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Some adjoints in homotopy categories

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Dedicated to Max Karoubi, who started me on this project by encouraging me to read the remarkable article [8]

Abstract

Let R be a ring. In a previous paper [11] we found a new description for the category $\mathbf{K}(R\text{-Proj})$; it is equivalent to the Verdier quotient $\mathbf{K}(R\text{-Flat})/\mathcal{S}$, for some suitable $\mathcal{S} \subset \mathbf{K}(R\text{-Flat})$. In this article we show that the quotient map from $\mathbf{K}(R\text{-Flat})$ to $\mathbf{K}(R\text{-Flat})/\mathcal{S}$ always has a right adjoint. This gives a new, fully faithful embedding of $\mathbf{K}(R\text{-Proj})$ into $\mathbf{K}(R\text{-Flat})$. Its virtue is that it generalizes to nonaffine schemes.

0. **Introduction**

Let R be a ring, let $\mathbf{K}(R\text{-Flat})$ be the homotopy category of cochain complexes of flat R-modules, and let $\mathbf{K}(R\text{-Proj})$ be the homotopy category of cochain complexes of projective R-modules. In [11] we found a novel, new description of $\mathbf{K}(R\text{-Proj})$; it is equivalent to the Verdier quotient $\mathbf{K}(R\text{-Flat})/\mathcal{S}$, where the full subcategory $\mathcal{S} \subset \mathbf{K}(R\text{-Flat})$ has many equivalent characterizations, six of which may be found in [11, Fact 2.14].

Let j^* be the Verdier quotient map from $\mathbf{K}(R\text{-Flat})$ to the category $\mathbf{K}(R\text{-Proj}) \cong \mathbf{K}(R\text{-Flat})/\mathcal{S}$. By the construction of [11] it is clear that j^* has a left adjoint $j_! : \mathbf{K}(R\text{-Proj}) \longrightarrow \mathbf{K}(R\text{-Flat})$; the functor $j_!$ is nothing other than the obvious inclusion. The key theorem of this article says:

THEOREM 0.1. *The functor j^* also has a* right *adjoint* $j_* : \mathbf{K}(R\text{-Proj}) \longrightarrow \mathbf{K}(R\text{-Flat})$.

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Remark 0.2. Several comments are in order.

(i) The functor j_* is fully faithful; any adjoint, right or left, of a Verdier quotient map must be fully faithful. It follows that j_* gives a nonobvious embedding of $\mathbf{K}(R\text{-Proj})$ into $\mathbf{K}(R\text{-Flat})$.

(ii) It so happens that the right adjoint j_* can be generalized to nonaffine schemes, while the left adjoint $j_!$ does not exist in general; $j_!$ is an affine phenomenon, which seems to depend on the existence of enough projectives. In the introduction to [11] we explained that the motivation, for studying the new description of $\mathbf{K}(R\text{-Proj})$, came from the idea that it might generalize to the global, nonaffine framework. This program was carried out in Daniel Murfet's Ph.D. thesis. It turns out that the existence of the adjoint j_* is key to his approach; it is the main tool which allows him to reduce global problems to affine ones.

(iii) Special cases of Theorem 0.1 were known. If R is left noetherian and right coherent, and if a dualizing complex exists for R, then there is a discussion of the existence of j_* in the closing paragraphs of [11, §2]. If R is commutative, noetherian and of finite Krull dimension, the existence of j_* follows easily from [4, Th. 4.6].

(iv) The relation with dualizing complexes is perhaps the most intriguing. We mentioned it already in (iii), and in [11, §2]; the functor j_* can be constructed using a dualizing complex. But now we know it to exist unconditionally, even when there is no dualizing complex for R. Some consequences of this are discussed at the end of [11, §2]; let us only remind the reader of one of them.

Suppose R is left noetherian and right coherent, and let $\mathcal{I}$ be a dualizing complex. It turns out that the unit of adjunction $R \longrightarrow j_*j^*R$ is nothing other than the natural map $R \longrightarrow \mathcal{H}om(\mathcal{I}, \mathcal{I})$. The curious aspect is that, by Theorem 0.1, the map $R \longrightarrow j_*j^*R$ exists unconditionally, even for rings R where there is no dualizing complex.

Theorem 0.1 is our main result. But there is one other theorem that perhaps deserves mention.

THEOREM 0.3. *The inclusion of* $\mathbf{K}(R\text{-Flat})$ *into* $\mathbf{K}(R\text{-Mod})$ *has a right adjoint. Here* $\mathbf{K}(R\text{-Mod})$ *is the homotopy category of all cochain complexes of* R*-modules.*

Remark 0.4. In Remark 3.3 we will see that the Flat Cover Conjecture follows immediately from Theorem 0.3. The reader should not be too excited by this; we use the Flat Cover Conjecture in the proof of Theorem 0.3.

