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Abstract

We show that the mapping class group of a compact orientable surface with higher complexity satisfies the following rigidity in the sense of measure equivalence: If the mapping class group is measure equivalent to a discrete group, then they are commensurable up to finite kernels. Moreover, we describe all locally compact second countable groups containing a lattice isomorphic to the mapping class group. We obtain similar results for finite direct products of mapping class groups.

1. Introduction

The purpose of this paper is to establish a rigidity theorem for the mapping class group in terms of measure equivalence. In this paper, by a discrete group we mean a discrete and countable one. Measure equivalence is introduced by Gromov [18] as follows.

Definition 1.1. We say that two discrete groups Γ and Λ are *measure equivalent* (ME) if there exists a measure-preserving action of $\Gamma \times \Lambda$ on a standard Borel space (Σ, m) with a σ-finite positive measure such that both of the actions of $\Gamma \times \{e\}$ and $\{e\} \times \Lambda$ on Σ are essentially free and have a fundamental domain of finite measure. The space (Σ, m) equipped with the $(\Gamma \times \Lambda)$-action is then called an ME *coupling* of Γ and Λ.

It is known that ME defines an equivalence relation between discrete groups (see §2 in [12]). One typical example of two ME groups is given by any two lattices in the same locally compact second countable group. This example motivates us to introduce ME. Commensurability up to finite kernels is the equivalence relation for discrete groups defined by declaring two groups in an exact sequence $1 \to A \to B \to C \to 1$ of discrete groups to be equivalent if the third group is finite. It is easy to see that any two discrete groups which are commensurable up to finite kernels are ME.

ME between two groups has another equivalent formulation in terms of orbit equivalence (OE) (see Theorem 2.12), which has been studied for a long time and is closely related to ergodic theory and the theory of von Neumann algebras. The first magnificent result on OE is due to Ornstein and Weiss [36] following Dye [5], [6] and it can be stated in terms of ME as follows: A discrete group is ME to $\mathbb{Z}$ if and only if it is an infinite amenable group. Connes, Feldman and Weiss [4] obtain a generalized result in terms of amenable discrete measured equivalence relations. Zimmer [44] extends the superrigidity theorem for semisimple Lie groups of noncompact type due to Margulis to the one in the context of OE, which is called his cocycle superrigidity theorem, and he classifies lattices in simple Lie groups of real rank at least two up to ME. In addition to this classification, his cocycle superrigidity theorem has many applications to various rigidity phenomena of higher rank lattices.

Recent studies on ME and OE are rapidly developing. By utilizing Zimmer's cocycle superrigidity theorem, Furman [12] obtains a beautiful rigidity result, which completely determines the class of discrete groups ME to higher rank lattices. Namely, if a discrete group Λ is ME to a lattice in a connected simple Lie group G with its center finite and its real rank at least two, then there exists a homomorphism from Λ onto a lattice in $\mathrm{Aut}(\mathrm{Ad}G)$ whose kernel is finite. Gaboriau's discovery in [16] that ℓ^2-Betti numbers for discrete groups are invariant under ME in a certain sense leads to surprising progress in the classification problem of ME because these numerical invariants are defined for all discrete groups and are computable for various discrete groups arising geometrically. Popa shows in [37] and [38] that the Bernoulli actions of groups satisfying Kazhdan's property (T) have various rigidity properties in terms of OE (see also [14]). It is remarkable that he treats all groups satisfying Kazhdan's property (T), which is a very large class of groups. The reader should be referred to [15], [40] and [42] for more details of recent development in the theory of ME and OE.

Let $M = M_{g,p}$ be a connected compact orientable surface of *type* (g, p), that is, of genus g and with p boundary components. Throughout the paper, a surface is assumed to be connected, compact and orientable unless otherwise stated. The *mapping class group* $\Gamma(M)$ of M is defined as the group of isotopy classes of all orientation-preserving diffeomorphisms of M. The *extended mapping class group* $\Gamma(M)^\diamond$ of M is the group of isotopy classes of all diffeomorphisms of M, which contains $\Gamma(M)$ as a subgroup of index two. Let $\kappa(M) = 3g + p - 4$ be the *complexity* of M. If $\kappa(M) > 0$, we say that M has higher complexity. Let $C = C(M)$ be the *curve complex* for a surface M. In [27], we obtain some classification result of $\Gamma(M)$ in terms of ME and give various examples of discrete groups not ME to $\Gamma(M)$. In the proof, we establish fundamental methods to study subrelations in a discrete measured equivalence relation arising from a standard action of $\Gamma(M)$, where a *standard* action means an essentially free, measure-preserving action on a standard Borel space with a finite positive measure. The curve complex plays one of the most important roles in the study of them. Using these methods, we show the following rigidity theorem for $\Gamma(M)$, which is the main result of this paper. Let $\mathrm{Aut}(C)$ denote the automorphism group of the simplicial complex C. Note that we have the natural homomorphism $\pi\colon \Gamma(M)^\diamond \to \mathrm{Aut}(C)$ such that $\pi(\Gamma(M)^\diamond)$ is a finite index subgroup of $\mathrm{Aut}(C)$ and $\ker(\pi)$ is finite (see Theorem 2.8).

THEOREM 1.1. *If a discrete group* Λ *is* ME *to the mapping class group* $\Gamma(M)$ *with* $\kappa(M) > 0$, *then there exists a surjective homomorphism* ρ *from* Λ *onto a finite index subgroup of* $\mathrm{Aut}(C)$ *with* $\ker(\rho)$ *finite.*

This theorem completely determines the class of discrete groups ME to $\Gamma(M)$ and provides the first example of infinite discrete groups satisfying such an extreme rigidity in the theory of ME. Remark that uniform and nonuniform lattices in Lie groups treated in Furman's rigidity result are not commensurable up to finite kernels. Hence, if Γ is a lattice in Furman's rigidity theorem, then there exists a discrete group which is ME to Γ and is not commensurable up to finite kernels with Γ.

Remark 1.1. Both Furman's and Popa's rigidity theorems are concerned with discrete groups satisfying (or related to) Kazhdan's property (T). On the other hand, the mapping class group of a surface of genus at most two does not satisfy Kazhdan's property (T), and moreover it contains a subgroup of finite index which admits a quotient isomorphic to a non-abelian free group of finite rank (see [31]). It is unknown whether the mapping class groups of other surfaces satisfy Kazhdan's property (T) or not.

Theorem 1.1 completes the classification of mapping class groups up to ME.

