 + c_0^{\frac{1}{2}} \sup_{0 \leq s \leq t} \frac{\dot{\lambda}^4}{\lambda^7}(s),$$

$$(113) \quad \left| \langle w, \ddot{J}_\lambda \rangle \lambda \right| \lesssim c_0 \varepsilon^2 + c_0^{\frac{1}{2}} \sup_{0 \leq s \leq t} \frac{\dot{\lambda}^4}{\lambda^7}(s),$$

$$(114) \quad \left| \left\langle w_0, \left( \ddot{\lambda} - 2 \frac{\dot{\lambda}^2}{\lambda} \right) (r \partial_r J)_\lambda \right\rangle \right| \lesssim c_0 \varepsilon^2 + c_0^{\frac{1}{2}} \sup_{0 \leq s \leq t} \frac{\dot{\lambda}^4}{\lambda^7}(s),$$

$$(115) \quad \left| \left\langle \frac{w \cdot (2w_0 + w)}{r^2}, \sin(2I_\lambda) J_\lambda \right\rangle \lambda \right| \lesssim c_0 \varepsilon^2 + c_0^{\frac{1}{2}} \sup_{0 \leq s \leq t} \frac{\dot{\lambda}^4}{\lambda^7}(s),$$

$$(116) \quad \left| \langle \tilde{\mathcal{N}}(u), J_\lambda \rangle \lambda \right| \lesssim c_0 \varepsilon^2 + c_0^{\frac{1}{2}} \sup_{0 \leq s \leq t} \frac{\dot{\lambda}^4}{\lambda^7}(s).$$

The proof of estimates (112)–(116) is similar to the proof of the estimates in Lemma 5.5 above. We will always follow the three-step strategy: 1) distribute correct powers of  $r$  and  $(1+r)$  inside the inner product; 2) apply the Cauchy-Schwartz inequality; 3) refer to estimates (60), (62), (94)–(95), and (73). We will be a bit more terse here than before and leave some of the details to the reader. Each proof will be written out under an individual heading.

*Proof of estimates (112).* We start with estimate (112). Using our abstract notation from line (58b), we have that:

$$\begin{aligned}
& |\langle w, \dot{J}_\lambda \dot{\lambda} \rangle| \\
& \lesssim \dot{\lambda}^2 \lambda^{-\frac{5}{2}} \|\lambda^{-\frac{1}{2}} (\lambda r)^{\frac{\delta}{2}} (1+r)^{-\frac{\delta}{2}} r^{-2} w\|_{L^2(rdr)} \|(1+r)^{\frac{\delta}{2}} (r^{2-\frac{\delta}{2}} F^4)_\lambda\|_{L^2(rdr)} \\
& \lesssim \dot{\lambda}^2 \lambda^{-\frac{7}{2}} \cdot (c_0^2 \varepsilon^2 + \varepsilon \sup_{0 \leq s \leq t} \dot{\lambda}^4 \lambda^{-7}(s))^{\frac{1}{2}} \\
& \lesssim c_0 \varepsilon^2 + c_0^{\frac{1}{2}} \sup_{0 \leq s \leq t} \dot{\lambda}^4 \lambda^{-7}(s).
\end{aligned}$$

We remark here that this and similar estimates (below) are the source of our restriction  $4 \leq k$  on the homotopy class in Theorem 1.3. Notice that one cannot arrive at the desired bound by simply applying the orbital stability estimate (60). It is crucial that we use (95) here, and this causes more weights to be placed on  $r \partial_r J_\lambda$ . It is likely that one can lower the value of  $k$  in these arguments through a more careful analysis. We will not pursue this here.  $\square$

*Proof of estimate (113).* This is similar to the proof of (112). Notice that from estimate (73) we have the following bound:

$$|\ddot{\lambda}| \lesssim \varepsilon^2 \lambda^2 + \sup_{0 \leq s \leq t} \dot{\lambda}^2 \lambda^{-1}(s).$$

Therefore, using the notation from line (58b) we have the chain of inequalities:

$$\begin{aligned}
|\langle w, \ddot{J}_\lambda \lambda \rangle| & \lesssim \varepsilon^2 |\langle w, F_\lambda^4 \rangle \lambda^2| + \sup_{0 \leq s \leq t} \dot{\lambda}^2 \lambda^{-1}(s) \cdot |\langle w, F_\lambda^4 \rangle| \\
& \lesssim \varepsilon^3 + \sup_{0 \leq s \leq t} \dot{\lambda}^2 \lambda^{-\frac{7}{2}}(s) \cdot (c_0^2 \varepsilon^2 + \varepsilon \sup_{0 \leq s \leq t} \dot{\lambda}^4 \lambda^{-7}(s))^{\frac{1}{2}} \\
& \lesssim c_0 \varepsilon^2 + c_0^{\frac{1}{2}} \sup_{0 \leq s \leq t} \dot{\lambda}^4 \lambda^{-7}(s).
\end{aligned}$$

Notice that we have again used the condition  $\varepsilon \leq c_0^2$  on this last line.  $\square$

*Proof of estimate (114).* This will follow by a direct application of estimate (73) and definition (56). Notice that (73) and (62) taken together imply that:

$$|\ddot{\lambda} - 2\dot{\lambda}^2 \lambda^{-1}| \lesssim \varepsilon^2 \lambda^2 + c_0^{\frac{1}{2}} \sup_{0 \leq s \leq t} \dot{\lambda}^2 \lambda^{-1}(s).$$

Therefore, a simple computation again using (62) shows that:

$$\begin{aligned}
|\langle w_0, (\ddot{\lambda} - 2\dot{\lambda}^2 \lambda^{-1})(r \partial_r J)_\lambda \rangle| & \lesssim \dot{\lambda}^2 \lambda^{-6} \cdot (\varepsilon^2 \lambda^2 + c_0^{\frac{1}{2}} \sup_{0 \leq s \leq t} \dot{\lambda}^2 \lambda^{-1}(s)) \\
& \lesssim \varepsilon^4 + c_0^{\frac{1}{2}} \sup_{0 \leq s \leq t} \dot{\lambda}^4 \lambda^{-7}(s),
\end{aligned}$$

which is enough to imply (114) since  $\varepsilon \leq c_0^2$ .  $\square$

*Proof of estimate (115).* The left-hand side of this estimate can be bounded by the inequality:

$$\begin{aligned} \left| \left\langle \frac{w^2 + |w_0| \cdot |w|}{r^2}, F_\lambda^4 \right\rangle \lambda \right| &\lesssim \|\lambda^{-\frac{1}{2}}(\lambda r)^{\frac{\delta}{2}}(1+r)^{-\frac{\delta}{2}}r^{-2}w\|_{L^2(rdr)}^2 \\ &+ \|\lambda^{-\frac{1}{2}}(\lambda r)^{\frac{\delta}{2}}(1+r)^{-\frac{\delta}{2}}r^{-2}w_0\|_{L^2(rdr)} \cdot \|\lambda^{-\frac{1}{2}}(\lambda r)^{\frac{\delta}{2}}(1+r)^{-\frac{\delta}{2}}r^{-2}w\|_{L^2(rdr)}. \end{aligned}$$

Estimate (115) now follows directly from (95) applied to the terms involving  $w$  in this last line above and the following bound which is a consequence of the explicit identity (56) (or the notation on line (58a)):

$$(117) \quad \|\lambda^{-\frac{1}{2}}(\lambda r)^{\frac{\delta}{2}}(1+r)^{-\frac{\delta}{2}}r^{-2}w_0\|_{L^2(rdr)} \lesssim \dot{\lambda}^2 \lambda^{-\frac{7}{2}}.$$

This concludes the proof of estimate (115). □

*Proof of the inequality (116).* To do this, we first note that by the inequality (86) and the orbital stability estimate (60), used in conjunction with the Poincaré type estimate (68) as well as the notation from line (58b), we have that:

$$|\langle \tilde{\mathcal{N}}(u), J_\lambda \rangle \lambda| \lesssim \varepsilon \|\lambda^{-\frac{1}{2}}(\lambda r)^{\frac{\delta}{2}}(1+r)^{-\frac{\delta}{2}}r^{-2}u\|_{L^2(rdr)}^2.$$

By adding together estimate (117) and the corresponding bound (95) for  $w$  in the decomposition  $u = w_0 + w$ , we have the single estimate for  $u$ :

$$\|\lambda^{-\frac{1}{2}}(\lambda r)^{\frac{\delta}{2}}(1+r)^{-\frac{\delta}{2}}r^{-2}u\|_{L^2(rdr)}^2 \lesssim c_0^2 \varepsilon^2 + \sup_{0 \leq s \leq t} \dot{\lambda}^4 \lambda^{-7}(s).$$

Substituting this into the right-hand side of the previous line, we obtain the desired bound (116). This completes our proof of Lemma 5.6. □

### 6. Space-time bounds and the proof of the Main Estimate

In this final section of the paper, we prove our main technical estimate (94). The crucial role in this will be played by the remarkable factorization property of the linearized Hamiltonian (46) which allows us to introduce the “conjugate” Hamiltonian (50). This new Hamiltonian  $\tilde{H}_\lambda$  possesses the striking properties (51)–(53) which are ultimately responsible for very strong estimates, proved dynamically and by means of simple yet quite precise physical space methods for the corresponding Cauchy problem  $(\partial_t^2 + \tilde{H}_\lambda)w = F$ . The key is the physical-space repulsive properties of the operator (50) which lead to the desired estimates *independent of how violently the scaling parameter  $\lambda$  grows*, so long as this growth is monotonic. This stands in stark contrast to the usual procedure in asymptotic stability analysis which attempts to estimate the linearized operator through *nondynamic* spectral analysis (see e.g. [5], [20]). Such a procedure is not as natural in the present context which represents a truly nonlinear situation not directly amenable to the standard perturbative techniques. From this point of view, the analysis we present here is close in spirit to the work of Merle-Raphael on the blowup for the critical NLS [28].

We again remind the reader that it is the precise form of the nonlinear (19), embodied by the first order Bogomol'nyi (25), that is the indispensable structure.

