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Abstract

Let k be any global field of characteristic not 2. We construct a k-variety X such that $X(k)$ is empty, but for which the emptiness cannot be explained by the Brauer-Manin obstruction or even by the Brauer-Manin obstruction applied to finite étale covers.

1. Introduction

1.1. *Background.* Call a variety *nice* if it is smooth, projective, and geometrically integral. (See §§2 and 3 for further terminology used here.) Let X be a nice variety over a global field k. If X has a k-point, then X has a k_v-point for every place v of k; i.e., the set $X(\mathbb{A})$ of adelic points is nonempty. The converse, known as the *Hasse principle*, does not always hold, as has been known at least since the 1940s: it can fail for genus-1 curves, for instance [Lin40], [Rei42]. Manin [Man71] showed that the Brauer group of X can often explain failures of the Hasse principle: one can define a subset $X(\mathbb{A})^{\mathrm{Br}}$ of $X(\mathbb{A})$ that contains $X(k)$, and $X(\mathbb{A})^{\mathrm{Br}}$ can be empty even when $X(\mathbb{A})$ is nonempty.

Conditional results [SW95], [Poo01] predicted that this Brauer-Manin obstruction was insufficient to explain all failures of the Hasse principle. But the insufficiency was proved only in 1999, when a ground-breaking paper of Skorobogatov [Sko99] constructed a variety for which one could prove $X(\mathbb{A})^{\mathrm{Br}} \neq \varnothing$ and $X(k) = \varnothing$. He showed that for a bielliptic surface X, the set $X(\mathbb{A})^{\mathrm{et,Br}}$ obtained by applying the Brauer-Manin obstruction to finite étale covers of X could be empty even when $X(\mathbb{A})^{\mathrm{Br}}$ was not.

1.2. *Our result.* We give a construction to show that even this combination of finite étale descent and the Brauer-Manin obstruction is insufficient to explain all failures of the Hasse principle. Combining our result with a result announced

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in [Dem09] shows that even the general "descent obstruction" cannot explain the nonexistence of rational points on our examples; see Remark 8.3. Our argument does not use [Sko99], so it also gives a new approach to constructing varieties for which the Brauer-Manin obstruction is insufficient to explain the failure of the Hasse principle.

The idea behind our construction can be described in a few lines, though the details will occupy the rest of the paper. Start with a nice curve C such that $C(k)$ is finite and nonempty. Construct a nice k-variety X with a morphism $\beta\colon X \to C$ such that

(i) For each $c \in C(k)$, the fiber $X_c := \beta^{-1}(c)$ violates the Hasse principle.
(ii) Every finite étale cover of X arises from a finite étale cover of C.
(iii) The map β induces an isomorphism $\operatorname{Br} C \xrightarrow{\sim} \operatorname{Br} X$, and this remains true after base extension by any finite étale morphism $C' \to C$.

Properties (ii) and (iii) imply that $X(\mathbb{A})^{\text{et,Br}}$ is the inverse image under β of $C(\mathbb{A})^{\text{et,Br}}$, which contains the nonempty set $C(k)$. Then by (i), we have $X(\mathbb{A})^{\text{et,Br}} \neq \varnothing$ but $X(k) = \varnothing$.

Our X will be a 3-fold, and the general fiber of β will be a Châtelet surface (a kind of conic bundle over $\mathbb{P}^1$).

1.3. *Commentary.* Suppose, in addition, that the Jacobian J of C is such that the Mordell-Weil group $J(k)$ and the Shafarevich-Tate group $\text{Ш}(J)$ are both finite. (For instance, these hypotheses are known to hold if C is any elliptic curve over $\mathbb{Q}$ of analytic rank 0.) Then $C(\mathbb{A})^{\text{Br}}$ is essentially (ignoring some technicalities regarding the connected components at archimedean places) equal to $C(k)$. (Scharaschkin and Skorobogatov independently observed that this follows from the comparison of the Cassels-Tate pairing with the Brauer evaluation pairing in [Man71]; see [Sko01, §6.2] for related results, and [Sto07, Th. 8.6] for a significant generalization.) Thus $X(\mathbb{A})^{\text{Br}}$ is essentially a subset of $\bigcup_{c \in C(k)} X_c(\mathbb{A})$.

Also, $X_c(\mathbb{A})^{\text{Br}} = \varnothing$ for each $c \in C(k)$ (all failures of the Hasse principle for Châtelet surfaces are explained by the Brauer-Manin obstruction [CTSSD87a], [CTSSD87b]). But the elements of $\operatorname{Br} X_c$ used to obstruct k-points on the fiber X_c do not extend to elements of $\operatorname{Br} X$, so it does not follow that $X(\mathbb{A})^{\text{Br}}$ is empty.