For the uninitiated reader: the Flat Cover Conjecture was an open problem for about twenty years until being proved, around 2000, by Eklof and Trlifaj [2] and by Bican, El Bashir and Enochs [1]. There is a slightly more extended discussion, of the history of the Flat Cover Conjecture, in Remark 2.10.

1. A general adjunction lemma

The main theorem of this article asserts that the natural projection, from $\mathbf{K}(R\text{-Flat})$ to its Verdier quotient $\mathbf{K}(R\text{-Flat})/\mathbf{K}(R\text{-Proj})^{\perp}$, has a right adjoint. By [10, Prop. 9.18] it suffices to show that the inclusion map $i_* : \mathbf{K}(R\text{-Proj})^{\perp} \longrightarrow \mathbf{K}(R\text{-Flat})$ has a right adjoint. We are therefore interested in general results, giving sufficient conditions for the inclusion of a triangulated subcategory to have a right adjoint. Before stating the helpful little proposition we will use, we remind ourselves of a couple of definitions.

Definition 1.1. Let $\mathcal{T}$ be a triangulated category, and let $\mathcal{S} \subset \mathcal{T}$ be a full subcategory. The subcategory $\mathcal{S}$ is called *thick* if it is a triangulated subcategory, and if every direct summand of any object in $\mathcal{S}$ lies in $\mathcal{S}$.

Definition 1.2. Let $\mathcal{T}$ be a category, and let $\mathcal{S}$ be a full subcategory. A morphism $s \longrightarrow t$ is called *an $\mathcal{S}$-precover of t* if s is an object in $\mathcal{S}$, and every morphism $\bar{s} \longrightarrow t$, with $\bar{s} \in \mathcal{S}$, factors (not necessarily uniquely) through $s \longrightarrow t$.

Remark 1.3. As the terminology might lead one to suspect, an $\mathcal{S}$-precover is a coarse version, the finer notion being an $\mathcal{S}$-cover. Every $\mathcal{S}$-cover is certainly an $\mathcal{S}$-precover. We do not need the refinement, and hence I will not explain it.

PROPOSITION 1.4. *Let $\mathcal{T}$ be a triangulated category, and let $\mathcal{S} \subset \mathcal{T}$ be a thick subcategory. Assume further that*

(i) *Every object $t \in \mathcal{T}$ admits an $\mathcal{S}$-precover.*

(ii) *Every idempotent in $\mathcal{T}$ splits.*

Then the inclusion $F : \mathcal{S} \longrightarrow \mathcal{T}$ has a right adjoint.

Proof. Let t be an object in $\mathcal{T}$; we need to show that there exists an object $S \in \mathcal{S}$ and a morphism $S \longrightarrow t$ which is universal. This means that every other morphism $\bar{s} \longrightarrow t$, $\bar{s} \in \mathcal{S}$ must factor *uniquely* through $S \longrightarrow t$. By hypothesis (i) we may choose an $\mathcal{S}$-precover $f : s \longrightarrow t$; every morphism $\bar{s} \longrightarrow t$ factors through f, but not necessarily uniquely. We will show how to choose a direct summand S of s for which the factorization is unique. The construction is as follows.

Complete $f : s \longrightarrow t$ to a triangle $r \xrightarrow{\alpha} s \xrightarrow{f} t \longrightarrow \Sigma r$ and then choose an $\mathcal{S}$-precover $\beta : s' \longrightarrow r$. We have a pair of composable morphism $s' \xrightarrow{\beta} r \xrightarrow{\alpha} s$, which we may complete to a morphism of triangles

$$(*)\qquad \begin{array}{ccccccc} s' & \xrightarrow{\alpha\beta} & s & \xrightarrow{g} & s'' & \longrightarrow & \Sigma s' \\ \beta\downarrow & & 1\downarrow & & w\downarrow & & \downarrow\Sigma\beta \\ r & \xrightarrow[\alpha]{} & s & \xrightarrow[f]{} & t & \longrightarrow & \Sigma r. \end{array}$$

We know that s' and s belong to the thick subcategory $\mathscr{S}$, and the top row of $(*)$, being a triangle, teaches us that s'' must also be in $\mathscr{S}$. Because $f : s \longrightarrow t$ is an $\mathscr{S}$-precover we know that the morphism $w : s'' \longrightarrow t$ must factor through it; we can write w as a composite $s'' \stackrel{h}{\longrightarrow} s \stackrel{f}{\longrightarrow} t$. Now let $e : s \longrightarrow s$ be the composite $s \stackrel{g}{\longrightarrow} s'' \stackrel{h}{\longrightarrow} s$.

The diagram $(*)$ informs us that $f = wg$, while the definition of h gives that $w = fh$. Combining these, we have

(iii) $f = fhg = fe$.