THEOREM 1.2. *Suppose that* M^1 *and* M^2 *are distinct surfaces of type* (g_1, p_1) *and of type* (g_2, p_2), *respectively*, *satisfying* $\kappa(M^1) > 0$, $\kappa(M^2) > 0$ *and* $g_1 \leq g_2$. *Moreover*, *assume that* $\Gamma(M^1)$ *and* $\Gamma(M^2)$ *are* ME. *Then the following only two possibilities occur*: $((g_1, p_1), (g_2, p_2)) = ((0, 5), (1, 2)), ((0, 6), (2, 0))$.

Remark 1.2. Note that if $\kappa(M) < 0$ and $M \neq M_{1,0}$, then $\Gamma(M)$ is finite. Both $\Gamma(M_{1,0})$ and $\Gamma(M_{1,1})$ are isomorphic to $SL(2,\mathbb{Z})$ and $\Gamma(M_{0,4})$ is commensurable up to finite kernels with $SL(2,\mathbb{Z})$. It is known that $\Gamma(M_{0,5})$ and $\Gamma(M_{1,2})$ (resp. $\Gamma(M_{0,6})$ and $\Gamma(M_{2,0})$) are commensurable up to finite kernels (see Theorem 2.8).

Let Γ be a lattice in a connected simple Lie group G with its center finite and its real rank at least two. In the proof of Furman's rigidity theorem in [12], the main ingredient is to prove the following (see Theorem 4.1 in [12]): Let (Ω, ω) be a self ME coupling of Γ (i.e., an ME coupling of Γ and Γ). Then there exists an essentially unique, almost $(\Gamma \times \Gamma)$-equivariant Borel map $\Psi\colon \Omega \to \mathrm{Aut}(\mathrm{Ad}G)$, where the equivariance means the equation

$$\Psi((\gamma, \gamma')x) = \mathrm{Ad}(\gamma)\Psi(x)\mathrm{Ad}(\gamma')^{-1}$$

for any $\gamma, \gamma' \in \Gamma$ and a.e. $x \in \Omega$. Furman used Zimmer's cocycle superrigidity theorem for the construction of Ψ. On the other hand, we will show that for any self ME coupling (Σ, m) of $\Gamma(M)$, there exists an essentially unique, almost $(\Gamma(M) \times \Gamma(M))$-equivariant Borel map $\Phi\colon \Sigma \to \mathrm{Aut}(C)$, where the equivariance means the equation

$$\Phi((\gamma, \gamma')x) = \pi(\gamma)\Phi(x)\pi(\gamma')^{-1}$$

for any $\gamma, \gamma' \in \Gamma(M)$ and a.e. $x \in \Sigma$. Here, $\pi\colon \Gamma(M)^{\diamond} \to \mathrm{Aut}(C)$ is the natural homomorphism. This construction of Φ is the heart of the proof of Theorem 1.1. In Section 3, we give an outline of the construction of Φ.

After the construction of the map Φ, we apply Furman's technique in [12] for higher rank lattices that is applicable to a more general situation. More precisely, given an ME coupling (Σ', m') of $\Gamma(M)$ and a discrete group Λ, we construct the self ME coupling of $\Gamma(M)$ associated with it. Using his technique for the equivariant Borel map from this self ME coupling into $\mathrm{Aut}(C)$, one can find the homomorphism ρ in Theorem 1.1.

Moreover, we consider the same problem as above for a finite direct product $\Gamma(M_1) \times \cdots \times \Gamma(M_n)$ of mapping class groups $\Gamma(M_i)$ with $\kappa(M_i) > 0$ for all i. Monod and Shalom introduced in [35] the class $\mathscr{C}$ consisting of discrete groups Γ which admit a mixing unitary representation π on a Hilbert space such that the second bounded cohomology group $H_b^2(\Gamma, \pi)$ of Γ with coefficient π does not vanish. They show in that paper that a nontrivial finite direct product of discrete groups in $\mathscr{C}$ satisfies various measurable rigidity properties. The class $\mathscr{C}$ contains a large number of discrete groups arising geometrically (e.g., word-hyperbolic groups) and whether a discrete group is in $\mathscr{C}$ or not is invariant under ME. Since Hamenstädt proves in [20] that the mapping class group of a surface with higher complexity belongs to $\mathscr{C}$, one obtains various measurable rigidity theorems as in [35] for direct products of mapping class groups.

Following Monod and Shalom's ingenious technique treating fundamental domains for actions on ME couplings, one can find an essentially unique, almost equivariant Borel map from a self ME coupling of a direct product of $\Gamma(M_i)$ into the automorphism group of the direct product of $\mathrm{Aut}(C(M_i))$. The following theorem is then proved.

THEOREM 1.3. *Let n be a positive integer. Let M_i be a surface with $\kappa(M_i) > 0$ for each $i \in \{1, \ldots, n\}$ and put*

$$G = \mathrm{Aut}(\mathrm{Aut}(C(M_1)) \times \cdots \times \mathrm{Aut}(C(M_n))).$$

If a discrete group Λ is ME *to the direct product $\Gamma(M_1) \times \cdots \times \Gamma(M_n)$, then there exists a surjective homomorphism ρ from Λ onto a finite index subgroup of G with* $\ker(\rho)$ *finite.*

Note that $\mathrm{Aut}(C(M_1)) \times \cdots \times \mathrm{Aut}(C(M_n))$ is naturally a finite index subgroup of its automorphism group G (see Corollary 7.3).

In [11], Furman gives another application of the map Ψ mentioned above. He explicitly describes all locally compact second countable (lcsc) groups containing a lattice isomorphic to a lattice in a simple Lie group of higher rank. Roughly speaking, such a lcsc group can be built from the ambient Lie group or from the lattice itself and their actions on a compact group. Following his argument, we describe a lcsc group containing a lattice isomorphic to the mapping class group. We fix the notation as follows: Let n be a positive integer and let M_i be a surface with $\kappa(M_i) > 0$ for each $i \in \{1, \ldots, n\}$. Put

$$G_0 = \Gamma(M_1)^{\diamond} \times \cdots \times \Gamma(M_n)^{\diamond}, \quad G = \mathrm{Aut}(\mathrm{Aut}(C(M_1)) \times \cdots \times \mathrm{Aut}(C(M_n))),$$

and let $\pi\colon G_0 \to G$ be the natural homomorphism.

