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The perverse filtration and the Lefschetz hyperplane theorem

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Abstract

We describe the perverse filtration in cohomology using the Lefschetz hyperplane theorem.

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1. **Introduction**

In this paper we give a geometric description of the middle perverse filtration on the cohomology and on the cohomology with compact supports of a complex with constructible cohomology sheaves of abelian groups on a quasi projective variety. The description is in terms of restriction to generic hyperplane sections and it is somewhat unexpected, especially if one views the constructions leading to perverse sheaves as transcendental and hyperplane sections as more algebro-geometric.

The results of this paper are listed in Section 4, and hold for a quasi projective variety. For the sake of simplicity, we describe here the case of the cohomology of an n-dimensional affine variety $Y \subseteq \mathbb{A}^N$ with coefficients in a complex K.

The theory of t-structures endows the (hyper)cohomology groups $H(Y, K)$ with a canonical filtration P, called the perverse filtration,

$$P^p H(Y, K) = \operatorname{Im}\{H(Y, {}^{\mathfrak{p}}\tau_{\leq -p} K) \longrightarrow H(Y, K)\},$$

which is the abutment of the perverse spectral sequence. Let

$$Y_* = \{Y \supseteq Y_{-1} \supseteq \cdots \supseteq Y_{-n}\}$$

be a sequence of closed subvarieties; we call this data an n-flag. Basic sheaf theory endows $H(Y, K)$ with the so-called flag filtration F, abutment of the spectral sequence $E_1^{p,q} = H^{p+q}(Y_p, Y_{p-1}, K_{|Y_p}) \Longrightarrow H^*(Y, K)$. We have $F^p H(Y, K) = \operatorname{Ker}\{H(Y, K) \to H(Y_{p-1}, K_{|Y_{p-1}})\}$. For an arbitrary n-flag, the perverse and flag filtrations are unrelated.

In terms of filtrations, the main result of this paper is that if the n-flag is obtained using n hyperplane sections in sufficiently general position, then

$$P^p H^j(Y, K) = F^{p+j} H^j(Y, K). \tag{1}$$

More precisely, we construct a complex $R\Gamma(Y, K)$ endowed with two filtrations P and F and we prove (Theorem 4.1.1) that there is a natural isomorphism in the filtered derived category $DF(\mathrm{Ab})$ of abelian groups

$$(R\Gamma(Y, K), P) = (R\Gamma(Y, K), \operatorname{Dec}(F)), \tag{2}$$

where $\operatorname{Dec}(F)$ is the shifted filtration associated with F. Then (1) follows from (2).

Our methods seem to break down in the non quasi projective case and also for other perversities.

The constructions and results are amenable to mixed Hodge theory. We offer the following application: let $f : X \to Y$ be any map of algebraic varieties, Y be quasi projective and C be a bounded complex with constructible cohomology sheaves on X. Then the perverse Leray spectral sequences can be identified with suitable flag spectral sequences on X. In the special case when $K = \mathbb{Q}_X$, we obtain the following result due to M. Saito: the perverse spectral sequences for $H(X, \mathbb{Q})$ and $H_c(X, \mathbb{Q})$, are spectral sequences of mixed Hodge structures. Further Hodge-theoretic applications concerning the decomposition theorem are mentioned in Remark 7.0.5 and will appear in [6].

The isomorphism (2) lifts to the bounded derived category $D^b(\mathcal{P}_Y)$ of perverse sheaves with rational coefficients. This was the basis of the proof of our results in an earlier version of this paper. The present formulation, which short-circuits $D^b(\mathcal{P}_Y)$, is based on the statement of Proposition 5.6.1 which has been suggested to us by an anonymous referee. We are deeply grateful for this suggestion. The main point is that a suitable strengthening of the Lefschetz hyperplane theorem yields cohomological vanishings for the bifiltered complex $(R\Gamma(Y, K), P, F)$ which yield (2). These vanishings are completely analogous to the ones occurring for topological cell complexes and, for example, one can fit the classical Leray spectral sequence of a fiber bundle in the framework of this paper.

The initial inspiration for this work comes from Arapura's paper [1], which deals with the standard filtration, versus the perverse one. In this case, the flag has to be special: it is obtained by using high degree hypersurfaces containing the bad loci of the ordinary cohomology sheaves. The methods of this paper are easily adapted to that setting; see [7].

The fact that the perverse filtration is related to general hyperplane sections confirms, in our opinion, the more fundamental role played by perverse sheaves with respect to ordinary sheaves. The paper [1] has also directed us to the beautiful [17] and the seminal [4]. The influence on this paper of the ideas contained in [4], [17] is hard to overestimate.

2. **Notation**

A variety is a separated scheme of finite type over the field of complex numbers $\mathbb{C}$. A map of varieties is a map of $\mathbb{C}$-schemes. The results of this paper hold for sheaves of R-modules, where R is a commutative ring with identity with finite global dimension, e.g. $R = \mathbb{Z}$, R a field, etc. For the sake of exposition we work with $R = \mathbb{Z}$, i.e. with sheaves of abelian groups.

The results of this paper hold, with routine adaptations of the proofs, in the case of varieties over an algebraically closed field and étale sheaves with the usual coefficients: $\mathbb{Z}/l^m\mathbb{Z}$, $\mathbb{Z}_l$, $\mathbb{Q}_l$, $\mathbb{Z}_l[E]$, $\mathbb{Q}_l[E]$ ($E \supseteq \mathbb{Q}_l$ a finite extension) and $\overline{\mathbb{Q}}_l$.

We do not discuss further these variants, except to mention that the issue of stratifications is addressed in [5, §§2.2 and 6]. The term *stratification* refers to an algebraic Whitney stratification [14]. Recall that any two stratifications admit a common refinement and that maps of varieties can be stratified.

Given a variety Y, there is the category $D_Y = D_Y(\mathbb{Z})$ which is the full subcategory of the derived category of the category Sh_Y of sheaves of abelian groups whose objects are the bounded complexes with constructible cohomology sheaves, i.e. bounded complexes K whose cohomology sheaves $\mathcal{H}^i(K)$, restricted to the strata of a suitable stratification Σ of Y, become locally constant with fiber a finitely generated abelian group. For a given Σ, a complex with this property is called Σ-constructible.

Given a stratification Σ of Y, there are the full subcategories $D_Y^{\Sigma} \subseteq D_Y$ of complexes which are Σ-constructible. Given a map $f : X \to Y$ of varieties, there are the usual four functors $(f^*, Rf_*, Rf_!, f^!)$. By abuse of notation, we denote Rf_* and $Rf_!$ simply by f_* and $f_!$. The four functors preserve stratifications; i.e. if $f : (X, \Sigma') \to (Y, \Sigma)$ is stratified, then $f_*, f_! : D_X^{\Sigma'} \to D_Y^{\Sigma}$ and $f^*, f^! : D_Y^{\Sigma} \to D_X^{\Sigma'}$.

The abelian categories Sh_Y and $\mathrm{Ab} := \mathrm{Sh}_{pt}$ have enough injectives. The right derived functor of global sections is denoted $R\Gamma(Y, -)$. Hypercohomology groups are denoted simply by $H(Y, K)$. Similarly, we have $R\Gamma_c(Y, -)$ and $H_c(Y, K)$.

We consider only the *middle* perversity t-structure on D_Y [5]. The truncation functors are denoted ${}^{\mathfrak{p}}\tau_{\leq i} : D_Y \to {}^{\mathfrak{p}}D_Y^{\leq i}$, ${}^{\mathfrak{p}}\tau_{\geq j} : D_Y \to {}^{\mathfrak{p}}D_Y^{\geq j}$, the heart $\mathcal{P}_Y := {}^{\mathfrak{p}}D_Y^{\leq 0} \cap {}^{\mathfrak{p}}D_Y^{\geq 0}$ is the abelian category of perverse sheaves on Y and we denote the perverse cohomology functors ${}^{\mathfrak{p}}\mathcal{H}^i := {}^{\mathfrak{p}}\tau_{\leq 0} \circ {}^{\mathfrak{p}}\tau_{\geq 0} \circ [i] : D_Y \longrightarrow \mathcal{P}_Y$.

The perverse t-structure is compatible with a fixed stratification, i.e. truncations preserve Σ-constructibility and we have ${}^{\mathfrak{p}}\mathcal{H}^i : D_Y^{\Sigma} \longrightarrow \mathcal{P}_Y^{\Sigma}$, etc.

In this paper, the results we prove in cohomology have a counterpart in cohomology with compact supports. If we employ field coefficients, then middle perversity is preserved by duality and the results in cohomology are equivalent to the ones in cohomology with compact supports by virtue of Poincaré-Verdier Duality.

Due to the integrality of the coefficients, middle perversity is not preserved by duality; see [5, §3.3]. However, we can prove the results in cohomology and in compactly supported cohomology using the same techniques. For expository reasons, we often emphasize cohomology.

Filtrations, on groups and complexes, are always *finite*, i.e. $F^i K = K$ for $i \ll 0$ and $F^j K = 0$ for $j \gg 0$, and *decreasing*, i.e. $F^i K \supseteq F^{i+1} K$. We say that F has type $[a, b]$, for $a \leq b \in \mathbb{Z}$, if $\mathrm{Gr}_F^p K \simeq 0$ for every $i \notin [a, b]$.

A standard reference for the filtered derived category $DF(A)$ of an abelian category is [15]. Useful complements can be found in [5, §3] and in [4, Appendix]. We denote the filtered version of D_Y by $D_Y F$. The objects are filtered complexes (K, F), with $K \in D_Y$. This is a full subcategory of $D^b F(\mathrm{Sh}_Y)$.