Next observe that, if $\bar{s}$ is an object of $\mathscr{S}$ and some composite $\bar{s} \stackrel{\rho}{\longrightarrow} s \stackrel{f}{\longrightarrow} t$ vanishes, then the map ρ must factor first through $\alpha : r \longrightarrow s$, and then through the $\mathscr{S}$-precover $\beta : s' \longrightarrow r$. The top row in the diagram $(*)$ tells us that $\bar{s} \stackrel{\rho}{\longrightarrow} s \stackrel{g}{\longrightarrow} s''$ must vanish. Therefore $e\rho = hg\rho = 0$. Summarizing:

(iv) If $\bar{s}$ belongs to $\mathscr{S}$ and the composite $\bar{s} \stackrel{\rho}{\longrightarrow} s \stackrel{f}{\longrightarrow} t$ vanishes, then so does $\bar{s} \stackrel{\rho}{\longrightarrow} s \stackrel{e}{\longrightarrow} s$.

From (iii) we know that $f(1-e) = 0$. Applying (iv), with $\rho = 1-e$, we conclude that $e(1-e) = 0$, that is

(v) The map $e : s \longrightarrow s$ is idempotent; that is $e^2 = e$.

By (ii) we know that idempotents in $\mathscr{T}$ split; the map $e : s \longrightarrow s$ has a factorization $s \stackrel{u}{\longrightarrow} S \stackrel{v}{\longrightarrow} s$ with uv being the identity $1_S : S \longrightarrow S$. By (i) we know that $\mathscr{S} \subset \mathscr{T}$ is thick, meaning that S must be an object in $\mathscr{S}$. I assert:

(vi) Let $e = vu$ be a splitting as above. The composite $fv : S \longrightarrow t$ has the property that any morphism $\bar{s} \longrightarrow t$, $\bar{s} \in \mathscr{S}$ factors *uniquely* through fv.

It remains to prove (vi).

Suppose we are given a morphism $\rho : \bar{s} \longrightarrow t$, $\bar{s} \in \mathscr{S}$. Let us first prove the existence of a factorization. Because $f : s \longrightarrow t$ is a precover the map ρ must factor as $\bar{s} \stackrel{\sigma}{\longrightarrow} s \stackrel{f}{\longrightarrow} t$. Now observe

$$\begin{aligned} \rho &= f\sigma \\ &= fe\sigma && \text{by (iii)} \\ &= fvu\sigma && \text{because } e = vu\text{; see (vi)} \end{aligned}$$

and we have factored ρ through fv.

It remains to prove the uniqueness. Suppose $\tau : \bar{s} \longrightarrow S$ is such that the composite $fv\tau$ vanishes. By (iv) we have that $ev\tau = 0$. Writing $e = vu$ this becomes $vuv\tau = 0$, hence certainly $uvuv\tau = 0$. But $uv = 1$; we conclude that $\tau = 0$. □

2. The existence of $\mathcal{S}$-precovers

We want to apply Proposition 1.4 to the inclusion of $\mathbf{K}(R\text{-Proj})^\perp$ into $\mathbf{K}(R\text{-Flat})$. To do this, we must prove that the hypotheses of Proposition 1.4 are satisfied. Most of these hypotheses are quite easy; the nontrivial one is that every object in $\mathbf{K}(R\text{-Flat})$ has a $\mathbf{K}(R\text{-Proj})^\perp$-precover. This section is devoted to the proof.

Remark 2.1. By its very nature, the notation $\mathbf{K}(R\text{-Proj})^\perp$ is relative. When we have a subcategory $\mathbf{K}(R\text{-Proj}) \subset \mathbf{K}(R\text{-Flat})$, we can speak about the orthogonal; we remind the reader that $\mathbf{K}(R\text{-Proj})^\perp$ is the full subcategory of all objects $Y \in \mathbf{K}(R\text{-Flat})$ so that $\mathrm{Hom}(X, Y)$ vanishes for every $X \in \mathbf{K}(R\text{-Proj})$. The notation $\mathbf{K}(R\text{-Proj})^\perp$ assumes that we know which Y's are permissible; we must be given the ambient category where we are embedding $\mathbf{K}(R\text{-Proj})$. If we embed $\mathbf{K}(R\text{-Proj})$ in the larger category $\mathbf{K}(R\text{-Mod})$, then the orthogonal could be expected to be much larger.

Notation 2.2. For the rest of this article, let us agree that $\mathcal{S}$ will be the orthogonal of $\mathbf{K}(R\text{-Proj}) \subset \mathbf{K}(R\text{-Flat})$; it is the category which, until now, we have been referring to simply as $\mathbf{K}(R\text{-Proj})^\perp$. By [11, Th. 8.6], we know that the objects in the category $\mathcal{S}$ are the complexes Z of flat R-modules, which satisfy either of the equivalent conditions:

(i) Z is a filtered direct limit of contractible complexes of finitely generated, projective R-modules.