THEOREM 1.4. *Suppose that Γ is a subgroup of finite index in G_0. Let $\sigma\colon \Gamma \to H$ be an injective homomorphism into a* lcsc *group H such that $\sigma(\Gamma)$ is a lattice in H. Then*

(i) *there exists a continuous homomorphism $\Phi_0\colon H \to G$ such that $\Phi_0(\sigma(\gamma)) = \pi(\gamma)$ for any $\gamma \in \Gamma$;*

(ii) *let K be the kernel of Φ_0 and let Γ act on K by conjugation via σ. Let $\rho\colon \Gamma \ltimes K \to H$ be the homomorphism defined by $\rho(k) = k$ for $k \in K$ and $\rho(\gamma) = \sigma(\gamma)$ for $\gamma \in \Gamma$. Then $[H : \rho(\Gamma \ltimes K)] \leq [G : \pi(\Gamma)] < \infty$ and* $\ker(\rho)$ *is finite.*

In the assertion (ii), *for $\gamma \in \Gamma$ and $k \in K$, we have $(\gamma, k) \in \ker(\rho)$ if and only if $\pi(\gamma) = e$ and $k = \sigma(\gamma)^{-1}$. In particular, if the kernel of the restriction of π to Γ is trivial, then ρ is an isomorphism onto its image.*

This theorem says that there exists no interesting lcsc group containing a lattice isomorphic to the mapping class group. The following is easily shown.

COROLLARY 1.5. *Let Γ be a subgroup of finite index in G_0 and suppose that Γ is isomorphic to a lattice in a* lcsc *group H. Then the lattice is cocompact in H and H has infinitely many connected components.*

It follows from this corollary that any subgroup of finite index of the mapping class group for a surface with higher complexity can not be isomorphic to a lattice in a connected semisimple Lie group, which is proved by Kaimanovich and Masur [25]. They show more generally that any sufficiently large subgroup of the mapping class group can not be isomorphic to a lattice in a semisimple Lie group.

In this direction, Farb and Masur [7], Bestvina and Fujiwara [2], and Yeung [43] study homomorphisms from a lattice in a semisimple Lie group into the mapping class group and concluded that their images are finite.

In a subsequent paper [28], we give an application of the existence of an equivariant Borel map from a self ME coupling of the mapping class group, following Furman [13]. In [28], we establish OE rigidity for ergodic standard actions of the mapping class group.

2. **Preliminaries**

2.1. *The mapping class group.* In this subsection, we recall fundamental facts on the mapping class group and several geometric objects related to it. We refer the reader to [8], [22], [24] or Sections 3.1, 3.2, 4.3, and 4.5 in [27] and the references therein for the material of this subsection.

Let $M = M_{g,p}$ be a surface of genus g and with p boundary components. Let $\Gamma(M)$ and $\Gamma(M)^\diamond$ be the mapping class group and the extended one of M, respectively, introduced in Section 1. Let $\kappa(M) = 3g + p - 4$ be the complexity of M. When $\kappa(M) > 0$, we say that M has higher complexity.

For a surface M, let $V(C) = V(C(M))$ be the set of all nontrivial isotopy classes of nonperipheral simple closed curves on M. Let $S(M)$ denote the set of all nonempty finite subsets of $V(C)$ which can be realized disjointly on M at the same time. When $\kappa(M) > 0$, the curve complex $C = C(M)$ is defined as a simplicial complex whose vertex set is $V(C)$ and simplex set is $S(M)$, which is introduced by Harvey [21]. Remark that when $\kappa(M) = 0$, the curve complex of M is defined in a slightly different way so that its vertex set $V(C)$ is given in the same way as above. If $\kappa(M) \geq 0$, then $\Gamma(M)^\diamond$ has a natural and simplicial action on C, and C is connected and has infinite diameter. Moreover, when C is equipped with a natural combinatorial metric, it is hyperbolic in the sense of Gromov (see [33]).

Let M be a surface with $\kappa(M) \geq 0$ and denote by $i\colon V(C) \times V(C) \to \mathbb{N}$ the geometric intersection number. Let $\mathcal{MF} = \mathcal{MF}(M)$ be the space of *measured foliations* on M and let $\mathcal{PMF} = \mathcal{PMF}(M)$ be the space of *projective measured foliations* on M. The space $\mathcal{PMF}$ is also called the *Thurston boundary* and is homeomorphic to the sphere of dimension $6g-7+2p$. Note that $S(M)$ is naturally embedded into $\mathcal{PMF}$. The function i can be continuously extended to a function $\mathcal{MF} \times \mathcal{MF} \to \mathbb{R}_{\geq 0}$ in the following manner:

$$i(r_1F_1, r_2F_2) = r_1r_2i(F_1, F_2)$$

for any $r_1, r_2 \in \mathbb{R}_{>0}$ and $F_1, F_2 \in \mathcal{MF}$. Hence, for two elements $F_1, F_2 \in \mathcal{PMF}$, whether $i(F_1, F_2) = 0$ or $\neq 0$ makes sense. It is known that $\Gamma(M)^\diamond$ acts continuously on both $\mathcal{MF}$ and $\mathcal{PMF}$ and

$$i(gF_1, gF_2) = i(F_1, F_2)$$

for any $g \in \Gamma(M)^\diamond$ and $F_1, F_2 \in \mathcal{MF}$ (or $\mathcal{PMF}$). Let

$$\mathcal{MIN} = \{F \in \mathcal{PMF} \mid i(F, \alpha) \neq 0 \text{ for any } \alpha \in V(C)\}$$

be the set of all *minimal measured foliations* on M, which is a $\Gamma(M)^\diamond$-invariant Borel subset of $\mathcal{PMF}$. The Thurston boundary $\mathcal{PMF}$ is an ideal boundary of the Teichmüller space $\mathcal{T} = \mathcal{T}(M)$ for M. The union $\overline{\mathcal{T}} = \mathcal{T} \cup \mathcal{PMF}$ is called the *Thurston compactification* of the Teichmüller space, which is homeomorphic to the closed Euclidean ball of dimension $6g - 6 + 2p$ whose boundary corresponds to $\mathcal{PMF}$.

For $g \in \Gamma(M)$, we denote by

$$\mathrm{Fix}(g) = \{x \in \overline{\mathcal{T}} \mid gx = x\}$$

the fixed point set of g. Each element $g \in \Gamma(M)$ is classified in terms of $\mathrm{Fix}(g)$ as follows (see Exposé 9, §V, Théorème and Exposé 11, §4, Théorème in [8]):

(i) g has finite order and has a fixed point on $\mathcal{T}$;

(ii) g is *pseudo-Anosov*, i.e., $\mathrm{Fix}(g)$ consists of exactly two points in $\mathcal{MIN}$;

(iii) g has infinite order and is *reducible*, i.e., there exists $\sigma \in S(M)$ with $g\sigma = \sigma$.