The first thing we will need here for the proof of (94) is a space-time estimate for general solutions to the conjugated linearized equation (50). For us this will take the form of a weighted  $L^2$  inequality involving integration over both space and time variables. These are commonly referred to as Morawetz estimates, and they have a rich history in both linear and nonlinear analysis of the dispersive properties of wave equations. The estimate we use here is based on the following energy, defined for sufficiently smooth and well decaying functions  $\psi$  on  $\mathbb{R}^+$ :

$$(118) \quad \mathbb{E}_\delta[\psi](t_0, t_1) = \sup_{t_0 \leq s \leq t_1} \int_{\mathbb{R}^+} \lambda^{-1} \frac{(\lambda r)^\delta}{1+r^\delta} \left[ (L\psi)^2 + \frac{\psi^2}{r^2} \right] (s) r dr \\ + \int_{t_0}^{t_1} \int_{\mathbb{R}^+} \lambda^{-1} \left[ \frac{(\lambda r)^\delta}{(1+r^\delta)^2 r} (L\psi)^2 + \frac{(\lambda r)^\delta \psi^2}{1+r^\delta r^3} \right] (s) r dr ds,$$

where  $0 < \delta \ll 1$  is a small fixed constant which will measure a loss in certain time integrations which appear in the sequel. Here we have set  $L = \partial_t + \partial_r$ . The main estimate we will use is contained in the following:

**PROPOSITION 6.1** (Morawetz estimate for  $\tilde{H}_\lambda$ ). *Consider the time-dependent Hamiltonian  $\tilde{H}_\lambda$ . Let  $\psi$  be a smooth function on  $[t_0, t_1] \times (0, \infty)$ , satisfying the following uniform bounds:*

$$(119) \quad |\psi| \leq C_{\psi(t)} r, \quad |\partial_t \psi| + |\partial_r \psi| \leq C_{\psi(t)}, \quad 0 \leq r \leq 1,$$

while decaying sufficiently rapidly at  $r = \infty$ . Furthermore, suppose that:

$$(120) \quad \partial_t^2 \psi + \tilde{H}_\lambda \psi = \partial_t G + H.$$

Then, if one has the pointwise inequalities  $0 \leq \dot{\lambda}$  and  $\dot{\lambda} \lambda^{-2} \lesssim \varepsilon$  for all times  $t_0 \leq s \leq t_1$ , one also has the following estimate:

$$(121) \quad \mathbb{E}_\delta[\psi](t_0, t_1) \lesssim \delta^{-1} \left[ \int_{t_0}^{t_1} \int_{\mathbb{R}^+} \lambda^{-1} (\lambda r)^\delta [(\partial_r G)^2 + \varepsilon^2 (\lambda G)^2 + H^2] (s) r^2 dr ds \right. \\ \left. + \sup_{t_0 \leq s \leq t_1} \int_{\mathbb{R}^+} \lambda^{-1} \frac{(\lambda r)^\delta}{1+r^\delta} G^2(s) r dr + \mathbb{E}_\delta[\psi](t_0, t_0) \right],$$

which holds with an implicit constant independent of  $\lambda$  and  $\delta$ .

*Remark 6.2.* The constant  $0 < \delta \ll 1$  will signify a small loss in time when we attempt to apply (121) in the proof of (94). This is ultimately why we are restricted to the time interval  $[0, \varepsilon^{-4}]$  in the statement of 5.3 and hence in Proposition 5.1. We also remark here that this small loss in time can in fact be avoided through a somewhat more careful analysis involving the precise form of the equation for  $\lambda$  given on line (69).

*Proof of estimate (121).* Notice that all of the weights in the inequality are time translation invariant. Therefore, we may normalize the discussion to  $t_0 = 0$ . We begin by conjugating equation (120) by  $r^{\frac{1}{2}}$ . Therefore, we denote the new variable:

$$\tilde{\psi} = r^{\frac{1}{2}} \psi.$$

We note here that the decay  $\tilde{\psi}$  at the origin ( $\sim r^{\frac{3}{2}}$ ) will be sufficient to perform the integration by parts to follow. We also observe that  $\partial_t \tilde{\psi}$  and  $\partial_r \tilde{\psi}$  may be assumed to be bounded at  $r = 0$ .

Next, recall that the original (super-symmetric conjugate) Hamiltonian has the form:

$$\tilde{H}_\lambda = -\partial_r^2 - \frac{1}{r} \partial_r + V_\lambda(r).$$

We define the one-dimensional Hamiltonian:

$$\mathcal{H}_\lambda = -\partial_r^2 - \frac{1/4}{r^2} + V_\lambda(r).$$

Then a quick computation shows that equation (120) becomes:

$$(122) \quad \partial_t^2 \tilde{\psi} + \mathcal{H}_\lambda \tilde{\psi} = r^{\frac{1}{2}} (\partial_t G + H).$$

The multiplier we use is the following:

$$X = \lambda^{-1+\delta} \frac{r^\delta}{1+r^\delta} L = \lambda^{-1+\delta} \left(1 - \frac{1}{1+r^\delta}\right) L.$$

Multiplying equation (122) by the quantity  $X \tilde{\psi}$  and integrating the resulting expression over the interval  $[0, t] \times (0, \infty)$ , we have the identity:

$$(123) \quad \frac{1}{2} \int_0^t \int_{\mathbb{R}^+} \lambda^{-1} \frac{(\lambda r)^\delta}{1+r^\delta} \left[ \underline{L}(L \tilde{\psi})^2 + (V_\lambda - \frac{1}{4} r^{-2}) L(\tilde{\psi})^2 \right] dr dt \\ = \int_0^t \int_{\mathbb{R}^+} \lambda^{-1} \frac{(\lambda r)^\delta}{1+r^\delta} (\underline{L}(G) + \partial_r(G) + H) \cdot L(\tilde{\psi}) r^{\frac{1}{2}} dr dt.$$

Here  $\underline{L} = \partial_t - \partial_r$  is the incoming null derivative. We integrate by parts on the left-hand side of this last expression, using the following lower bounds for terms involving the potential:

$$C \frac{k^2}{r^2} \geq \left( V_\lambda - \frac{1}{4} r^{-2} \right) \geq c \frac{k^2}{r^2}, \\ -L \left[ \lambda^{-1} \frac{(\lambda r)^\delta}{1+r^\delta} \left( V_\lambda - \frac{1}{4} r^{-2} \right) \right] \geq c \lambda^{-1} \frac{(\lambda r)^\delta}{1+r^\delta} \frac{k^2}{r^3},$$

which follow from (51)–(53), the condition  $4 \leq k$ , and the positivity of  $\dot{\lambda}$ . Applying a couple of Cauchy-Schwartz inequalities to the last two terms on the right-hand side of (123) and using the positivity condition  $\dot{\lambda} \geq 0$ , we then we arrive at the

bound:

$$\begin{aligned}
 (124) \quad & \int_{\mathbb{R}^+} \lambda^{-1} \frac{(\lambda r)^\delta}{1+r^\delta} \left[ (L\tilde{\psi})^2 + k^2 \frac{\tilde{\psi}^2}{r^2} \right] (t) dr \\
 & + \delta \int_0^t \int_{\mathbb{R}^+} \lambda^{-1} \frac{(\lambda r)^\delta}{(1+r^\delta)^2 r} (L\tilde{\psi})^2 dr ds \\
 & + k^2 \int_0^t \int_{\mathbb{R}^+} \lambda^{-1} \frac{(\lambda r)^\delta}{1+r^\delta} \frac{(\tilde{\psi})^2}{r^3} dr ds \\
 & \lesssim \left( \int_0^t \int_{\mathbb{R}^+} \lambda^{-1} (\lambda r)^\delta ((\partial_r G)^2 + H^2) r^2 dr ds \right)^{\frac{1}{2}} \\
 & \cdot \left( \int_0^t \int_{\mathbb{R}^+} \lambda^{-1} \frac{(\lambda r)^\delta}{(1+r^\delta)^2 r} (L\tilde{\psi})^2 dr ds \right)^{\frac{1}{2}} \\
 & + \int_{\mathbb{R}^+} \lambda^{-1} \frac{(\lambda r)^\delta}{1+r^\delta} \left[ (L\tilde{\psi})^2 + k^2 \frac{\tilde{\psi}^2}{r^2} \right] (0) dr \\
 & + \int_0^t \int_{\mathbb{R}^+} \lambda^{-1} \frac{(\lambda r)^\delta}{1+r^\delta} \underline{L}(G) \cdot L(\tilde{\psi}) r^{\frac{1}{2}} dr ds.
 \end{aligned}$$

It remains to deal with the last integral on the right-hand side of the above expression. To do this, we integrate by parts with respect to the incoming derivative  $\underline{L}$ . Employing the pointwise bound:

$$\begin{aligned}
 \left| \underline{L} \left( \lambda^{-1} \frac{(\lambda r)^\delta}{1+r^\delta} r^{\frac{1}{2}} \right) \right| & \lesssim \left| \frac{\dot{\lambda}}{\lambda^2} \right| \cdot \frac{(\lambda r)^\delta}{1+r^\delta} r^{\frac{1}{2}} + \lambda^{-1} \frac{(\lambda r)^\delta}{1+r^\delta} r^{-\frac{1}{2}} \\
 & \lesssim \varepsilon \frac{(\lambda r)^\delta}{1+r^\delta} r^{\frac{1}{2}} + \lambda^{-1} \frac{(\lambda r)^\delta}{(1+r^\delta)r} r^{\frac{1}{2}}
 \end{aligned}$$

and using equation (122) together with the upper bound  $|V_\lambda| \lesssim r^{-2}$ , we have that:

$$\begin{aligned}
 (125) \quad & \int_0^t \int_{\mathbb{R}^+} \lambda^{-1} \frac{(\lambda r)^\delta}{1+r^\delta} \underline{L}(G) \cdot L(\tilde{\psi}) r^{\frac{1}{2}} dr ds \\
 & \lesssim \left( \int_0^t \int_{\mathbb{R}^+} \lambda^{-1} (\lambda r)^\delta \left[ \frac{G^2}{r^2} + \varepsilon^2 (\lambda G)^2 + H^2 \right] (s) r^2 dr ds \right)^{\frac{1}{2}} \\
 & \cdot \left( \int_0^t \int_{\mathbb{R}^+} \lambda^{-1} \left[ \frac{(\lambda r)^\delta}{(1+r^\delta)^2 r} (L\tilde{\psi})^2 + \frac{(\lambda r)^\delta}{1+r^\delta} \frac{(\tilde{\psi})^2}{r^3} \right] dr ds \right)^{\frac{1}{2}} \\
 & - \frac{1}{2} \int_0^t \int_{\mathbb{R}^+} \lambda^{-1} \frac{(\lambda r)^\delta}{1+r^\delta} \partial_t (G^2) r dr ds - \int_0^t \int_{\mathbb{R}^+} \lambda^{-1} \frac{(\lambda r)^\delta}{1+r^\delta} GH r dr ds \\
 & + \left( \sup_{0 \leq s \leq t} \int_{\mathbb{R}^+} \lambda^{-1} \frac{(\lambda r)^\delta}{1+r^\delta} G^2(s) r dr \right)^{\frac{1}{2}} \cdot \left( \sup_{0 \leq s \leq t} \int_{\mathbb{R}^+} \lambda^{-1} \frac{(\lambda r)^\delta}{1+r^\delta} (L\tilde{\psi})^2(s) dr \right)^{\frac{1}{2}}.
 \end{aligned}$$



