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Drift diffusion equations with fractional diffusion and the quasi-geostrophic equation

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Abstract

Motivated by the critical dissipative quasi-geostrophic equation, we prove that drift-diffusion equations with L^2 initial data and minimal assumptions on the drift are locally Hölder continuous. As an application we show that solutions of the quasi-geostrophic equation with initial L^2 data and critical diffusion $(-\Delta)^{1/2}$ are locally smooth for any space dimension.

1. **Introduction**

Nonlinear evolution equations with fractional diffusion arise in many contexts: In the quasi-geostrophic flow model (Constantin [4]), in boundary control problems (Duvaut-Lions [9]), in surface flame propagation and in financial mathematics. In this paper, motivated by the quasi-geostrophic model, we study the equation:

$$\begin{aligned} \partial_t \theta + v\cdot\nabla\theta &= -\Lambda\theta, \qquad x\in\mathbb{R}^N, \\ \operatorname{div} v &= 0, \end{aligned} \tag{1}$$

where $\Lambda\theta = (-\Delta)^{1/2}\theta$. The main two theorems are roughly the following *a priori* estimates:

THEOREM 1 (from L^2 to L^∞). *Let $\theta(t,x)$ be a function in*

$$L^\infty(0,T;L^2(\mathbb{R}^N))\cap L^2(0,T;H^{1/2}(\mathbb{R}^N)).$$

For every $\lambda>0$,

$$\theta_\lambda = (\theta-\lambda)_+.$$

If θ (*and* $-\theta$) *satisfies for every* $\lambda > 0$ *the level set energy inequalities*:

$$\int_{\mathbb{R}^N} \theta_\lambda^2(t_2, x)\,dx + 2\int_{t_1}^{t_2}\int_{\mathbb{R}^N} |\Lambda^{1/2}\theta_\lambda|^2\,dx\,dt \leq \int_{\mathbb{R}^N} \theta_\lambda^2(t_1, x)\,dx, \qquad 0 < t_1 < t_2,$$

then:

$$\sup_{x\in\mathbb{R}^N} |\theta(T,x)| \leq C^* \frac{\|\theta_0\|_{L^2}}{T^{N/2}}.$$

Remark. That solutions to equation (1) are expected to satisfy the energy inequality follows from writing Λ as the normal derivative of the harmonic extension of θ to the upper half space. Existence theory is sketched in appendix C. In the case of the quasi-geostrophic equation it can also be seen as a corollary of Córdoba and Córdoba [7].

Those energy inequalities are reminiscent of the notion of entropic solutions for scalar conservation laws. Consider a weak solution of (1) lying in $L^2(H^{1/2})$ for which we can define the equality (in the sense of distribution for example):

$$\phi'(\theta)v \cdot \nabla\theta = \operatorname{div}(v\phi(\theta)),$$

for any Lipschitz function ϕ. Then θ verifies the level set energy inequalities. In the case of the quasi-geostrophic equation, $v \in L^2(H^{1/2})$ and we can give meaning to:

$$v \cdot \nabla\phi(\theta).$$

Indeed, using the harmonic extension, we can show that if θ lies in $L^2(H^{1/2})$ so does $\phi(\theta)$. and so $\nabla\phi(\theta)$ lies in $L^2(H^{-1/2})$.

For the second theorem, (from L^∞ to C^α), we need better control of v:

THEOREM 2 (from L^∞ to C^α). *We define* $Q_r = [-r, 0] \times [-r, r]^N$, *for* $r > 0$. *Assume now that* $\theta(t, x)$ *is bounded in* $[-1, 0] \times \mathbb{R}^N$ *and* $v|_{Q_1} \in L^\infty(-1, 0; \mathrm{BMO})$; *then* θ *is* C^α *in* $Q_{1/2}$.

Remark 1. The global bound of θ is not really necessary, only local L^∞ and integrability at infinity against the Poisson kernel, as we will see later.

Remark 2. Note that both theorems depend only on the resulting energy inequality and not on the special form of Λ.

From these two theorems, the regularity of solutions to the quasi-geostrophic equation follows.

THEOREM 3. *Let* θ *be a solution to an equation*

$$\begin{aligned} \partial_t\theta + u \cdot \nabla\theta &= -\Lambda\theta, \qquad x \in \mathbb{R}^N, \\ \operatorname{div} u &= 0, \end{aligned} \tag{2}$$

with

$$u_j = \bar{R}_j[\theta], \tag{3}$$

$\bar{R}_j$ a singular integral operator. Assume also that θ verifies the level set energy inequalities stated in Theorem 1. Then, for every $t_0 > 0$ there exists α such that θ is bounded in $C^\alpha([t_0,\infty[\times\mathbb{R}^N)$.

Indeed, Theorem 1 gives that θ is uniformly bounded on $[t_0,\infty[$ for every $t_0 > 0$. Singular integral operators are bounded from L^∞ to BMO. This gives that $u \in L^\infty(t_0,\infty;\mathrm{BMO}(\mathbb{R}^N))$ and, after proper scaling, Theorem 2 gives the result of Theorem 3.

Remark 1. Higher regularity then follows from standard potential theory, when we notice that the fundamental solution of the operator:

$$\partial_t + \Lambda\theta = 0$$

is the Poisson kernel and that in the nonlinear term we can subtract a constant both θ_0 from θ and u_0 from u, this last one by a change of coordinates:

$$x^* = x - tu_0,$$

doubling its Hölder decay (see appendix).

Strictly speaking, the dissipative quasi-geostrophic flow model in the critical case corresponds to the case $N = 2$ and

$$u_1 = -R_2\theta, \qquad u_2 = R_1\theta,$$

where R_i is the usual Riesz transform defined from the Fourier transform: $\widehat{R_i\theta} = \frac{i\xi_i}{|\xi|}\hat{\theta}$. This model was introduced by some authors as a toy model to investigate the global regularity of solutions to 3D fluid mechanics (see for instance [4]). When replacing the diffusion term $-\Lambda$ by $-\Lambda^\beta$, $0 \le \beta \le 2$, the situation is classically decomposed into three cases according to the order of diffusion versus transport: The subcritical case for $\beta > 1$, the critical case for $\beta = 1$ and the supercritical case for $\beta < 1$.

Weak solutions have been constructed by Resnick in [12]. Constantin and Wu showed in [6] that in the subcritical case any solution with smooth initial value is smooth for all time. Constantin Córdoba and Wu showed in [5] that the regularity is conserved for all time in the critical case provided that the initial value is small in L^∞. In both the critical case and supercritical cases, Chae and Lee considered in [3] the well-posedness of solutions with initial conditions small in Besov spaces (see also Wu [16]).

Notice that our case corresponds to the critical case and global regularity in $C^{1,\beta}$, $\beta < 1$ is shown for any initial value in the energy space without hypothesis of smallness. This ensures that the solutions are classical.

Let us also cite a result of the maximum principle due to Córdoba and Córdoba [7], results of behavior in large time due to Schonbek and Schonbek [13], [14], and a criterion for blow-up in Chae [2].

Remark 2. In a recently posted preprint in arXiv, Kiselev, Nazarov, and Volberg present a very elegant proof of the fact that in 2D, solutions with periodic C^∞ data for the quasi-geostrophic equation remain C^∞ for all time ([10]).

We conclude our introduction by pointing out that our techniques also can be seen as a parabolic De Giorgi-Nash-Moser method to treat "boundary parabolic problems" of the type:

$$\begin{aligned} \operatorname{div}(a\nabla\theta) &= 0, && \text{in} \quad \Omega\times[0,T], \\ [f(\theta)]_t &= \theta_\nu, && \text{on} \quad \partial\Omega\times[0,T] \end{aligned}$$

that arise in boundary control (see Duvaut Lions [9]). Note also that results similar to Theorem 1 can be obtained even for systems. (See Vasseur [15] and Mellet, Vasseur [11] for applications of the method in fluid mechanics.)

2. L^∞ bounds

This section is devoted to the proof of Theorem 1. The simple proof is based on a recurrence nonlinear relation between consecutive truncations of θ at an increasing sequence of levels. Following the ideas of De Giorgi, this is attained thanks to the interplay between the energy inequality that controls $|\nabla\theta|$ by θ, and the opposite effect of the Sobolev inequality that controls θ by $\nabla\theta$, and the different homogeneity of these inequalities.

We use the truncation energy inequality for the levels:

$$\lambda = C_k = M(1-2^{-k}),$$

where M will be chosen later. This leads to the following energy inequality for the truncation function $\theta_k = (\theta - C_k)_+$:

$$\partial_t \int_{\mathbb{R}^N} {\theta_k}^2\,dx + 2\int_{\mathbb{R}^N} |\Lambda^{1/2}\theta_k|^2\,dx \le 0. \tag{4}$$

Let us fix a $t_0 > 0$, we want to show that θ is bounded for $t > t_0$. We introduce $T_k = t_0(1-2^{-k})$, and the level set of energy/dissipation of energy:

$$U_k = \sup_{t\ge T_k}\left(\int_{\mathbb{R}^N} {\theta_k}^2\,dx\right) + 2\int_{T_k}^{\infty}\int_{\mathbb{R}^N} |\Lambda^{1/2}\theta_k|^2\,dx\,dt.$$

Integrating (4) in time between s, $T_{k-1} < s < T_k$, and $t > T_k$ and between s and $+\infty$ we find:

$$U_k \le 2\int_{\mathbb{R}^N} {\theta_k}^2(s)\,dx.$$

Taking the mean value in s on $[T_{k-1}, T_k]$ we find:

$$U_k \le \frac{2^{k+1}}{t_0} \int_{T_{k-1}}^{\infty} \int_{\mathbb{R}^N} {\theta_k}^2 \, dx \, dt. \tag{5}$$

We want to control the right-hand side by U_{k-1} in a nonlinear way. Sobolev and Hölder inequalities give:

$$U_{k-1} \ge C \|\theta_{k-1}\|^2_{L^{\frac{2(N+1)}{N}}(]T_{k-1},\infty[\times\mathbb{R}^N)}.$$