These three types are mutually exclusive. We say that $F \in \mathcal{PMF}$ is a *pseudo-Anosov foliation* if F is a fixed point for some pseudo-Anosov element. The set of all pseudo-Anosov foliations is known to be dense in $\mathcal{PMF}$. A pseudo-Anosov element $g \in \Gamma(M)$ has the following remarkable dynamics on $\overline{\mathcal{T}}$ (see Theorem 7.3.A in [24]): The two fixed points $F_\pm(g) \in \mathcal{MIN}$ of g satisfy that if U is any neighborhood of $F_+(g)$ in $\overline{\mathcal{T}}$ and K is any compact set in $\overline{\mathcal{T}} \setminus \{F_-(g)\}$, then $g^n(K) \subset U$ for all sufficiently large $n \in \mathbb{N}$.

Since the curve complex C is hyperbolic, we can consider its boundary ∂C at infinity, which is not compact. There exists a natural $\Gamma(M)$-equivariant continuous map $\mathcal{MIN} \to \partial C$, which is injective on the set of all uniquely ergodic measured foliations. This set contains all pseudo-Anosov foliations (see [19], [29], and Section 3.2 in [27]).

McCarthy and Papadopoulos [34] classify subgroups of $\Gamma(M)$ into four types by using the above classification of elements of $\Gamma(M)$.

THEOREM 2.1 ([34]). *Each subgroup Γ of $\Gamma(M)$ is classified into the following four cases*:

(i) Γ *is finite*;

(ii) *There exists a pseudo-Anosov element $g \in \Gamma$ such that $h\{F_\pm(g)\} = \{F_\pm(g)\}$ for any $h \in \Gamma$. In this case, Γ is virtually cyclic and is said to be* IA (= *infinite, irreducible, and amenable*);

(iii) Γ *is infinite and* reducible, *i.e.*, *there exists* $\sigma \in S(M)$ *such that* $g\sigma = \sigma$ *for any* $g \in \Gamma$;

(iv) *There exist two pseudo-Anosov elements* $g_1, g_2 \in \Gamma$ *such that* $\{F_{\pm}(g_1)\} \cap \{F_{\pm}(g_2)\} = \varnothing$. *In this case*, Γ *contains a non-abelian free subgroup and is said to be* sufficiently large.

We next recall the canonical reduction system (CRS) for a subgroup of $\Gamma(M)$, which plays an important role in the study of reducible subgroups. We refer the reader to Chapter 7 in [22] for more details on CRS's. For $\sigma \in S(M)$, we denote by M_σ the surface obtained by cutting M along a realization of curves in σ. For an integer $m \geq 3$, let $\Gamma(M;m)$ be the subgroup of $\Gamma(M)$ consisting of all elements which act trivially on the homology group $H_1(M;\mathbb{Z}/m\mathbb{Z})$. This subgroup satisfies the following notable properties (see Theorem 1.2 and Corollaries 1.5, 1.8, 3.6 in [22]).

THEOREM 2.2 ([22]). *In the above notation*, *the following assertions hold*:

(i) $\Gamma(M;m)$ *is a torsion-free subgroup of finite index in* $\Gamma(M)$.

(ii) *If* $g \in \Gamma(M;m)$ *and* $F \in \mathcal{PMF}$ *satisfy* $g^n F = F$ *for some* $n \in \mathbb{Z} \setminus \{0\}$, *then* $gF = F$.

(iii) *If* $g \in \Gamma(M;m)$ *and* $\sigma \in S(M)$ *satisfy* $g^n\sigma = \sigma$ *for some* $n \in \mathbb{Z} \setminus \{0\}$, *then* $g\alpha = \alpha$ *for any* $\alpha \in \sigma$ *and* g *preserves each component of* M_σ *and of the boundary of* M.

When we consider the problem of ME in the sections that follow, we study measure-preserving actions of (a finite index subgroup of) $\Gamma(M;m)$ instead of $\Gamma(M)$.

Definition 2.1 ([22, Chap. 7]). Let M be a surface with $\kappa(M) \geq 0$ and let $m \geq 3$ be an integer. Let Γ be a subgroup of $\Gamma(M;m)$. A curve $\alpha \in V(C)$ is called an *essential reduction class* for Γ if the following two conditions are satisfied:

(i) $g\alpha = \alpha$ for any $g \in \Gamma$;

(ii) If $\beta \in V(C)$ satisfies $i(\alpha, \beta) \neq 0$, then there exists $g \in \Gamma$ such that $g\beta \neq \beta$.

The *canonical reduction system* (CRS) $\sigma(\Gamma)$ for Γ is defined to be the set of all essential reduction classes for Γ, which is either an element of $S(M)$ or empty. We define the CRS for a general subgroup Γ of $\Gamma(M)$ as the CRS for $\Gamma \cap \Gamma(M;m)$, which is independent of m.

The following theorem is fundamental in the study of reducible subgroups.

THEOREM 2.3 ([22, Cor. 7.17]). *An infinite subgroup* Γ *of* $\Gamma(M)$ *is reducible if and only if* $\sigma(\Gamma)$ *is nonempty.*

Given a subgroup Γ of $\Gamma(M;m)$ and $\sigma \in S(M)$ with $g\sigma = \sigma$ for any $g \in \Gamma$, thanks to Theorem 2.2 (iii), we have the natural homomorphism

$$p_\sigma \colon \Gamma \to \prod_Q \Gamma(Q),$$

where Q runs through all components of M_σ.

LEMMA 2.4 ([3, Lemma 2.1 (1)], [24, Cor. 4.1.B]). *The kernel of p_σ is contained in the subgroup of $\Gamma(M)$ generated by Dehn twists about all curves in σ.*

For each component Q of M_σ, let $p_Q \colon \Gamma \to \Gamma(Q)$ be the composition of p_σ with the projection onto $\Gamma(Q)$. As for the quotient group $p_Q(\Gamma)$, the following theorem is known.