Integrating by parts one more time in the term involving  $\partial_t(G)^2$  above, again using the fact that  $\dot{\lambda} \geq 0$ , and using also the following fixed time Poincaré type estimate:

$$(126) \quad \int_{\mathbb{R}^+} \lambda^{-1}(\lambda r)^\delta G^2 dr \lesssim \int_{\mathbb{R}^+} \lambda^{-1}(\lambda r)^\delta (\partial_r G)^2 r^2 dr,$$

we add together estimates (124)–(126) and take the sup over different times to achieve the bound:

$$(127) \quad \begin{aligned} \tilde{\mathbb{E}}_\delta[\tilde{\psi}](0, t) \lesssim \delta^{-1} & \left[ \int_0^t \int_{\mathbb{R}^+} \lambda^{-1}(\lambda r)^\delta [(\partial_r G)^2 + \varepsilon^2(\lambda G)^2 + H^2](s) r^2 dr ds \right. \\ & \left. + \sup_{0 \leq s \leq t} \int_{\mathbb{R}^+} \lambda^{-1} \frac{(\lambda r)^\delta}{1+r^\delta} G^2(s) r dr + \tilde{\mathbb{E}}_\delta[\tilde{\psi}](0, 0) \right], \end{aligned}$$

where we define the one-dimensional energy analogous to (118):

$$\begin{aligned} \tilde{\mathbb{E}}_\delta[\tilde{\psi}](0, t) = \sup_{0 \leq s \leq t} & \int_{\mathbb{R}^+} \lambda^{-1} \frac{(\lambda r)^\delta}{1+r^\delta} \left[ (L\tilde{\psi})^2 + k^2 \frac{\tilde{\psi}^2}{r^2} \right](s) dr \\ & + \int_0^t \int_{\mathbb{R}^+} \lambda^{-1} \left[ \frac{(\lambda r)^\delta}{(1+r^\delta)^2 r} (L\tilde{\psi})^2 + k^2 \frac{(\lambda r)^\delta}{1+r^\delta} \frac{\tilde{\psi}^2}{r^3} \right](s) dr ds. \end{aligned}$$

Finally, to complete the proof, we use the expansion:

$$r^{-\frac{1}{2}} L\tilde{\psi} = L\psi + \frac{1}{2} r^{-1} \psi,$$

and the fact that  $4 \leq k$  to bound the energy  $\tilde{\mathbb{E}}_\delta$  from above and below by  $\mathbb{E}_\delta$ . This completes our proof of estimate (121).  $\square$

We now turn to the proof of (94). The precise statement of what we need to show is the following:

**PROPOSITION 6.3** (Energy estimates for the quantity  $w$ ). *Let  $u = w_0 + w$  be the decomposition of  $u$  given on line (92), where  $u$  itself is part of the decomposition (44) of the full field  $\phi$ . In particular  $u$  solves equation (45). Suppose also that the initial conditions for  $\phi$  are given as to satisfy (28)–(29) and that  $u$  obeys the extra decay estimate (147) (this will be proved shortly). Furthermore, assume that the main assumptions of Proposition 5.3 hold; in particular, we have that (93) and  $\dot{\lambda} \geq 0$ . Then the following estimate holds for  $t \in [0, \varepsilon^{-4}]$ :*

$$(128) \quad \mathbb{E}_\delta[A_\lambda w](0, t) \lesssim \delta^{-1} \left( c_0^2 \varepsilon^2 + \varepsilon \sup_{0 \leq s \leq t} \frac{\dot{\lambda}^4}{\lambda^7}(s) \right).$$

*The implicit constant depends on  $C$  from line (93) but is independent of the size of  $c_0, \varepsilon$ , or  $\delta$ . In particular, for a fixed  $0 < \delta \ll 1$  we have that estimate (94) holds.*

*Proof of estimate (128).* The first order of business is to reduce the proof to simpler bounds. In the sequel, we will only show that:

$$(129) \quad \mathbb{E}_\delta[A_\lambda w](t_0, t_1) \lesssim \delta^{-1} \left( \mathbb{E}_\delta[A_\lambda w](t_0, t_0) + \varepsilon^3 + \varepsilon \sup_{t_0 \leq s \leq t_1} \frac{\dot{\lambda}^4}{\lambda^7}(s) \right),$$

for all time intervals  $[t_0, t_1]$  inside the regular interval  $[0, T^{**})$ , where again  $T^{**}$  is the blowup time, provided that one *also* has the inequality:

$$(130) \quad \int_{t_0}^{t_1} \frac{\dot{\lambda}^4}{\lambda^7}(s) ds \leq \varepsilon.$$

We claim that along with the bootstrapping assumption (93) and the analysis done in Section 5.1, this is enough to establish (128).

To verify this claim, first notice that if we are in the time interval  $[0, T^*]$  where  $T^*$  is defined as on line (78), then we automatically have (130) on account of (79). Therefore, we may work inside intervals of the form  $[T^*, t]$ , and we are only trying to establish:

$$(131) \quad \mathbb{E}_\delta[A_\lambda w](T^*, t) \lesssim \delta^{-1} \varepsilon \frac{\dot{\lambda}^4}{\lambda^7}(t).$$

Notice that we are using the monotonicity of  $\dot{\lambda}^4 \lambda^{-7}$  established in Section 5.1. We now claim that (131) easily follows from (129) and the bootstrapping assumption (93). Indeed, let  $[t_{k-1}, t_k]$  be any interval where *equality* in (130) holds. Then we have:

$$(132) \quad \int_{t_{k-1}}^{t_k} \frac{d}{ds} \ln(\dot{\lambda}^4 \lambda^{-7}) ds \gtrsim \int_{t_{k-1}}^{t_k} \frac{\dot{\lambda}}{\lambda} ds \gtrsim \varepsilon^{-3} \int_{t_{k-1}}^{t_k} \frac{\dot{\lambda}^4}{\lambda^7} ds = \varepsilon^{-2}.$$

Here we have used the bootstrapping bound (93) in the simple form  $|\ddot{\lambda} - 2\dot{\lambda}^2 \lambda^{-1}| \ll \dot{\lambda}^2 \lambda^{-1}$ , which holds as long as we are in the region past  $[0, T^*]$  (in particular, one has access to a lower bound consistent with (78) which allows one to uniformize the right-hand side of (93)). Notice that we have also used the orbital stability bound (62) several times in deriving the inequalities. Integrating the inequality (132) we see that:

$$(133) \quad \frac{\dot{\lambda}^4}{\lambda^7}(t_{k-1}) \lesssim e^{-\varepsilon^{-2}} \frac{\dot{\lambda}^4}{\lambda^7}(t_k),$$

on any time interval  $[t_{k-1}, t_k]$  past  $[0, T^*]$  where (130) also holds. It is now a simple matter to derive (131) from (129). We first decompose the interval  $[T^*, t]$  into a finite collection of  $N$  subintervals  $[t_{k-1}, t_k]$  where equality in (130) holds.<sup>12</sup> On each of these intervals, we have estimate (129). By repeatedly using the bound

<sup>12</sup>On the last interval there may be a strict inequality in (130), but this single interval may also be estimated with (129), and the answer may then be directly added into the final bound.

(133), each of these estimates may be inductively expanded to yield:

$$\mathbb{E}_\delta[A_\lambda w](t_{k-1}, t_k) \lesssim \delta^{-1} C \sum_{i=0}^k \left( \frac{C\delta^{-1}}{C_1} \right)^i \cdot \varepsilon \frac{\dot{\lambda}^4}{\lambda^7}(t_k) \lesssim \frac{\delta^{-1} C}{C_1^{N-k}} \cdot \varepsilon \frac{\dot{\lambda}^4}{\lambda^7}(t),$$

where  $C_1 \sim e^{\varepsilon^{-2}}$  is some incredibly large constant that beats the (uniform) implicit constant  $C\delta^{-1}$  appearing in estimates (129). Summing this last line over  $0 \leq k \leq N$ , we have the bound (131).

We now prove (129) under the additional assumption that (130) also holds. We start by providing the general setup and then reduce the proof to a number of separate estimates to be dealt with under their own boldfaced headings. We first record the equation for  $w$ . Recall that the purpose of the decomposition (92) is to eliminate the main source term  $A_\lambda(\ddot{I}_\lambda)$  on the right-hand side of (45), obtained after applying  $A_\lambda$ . Therefore, we have that:

$$A_\lambda[\partial_t^2 w + H_\lambda w] = -A_\lambda(\ddot{w}_0) + A_\lambda \mathcal{N}(u).$$

To put things in the form where estimate (121) can be used, we commute the  $A_\lambda$  operator with  $\partial_t^2$  on the left-hand side of this last equation which yields the expression:

$$\begin{aligned} (134) \quad & \partial_t^2 W + \tilde{H}_\lambda W = -A_\lambda(\ddot{w}_0) + [\partial_t^2, A_\lambda]w + A_\lambda \mathcal{N}(u) \\ & = \partial_t(A_\lambda)\partial_t(w_0) - \partial_t(A_\lambda\partial_t(w_0)) + 2\partial_t(\partial_t(A_\lambda) \cdot w) - \partial_t^2(A_\lambda) \cdot w + A_\lambda \mathcal{N}(u) \\ & = M_1 + \partial_t M_2 + \partial_t R_1 + R_2 + R_3, \end{aligned}$$

where we have set  $W = A_\lambda w$ . The  $M$  terms above constitute the “main source” which feeds the quantity  $W$  through the wave-flow of the Hamiltonian  $\tilde{H}_\lambda$ . By contrast, the  $R$  terms on line (134) are for the most part “errors” which will be reabsorbed back onto the left-hand side of estimate (128). This is where the limits on the time interval and the decay estimate (147) will come into play. We now turn to the details of all of this.

To estimate the terms  $M_1$  and  $M_2$  via the general bound (121), we will show the following four estimates:

$$(135) \quad \int_{t_0}^{t_1} \int_{\mathbb{R}^+} \lambda^{-1}(\lambda r)^\delta |M_1|^2(s) r^2 dr ds \lesssim \varepsilon^3 + \varepsilon \sup_{t_0 \leq s \leq t_1} \frac{\dot{\lambda}^4}{\lambda^7}(s),$$

$$(136) \quad \int_{t_0}^{t_1} \int_{\mathbb{R}^+} \lambda^{-1}(\lambda r)^\delta |\partial_r M_2|^2(s) r^2 dr ds \lesssim \varepsilon^3 + \varepsilon \sup_{t_0 \leq s \leq t_1} \frac{\dot{\lambda}^4}{\lambda^7}(s),$$

$$(137) \quad \int_{t_0}^{t_1} \int_{\mathbb{R}^+} \lambda^{-1}(\lambda r)^\delta |\lambda M_2|^2(s) r^2 dr ds \lesssim \varepsilon^3 + \varepsilon \sup_{t_0 \leq s \leq t_1} \frac{\dot{\lambda}^4}{\lambda^7}(s),$$

$$(138) \quad \sup_{t_0 \leq s \leq t_1} \int_{\mathbb{R}^+} \lambda^{-1} \frac{(\lambda r)^\delta}{1+r^\delta} |M_2|^2(s) r dr \lesssim \varepsilon^3 + \varepsilon \sup_{t_0 \leq s \leq t_1} \frac{\dot{\lambda}^4}{\lambda^7}(s).$$

Recall we are assuming that  $\varepsilon \leq c_0^2$ , so these estimates will be enough to generate the right-hand side of (128) for the  $M$  terms.