Note that if $\theta_k > 0$ then $\theta_{k-1} \ge 2^{-k} M$. So,

$$\mathbf{1}_{\{\theta_k>0\}} \le \left(\frac{2^k}{M}\theta_{k-1}\right)^{2/N}.$$

Hence:

$$\begin{aligned} U_k &\le \frac{2^{k+1}}{t_0} \int_{T_{k-1}}^{\infty} \int_{\mathbb{R}^N} \theta_{k-1}^2 \mathbf{1}_{\{\theta_k>0\}} \, dx \, dt \\ &\le 2 \frac{2^{\frac{N+2}{N}k}}{t_0 M^{2/N}} \int_{T_{k-1}}^{\infty} \int_{\mathbb{R}^N} \theta_{k-1}^{2\frac{N+1}{N}} \, dx \, dt \le 2C \frac{2^{\frac{N+2}{N}k}}{t_0 M^{2/N}} U_{k-1}^{\frac{N+1}{N}}. \end{aligned}$$

For M such that $M/t_0^{N/2}$ is big enough (depending on U_0) we have U_k which converges to 0. This gives $\theta \le M$ for $t \ge t_0$. The same proof on $-\theta$ gives the same bound for $|\theta|$. Note that $U_0 \le \|\theta_0\|^2_{L^2}$. The scaling invariance $\theta_\varepsilon(s, y) = \theta(\varepsilon s, \varepsilon y)$ gives the final dependence with respect to $\|\theta_0\|_{L^2}$. □

This theorem leads to the following corollary.

COROLLARY 4. *There exists a constant $C^* > 0$ such that any solution θ of* (2), (3) *verifies*:

$$\sup_{x\in\mathbb{R}^N} |\theta(T, x)| \le C^* \frac{\|\theta_0\|_{L^2(\mathbb{R}^N)}}{T^{N/2}},$$

$$\|u(T, \cdot)\|_{\mathrm{BMO}(\mathbb{R}^N)} \le C^* \frac{\|\theta_0\|_{L^2(\mathbb{R}^N)}}{T^{N/2}}.$$

Proof. First note that the property on u follows directly from the property on θ and the imbedding of the Riesz function from L^∞ to BMO. We make use of the following result of Córdoba and Córdoba (see [7]): for any convex function ϕ we have the pointwise inequality:

$$-\phi'(\theta)\Lambda\theta \le -\Lambda(\phi(\theta)).$$

Making use of this inequality with:

$$\phi_k(\theta) = (\theta - C_k)_+ = \theta_k$$

leads to:

$$\partial_t \theta_k + u \cdot \nabla \theta_k \leq -\Lambda \theta_k.$$

Multiplying by θ_k and integrating in x give (4), when u is divergence free. □

Remark. We point out that the level set energy inequalities we assume in Theorem 1 are heuristically general facts (see Appendix C).

3. Local energy inequality

In order to develop the Hölder regularity method, it is necessary to localize by space and time truncation the energy inequality above. Due to the nonlocality of the diffusion operator, this appears complicated. Fortunately, $\Lambda\theta$ can be thought as the normal derivative of a harmonic extension of θ (the Dirichlet to Neumann operator of θ). This allows us to realize the truncation as a standard local one in one more dimension: We introduce first the harmonic extension L defined from $C_0^\infty(\mathbb{R}^N)$ to $C_0^\infty(\mathbb{R}^N \times \mathbb{R}^+)$ by:

$$\begin{aligned} -\Delta L(\theta) &= 0 \qquad && \text{in } \mathbb{R}^N \times (0,\infty), \\ L(\theta)(x,0) &= \theta(x) && \text{for } x \in \mathbb{R}^N. \end{aligned}$$

(This extension consists simply in convolving θ with the Poisson kernel of the upper half space in one more variable. See [1] for a general discussion.) Then the following result holds true: consider θ defined on $\mathbb{R}^N$. Then:

$$\Lambda\theta(x) = \partial_\nu [L\theta](x), \tag{6}$$

where we denote $\partial_\nu[L\theta]$ the normal derivative of $L\theta$ on the boundary

$$\left\{(x,0) | x \in \mathbb{R}^N\right\}.$$

In the following, we will denote the harmonic extension of θ by:

$$\theta^*(t,x,z) = L(\theta(t,\cdot))(x,z). \tag{7}$$

We denote $B_r = [-r,r]^N$ a cube in the x variable only, $B_r^* = B_r \times (0,r) \in \mathbb{R}^N \times (0,\infty)$ a cube in the x, z variables, sitting on the $z = 0$ plane, and $[y]_+ = \sup(0, y)$.

The rest of this section is devoted to the proof of the following proposition, a local energy inequality in the x, z variables. The effect of the nonlocal part of Λ becomes encoded locally in the extra variable. At this point, we know already that for any positive time $T > 0$:

$$\|\theta\|_{L^2(\mathbb{R}^N)} + \|\theta\|_{L^\infty(\mathbb{R}^N)} \leq C$$

uniformly in $t > T$, and for the application we have in mind, the quasi-geostrophic equation, this implies that

$$\|v\|_{L^2(\mathbb{R}^N)} + \|v\|_{\mathrm{BMO}(\mathbb{R}^N)} \le C,$$

uniformly in $t > T$. This is, therefore, the main hypothesis below.

PROPOSITION 5. *Let t_1, t_2 be such that $t_1 < t_2$ and let $\theta \in L^\infty(t_1, t_2; L^2(\mathbb{R}^N))$ with $\Lambda^{1/2}\theta \in L^2((t_1, t_2) \times \mathbb{R}^N)$, be solution to (1) with a velocity v satisfying*:

$$\|v\|_{L^\infty(t_1,t_2;\mathrm{BMO}(\mathbb{R}^N))} + \sup_{t_1 \le t \le t_2} \left| \int_{B_2} v(t,x)\,dx \right| \le C_v. \tag{8}$$

Then there exists a constant Φ (depending only on C_v) such that for every $t_1 \le t \le t_2$ and cut-off function η such that the restriction of $\eta[\theta^*]_+$ on B_2^* is compactly supported in $B_2 \times (-2, 2)$:

$$\begin{aligned}\int_{t_1}^{t_2} \int_{B_2^*} |\nabla(\eta[\theta^*]_+)|^2\,dx\,dz\,dt + \int_{B_2} (\eta[\theta]_+)^2(t_2, x)\,dx \\ \le \int_{B_2} (\eta[\theta]_+)^2(t_1, x)\,dx + \Phi \int_{t_1}^{t_2} \int_{B_2} ([\nabla\eta][\theta]_+)^2\,dx\,dt \\ + 2\int_{t_1}^{t_2} \int_{B_2^*} ([\nabla\eta][\theta^*]_+)^2\,dx\,dz\,dt.\end{aligned} \tag{9}$$

Remark. Note that, as a difference with the standard parabolic estimates, this energy inequality controls $\|\eta\theta\|_{L_t^\infty(L_x^2)}$ and $\|\nabla(\eta\theta^*)\|_{L_t^2(L_{xz}^2)}$. We are missing, in some sense, $\|\eta\theta^*\|_{L_t^p(L_{xz}^2)}$ that would provide the link between $\nabla(\eta\theta^*)$ and $\eta\theta$. In order to control $\eta\theta^*$ we will have to make a careful decomposition of $\eta\theta^*$ as the part coming from $\eta\theta$ as boundary value, and the rest coming from "far away". (Step 5: Propagation of the support property (12) and the proof of Lemma 6, below.)

Proof. We have for every $t_1 < t < t_2$:

$$\begin{aligned}0 &= \int_{B_2^*} \eta^2[\theta^*]_+ \Delta\theta^*\,dx\,dz \\ &= -\int_{B_2^*} |\nabla(\eta[\theta^*]_+)|^2\,dx\,dz + \int_{B_2^*} |\nabla\eta|^2[\theta^*]_+^2\,dx\,dz + \int_{B_2} \eta^2[\theta]_+\Lambda\theta\,dx.\end{aligned}$$

Using equation (1), we find that:

$$-\int_{B_2} \eta^2[\theta]_+\Lambda\theta\,dx = \frac{\partial}{\partial t}\left(\int_{B_2} \eta^2 \frac{[\theta]_+^2}{2}\,dx\right) - \int_{B_2} \nabla\eta^2 \cdot v \frac{[\theta]_+^2}{2}\,dx.$$

This leads to:

$$\int_{t_1}^{t_2}\int_{B_2^*}|\nabla(\eta[\theta^*]_+)|^2\,dx\,dz\,ds+\int_{B_2}\eta^2\frac{[\theta]_+^2(t_2)}{2}\,dx$$
$$\leq\int_{B_2}\eta^2\frac{[\theta]_+^2(t_1)}{2}\,dx+\int_{t_1}^{t_2}\int_{B_2^*}|\nabla\eta|^2[\theta^*]_+^2\,dx\,dz\,ds$$
$$+\left|\int_{t_1}^{t_2}\int_{B_2}\eta\nabla\eta\cdot v[\theta]_+^2\,dx\,ds\right|.$$

To dominate the last term, we first use the trace theorem and Sobolev imbedding to find:

$$\begin{aligned}\|\mathbf{1}_{\{B_2\}}\eta\theta_+\|^2_{L^{\frac{2N}{N-1}}(\mathbb{R}^N)}\leq C\|\mathbf{1}_{\{B_2\}}\eta\theta_+\|^2_{H^{1/2}(\mathbb{R}^N)}&=C\int_{B_2}(\eta\theta_+)\Lambda(\mathbf{1}_{\{B_2\}}\eta\theta_+)\,dx\\ &= C\int_0^\infty\int_{\mathbb{R}^N}|\nabla L(\mathbf{1}_{\{B_2\}}\eta\theta_+)|^2\,dx\,dz\\ &\leq C\int_0^\infty\int_{\mathbb{R}^N}|\nabla[\mathbf{1}_{\{B_2^*\}}\eta(\theta^*)_+]|^2\,dx\,dz\\ &= C\int_{B_2^*}|\nabla[\eta(\theta^*)_+]|^2\,dx\,dz.\end{aligned}$$