(ii) Z is an acyclic complex of flat R-modules

$$\cdots \longrightarrow Z^{i-1} \xrightarrow{\partial^{i-1}} Z^i \xrightarrow{\partial^i} Z^{i+1} \xrightarrow{\partial^{i+1}} Z^{i+2} \longrightarrow \cdots$$

where the images I^i of the maps $\partial^i : Z^i \longrightarrow Z^{i+1}$ are all flat R-modules.

We want to show that every object in $\mathbf{K}(R\text{-Flat})$ admits an $\mathcal{S}$-precover; see Definition 1.2 for what constitutes an $\mathcal{S}$-precover. We will actually prove a stronger assertion, showing that every object in the larger category $\mathbf{K}(R\text{-Mod})$ admits an $\mathcal{S}$-precover. To give the proof, it is helpful to consider an auxiliary ring. Out of the ring R we will now construct a ring $S = S(R)$.

Notation 2.3. Let R be a ring. We wish to consider the R-algebra of the quiver

$$\cdots \xrightarrow[\partial^{i-2}]{} \overset{i-1}{\bullet} \xrightarrow[\partial^{i-1}]{} \overset{i}{\bullet} \xrightarrow[\partial^{i}]{} \overset{i+1}{\bullet} \xrightarrow[\partial^{i+1}]{} \overset{i+2}{\bullet} \xrightarrow[\partial^{i+2}]{} \cdots$$

with the relation that $\partial^{i+1}\partial^i = 0$. That is, we let S be the free R-module with basis $\{1, \partial^i, e^j\}$, with $i, j \in \mathbb{Z}$. We make it into an algebra by declaring that R commutes with all the basis elements. The relations among the basis elements assert first that 1 is the identity element, that is

$$1\partial^i = \partial^i 1 = \partial^i, \qquad 1e^j = e^j 1 = e^j, \qquad 1 \cdot 1 = 1.$$

Next they say that the e^j are orthogonal idempotents; that is

$$e^i e^j = 0 \text{ if } i \neq j, \qquad e^i e^i = e^i.$$

The ∂^i multiply by the following rules

$$\partial^i \partial^j = 0, \qquad e^{i+1}\partial^i = \partial^i e^i = \partial^i$$

and finally

$$e^j \partial^i = 0 \text{ unless } j = i+1, \qquad\qquad \partial^i e^j = 0 \text{ unless } j = i.$$

Remark 2.4. If Z is a complex of R-modules

$$\cdots \longrightarrow Z^{i-1} \longrightarrow Z^i \longrightarrow Z^{i+1} \longrightarrow \cdots,$$

then it is obvious that

$$\operatorname{inc}(Z) \quad = \quad \bigoplus_{i=-\infty}^{\infty} Z^i$$

is, in a natural way, a module over the algebra S. We have a fully faithful functor from the category of complexes of R-modules, which we will denote $\mathbf{C}(R\text{-Mod})$, to the category of S-modules, which we have been denoting S-Mod. That is, we have an inclusion functor

$$\operatorname{inc} \ : \ \mathbf{C}(R\text{-Mod}) \longrightarrow S\text{-Mod}.$$

This functor has a right adjoint, which we will denote

$$C \ : \ S\text{-Mod} \longrightarrow \mathbf{C}(R\text{-Mod});$$

the functor C takes the S-module M to the complex

$$\cdots \xrightarrow{\partial^{i-2}} e^{i-1}M \xrightarrow{\partial^{i-1}} e^i M \xrightarrow{\partial^i} e^{i+1}M \xrightarrow{\partial^{i+1}} \cdots.$$

We will study the functors inc and C a little.

LEMMA 2.5. *The functor* $C : S\text{-Mod} \longrightarrow \mathbf{C}(R\text{-Mod})$ *is exact and preserves colimits.*

Proof. Suppose

$$0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0$$

is a short exact sequence in S-Mod. Then its two direct summands

$$\begin{array}{ccccccccc} 0 & \longrightarrow & e^i L & \longrightarrow & e^i M & \longrightarrow & e^i N & \longrightarrow & 0 \\ 0 & \longrightarrow & (1-e^i)L & \longrightarrow & (1-e^i)M & \longrightarrow & (1-e^i)N & \longrightarrow & 0 \end{array}$$

must both be exact. In particular, the top row is exact, for every integer i. This means that

$$0 \longrightarrow C(L) \longrightarrow C(M) \longrightarrow C(N) \longrightarrow 0$$

is a short exact sequence of complexes.

As for the assertion about colimits, in view of the right exactness of C it suffices to consider coproducts. Let $\{M_\lambda, \lambda \in \Lambda\}$ be a family of S-modules. Then

$$e^i \left\{ \bigoplus_{\lambda\in\Lambda} M_\lambda \right\} \quad = \quad \bigoplus_{\lambda\in\Lambda} e^i M_\lambda,$$

and the lemma follows. □

LEMMA 2.6. *Let P be a projective S-module. Then the complex $C(P)$ is a contractible complex of projective R-modules.*

Proof. Because P is projective, there exists an S-module Q so that $P \oplus Q$ is free. Since

$$C(P \oplus Q) = C(P) \oplus C(Q),$$

it suffices to show that $C(P \oplus Q)$ is a contractible complex of projective R-modules. Thus we are reduced to the case where P is free.