We shall estimate the  $R$  terms on line (134) in a nonlinear fashion. Specifically, we will bound them in terms of a small constant times the energy on the left-hand side of (128), plus one term involving the nonlinearity  $\mathcal{N}$  applied to  $w_0$  which fits into the pattern of the right-hand side of (135)–(138) above. What we propose to show is the following:

$$(139) \quad \int_{t_0}^{t_1} \int_{\mathbb{R}^+} \lambda^{-1}(\lambda r)^\delta |\partial_r R_1|^2(s) r^2 dr ds \lesssim \varepsilon^2 \mathbb{E}_\delta[W](t_0, t_1),$$

$$(140) \quad \int_{t_0}^{t_1} \int_{\mathbb{R}^+} \lambda^{-1}(\lambda r)^\delta |\lambda R_1|^2(s) r^2 dr ds \lesssim \varepsilon^2 \mathbb{E}_\delta[W](t_0, t_1),$$

$$(141) \quad \sup_{t_0 \leq s \leq t_1} \int_{\mathbb{R}^+} \lambda^{-1} \frac{(\lambda r)^\delta}{1+r^\delta} |R_1|^2(s) r dr \lesssim \varepsilon^2 \mathbb{E}_\delta[W](t_0, t_1),$$

$$(142) \quad \int_{t_0}^{t_1} \int_{\mathbb{R}^+} \lambda^{-1}(\lambda r)^\delta |R_2|^2(s) r^2 dr ds \lesssim \varepsilon^2 \mathbb{E}_\delta[W](t_0, t_1),$$

$$(143) \quad \int_{t_0}^{t_1} \int_{\mathbb{R}^+} \lambda^{-1}(\lambda r)^\delta |R_3|^2(s) r^2 dr ds \lesssim \varepsilon \sup_{t_0 \leq s \leq t_1} \frac{\dot{\lambda}^4}{\lambda^7}(s) + t_1^\delta \varepsilon^2 \mathbb{E}_\delta[W](t_0, t_1).$$

Using the conditions  $\varepsilon$  is sufficiently small and that  $t_1 \leq \varepsilon^{-4}$ , all of the estimates added together will imply estimate (128) for the  $R$  terms on line (134) above.

We now turn to the details of the proofs of estimates (135)–(138) and (139)–(143). We will do each of these separately and in order.

In what follows, we will consistently use the following “identities” which are in accordance with our notation from Section 2.1:

$$(\lambda r)^\delta F_\lambda^m = F_\lambda^{m-\delta}, \quad (1+r^\delta) F_\lambda^m \leq F_\lambda^{m-\delta},$$

where the second “inequality” holds provided that  $\lambda \geq 1$ .

*Proof of estimate (135).* Here and throughout the sequel we will rely heavily on the abstract function notation from lines (58) above. Multiplying together estimates from lines (58a) and (58c), we have that:

$$|\partial_t(A_\lambda) \partial_t(w_0)| \lesssim \left( \dot{\lambda}^4 \lambda^{-5} + |\ddot{\lambda}| \dot{\lambda}^2 \lambda^{-4} \right) F_\lambda^9.$$

To resolve the second term above which contains the expression  $\ddot{\lambda}$ , we do the following. Notice that the orbital stability bound  $\dot{\lambda} \lambda^{-2} \lesssim \varepsilon$  and the bootstrapping estimate (93) give the rough *pointwise* bound:

$$(144) \quad \begin{aligned} |\ddot{\lambda}| &\lesssim \dot{\lambda}^2 \lambda^{-1} + (\varepsilon^2 + \dot{\lambda}^4 \lambda^{-7}) \lambda^2 \\ &\lesssim \dot{\lambda}^2 \lambda^{-1} + \varepsilon^2 \lambda^2. \end{aligned}$$

Therefore, substituting this estimate back into the previous line we have that:

$$|\partial_t(A_\lambda)\partial_t(w_0)| \lesssim \left(\dot{\lambda}^4\lambda^{-5} + \varepsilon^2\dot{\lambda}^2\lambda^{-2}\right)F_\lambda^9.$$

Using this last line we can now estimate:

$$\begin{aligned} \text{(L.H.S.)}(135) &\lesssim \int_{t_0}^{t_1} \int_{\mathbb{R}^+} \lambda^{-1} \left(\dot{\lambda}^4\lambda^{-5} + \varepsilon^2\dot{\lambda}^2\lambda^{-2}\right)^2 F_\lambda^4 r^2 dr ds \\ &\lesssim \int_{t_0}^{t_1} \dot{\lambda}^8\lambda^{-14}(s) ds + \varepsilon^4 \int_{t_0}^{t_1} \dot{\lambda}^4\lambda^{-8}(s) ds \\ &\lesssim \sup_{t_0 \leq s \leq t_1} \dot{\lambda}^4\lambda^{-7}(s) \cdot \int_{t_0}^{t_1} \dot{\lambda}^4\lambda^{-7}(s) ds \\ &\quad + \varepsilon^4 \sup_{t_0 \leq s \leq t_1} \dot{\lambda}^3\lambda^{-6}(s) \cdot \int_{t_0}^{t_1} \dot{\lambda}\lambda^{-2}(s) ds \\ &\lesssim \varepsilon \sup_{t_0 \leq s \leq t_1} \dot{\lambda}^4\lambda^{-7}(s) + \varepsilon^7. \end{aligned}$$

To obtain the last line, we have used both estimate (130), the bound (62), and the assumption that  $\lambda \geq 1$ .

*Proof of estimate (136).* This is very similar to the analysis above, with an addition of a small twist. First of all, by combining lines (58a), (58c), (38), and then (144), we have the abstract notational bound:

$$\begin{aligned} |\partial_r(A_\lambda\partial_t(w_0))| &\lesssim \left(\dot{\lambda}^3\lambda^{-3} + |\ddot{\lambda}|\dot{\lambda}\lambda^{-2}\right)F_\lambda^4 \\ &\lesssim \left(\dot{\lambda}^3\lambda^{-3} + \varepsilon^2\dot{\lambda}\right)F_\lambda^4. \end{aligned}$$

Substituting this into the left-hand side of (136), we have the chain of inequalities:

$$\begin{aligned} \text{(L.H.S.)}(136) &\lesssim \int_{t_0}^{t_1} \int_{\mathbb{R}^+} \lambda^{-1} \left(\dot{\lambda}^3\lambda^{-3} + \varepsilon^2\dot{\lambda}\right)^2 F_\lambda^4 r^2 dr ds \\ &\lesssim \int_{t_0}^{t_1} \dot{\lambda}^6\lambda^{-10}(s) ds + \varepsilon^4 \int_{t_0}^{t_1} \dot{\lambda}^2\lambda^{-4}(s) ds \\ &\lesssim \int_{t_0}^{t_1} \dot{\lambda}^6\lambda^{-10}(s) ds + \varepsilon^4 \sup_{t_0 \leq s \leq t_1} \dot{\lambda}\lambda^{-2}(s) \cdot \int_{t_0}^{t_1} \dot{\lambda}\lambda^{-2}(s) ds \\ &\lesssim \int_{t_0}^{t_1} \dot{\lambda}^6\lambda^{-10}(s) ds + \varepsilon^5. \end{aligned}$$

We now need to do a little work, because the first term on the right-hand side of this last line above is not manifestly of the correct form. To correct it, we first integrate by parts with respect to time which yields the identity:

$$\int_{t_0}^{t_1} \dot{\lambda}^6\lambda^{-10} ds = \dot{\lambda}^5\lambda^{-9}(t_1) - \dot{\lambda}^5\lambda^{-9}(t_0)$$

$$+10 \int_{t_0}^{t_1} \dot{\lambda}^6 \lambda^{-10} ds - 5 \int_{t_0}^{t_1} \ddot{\lambda} \dot{\lambda}^4 \lambda^{-9} ds.$$

Now, using the bootstrapping assumption (93) as well as estimate (62), this last expression leads to the following nonlinear bound:

$$\begin{aligned} (145) \quad \int_{t_0}^{t_1} \dot{\lambda}^6 \lambda^{-10} ds &\lesssim c_0^{\frac{1}{2}} \int_{t_0}^{t_1} \dot{\lambda}^6 \lambda^{-10}(s) ds + \varepsilon \sup_{t_0 \leq s \leq t_1} \dot{\lambda}^4 \lambda^{-7}(s) \\ &\quad + \int_{t_0}^{t_1} (\varepsilon^2 + \sup_{t_0 \leq \cdot \leq s} \dot{\lambda}^4 \lambda^{-7}) \cdot \dot{\lambda}^4 \lambda^{-7}(s) ds \\ &\lesssim c_0^{\frac{1}{2}} \int_{t_0}^{t_1} \dot{\lambda}^6 \lambda^{-10}(s) ds + \varepsilon \sup_{t_0 \leq s \leq t_1} \dot{\lambda}^4 \lambda^{-7}(s) + \varepsilon^2 \int_{t_0}^{t_1} \dot{\lambda}^4 \lambda^{-7}(s) ds \\ &\lesssim c_0^{\frac{1}{2}} \int_{t_0}^{t_1} \dot{\lambda}^6 \lambda^{-10}(s) ds + \varepsilon \sup_{t_0 \leq s \leq t_1} \dot{\lambda}^4 \lambda^{-7}(s) + \varepsilon^3. \end{aligned}$$

In the first inequality above, we have used (73) and the monotonicity established in Section 5 (this works as long as  $T^* \leq t_0$ , whereas in the other case we may as well assume that  $t_0 = 0$ ). Notice also that in the last two lines above we made several uses of the assumption (130).

*Proof of estimate (137).* This is virtually identical to the proof of (136). Another simple calculation using lines (58a), (58c), and then (144) gives us:

$$\begin{aligned} |\lambda(A_\lambda \partial_t(w_0))| &\lesssim (\dot{\lambda}^3 \lambda^{-3} + |\ddot{\lambda}| \dot{\lambda} \lambda^{-2}) F_\lambda^3 \\ &\lesssim (\dot{\lambda}^3 \lambda^{-3} + \varepsilon^2 \dot{\lambda}) F_\lambda^3. \end{aligned}$$

Substituting this into the left-hand side of (137), the proof follows verbatim from the calculations done in the previous paragraph.

*Proof of estimate (138).* Once again using lines (58a), (58c), and then (144) we have that:

$$\begin{aligned} |(A_\lambda \partial_t(w_0))| &\lesssim (\dot{\lambda}^3 \lambda^{-4} + |\ddot{\lambda}| \dot{\lambda} \lambda^{-3}) F_\lambda^3 \\ &\lesssim (\dot{\lambda}^3 \lambda^{-4} + \varepsilon^2 \dot{\lambda} \lambda^{-1}) F_\lambda^3. \end{aligned}$$

Plugging this last line into the left-hand side of (138) and simply using the bound (62), we arrive at the chain of inequalities:

$$\begin{aligned} (\text{L.H.S.})(138) &\lesssim \sup_{t_0 \leq s \leq t_1} \int_{\mathbb{R}^+} \lambda^{-1} (\dot{\lambda}^3 \lambda^{-4} + \varepsilon^2 \dot{\lambda} \lambda^{-1})^2 F_\lambda^4 r dr \\ &\lesssim \sup_{t_0 \leq s \leq t_1} \dot{\lambda}^6 \lambda^{-11} + \varepsilon^4 \sup_{t_0 \leq s \leq t_1} \dot{\lambda}^2 \lambda^{-5} \\ &\lesssim \varepsilon^2 \sup_{t_0 \leq s \leq t_1} \dot{\lambda}^4 \lambda^{-7} + \varepsilon^6. \end{aligned}$$

This concludes our proof of the first list of estimates (135)–(138) above.

We now turn our attention to the proofs of estimates (139)–(143).