1.4. *Outline of the paper.* Section 2 introduces some basic notation. Section 3 recalls some cohomological obstructions to rational points, and discusses how they relate to one another. Our X, a Châtelet surface bundle over C, will be constructed as a conic bundle over $C \times \mathbb{P}^1$; Section 4 describes the type of conic bundle we need, and Section 5 computes the Brauer group of this conic bundle. The Brauer group calculations involve some group cohomology lemmas, which have been relegated to an appendix. Section 6 constructs the particular X, and Sections 7 and 8 compute $X(\mathbb{A})^{\text{Br}}$ and $X(\mathbb{A})^{\text{et,Br}}$, respectively.

2. Notation

Given a field k, we fix a separable closure $\bar{k}$ of k and define $G_k := \operatorname{Gal}(\bar{k}/k)$. For any k-variety V, define $\overline{V} := V \times_k \bar{k}$. For any integral variety V, let $\kappa(V)$ be the function field. If D is a divisor on a nice variety V, let $[D]$ be its class in $\operatorname{Pic} V$.

An *algebraic group* over k is a smooth group scheme of finite type over k. Suppose that G is an algebraic group over k and X is a k-variety. Let $H^1(X, G)$ be the cohomology set defined using Čech 1-cocycles for the étale topology. There is an injection

$$\{\text{isomorphism classes of torsors over } X \text{ under } G\} \hookrightarrow H^1(X, G);$$

descent theory shows that this is a bijection, at least if G is affine. If G is commutative, then for any $i \in \mathbb{Z}_{\geq 0}$ define $H^i(X, G)$ as the usual étale cohomology group; this is compatible with the Čech cocycle definition when $i = 1$. Let $\operatorname{Br} X$ be the cohomological Brauer group $H^2(X, \mathbb{G}_m)$.

By a *global field* we mean either a finite extension of $\mathbb{Q}$ or the function field of a nice curve over a finite field. If k is a global field, let $\mathbb{A}$ be its adèle ring.

3. Cohomological obstructions to rational points

Let k be a global field. Let X be a k-variety.

3.1. *Brauer-Manin obstruction.* (See [Sko01, §5.2]; there it is assumed that k is a number field and X is smooth and geometrically integral, but the definitions and statements we use in this section do not require these extra hypotheses.) There is an evaluation pairing

$$\operatorname{Br} X \times X(\mathbb{A}) \to \mathbb{Q}/\mathbb{Z},$$

and $X(\mathbb{A})^{\mathrm{Br}}$ is defined as the set of elements of $X(\mathbb{A})$ that pair with every element of $\operatorname{Br} X$ to give 0. The reciprocity law for $\operatorname{Br} k$ implies $X(k) \subseteq X(\mathbb{A})^{\mathrm{Br}}$. In particular, if $X(\mathbb{A}) \neq \varnothing$ but $X(\mathbb{A})^{\mathrm{Br}} = \varnothing$, then X violates the Hasse principle.

3.2. *Descent obstruction.* (See [Sko01, §5.3].) If G is a (not necessarily connected) linear algebraic group over k, and $f\colon Y \to X$ is a right torsor under G, then any 1-cocycle $\sigma \in Z^1(k, G)$ gives rise to a "twisted" right torsor $f^\sigma\colon Y^\sigma \to X$ under a twisted form G^σ of G. Moreover, the isomorphism type of the torsor depends only on the cohomology class $[\sigma] \in H^1(k, G)$. It is not hard to show that

$$X(k) = \bigcup_{[\sigma] \in H^1(k,G)} f^\sigma(Y^\sigma(k)). \tag{1}$$

Therefore $X(k)$ is contained in the set

$$X(\mathbb{A})^f := \bigcup_{[\sigma] \in H^1(k,G)} f^\sigma(Y^\sigma(\mathbb{A})).$$

Define

$$X(\mathbb{A})^{\text{descent}} := \bigcap X(\mathbb{A})^{f}$$

where the intersection is taken over all linear algebraic groups G and all right torsors $Y \to X$ under G. If $X(\mathbb{A})^{\text{descent}} = \varnothing$, then we say that there is a *descent obstruction* to the existence of a rational point.

3.3. *Brauer-Manin obstruction applied to étale covers.* For reasons that will be clearer in Section 3.4, it is interesting to combine descent for torsors under finite étale group schemes with the Brauer-Manin obstruction. Define

$$X(\mathbb{A})^{\text{et,Br}} := \bigcap_{\substack{G \text{ finite} \\ f: Y \to X}} \ \bigcup_{[\sigma] \in H^1(k,G)} f^{\sigma}(Y^{\sigma}(\mathbb{A})^{\text{Br}}),$$

where the intersection is taken over all finite étale group schemes G over k and all right torsors $f: Y \to X$ under G. We have $X(k) \subseteq X(\mathbb{A})^{\text{et,Br}} \subseteq X(\mathbb{A})^{\text{Br}}$, where the first inclusion follows from (1), and the second follows from taking $G = \{1\}$ and $Y = X$ in the definition of $X(\mathbb{A})^{\text{et,Br}}$.