Any free module is a coproduct of free modules of rank 1; since the functor C respects coproducts, we are reduced to the case where P is free and of rank 1. But in this case we know $C(P)$ explicitly; it is the coproduct of all the suspensions of the complex

$$\cdots \longrightarrow 0 \longrightarrow R \xrightarrow{\ 1\ } R \longrightarrow 0 \longrightarrow \cdots$$

which is clearly a contractible complex of projective R-modules. □

LEMMA 2.7. *Let Z be a contractible complex of projective R-modules. Then* $\mathrm{inc}(Z)$ *is a projective S-module.*

Proof. The complex Z, being a contractible complex of projectives, decomposes as a coproduct of complexes

$$\cdots \longrightarrow 0 \longrightarrow P \xrightarrow{\ 1\ } P \longrightarrow 0 \longrightarrow \cdots$$

with P a projective R-module. The functor inc has a right adjoint, and therefore respects coproducts. It suffices therefore to prove the Lemma in the special case, where Z is of the form

$$\cdots \longrightarrow 0 \longrightarrow P \xrightarrow{\ 1\ } P \longrightarrow 0 \longrightarrow \cdots$$

as above.

Next observe that P is projective, and hence there exists a Q with $P \oplus Q$ free. Let Z' be the complex

$$\cdots \longrightarrow 0 \longrightarrow Q \stackrel{1}{\longrightarrow} Q \longrightarrow 0 \longrightarrow \cdots.$$

We observe that $\mathrm{inc}(Z \oplus Z') = \mathrm{inc}(Z) \oplus \mathrm{inc}(Z')$, and it therefore suffices to prove $\mathrm{inc}(Z \oplus Z')$ projective. We may therefore assume that Z is a complex

$$\cdots \longrightarrow 0 \longrightarrow P \stackrel{1}{\longrightarrow} P \longrightarrow 0 \longrightarrow \cdots$$

with P free. But then Z is a coproduct of complexes

$$\cdots \longrightarrow 0 \longrightarrow R \stackrel{1}{\longrightarrow} R \longrightarrow 0 \longrightarrow \cdots$$

and it suffices to consider the case where Z is a complex as above. But now we observe that the ring S decomposes, as a left S-module, into the direct sum

$$S \quad = \quad Se^i \oplus S(1-e^i).$$

The module Se^i is therefore projective, and the reader can easily check that it agrees with $\mathrm{inc}(Z)$, in the special case where Z is the complex

$$\cdots \longrightarrow 0 \longrightarrow R \stackrel{1}{\longrightarrow} R \longrightarrow 0 \longrightarrow \cdots$$

with the nonzero modules in dimensions i and $i+1$. □

PROPOSITION 2.8. *If an S-module M is flat, then the complex $C(M)$ belongs to the category $\mathcal{S}$ of Notation* 2.2. *If a complex Z belongs to $\mathcal{S}$, then the S-module* $\mathrm{inc}(Z)$ *is flat.*

Proof. If M is a flat S-module, then M is a filtered direct limit of finitely generated projective S-modules F_λ. Lemma 2.6 tells us that the complexes $C(F_\lambda)$ are contractible complexes of projective R-modules, and Lemma 2.5 asserts that the functor C respects colimits. Hence $C(M)$ is a filtered direct limit of the complexes $C(F_\lambda)$, each of which is a contractible complex of projectives. By [11, Th. 8.6] we conclude that $C(M)$ lies in the subcategory $\mathcal{S}$.

Suppose now that Z is an object of $\mathcal{S}$. From [11, Th. 8.6] we know that Z is a filtered colimit of contractible complexes Z_λ of projectives. Lemma 2.7 says that each $\mathrm{inc}(Z_\lambda)$ is a projective S-module. Since the functor inc has a right adjoint, it respects colimits; we have that $\mathrm{inc}(Z)$ is a filtered direct limit of the projective S-modules $\mathrm{inc}(Z_\lambda)$. Hence $\mathrm{inc}(Z)$ must be flat. □

LEMMA 2.9. *Every object in the category* $\mathbf{K}(R\text{-Mod})$ *has an $\mathcal{S}$-precover, where $\mathcal{S} \subset$* $\mathbf{K}(R\text{-Mod})$ *is the category of Notation* 2.2.

Proof. Let Z be any object in the category $\mathbf{K}(R\text{-Mod})$; that is, Z is any complex of R-modules. Then $\mathrm{inc}(Z)$ is a module over the ring S. By [1] we know that $\mathrm{inc}(Z)$ has a flat precover. Let us choose one; we select a flat precover

$F \longrightarrow \mathrm{inc}(Z)$. I assert that the map $C(F) \longrightarrow C\big(\mathrm{inc}(Z)\big) \cong Z$ is an $\mathcal{S}$-precover for the complex Z.