We denote a “canonical” isomorphism with the symbol “$=$.”

3. The perverse and flag spectral sequences

In this paper, we relate the perverse spectral sequences with certain classical objects that we call flag spectral sequences.

In order to do so, we exhibit these spectral sequences as the ones associated with a collection of filtered complexes of abelian groups. These, in turn, arise by taking the global sections of (a suitable injective model of) the complex K endowed with the filtrations P, F, G and δ which we are about to define.

In this section, starting with a variety Y and a complex $K \in D_Y$, we construct the multi-filtered complex (K, P, F, G, δ) and we list its relevant properties. By passing to global sections, we identify the ensuing spectral sequences of filtered complexes with the perverse and flag ones.

3.1. (K, P). The system of truncation maps $\cdots \to {}^{\mathfrak{p}}\tau_{\leq -p}K \to {}^{\mathfrak{p}}\tau_{\leq -p+1}K \to \dots$ is isomorphic in D_Y to a system of inclusion maps $\cdots \to P^pK' \to P^{p-1}K' \to \dots$, where the filtered complex (K', P) is of injective type, i.e. all $\mathrm{Gr}^p_P K'$, hence all P^pK' and K', have injective entries; see [5, 3.1.2.7]. The filtered complex (K', P) is well-defined up to unique isomorphism in the filtered D_YF by virtue of [5, Prop. 3.1.4.(i)] coupled with the second axiom, "$\mathrm{Hom}^{-1} = 0$," of t-structures. We replace K with K' and obtain (K, P). In particular, from now on, K is injective.

3.2. *Flags.* The smooth irreducible projective variety $F(N, n)$ of n-flags on the N-dimensional projective space $\mathbb{P}^N$ parametrizes linear n-flags $\mathfrak{F} = \{\Lambda_{-1} \subseteq \cdots \subseteq \Lambda_{-n}\}$, where $\Lambda_{-p} \subseteq \mathbb{P}^N$ is a codimension p linear subspace.

A linear n-flag $\mathfrak{F}$ on $\mathbb{P}^N$ is said to be *general* if it belongs to a suitable Zariski dense open subset of the variety of flags $F(N, n)$. We say that a *pair of flags is general* if the same is true for the pair with respect to $F(N, n) \times F(N, n)$. In this paper, this open set depends on the complex K and on the fixed chosen embedding $Y \subseteq \mathbb{P}^N$. We discuss this dependence in Section 5.2.

A linear n-flag $\mathfrak{F}$ on $\mathbb{P}^N$ gives rise to an *n-flag* on $Y \subseteq \mathbb{P}^N$, i.e. an increasing sequence of closed subvarieties of Y:

$$Y_* = Y_*(\mathfrak{F}): \quad Y = Y_0 \supseteq Y_{-1} \supseteq \cdots \supseteq Y_{-n}, \qquad Y_p := \Lambda_p \cap Y. \tag{3}$$

We set $Y_{-n-1} := \varnothing$ and we have the (resp., closed, open and locally closed) embeddings:

$$i_p : Y_p \longrightarrow Y, \qquad j_p : Y \setminus Y_{p-1} \longrightarrow Y, \quad k_p : Y_p \setminus Y_{p-1} \longrightarrow Y. \tag{4}$$

Let $h : Z \to Y$ be a locally closed embedding. There are the exact functor $(-)_Z = h_! h^*(-)$, which preserves c-softness, and the left exact functor Γ_Z, which preserves injectivity and satisfies $H_Z(Y, K) = H(Y, R\Gamma_Z K)$; see [16]. If h is closed, then $R\Gamma_Z = h_! h^! = h_* h^!$; and, since K is injective, $R\Gamma_Z K = \Gamma_Z K$. If

$Z' \subseteq Z$ is closed, then we have the distinguished triangle $R\Gamma_{Z'}K \to R\Gamma_Z K \to R\Gamma_{Z-Z'}K \stackrel{+}{\to}$ which, again by the injectivity of K, is the triangle associated with the exact sequence $0 \to \Gamma_{Z'}K \to \Gamma_Z K \to \Gamma_{Z-Z'}K \to 0$.

3.3. (K, F, G, δ). We have constructed (K, P) of injective type. Let $Y \subseteq \mathbb{P}^N$ be an embedding of the quasi projective variety Y. Let $\mathfrak{F}, \mathfrak{F}'$ be two, possibly identical, linear n-flags on $\mathbb{P}^N$ with associated flags, Y_* and Z_* on Y. We denote the corresponding maps (4) by i', j', k'.

We define the three filtrations F, G and δ on K. They are well-defined, up to unique isomorphism, in the filtered $D_Y F$.

The *flag filtration* $F = F_{Y_*} = F_{Y_*(\mathfrak{F})}$, of type $[-n, 0]$, is defined by setting $F^p K := K_{Y-Y_{p-1}}$:

$$0 \subseteq K_{Y-Y_{-1}} \subseteq \ldots \subseteq K_{Y-Y_{p-1}} \subseteq K_{Y-Y_{-n}} \subseteq K. \tag{5}$$

The *flag filtration* $G = G_{Z_*} = G_{Z_*(\mathfrak{F}')}$, of type $[0, n]$, is defined by setting $G^p K := \Gamma_{Z_{-p}} K$:

$$0 \subseteq \Gamma_{Z_{-n}}K \subseteq \ldots \subseteq \Gamma_{Z_{-p}}K \subseteq \Gamma_{Z_{-1}} K \subseteq K. \tag{6}$$

The *flag filtration* $\delta = \delta(F_{Y_*(\mathfrak{F})} G_{Z_*(\mathfrak{F}')})$, of type $[-n, n]$, is the *diagonal filtration* defined by $\delta^p K = \sum_{i+j=p} F^i K \cap G^j K$.

Note that one does not need injectivity to define the filtrations. However, without this assumption, the resulting filtration G and δ would not be canonically defined in $D_Y F$. Moreover, injectivity yields $\Gamma_Z K = R\Gamma_Z K$ for every locally closed $Z \subseteq Y$, a fact we use throughout without further mention.

3.4. *The graded complexes associated with* (K, P, F, G, δ). Recall that $\mathrm{Gr}^p_\delta = \oplus_{i+j=p} \mathrm{Gr}^i_F \mathrm{Gr}^j_G$ and that the Zassenhaus Lemma implies $\mathrm{Gr}^i_F \mathrm{Gr}^j_G = \mathrm{Gr}^j_G \mathrm{Gr}^i_F$. Since the formation of F is an exact functor, the formation of G is exact when applied to injective sheaves, and injective sheaves are c-soft, we have

$$\mathrm{Gr}^p_P K, \quad \mathrm{Gr}^p_G K, \quad \mathrm{Gr}^j_G \mathrm{Gr}^p_P \qquad \text{are injective,} \tag{7}$$

$$\begin{gathered} \mathrm{Gr}^p_F K, \quad \mathrm{Gr}^i_F \mathrm{Gr}^p_P K, \quad \mathrm{Gr}^i_F \mathrm{Gr}^j_G K, \\ \mathrm{Gr}^i_F \mathrm{Gr}^j_G \mathrm{Gr}^p_P K, \quad \mathrm{Gr}^p_\delta K, \quad \mathrm{Gr}^a_\delta \mathrm{Gr}^p_P K \qquad \text{are } c\text{-soft.} \end{gathered} \tag{8}$$

In particular, $P^p K$, $G^p K$ and $P^p K \cap G^j K$ are injective. We have the analogous c-softness statement for (8), e.g. the $F^i K \cap G^j \cap P^p K$ are c-soft. By construction (§3.1), the filtration P splits in each degree, and the formation of F and G is compatible with direct sums. Hence, we have the following list of natural isomorphisms:

1) $\mathrm{Gr}_P^p K = {}^{\mathfrak{p}}\mathcal{H}^{-p}(K)[p]$,

2) $\mathrm{Gr}_F^p K = K_{Y_p - Y_{p-1}} = k_{p!} k_p^* K$,

3) $\mathrm{Gr}_G^p K = \Gamma_{Z_{-p} - Z_{-p-1}} K = k'_{-p*} k'^{!}_{-p} K$.

4) $\mathrm{Gr}_G^j \mathrm{Gr}_F^i K = \mathrm{Gr}_F^i \mathrm{Gr}_G^j K = (\Gamma_{Z_{-j}-Z_{-j-1}} K)_{Y_i - Y_{i-1}} = (k'_{-j*} k'^{!}_{-j} K)_{Y_i - Y_{i-1}}$.

5) $\mathrm{Gr}_\delta^p K = \oplus_{i+j=a} (\Gamma_{Z_{-j}-Z_{-j-1}} K)_{Y_i - Y_{i-1}}$.

6) $\mathrm{Gr}_F^i \mathrm{Gr}_P^p = ({}^{\mathfrak{p}}\mathcal{H}^{-p}(K)[p])_{Y_p - Y_{p-1}}$.

7) $\mathrm{Gr}_G^j \mathrm{Gr}_P^p = \Gamma_{Z_{-j}-Z_{-j-1}} ({}^{\mathfrak{p}}\mathcal{H}^{-p}(K)[p]) = k'_{-j*} k'^{!}_{-j}\, {}^{\mathfrak{p}}\mathcal{H}^{-p}(K)[p]$.