In the last inequality, we have used the support property of $\eta(\theta^*)_+$. In the second to the last inequality we have used the fact that $L(\mathbf{1}_{\{\mathbf{B_2}\}}\eta\theta_+)$ is harmonic and has the same trace as $\mathbf{1}_{\{B_2^*\}}\eta(\theta^*)_+$ at $z=0$. Therefore we split:

$$\left|\int_{t_1}^{t_2}\int_{\mathbb{R}^N}\nabla\eta^2\cdot v\frac{[\theta]_+^2}{2}\,dx\,ds\right|\leq\varepsilon\int_{t_1}^{t_2}\|\eta\theta_+\|^2_{L^{\frac{2N}{N-1}}(\mathbb{R}^N)}\,ds$$
$$+\frac{1}{\varepsilon}\int_{t_1}^{t_2}\|[\nabla\eta]v[\theta]_+\|^2_{L^{\frac{2N}{N+1}}(\mathbb{R}^N)}\,ds.$$

The first term is absorbed by the left. The second can be bounded, using Hölder inequality, by:

$$\frac{1}{\varepsilon}\|v\|^2_{L^\infty(t_1,t_2;L^{2N}(B_2))}\int_{t_1}^{t_2}\int_{\mathbb{R}^N}|[\nabla\eta][\theta]_+|^2\,dx\,ds,$$

which gives the desired result. □

4. From L^2 to L^∞

In these two sections (4 and 5) we follow De Giorgi's ideas in his classical proof of the Hölder continuity of solutions to elliptic equations (see [8]). The first step is a local, scalable version of the L^∞ bound above. It establishes that the space time localization of a level truncation of θ is bounded by the L^2 norm. The second step is the so-called “oscillation lemma”. We give a rough description of this lemma. Suppose that θ oscillates in $Q_1 = [-1,0] \times B_1$ between -2 and 2, but is negative most of the time. In particular, if $\|\theta_+\|_{L^2}$ is very small, then the local L^2 to L^∞ bound mentioned above, will imply that (in a small domain) θ^+ is very small. In particular, we prove that $\theta_+|_{Q_{1/2}=[-1/2,0]\times B_{1/2}} \le 2-\lambda$, effectively reducing the oscillations of θ by λ (see Lemma 6). Of course, we do not know *a priori* that $\|\theta_+\|_{L^2}$ is very small.

But we do know that in Q_1, θ is at least half of the time positive, or negative, say negative. We then have to reproduce a version of De Giorgi’s isoperimetric inequality which says that to go from zero to one θ needs “some room” (§5). Therefore the set $\{\theta \le 1\}$ is “strictly larger” than the set $\{\theta \le 0\}$ (see Lemma 8). Repeating this argument at truncation levels $C_k = 2 - 2^{-k}$, we fall, after a finite number of steps, k_0, into the first case, effectively diminishing the oscillations of θ by $\lambda 2^{-k_0}$. This implies Hölder continuity (§§6 and 7).

This section is devoted to the first step of the proof, the L^2 to L^∞ lemma. It says that, under suitable conditions on v, we can control the L^∞ norm of θ from the L^2 norm of both θ and θ^* locally.

LEMMA 6. *We assume, as in Section 3, that*

$$\|v\|_{L^\infty(-4,0;\mathrm{BMO}(\mathbb{R}^N))} + \sup_{-4\le t\le 0} \left| \int_{B_4} v(t,x)\,dx \right| \le C_v.$$

Then, there exists $\varepsilon_0 > 0$ (depending only on N and C_v), and $\lambda > 0$ (depending only on N) such that for every θ solution to (1) the following property holds true.

If

$$\theta^* \le 2 \qquad \text{in } [-4,0] \times B_4^*,$$

and

$$\int_{-4}^{0} \int_{B_4^*} (\theta^*)_+^2 \,dx\,dz\,ds + \int_{-4}^{0} \int_{B_4} (\theta)_+^2 \,dx\,ds \le \varepsilon_0,$$

then:

$$(\theta)_+ \le 2-\lambda \qquad \text{on } [-1,0] \times B_1.$$

Remark. Note that this is not a “pure” L^2 to L^∞ estimate, since we assume that θ^* is already bounded by 2. Nevertheless, since our final objective is an improvement of the oscillation of θ^*, the gain from 2 to $2-\lambda$ will suffice.

Proof. We split the proof of the lemma into several steps. Steps 1 and 2 are preliminary views. We construct auxiliary barriers and recurrence constants. Step 3 describes which is the recurrence relation we are aiming for. The actual proof really starts in Step 4. As in the proof of Theorem 1 at the beginning of the paper, we will now consider a sequence of truncations for an increasing sequence of levels C_k converging to $2-\lambda$, and will prove that by the time we reach $2-\lambda$, the corresponding truncation has zero energy and is, thus, identically zero. Before giving into the proof, we give an informal description of the arguments involved. In principle, we would like to prove that if θ_+^* is less than 2 in the cylinder $B_4^* \times [-4, 0]$ and both θ_+^* and θ_+ have very small L^2 norm, then they are both less than $2-\lambda$ in $B_1^* \times [-1, 0]$. Thanks to the barrier b_1 below, we notice that it is enough that θ_+ be below $2-\lambda$, since θ_+^* is bounded by b_1 plus the harmonic extension of θ_+. The second observation we make is that if the L^2 norm of θ_+ is small, θ_+^* dips to very small values for z small (proportionally in some way to $\|\theta_+\|_{L^2}$). Indeed, the part corresponding to b_1 goes linearly to zero while the Poisson kernel smoothes θ_+. In other words the influence of the global part of Λ, reflected by b_1, decays linearly as z goes to zero, and we can almost eliminate as we truncate θ at increasing levels (Step 5, propagation of the support property). This eliminates the need of a truncation in z and almost eliminates the influence of the global part of Λ. It only remains on the small "lateral edges", that is where the space truncation takes place and for very small z, and that has exponential decay (barrier b_2). That is what allows us to obtain the appropriate recurrence relation for A_k, that implies that "$\theta_\infty = 0$" on $B_1 \times [-1, 0]$, i.e. $\theta \le 2-\lambda$.

Step 4 is a first "long jump" to $k \ge 12N$, that puts us into the appropriate configuration described above, to start the inductive process provided θ_+ is small enough.

Step 1. *Useful barrier functions.* The following two barrier functions, b_1 and b_2, will be used to control how the values of θ^* far from the "disc" $D_1^* = B_1 \times \{0\}$ influence θ^* near D_1^*.

Consider the function b_1, defined by:

$$\begin{aligned} \Delta b_1 &= 0 && \text{in } B_4^* \\ b_1 &= 2 && \text{on the sides of the cube } B_4^* \text{ except for } z = 0 \\ b_1 &= 0 && \text{for } z = 0. \end{aligned}$$

Then there exists $\lambda > 0$ such that:

$$b_1(x, z) \le 2 - 4\lambda \text{ on } B_2^*.$$

This result follows directly from the maximum principle.

We consider now b_2 harmonic function defined by:

$$\begin{aligned} \Delta b_2 &= 0 && \text{in } [0, \infty[\times[0, 1], \\ b_2(0, z) &= 2 && 0 \le z \le 1, \\ b_2(x, 0) = b_2(x, 1) &= 0 && 0 < x < \infty. \end{aligned}$$

Then there exists $\bar{C} > 0$ such that:

$$|b_2(x, z)| \le \bar{C} e^{-x/2}. \tag{10}$$

Notice that $\bar{C}$ is universal. Indeed we can see easily that

$$b_2(x,z) \le 2\sqrt{2}\cos(z/2)e^{-x/2},$$

since this function is harmonic and bigger than b_2 on the boundary.

Step 2. *Setting of constants.* In this step we fix a set of constants. We make the choice to set them right away to convince the reader that the proof is not circular.

LEMMA 7. *There exist* $0<\delta<1$ *and* $M>1$ *such that for every* $k>0$:

$$2N\bar{C}e^{-\frac{2^{-k}}{4(\sqrt{2}+1)\delta^k}} \le \lambda 2^{-k-2},$$
$$\frac{M^{-k/2}}{\delta^{\frac{N(k+1)}{2}}}\|P(1)\|_{L^2} \le \lambda 2^{-k-2},$$
$$M^{-k} \ge C_0^k M^{-(1+1/N)(k-3)} \qquad k \ge 12N,$$

where $\bar{C}$ *is defined from Step* 1, $P(1)$ *is the the value at* $z=1$ *of the Poisson kernel* $P(z)(x)$, *and* C_0 *is defined by* (16).

The proof is easy. We first construct δ to verify the first inequality in the following way. If $\delta<1/4$, the inequality is true for $k>k_0$ due to the exponential decay. If necessary, we then choose δ smaller to make the inequality also valid for $k<k_0$. Now that δ has been fixed, we have to choose M large to satisfy the remaining inequalities. Note that the second inequality is equivalent to:

$$\left(\frac{2}{\delta^{N/2}\sqrt{M}}\right)^k \le \frac{\lambda\delta^{N/2}}{4\|P(1)\|_{L^2}}.$$

It is, thus, sufficient to take:

$$M \ge \sup\left(\frac{2}{\delta^{N/2}}, \frac{8\|P(1)\|_{L^2}}{\lambda\delta^N}\right)^2.$$

The third inequality is equivalent to:

$$\left(\frac{M}{C_0^N}\right)^{k/N} \ge M^{3(1+1/N)}.$$

For this case it is sufficient to take $M \ge \sup(1, C_0^{2N})$. Indeed, this ensures $M^2/C_0^{2N} \ge M$ and so:

$$\left(\frac{M}{C_0^N}\right)^{k/N} \ge M^{k/(2N)} \ge M^6,$$

for $k \ge 12N$. But $M^6 \ge M^{3(1+1/N)}$ for $M \ge 1$ and $N \ge 2$.