Observe first of all that F is flat, and Proposition 2.8 therefore tells us that $C(F)$ belongs to the category $\mathcal{S}$. The map $C(F) \longrightarrow Z$ therefore is an object in the category $\mathcal{S}/Z$. We need to prove it a precover.

Suppose therefore that we are given some chain map $X \longrightarrow Z$, with X an object of $\mathcal{S}$. Proposition 2.8 tells us that $\mathrm{inc}(X)$ is a flat S-module, and hence the map $\mathrm{inc}(X) \longrightarrow \mathrm{inc}(Z)$ is a map from the flat S-module $\mathrm{inc}(X)$ to the S-module $\mathrm{inc}(Z)$. It must therefore factor through the flat precover $F \longrightarrow \mathrm{inc}(Z)$. In the category S-Mod we have a factorization

$$\mathrm{inc}(X) \longrightarrow F \longrightarrow \mathrm{inc}(Z).$$

Applying the functor C, and noting that $C \circ \{\mathrm{inc}\}$ is naturally isomorphic to the identity, we deduce a factorization

$$X \longrightarrow C(F) \longrightarrow Z,$$

as required. □

Remark 2.10. In the proof of Lemma 2.9 we appealed to the fact that every module has a flat precover; this may be found in [1]. This result has a long history. It was first conjectured by Enochs [3]; Enochs pointed out that the existence of a flat precover implies the existence of a flat cover (whatever that means); remember that, in Remark 1.3, we decided not to worry about flat covers in this article. Enochs then conjectured that every module should always have a flat cover or, equivalently, a flat precover. The problem took almost twenty years to solve; the article [1] contains two proofs of the fact. One proof, due to Enochs, relies on a result of Eklof and Trlifaj [2]. The second proof is different and was produced, at about the same time, by Bican and El Bashir.

Since then there has been another approach to the Flat Cover Conjecture, by way of model category methods; see Hovey [7] and Gillespie [5], [6].

3. Adjoints to inclusions of $\mathbf{K}(R\text{-Proj})^{\perp}$

Let the notation be as in Notation 2.2. We remind the reader; we have inclusions of categories $\mathbf{K}(R\text{-Proj}) \subset \mathbf{K}(R\text{-Flat}) \subset \mathbf{K}(R\text{-Mod})$. The category $\mathcal{S}$ was defined to be the orthogonal category to the inclusion $\mathbf{K}(R\text{-Proj}) \subset \mathbf{K}(R\text{-Flat})$. That is, the objects are all the objects Y in $\mathbf{K}(R\text{-Flat})$, so that $\mathrm{Hom}(X, Y) = 0$ for all $X \in \mathbf{K}(R\text{-Proj})$. We arrive now at the main theorem of this article:

THEOREM 3.1. *The inclusions of $\mathcal{S}$, into either of the categories* $\mathbf{K}(R\text{-Flat})$ *and* $\mathbf{K}(R\text{-Mod})$, *have right adjoints.*

Proof. The idea is to apply Proposition 1.4. The categories $\mathcal{S}$, $\mathbf{K}(R\text{-Flat})$ and $\mathbf{K}(R\text{-Mod})$ all have coproducts; hence [10, Prop. 1.6.8] shows that idempotents split in all three categories. It follows that $\mathcal{S}$ is equivalent to a thick subcategory, of either $\mathbf{K}(R\text{-Flat})$ or $\mathbf{K}(R\text{-Mod})$.

To prove the existence of an adjoint it therefore suffices to show that every object in $\mathbf{K}(R\text{-Mod})$ admits an $\mathcal{S}$-precover. The statement that every object in $\mathbf{K}(R\text{-Flat}) \subset \mathbf{K}(R\text{-Mod})$ has a flat precover is weaker; there are fewer objects in the smaller category $\mathbf{K}(R\text{-Flat})$. The existence of $\mathcal{S}$-precovers in $\mathbf{K}(R\text{-Mod})$ was verified in Lemma 2.9. $\square$

Combining Theorem 3.1 with [10, Th. 9.18], we deduce that the projection $j^* : \mathbf{K}(R\text{-Flat}) \longrightarrow \mathbf{K}(R\text{-Flat})/\mathcal{S}$ has a right adjoint j_*. That is we have proved Theorem 0.1. In the introduction we promised the reader another result; it is time to deliver.