*Proof of (139).* We first provide a pointwise bound for the term  $\partial_r R_1$  from line (134) above. This involves a simple application of the abstract notation (58c), the rules from line (38), and the decomposition:

$$\partial_r w = -W + \frac{k}{r} \cos(I_\lambda) \cdot w.$$

Together, these give us the following estimate:

$$|\partial_r(\partial_t(A_\lambda) \cdot w)| \lesssim (|\dot{\lambda} \lambda w| + |\dot{\lambda} W|) F_\lambda^6.$$

Plugging this identity in the left-hand side of (139) leads us to the estimates (also using (62)):

$$\begin{aligned} & \int_{t_0}^{t_1} \int_{\mathbb{R}^+} \lambda^{-1} (\lambda r)^\delta |\partial_r R_1|^2 r^2 dr ds \\ & \lesssim \int_{t_0}^{t_1} \int_{\mathbb{R}^+} \frac{\dot{\lambda}^2}{\lambda^5} (\lambda r)^\delta \left[ \frac{w^2}{r^5} + \frac{W^2}{r^3} \right] ((r^4 + r^6) F^{12})_\lambda r dr dt \\ & \lesssim \varepsilon^2 \int_{t_0}^{t_1} \int_{\mathbb{R}^+} \lambda^{-1} \frac{(\lambda r)^\delta}{(1+r)^\delta} \left[ \frac{w^2}{r^5} + \frac{W^2}{r^3} \right] r dr dt \\ & \lesssim \varepsilon^2 \int_{t_0}^{t_1} \int_{\mathbb{R}^+} \lambda^{-1} \frac{(\lambda r)^\delta}{(1+r)^\delta} \frac{W^2}{r^3} r dr dt, \\ & \lesssim \varepsilon^2 \mathbb{E}_\delta[W](t_0, t_1). \end{aligned}$$

To obtain the second to last line above, we have used the comparison estimate (164) from Appendix B on the term involving  $w$ .

*Proof of estimate (140).* This is virtually identical to the proof of (139) in the previous paragraph. A simple calculation using the notation from lines (58) and line (38) gives us the bound:

$$|\lambda(\partial_t(A_\lambda) \cdot w)| \lesssim \dot{\lambda} \lambda |w| F_\lambda^7.$$

The proof now follows line for line from the calculations performed above.

*Proof of estimate (141).* We again use the formulas on line (58c) which give us:

$$|(\partial_t(A_\lambda) \cdot w)| \lesssim \dot{\lambda} |w| F_\lambda^7.$$

Substituting this last line in the left-hand side of (141), we have the following chain of inequalities where the second to last line involves the bound (164):

$$\begin{aligned} \sup_{t_0 \leq s \leq t_1} \int_{\mathbb{R}^+} \lambda^{-1} \frac{(\lambda r)^\delta}{1+r^\delta} |R_1|^2 r dr & \lesssim \sup_{t_0 \leq s \leq t_1} \int_{\mathbb{R}^+} \frac{\dot{\lambda}^2}{\lambda^5} \frac{(\lambda r)^\delta}{1+r^\delta} \frac{w^2}{r^4} (r^4 F^{14})_\lambda r dr dt \\ & \lesssim \varepsilon^2 \sup_{t_0 \leq s \leq t_1} \int_{\mathbb{R}^+} \lambda^{-1} \frac{(\lambda r)^\delta}{(1+r)^\delta} \frac{w^2}{r^4} r dr dt \end{aligned}$$

$$\begin{aligned} &\lesssim \varepsilon^2 \sup_{t_0 \leq s \leq t_1} \int_{\mathbb{R}^+} \lambda^{-1} \frac{(\lambda r)^\delta}{(1+r)^\delta} \frac{W^2}{r^2} r dr dt \\ &\lesssim \varepsilon^2 \mathbb{E}_\delta[W](t_0, t_1). \end{aligned}$$

*Proof of estimate (142).* We estimate the  $R_2$  term from the line (134). By using the second derivative identity from line (58c), estimate (144), and then the bound (62), we have the following pointwise estimate:

$$\begin{aligned} |\partial_t^2(A_\lambda) \cdot w| &\lesssim (\dot{\lambda}^2 \lambda^{-1} + \varepsilon^2 \lambda^2) F_\lambda^7 \cdot |w| \\ &\lesssim \varepsilon^2 \lambda^3 F_\lambda^7 \cdot |w|. \end{aligned}$$

We now substitute this estimate into the left-hand side of (142) which allows us to estimate:

$$\begin{aligned} \int_{t_0}^{t_1} \int_{\mathbb{R}^+} \lambda^{-1} (\lambda r)^\delta |R_2|^2 r^2 dr ds &\lesssim \varepsilon^4 \int_{t_0}^{t_1} \int_{\mathbb{R}^+} \lambda^5 (\lambda r)^\delta |w|^2 \cdot F_\lambda^{14} r^2 dr ds \\ &\lesssim \varepsilon^4 \int_{t_0}^{t_1} \int_{\mathbb{R}^+} \lambda^{-1} (\lambda r)^\delta \frac{w^2}{r^5} \cdot (r^6 F^{14})_\lambda r dr ds \\ &\lesssim \varepsilon^4 \int_{t_0}^{t_1} \int_{\mathbb{R}^+} \lambda^{-1} \frac{(\lambda r)^\delta}{(1+r)^\delta} \frac{w^2}{r^5} r dr ds \\ &\lesssim \varepsilon^4 \int_{t_0}^{t_1} \int_{\mathbb{R}^+} \lambda^{-1} \frac{(\lambda r)^\delta}{(1+r)^\delta} \frac{W^2}{r^3} r dr ds \\ &\lesssim \varepsilon^4 \mathbb{E}_\delta[W](t_0, t_1). \end{aligned}$$

*Proof of estimate (143).* First of all, using the formula for the nonlinearity  $\mathcal{N}(u)$  given on line (85) and making use of the formula (47) for the operator  $A_\lambda$  as well as the formula (56) for  $w_0$ , we easily have the pointwise bound:

$$(146) \quad |A_\lambda \mathcal{N}(u)| \lesssim \dot{\lambda}^4 \lambda^{-5} F_\lambda^7 + \frac{(|u| + |w_0|) \cdot |w|}{r^3} + \frac{(|u| + |w_0|) \cdot |W|}{r^2}.$$

We will deal with the first term on the right-hand side above by itself. The other two terms can be handled together.

We now substitute the first term on the right-hand side of (146) for  $R_3$  on the left-hand side of (143). Doing this we are left with estimating:

$$\begin{aligned} \int_{t_0}^{t_1} \int_{\mathbb{R}^+} \dot{\lambda}^8 \lambda^{-11} (\lambda r)^\delta F_\lambda^{12} r^2 dr ds &\lesssim \int_{t_0}^{t_1} \dot{\lambda}^8 \lambda^{-14}(s) ds \\ &\lesssim \sup_{t_0 \leq s \leq t_1} \dot{\lambda}^4 \lambda^{-7}(s) \cdot \int_{t_0}^{t_1} \dot{\lambda}^4 \lambda^{-7}(s) ds \\ &\lesssim \varepsilon \sup_{t_0 \leq s \leq t_1} \dot{\lambda}^4 \lambda^{-7}(s). \end{aligned}$$

This proves estimate (143) for the  $w_0$  portion of  $R_3$ .



It remains to deal with (143) for the last two terms on the right-hand side of (146). Upon substitution of these into the right-hand side of (143) we have that:

$$\begin{aligned}
 & \int_{t_0}^{t_1} \int_{\mathbb{R}^+} \lambda^{-1} (\lambda r)^\delta (|u|^2 + |w_0|^2) \cdot \left[ \frac{w^2}{r^6} + \frac{W^2}{r^4} \right] r^2 dr ds \\
 & \lesssim \int_{t_0}^{t_1} \int_{\mathbb{R}^+} \lambda^{-1} \frac{(\lambda r)^\delta}{(1+r)^\delta} \left[ \frac{w^2}{r^5} + \frac{W^2}{r^3} \right] \cdot (1+r)^\delta (|u|^2 + |w_0|^2) r dr ds \\
 & \lesssim \left( \sup_{\substack{r \\ t_0 \leq s \leq t_1}} (1+r)^\delta |u|^2 + \varepsilon^2 \right) \cdot \int_{t_0}^{t_1} \int_{\mathbb{R}^+} \lambda^{-1} \frac{(\lambda r)^\delta}{(1+r)^\delta} \left[ \frac{w^2}{r^5} + \frac{W^2}{r^3} \right] r dr ds \\
 & \lesssim t_1^\delta \varepsilon^2 \int_{t_0}^{t_1} \int_{\mathbb{R}^+} \lambda^{-1} \frac{(\lambda r)^\delta}{(1+r)^\delta} \frac{W^2}{r^3} r dr ds \\
 & \lesssim t_1^\delta \varepsilon^2 \mathbb{E}_\delta[W](t_0, t_1).
 \end{aligned}$$

Notice that in the above estimates we have made crucial use of the special pointwise estimate (147) proved below. This is the only place in the paper which requires the extra decay of the initial data. This completes our proof of estimate (143), and thus our proof of Proposition 6.3.  $\square$

6.1. *A simple decay estimate.* In this subsection we will prove the rough decay estimate:

$$(147) \quad \sup_r (1+r)^\delta |u|^2 \lesssim t^\delta \varepsilon^2.$$

That is, our aim is to show that the reduced field quantity  $u$  enjoys some amount of *pointwise* decay outside of a sufficiently large cone centered at the space-time origin  $t = 0$  and  $r = 0$ .

LEMMA 6.4 (Decay of  $u$  at space-like infinity). *Let  $u = \phi - I_\lambda$  be the reduced field quantity as defined in Lemma 4.1 which in addition satisfies the initial conditions of Theorem 1.4. In particular,  $u$  is a solution to equation (45) with initial data (28)–(29) and obeys the estimate (60) on the time interval  $[0, T]$  where  $\phi$  exists and remains smooth. Then  $u$  also obeys the following stronger energy type estimate for any time  $t \in [0, T]$ , for which in addition  $\lambda \geq 1$ :*

$$(148) \quad \int_{2t \leq r} r^2 \left[ (\partial_t \phi)^2 + (\partial_r u)^2 + \frac{k^2}{r^2} u^2 \right] r dr \lesssim \varepsilon^2.$$

*Remark 6.5.* To transform estimate (148) into an  $L^\infty$  bound can be done in an elementary way by applying the Poincaré type estimate (68) to the quantity  $r \chi_{3t \leq r} u$  and then using the bound (148) to estimate the resulting right-hand side. Here  $\chi_{3t \leq r}$  is a smooth cutoff onto the region where  $3t \leq r$  which satisfies the homogeneity bound  $|\chi'_{3t \leq r}| \lesssim r^{-1}$ . Therefore, we arrive at the estimate:  $\sup_{3t \leq r} r |u| \lesssim \varepsilon$ . By combining this with the pointwise bound  $|u| \leq \varepsilon$  which holds everywhere, we easily have (147) whenever  $\delta \leq 1$ .