Therefore we can fix:

$$M = \sup\left(1, C_0^{2N}, \left(\frac{2}{\delta^{N/2}}\right)^2, \left(\frac{8\|P(1)\|_{L^2}}{\lambda\delta^N}\right)^2\right).$$

The constant λ, δ, and M are now fixed for the rest of the proof. The constant ε_0 will be constructed from these.

Step 3. *Induction.* We set:

$$\theta_k = (\theta - C_k)_+, \qquad \theta_k{}^* = (\theta^* - C_k)_+,$$

with $C_k = 2 - \lambda(1+2^{-k})$. Note that $\theta_k{}^* \neq (\theta_k)^*$. We consider a cut-off function in x only such that:

$$\mathbf{1}_{\{B_{1+2^{-k-1/2}}\}} \le \eta_k \le \mathbf{1}_{\{B_{1+2^{-k}}\}} \text{ and } |\nabla \eta_k| \le C2^k.$$

We denote:

$$A_k = \int_{-1-2^{-k}}^{0} \int_0^{\delta^k} \int_{\mathbb{R}^N} |\nabla(\eta_k \theta_k^*)|^2 \, dx\, dz\, dt + \sup_{t\in[-1-2^{-k},0]} \int_{\mathbb{R}^N} (\eta_k \theta_k)^2 \, dx.$$

We want to prove simultaneously that for every $k \ge 0$:

$$A_k \le M^{-k} \tag{11}$$

$$\eta_k \theta_k^* = 0 \qquad \text{for } \delta^k \le z \le \inf(2, \delta^{k-1}). \tag{12}$$

Step 4. (*Initial step*). We prove in this step that if ε_0 is small enough, then (11) is verified for $0 \le k \le 12N$, and that (12) is verified for $k=0$. We use the energy inequality (9) with cut-off function $\eta_k(x)\psi(z)$ where ψ is a fixed cut-off function in z only. Taking the mean value of (9) in t_1 between -4 and -2, we find that (11) is verified for $0 \le k \le 12N$ if ε_0 is taken such that:

$$C2^{24N}(1+\Phi)\varepsilon_0 \le M^{-12N}. \tag{13}$$

We have used that $|\nabla \eta_k|^2 \le C2^{24N}$ for $0 \le k \le 12N$. Let us consider now the support property (12). By the maximum principle, we have:

$$\theta^* \le (\theta_+ \mathbf{1}_{B_4}) * P(z) + b_1(x,z),$$

in $\mathbb{R}^+ \times B_4^*$, where $P(z)$ is the Poisson kernel. Indeed, the right-hand side function is harmonic, positive and the trace on the boundary is bigger that the one of θ^*.

From Step 1 we have: $b_1(x,z) \le 2 - 4\lambda$. Moreover:

$$\|\theta_+ \mathbf{1}_{B_4} * P(z)\|_{L^\infty(z\ge 1)} \le C\|P(1)\|_{L^2}\sqrt{\varepsilon_0} \le C\sqrt{\varepsilon_0}.$$

Choosing ε_0 small enough such that this constant is smaller that 2λ gives:

$$\theta^* \le 2 - 2\lambda \qquad \text{for } 1 \le z \le 2, t \ge 0, x \in B_2,$$

and so:

$$\theta_0^* = (\theta^* - (2-2\lambda))_+ \le 0 \qquad \text{for } 1 \le z \le 2, t \ge 0, x \in B_2.$$

Hence $\eta_0 \theta_0^*$ vanishes for $1 = \delta^0 \le z \le 2$.

Step 5. *Propagation of the support property* (12). Assume that (11) and (12) are verified at k. We want to show that (12) is verified at $(k+1)$. We will show also that the following is verified at k:

$$\eta_{k+1}\theta^*_{k+1} \leq [(\eta_k\theta_k) * P(z)]\eta_{k+1}, \qquad \text{on } \bar{B}^*_k, \tag{14}$$

where $\bar{B}^*_k = B_{1+2^{-k}} \times [0, \delta^k]$. We want to control θ^*_k on this set by harmonic functions taking into account the contributions of the sides one by one. Consider $B_{1+2^{-k-1/2}} \times [0, \delta^k]$. On $z = \delta^k$ we have no contribution thanks to the induction property (12) at k (the trace is equal to 0). The contribution of the side $z = 0$ can be controlled by: $\eta_k\theta_k * P(z)$. (It has the same trace as θ_k on $B_{1+2^{-k-1/2}}$.)

On each of the other sides we control the contribution by the function of $x = (x_1, \cdot\cdot, x_N)$:

$$b_2((x_i - x^+)/\delta^k, z/\delta^k) + b_2((-x_i + x^-)/\delta^k, z/\delta^k),$$

where $x^+ = (1 + 2^{-k-1/2})$ and $x^- = -x^+$. Indeed, b_2 is harmonic, and on the side x_i^+ and x_i^- it is bigger than 2. Finally, by the maximum principle:

$$\theta^*_k \leq \sum_{i=1}^{N} \Big[b_2((x_i - x^+)/\delta^k, z/\delta^k) + b_2((-x_i + x^-)/\delta^k, z/\delta^k)\Big] + (\eta_k\theta_k) * P(z).$$

From Step 1, for $x \in B_{1+2^{-k-1}}$:

$$\sum_{i=1}^{N} \Big[b_2((x_i - x^+)/\delta^k, z/\delta^k) + b_2((-x_i + x^-)/\delta^k, z/\delta^k)\Big]$$
$$\leq 2N\bar{C}e^{-\frac{2^{-k}}{4(\sqrt{2}+1)\delta^k}} \leq \lambda 2^{-k-2},$$

(thanks to Step 2). This gives (14) since:

$$\theta^*_{k+1} \leq (\theta^*_k - \lambda 2^{-k-1})_+.$$

More precisely,

$$\theta^*_{k+1} \leq ((\eta_k\theta_k) * P(z) - \lambda 2^{-k-2})_+.$$

Now,

$$\eta_{k+1}\theta^*_{k+1} \leq ((\eta_k\theta_k) * P(z) - \lambda 2^{-k-2})_+.$$

From the second property of Step 2, we find for $\delta^{k+1} \leq z \leq \delta^k$:

$$|(\eta_k\theta_k) * P(z)| \leq \sqrt{A_k}\|P(z)\|_{L^2}$$
$$\leq \frac{M^{-k/2}}{\delta^{(k+1)N/2}}\|P(1)\|_{L^2} \leq \lambda 2^{-k-2}.$$

The last inequality makes use of Step 2. Therefore:

$$\eta_{k+1}\theta^*_{k+1} \leq 0 \qquad \text{for } \delta^{k+1} \leq z \leq \delta^k.$$

Note, in particular, that with Step 4 this gives that (12) is verified up to $k = 12N + 1$ and (14) up to $k = 12N$.

Step 6. *Propagation of Property* (11). We show in this step that if (12) is true for $k-3$ and (11) is true for $k-3$, $k-2$ and $k-1$ then (11) is true for k.

First notice that from Step 5, (12) is true at $k-2$, $k-1$, and k. We just need to show that:

$$A_k \le C_0^k (A_{k-3})^{1+1/N} \qquad \text{for } k \ge 12N+1, \tag{15}$$

with:

$$C_0 = C\frac{2^{1+2/N}}{\lambda^{2/N}}. \tag{16}$$

Indeed, if we use the third inequality of Step 2, this will give us the result.

Step 7. *Proof of* (15). Since $\eta_k \theta_k^* \mathbf{1}_{\{0<z<\delta^{k-1}\}}$ has the same trace at $z=0$ as $(\eta_k\theta_k)^*$ and the latter is harmonic we have:

$$\begin{aligned}\int_0^{\delta^{k-1}} \int_{\mathbb{R}^N} \int |\nabla(\eta_k\theta_k^*)|^2 &= \int_0^\infty \int_{\mathbb{R}^N} |\nabla(\eta_k\theta_k^* \mathbf{1}_{\{0<z<\delta^{k-1}\}})|^2 \\ &\ge \int_0^\infty \int_{\mathbb{R}^N} |\nabla(\eta_k\theta_k)^*|^2 = \int_{\mathbb{R}^N} |\Lambda^{1/2}(\eta_k\theta_k)|^2.\end{aligned}$$

Note that we have used (12) in the first equality. Sobolev and Hölder inequalities give:

$$A_{k-3} \ge C\|\eta_{k-3}\theta_{k-3}\|^2_{L^{\frac{2(N+1)}{N}}([-1-2^{-k-3},0]\times\mathbb{R}^N)}.$$

From (14):

$$\|\eta_{k-2}\theta_{k-2}^*\|^2_{L^{\frac{2(N+1)}{N}}} \le \|P(1)\|^2_{L^1}\|\eta_{k-3}\theta_{k-3}\|^2_{L^{\frac{2(N+1)}{N}}}.$$

Thus,

$$\begin{aligned}A_{k-3} &\ge C\|\eta_{k-2}\theta_{k-2}^*\|^2_{L^{\frac{2(N+1)}{N}}} + C\|\eta_{k-3}\theta_{k-3}\|^2_{L^{\frac{2(N+1)}{N}}} \\ &\ge C\left(\|\eta_{k-2}\theta_{k-1}^*\|^2_{L^{\frac{2(N+1)}{N}}} + \|\eta_{k-2}\theta_{k-1}\|^2_{L^{\frac{2(N+1)}{N}}}\right).\end{aligned}$$