3.4. *Comparisons.* Let $X(\mathbb{A})^{\text{connected}}$ be defined in the same way as $X(\mathbb{A})^{\text{descent}}$, but using only *connected* linear algebraic groups instead of all linear algebraic groups. Harari [Har02, Th. 2(2)] showed that $X(\mathbb{A})^{\text{Br}} \subseteq X(\mathbb{A})^{\text{connected}}$ for any geometrically integral variety X over a number field k. In other words, the Brauer-Manin obstruction is strong enough to subsume all descent obstructions from connected linear algebraic groups. Also, an arbitrary linear algebraic group is an extension of a finite étale group scheme by a connected linear algebraic group, so one might ask:

Question 3.1. Does $X(\mathbb{A})^{\text{et,Br}} \subseteq X(\mathbb{A})^{\text{descent}}$ hold for every nice variety X over a number field?

This does not seem to follow formally from Harari's result. But, in response to an early draft of this paper, Demarche has announced a positive answer [Dem09], and Skorobogatov has proved the opposite inclusion [Sko09, Cor. 1.2] by generalizing the proof of [Sto07, Prop. 5.17]. Combining these results shows that $X(\mathbb{A})^{\text{et,Br}} = X(\mathbb{A})^{\text{descent}}$ for any nice variety X over a number field.

4. **Conic bundles**

In this section, k is any field of characteristic not 2. Let B be a nice k-variety. Let $\mathscr{L}$ be a line sheaf on B. Let $\mathscr{E}$ be the rank-3 vector sheaf $\mathscr{O} \oplus \mathscr{O} \oplus \mathscr{L}$ on B. Let $a \in k^{\times}$ and let $s \in \Gamma(B, \mathscr{L}^{\otimes 2})$ be a nonzero global section. The zero locus of

$$1 \oplus (-a) \oplus (-s) \in \Gamma(B, \mathscr{O} \oplus \mathscr{O} \oplus \mathscr{L}^{\otimes 2}) \subseteq \Gamma(B, \operatorname{Sym}^2 \mathscr{E})$$

in $\mathbb{P}\mathscr{E}$ is a projective geometrically integral scheme X with a morphism $\alpha\colon X \to B$. If U is a dense open subscheme of B with a trivialization $\mathscr{L}|_U \simeq \mathbb{O}_U$ and we identify $s|_U$ with an element of $\Gamma(U, \mathbb{O}_U)$, then the affine scheme defined by $y^2 - az^2 = s|_U$ in $\mathbb{A}^2_U$ is a dense open subscheme of X. Therefore we call X the *conic bundle given by* $y^2 - az^2 = s$. In the special case where $B = \mathbb{P}^1$, $\mathscr{L} = \mathbb{O}(2)$, and the homogeneous form $s \in \Gamma(\mathbb{P}^1, \mathbb{O}(4))$ is separable, X is called the *Châtelet surface given by* $y^2 - az^2 = s(x)$, where $s(x) \in k[x]$ denotes a dehomogenization of s.

Returning to the general case, we let Z be the subscheme $s = 0$ of B. Call Z the *degeneracy locus* of the conic bundle. Each fiber of α above a point of $B - Z$ is a smooth plane conic, and each fiber above a geometric point of Z is a union of two projective lines crossing transversely at a point. A local calculation shows that if Z is smooth over k, then X is smooth over k.

LEMMA 4.1. *The generic fiber $\overline{X}_\eta$ of $\overline{X} \to \overline{B}$ is isomorphic to $\mathbb{P}^1_{\kappa(\overline{B})}$.*

Proof. It is a smooth plane conic, and it has a rational point since a is a square in $\bar{k} \subseteq \kappa(\overline{B})$. □

5. Brauer group of conic bundles

The calculations of this section are similar to well-known calculations that have been done for conic bundles over $\mathbb{P}^1$; see [Sko01, §7.1], for instance.

LEMMA 5.1. *Let $X \to B$ be as in Section 4. If the degeneracy locus Z is nice, then the homomorphism $H^1(k, \operatorname{Pic} \overline{B}) \xrightarrow{\alpha^*} H^1(k, \operatorname{Pic} \overline{X})$ is an isomorphism.*

Proof. We compute $\operatorname{Pic} \overline{X}$ in the following paragraphs by constructing a commutative diagram of G_k-modules

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{Z} \arrow[r, "\lambda_1"] \arrow[d, equal] & \widetilde{\mathbb{Z}}^2 \arrow[r, "\lambda_2"] \arrow[d, "\tau_1"] & \widetilde{\mathbb{Z}}^2 \arrow[r, "\lambda_3"] \arrow[d, "\tau_2"] & \mathbb{Z} \arrow[r] \arrow[d, equal, "\deg"'] & 0 \\
0 \arrow[r] & \mathbb{Z} \arrow[r, "\rho_1"] & \operatorname{Pic} \overline{B} \oplus \widetilde{\mathbb{Z}}^2 \arrow[r, "\rho_2"] & \operatorname{Pic} \overline{X} \arrow[r, "\rho_3"] & \operatorname{Pic} \overline{X}_\eta \arrow[r] & 0
\end{tikzcd} \tag{2}$$

with exact rows.