THEOREM 2.5 ([22, Cor. 7.18]). *If a subgroup Γ of $\Gamma(M;m)$ is reducible and Q is a component of the disconnected surface $M_{\sigma(\Gamma)}$ obtained by cutting M along the* CRS *$\sigma(\Gamma)$ for Γ, then the image $p_Q(\Gamma)$ either is trivial or contains a pseudo-Anosov element in $\Gamma(Q)$. In particular, $p_Q(\Gamma)$ cannot be infinite reducible.*

If $p_Q(\Gamma)$ is trivial, IA or sufficiently large, then we say that Q is *T* (= trivial), *IA* (= infinite, irreducible, and amenable) or *IN* (= irreducible and nonamenable), respectively, for the reducible subgroup Γ.

LEMMA 2.6. *Let Γ be a finite index subgroup of $\Gamma(M)$ and define the subgroup*

$$\Gamma_\sigma = \{g \in \Gamma \mid g\sigma = \sigma\}$$

for $\sigma \in S(M)$. Then the CRS *for Γ_σ is equal to σ.*

This lemma easily follows from Theorem 7.16 in [22] because any component of M_σ which is not a pair of pants is IN for Γ_σ if Γ is a finite index subgroup of $\Gamma(M;m)$.

LEMMA 2.7. *Let Γ be an infinite subgroup of $\Gamma(M;m)$ and let $\alpha \in V(C(M))$. Assume that $g\alpha = \alpha$ for all $g \in \Gamma$. If for each component Q of M_α, we have $g\beta = \beta$ for any $\beta \in V(C(Q))$ and any $g \in \Gamma$, then the* CRS *for Γ is $\{\alpha\}$.*

Proof. Since Γ is infinite and reducible, the CRS $\sigma(\Gamma)$ for Γ is nonempty. Let $\delta \in \sigma(\Gamma)$. We show that $\delta = \alpha$. Let Q be a component of M_α. If $\delta \in V(C(Q))$, then there exists $\beta \in V(C(Q))$ with $i(\beta, \delta) \neq 0$. By assumption, β is invariant for Γ, which contradicts the assumption that δ is an essential reduction class for Γ. Thus, either $i(\delta, \alpha) \neq 0$ or $\delta = \alpha$. The former case can not happen because α is invariant for Γ and $\delta \in \sigma(\Gamma)$. □

2.2. *The automorphism group of the curve complex.* Let M be a surface with $\kappa(M) > 0$. Then we have the natural homomorphism $\pi\colon \Gamma(M)^\diamond \to \mathrm{Aut}(C)$. It is a natural question whether π is an isomorphism or not. The following theorem says that π is in fact an isomorphism for almost all surfaces M. In [23], Ivanov sketches a proof of this statement for surfaces of genus at least two, and Korkmaz [30] gives a proof for some surfaces of genus less than two. Luo [32] suggests another approach for this question, which does not distinguish the cases of surfaces of higher and lower genus, and finally concludes the following

THEOREM 2.8. *Let M be a surface with $\kappa(M) > 0$.*

(i) *If M is neither $M_{1,2}$ nor $M_{2,0}$, then π is an isomorphism.*

(ii) *If $M = M_{1,2}$, then the image of π is a subgroup of* $\mathrm{Aut}(C)$ *with its index five and* $\ker(\pi)$ *is the subgroup generated by a hyperelliptic involution, which is isomorphic to $\mathbb{Z}/2\mathbb{Z}$.*

(iii) *If $M = M_{2,0}$, then π is surjective and* $\ker(\pi)$ *is the subgroup generated by a hyperelliptic involution, which is isomorphic to $\mathbb{Z}/2\mathbb{Z}$.*

(iv) *The two simplicial complexes $C(M_{0,5})$ and $C(M_{1,2})$ (resp. $C(M_{0,6})$ and $C(M_{2,0})$) are isomorphic.*

THEOREM 2.9. *Let Γ be a subgroup of finite index in* $\mathrm{Aut}(C)$. *For each $g_0 \in \mathrm{Aut}(C) \setminus \{e\}$, the set $\{gg_0g^{-1} \mid g \in \Gamma\}$ consists of infinitely many elements.*

Proof. We may assume that $M \neq M_{1,2}, M_{2,0}$ by Theorem 2.8 (iv). By Theorem 2.8 (i), we identify $\mathrm{Aut}(C)$ with $\Gamma(M)^\diamond$. Let $g_0 \in \Gamma(M)^\diamond$ and assume that the set $\{gg_0g^{-1} \mid g \in \Gamma\}$ consists of only finitely many elements. Then note that for any infinite subset $\{h_n\}_{n\in\mathbb{N}}$ of Γ, there exists an infinite subsequence $\{n_i\}_{i\in\mathbb{N}}$ of $\mathbb{N}$ such that $h_{n_i} g_0 h_{n_i}^{-1} = h_{n_j} g_0 h_{n_j}^{-1}$ for each i, j. Put

$$\mathrm{Fix}(g_0) = \{x \in \overline{\mathcal{T}} \mid g_0 x = x\},$$

which is a nonempty closed subset of $\overline{\mathcal{T}}$.

Assume $\mathrm{Fix}(g_0) \not\supset \mathcal{PMF}$. If we deduce a contradiction, then the inclusion $\mathrm{Fix}(g_0) \supset \mathcal{PMF}$ holds, and this implies $g_0 = e$ and completes the proof. Since the set of pseudo-Anosov foliations is dense in $\mathcal{PMF}$, there exist pseudo-Anosov elements $g_1, g_2 \in \Gamma(M)$ such that $\{F_\pm(g_1)\} \cap \{F_\pm(g_2)\} = \emptyset$ and $F_+(g_1), F_+(g_2) \in \mathcal{PMF} \setminus \mathrm{Fix}(g_0)$. Using the assumption that Γ is a subgroup of finite index in $\Gamma(M)^\diamond$, we may assume that $g_1, g_2 \in \Gamma$.