Using (12) and the fact that η_k is a cut-off function in x, we have that $\eta_k\theta_k^*$ vanishes on the boundary of $B_{1+2^{-k}} \times [-\delta^k, \delta^k]$. We can then apply Proposition 5 on $\eta_k\theta_k^*\mathbf{1}_{\{0<z<\delta^{k-1}\}}$. Taking the mean value of (9) in t_1 between $-1-2^{-k-1}$ and $-1-2^{-k}$, we find:

$$A_k \le C2^{2k}(\Phi+2)\left(\int_0^{\delta^k}\int \eta_{k-1}^2 {\theta_k}^2 + \int_0^{\delta^k}\int \eta_{k-1}^2 {\theta_k^*}^2\right).$$

We have used here the fact that $|\nabla\eta|^2 \le C2^{2k}\eta_{k-1}^2$. If $\theta_k > 0$ then $\theta_{k-1} \ge 2^{-k}\lambda$. Now,

$$\mathbf{1}_{\{\theta_k>0\}} \le \frac{C2^k}{\lambda}\theta_{k-1},$$

and

$$\mathbf{1}_{\{\eta_{k-1}>0\}}\mathbf{1}_{\{\theta_k>0\}} \le \frac{C2^k}{\lambda}\eta_{k-2}\theta_{k-1}.$$

Therefore:

$$\int \eta_{k-1}^2{\theta_k}^2 + \int \eta_{k-1}^2\theta_k^{*2}$$
$$\le \frac{C2^{2k/N}}{\lambda^{2/N}}\left[\int(\eta_{k-2}\theta_{k-1})^{\frac{2(N+1)}{N}} + \int(\eta_{k-2}\theta_{k-1}^*)^{\frac{2(N+1)}{N}}\right].$$

and so:

$$A_k \le \frac{C2^{k(2+2/N)}}{\lambda^{2/N}}A_{k-3}^{1+1/N}.$$

This gives (15), for C big enough compared to $\lambda^{2/N}$. □

5. The second technical lemma

At this moment we have proven that if θ, θ^* are less than 2, and their L^2 norm is very small in $B_4 \times [-4, 0]$ then both θ, θ^* are less than $2-\lambda$ on $B_1 \times [-1, 0]$. That is, their oscillation decreased. We need to prove now that it is enough that

$$|\{\theta^* \le 0\}| \ge \frac{|Q_4|}{2}$$

to imply that θ, θ^* are less than $2-\lambda$ in $B_1 \times [-1, 0]$, since this is our induction hypothesis. This is based on the fact that, due to the energy inequality, there must be a quantitative decay in measure between the consecutive level sets $\{\theta \ge 0\}$, $\{\theta \ge 1\}$, $\{\theta \ge 2 - 1/2\}$, $\{\theta \ge 2 - 1/4\}$, etc.... Lemma 8 below measures this quantitative separation between $\{\theta \le 0\}$ and $\{\theta \ge 1\}$, i.e. if $\{0 < \theta^* < 1\}$ is small then $(\theta^* - 1)_+$ is very small.

We set $Q_r = B_r \times [-r, 0]$ and $Q_r^* = B_r^* \times [-r, 0]$.

LEMMA 8. *For every $\varepsilon_1 > 0$, there exists a constant $\delta_1 > 0$ with the following property*: *For every solution θ to* (1) *with v verifying* (8) *and*:

$$\theta^* \le 2 \qquad \text{in } Q_4^*,$$
$$\left|\{(x,z,t) \in Q_4^*;\ \theta^*(x,z,t) \le 0\}\right| \ge \frac{|Q_4^*|}{2},$$

we have the following implication: *If*

$$\left|(x,z,t) \in Q_4^*;\ 0 < \{\theta^*(x,z,t) < 1\}\right| \le \delta_1,$$

then:

$$\int_{Q_1} (\theta-1)_+^2 \, dx \, dt + \int_{Q_1^*} (\theta^*-1)_+^2 \, dx \, dz \, dt \le C\sqrt{\varepsilon_1}.$$

Note that δ_1 depends only on N and C_v in (8).

Proof. Take $\varepsilon_1 \ll 1$. From the energy inequality (9) and using that $\theta^* \le 2$ in Q_4^*, we get:

$$\int_{-4}^0 \int_{B_1^*} |\nabla \theta_+^*|^2 \, dx \, dz \, dt \le C.$$

Let:

$$K = \frac{4 \int |\nabla \theta_+^*|^2 \, dx \, dz \, dt}{\varepsilon_1}.$$

Then:

$$\left| \{t \mid \int_{B_1^*} |\nabla \theta_+^*|^2(t) \, dx \, dz \ge K\} \right| \le \frac{\varepsilon_1}{4}. \tag{17}$$

For all $t \in \{t \mid \int_{B_1^*} |\nabla \theta_+^*|^2(t) \, dx \, dz \le K\}$, the De Giorgi lemma (see appendix) gives that:

$$|\mathscr{A}(t)||\mathscr{B}(t)| \le |\mathscr{C}(t)|^{1/2} K^{1/2},$$

where:

$$\begin{aligned}
\mathscr{A}(t) &= \{(x,z) \in B_1^* \mid \theta^*(t,x,z) \le 0\},\\
\mathscr{B}(t) &= \{(x,z) \in B_1^* \mid \theta^*(t,x,z) \ge 1\},\\
\mathscr{C}(t) &= \{(x,z) \in B_1^* \mid 0 < \theta^*(t,x,z) < 1\}.
\end{aligned}$$

Let us set

$$\delta_1 = \varepsilon_1^8,$$

$$I = \{t \in [-4,0];\ |\mathscr{C}(t)|^{1/2} \le \varepsilon_1^3 \text{ and } \int_{B_1^*} |\nabla \theta_+^*|^2(t) \, dx \, dz \le K\}.$$

First we have, using the Tchebichev inequality:

$$\left|\{t \in [-4,0];\ |\mathscr{C}(t)|^{1/2} \ge \varepsilon_1^3\}\right| \le \frac{|\{(t,x,z) \mid 0 < \theta^* < 1\}|}{\varepsilon_1^6} \le \frac{\delta_1}{\varepsilon_1^6} \le \varepsilon_1^2 \le \varepsilon_1/4.$$

Hence $|[-4,0] \setminus I| \le \varepsilon_1/2$. Secondly we get for every $t \in I$ such that $|\mathscr{A}(t)| \ge 1/4$:

$$|\mathscr{B}(t)| \le \frac{|\mathscr{C}(t)|^{1/2} K^{1/2}}{|\mathscr{A}(t)|} \le 4C\varepsilon_1^{5/2} \le \varepsilon_1^2. \tag{18}$$

In particular:

$$\int_{B_1^*} {\theta_+^*}^2(t) \, dx \, dz \le 4(|\mathscr{B}(t)| + |\mathscr{C}(t)|) \le 8\varepsilon_1^2.$$

But

$$\int_{B_1} \theta_+^2(t)\,dx = \int_{B_1} \theta_+^{*\,2}(t,x,z)\,dx - 2\int_0^z \int_{B_1} \theta_+^*(t)\partial_z\theta^*\,dx\,d\bar{z},$$

for any z. And so, integrating in z on [0, 1], we find:

$$\int_{B_1} \theta_+^2(t)\,dx \le \int_{B_1^*} \theta_+^{*\,2}(t,x,z)\,dx\,dz + 2\sqrt{K}\sqrt{\int_{B_1^*} \theta_+^{*\,2}(t)\,dx\,dz} \le C\sqrt{\varepsilon_1}.$$

We want to show that $|\mathcal{A}(t)| \ge 1/4$ for every $t \in I \cap [-1,0]$. First, since

$$|\{(t,x,z) \mid \theta^* \le 0\}| \ge |Q_4^*|/2,$$

there exists $t_0 \le -1$ such that $|\mathcal{A}(t_0)| \ge 1/4$. So for this t_0, $\int \theta_+^2(t_0)\,dx \le C\sqrt{\varepsilon_1}$. Using the energy inequality (9), for any $r > 0$ (where $\nabla\eta$ is of order $1/r$), we have for every $t \ge t_0$:

$$\int_{B_1} \theta_+^2(t)\,dx \le \int_{B_1} \theta_+^2(t_0)\,dx + \frac{C(t-t_0)}{r} + Cr.$$

Let us choose r such that

$$Cr + C\sqrt{\varepsilon_1} \le 1/128.$$

So for $t - t_0 \le \delta^* = r/(128C)$ we have:

$$\int_{B_1} \theta_+^2(t)\,dx \le \frac{1}{64}.$$

(Note that the δ^* do not depend on ε_1. Hence we can suppose $\varepsilon_1 \ll \delta^*$.) We have:

$$\begin{aligned}\theta_+^*(z) &= \theta_+ + \int_0^z \partial_z\theta_+^*\,d\bar{z}\\ &\le \theta_+ + \sqrt{z}\left(\int_0^z |\partial_z\theta_+^*|^2\,d\bar{z}\right)^{1/2}.\end{aligned}$$

So, for $t - t_0 \le \delta^*$, $t \in I$ and $z \le \varepsilon_1^2$ we have for each x:

$$\theta_+^*(t,x,z) \le \theta_+(t,x) + \left(\varepsilon_1^2 \int_0^\infty |\partial_z\theta_+^*|^2\,d\bar{z}\right)^{1/2}.$$

The integral, in x only, of the square of the right-hand side term is less than $1/8 + C\sqrt{\varepsilon_1} \le 1/4$. So by Tchebichev, for every fixed $z \le \varepsilon_1$:

$$|\{x \in B_1,\ \theta_+^*(t,x,z) \ge 1\}| \le \frac{1}{4}.$$

Integrating in z on $[0,\varepsilon_1^2]$ gives:

$$|\{z\leq\varepsilon_1^2,\ x\in B_1,\ \theta_+^*(t)\geq 1\}|\leq\frac{\varepsilon_1^2}{4}.$$