Let $\widetilde{\mathbb{Z}}^2$ be the induced module $\operatorname{Ind}_{G_{k(\sqrt{a})}}^{G_k} \mathbb{Z}$: as a group it is $\mathbb{Z}^2$, and an element $\sigma \in G_k$ acts on an element of it either trivially or by interchanging the coordinates, according to whether σ fixes $\sqrt{a}$ or not.

Call a divisor of $\overline{X}$ *vertical* if it is supported on prime divisors lying above prime divisors of $\overline{B}$, and *horizontal* otherwise. The fiber of α above the generic point of $\overline{Z}$ consists of two intersecting copies of $\mathbb{P}^1_{\kappa(\overline{Z})}$, so $\alpha^{-1}(\overline{Z})$ is a union of two prime divisors F_1 and F_2 of $\overline{X}$.

Choose $L \in \operatorname{Div} B$ with $[L] = \mathscr{L}$. Since Z is the zero locus of $s \in \Gamma(B, \mathscr{L}^{\otimes 2})$, the divisor $Z - 2L$ is the divisor of some function $g \in \kappa(B)^\times$. Let $U := B - \operatorname{supp}(L)$.

Then X has an open subscheme X' given by $y^2 - a = gt^2$ in the affine space $\mathbb{A}^2_U$ with coordinates t and y. The restrictions of F_1 and F_2 in $\operatorname{Div} X'$ are given by $y - \sqrt{a} = g = 0$ and $y + \sqrt{a} = g = 0$; we may assume that the former is F_1. The Zariski closures in $\overline{X}$ of the divisors given by $y - \sqrt{a} = t = 0$ and $y + \sqrt{a} = t = 0$ are horizontal; call them H_1 and H_2. We choose a function $f \in \kappa(\overline{X})^\times$ that on the generic fiber induces an isomorphism $\overline{X}_\eta \to \mathbb{P}^1_{\kappa(\overline{B})}$ (the usual parametrization of a conic); explicitly, we take

$$f := \frac{y - \sqrt{a}}{t} = \frac{gt}{y + \sqrt{a}}.$$

A straightforward calculation shows that the divisor of f on $\overline{X}$ is

$$(f) = H_1 - H_2 + F_1 - \alpha^* L. \tag{3}$$

Bottom row: Define ρ_1 by $\rho_1(1) = (-2\mathscr{L}, (1,1))$. Define $\rho_2(\mathscr{M}, (m,n)) = \alpha^*\mathscr{M} + m[F_1] + n[F_2]$. Let ρ_3 be restriction. Each ρ_i is G_k-equivariant. Given a prime divisor D on $\overline{X}_\eta$, its Zariski closure in $\overline{X}$ restricts to give D on $\overline{X}_\eta$, so ρ_3 is surjective. The kernel of ρ_3 is generated by the classes of vertical prime divisors of $\overline{X}$; in fact, there is exactly one above each prime divisor of $\overline{B}$ except that above $\overline{Z} \in \operatorname{Div} \overline{B}$ we have $F_1, F_2 \in \operatorname{Div} \overline{X}$. This proves exactness at $\operatorname{Pic} \overline{X}$ of the bottom row. Since $s \in \Gamma(B, \mathscr{L}^{\otimes 2})$, we have $[Z] = 2\mathscr{L}$, and $[F_1] + [F_2] = \alpha^*[Z] = 2\alpha^*\mathscr{L}$. Also, a rational function on $\overline{X}$ with vertical divisor must be the pullback of a rational function on $\overline{B}$. The previous two sentences prove exactness at $\operatorname{Pic} \overline{B} \oplus \widetilde{\mathbb{Z}}^2$. Injectivity of ρ_1 is trivial, so this completes the proof that the bottom row of (2) is exact.

Top row: Define

$$\begin{aligned}
\lambda_1(m) &= (m,m) \\
\lambda_2(m,n) &= (n-m, m-n) \\
\lambda_3(m,n) &= m+n.
\end{aligned}$$

These maps are G_k-equivariant and they make the top row of (2) exact.