8) $\mathrm{Gr}_F^i \mathrm{Gr}_G^j \mathrm{Gr}_P^p K = (\Gamma_{Z_{-j}-Z_{-j-1}} ({}^{\mathfrak{p}}\mathcal{H}^{-p}(K)[p]))_{Y_i - Y_{i-1}}$.

9) $\mathrm{Gr}_\delta^a \mathrm{Gr}_P^p K = \oplus_{i+j=a} (\Gamma_{Z_{-j}-Z_{-j-1}} ({}^{\mathfrak{p}}\mathcal{H}^{-p}(K)[p]))_{Y_i - Y_{i-1}}$.

Remark 3.4.1. If the pair of flags is general, then (cf. §5.2, or [4, Complement to §3])

$$(\Gamma_{Z_{-j}-Z_{-j-1}} K)_{Y_i - Y_{i-1}} = \Gamma_{Z_{-j}-Z_{-j-1}} (K_{Y_i - Y_{i-1}}).$$

In general, the two sides differ, for the left-hand side is zero on Y_{i-1}.

3.5. *$(R\Gamma(Y,K), P, F, \delta)$ and $(R\Gamma_c(Y,K), P, G, \delta)$.* Since K is injective, we have $R\Gamma(Y,K) = \Gamma(Y,K)$, $R\Gamma_c(Y,K) = \Gamma_c(Y,K)$. We keep "$R$" in the notation.

By applying the left exact functors Γ and Γ_c, we obtain the multi-filtered complexes of abelian groups

$$(R\Gamma(Y,K), P, F, \delta), \qquad (R\Gamma_c(Y,K), P, G, \delta), \tag{9}$$

by setting, for example, $P^p R\Gamma(Y,K) := \Gamma(Y, P^p K)$, etc.

Since injective sheaves and c-soft sheaves are Γ and Γ_c-injective, we have

$$R\Gamma(Y, {}^{\mathfrak{p}}\mathcal{H}^{-p}(K)[p]) = \Gamma(Y, \mathrm{Gr}_P^p K) = \mathrm{Gr}_P^p\, \Gamma(Y,K), \tag{10}$$

$$R\Gamma_c(Y, {}^{\mathfrak{p}}\mathcal{H}^{-p}(K)[p]) = \Gamma_c(Y, \mathrm{Gr}_P^p K) = \mathrm{Gr}_P^p\, \Gamma_c(Y,K), \tag{11}$$

with analogous formulæ for the following graded objects

$$\mathrm{Gr}_F^p,\ \mathrm{Gr}_G^p,\ \mathrm{Gr}_F^a \mathrm{Gr}_P^b,\ \mathrm{Gr}_G^a \mathrm{Gr}_P^b,\ \mathrm{Gr}_F^i \mathrm{Gr}_G^j,\ \mathrm{Gr}_\delta^p,\ \mathrm{Gr}_F^i \mathrm{Gr}_G^j \mathrm{Gr}_P^b,\ \mathrm{Gr}_\delta^p \mathrm{Gr}_P^b. \tag{12}$$

Remark 3.5.1. Though the formation of F does not preserve injectivity, one can always take filtered injective resolutions. In that case, the resulting $F^p K$ is not exactly $K_{Y-Y_{p-1}}$, etc., but rather an injective resolution of it. This would allow us to drop the mention of c-softness. On the other hand, the F-construction is exact and formulæ like the ones in Section 3.4 are readily proved.

3.6. *The perverse and flag spectral sequences.* With the aid of Sections 3.4, 3.5 it is immediate to recognize the E_1-terms of the spectral sequences associated with the filtered complexes $(R\Gamma(Y,K), P, F, \delta)$ and $(R\Gamma_c(Y,K), P, G, \delta)$.

Definition 3.6.1 (Perverse spectral sequence and filtration). The *perverse spectral sequence* for $H(Y,K)$ is the spectral sequences of the filtered complexes $(R\Gamma(Y,K), P)$:

$$E_1^{p,q} = H^{2p+q}(Y, {}^{\mathfrak{p}}\mathcal{H}^{-p}(K)) \Longrightarrow H^*(Y,K) \tag{13}$$

and the abutment is the *perverse filtration* P on $H^*(Y,K)$ defined by

$$P^p H^*(Y,K) = \operatorname{Im}\{H^*(Y, {}^{\mathfrak{p}}\tau_{\leq -p}K) \longrightarrow H^*(Y,K)\}, \tag{14}$$

similarly, for $H_c(Y,K)$ using $(R\Gamma_c(Y,K), P)$.

Let $f : X \to Y$ be a map of algebraic varieties and $C \in D_X$.

Definition 3.6.2. The *perverse Leray spectral sequences* for $H(X,C)$ (resp. $H_c(X,C)$) are the corresponding perverse spectral sequences on Y for $K := f_*C$ (resp. $K := f_!C$).

Let $Y \subseteq \mathbb{P}^N$ be an embedding of the quasi projective variety Y, $\mathfrak{F}, \mathfrak{F}'$ be two linear flags on $\mathbb{P}^N$ and Y_* and Z_* be the corresponding flags on Y.

Definition 3.6.3 (Flag spectral sequence and filtration (F-version)). The F *flag spectral sequence* associated with Y_* is the spectral sequence associated with the filtered complex $(R\Gamma(Y,K), F)$:

$$E_1^{p,q} = H^{p+q}(Y, K_{Y_p - Y_{p-1}}) \Longrightarrow H^*(Y,K) \tag{15}$$

and its abutment is the *flag filtration* $F = F_{Y_*}$ on $H^*(Y,K)$ defined by

$$F^p H^*(Y,K) = \operatorname{Ker}\{H^*(Y,K) \longrightarrow H^*(Y_{p-1}, K_{|Y_{p-1}})\}. \tag{16}$$

Definition 3.6.4 (Flag spectral sequence and filtration (G-version)). The G *flag spectral sequence* associated with Z_* is the spectral sequence associated with the filtered complex $(R\Gamma_c(Y,K), G)$:

$$E_1^{p,q} = H_c^{p+q}(Y, k_{-p*}k_{-p}^! K) \Longrightarrow H_c^*(Y,K) \tag{17}$$

and its abutment is the *flag filtration* $G = G_{Z_*}$ on $H_c^*(Y,K)$ defined by

$$G^p H_c^*(Y,K) = \operatorname{Im}\{H_c^*(Y, \Gamma_{Z_{-p}}K) \longrightarrow H_c^*(Y,K)\}. \tag{18}$$

Definition 3.6.5 (Flag spectral sequence and filtration (δ-version)). The δ *flag spectral sequences* associated with (Y_*, Z_*) are the spectral sequences associated with the filtered complexes $(R\Gamma(Y,K), \delta)$ and $(R\Gamma_c(Y,K), \delta)$.

Remark 3.6.6. We omit displaying these spectral sequences since, due to Remark 3.4.1, they do not have familiar E_1-terms. If the pair of flags is general, or merely in good position with respect to Σ and each other (cf. §5.2), then we have equality in Remark 3.4.1 and the E_1-terms take the following form (we write $H_{c,Z}$ for $H_c \circ R\Gamma_Z$):

$$E_1^{p,q} = \bigoplus_{i+j=p} H^{p+q}_{Z_{-j}-Z_{-j-1}}(Y, K_{Y_i - Y_{i-1}}) \Longrightarrow H^*(Y, K), \tag{19}$$

$$E_1^{p,q} = \bigoplus_{i+j=p} H^{p+q}_{c,Z_{-j}-Z_{-j-1}}(Y, K_{Y_i - Y_{i-1}}) \Longrightarrow H^*_c(Y, K) \tag{20}$$

and their abutments are the *flag filtrations* $\delta = \delta(Y_*, Z_*)$ on $H(Y, K)$ and on $H^*_c(Y, K)$ defined by

$$\delta^p H^*(Y, K) = \operatorname{Im}\left\{ \bigoplus_{i+j=p} H^*_{Z_{-j}}(Y, K_{Y-Y_i}) \longrightarrow H^*(Y, K) \right\}, \tag{21}$$

$$\delta^p H^*_c(Y, K) = \operatorname{Im}\left\{ \bigoplus_{i+j=p} H^*_{c,Z_{-j}}(Y, K_{Y-Y_i}) \longrightarrow H^*(Y, K) \right\}. \tag{22}$$

3.7. *The shifted filtration and spectral sequence.* We need the notion and basic properties ([10]) of the *shifted filtration* for a filtered complex (L, F) in an abelian category. We make the definition explicit in Ab.

The *shifted filtration* $\operatorname{Dec}(F)$ on L is:

$$\operatorname{Dec}(F)^p L^l := \left\{ x \in F^{p+l} K^l \mid dx \in F^{p+l+1} L^{l+1} \right\}.$$

The *shifted spectral sequence of* (L, F) is the one for $(L, \operatorname{Dec}(F))$ and we have

$$\operatorname{Dec}(F)^p H^l(L) = F^{p+l} H^l(L), \qquad E_r^{p,q}(L, \operatorname{Dec}(F)) = E_{r+1}^{2p+q,-p}(L, F). \tag{23}$$

4. **Results**

We prove results for Y quasi projective. The statements and the proofs are more transparent when Y is affine. We state and prove the results in the affine case first. The multi-filtered complexes of abelian groups $(R\Gamma(Y, K), P, F, \delta)$ and $(R\Gamma_c(Y, K), P, G, \delta)$, which give rise to the spectral sequences and filtrations we are interested in, are defined in Section 3.