First we work in $B_1\times[0,\varepsilon_1^2]$. Since $|\mathscr{C}(t)|\leq\varepsilon_1^6$, this gives

$$\begin{aligned}|\mathscr{A}(t)|&\geq|B_1|\varepsilon_1^2-|\{z\leq\varepsilon_1^2,\ x\in B_1,\ \theta_+^*(t)\geq 1\}|-|\mathscr{C}(t)|\\&\geq\varepsilon_1^2(1-1/4)-\varepsilon_1^6\geq\varepsilon_1^2/2.\end{aligned}$$

In the same way as in (18) we find:

$$|\mathscr{B}(t)|\leq\frac{|\mathscr{C}(t)|^{1/2}K^{1/2}}{|\mathscr{A}(t)|}\leq C\sqrt{\varepsilon_1},$$

and:

$$\begin{aligned}|\mathscr{A}(t)|&\geq 1-|\mathscr{B}(t)|-|\mathscr{C}(t)|\\&\geq 1-2\sqrt{\varepsilon_1}-\varepsilon_1^6\geq 1/4.\end{aligned}$$

Hence, for every $t\in[t_0,t_0+\delta^*]\cap I$ we have: $|\mathscr{A}(t)|\geq 1/4$. On $[t_0+\delta^*/2,t_0+\delta^*]$ there exists $t_1\in I$ $(\delta^*\geq\varepsilon_1/4)$. And so, we can construct an increasing sequence t_n, $0\geq t_n\geq t_0+n\delta^*/2$ such that $|\mathscr{A}(t)|\geq 1/4$ on $[t_n,t_n+\delta^*]\cap I\supset[t_n,t_{n+1}]\cap I$. Finally on $I\cap[-1,0]$ we have $|\mathscr{A}(t)|\geq 1/4$. This gives from (18) that for every $t\in I\cap[-1,0]$: $|\mathscr{B}(t)|\leq\varepsilon_1/16$. Hence:

$$|\{\theta^*\geq 1\}|\leq\varepsilon_1/16+\varepsilon_1/2\leq\varepsilon_1.$$

Since $(\theta^*-1)_+\leq 1$, this gives that:

$$\int_{Q_1^*}(\theta^*-1)_+^2\,dx\,dz\,dt\leq\varepsilon_1.$$

We have for every t,x fixed:

$$\theta-\theta^*(z)=-\int_0^z\partial_z\theta^*\,dz.$$

So:

$$(\theta-1)_+^2\leq 2\left((\theta^*(z)-1)_+^2+\left(\int_0^z|\nabla\theta^*|\,dz\right)^2\right)$$

for any z. Hence

$$(\theta-1)_+^2\leq\frac{2}{\sqrt{\varepsilon_1}}\int_0^{\sqrt{\varepsilon_1}}(\theta^*-1)_+^2\,dz+2\sqrt{\varepsilon_1}\int_0^{\sqrt{\varepsilon_1}}|\nabla\theta^*|^2\,dz.$$

Therefore:

$$\int_{Q_1}(\theta-1)_+^2\,dx\,ds\leq C\sqrt{\varepsilon_1}.\qquad\square$$

6. **Oscillation lemma**

In the first technical lemma, we have established that if $0 \leq \theta^* \leq 2$ and its energy or norm is very small, in B_4^*, then, $\theta^* \leq 2-\lambda$ in B_1; i.e., the oscillation of θ actually decays. We want now to get rid of the "very small" hypothesis. This second lemma proves that if $\theta^* \leq 0$ "half of the time" and it needs very little room, δ, to go from $\{\theta^* \leq 0\}$ to $\{\theta \geq 1\}$, it is because $(\theta - 1)_+$ has very small norm to start with. This produces a dichotomy: or the support of θ decreases substantially, or θ becomes small anyway.

PROPOSITION 9. *There exists $\lambda^* > 0$ such that for every solution θ of* (1) *with v verifying* (8), *if*:

$$\theta^* \leq 2 \text{ in } Q_1^*, \qquad |\{(t,x,z) \in Q_1^*;\ \theta^* \leq 0\}| \geq \frac{1}{2},$$

then:

$$\theta^* \leq 2-\lambda^* \qquad \text{in } Q_{1/16}^*.$$

Note that λ^* depends only on N and C_v in (8).

Proof. For every $k \in \mathbb{N}$, $k \leq K_+ = E(1/\delta_1 + 1)$ (where δ_1 is as defined in Lemma 8 for ε_1 such that $4C\sqrt{\varepsilon_1} \leq \varepsilon_0$, with ε_0 defined as in Lemma 6), we define:

$$\bar{\theta}_k = 2(\bar{\theta}_{k-1} - 1) \qquad \text{with} \qquad \bar{\theta}_0 = \theta.$$

So we have: $\bar{\theta}_k = 2^k(\theta - 2) + 2$. Note that for every k, $\bar{\theta}_k$ verifies (1), $\bar{\theta}_k \leq 2$ and $|\{(t,x,z) \in Q_1^* \mid \bar{\theta}_k \leq 0\}| \geq \frac{1}{2}$. Assume that for all those k, $|\{0 < \bar{\theta}_k^* < 1\}| \geq \delta_1$. Then, for every k:

$$|\{\bar{\theta}_k^* < 0\}| = |\{\bar{\theta}_{k-1}^* < 1\}| \geq |\{\bar{\theta}_{k-1}^* < 0\}| + \delta_1.$$

Hence:

$$|\{\bar{\theta}_{K_+}^* \leq 0\}| \geq 1,$$

and $\bar{\theta}_{K_+}^* < 0$ almost everywhere, which means: $2^{K_+}(\theta^* - 2) + 2 < 0$ or

$$\theta^* < 2 - 2^{-K_+}.$$

And in this case we are done.

Otherwise, there exists $0 \leq k_0 \leq K_+$ such that: $|\{0 < \bar{\theta}_{k_0}^* < 1\}| \leq \delta_1$. From Lemma 8 and Lemma 6 (applied on $\bar{\theta}_{k_0+1}$) we get $(\bar{\theta}_{k_0+1})_+ \leq 2-\lambda$ which means:

$$\theta \leq 2 - 2^{-(k_0+1)}\lambda \leq 2 - 2^{-K_+}\lambda,$$

in $Q_{1/8}$.

Consider the function b_3 defined by:

$$\begin{aligned} &\Delta b_3 = 0 && \text{in } B^*_{1/8}, \\ &b_3 = 2 && \text{on the sides of the cube except for } z = 0 \\ &b_3 = 2 - 2^{-K_+} \inf(\lambda, 1) && \text{on } z = 0. \end{aligned}$$

We have $b_3 < 2 - \lambda^*$ in $B^*_{1/16}$. And from the maximum principle we get $\theta^* \leq b_3$. □

7. Proof of Theorem 2

We fix $t_0 > 0$ and consider $t \in [t_0, \infty[\times \mathbb{R}^N$. We define:

$$F_0(s, y) = \theta(t + st_0/4, x + t_0/4(y - x_0(s))),$$

where $x_0(s)$ is solution to:

$$\begin{aligned} &\dot{x}_0(s) = \frac{1}{|B_4|} \int_{x_0(s)+B_4} v(t + st_0/4, x + yt_0/4)\, dy \\ &x_0(0) = 0. \end{aligned}$$

Note that $x_0(s)$ is uniquely defined from the Cauchy-Lipschitz theorem. We set:

$$\begin{aligned} &\tilde{\theta}^*_0(s, y) = \frac{4}{\sup_{Q^*_4} F^*_0 - \inf_{Q^*_4} F^*_0} \left(F^*_0 - \frac{\sup_{Q^*_4} F^*_0 + \inf_{Q^*_4} F^*_0}{2} \right). \\ &v_0(s, y) = v(t + st_0/4, x + t_0/4(y - x_0(s))) - \dot{x}_0(s), \end{aligned}$$

and then for every $k > 0$:

$$\begin{aligned} &F_k(s, y) = F_{k-1}(\tilde{\mu} s, \tilde{\mu}(y - x_k(s))), \\ &\tilde{\theta}^*_k(s, y) = \frac{4}{\sup_{Q^*_4} F^*_k - \inf_{Q^*_4} F^*_k} \left(F^*_k - \frac{\sup_{Q^*_4} F^*_k + \inf_{Q^*_4} F^*_k}{2} \right), \end{aligned}$$

$$\begin{aligned} &\dot{x}_k(s) = \frac{1}{|B_4|} \int_{x_k(s)+B_4} v_{k-1}(\tilde{\mu} s, \tilde{\mu} y)\, dy, \\ &x_k(0) = 0, \\ &v_k(s, y) = v_{k-1}(\tilde{\mu} s, \tilde{\mu}(y - x_k(s))) - \dot{x}_k(s), \end{aligned}$$

where $\tilde{\mu}$ will be chosen later. We divide the proof in several steps.

Step 1. For k=0, $\tilde{\theta}_0$ is solution to (2) in $[-4, 0] \times \mathbb{R}^N$, $\|v_0\|_{\mathrm{BMO}} = \|v\|_{\mathrm{BMO}}$, $\int v_0(s)\, dy = 0$ for every s and $|\tilde{\theta}_0| \leq 2$. Assume that it is true at $k-1$. Then:

$$\partial_s F_k = \tilde{\mu} \partial_s \tilde{\theta}_{k-1}() - \tilde{\mu} \dot{x}_k(s) \cdot \nabla \tilde{\theta}_{k-1}.$$

So $\tilde{\theta}_k$ is solution of (2) and $|\tilde{\theta}_k| \le 2$. By construction, for every s we have $\int_{B_4} v_k(s,y)\,dy = 0$ and $\|v_k\|_{\mathrm{BMO}} = \|v_{k-1}\|_{\mathrm{BMO}} = \|v\|_{\mathrm{BMO}}$. Moreover we have:

$$\begin{aligned}|\dot{x}_k(s)| &\le \int_{B_4} v_{k-1}(\tilde{\mu}(y - x_k(s)))\,dy \le C\|v_{k-1}(\tilde{\mu}y)\|_{L^p} \\ &\le C\tilde{\mu}^{-N/p}\|v_{k-1}\|_{L^p} \le C_p\tilde{\mu}^{-N/p}\|v_{k-1}\|_{\mathrm{BMO}}.\end{aligned}$$

So, for $0 \le s \le 1$, $y \in B_4$ and $p > N$:

$$|\tilde{\mu}(y - x_k(s))| \le 4\tilde{\mu}(1 + C_p\tilde{\mu}^{-N/p}) \le C\tilde{\mu}^{1-N/p}.$$

For $\tilde{\mu}$ small enough this is smaller than 1.