Vertical maps: By Lemma 4.1, we have an isomorphism $\deg\colon \operatorname{Pic} \overline{X}_\eta \simeq \mathbb{Z}$ of G_k-modules; this defines the rightmost vertical map in (2). Define

$$\begin{aligned}
\tau_1(m,n) &= (-(m+n)\mathscr{L}, (m,n)) \\
\tau_2(m,n) &= m[H_1] + n[H_2].
\end{aligned}$$

These too are G_k-equivariant.

Commutativity of the first square is immediate from the definitions. Commutativity of the second square follows from (3). Commutativity of the third square follows since H_1 and H_2 each meet the generic fiber $\overline{X}_\eta$ in a single $\kappa(\overline{B})$-rational point. This completes the construction of (2).

We now take cohomology by applying results of Appendix A. Because of the vertical isomorphisms at the left and right ends of (2), the two rows define the same class $\xi \in H^2(G_k, \mathbb{Z})$. We have $H^0(G_k, \widetilde{\mathbb{Z}}^2) = \mathbb{Z}\cdot(1,1)$, and Shapiro's lemma yields

$$H^1(G_k, \widetilde{\mathbb{Z}}^2) = H^1(G_{k(\sqrt{a})}, \mathbb{Z}) = 0, \tag{4}$$

so Lemma A.1 implies $\xi \neq 0$. We are almost ready to apply Lemma A.2 to the bottom row of (2), but first we must check the splitting hypotheses. After restricting from G_k to $G_{k(\sqrt{a})}$, the injection ρ_1 is split by the projection $\operatorname{Pic} \overline{B} \oplus \mathbb{Z}^2 \to \mathbb{Z}$ onto the last factor, and the surjection ρ_3 is split by the map sending a positive generator of $\operatorname{Pic} \overline{X}_\eta$ to $[H_1] \in \operatorname{Pic} \overline{X}$. Now Lemma A.2 yields an isomorphism

$$H^1(G_k, \operatorname{Pic} \overline{B} \oplus \widetilde{\mathbb{Z}}^2) \to H^1(G_k, \operatorname{Pic} \overline{X})$$

and the first group equals $H^1(G_k, \operatorname{Pic} \overline{B})$ by (4). □

LEMMA 5.2. *If W and Y are nice k-varieties, and W is birational to $Y \times \mathbb{P}^1$, then the homomorphism* $\operatorname{Br} Y \to \operatorname{Br} W$ *induced by the composition* $W \dashrightarrow Y \times \mathbb{P}^1 \to Y$ *is an isomorphism.*

Proof. Use the birational invariance of the Brauer group and the isomorphism $\operatorname{Br}(Y \times \mathbb{P}^1) \simeq \operatorname{Br} Y$. □

LEMMA 5.3. *Let $X \to B$ be as in Section* 4. *If* $\operatorname{Br} \overline{B} = 0$, *then* $\operatorname{Br} \overline{X} = 0$.

Proof. Apply Lemmas 4.1 and 5.2. □

PROPOSITION 5.4. *Let $X \to B$ be as in Section* 4. *Suppose in addition that*

- *k is a global field* (*still of characteristic not* 2),
- *the degeneracy locus Z is nice*,
- $\operatorname{Br} \overline{B} = 0$, *and*
- $X(\mathbb{A}) \neq \varnothing$.

Then $\alpha^*\colon \operatorname{Br} B \to \operatorname{Br} X$ *is an isomorphism.*

Proof. The Hochschild-Serre spectral sequence yields an exact sequence

$$\operatorname{Br} k \to \ker\left(\operatorname{Br} X \to \operatorname{Br} \overline{X}\right) \to H^1(k, \operatorname{Pic} \overline{X}) \to H^3(k, \mathbb{G}_m).$$

Since $\operatorname{Br} k \to \bigoplus_v \operatorname{Br} k_v$ is injective and $X(\mathbb{A}) \neq \varnothing$, the homomorphism $\operatorname{Br} k \to \operatorname{Br} X$ is injective. By Lemma 5.3, we have $\operatorname{Br} \overline{X} = 0$. Finally, $H^3(k, \mathbb{G}_m) = 0$. Thus we obtain a short exact sequence, the second row of

$$\begin{tikzcd}
0 \arrow[r] & \operatorname{Br} k \arrow[r] \arrow[d, equal] & \operatorname{Br} B \arrow[r] \arrow[d] & H^1(k, \operatorname{Pic} \overline{B}) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \operatorname{Br} k \arrow[r] & \operatorname{Br} X \arrow[r] & H^1(k, \operatorname{Pic} \overline{X}) \arrow[r] & 0.
\end{tikzcd}$$

The first row is obtained in the same way, and the vertical maps are induced by α. The result now follows from Lemma 5.1. □

Remark 5.5. In response to an earlier draft of this paper, Colliot-Thélène has found an alternative proof of Proposition 5.4; see [CT09, Prop. 2.1]. This proof, which is a little shorter and works in slightly greater generality, compares $\operatorname{Br} X$ and $\operatorname{Br} B$ using residue maps instead of going through $H^1(k, \operatorname{Pic} \overline{X})$ and $H^1(k, \operatorname{Pic} \overline{B})$.