THEOREM 3.2. *Let R be a ring. The inclusion* $\mathbf{K}(R\text{-Flat}) \longrightarrow \mathbf{K}(R\text{-Mod})$ *has a right adjoint.*

Proof. The subcategory $\mathbf{K}(R\text{-Proj}) \subset \mathbf{K}(R\text{-Mod})$ is a localizing subcategory, and from [11, Cor. 5.10] we know that it satisfies Brown representability. Therefore, by [10, Th. 8.4.4], the inclusion has a right adjoint. From the formalism of Bousfield localizations (see, for example, [10, Prop. 9.1.8]) we deduce that every object Y in $\mathbf{K}(R\text{-Mod})$ admits a triangle

$$X \longrightarrow Y \longrightarrow Z \longrightarrow \Sigma X,$$

with $X \in \mathbf{K}(R\text{-Proj})$ and Z orthogonal to $\mathbf{K}(R\text{-Proj})$.

Here is where our notation could begin to haunt us. I could say that Z belongs to $\mathbf{K}(R\text{-Proj})^\perp$, meaning that Z is an object in $\mathbf{K}(R\text{-Mod})$, and $\mathrm{Hom}(A, Z) = 0$ for all $A \in \mathbf{K}(R\text{-Proj})$. This notation has a problem, as we already discussed in Remark 2.1. Until now, whenever we wrote $\mathbf{K}(R\text{-Proj})^\perp$, what we meant was the category of all $Z \in \mathbf{K}(R\text{-Flat})$, satisfying the orthogonality condition. Now we are permitting Z to lie in the larger category $\mathbf{K}(R\text{-Mod})$. For this proof we will therefore avoid the notation $\mathbf{K}(R\text{-Proj})^\perp$.

Anyway, let us return to our triangle $X \longrightarrow Y \longrightarrow Z \longrightarrow \Sigma X$ above. The object Z belongs to $\mathbf{K}(R\text{-Mod})$, and happens to be orthogonal to $\mathbf{K}(R\text{-Proj})$. Theorem 3.1 tells us that the inclusion $\mathcal{S} \longrightarrow \mathbf{K}(R\text{-Mod})$ has a right adjoint. Just by virtue of the fact that Z belongs to $\mathbf{K}(R\text{-Mod})$, there is a distinguished triangle

$$S \longrightarrow Z \longrightarrow W \longrightarrow \Sigma S,$$

with S in $\mathcal{S}$ and W orthogonal to $\mathcal{S}$. Now note that S is an object of $\mathcal{S}$, which was the orthogonal of the inclusion $\mathbf{K}(R\text{-Proj}) \longrightarrow \mathbf{K}(R\text{-Flat})$; this means that $\mathrm{Hom}(A, S) = 0$ for all objects $A \in \mathbf{K}(R\text{-Proj})$. We have, for all objects $A \in \mathbf{K}(R\text{-Proj})$, that

$$\mathrm{Hom}(A, S) \quad = \quad 0 \quad = \quad \mathrm{Hom}(A, Z);$$

from the distinguished triangle $S \longrightarrow Z \longrightarrow W \longrightarrow \Sigma S$ we conclude that $\mathrm{Hom}(A, W)$ must also vanish. That is, for any object $A \in \mathbf{K}(R\text{-Proj})$ and for any object $B \in \mathcal{S}$, we have that

$$\mathrm{Hom}(A, W) \quad = \quad 0 \quad = \quad \mathrm{Hom}(B, W).$$

But now recall [11, Prop. 8.1]; it says that the inclusion $j_! : \mathbf{K}(R\text{-Proj}) \longrightarrow \mathbf{K}(R\text{-Flat})$ has a right adjoint. It follows that every object $F \in \mathbf{K}(R\text{-Flat})$ admits a distinguished triangle

$$A \longrightarrow F \longrightarrow B \longrightarrow \Sigma A,$$

with $A \in \mathbf{K}(R\text{-Proj})$ and $B \in \mathcal{S}$. We know that $\mathrm{Hom}(A, W) = 0 = \mathrm{Hom}(B, W)$; from the triangle we conclude that $\mathrm{Hom}(F, W)$ vanishes, for every $F \in \mathbf{K}(R\text{-Flat})$.

We have constructed two distinguished triangles in the category $\mathbf{K}(R\text{-Mod})$, namely $X \longrightarrow Y \longrightarrow Z \longrightarrow \Sigma X$ and $S \longrightarrow Z \longrightarrow W \longrightarrow \Sigma S$. That is, we have two composable morphisms $Y \longrightarrow Z \longrightarrow W$, and we know how to complete each to a triangle. The octahedral lemma gives us two more distinguished triangles

$$F \longrightarrow Y \longrightarrow W \longrightarrow \Sigma F$$

$$X \longrightarrow F \longrightarrow S \longrightarrow \Sigma X.$$

In the second of these we know that $X \in \mathbf{K}(R\text{-Proj}) \subset \mathbf{K}(R\text{-Flat})$ and $S \in \mathcal{S} \subset \mathbf{K}(R\text{-Flat})$, and hence the entire triangle must lie in the category $\mathbf{K}(R\text{-Flat})$. Thus F lies in $\mathbf{K}(R\text{-Flat})$, while W is orthogonal to $\mathbf{K}(R\text{-Flat})$. The existence of the distinguished triangle $F \longrightarrow Y \longrightarrow W \longrightarrow \Sigma F$, for every $Y \in \mathbf{K}(R\text{-Mod})$, establishes that the inclusion $\mathbf{K}(R\text{-Flat}) \longrightarrow \mathbf{K}(R\text{-Mod})$ has a right adjoint; see [10, Th. 9.1.13]. □