Step 2. For every k we can use the oscillation lemma. If $|\{\tilde{\theta}_k^* \le 0\}| \ge \frac{1}{2}|Q_4^*|$ then we have $\tilde{\theta}_k^* \le 2 - \lambda^*$. Otherwise, we have $|\{-\tilde{\theta}_k^* \le 0\}| \ge \frac{1}{2}|Q_4^*|$ and applying the oscillation lemma on $-\tilde{\theta}_k^*$ gives $\tilde{\theta}_k^* \ge -2 + \lambda^*$. In both cases this gives:

$$|\sup \tilde{\theta}_k^* - \inf \tilde{\theta}_k^*| \le 2 - \lambda^*.$$

and so:

$$|\sup_{Q_1^*} F_k^* - \inf_{Q_1^*} F_k^*| \le (1 - \lambda^*/2)^k |\sup_{Q_1^*} F_0^* - \inf_{Q_1^*} F_0^*|.$$

Step 3. For $s \le \tilde{\mu}^{2n}$:

$$\sum_{k=0}^{n} \tilde{\mu}^{n-k} x_k(s) \le \tilde{\mu}^{2n} \sum_{k=0}^{n} \frac{\tilde{\mu}^{n-k}}{\tilde{\mu}^{-N/p}} \le \frac{\tilde{\mu}^n}{2},$$

for $\tilde{\mu}$ small enough that

$$\left| \sup_{[-\tilde{\mu}^{2n},0]\times B^*_{\tilde{\mu}^n/2}} \theta^* - \inf_{[-\tilde{\mu}^{2n},0]\times B^*_{\tilde{\mu}^n/2}} \theta^* \right| \le (1 - \lambda^*/2)^n.$$

This gives that θ^* is C^α at $(t, x, 0)$, and so θ is C^α at (t, x). □

Appendix A. Proof of the De Giorgi isoperimetric lemma

Let $\omega \in H^1([-1,1]^{N+1})$. We denote:

$$\begin{aligned}\mathscr{A} &= \{x;\ \omega(x) \le 0)\}, \\ \mathscr{B} &= \{x;\ \omega(x) \ge 1)\}, \\ \mathscr{C} &= \{x;\ 0 < \omega(x) < 1)\},\end{aligned}$$

and

$$\chi = \mathbf{1}_{\{y_1 + s(y_1 - y_2)/|y_1 - y_2| \in \mathscr{C}\}}.$$

We have:

$$
\begin{aligned}
|\mathcal{A}||\mathcal{B}| &\le \int_{\mathcal{A}}\int_{\mathcal{B}} (\omega(y_1)-\omega(y_2))\,dy_1\,dy_2\\
&= \int_{\mathcal{A}}\int_{\mathcal{B}}\int_0^{|y_1-y_2|} \nabla\omega\Big(y_1+s\frac{y_1-y_2}{|y_1-y_2|}\Big)\cdot\frac{y_1-y_2}{|y_1-y_2|}\,ds\,dy_1\,dy_2\\
&= \int_{\mathcal{A}}\int_{\mathcal{B}}\int_0^{|y_1-y_2|} \chi\nabla\omega\Big(y_1+s\frac{y_1-y_2}{|y_1-y_2|}\Big)\cdot\frac{y_1-y_2}{|y_1-y_2|}\,ds\,dy_1\,dy_2\\
&\le \int_{\mathcal{A}}\int_{\mathcal{B}}\int_0^{\infty} \chi\left|\nabla\omega\Big(y_1+s\frac{y_1-y_2}{|y_1-y_2|}\Big)\right|\,ds\,dy_1\,dy_2\\
&\le \int_{B_1}\int_{B_1}\int_0^{\infty} \chi\left|\nabla\omega\Big(y_1+s\frac{y_1-y_2}{|y_1-y_2|}\Big)\right|\,ds\,dy_1\,dy_2\\
&\le \int_{S_{N-1}}\int_{B_1}\int_0^{\infty} \frac{|\nabla\omega(y_1+s\nu)|}{s^{N-1}}\mathbf{1}_{\{(y_1+s\nu)\in\mathcal{C}\}}s^{N-1}\,d\nu\,ds\,dy_1\\
&\le C\int_{B_1}\int_{B_1} \frac{|\nabla\omega(y_1+y_2)|}{|y_2|^{N-1}}\mathbf{1}_{\{y_1+y_2\in\mathcal{C}\}}\,dy_2\,dy_1\\
&\le C\|\nabla\omega\|_{L^2}|\mathcal{C}|^{1/2}. \qquad\square
\end{aligned}
$$

Appendix B. Higher regularity

We give the proof of the following theorem.

THEOREM 10. *Let θ be a solution of the quasi-geostrophic equations (2), (3) satisfying the regularity properties of Theorem 3:*

$$
\begin{aligned}
\theta\in L^\infty(0,\infty;L^2)\cap L^2(0,\infty;H^{1/2})\\
\cap L^\infty([t_0,\infty[\times\mathbb{R}^N)\cap C^\alpha([t_0,\infty[\times\mathbb{R}^N),
\end{aligned}
$$

for every $t_0>0$. Then θ belongs to $C^{1,\beta}([t_0,\infty[\times\mathbb{R}^N)$ for every $\beta<1$ and $t_0>0$ and is therefore a classical solution.

Proof. We want to show the regularity at a fixed point

$$
y_0=(t_0,x_0)\in]0,\infty[\times\mathbb{R}^N\subset\mathbb{R}^m
$$

where $m=N+1$. Note that changing $\theta(t,x)$ by $\theta(t,x-u(t_0,x_0)t)-\theta(t_0,x_0)$ if necessary, we can assume without loss of generality that $\theta(y_0)=0$ and $u(y_0)=0$. The fundamental solution of:

$$
\partial_t\theta+\Lambda\theta=0
$$

is the Poisson kernel:

$$
P(t,x)=\frac{Ct}{(|x^2|+t^2)^{\frac{N+1}{2}}},
$$

a homogeneous function of order $-N$ if extended for t negative. The solution θ of (2) can be represented as the sum of two terms.

$$\theta(t,x) = P(t,\cdot) * \theta_0 - g(t,x), \tag{19}$$

where:

$$\begin{aligned} g(t,x) &= \int_0^t \int_{\mathbb{R}^N} P(t-t_1, x-x_1) \operatorname{div}(u(t_1,x_1)\theta(t_1,x_1))\, dt_1\, dx_1 \\ &= \int_0^\infty \int_{\mathbb{R}^N} \nabla_x \widetilde{P}(y-y_1) \cdot u(y_1)\theta(y_1)\, dy_1. \end{aligned}$$

In the last inequality, we denoted $y=(t,x)$, $\widetilde{P}$ the extension of P for negative t with value 0, and we passed the divergence on $\widetilde{P}$, which becomes a singular integral. The first term in (19) is smooth for $t>0$ and depends only on the initial data. Focusing on the second one $g(y)$, we fix $e \in \mathbf{S}_m$, and estimate $g(y_0+he)-g(y_0)$ for $h>0$ in the standard way. We split the integral:

$$g(y_0) - g(y_0+he) = \int_0^\infty \int_{\mathbb{R}^N} Q_0(y_0-y_1, he) u(y_1)\theta(y_1)\, dy_1 \tag{20}$$

where:

$$Q_0(y,he) = \nabla_x \widetilde{P}(y) - \nabla_x \widetilde{P}(y+he),$$

into two parts, one on the ball B_{10h} centered to y_0 and radius $10h$, and the second on the complement. The first part has no cancellation so we separate the integrals:

$$\begin{aligned} &\int_{B_{10h}} \mathbf{1}_{\{t_1 \geq 0\}} [\nabla_x \widetilde{P}(y_0-y_1) - \nabla_x \widetilde{P}(y_0+he-y_1)] u(y_1)\theta(y_1)\, dy_1 \\ &\qquad = \int_{B_{10h}} \mathbf{1}_{\{t_1 \geq 0\}} \nabla_x \widetilde{P}(y_0-y_1) u(y_1)\theta(y_1)\, dy_1 \\ &\qquad\quad - \int_{B_{10h}} \mathbf{1}_{\{t \geq 0\}} \nabla_x \widetilde{P}(y_0+he-y_1) u(y_1)\theta(y_1)\, dy_1. \end{aligned}$$

If θ is C^α, $\alpha>0$, from the Riesz transform u, is also C^α, and since $\theta(y_0) = u(y_0) = 0$, we have:

$$|u(y_1)\theta(y_1)| \leq \inf(|y_1-y_0|^{2\alpha}, C). \tag{21}$$

So the first integral is convergent and bounded by $Ch^{2\alpha}$. To deal with the second one, notice that $\nabla_x \widetilde{P}$ have mean value zero on any slice $t=C$ of B_{10h}, so we can add and subtract $\theta(y_0+he)u(y_0+he)$. Now,

$$|\theta(y_1)u(y_1) - \theta(y_0+he)u(y_0+he)| \leq Ch^\alpha |y_0+he-y_1|^\alpha,$$

where we have used again that $u(y_0)=\theta(y_0)=0$. Hence the integral is also convergent and bounded by $Ch^{2\alpha}$. Thus, the contribution of B_{10h} on (20) is smaller that $Ch^{2\alpha}$.