4.1. *The results over an affine base.* In this section Y is *affine* of dimension n and $K \in D_Y$. Let $Y \subseteq \mathbb{P}^N$ be a fixed embedding and $\mathfrak{F}, \mathfrak{F}'$ be a pair of linear n-flags on $\mathbb{P}^N$.

THEOREM 4.1.1 (Perverse filtration on cohomology for affine varieties). *Let $\mathfrak{F}$ be general. There is a natural isomorphism in the filtered derived category*

$DF(\mathrm{Ab})$:

$$(R\Gamma(Y,K),P) \simeq (R\Gamma(Y,K),\mathrm{Dec}(F))$$

identifying the perverse spectral sequence with the shifted flag spectral sequence so that

$$P^pH^l(Y,K) = F^{p+l}H^l(Y,K) = \mathrm{Ker}\{H^l(Y,K) \longrightarrow H^l(Y_{p+l-1}, K_{|Y_{p+l-1}})\}.$$

THEOREM 4.1.2 (Perverse filtration on H_c and affine varieties). *Let $\mathfrak{F}'$ be general. There is a natural isomorphism in the filtered derived category* $DF(\mathrm{Ab})$:

$$(R\Gamma_c(Y,K),P) \simeq (R\Gamma_c(Y,K),\mathrm{Dec}(G))$$

identifying the perverse spectral sequence with the shifted flag spectral sequence so that

$$P^pH^l_c(Y,K) = G^{p+l}H^l_c(Y,K) = \mathrm{Im}\{H^l_c(Y,R\Gamma_{Z_{-p-l}}K) \longrightarrow H^l_c(Y,K)\}.$$

In what follows, $f: X \to Y$ is an algebraic map, with Y affine, $C \in D_X$, and given a linear n-flag $\mathfrak{F}$ on $\mathbb{P}^N$, we denote by $X_* = f^{-1}Y_*$ the corresponding pre-image n-flag on X.

THEOREM 4.1.3 (Perverse Leray and affine varieties). *Let $\mathfrak{F}$ be general. The perverse Leray spectral sequence for $H(X,C)$ is the corresponding shifted X_* flag spectral sequence. The analogous statement for $H_c(X,C)$ holds.*

Remark 4.1.4. The δ-variants of Theorems 4.1.1, 4.1.2, and 4.1.3 for cohomology and for cohomology with compact supports, hold for the δ filtration as well, and with the same proof. In this case one requires the pair of flags to be general.

Remark 4.1.5. Rather surprisingly, the differentials of the perverse (Leray) spectral sequences can be identified with the differentials of a flag spectral sequence. In turn, these are classical algebraic topology objects stemming from a filtration by closed subsets, i.e. from the cohomology sequences associated with the triples (Y_p, Y_{p-1}, Y_{p-2}).

4.2. *The results over a quasi projective base.* In this section, Y is a quasi projective variety of dimension n and $K \in D_Y$.

There are several ways to state and prove generalizations of the results in Section 4 to the quasi projective case. We thank an anonymous referee for suggesting this line of argument as an alternative to our original two arguments that used Jouanolou's trick (as in [1]), and finite and affine Čech coverings. For an approach via Verdier's spectral objects see [6].

Let Y be quasi projective, $Y \subseteq \mathbb{P}^N$ be a fixed *affine* embedding and $(\mathfrak{F},\mathfrak{F}')$ be a pair of linear n-flags on $\mathbb{P}^N$. The notion of a δ flag spectral sequence is defined in Definition 3.6.3; see also Remark 3.6.6.

THEOREM 4.2.1 (Quasi projective case via two flags). *Let the pair of flags be general. There are natural isomorphisms in $DF(\mathrm{Ab})$:*

$$(R\Gamma(Y, K), P) \simeq (R\Gamma(Y, K), \mathrm{Dec}(\delta)), \quad (R\Gamma_c(Y, K), P) \simeq (R\Gamma_c(Y, K), \mathrm{Dec}(\delta))$$

identifying the perverse and the shifted δ flag spectral sequence, inducing the identity on the abutted filtered spaces.

Moreover, if $f : X \to Y$ and $C \in D_X$ are given, then the perverse Leray spectral sequences coincide with the shifted δ flag spectral sequences associated with the preimage flags on X.

5. **Preparatory material**

5.1. *Vanishing results.*

THEOREM 5.1.1 (Cohomological dimension of affine varieties). *Let Y be affine and $Q \in \mathcal{P}_Y$ be a perverse sheaf on Y. Then*

$$H^r(Y, Q) = 0, \quad \textit{for all } r > 0, \qquad H^r_c(Y, Q) = 0, \quad \textit{for all } r < 0.$$

Proof. We give several references. The original proof of the first statement is due to Michael Artin [2], XIV and is valid in the étale context. [14, §2.5] proved the theorem for intersection homology with compact supports and with twisted coefficients on a pure-dimensional variety; the reader can translate the results in intersection cohomology and intersection cohomology with compact supports on a pure-dimensional variety; a standard devissage argument implies the result for a perverse sheaf Q on arbitrary varieties; Q is a finite extension of intersection cohomology complexes with twisted, not necessarily semisimple, coefficients on the irreducible components. In [5, Th. 4.1.1] the case of H is proved directly; the case of H_c is proved for field coefficients by invoking duality; however, one can prove it directly and for arbitrary coefficients. The textbook [16] proves it for Stein manifolds (see loc. cit., Th. 10.3.8); the general case follows by embedding Y as a closed subset of an affine space $i : Y \to \mathbb{C}^n$ and by applying the statement to the perverse sheaf i_*Q. □

Let Y be quasi projective. Fix an *affine* embedding $Y \subseteq \mathbb{P}^N$. Let $\Lambda, \Lambda' \subseteq \mathbb{P}^N$ be two hyperplanes, $H := Y \cap \Lambda \subseteq Y$ and $j : Y \setminus H \to Y \leftarrow H : i$ be the corresponding open and closed immersions. Note that $j^! = j^*$. Similarly, for Λ'.

THEOREM 5.1.2 (Strong Weak Lefschetz). *Let Y be quasi projective and $Q \in \mathcal{P}_Y$. If Λ is general, then*

$$H^r(Y, j_! j^! Q) = 0, \ \forall r < 0, \qquad H^r_c(Y, j_* j^* Q) = 0, \ \forall r > 0.$$

Let (Λ, Λ') be a general pair. Then $j_! j^* j'_* j'^! Q = j'_* j'^! j_! j^* Q$ and

$$H^r(Y, j_! j^* j'_* j'^! Q) = H^r_c(Y, j'_* j'^! j_! j^* Q) = 0, \ \forall r \neq 0.$$

Proof. We give several references for the first statement. In [4, Lemma 3.3]; this proof is valid in the étale context. The second statement is observed in [4, Complement to §3], [14, §2.5]. M. Goresky has informed us that P. Deligne has also proved this result (unpublished). □

We include a sketch of the proof of this result, following [4], in Section 5.2, where we also complement the arguments in [4] needed in the sequel of the paper.

Remark 5.1.3. Since $j^! = j^*$, we may reformulate the first statement of Theorem 5.1.2 as follows

$$H^r(Y, Q_{Y-H}) = 0, \ \forall r < 0, \qquad H^r_c(Y, R\Gamma_{Y-H} Q) = 0, \ \forall r > 0$$

and similarly for the second one. Moreover, by Theorem 5.1.1, if Y is affine, then the vanishing results hold for every $r \neq 0$.

Remark 5.1.4. It is essential that the embedding $Y \subseteq \mathbb{P}^N$ be affine. For example, the conclusion does not hold in the case when $Y = \mathbb{A}^2 \setminus \{0\} \subseteq \mathbb{P}^2$ and $Q = \mathbb{Z}_Y[2]$.

5.2. *Transversality, base change and choosing good flags.* In this section we highlight the role of transversality in the proof of Theorem 5.1.2. In fact, transversality implies several base change equalities which we use throughout the paper in order to prove the vanishing results in Theorem 5.1.2, its iteration Lemma 6.1.1, its "two-flag-extension" (27) and to observe (29). While the vanishing results are used to realize condition (32), which is the key to the main results of this paper, the base change equality (29) is used to reduce the results for the perverse spectral sequences Theorems 4.1.3, 4.2.1 with respect to a map $X \to Y$, to analogous results for perverse spectral sequences on Y.

These base change properties hold generically by virtue of the generic base change theorem [11], and this is enough for the purposes of this paper. On the other hand, it is possible to pinpoint the conditions one needs to impose on flags; see Definition 5.2.4 and Remark 5.2.6.

Let $Y \subseteq \mathbb{P}^N$ be an affine embedding of the quasi projective variety Y and $Y \subseteq \overline{Y}$ be the resulting projective completion. There is a natural decomposition into locally closed subsets $\mathbb{P}^N = (\mathbb{P}^N \setminus \overline{Y}) \coprod (\overline{Y} \setminus Y) \coprod Y$. Let $K \in D_Y$.

Definition 5.2.1 (Stratifications adapted to the complex and to the embedding). We say that a stratification Σ of $\mathbb{P}^N$ is *adapted to the embedding* $Y \subseteq \mathbb{P}^N$ if $Y, \overline{Y} \setminus Y$, hence $\overline{Y}$, and $\mathbb{P}^N \setminus \overline{Y}$ are unions of strata, and Σ induces by restriction stratifications on $\mathbb{P}^N, \mathbb{P}^N \setminus \overline{Y}, \overline{Y}, Y, \overline{Y} \setminus Y$ with respect to which all possible inclusions among these varieties are stratified maps. We denote these induced stratifications by Σ_Y, etc. and we say that Σ is *adapted to* K if K is Σ_Y-constructible.