*Proof of estimate (148).* The proof is an integration by parts argument with a certain multiplier. We denote by  $\alpha(y)$  a smooth increasing function, supported where  $10 \leq y$ , satisfying  $\alpha' \leq 3y$  and the homogeneity bound  $y^{-1}\alpha \leq \alpha'$ . The desired result will now follow from computing the left-hand side of the identity:

$$(149) \quad \int_0^t \int_{\mathbb{R}^+} [\partial_t^2 \phi + H_\lambda u - \mathcal{N}(u)] \partial_t \phi \cdot \alpha(r-2s) r dr ds = 0,$$

where  $\mathcal{N}(u) = \text{R.H.S.}(45)$ . Also, we will write the Hamiltonian from line (46) as  $H_\lambda = -\partial_r^2 - r^{-1}\partial_r + Q_\lambda$ . Notice that we have  $Q_\lambda \geq ck^2r^{-2}$  on the support of  $\alpha(r-2s)$ .

Using the factorization (46) as well as the fact that  $A_\lambda(\dot{I}_\lambda) = 0$ , thanks to (48)–(49), we may transform (149) into the identity:

$$(150) \quad \begin{aligned} & -\frac{1}{2} \int_{\mathbb{R}^+} [(\partial_t \phi)^2 + (\partial_r u)^2 + Q_\lambda u^2] \cdot \alpha r dr \Big|_0^t \\ &= \int_0^t \int_{\mathbb{R}^+} [(\partial_t \phi)^2 + \partial_t \phi \partial_r u + (\partial_r u)^2 + Q_\lambda u^2] \cdot \alpha'(r-2s) r dr ds \\ &\quad - k \int_0^t \int_{\mathbb{R}^+} \frac{u}{r} \cos(I_\lambda) \dot{I}_\lambda \cdot \alpha' r dr ds - \frac{1}{2} \int_0^t \int_{\mathbb{R}^+} \dot{Q}_\lambda u^2 \cdot \alpha r dr ds \\ &\quad - \int_0^t \int_{\mathbb{R}^+} \mathcal{N}(u) \partial_t \phi \cdot \alpha r dr ds \\ &= T_1 + T_2 + T_3 + T_4. \end{aligned}$$

The proof will be complete once we show that the terms on the right-hand side of this last expression are either nonnegative or are bounded in absolute value by  $C\varepsilon^2$ . In fact, it is more or less immediate that we have:

$$(151) \quad |T_2| \lesssim \varepsilon^2, \quad T_1 + T_3 + T_4 \geq 0.$$

The first estimate above is a consequence of the Cauchy-Schwartz inequality, the orbital stability bound (60), and the following fixed time estimate valid for  $1 \leq \lambda$ :

$$\|\dot{I}_\lambda \cdot \alpha'\|_{L^2(r dr)} \lesssim \varepsilon.$$

This last line uses our assumptions that  $4 \leq k$  and  $1 \leq \lambda$ . Specifically, the ODE bound (62)  $|\dot{\lambda}\lambda^{-1}| \lesssim \varepsilon\lambda$  (from (62) above) and a simple calculation, using the assumption that  $1 \leq \lambda$  and involving lines (48) and (34), give us the bound  $|\dot{I}_\lambda \cdot \alpha'| \lesssim |r\dot{I}| \lesssim \varepsilon(1+r)^{-3}$ .

The second bound on line (151) will follow from the estimate  $|T_3 + T_4| \lesssim \varepsilon T_1$ . The desired result is then a consequence of the homogeneity property of  $\alpha$  and bounds:

$$|\mathcal{N}(u)| \lesssim \varepsilon \frac{|u|}{r^2}, \quad |\dot{Q}_\lambda| \lesssim \varepsilon \frac{1}{r^3}.$$

The first bound above is a simple consequence of the orbital stability estimate (60) together with (68). The second bound follows again from the estimate  $|\dot{\lambda}\lambda^{-1}| \lesssim \varepsilon\lambda$  of (62) and the explicit formulas on lines (34)–(35).

The estimate (148) now follows from the form of the left-hand side of (150) and the smallness condition (60).  $\square$

### Appendix A. Computation of the constant $C_*$

The purpose of the appendix is to derive an explicit formula  $C_* = 0$  for the special constant  $C_*$  which appeared on line (90). Here we have written  $J = J_1$  according to previous notation. In what follows we shall also denote  $I = I_1 = I^k$ . Rescaling we have that:

$$C_* = T_1 + T_2 + T_3,$$

where:

$$T_1 = -k^2 \left\langle \frac{(aJ + br^2J)^2}{r^2}, \sin(2I) \cdot J \right\rangle,$$

$$T_2 = \left\langle aJ + br^2J, r\partial_r J \right\rangle,$$

$$T_3 = - \left\langle r\partial_r (aJ + br^2J), r\partial_r J \right\rangle.$$

Recall that the constants  $a$  and  $b$  are given on line (57). Using now the identity  $\sin(2I) = 2\sin(I)\cos(I)$  as well as (48) and (25), we have that:

$$\begin{aligned} (152) \quad T_2 &= -k^2 \int_{\mathbb{R}^+} (aJ + br^2J)^2 \partial_r (\sin^2(I)) \, dr \\ &= - \int_{\mathbb{R}^+} (a + br^2)^2 J^2 \partial_r (J^2) \, dr \\ &= 2ab \int_{\mathbb{R}^+} J^4 \, r \, dr + 2b^2 \int_{\mathbb{R}^+} J^4 \, r^3 \, dr. \end{aligned}$$

To obtain the last line above, we have used the expansion  $J^4 = k^2 J^2 \sin^2(I)$ , the Pythagorean identity, and the definitions of  $a, b$ .

We now move on the term  $T_2$  above. Here we have directly that:

$$\begin{aligned} (153) \quad T_2 &= \int_{\mathbb{R}^+} (aJ + br^2J) \partial_r J \, r^2 \, dr \\ &= -a \int_{\mathbb{R}^+} J^2 \, r \, dr - 2b \int_{\mathbb{R}^+} J^2 \, r^3 \, dr = -\frac{1}{4} \int_{\mathbb{R}^+} J^2 \, r^3 \, dr. \end{aligned}$$

Finally, we compute that:

(154)

$$\begin{aligned} T_3 &= -a \int_{\mathbb{R}^+} (r \partial_r J)^2 r dr \lesssim -b \int_{\mathbb{R}^+} (r \partial_r J)^2 r^3 dr \lesssim -2b \int_{\mathbb{R}^+} J \partial_r J r^4 dr \\ &= -ak^2 \int_{\mathbb{R}^+} J^2 \cos^2(I) r dr \\ &\lesssim -bk^2 \int_{\mathbb{R}^+} J^2 \cos^2(I) r^3 dr + 4b \int_{\mathbb{R}^+} J^2 r^3 dr \\ &= a \int_{\mathbb{R}^+} J^4 r dr + b \int_{\mathbb{R}^+} J^4 r^3 dr + 4b \int_{\mathbb{R}^+} J^2 r^3 dr. \end{aligned}$$

We now add together equations (152)–(154) into the single formula:

$$(155) \quad C_* = \frac{3}{2}a \int_{\mathbb{R}^+} J^4 r dr + \frac{3}{2}b \int_{\mathbb{R}^+} J^4 r^3 dr + 3b \int_{\mathbb{R}^+} J^2 r^3 dr.$$

It remains to compute the first two integrals in this last expression.

$$\begin{aligned} \int_{\mathbb{R}^+} J^4 r dr &= -k \int_{\mathbb{R}^+} r \partial_r (\cos(I)) J^2 r dr \\ &= 2k \int_{\mathbb{R}^+} \cos(I) J^2 r dr + 2k^2 \int_{\mathbb{R}^+} \cos^2(I) J^2 r dr \\ &= k^2 \int_{\mathbb{R}^+} r \partial_r (\sin^2(I)) r dr + 2k^2 \int_{\mathbb{R}^+} J^2 r dr - 2 \int_{\mathbb{R}^+} J^4 r dr \\ &= (2k^2 - 2) \int_{\mathbb{R}^+} J^2 r dr - 2 \int_{\mathbb{R}^+} J^4 r dr. \end{aligned}$$

An almost identical calculation also shows that:

$$\int_{\mathbb{R}^+} J^4 r^3 dr = (2k^2 - 5) \int_{\mathbb{R}^+} J^2 r^3 dr - 2 \int_{\mathbb{R}^+} J^4 r^3 dr.$$

Therefore, recalling the definition of  $a$  and  $b$ , these last two calculations together give:

$$a \int_{\mathbb{R}^+} J^4 r dr + b \int_{\mathbb{R}^+} J^4 r^3 dr = -\frac{1}{3}b \int_{\mathbb{R}^+} J^2 r^3 dr.$$

Inserting the last line into (155) and using that  $b = \frac{1}{4}$ , we have  $C_* = 0$  as desired.

### Appendix B. A general functional analysis lemma

In this appendix, we prove a general form of a coercive estimate we need throughout the paper. This turns out to be more expedient, because the required structure is simply a matter of compactness and weak convergence in various weighted Sobolev spaces. The general result which we propose to prove here is the following:

**LEMMA B.1** (Coercive bounds for first order operators). *Let  $B_\ell$  be a sequence of first order differential operators with real smooth (but not necessarily bounded!) coefficients.*

coefficients on the half line  $(0, \infty)$ , continuously indexed (in the weighted  $L^2$  space defined by the left-hand side of (156) below) by  $\ell \in [0, 1]$  and such that the following sub-coercivity holds for some continuously indexed (for functions in the norm (157)) function  $|f_\ell| \lesssim r^{-2-\gamma+\sigma} (1+r)^{-2\sigma}$  with  $\sigma > 0$ :

$$(156) \quad \int_{\mathbb{R}^+} \frac{(B_\ell \psi)^2}{r^\gamma} r dr = \int_{\mathbb{R}^+} \left[ \frac{(\partial_r \psi)^2}{r^\gamma} + h_\ell \frac{\psi^2}{r^{2+\gamma}} + f_\ell \psi^2 \right] r dr,$$

for any real valued function  $\psi$  with finite norm:

$$(157) \quad \|\psi\|_{\mathcal{H}^\gamma}^2 = \int_{\mathbb{R}^+} \left[ \frac{(\partial_r \psi)^2}{r^\gamma} + \frac{\psi^2}{r^{2+\gamma}} \right] r dr.$$

Here  $0 \leq \gamma$  is a fixed parameter, and  $0 < C \leq h_\ell$  is some strictly positive function. Then there exists a universal constant, uniform in  $\ell$ , such that the following bound holds:

$$(158) \quad \|\psi\|_{\mathcal{H}^\gamma}^2 \lesssim \int_{\mathbb{R}^+} \frac{(B_\ell \psi)^2}{r^\gamma} r dr,$$

for any real  $\mathcal{H}^\gamma$  function  $\psi$  which also satisfies:

$$(159) \quad \int_{\mathbb{R}^+} \psi \cdot J^\ell m_\ell r dr = 0,$$

for some positive weight function  $0 < m_\ell$ . Here the function  $J^\ell$  is the (nontrivial) “ground-state” given by  $B_\ell J^\ell = 0$ , and we are assuming  $m_\ell J^\ell \in (\mathcal{H}^\gamma)^*$  depends continuously on  $\ell$ .