Remark 3.3. Theorem 3.2 tells us that the inclusion $I : \mathbf{K}(R\text{-Flat}) \to \mathbf{K}(R\text{-Mod})$ has a right adjoint $J : \mathbf{K}(R\text{-Mod}) \longrightarrow \mathbf{K}(R\text{-Flat})$. Let us see what this means in a special case.

Let M be any R-module, and consider the complex $A = A(M)$ below

$$\cdots \longrightarrow 0 \longrightarrow M \longrightarrow 0 \longrightarrow \cdots .$$

The existence of a right adjoint to the inclusion I gives us a morphism, in the category $\mathbf{K}(R\text{-Mod})$, of the form $\varepsilon_A : IJA \longrightarrow A$. That is, we have a chain map

$$\begin{array}{ccccccccc}
\cdots & \longrightarrow & Z^{-1} & \longrightarrow & Z^{0} & \longrightarrow & Z^{1} & \longrightarrow & \cdots \\
 & & \big\downarrow & & {\scriptstyle\rho}\big\downarrow & & \big\downarrow & & \\
\cdots & \longrightarrow & 0 & \longrightarrow & M & \longrightarrow & 0 & \longrightarrow & \cdots ,
\end{array}$$

where the complex $Z = IJA$ is a complex of flat R-modules. Furthermore, given any map $\varphi : F \longrightarrow M$, with F a flat R-module, we have a factorization of φ

through chain maps of complexes

$$\begin{array}{ccccccccc}
\cdots & \longrightarrow & 0 & \longrightarrow & F & \longrightarrow & 0 & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & Z^{-1} & \longrightarrow & Z^0 & \longrightarrow & Z^1 & \longrightarrow & \cdots \\
 & & \downarrow & & {\scriptstyle\rho}\downarrow & & \downarrow & & \\
\cdots & \longrightarrow & 0 & \longrightarrow & M & \longrightarrow & 0 & \longrightarrow & \cdots .
\end{array}$$

This factorization, in the homotopy category, happens to be unique up to homotopy, but we do not care. What is important is that $\varphi : F \longrightarrow M$ factors through $\rho : Z^0 \longrightarrow M$. The map $\rho : Z^0 \longrightarrow M$ is a flat precover for M.

In other words, from the existence of the right adjoint to the inclusion I : $\mathbf{K}(R\text{-Flat}) \longrightarrow \mathbf{K}(R\text{-Mod})$, one deduces in a few lines that flat precovers exist. At the moment this is no great discovery; to prove the existence of a right adjoint we used the existence of flat precovers, for an associated ring S. See the proof of Lemma 2.9.

Remark 3.4. The main theorem in this article is Theorem 0.1, and in Remark 0.2(ii) we mentioned that it has a generalization to schemes. This goes as follows.

Let X be a quasicompact, separated scheme, and let $\mathbf{K}(\text{Flat}/X)$ be the homotopy category of all chain complexes of flat, quasicoherent $\mathcal{O}_X$-modules. Let $\mathcal{S} \subset \mathbf{K}(\text{Flat}/X)$ be the subcategory of chain complexes as in Notation 2.2(ii); the reader should be cautioned that, on nonaffine schemes X, conditions (i) and (ii) of Notation 2.2 are not in general equivalent. In his thesis [9] Murfet defines the functor $j^* : \mathbf{K}(\text{Flat}/X) \longrightarrow \mathbf{K}_m(\text{Proj}/X)$ to be the Verdier quotient map

$$j^* \ : \ \mathbf{K}(\text{Flat}/X) \longrightarrow \frac{\mathbf{K}(\text{Flat}/X)}{\mathcal{S}} \quad \doteqdot \quad \mathbf{K}_m(\text{Proj}/X).$$

When $X = \text{Spec}(R)$ is affine we know that $\mathbf{K}_m(\text{Proj}/X)$ is naturally isomorphic to $\mathbf{K}(R\text{-Proj})$. Murfet's thesis explores the category $\mathbf{K}_m(\text{Proj}/X)$ and shows that, even on nonaffine X, this category has many of the good properties of $\mathbf{K}(R\text{-Proj})$.

The precise assertion of Remark 0.2(ii) is that the functor $j^* : \mathbf{K}(\text{Flat}/X) \longrightarrow \mathbf{K}_m(\text{Proj}/X)$ always has a right adjoint; the proof may be found in [9, Th. 3.16].

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