Let $s \in \mathrm{Fix}(g_0)$. It follows from the remark in the first paragraph that there exists an infinite increasing subsequence $\{n_i\}$ of $\mathbb{N}$ such that $g_1^{-n_i} g_0 g_1^{n_i} = g_0$ for each i. Then we have $s = g_0 s = g_1^{-n_i} g_0 g_1^{n_i} s$, which implies $g_1^{n_i} s \in \mathrm{Fix}(g_0)$ for each i. If $s \neq F_-(g_1)$, then $g_1^{n_i} s \to F_+(g_1) \in \mathcal{PMF} \setminus \mathrm{Fix}(g_0)$ as $i \to \infty$. This is a contradiction. Thus, $s = F_-(g_1)$. Similarly, we can show that if $s \in \mathrm{Fix}(g_0)$, then $s = F_-(g_2)$. This contradicts $\{F_\pm(g_1)\} \cap \{F_\pm(g_2)\} = \varnothing$. $\square$

2.3. *Measure equivalence and orbit equivalence.* In this subsection, we recall the construction of weakly orbit equivalent actions from an ME coupling given in Section 3 in [13]. We refer the reader to [1] and Chapter XIII, Section 3 in [41] for the terminology of a discrete measured groupoid and its amenability. We fix the notation as follows: Given a discrete measured groupoid $\mathcal{G}$ on a *standard finite measure space* (X, μ) (i.e., a standard Borel space with a finite positive measure) and a Borel subset $A \subset X$ with positive measure, we denote by

$$(\mathcal{G})_A = \{\gamma \in \mathcal{G} \mid r(\gamma), s(\gamma) \in A\}$$

the groupoid restricted to A, where $r, s \colon \mathcal{G} \to X$ are the range and source maps, respectively. For $x, y \in X$, let

$$\mathcal{G}_y^x = \{\gamma \in \mathcal{G} \mid r(\gamma) = x,\ s(\gamma) = y\}$$

and let $e_x \in \mathcal{G}_x^x$ denote the unit. If A is a Borel subset of X, then $\mathcal{G}A$ denotes the saturation defined by

$$\mathcal{G}A = \{r(\gamma) \in X \mid \gamma \in \mathcal{G},\ s(\gamma) \in A\},$$

which is a Borel subset of X.

Suppose that a discrete group G admits a measure-preserving action on (X, μ). Then the product space $G \times X$ has the following groupoid structure:

- The range and source maps are given by $r(g, x) = gx$ and $s(g, x) = x$, respectively, for $g \in G$ and $x \in X$.
- The operation of products is given by $(g_1, g_2x)(g_2, x) = (g_1g_2, x)$ for g_1, $g_2 \in G$ and $x \in X$.
- (e, x) is the unit element at $x \in X$.
- The inverse of $(g, x) \in G \times X$ is given by (g^{-1}, gx).

When $G \times X$ is equipped with this groupoid structure, we denote it by $G \ltimes X$.

Let (Σ, m) be an ME coupling of discrete groups Γ and Λ, and choose fundamental domains $Y, X \subset \Sigma$ for the Γ-action and Λ-action, respectively. Remark that we have a natural Γ-action on the space X equipped with the restricted measure of m because X can be identified with the quotient space Σ/Λ as a Borel space. Similarly, we have a natural Λ-action on Y. In order to distinguish from the original Γ-action and Λ-action on Σ, we denote the Γ-action on X and the Λ-action on Y by $\gamma \cdot x$ and $\lambda \cdot y$, respectively, using a dot. Note that one can choose X and Y so that $A = X \cap Y$ satisfies $\Gamma \cdot A = X$ and $\Lambda \cdot A = Y$ up to null sets. In what follows, we suppose that X and Y satisfy this condition.

Let $\mathcal{G} = \Gamma \ltimes X$ (resp. $\mathcal{H} = \Lambda \ltimes Y$) be the discrete measured groupoid on (X, μ) (resp. (Y, ν)) constructed from the above action. We can define cocycles

$$\alpha \colon \Gamma \times X \to \Lambda, \quad \beta \colon \Lambda \times Y \to \Gamma$$

so that $\gamma \cdot x = (\gamma, \alpha(\gamma, x))x \in X$ and $\lambda \cdot y = (\beta(\lambda, y), \lambda)y \in Y$ for any $\gamma \in \Gamma$, $\lambda \in \Lambda$, and a.e. $x \in X$, $y \in Y$. Let

$$p \colon X \to Y, \quad q \colon Y \to X$$

be the Borel maps defined by

$$p(x) = \Gamma x \cap Y, \quad q(y) = \Lambda y \cap X$$

for $x \in X$ and $y \in Y$. Note that both p and q are the identity on $A = X \cap Y$ and

$$p(\gamma \cdot x) = \alpha(\gamma, x) \cdot p(x), \quad q(\lambda \cdot y) = \beta(\lambda, y) \cdot q(y)$$

for any $\gamma \in \Gamma$, $\lambda \in \Lambda$, and a.e. $x \in X$, $y \in Y$. Define groupoid homomorphisms

$$f \colon \mathcal{G} \ni (\gamma, x) \mapsto (\alpha(\gamma, x), p(x)) \in \mathcal{H},$$
$$g \colon \mathcal{H} \ni (\lambda, y) \mapsto (\beta(\lambda, y), q(y)) \in \mathcal{G}.$$

Note that $\beta(\alpha(\gamma, x), x) = \gamma$ for any $\gamma \in \Gamma$ and a.e. $x \in A$ with $\gamma \cdot x \in A$, and $\alpha(\beta(\lambda, y), y) = \lambda$ for any $\lambda \in \Lambda$ and a.e. $y \in A$ with $\lambda \cdot y \in A$. Therefore, we obtain the following

PROPOSITION 2.10. *The groupoid homomorphisms*

$$f \colon (\mathcal{G})_A \to (\mathcal{H})_A, \quad g \colon (\mathcal{H})_A \to (\mathcal{G})_A$$

satisfy $g \circ f = \mathrm{id}$ *and* $f \circ g = \mathrm{id}$.

This proposition implies that the two actions of Γ on X and of Λ on Y are weakly orbit equivalent, that is, $(\Gamma \cdot x) \cap A = (\Lambda \cdot x) \cap A$ for a.e. $x \in A$.

Consider the $(\Gamma \times \Lambda)$-action on $X \times \Lambda$ defined by

$$(\gamma, \lambda)(x, \lambda') = (\gamma \cdot x, \alpha(\gamma, x)\lambda'\lambda^{-1})$$

for $\gamma \in \Gamma$, $\lambda, \lambda' \in \Lambda$, and $x \in X$. It is easy to check the following

LEMMA 2.11. *The Borel map* $\Sigma \to X \times \Lambda$ *defined by* $\lambda x \mapsto (x, \lambda^{-1})$ *for* $x \in X$ *and* $\lambda \in \Lambda$ *is a* $(\Gamma \times \Lambda)$*-equivariant Borel isomorphism.*

Conversely, we know the following theorem, which will not be used in the sequel. For simplicity, a *standard* action of a discrete group means an essentially free, measure-preserving Borel action of it on a standard finite measure space.