Remark 5.2.2. Since maps of varieties can be stratified and a finite collection of stratifications admits a common refinement, stratifications which are adapted to the complex and to the embedding exist.

Let Σ be a stratification of $\mathbb{P}^N$ adapted to K and to the embedding $Y \subseteq \mathbb{P}^N$. Let $\Lambda \subseteq \mathbb{P}$ be a hyperplane, $H := \Lambda \cap Y$ and $\overline{H} = \Lambda \cap \overline{Y}$. Set $\overline{U} := (\overline{Y} \setminus \overline{H})$ and $U := (Y \setminus H)$. Consider the cartesian diagram

(24)
$$\begin{tikzcd} H \arrow[r, "i"] \arrow[d, "J"] & Y \arrow[d, "J"] & U \arrow[l, "j"'] \arrow[d, "J"] \\ \overline{H} \arrow[r, "i"] & \overline{Y} & \overline{U}. \arrow[l, "j"'] \end{tikzcd}$$

We address the following question: when is the natural map

(25)
$$J_! j_* j^* K \to j_* J_! j^* K$$

an isomorphism? In general the two differ on $\overline{H} \cap (\overline{Y} \setminus Y)$. By the octahedron axiom, the map (25) is an isomorphism if and only if the natural base change map $J_* i^* K \to i^* J_* K$ is an isomorphism. This latter condition is met if Λ is general ([4, Lemma 3.3]). In fact it is sufficient that Λ meets transversally the strata in $\Sigma_{\overline{Y}-Y}$. This is a condition on the stratification, not on K. It follows that the analogous map $j_! J_* J^! K \to J_* j_! J^! K$ is also an isomorphism under the same conditions.

Proof of Theorem 5.1.2 (see [4]). We prove the first statement for cohomology. The point is that a general linear section produces the isomorphism (25) and this identifies the cohomology groups in question with compactly supported cohomology groups on affine varieties where one uses Theorem 5.1.1. Note that since the maps of type j and J are affine, all the complexes appearing below are perverse. We have the following chain of equalities:

$$\begin{aligned} H^r(Y, j_! j^* Q) &= H^r(\overline{Y}, J_* j_! j^* Q) = H^r(\overline{Y}, j_! J_* j^* Q), \\ &= H^r_c(\overline{Y}, j_! J_* j^! Q) = H^r_c(\overline{U}, J_* j^! Q) \end{aligned}$$

and, since $\overline{U}$ is affine and $J_* j^! Q$ is perverse, the last group is zero for $r < 0$ and the first statement for cohomology follows. The one for compactly supported cohomology is proved in a similar way.

In order to prove the second statement, we consider the Cartesian diagram

(26)
$$\begin{tikzcd} U \cap U' \arrow[r, "j'"] \arrow[d, "j"] & U \arrow[d, "j"] \\ U' \arrow[r, "j'"] & Y. \end{tikzcd}$$

Since the embedding $Y \subseteq \mathbb{P}^N$ is affine, the open sets U, U' and $U \cap U'$ are affine. Note that this fails if the embedding is not affine. We have that $j_!, j_*, j^! = j^*$ are all t-exact and preserve perverse sheaves. The same is true for j'.

The equality $j_! j^* j'_* j'^! Q = j'_* j'^! j_! j^* Q$ is proved using base change considerations similar to the ones we have made for (25).

We prove the vanishing in cohomology. The case of cohomology with compact supports is proved in a similar way. The case $r < 0$ is covered by the first statement. We need suitable "reciprocal" transversality conditions which are the obvious generalization of the ones mentioned in discussion of (25). We leave the formulation of these conditions to the reader. It will suffice to say that they are met by a general pair (Λ, Λ'). The case $r > 0$ follows from Theorem 5.1.1 applied to the affine U': $H(Y, j'_* j'^! j_! j^* Q) = H(U', j'^! j_! j^* Q)$. $\square$

Remark 5.2.3. Let $Q \in \mathcal{P}_Y$ and Σ be a stratification of $\mathbb{P}^N$ adapted to $Y \subseteq \mathbb{P}^N$ and such that $Q \in \mathcal{P}_Y^{\Sigma_Y}$. An inspection of the proof of Theorem 5.1.2 reveals that it is sufficient to choose Λ so that it meets transversally the strata in $\overline{Y} \setminus Y$. It is not relevant how H meets the strata in Σ_Y. A similar remark holds in the case of a pair of hyperplanes.

We now introduce a kind of transversality notion that is sufficient for the purpose of this paper. Let Σ be as above.

Definition 5.2.4 (Flag in good position wrt Σ). A linear n-flag $\mathfrak{F} = \Lambda_0 \subseteq \Lambda_{-1} \subseteq \cdots \subseteq \Lambda_{-n}$ on $\mathbb{P}^N$ is in *good position with respect to* Σ if it is subject to the following inductively defined conditions:

1) Λ_{-1} meets all the strata of $\Sigma_0 := \Sigma$ transversally; let Σ_{-1} be a refinement of Σ such that its restriction to $\Lambda_{-1} \simeq \mathbb{P}^{N-1}$ is adapted to the embedding $Y_{-1} \subseteq \Lambda_{-1}$;
2) Λ_{-2} meets all the strata of Σ_{-1} transversally; we iterate these conditions and constructions and introduce $\Sigma_{-2}, \Lambda_{-3}, \ldots, \Sigma_{-n+1}, \Lambda_{-n}$ and we require that, for every $i = 1, \ldots, n$;
   i) Λ_{-i} meets all the strata of Σ_{-i+1} transversally.

We define $\Sigma' := \Sigma_{-n}$.

Remark 5.2.5. By the Bertini theorem, it is clear that a general linear n-flag $\mathfrak{F}$ is in good position with respect to a fixed Σ. Of course, "general" depends on Σ. Note also that if $\mathfrak{F}$ is in good position, then Y_p has pure codimension p in Y.

Remark 5.2.6. There is the companion notion of a pair of linear n-flags $(\mathfrak{F}, \mathfrak{F}')$ being *in good position with respect to* Σ *and to each other*. We leave the task of writing down the precise formulation to the reader. The notion is again inductive and proceeds, also by imposing mutual transversality, in the following order: $\Lambda_{-1}, \Lambda'_{-1}, \Lambda_{-2}, \Lambda'_{-2}$, etc. It suffices to say that a general pair of flags will do.

The proof of Theorem 5.1.2 works well inductively with the elements of a linear flag $\mathfrak{F}$ on $\mathbb{P}^N$ in good position with respect to the embedding and to the perverse sheaf Q. We use this fact in the proof of Lemma 6.1.1. Similarly, this kind of argument works well with a pair of flags in good position with respect to the Q, the embedding and each other. In particular, it works for a general pair of flags. In these cases, we have the equality $\Gamma_{Z_{-j}-Z_{-j-1}}(Q_{Y_i-Y_{i-1}}) = (\Gamma_{Z_{-j}-Z_{-j-1}}Q)_{Y_i-Y_{i-1}}$ which follows from repeated use of the equality $j_!j^*j'_*j'^!Q = j'_*j'^!j_!j^*Q$ of Theorem 5.1.2. By transversality, the shift $[i+j]$ of these complexes are perverse. This allows us to apply the vanishing results of Theorem 5.1.2 and deduce, for general pairs of flags on the *quasi projective* variety Y, that

(27)

$$H^r(Y, \Gamma_{Z_{-j}-Z_{-j-1}}(Q_{Y_i-Y_{i-1}})) = H^r_c(Y, \Gamma_{Z_{-j}-Z_{-j-1}}(Q_{Y_i-Y_{i-1}})) = 0, \ \forall\, r \neq 0.$$

Let $f : X \to Y$ be a map of varieties. The diagram (24) induces the cartesian diagram:

(28)
$$\begin{tikzcd} X_H \arrow[r, "i"] \arrow[d, "f"] & X \arrow[d, "f"] & X_U \arrow[l, "j"'] \arrow[d, "f"] \\ H \arrow[r, "i"] & Y & U. \arrow[l, "j"'] \end{tikzcd}$$

The previous base change discussion implies, for Λ meeting all strata of Σ transversally, that

(29) $$f_*j_!j^*C = j_!j^*f_*C, \qquad f_!j_*j^!C = j_*j^!f_!C.$$

Similar base change equations hold for a linear flag $\mathfrak{F}$ on $\mathbb{P}^N$ in good position with respect to $f_!C$ and to the embedding $Y \subseteq \mathbb{P}^N$ (e.g. general) and also for a pair of flags in good position with respect to $f_!C$, the embedding and each other (e.g. a general pair).

5.3. *Two short exact sequences.*

LEMMA 5.3.1. *Let $Y \subseteq \mathbb{P}^N$ be quasi projective and $Q \in \mathcal{P}_Y$. If $\Lambda \subseteq \mathbb{P}^N$ is a general linear section, then there are natural exact sequences in*

(30) $$0 \longrightarrow i_*i^*Q[-1] \longrightarrow j_!j^!Q \longrightarrow Q \longrightarrow 0,$$

(31) $$0 \longrightarrow Q \longrightarrow j_*j^*Q \longrightarrow i_!i^!Q[1] \longrightarrow 0.$$

Proof. There are the distinguished triangles in D_Y:

$$j_!j^!Q \longrightarrow Q \longrightarrow i_*i^*Q \xrightarrow{+}, \qquad i_!i^!Q \longrightarrow Q \longrightarrow j_*j^*Q \xrightarrow{+}.$$

Since j is affine, $j_!$ and j_* are t-exact and $j_!j^!Q$ and j_*j^*Q are perverse. We choose Λ so that it is transverse to the strata of a stratification for Q. It follows that $i_*i^*Q[-1] = i_!i^!Q[1]$ is perverse. Each conclusion follows from the long exact sequence of perverse cohomology of the corresponding distinguished triangles. □

5.4. *The forget-the-filtration map.* Let A be an abelian category. [5, Prop. 3.1.4.(i)] is a sufficient condition for the natural forget-the-filtration map

$$\mathrm{Hom}_{DF(A)} \to \mathrm{Hom}_{D(A)}$$

to be an isomorphism. We need the bifiltered counterpart of this sufficient condition.