*Proof of Lemma B.1.* The proof is based on a contradiction argument centered around weak convergence. Suppose that estimate (158) was not true. Then there would exist a sequence of  $\psi_n$  and  $\ell_n$  such that:

$$(160) \quad \int_{\mathbb{R}^+} \frac{(B_{\ell_n} \psi_n)^2}{r^\gamma} r dr \leq c_n \|\psi_n\|_{\mathcal{H}^\gamma}^2,$$

where  $c_n \rightarrow 0$  is some sequence of constants. We assume that this sequence is normalized so that  $\|\psi_n\|_{\mathcal{H}^\gamma} = 1$ . The space  $\mathcal{H}^\gamma$  is a Hilbert space (with an obvious scalar product) defined as a closure of  $C_0^\infty(\mathbb{R}_+)$  functions in the  $\mathcal{H}^\gamma$  norm. Therefore, we can choose a subsequence  $\psi_{n_k}$  which converges weakly in  $\mathcal{H}^\gamma$  to  $\psi_\infty \in \mathcal{H}^\gamma$ . Furthermore, we may assume (by perhaps taking another subsequence) that  $\ell_{n_k} \rightarrow \ell_\infty$  for some  $\ell_\infty \in [0, 1]$ . We now use  $\psi_n$  and  $\ell_n$  to denote this subsequence. Also, note that by Cauchy-Schwartz the unit normalization implies that  $\|\psi_\infty\|_{\mathcal{H}^\gamma} \leq 1$ .

By the continuity of the  $J^\ell$  and the uniform boundedness of the  $\psi_n$ , we have from the identity (159) that the limiting function satisfies:

$$(161) \quad \int_{\mathbb{R}^+} \psi_\infty \cdot J^{\ell_\infty} m_{\ell_\infty} r dr = 0.$$

Therefore, since the “ground-state”  $J^{\ell\infty}$  is unique (it satisfies a first order ODE) and the measure  $m_{\ell\infty} r dr$  is strictly positive on  $(0, \infty)$ , we will have a contradiction if we can establish that  $\psi_\infty$  is nontrivial. This contradiction would come from again invoking uniform boundedness and the continuity of the operators  $B_\ell$  which implies that:

$$\int_{\mathbb{R}^+} \frac{(B_{\ell\infty} \psi_n)^2}{r^\gamma} r dr \rightarrow 0,$$

so that  $\psi_\infty$  is a weak solution of  $B_{\ell\infty} \psi_\infty = 0$  and, hence, a smooth solution via ODE regularity, thus violating uniqueness as (161) implies  $\psi_\infty \neq \beta J^{\ell\infty}$  for any constant  $\beta \neq 0$ .

To show that  $\psi_\infty$  is nontrivial, we make crucial use of the sub-coercivity condition (156). By the unit normalization, the universal lower bound on  $h_\ell$ , and the assumption of (160), we have that there exists a universal lower bound to the limit:

$$0 > \overline{\lim}_{n \rightarrow \infty} \int_{\mathbb{R}^+} f_{\ell_n} \psi_n^2 r dr.$$

Therefore, we shall have that  $\psi_\infty$  is not everywhere zero if we can show that the sequence  $f_{\ell_n} \psi_n^2$  converges strongly in  $L^1(r dr)$ . This in turn follows from the universal bounds on and continuity of  $f_\ell$  and the fact that  $\psi_n$  converges strongly in the weighted space:

$$\int_{\mathbb{R}^+} r^{-2-\gamma+\sigma} (1+r)^{-2\sigma} \psi^2 r dr = \|\psi\|_{\mathcal{L}_{\gamma,\sigma}^2}^2.$$

This latter strong convergence is provided via uniform boundedness and the compact inclusion  $\mathcal{H}^\gamma \Subset \mathcal{L}_{\gamma,\sigma}^2$  whenever  $0 < \sigma$ . □

In practice, we will only need two special cases of equation Lemma B.1 above. The first case is where  $B_\ell \equiv A_1$  and  $\gamma = 0$ , where  $A_1$  is the first order operator from line (47) above. The second cases are when we set  $\ell = \lambda^{-1}$  with  $1 \leq \lambda$  and:

$$B_\ell = (1 + (\lambda^{-1}r))^{-\frac{\delta}{2}} A_1 (1 + (\lambda^{-1}r))^{\frac{\delta}{2}},$$

where in this case we set  $\psi = (1 + (\lambda^{-1}r))^{-\frac{\delta}{2}} u$ , as well as  $J^\ell = (1 + (\lambda^{-1}r))^{-\frac{\delta}{2}} J_1$ . Finally, in this case we set  $m_\ell = (1 + (\lambda^{-1}r))^\delta$ . We apply this to  $\gamma = 2 - \delta$  and  $\gamma = 3 - \delta$ .

In all of these cases we leave it to the reader to prove that the condition (156) holds (the continuity is obvious). This is a simple matter of integration by parts, the fact that  $4 \leq k$  (notice that this works for our range of  $\gamma$ , which is the main thing to check here), and also that we have chosen  $\delta \ll 1$ . However, we do call the reader’s attention to an important and perhaps subtle point. In order for the integration by parts to work, it is necessary to show that a boundary term of the form  $\lim_{r \rightarrow 0} r^{-\gamma} \psi^2$  vanishes for any function  $\psi$  in the space  $\mathcal{H}^\gamma$ . This follows from the finiteness of that norm and the fact that the Poincaré type estimate (68) above applied to  $r^{-\frac{\gamma}{2}} \psi$  implies that this function is *continuous* on the closed interval  $[0, 1]$ .

This latter fact is perhaps a bit subtle, and it is crucial for showing  $r^{-\frac{\lambda}{2}}\psi$  vanishes at  $r = 0$  via the finiteness of the weighted  $L^2$  norm (no derivative) contained in  $\mathcal{H}^\gamma$ . Again, we leave the reader to check the details of all this.

Now, applying the above result in these two cases and rescaling by  $\lambda$ , we have that:

LEMMA B.2 (Applied version of Lemma B.1). *Suppose that  $4 \leq k$  and  $A_\lambda$  is the operator defined on line (47). Then if  $u$  is a function which satisfies the admissibility condition of Lemma 1.9, and is such that the energy from line (60) is finite, and such that the orthogonality relation holds:*

$$\int_{\mathbb{R}^+} u \cdot J_\lambda r dr = 0,$$

then one has the following universal bounds whenever  $\delta \ll 1$  is small enough (and  $1 \leq \lambda$  in the last two cases):

$$(162) \quad \int_{\mathbb{R}^+} [(\partial_r u)^2 + \frac{u^2}{r^2}] r dr \lesssim \int_{\mathbb{R}^+} (A_\lambda u)^2 r dr,$$

$$(163) \quad \int_{\mathbb{R}^+} \frac{(\lambda r)^\delta}{(1+r)^\delta} \frac{u^2}{r^4} r dr \lesssim \int_{\mathbb{R}^+} \frac{(\lambda r)^\delta}{(1+r)^\delta} \frac{(A_\lambda u)^2}{r^2} r dr,$$

$$(164) \quad \int_{\mathbb{R}^+} \frac{(\lambda r)^\delta}{(1+r)^\delta} \frac{u^2}{r^5} r dr \lesssim \int_{\mathbb{R}^+} \frac{(\lambda r)^\delta}{(1+r)^\delta} \frac{(A_\lambda u)^2}{r^3} r dr.$$

*Acknowledgments.* The authors would like to thank M. Grillakis and J. Shatah for valuable discussions of the symmetric wave-map problem, and J. Krieger and W. Schlag for valuable discussions regarding their recent work [20] and related stability problems of wave equation solitons. We are also deeply indebted to Pierre Raphael for pointing out to us a subtle miscalculation of the blowup rate in an earlier version of the paper and for other valuable discussions. The authors would like to thank the MIT and UCSD mathematics departments for their hospitality while this work was being conducted.

## References

- [1] A. A. BELAVIN and A. M. POLYAKOV, Metastable states of two-dimensional isotropic ferromagnets (Russian), *JETP Lett.* **22** (1975), 245–247.
- [2] P. BIZOŃ, Y. N. OVCHINNIKOV, and I. M. SIGAL, Collapse of an instanton, *Nonlinearity* **17** (2004), 1179–1191. MR 2005e:35145 Zbl 1059.35081
- [3] P. BIZOŃ, T. CHMAJ, and Z. TABOR, Formation of singularities for equivariant  $(2 + 1)$ -dimensional wave maps into the 2-sphere, *Nonlinearity* **14** (2001), 1041–1053. MR 2003b:58043 Zbl 0988.35010
- [4] E. B. BOGOMOL'NYI, The stability of classical solutions, *Jadernaja Fiz.* **24** (1976), 861–870. MR 56 #2082
- [5] V. S. BUSLAEV and G. S. PERELMAN, Scattering for the nonlinear Schrödinger equation: states that are close to a soliton, *Algebra i Analiz* **4** (1992), 63–102. MR 94b:35256 Zbl 0853.35112