THEOREM 2.12 ([13, Th. 3.3]). *If two discrete groups* Γ *and* Λ *have ergodic standard actions on* (X, μ) *and* (Y, ν) *which are weakly orbit equivalent, then there* *exists an* ME *coupling* (Σ, m) *of* Γ *and* Λ *such that the* Γ*-actions on* X *and* $\Lambda\backslash\Sigma$ (*resp. the* Λ*-actions on* Y *and* $\Gamma\backslash\Sigma$) *are conjugate.*

2.4. *Normal subgroupoids.* In this subsection, we introduce the notion of normal subgroupoids of a discrete measured groupoid, based on [9] and Subsection 4.6.1 in [27]. This notion is a generalization of normal subrelations of a discrete measured equivalence relation and also a generalization of normal subgroups of a discrete group.

Let $\mathcal{G}$ be a discrete measured groupoid on a standard finite measure space (X, μ) and denote by $r, s\colon \mathcal{G} \to X$ the range and source maps, respectively. Let $\mathcal{S}$ be a subgroupoid of $\mathcal{G}$. In this paper, we mean by a subgroupoid of $\mathcal{G}$ a Borel subgroupoid of $\mathcal{G}$ whose unit space is the same as the one for $\mathcal{G}$. We denote by $\mathrm{End}_{\mathcal{G}}(\mathcal{S})$ the set of all Borel maps $\varphi\colon \mathrm{dom}(\varphi) \to \mathcal{G}$ from a Borel subset $\mathrm{dom}(\varphi)$ of X such that

(i) $s(\varphi(x)) = x$ for a.e. $x \in \mathrm{dom}(\varphi)$;

(ii) for a.e. $\gamma \in (\mathcal{G})_{\mathrm{dom}(\varphi)}$, $\gamma \in \mathcal{S}$ if and only if $\varphi(r(\gamma))\gamma\varphi(s(\gamma))^{-1} \in \mathcal{S}$.

We define the composition $\psi \circ \varphi\colon \mathrm{dom}(\psi \circ \varphi) \to \mathcal{G}$ of two elements $\varphi, \psi \in \mathrm{End}_{\mathcal{G}}(\mathcal{S})$ by putting

$$\mathrm{dom}(\psi \circ \varphi) = \{x \in \mathrm{dom}(\varphi) \mid r(\varphi(x)) \in \mathrm{dom}(\psi)\},$$
$$\psi \circ \varphi(x) = \psi(r(\varphi(x)))\varphi(x)$$

for $x \in \mathrm{dom}(\psi \circ \varphi)$. It is easy to check that $\psi \circ \varphi \in \mathrm{End}_{\mathcal{G}}(\mathcal{S})$.

Definition 2.2. A subgroupoid $\mathcal{S}$ of a discrete measured groupoid $\mathcal{G}$ on a standard finite measure space (X, μ) is said to be *normal* in $\mathcal{G}$ if the following condition is satisfied: There exists a countable family $\{\phi_n\}$ of maps in $\mathrm{End}_{\mathcal{G}}(\mathcal{S})$ such that for a.e. $\gamma \in \mathcal{G}$, we can find ϕ_n in the family satisfying $r(\gamma) \in \mathrm{dom}(\phi_n)$ and $\phi_n(r(\gamma))\gamma \in \mathcal{S}$. In this case, we write $\mathcal{S} \lhd \mathcal{G}$ and we call $\{\phi_n\}$ a family of normal choice functions for the pair $(\mathcal{G}, \mathcal{S})$.

The following two lemmas give natural examples of normal subgroupoids. The proof of Lemma 2.13 is straightforward. Lemmas 2.14 and 2.15 can be proved by using 18.14 in [26].

LEMMA 2.13. *Suppose that a discrete group* G *has a measure-preserving action on* (X, μ). *Let* H *be a normal subgroup of* G. *Let* $\mathcal{G}$ *and* $\mathcal{H}$ *be the groupoids generated by the actions of* G *and* H, *respectively. Then the subgroupoid* $\mathcal{H}$ *is normal in* $\mathcal{G}$.

LEMMA 2.14. *Let $\mathcal{G}$ be a discrete measured groupoid on (X, μ). Then the isotropy groupoid*

$$\mathcal{G}_0 = \{\gamma \in \mathcal{G} \mid r(\gamma) = s(\gamma)\}$$

is normal in $\mathcal{G}$.

LEMMA 2.15. *Let $\mathcal{G}$ be a discrete measured groupoid on (X, μ) and let A be a Borel subset of X. Then there exists a Borel map $f: \mathcal{G}A \to \mathcal{G}$ such that*

(i) *$s(f(x)) = x$ and $r(f(x)) \in A$ for a.e. $x \in \mathcal{G}A$;*

(ii) *$f(x) = e_x \in \mathcal{G}_x^x$ for a.e. $x \in A$, where $\mathcal{G}_x^x = \{\gamma \in \mathcal{G} \mid r(\gamma) = s(\gamma) = x\}$ and e_x is the unit element of the isotropy group $\mathcal{G}_x^x$.*

LEMMA 2.16. *Let $\mathcal{G}$ be a discrete measured groupoid on (X, μ) and let $\mathcal{S}$ be a normal subgroupoid of $\mathcal{G}$. If A is a Borel subset of X with positive measure, then $(\mathcal{S})_A$ is normal in $(\mathcal{G})_A$.*

Proof. Let $\{\phi_n\}$ be a family of normal choice functions for the pair $(\mathcal{G}, \mathcal{S})$. We put

$$D_n = \{x \in A \cap \mathrm{dom}(\phi_n) \mid r(\phi_n(x)) \in \mathcal{S}A\}.$$

Define a Borel map $\phi_n': D_n \to (\mathcal{G})_A$ by $\phi_n'(x) = f(r(\phi_n(x)))\phi_n(x)$ for $x \in D_n$, where $f: \mathcal{S}A \to \mathcal{S}$ is a Borel map given by Lemma 2.15 such that

- $s(f(x)) = x$ and $r(f(x)) \in A$ for a.e. $x \in \mathcal{S}A$;
- $f(x) = e_x \in \mathcal{S}_x^x$ for a.e. $x \in A$.