6. **Construction**

From now on, k is a global field of characteristic not 2. Fix $a \in k^\times$, and fix relatively prime separable degree-4 polynomials $P_\infty(x)$, $P_0(x) \in k[x]$ such that the (nice) Châtelet surface $\mathscr{V}_\infty$ given by

$$y^2 - az^2 = P_\infty(x)$$

over k satisfies $\mathscr{V}_\infty(\mathbb{A}) \neq \varnothing$ but $\mathscr{V}_\infty(k) = \varnothing$. (Such Châtelet surfaces exist over any global field k of characteristic not 2; see [Poo09, Prop. 5.1 and §11]. If $k = \mathbb{Q}$, then one may use the original example from [Isk71], with $a := -1$ and $P_\infty(x) := (x^2-2)(3-x^2)$.) Let $\tilde{P}_\infty(w,x)$ and $\tilde{P}_0(w,x)$ be the homogenizations of P_∞ and P_0. Define $\mathscr{L} := \mathscr{O}(1,2)$ on $\mathbb{P}^1 \times \mathbb{P}^1$ and define

$$s_1 := u^2 \tilde{P}_\infty(w,x) + v^2 \tilde{P}_0(w,x) \in \Gamma(\mathbb{P}^1 \times \mathbb{P}^1, \mathscr{L}^{\otimes 2}),$$

where the two copies of $\mathbb{P}^1$ have homogeneous coordinates (u,v) and (w,x), respectively. Let $Z_1 \subset \mathbb{P}^1 \times \mathbb{P}^1$ be the zero locus of s_1. Let $F \subset \mathbb{P}^1$ be the (finite) branch locus of the first projection $Z_1 \to \mathbb{P}^1$.

Let $\alpha_1 : \mathscr{V} \to \mathbb{P}^1 \times \mathbb{P}^1$ be the conic bundle given by $y^2 - az^2 = s_1$, in the terminology of Section 4. Composing α_1 with the first projection $\mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1$ yields a morphism $\beta_1 : \mathscr{V} \to \mathbb{P}^1$ whose fiber above $\infty := (1:0)$ is the Châtelet surface $\mathscr{V}_\infty$ defined earlier.

Let C be a nice curve over k such that $C(k)$ is finite and nonempty. Choose a dominant morphism $\gamma : C \to \mathbb{P}^1$, étale above F, such that $\gamma(C(k)) = \{\infty\}$. Define the fiber product $X := \mathscr{V} \times_{\mathbb{P}^1} C$ and morphisms α and β as in the diagram

$$\begin{tikzcd}
X \arrow[r] \arrow[d, "\alpha"'] \arrow[dd, bend left, "\beta"] & \mathscr{V} \arrow[d, "\alpha_1"'] \arrow[dd, bend left, "\beta_1"] \\
C \times \mathbb{P}^1 \arrow[r, "(\gamma,1)"] \arrow[d, "1^{\mathrm{st}}"'] & \mathbb{P}^1 \times \mathbb{P}^1 \arrow[d, "1^{\mathrm{st}}"'] \\
C \arrow[r, "\gamma"] & \mathbb{P}^1 .
\end{tikzcd}$$

Each map labeled 1^{st} is the first projection. Define $B := C \times \mathbb{P}^1$ and $s := (\gamma,1)^* s_1 \in \Gamma(B, (\gamma,1)^* \mathscr{O}(2,4))$. Thus $X \stackrel{\alpha}{\to} B$ can alternatively be described as the conic bundle given by $y^2 - az^2 = s$. Its degeneracy locus Z is $(\gamma,1)^* Z_1 \subset B$.

7. No Brauer-Manin obstruction

We continue with the notation of Section 6.

LEMMA 7.1. *The curve Z is nice.*

Proof. Since $P_0(x)$ and $P_\infty(x)$ are separable and have no common factor, a short calculation shows that Z_1 is smooth over k. We have $Z = Z_1 \times_{\mathbb{P}^1} C$. Since Z_1 and C are smooth over k and the branch loci of $Z_1 \to \mathbb{P}^1$ and $C \to \mathbb{P}^1$ do not intersect, Z is smooth too. Since Z_1 is ample on $\mathbb{P}^1 \times \mathbb{P}^1$ and γ is finite, Z is ample on $C \times \mathbb{P}^1$. Therefore, Z is geometrically connected by [Har77, Cor. III.7.9]. Since Z is also smooth, it is geometrically integral. □

Lemma 7.1 and the sentence before Lemma 4.1 imply that the 3-fold X is nice.