- [6] T. CAZENAVE, J. SHATAH, and A. S. TAHVILDAR-ZADEH, Harmonic maps of the hyperbolic space and development of singularities in wave maps and Yang-Mills fields, *Ann. Inst. H. Poincaré Phys. Théor.* **68** (1998), 315–349. MR 2000g:58042 Zbl 0918.58074
- [7] K.-C. CHANG, W. Y. DING, and R. YE, Finite-time blow-up of the heat flow of harmonic maps from surfaces, *J. Differential Geom.* **36** (1992), 507–515. MR 93h:58043 Zbl 0765.53026
- [8] Y. M. CHEN and M. STRUWE, Existence and partial regularity results for the heat flow for harmonic maps, *Math. Z.* **201** (1989), 83–103. MR 90i:58031 Zbl 0652.58024
- [9] D. CHRISTODOULOU and A. S. TAHVILDAR-ZADEH, On the asymptotic behavior of spherically symmetric wave maps, *Duke Math. J.* **71** (1993), 31–69. MR 94j:58044 Zbl 0791.58105
- [10] ———, On the regularity of spherically symmetric wave maps, *Comm. Pure Appl. Math.* **46** (1993), 1041–1091. MR 94e:58030 Zbl 0744.58071
- [11] R. CÔTE, Instability of nonconstant harmonic maps for the  $(1 + 2)$ -dimensional equivariant wave map system, *Internat. Math. Res. Not.* (2005), 3525–3549. MR 2006k:58023 Zbl 1101.35055
- [12] J. EELLS, JR. and J. H. SAMPSON, Harmonic mappings of Riemannian manifolds, *Amer. J. Math.* **86** (1964), 109–160. MR 29 #1603 Zbl 0122.40102
- [13] R. T. GLASSEY, On the blowing up of solutions to the Cauchy problem for nonlinear Schrödinger equations, *J. Math. Phys.* **18** (1977), 1794–1797. MR 57 #842 Zbl 0372.35009
- [14] J. F. GROTHOWSKI and J. SHATAH, Geometric evolution equations in critical dimensions, *Calc. Var. Partial Differential Equations* **30** (2007), 499–512. MR 2008e:53122 Zbl 1127.58003
- [15] F. HÉLEIN, *Constant Mean Curvature Surfaces, Harmonic Maps and Integrable Systems, Lectures in Mathematics ETH Zürich*, Birkhäuser, Basel, 2001, Notes taken by Roger Moser. MR 2002f:53009 Zbl 1158.53301
- [16] F. HÉLEIN, *Harmonic Maps, Conservation Laws and Moving Frames*, second ed., *Cambridge Tracts in Mathematics* **150**, Cambridge Univ. Press, Cambridge, 2002. MR 2003g:58024 Zbl 1010.58010
- [17] J. ISENBERG and S. L. LIEBLING, Singularity formation in  $2 + 1$  wave maps, *J. Math. Phys.* **43** (2002), 678–683. MR 2002i:58034 Zbl 1052.58032
- [18] J. JOST, *Riemannian Geometry and Geometric Analysis*, fourth ed., *Universitext*, Springer-Verlag, New York, 2005. MR 2006c:53002 Zbl 1083.53001
- [19] S. KLAINERMAN and S. SELBERG, Remark on the optimal regularity for equations of wave maps type, *Comm. Partial Differential Equations* **22** (1997), 901–918. MR 99c:35163 Zbl 0884.35102
- [20] J. KRIEGER and W. SCHLAG, On the focusing critical semi-linear wave equation, *Amer. J. Math.* **129** (2007), 843–913. MR 2009f:35231 Zbl 05170329
- [21] J. KRIEGER, Global regularity of wave maps from  $\mathbf{R}^{2+1}$  to  $H^2$ . Small energy, *Comm. Math. Phys.* **250** (2004), 507–580. MR 2005f:58049 Zbl 1099.58010
- [22] ———, Stability of spherically symmetric wave maps, *Mem. Amer. Math. Soc.* **181** (2006), viii+80. MR 2008j:35109 Zbl 05030283
- [23] R. A. LEESE, M. PEYRARD, and W. J. ZAKRZEWSKI, Soliton stability in the  $O(3)$   $\sigma$ -model in  $(2 + 1)$  dimensions, *Nonlinearity* **3** (1990), 387–412. MR 91i:58033 Zbl 0721.35068
- [24] H. A. LEVINE, Instability and nonexistence of global solutions to nonlinear wave equations of the form  $Pu_{tt} = -Au + \mathcal{F}(u)$ , *Trans. Amer. Math. Soc.* **192** (1974), 1–21. MR 49 #9436 Zbl 0288.35003
- [25] J. M. LINHART and L. A. SADUN, Fast and slow blowup in the  $S^2$   $\sigma$ -model and the  $(4 + 1)$ -dimensional Yang-Mills model, *Nonlinearity* **15** (2002), 219–238. MR 2002m:81156 Zbl 1006.81052



- [26] N. MANTON and P. SUTCLIFFE, *Topological Solitons, Cambridge Monogr. Math. Phys.*, Cambridge Univ. Press, Cambridge, 2004. MR 2006d:58020 Zbl 1100.37044
- [27] Y. MARTEL and F. MERLE, Blow up in finite time and dynamics of blow up solutions for the  $L^2$ -critical generalized KdV equation, *J. Amer. Math. Soc.* **15** (2002), 617–664. MR 2003c:35142 Zbl 0996.35064
- [28] F. MERLE and P. RAPHAEL, The blow-up dynamic and upper bound on the blow-up rate for critical nonlinear Schrödinger equation, *Ann. of Math.* **161** (2005), 157–222. MR 2006k:35277
- [29] B. PIETTE and W. J. ZAKRZEWSKI, Shrinking of solitons in the  $(2 + 1)$ -dimensional  $S^2$  sigma model, *Nonlinearity* **9** (1996), 897–910. MR 97g:81071 Zbl 0895.58030
- [30] J. SHATAH and A. TAHVILDAR-ZADEH, Regularity of harmonic maps from the Minkowski space into rotationally symmetric manifolds, *Comm. Pure Appl. Math.* **45** (1992), 947–971. MR 93c:58056 Zbl 0769.58015
- [31] J. SHATAH, Weak solutions and development of singularities of the  $SU(2)$   $\sigma$ -model, *Comm. Pure Appl. Math.* **41** (1988), 459–469. MR 89f:58044 Zbl 0686.35081
- [32] J. SHATAH and A. S. TAHVILDAR-ZADEH, On the Cauchy problem for equivariant wave maps, *Comm. Pure Appl. Math.* **47** (1994), 719–754. MR 96c:58049 Zbl 0811.58059
- [33] M. STRUWE, Radially symmetric wave maps from  $(1 + 2)$ -dimensional Minkowski space to the sphere, *Math. Z.* **242** (2002), 407–414. MR 2004d:58040 Zbl 1052.58034
- [34] ———, Equivariant wave maps in two space dimensions, *Comm. Pure Appl. Math.* **56** (2003), 815–823. MR 2004c:58061 Zbl 1033.53019
- [35] T. TAO, Global regularity of wave maps. II. Small energy in two dimensions, *Comm. Math. Phys.* **224** (2001), 443–544. MR 2002h:58052 Zbl 1020.35046
- [36] D. TATARU, On global existence and scattering for the wave maps equation, *Amer. J. Math.* **123** (2001), 37–77. MR 2002c:58045 Zbl 0979.35100
- [37] ———, Rough solutions for the wave maps equation, *Amer. J. Math.* **127** (2005), 293–377. MR 2006a:58034 Zbl 02164530
- [38] M. I. WEINSTEIN, Modulational stability of ground states of nonlinear Schrödinger equations, *SIAM J. Math. Anal.* **16** (1985), 472–491. MR 86i:35130 Zbl 0583.35028

(Received April 28, 2006)

(Revised July 15, 2008)

*E-mail address:* irod@math.princeton.edu

DEPARTMENT OF MATHEMATICS, FINE HALL, WASHINGTON ROAD, PRINCETON UNIVERSITY,  
PRINCETON, NJ 08544-1000, UNITED STATES

*E-mail address:* jsterben@math.ucsd.edu

UNIVERSITY OF CALIFORNIA AT SAN DIEGO, DEPARTMENT OF MATHEMATICS,  
9500 GILMAN DRIVE, DEPT. 0112, LA JOLLA, 92093-0112, UNITED STATES

<http://www.math.ucsd.edu/~jsterben/>

This periodical is published bimonthly by the Department of Mathematics at Princeton University with the cooperation of the Institute for Advanced Study. *Annals* is typeset in  $\text{T}_{\text{E}}\text{X}$  by Sarah R. Warren and produced by Mathematical Sciences Publishers. The six numbers each year are divided into two volumes of three numbers each.

#### **Editorial correspondence**

Papers submitted for publication and editorial correspondence should be addressed to Maureen Schupsky, *Annals of Mathematics*, Fine Hall-Washington Road, Princeton University, Princeton, NJ, 08544-1000 U.S.A. The e-mail address is [annals@math.princeton.edu](mailto:annals@math.princeton.edu).

#### **Preparing and submitting papers**

The *Annals* requests that all papers include an abstract of about 150 words which explains to the nonspecialist mathematician what the paper is about. It should not make any reference to the bibliography. Authors are encouraged to initially submit their papers electronically and in PDF format. Please send the file to: [annals@math.princeton.edu](mailto:annals@math.princeton.edu) or to the Mathematics e-print arXiv: [front.math.ucdavis.edu/submissions](http://front.math.ucdavis.edu/submissions). If a paper is submitted through the arXiv, then please e-mail us with the arXiv number of the paper.

#### **Proofs**

A PDF file of the galley proof will be sent to the corresponding author for correction. If requested, a paper copy will also be sent to the author.

#### **Offprints**

Authors of single-authored papers will receive 30 offprints. (Authors of papers with one co-author will receive 15 offprints, and authors of papers with two or more co-authors will receive 10 offprints.) Extra offprints may be purchased through the editorial office.

#### **Subscriptions**

The price for a print and online subscription, or an online-only subscription, is \$390 per year for institutions. In addition, there is a postage surcharge of \$40 for print subscriptions that are mailed to countries outside of the United States. Individuals interested in subscriptions for their own personal use should contact the publisher at the address below. Subscriptions and changes of address should be sent to Mathematical Sciences Publishers, Department of Mathematics, University of California, Berkeley, CA 94720-3840 (e-mail: [contact@mathscipub.org](mailto:contact@mathscipub.org); phone: 1-510-643-8638; fax: 1-510-295-2608). (Checks should be made payable to "Mathematical Sciences Publishers".)

#### **Back issues and reprints**

Orders for missing issues and back issues should be sent to Mathematical Sciences Publishers at the above address. Claims for missing issues must be made within 12 months of the publication date. Online versions of papers published five or more years ago are available through JSTOR ([www.jstor.org](http://www.jstor.org)).

#### **Microfilm**

Beginning with Volume 1, microfilm may be purchased from NA Publishing, Inc., 4750 Venture Drive, Suite 400, PO Box 998, Ann Arbor, MI 48106-0998; phone: 1-800-420-6272 or 734-302-6500; email: [info@napubco.com](mailto:info@napubco.com), website: [www.napubco.com/contact.html](http://www.napubco.com/contact.html).

ALL RIGHTS RESERVED UNDER THE BERNE CONVENTION AND  
THE UNIVERSAL COPYRIGHT CONVENTION

## TABLE OF CONTENTS

DENIS OSIN. Small cancellations over relatively hyperbolic groups and embedding theorems .....	1–39
HARUZO HIDA. The Iwasawa $\mu$ -invariant of $p$ -adic Hecke $L$ -functions.....	41–137
IRENE I. BOUW and MARTIN MÖLLER. Teichmüller curves, triangle groups, and Lyapunov exponents .....	139–185
IGOR RODNIANSKI and JACOB STERBENZ. On the formation of singularities in the critical $O(3)$ $\sigma$ -model .....	187–242
YUAN-PIN LEE, HUI-WEN LIN and CHIN-LUNG WANG. Flops, motives, and invariance of quantum rings .....	243–290
LÁSZLÓ ERDŐS, BENJAMIN SCHLEIN and HORNG-TZER YAU. Derivation of the Gross-Pitaevskii equation for the dynamics of Bose-Einstein condensate .....	291–370
L. HAKAN ELIASSON and SERGEI B. KUKSIN. KAM for the nonlinear Schrödinger equation .....	371–435
MIRIAM BRISKIN, NINA ROYTVARF and YOSEF YOMDIN. Center conditions at infinity for Abel differential equations.....	437–483
IGOR KRICHEVER. Characterizing Jacobians via trisecants of the Kummer variety.....	485–516
MARC BURGER, ALESSANDRA IOZZI and ANNA WIENHARD. Surface group representations with maximal Toledo invariant .....	517–566
TIM DOKCHITSER and VLADIMIR DOKCHITSER. On the Birch-Swinnerton-Dyer quotients modulo squares .....	567–596
DMITRI BELIAEV and STANISLAV SMIRNOV. Random conformal snowflakes.....	597–615
SOURAV CHATTERJEE, RON PELED, YUVAL PERES and DAN ROMIK. Gravitational allocation to Poisson points.....	617–671
RADU LAZA. The moduli space of cubic fourfolds via the period map.....	673–711
NARUTAKA OZAWA and SORIN POPA. On a class of $\mathrm{II}_1$ factors with at most one Cartan subalgebra.....	713–749
MINHYONG KIM. $p$ -adic $L$ -functions and Selmer varieties associated to elliptic curves with complex multiplication .....	751–759