The objects (L, F, G) of the bifiltered derived category $DF_2(A)$ are complexes L endowed with two filtrations. The homotopies must respect both filtrations and one inverts bifiltered quasi isomorphisms, i.e. (homotopy classes of) maps inducing quasi isomorphisms on the bigraded objects $\mathrm{Gr}^a_F \mathrm{Gr}^b_G$. It is a routine matter to adapt Illusie's treatment of $DF(A)$ to the bifiltered setting and then to adapt the proof of [5, Prop. 3.1.4.(i)] to yield a proof of

PROPOSITION 5.4.1. *Assume that A has enough injectives and that (L, F, G), $(M, F, G) \in D^+F_2(A)$ are such that*

$$\mathrm{Hom}^n_{DFA}((\mathrm{Gr}^i_G L[-i], F), (\mathrm{Gr}^j_G M[-j], F)) = 0, \qquad \forall n < 0, \quad \forall i < j.$$

The "forget-the-second filtration" map is an isomorphism:

$$\mathrm{Hom}_{DF_2A}((L, F, G), (M, F, G)) \xrightarrow{\simeq} \mathrm{Hom}_{DFA}((L, F), (M, F)).$$

5.5. *The canonical lift of a t-structure.* The following is a mere special case of [4, App.]. Let A be an abelian category. The derived category $D(A)$ admits the standard t-structure, i.e. usual truncation. The filtered derived category $DF(A)$ admits a canonical t-structure which lifts (in a suitable sense which we do not need here) the given one on $D(A)$. This canonical t-structure on $DF(A)$ is described as follows. There are the two full subcategories

$$DF(A)^{\leq 0} := \{(L, F) \mid \mathrm{Gr}^i_F L \in D(A)^{\leq i}\},$$
$$DF(A)^{\geq 0} := \{(L, F) \mid \mathrm{Gr}^i_F L \in D(A)^{\geq i}\}.$$

The heart $DF(A)^{\leq 0} \cap DF(A)^{\geq 0}$ is

$$DF_\beta(A) = \{(L, F) \mid \mathrm{Gr}^i_F L[i] \in A\},$$

where β is for bête (see [5, 3.1.7]). The reader can verify the second axiom of t-structure, i.e. $\mathrm{Hom}^{-1}(DFA^{\leq 0}, DFA^{\geq 1}) = 0$, by a simple induction on the length of the filtrations, and the third axiom, i.e. the existence of the truncation triangles, by simple induction on the length of the filtration coupled with the use of Verdier's "Lemma of nine" (see [5, Prop. 1.1.11]).

5.6. *The key lemma on bifiltered complexes.* Let A be an abelian category and (L, P, F) be a bifiltered complex, i.e. an object in the bifiltered derived category $DF_2(A)$. Recall the existence of the shifted filtration $\mathrm{Dec}(F)$ associated with (L, F).

The purpose of this section is to prove the following result, the formulation of which has been suggested to us by an anonymous referee. This result is key to the approach presented in this paper.

PROPOSITION 5.6.1. *Let (L, P, F) be a bifiltered complex such that*

$$H^r(\mathrm{Gr}_F^a \mathrm{Gr}_P^b L) = 0, \quad \forall r \neq a - b. \tag{32}$$

Assume that L is bounded below and that A has enough injectives. There is a natural isomorphism in the filtered derived category $DF(A)$,

$$(L, P) \simeq (L, \mathrm{Dec}(F))$$

that induces the identity on L and thus identifies

$$(H^*(L), P) = (H^*(L), \mathrm{Dec}(F)).$$

In particular, there is a natural isomorphism between the spectral sequences associated with (L, P) and $(L, \mathrm{Dec}(F))$ inducing the identity on the abutments.

In order to prove Proposition 5.6.1, we need the following two lemmata.

LEMMA 5.6.2. *Let (L, F) be any filtered complex. Then the bifiltered complex $(K, F, \mathrm{Dec}(F))$ satisfies* (32).

Proof. This is a formal routine verification. □

LEMMA 5.6.3. *With things as in Proposition* 5.6.1, *the natural map*

$$\mathrm{Hom}_{DF_2(A)}((L, P, F), (L, \mathrm{Dec}(F), F)) \longrightarrow \mathrm{Hom}_{DF(A)}((L, F), (L, F))$$

induced by forgetting the first filtration is an isomorphism. The same is true with the roles of the filtrations P and $\mathrm{Dec}(F)$ *switched.*

Proof. Endow $D(A)$ with the standard t-structure (i.e. usual truncation). Endow $DF(A)$ with the canonical lift of this t-structure (see §5.5). The hypothesis (32) implies that, for every $b \in \mathbb{Z}$,

$$(\mathrm{Gr}_P^b L[-b], F) \in DF_\beta(A),$$

i.e., it is in the heart of the canonical lift of the standard t-structure to $DF(A)$. Similarly, Lemma 5.6.2 implies that, for every $b \in \mathbb{Z}$,

$$(\mathrm{Gr}_{\mathrm{Dec}(F)}^b L[-b], F) \in DF_\beta(A).$$

The hypotheses of Proposition 5.4.1 are met: in fact they are met for every i, j, due to the second axiom of t-structure. The first statement follows.

If we switch P and $\mathrm{Dec}(F)$, then the hypotheses of Proposition 5.4.1 are still met, for the same reason, and the second statement follows. □

Proof of Proposition 5.6.1. By Lemma 5.6.3, the identity on (L, F) admits natural lifts

$$\iota_P \in \mathrm{Hom}_{DF_2A}((L, P, F), (L, \mathrm{Dec}(F), F)),$$
$$\iota_{\mathrm{Dec}(F)} \in \mathrm{Hom}_{DF_2A}((L, \mathrm{Dec}(F), F), (L, P, F))$$

which are inverse to each other and hence isomorphisms.

By forgetting the second filtration, we obtain a pair of maps in

$$\mathrm{Hom}_{DFA}((L, P), (L, \mathrm{Dec}(F)) \text{ and } \mathrm{Hom}_{DFA}((L, \mathrm{Dec}(F)), (L, P)$$

which are inverse to each other. By forgetting both filtrations, both maps yield the identity on L. □

Remark 5.6.4. The results of this section hold if we replace $\mathrm{Dec}(F)$ with any filtration P' satisfying (32).

6. **Proof of the results**

In this section, we prove the main results of this paper and make a connection with Beilinson's equivalence theorem [4].

6.1. *Verifying the vanishing* (32) *for general flags.* Recall the set-up: Y is quasi projective of dimension n, $K \in D_Y$, $Y \subseteq \mathbb{P}^N$ is an affine embedding, $\mathfrak{F}, \mathfrak{F}'$ is a pair of linear n-flags on $\mathbb{P}^N$. We have the bounded multi-filtered complexes of abelian groups $(R\Gamma(Y, K), P, F, \delta)$, $(R\Gamma_c(Y, K), P, G, \delta)$ obtained using suitably acyclic resolutions. The perverse spectral sequences are the spectral sequences for the filtration P, the flag spectral sequences are the ones for the filtrations F, G, δ, similarly, for the perverse Leray spectral sequences. If the flags are arbitrary, then the perverse and the flag spectral sequences seem unrelated.

Let Σ be a stratification of $\mathbb{P}^N$ adapted to K and to the embedding $Y \subseteq \mathbb{P}^N$. The proof of Theorem 4.1.1 consists of showing that if the flag $\mathfrak{F}$ is in good position with respect to Σ (see Definition 5.2.4), then the vanishing conditions (32) hold for the bifiltered complexes $(R\Gamma(Y, K), P, F))$ by virtue of a repeated application of the strong weak Lefschetz Theorem 5.1.2, so that Proposition 5.6.1 applies and there is a natural identification of filtered complexes $(R\Gamma(Y, K), P) = (R\Gamma(Y, K), \mathrm{Dec}(F))$ and of the ensuing spectral sequences. The other results are proved in a similar way.

The key to the proof is Lemma 6.1.1 (below) which is suggested by a construction due to Beilinson [4, Lemma 3.3 and Complement to §3], which yields a technique to construct resolutions of perverse sheaves on varieties by using suitably transverse flags. The entries of the resolutions satisfy strong vanishing conditions and realize the wanted condition (32). There are three versions, left, right and bi-sided resolutions.