THEOREM 7.2. *We have $X(k) = \varnothing$, but $X(\mathbb{A})^{\mathrm{Br}}$ contains $\mathscr{V}_\infty(\mathbb{A}) \times C(k)$ and hence is nonempty.*

Proof. Since $\gamma(C(k)) = \{\infty\}$ and $\mathscr{V}_\infty(k) = \varnothing$, we have $X(k) = \varnothing$. We have $\operatorname{Br} \overline{B} = \operatorname{Br}(\overline{C} \times \overline{\mathbb{P}^1}) = \operatorname{Br} \overline{C} = 0$, by Lemma 5.2 and [Gro68, Cor. 5.8]. Also, $X(\mathbb{A})$ contains $\mathscr{V}_\infty(\mathbb{A}) \times C(k)$, so $X(\mathbb{A}) \neq \varnothing$. Thus Proposition 5.4 implies that $\operatorname{Br} B \to \operatorname{Br} X$ is an isomorphism. Composing with the isomorphism $\operatorname{Br} C \to \operatorname{Br} B$ of Lemma 5.2 shows that $\beta^*\colon \operatorname{Br} C \to \operatorname{Br} X$ is an isomorphism. Hence, if $\beta_{\mathbb{A}}\colon X(\mathbb{A}) \to C(\mathbb{A})$ is the map induced by β, then

$$X(\mathbb{A})^{\mathrm{Br}} = \beta_{\mathbb{A}}^{-1}(C(\mathbb{A})^{\mathrm{Br}}) \supseteq \beta_{\mathbb{A}}^{-1}(C(k)) = \mathscr{V}_\infty(\mathbb{A}) \times C(k). \qquad \square$$

8. No Brauer-Manin obstruction applied to étale covers

We continue with the notation of Section 6; in particular, X is the nice 3-fold defined there. For any variety V, let $\mathbf{Et}(V)$ be the category of finite étale covers of V.

LEMMA 8.1. *The morphism $X \to C$ induces an equivalence of categories $\mathbf{Et}(C) \to \mathbf{Et}(X)$.*

Proof. The geometric fibers of $X \to B = C \times \mathbb{P}^1$ are isomorphic to either $\mathbb{P}^1$ or two copies of $\mathbb{P}^1$ crossing at a point, so they have no nontrivial finite étale covers. Therefore [SGA1, IX.6.8] applies to show that $\mathbf{Et}(C \times \mathbb{P}^1) \to \mathbf{Et}(X)$ is an equivalence of categories. The same argument applies to $\mathbf{Et}(C) \to \mathbf{Et}(C \times \mathbb{P}^1)$. □

THEOREM 8.2. *The set $X(\mathbb{A})^{\mathrm{et,Br}}$ contains $\mathscr{V}_\infty(\mathbb{A}) \times C(k)$ and hence is nonempty.*

Proof. Let G be a finite étale group scheme over k, and let $f\colon Y \to X$ be a right torsor under G. Lemma 8.1 implies that f arises from a right torsor $h\colon \mathscr{C} \to C$

under G. In other words, we have a cartesian square

$$\begin{tikzcd} Y \arrow[r, "f"] \arrow[d, "b"] & X \arrow[d, "\beta"] \\ \mathscr{C} \arrow[r, "h"] & C. \end{tikzcd}$$

For any $\sigma \in Z^1(k, G)$ with $\mathscr{C}^\sigma(k) \neq \varnothing$, the twisted morphism $b^\sigma \colon Y^\sigma \to \mathscr{C}^\sigma$ is just like $\beta \colon X \to C$, since in Section 6 we could have replaced γ with the composition $D \hookrightarrow \mathscr{C}^\sigma \xrightarrow{h^\sigma} C \xrightarrow{\gamma} \mathbb{P}^1$ for any connected component D of $\mathscr{C}^\sigma$ containing a k-point; thus $Y^\sigma(\mathbb{A})^{\mathrm{Br}}$ contains $\mathscr{V}_\infty(\mathbb{A}) \times \mathscr{C}^\sigma(k)$ and $f^\sigma(Y^\sigma(\mathbb{A})^{\mathrm{Br}})$ contains $\mathscr{V}_\infty(\mathbb{A}) \times h(\mathscr{C}^\sigma(k))$. Taking the union over all such σ, and applying the analogue of (1) for $h \colon \mathscr{C} \to C$, we see that

$$\bigcup_{[\sigma] \in H^1(k,G)} f^\sigma(Y^\sigma(\mathbb{A})^{\mathrm{Br}})$$

contains $\mathscr{V}_\infty(\mathbb{A}) \times C(k)$. Finally, intersect over all G and all $f \colon Y \to X$. $\square$

Remark 8.3. Suppose that k is a number field. As mentioned in Section 3.4, Demarche has announced a proof that $X(\mathbb{A})^{\mathrm{et,Br}} \subseteq X(\mathbb{A})^{\mathrm{descent}}$ holds for every nice k-variety X [Dem09]. Assuming this, Theorem 8.2 implies that $X(\mathbb{A})^{\mathrm{descent}}$ is nonempty for our X, and in particular that even the descent obstruction is insufficient to explain all failures of the Hasse principle.