Outside of a neighborhood of size $10h$ we use the cancellation of $\nabla_x \widetilde{P}$. Up to Lipschitz regularity

$$|\nabla_x[\widetilde{P}(y_1 - y_0) - \widetilde{P}(y_1 + he - y_0)]| \leq \frac{h}{|y_1 - y_0|^{m+1}},$$

and we integrate against $|u\theta|$ which verifies (21). This gives the bound:

$$\int_{|y_1 - y_0| \geq 10h} \frac{h}{|y_1 - y_0|^{m+1-2\alpha}}\, dy_1 \leq C h^{2\alpha},$$

provided that $2\alpha < 1$. Altogether, this gives that if $\theta \in C^\alpha$ with $2\alpha < 1$, then

$$|g(y_0) - g(y_0 + he)| \leq C h^{2\alpha}.$$

Bootstrapping the argument gives that θ is C^α for any $\alpha < 1$.

To go beyond Lipschitz we consider a second order increment quotient:

$$Q_1(y, he) = |\nabla[\widetilde{P}(y + he) + \widetilde{P}(y - he) - 2\widetilde{P}(y)]|.$$

Now,

$$g(y_0 + he) + g(y_0 - he) - 2g(y_0) = \int_{\mathbb{R}^m} \mathbf{1}_{\{t_1 \geq 0\}} Q_1(y_0 - y_1, he) u(y_1)\theta(y_1)\, dy_1.$$

Note that $Q_1(y, he) = Q_0(y, he) - Q_0(y - he, he)$, so that for $|y| < 20h$, the local estimate of the previous argument together with the C^α property of θ and u gives:

$$\int_{B_{20h}} |Q_1(y_0 - y_1) u(y_1)\theta(y_1)|\, dy_1 \leq C h^{2\alpha}.$$

For $|y| > 20h$ and y not in the strip $\mathcal{T}_h = [t_0 - h, t_0 + h] \times \mathbb{R}^N$, we have:

$$|Q_1(y_0 - y_1, he)| \leq C \frac{h^2}{|y_0 - y_1|^{m+2}}.$$

and the corresponding integral:

$$\begin{aligned}\int_{|y_0 - y_1| \geq 20h} & \mathbf{1}_{\{y_1 \notin \mathcal{T}_h\}} |Q_1(y_0 - y_1) u(y_1)\theta(y_1)|\, dy_1 \\ & \leq C \int_{|y| \geq 20h} \frac{h^2}{|y|^{m+2}} (|y|^{2\alpha} \wedge 1)\, dy \leq C h^{2\alpha},\end{aligned}$$

whenever $2\alpha < 2$. It remains to control the contribution of the strip $\mathcal{T}_h \setminus B_{20h}$. The estimate on Q_0 gives that on this strip:

$$|Q_1(y_1 - y_0, he)| \leq C \frac{h}{|y_1 - y_0|^{N+2}} \leq C \frac{h}{|x_1 - x_0|^{N+2}}.$$

Note that on $\mathcal{T}_h \setminus B_{20h}$ we have $|x_1 - x_0| \geq h$. So the contribution of this strip is bounded by

$$\int_{t_0-h}^{t_0+h} \int_{|x_1-x_0|\geq h} \frac{h}{|x_1 - x_0|^{N+2-2\alpha}}\, dx_1\, dt_1 \leq C\frac{h^{2\alpha}}{h} \int_{t_0-h}^{t_0+h} dt_1 \leq Ch^{2\alpha},$$

whenever $2\alpha < 2$. That goes all the way to $C^{1,\beta}$ for every $\beta < 1$. □

Appendix C. Existence of solutions to equation (1)

In this appendix we sketch the existence theory of the approximate solution of the equation (1) satisfying the truncated energy inequalities in the hypothesis of Theorem 1. We start by restricting the problem to $B_1 \times [0, \infty]$ and adding an artificial diffusion term $\varepsilon\Delta$. We will use the eigenfunctions σ_k and eigenvalues λ_k^2 of the Laplacian in B_1; that is:

$$\Delta\sigma_k + \lambda_k^2\sigma_k = 0.$$

Note that $\sigma_k^*(x, z) = \sigma_k(x)e^{-\lambda_k z}$ is the harmonic extension of σ_k for the semi-infinite cylinder $Q_1 = B_1 \times [0, \infty]$ with data 0 in the lateral boundary, and:

$$\lambda_k\sigma_k(x) = \partial_\nu\sigma_k^*(x, 0),$$

where ∂_ν is the normal derivative. Also:

$$\int_{Q_1} \lambda_k\sigma_k^2\, dx\, dz = \int_{\partial Q_1} \sigma_k^*\partial_\nu\sigma_k^*\, dx = \int_{Q_1} |\nabla\sigma_k^*|^2\, dx\, dz,$$

and this formula is also correct for any series

$$g(x) = \sum f_k\sigma_k(x),$$

provided that $\sum f_k^2\lambda_k$ converges, i.e., $g \in H^{1/2}(B_1)$.

We want to solve then in $[0, \infty[\times B_1$ the equation:

$$\partial_t\theta + \operatorname{div}(v\theta) = \varepsilon\Delta\theta - (-\Delta^{1/2})\theta, \tag{22}$$

where $-\Delta^{1/2}\theta$ is understood as the operator that maps σ_k to $\lambda_k\sigma_k = \partial_\nu\sigma_k^*$. For, say, v bounded and divergence-free, this is straightforward by the Galerkin method: Let us restrict (22) to σ_k, with $1 \leq k \leq k_0$; i.e., we seek a function:

$$\theta = \theta_{\varepsilon,k_0} = \sum_1^{k_0} f_k(t)\sigma_k(x)$$

that is a solution of the equation when tested against σ_k, $1 \le k \le k_0$. The functions f_k are solutions to the following system of ODEs:

$$f_k'(t) = -[\varepsilon\lambda_k^2 + \lambda_k] f_k(t) + \sum_{l=1}^{k_0} a_{kl} f_l(t), \qquad 1 \le k \le k_0,$$

with initial value:

$$f_k(0) = \int_{B_1} \theta_0(x)\sigma_k(x)\, dx,$$

where:

$$a_{kl} = \int_{B_1} v(t,x) \cdot \nabla\sigma_k(x)\sigma_l(x)\, dx.$$

Note that, since v is divergence-free, the matrix a_{kl} is antisymmetric. This leads to the estimate:

$$\sum_{k=1}^{k_0} f_k^2(t_2) + \int_{t_1}^{t_2} \sum_{k=1}^{k_0} (\varepsilon\lambda_k^2 + \lambda_k) f_k^2(s)\, ds = \sum_{k=1}^{k_0} f_k^2(t_1).$$

In particular θ_{ε,k_0} satisfies the energy inequality:

$$\|\theta_{\varepsilon,k_0}(t_2)\|^2_{L^2(B_1)} + \int_{t_1}^{t_2} \left(\|\theta_{\varepsilon,k_0}(s)\|^2_{\dot H^{1/2}(B_1)} + \varepsilon \|\theta_{\varepsilon,k_0}(s)\|^2_{\dot H^1(B_1)} \right) ds \le \|\theta_{\varepsilon,k_0}(t_1)\|^2_{L^2(B_1)}.$$

Notice also that what we call $H^{1/2}(B_1)$ corresponds to the extension of θ to the half cylinder, and is such that

$$\|\theta\|_{\dot H^{1/2}(B_1)} \ge \|\theta\|_{\dot H^{1/2}(\mathbb{R}^N)}.$$

We now pass to the limit in k_0 and denote θ_ε the limit. If we test θ_{ε,k_0} with a function $\gamma \in L^\infty(0,T; L^2(B_1)) \cap L^2(0,T; H^1(B_1))$, there is no problem in passing to the limit in the term:

$$\int_{t_1}^{t_2} \int_{B_1} (\nabla\gamma) v \theta_{\varepsilon,k_0}\, dx\, ds,$$

since θ_{ε,k_0} converges strongly in $L^2([0,T] \times B_1)$. In particular, for $\gamma = (\theta_\varepsilon - \lambda)_+ = \theta_{\varepsilon,\lambda}$ the term converges to:

$$\int_{t_1}^{t_2} \int_{B_1} \nabla[\theta_{\varepsilon,\lambda}]^2 v\, dx\, ds = 0,$$

provided that v is divergence-free. This leads to the following corollaries:

COROLLARY 11. *The function θ_ε satisfies the hypothesis of Lemma 6 independently of ε, and therefore*:

$$\|\theta_\varepsilon(T)\|_{L^\infty(B_1)} \leq \frac{C}{T^{N/2}} \|\theta_\varepsilon(0)\|_{L^2(B_1)}.$$

COROLLARY 12. *The same theorem is true for $v \in L^2([0,T]\times B_1)$ independently of the L^2 norm of v.*

Proof. We approximate v by a mollification v_δ. □

COROLLARY 13. *For θ_0 prescribed in $L^2(\mathbb{R}^N)$, the same result is true in* $[0,T]\times\mathbb{R}^N$.

Proof. We may rescale the previous theorems to the ball of radius M by applying them to $\overline{\theta}(t,x) = M^{N/2}\theta(Mt, Mx)$. This change preserves the L^2 norm, and so we get:

$$\sup_{B_M} M^{N/2}\theta(s,y) \leq \frac{C}{(s/M)^{N/2}},$$

or

$$\sup_{B_M} \theta(s,y) \leq \frac{C}{(s)^{N/2}},$$

provided that $v \in L^2(B_M)$ is divergence free. Then letting M go to infinity gives the result. □

Final remark. Since all the estimates are independent of ε we may let ε go to zero for the limit to be a weak solution of the limiting equation, and satisfying the truncated energy inequalities.

Note also that the same approach can be taken for higher regularity. Indeed, the proof of higher regularity depends only on the truncated and localized energy inequality that is also satisfied by the ε-problem. We may then pass to the limit in ε and find a classical solution of the limiting problem.

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