The resolutions are complexes obtained by using the following general construction. Let $Q \in \mathcal{P}_Y^{\Sigma_Y}$, where Σ_Y is the trace of Σ on Y. The connecting maps associated with the short exact sequences $0 \to \mathrm{Gr}_F^{*+1} Q \to F^* Q / F^{*+2} Q \to \mathrm{Gr}^* Q \to 0$ give rise to a sequence of maps in D_Y

$$\mathrm{Gr}_F^{-n} Q \xrightarrow{d} \cdots \xrightarrow{d} \mathrm{Gr}_F^0 Q, \tag{33}$$

with $d^2 = 0$. We call this a complex in D_Y. The same is true for the G filtration: $\mathrm{Gr}_G^0 Q \to \cdots \to \mathrm{Gr}_G^n Q$ is a complex in D_Y. The bigraded objects $\mathrm{Gr}_F \mathrm{Gr}_G$ give rise to a double complex with associated single complex $\mathrm{Gr}_\delta^{-n} Q \to \cdots \to \mathrm{Gr}_\delta^n Q$ in D_Y. The transversality assumptions on the flags ensure that these are complexes of perverse sheaves resolving Q, that they are suitably acyclic and that their formation is an exact functor. More precisely, we have the following:

LEMMA 6.1.1 (Acyclic resolutions of perverse sheaves). *Let Y be quasi projective, Σ be a stratification adapted to the affine embedding $Y \subseteq \mathbb{P}^N$, $Q \in \mathcal{P}_Y^{\Sigma_Y}$ be a Σ_Y-constructible perverse sheaf on Y.*

Let $\mathfrak{F}$ be a linear n-flag on $\mathbb{P}^N$ in good position with respect to Σ. Then

(i) *There is the short exact sequence in $\mathcal{P}_Y^{\Sigma'_Y} \subseteq \mathcal{P}_Y$:*

$$0 \longrightarrow Q_{Y_{-n}-\varnothing}[-n] \longrightarrow \ldots \longrightarrow Q_{Y_{-1}-Y_{-2}}[-1] \longrightarrow Q_{Y-Y_{-1}}[0] \longrightarrow Q \longrightarrow 0;$$

(i′) *If, in addition, Y is affine, then $H^r(Y, Q_{Y_p - Y_{p-1}}) = 0$, $\forall r \neq p$.*

(ii) *There exists the short exact sequence in $\mathcal{P}_Y^{\Sigma'_Y} \subseteq \mathcal{P}_Y$:*

$$0 \longrightarrow Q \longrightarrow k_{0*}k_0^! Q \longrightarrow k_{-1*}k_{-1}^! Q[1] \longrightarrow \ldots \longrightarrow k_{-n*}k_{-n}^! Q[n] \longrightarrow 0;$$

(ii′) *If, in addition, Y is affine, then $H_c^r(Y, k_{-p_*}k_{-p}^! Q) = 0$, $\forall r \neq p$.*

Let $(\mathfrak{F}, \mathfrak{F}')$ be a pair of linear n-flags which are in good position with respect to Σ and to each other. Then

(iii) *The single complex $\mathrm{Gr}_\delta^* Q[*]$ associated with the double complex of perverse sheaves $\mathrm{Gr}_F^i \mathrm{Gr}_G^j Q[i+j]$ is canonically isomorphic to Q in $D^b(\mathcal{P}_Y)$.*

(iii′) $H^r(Y, (k'_{-j_*} k'^!_{-j} Q)_{Y_i - Y_{i-1}}) = H_c^r(Y, (k'_{-j_*} k'^!_{-j} Q)_{Y_i - Y_{i-1}}) = 0$ *for every* $r \neq i + j$.

Proof. Note that, in Lemma 5.3.1,

$$Y = Y_0, \quad j_! j^* Q = k_{0!} k_0^* Q = Q_{Y_0 - Y_{-1}}, \quad \text{and } j_* j^! Q = k_{0*} k_0^! Q.$$

More generally,

$$R\Gamma_{Z_{-j} - Z_{-j-1}} = k'_{-j_*} k'^!_{-j} \quad \text{and} \quad (-)_{Y_i - Y_{i-1}} = k_{i!} k_i^*.$$

Statement (i) follows by a simple iteration of Lemma 5.3.1, where one uses at each step the fact that $\mathfrak{F}$ is in good position with respect to the initial Σ. In this step, the relative position of the linear sections and the strata at infinity are unimportant.

Statement (i′) follows from an iterated use of Theorem 5.1.2 and Remark 5.2.3. Here it is important that the linear sections meet the strata at infinity transversally.

Statements (ii) and (ii′) are proved in a similar way.

The double complex is obtained as follows: first resolve Q as in (ii), then resolve each resulting entry as in (i). We thus have quasi isomorphisms in $C^b(\mathcal{P}_Y)$: $Q \to \mathrm{Gr}^*_G Q[*] \leftarrow \mathrm{Gr}^\bullet_\delta Q[\bullet]$ and (iii) follows. Finally, (iii′) now follows from (27). □

Remark 6.1.2. The formation of the left, right and bi-sided resolutions of $Q \in \mathcal{P}_Y^{\Sigma_Y}$ in Lemma 6.1.1 are exact functor with values in $C^b(\mathcal{P}_Y)$.

Assumption 6.1.3 (Choice of the pair of linear flags $\mathfrak{F}, \mathfrak{F}'$). We fix a pair of linear n-flags $\mathfrak{F}, \mathfrak{F}'$ on $\mathbb{P}^N$ in good position with respect to Σ and to each other. A general pair in $F(N,n) \times F(N,n)$ will do.

Remark 6.1.4. Since $K \in D_Y^{\Sigma_Y}$, the perverse sheaves ${}^{\mathfrak{p}}\mathcal{H}^b(K) \in D_Y^{\Sigma_Y}$ and, with our choice of the pair $(\mathfrak{F}, \mathfrak{F}')$, the conclusions of Lemma 6.1.1 hold for all the ${}^{\mathfrak{p}}\mathcal{H}^b(K)$.

LEMMA 6.1.5. *If Y is affine, then*

$$H^r(\mathrm{Gr}^a_F \mathrm{Gr}^b_P\, R\Gamma(Y,K)) = 0, \qquad \forall r \neq a-b,$$

$$H^r(\mathrm{Gr}^a_G \mathrm{Gr}^b_P\, R\Gamma_c(Y,K)) = 0, \qquad \forall r \neq a-b.$$

If Y is quasi projective, then

$$H^r(Y, \mathrm{Gr}^a_\delta \mathrm{Gr}^b_P R\Gamma(Y,K)) = H^r(Y, \mathrm{Gr}^a_\delta \mathrm{Gr}^b_P R\Gamma_c(Y,K)) = 0 \qquad \forall r \neq a-b.$$

Proof. We prove the first assertion and the second and third are proved in a similar way. By (12), the group in question is

$$H^{r-(a-b)}(Y, {}^{\mathfrak{p}}\mathcal{H}^{-b}(K)_{Y_a - Y_{a-1}}[a])$$

and the required vanishing follows from Assumption 6.1.3, Remark 6.1.4, and Lemma 6.1.1(i′). □

6.2. *Proofs of Theorems* 4.1.1, 4.1.2, 4.1.3 *and* 4.2.1.

Proof of Theorems 4.1.1 *and* 4.1.2. By the first two assertions of Lemma 6.1.5, we can apply Proposition 5.6.1 to $(R\Gamma(Y,K), P, F)$ and to $(R\Gamma_c(Y,K), P, G)$ and prove the first two theorems. □

Proof of Theorem 4.1.3. We prove the version for $H(X,C)$. The case of $H_c(X,C)$ is proved in a similar way. Given the fixed embedding $Y \subseteq \mathbb{P}^N$, pick a stratification Σ of $\mathbb{P}^N$ adapted to f_*C and to the embedding. Choose a linear n-flag $\mathfrak{F}$ on $\mathbb{P}^N$ in good position with respect to Σ, e.g. general. Let Y_* be the corresponding n-flag on Y and set $X_* := f^{-1}Y_*$. Denote by $\widetilde{i}, \widetilde{j}, \widetilde{k}$ the associated embeddings as in (4).

By Theorem 4.1.1, the perverse spectral sequence for $H(Y, f_*C)$, i.e. the perverse Leray spectral sequence for $H(X, C)$, is the shifted Y_* flag spectral sequence for $H(Y, f_*C)$ (F-version).

Our goal is to identify the Y_* flag spectral sequence for $H(Y, f_*C)$ with the X_* spectral sequence for $H(X, C)$. It is sufficient to show that

$$(R\Gamma(X, C), F_{X_*}) = (R\Gamma(Y, f_*C), F_{Y_*}); \tag{34}$$

in fact, the shifted versions would also coincide and we would be done. In general, the two filtered complexes for Y_* and X_* do not coincide, due to the failure of the base change theorem. In the present case, transversality prevents this from happening.

We assume that C is injective. The filtered complex (C, F) is of c-soft type. On varieties c-soft and soft are equivalent notions and soft sheaves are $f_!$ and f_*-injective.

We have the filtered complex $(R\Gamma(X, C), F_{X_*})$, i.e. the result of applying $\Gamma(X, -)$ to the C-analogue of (5).

Transversality ensures that we have the first equality in (29): $\widetilde{j_{p!}}\widetilde{j}_p^* f_*C = f_* j_{p!} j_p^* C$. This implies that, by applying f_* to the C-analogue of (5), we obtain the f_*C analogue of (5) on Y with respect to Y_*, i.e. (34) holds and we are done. □

Proof of Theorem 4.2.1. In view of the third assertion of Lemma 6.1.5, the proof is analogous to the proofs given above. □

6.3. *Resolutions in* $D^b(\mathcal{P}_Y)$. In an earlier version of this paper, we worked in the derived category of perverse sheaves $D^b(\mathcal{P}_Y)$ which, in the case of field coefficients, is equivalent to D_Y ([4]). We are very thankful to one of the anonymous referees for suggesting the considerably more elementary approach contained in this paper which takes place in $D(\mathrm{Ab})$. On the other hand, the approach in $D^b(\mathcal{P}_Y)$ explains the relation "$P = \mathrm{Dec}(F)$" at the level of complexes of (perverse) sheaves, i.e. before taking cohomology. We outline this approach in the case of the F-construction on Y affine. We omit writing down the similar details in the case of the G-construction in the affine case and in the case of the $\delta(F, G)$-construction in the quasi projective case.