Remark 8.4. It is not true that β induces an isomorphism

$$H^1(C, G) \to H^1(X, G)$$

for every linear algebraic group G. It fails for $G = \mathbb{G}_m$, for instance, as pointed out to me by Colliot-Thélène: the composition $\operatorname{Pic} C \to \operatorname{Pic} X \to \operatorname{Pic} X_\eta \simeq \mathbb{Z}$ is zero but $\operatorname{Pic} X \to \operatorname{Pic} X_\eta$ is nonzero. So the proof of Theorem 8.2 does not directly generalize to prove $X(\mathbb{A})^{\mathrm{descent}} \neq \varnothing$.

Remark 8.5. In [CT99] it is conjectured that for every nice variety over a number field, the Brauer-Manin obstruction is the only obstruction to the existence of a zero-cycle of degree 1. In response to an early draft of this paper, Colliot-Thélène has verified this conjecture for the 3-folds X we constructed [CT09, Th. 3.1].

Appendix A. Group cohomology

In this section, G is a profinite group and

(5) $$0 \to \mathbb{Z} \to A \xrightarrow{\phi} B \to \mathbb{Z} \to 0$$

is an exact sequence of discrete G-modules, with G acting trivially on each copy of $\mathbb{Z}$. Let $C = \phi(A)$. Split (5) into short exact sequences

$$0 \to \mathbb{Z} \to A \to C \to 0$$
$$0 \to C \to B \to \mathbb{Z} \to 0.$$

Take cohomology and use $H^1(G, \mathbb{Z}) = \mathrm{Hom}_{\mathrm{conts}}(G, \mathbb{Z}) = 0$ to obtain exact sequences

$$0 \to H^1(G, A) \to H^1(G, C) \xrightarrow{\delta_2} H^2(G, \mathbb{Z}) \tag{6}$$
$$H^0(G, B) \to \mathbb{Z} \xrightarrow{\delta_1} H^1(G, C) \to H^1(G, B) \to 0.$$

Let $\xi = \delta_2(\delta_1(1)) \in H^2(G, \mathbb{Z})$. (Thus ξ is the class of the 2-extension (5).)

LEMMA A.1. *If $H^0(G, B) \to \mathbb{Z}$ is not surjective and $H^1(G, A) = 0$, then $\xi \neq 0$.*

Proof. Nonsurjectivity of $H^0(G, B) \to \mathbb{Z}$ implies $\delta_1(1) \neq 0$, and $H^1(G, A) = 0$ implies that δ_2 is injective. Thus $\xi \neq 0$. □

LEMMA A.2. *If $\xi \neq 0$, and the injection $\mathbb{Z} \to A$ and the surjection $B \to \mathbb{Z}$ are both split after restriction to an index-2 open subgroup H of G, then the homomorphism $H^1(G, A) \to H^1(G, B)$ induced by ϕ is an isomorphism.*

Proof. Let I be the image of δ_2 in $H^2(G, \mathbb{Z}) \simeq \mathrm{Hom}_{\mathrm{conts}}(G, \mathbb{Q}/\mathbb{Z})$. Let J be the image of δ_1. The splitting of the injection implies that

$$I \subseteq \ker\left(\mathrm{Hom}_{\mathrm{conts}}(G, \mathbb{Q}/\mathbb{Z}) \to \mathrm{Hom}_{\mathrm{conts}}(H, \mathbb{Q}/\mathbb{Z})\right) \simeq \mathrm{Hom}(G/H, \mathbb{Q}/\mathbb{Z}),$$

so $\#I \leq 2$. The splitting of the surjection implies that $2J = 0$, but J is cyclic, so $\#J \leq 2$. Since $\xi \neq 0$, the composition $\mathbb{Z} \xrightarrow{\delta_1} H^1(G, C) \xrightarrow{\delta_2} H^2(G, \mathbb{Z})$ is nonzero, so the induced map $J \to I$ is nonzero. Therefore $\#I = \#J = 2$ and $J \to I$ is an isomorphism. In particular, $H^1(G, C) \simeq H^1(G, A) \oplus J$, and (6) then yields

$$0 \to J \to H^1(G, A) \oplus J \to H^1(G, B) \to 0,$$

with J mapping identically to $0 \oplus J$. Thus the map $H^1(G, A) \to H^1(G, B)$ is an isomorphism. □

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