The approach is based on Beilinson's Equivalence Theorem [4].

In what follows, Y is affine; we work with field coefficients, for example $\mathbb{Q}$, $D^b(\mathcal{P}_Y)$ is endowed with the standard t-structure, D_Y with the perverse t-structure.

An equivalence of t-categories is a functor, between triangulated categories with t-structures, which is additive, commutes with translations, preserves distinguished triangles, is t-exact (i.e. it preserves the hearts) and is an equivalence.

THEOREM 6.3.1 ([4]). *There is an equivalence of t-categories, called the realization functor*

$$r_Y : D^b(\mathcal{P}_Y) \xrightarrow{\simeq} D_Y.$$

An outcome of this result is that it implies that, up to replacing $K \in D_Y$ with a complex naturally isomorphic to it, there is a filtration B on K such that $\mathrm{Gr}^b_B K[b] \in \mathcal{P}_Y$. When we recall Section 5.5, this means that (K, B) is in the heart $D_Y F_\beta$ of the canonical lift to $D_Y F$ of the perverse t-structure on D_Y. This circumstance, coupled with the construction (33), allows us to describe an inverse s_Y to r_Y, i.e. to assign to K a complex of perverse sheaves

$$s_Y(K) = s_Y(K, B) = \mathrm{Gr}^*_B K[*] =: \mathcal{K}^* \in D^b(\mathcal{P}_Y).$$

Fix a stratification $\mathfrak{S}$ of Y such that all the finitely many nonzero $\mathrm{Gr}^b_B K$ are $\mathfrak{S}$-constructible. Note that if $K \in D_Y^{\Sigma'}$, then it is possible that $\mathrm{Gr}^b_B K \notin D^{\Sigma'}$ and one may need to refine. Choose an embedding $Y \subseteq \mathbb{P}^N$, a stratification Σ on $\mathbb{P}^N$ adapted (cf. Definition 5.2.1) to $\mathfrak{S}$ and to the embedding, and a linear n-flag $\mathfrak{F}$ on $\mathbb{P}^N$ in good position (cf. Definition 5.2.4; a general one will do) with respect to Σ.

Let $\Delta = \Delta(F, P)$ be the diagonal filtration. By transversality, we have that $(K, \Delta) \in D_Y F_\beta$. We obtain the double complex $\mathcal{K}^{*,*} := \mathrm{Gr}^*_F \mathrm{Gr}^*_P K$, with associated single complex $s(\mathcal{K}^{*,*})^* = s_Y(K, \Delta)$ that maps quasi isomorphically onto $\mathcal{K}^*$. We also have $H^r(Y, \mathcal{K}^{*,*}) = 0$ for every $r \neq 0$ so that we have obtained a resolution with $H(Y, -)$-acyclic entries.

The single complex $s(\mathcal{K}^{*,*})$ admits the bête filtration by rows B_{row}, where $B^q_{\mathrm{row}} s(\mathcal{K}^{*,*})$ is the single complex associated with the double complex $\mathcal{K}^{*,*\leq q}$. i.e. the result of replacing with zeroes the entries strictly above the q-th row.

There is another filtration, $\mathrm{Std}_{\mathrm{col}}$, where $\mathrm{Std}^p_{\mathrm{col}}\, s(\mathcal{K}^{*,*})$ is the single complex associated with the double complex obtained by keeping the columns $p' < -p$, replacing the columns $p' > -p$ with zeroes, and replacing the entries $\mathcal{K}^{p,q}$ in the $(-p)$-th column by $\mathrm{Ker}\{\mathcal{K}^{p,q} \to \mathcal{K}^{p+1,q}\}$.

By Remark 6.1.2, the exactness properties of the construction of the resolution of Lemma 6.1.1 ensure that the natural map $(s(\mathcal{K}^{*,*}), \mathrm{Std}_{\mathrm{col}}) \to (\mathcal{K}^*, \mathrm{Std})$ is a filtered quasi isomorphism.

It is via this construction that the relation $\mathrm{Dec}(F) = P$ becomes transparent: it holds in $D^b(\mathcal{P}_Y) \simeq D_Y$ and it descends to $D(\mathrm{Ab})$:

1) It is elementary to verify that $\mathrm{Dec}(B_{\mathrm{row}}) = \mathrm{Std}_{\mathrm{col}}$ (cf. [3, Rem. 3.11.1]);
2) The filtered complex of perverse sheaves $(s(\mathcal{K}^{*,*}), B_{\mathrm{row}})$ corresponds to (K, F) under the equivalence r_Y;

3) The t-exactness of r_Y ensures that the filtered complex $(s(\mathcal{K}^{*,*}), \mathrm{Std}_{\mathrm{col}}) \simeq (\mathcal{K}, \mathrm{Std})$ corresponds to (K, P);
4) By the exactness of the construction, the complex $s(H^0(Y, \mathcal{K}^{*,*}))$ inherits the relation $\mathrm{Dec}(B_{\mathrm{row}}) = \mathrm{Std}_{\mathrm{col}}$.
5) The bifiltered complex $(s(H^0(Y, \mathcal{K}^{*,*})), \mathrm{Std}_{\mathrm{col}}, B_{\mathrm{row}})$ is a realization in the bifiltered derived category $DF_2(Ab)$ of $(R\Gamma(Y, K), P, F)$ and by 4) the perverse spectral sequence is identified with the shifted flag spectral sequence.

Remark 6.3.2. On the affine Y, the functor $H^0_{\mathcal{P}_Y} : \mathcal{P}_Y \to Ab$, $Q \mapsto H^0(Y, Q)$ is right-exact. By [4, §3], this right-exact functor admits a left-derived functor $LH^0_{\mathcal{P}_Y} : D^-(\mathcal{P}_Y) \to D^-(\mathrm{Ab})$. The complex in Step 4) realizes $LH^0_{\mathcal{P}_Y}(\mathcal{K})$.

7. **Applications**

The following results are due to M. Saito [18] who used his own mixed Hodge modules. We offer a proof based on the methods of this paper.

THEOREM 7.0.3. *Let Y be quasi projective. The perverse spectral sequences*

$$E_1^{p,q} = H^{2p+q}(Y, {}^{\mathfrak{p}}\mathcal{H}^{-p}(\mathbb{Z}_Y)) \Longrightarrow H^*(Y, \mathbb{Z}),$$

$$E_1^{p,q} = H_c^{2p+q}(Y, {}^{\mathfrak{p}}\mathcal{H}^{-p}(\mathbb{Z}_Y)) \Longrightarrow H_c^*(Y, \mathbb{Z})$$

are spectral sequences in the category of mixed Hodge structures.

Proof. We prove the first statement when Y is affine. The other variants are proved in similar ways. By Theorem 4.1.1, there is an n-flag Y_* on Y such that the perverse spectral sequence for $H(Y, \mathbb{Z})$ is the shifted spectral sequence of the flag spectral sequence

$$E_1^{p,q} = H^{p+q}(Y_p, Y_{p-1}, \mathbb{Z}) \Longrightarrow H^*(Y, \mathbb{Z}),$$

which is in the category of mixed Hodge structures. □

THEOREM 7.0.4. *Let $f : X \to Y$ be a map of varieties where Y is quasi projective. The perverse Leray spectral sequences*

$$E_1^{p,q} = H^{2p+q}(Y, {}^{\mathfrak{p}}\mathcal{H}^{-p}(f_*\mathbb{Z}_Y)) \Longrightarrow H^*(X, \mathbb{Z}),$$

$$E_1^{p,q} = H_c^{2p+q}(Y, {}^{\mathfrak{p}}\mathcal{H}^{-p}(f_!\mathbb{Z}_Y)) \Longrightarrow H_c^*(X, \mathbb{Z})$$

are spectral sequences in the category of mixed Hodge structures.

Proof. We prove the case of cohomology over an affine base Y and leave the rest to the reader. By Theorem 4.1.3, the perverse Leray spectral sequence for $H(X, \mathbb{Z})$ is the shifted X_* flag spectral sequence with respect to a suitable n-flag X_* on X. This latter is in the category of mixed Hodge structures. □

Remark 7.0.5 (Mixed Hodge structures and the decomposition theorem). In the paper [8], we endow the cohomology of the direct summands, appearing in the decomposition theorem for the proper push forward of the intersection cohomology complex of a proper variety, with natural, pure polarized Hodge structures. These structures arise as subquotients of the pure Hodge structure of the cohomology of a resolution of the singularities of the domain of the map. In particular, this endows the intersection cohomology groups of proper varieties with pure polarized Hodge structures. In the paper [9], we prove that for projective morphisms of projective varieties, one can realize the direct sum splitting mentioned above in the category of pure Hodge structures. The methods of this paper allow us to endow the intersection cohomology groups $IH(Y,\mathbb{Z})$ and $IH_c(Y,\mathbb{Z})$ of a quasi projective variety with a mixed Hodge structure and to extend all the results of [8] to the case of quasi projective varieties. Furthermore, we compare the resulting mixed Hodge structures with the ones arising from M. Saito's work and we show that they coincide. Details will appear in [6].

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