We show that $\{\phi_n'\}$ is a family of normal choice functions for $((\mathcal{G})_A, (\mathcal{S})_A)$. Since ϕ_n' is the composition of ϕ_n and f, we see that $\phi_n' \in \mathrm{End}_{\mathcal{G}}(\mathcal{S})$. Let $\gamma \in (\mathcal{G})_A$. Then there exists ϕ_n such that $r(\gamma) \in \mathrm{dom}(\phi_n)$ and $\phi_n(r(\gamma))\gamma \in \mathcal{S}$. Note that $r(\gamma) \in A \cap \mathrm{dom}(\phi_n)$ and $r(\phi_n(r(\gamma))) = r(\phi_n(r(\gamma))\gamma) \in \mathcal{S}A$. Therefore, $r(\gamma) \in D_n$ and

$$\phi_n'(r(\gamma))\gamma = f(r(\phi_n(r(\gamma))))\phi_n(r(\gamma))\gamma \in (\mathcal{S})_A,$$

which completes the proof. □

LEMMA 2.17. *Let G be a discrete group generated by two subgroups G_1 and G_2 so that G_1 is normal in G, and assume that we have a measure-preserving action of G on a standard finite measure space (X, μ). We denote by $\mathcal{G}$, $\mathcal{G}_1$ and $\mathcal{G}_2$ the groupoids arising from the actions of G, G_1 and G_2, respectively. Let $A \subset X$ be a Borel subset with positive measure. Then $(\mathcal{G}_1)_A$ is normal in the subgroupoid $\mathcal{H} = (\mathcal{G}_1)_A \vee (\mathcal{G}_2)_A$ of $(\mathcal{G})_A$ generated by the two subgroupoids $(\mathcal{G}_1)_A$ and $(\mathcal{G}_2)_A$.*

Proof. For each $i = 1, 2$ and $g \in G_i$, define $A_g = A \cap g^{-1}A$ and $\psi_g: A_g \to (\mathcal{G})_A$ by $\psi_g(x) = (g, x)$ for $x \in A_g$. It is easy to check that $\psi_g \in \mathrm{End}_{\mathcal{H}}((\mathcal{G}_1)_A)$. For each word ω of elements in G_1 and G_2, we can naturally define the composition $\psi_\omega \in \mathrm{End}_{\mathcal{H}}((\mathcal{G}_1)_A)$ of ψ_g's. It is clear that $\{\psi_\omega\}_\omega$ forms a family of normal choice functions for $(\mathcal{H}, (\mathcal{G}_1)_A)$. □

3. **Groupoids associated with measure-preserving actions of the mapping class group**

3.1. *Subgroupoids defined geometrically.* In Sections 4 and 5, we consider mainly the groupoid generated by a measure-preserving action of the mapping class group and its subgroupoids. In this subsection, we collect fundamental facts on them. Most of the following results can be shown in the same way as in [27], where we assume that the action is essentially free.

Definition 3.1. A discrete measured groupoid $\mathcal{G}$ on a standard finite measure space (X, μ) is said to be of *infinite type* if there exists a Borel partition $X = A_1 \sqcup A_2$ such that

(i) for a.e. $x \in A_1$, the isotropy group $\mathcal{G}_x^x$ is infinite;

(ii) the associated principal groupoid of $(\mathcal{G})_{A_2}$ defined by

$$\{(r(\gamma), s(\gamma)) \in A_2 \times A_2 \mid \gamma \in (\mathcal{G})_{A_2}\}$$

is recurrent. Namely, the restriction of it to any Borel subset of A_2 with positive measure does not admit a Borel fundamental domain.

Note that for any $n \in \mathbb{N} \cup \{\infty\}$, the subset

$$X_n = \{x \in X \mid |\mathcal{G}_x^x| = n\}$$

is Borel and satisfies $\mathcal{G}X_n = X_n$.

Let $\mathcal{G}$ be a discrete measured groupoid on (X, μ) and let $\rho: \mathcal{G} \to G$ be a groupoid homomorphism into a standard Borel group G. Let S be a Borel G-space. Recall that a Borel map $\varphi: A \to S$ from a Borel subset $A \subset X$ is said to be *ρ-invariant* for $\mathcal{G}$ if $\rho(\gamma)\varphi(s(\gamma)) = \varphi(r(\gamma))$ for a.e. $\gamma \in (\mathcal{G})_A$. The following lemma is used to extend a ρ-invariant Borel map $\varphi: A \to S$ to a ρ-invariant Borel map defined on the saturation $\mathcal{G}A$.

LEMMA 3.1. *Let $\varphi: A \to S$ be a ρ-invariant Borel map for $\mathcal{G}$ as above. Define a Borel map $\varphi': \mathcal{G}A \to S$ by $\varphi'(x) = \rho(f(x)^{-1})\varphi(r(f(x)))$ for $x \in \mathcal{G}A$, where $f: \mathcal{G}A \to \mathcal{G}$ is the Borel map constructed in Lemma* 2.15. *Then φ' is also ρ-invariant for $\mathcal{G}$.*

Proof. Let $\gamma \in (\mathcal{G})_{\mathcal{G}A}$ and put $y = r(\gamma)$, $x = s(\gamma) \in \mathcal{G}A$. Then

$$\begin{aligned}\rho(\gamma)\varphi'(x) &= \rho(f(y)^{-1})\rho(f(y)\gamma f(x)^{-1})\varphi(r(f(x))) \\ &= \rho(f(y)^{-1})\varphi(r(f(y))) = \varphi'(y)\end{aligned}$$

since $f(y)\gamma f(x)^{-1} \in (\mathcal{G})_A$ and $r(f(y)\gamma f(x)^{-1}) = r(f(y))$. □

Assumption 3.1. We refer the following assumption as $(\star)$: Let Γ be a subgroup of $\Gamma(M; m)$, where M is a surface with $\kappa(M) > 0$ and $m \geq 3$ is an integer.

Let (X, μ) be a standard finite measure space. Assume that we have a measure-preserving action of Γ on (X, μ), which generates the groupoid

$$\mathcal{G} = \mathcal{G}_\Gamma = \{(\gamma, x) \in \Gamma \times X \mid \gamma \in \Gamma,\ x \in X\}.$$

Define the induced cocycle $\rho: \mathcal{G} \to \Gamma$ by $(\gamma, x) \mapsto \gamma$ for $\gamma \in \Gamma$ and $x \in X$.

Under the above assumption, we often use the following notation:

- For a subgroup Γ' of Γ, let $\mathcal{G}_{\Gamma'}$ denote the subgroupoid of $\mathcal{G}$ generated by the action of Γ':
$$\mathcal{G}_{\Gamma'} = \{(\gamma, x) \in \mathcal{G} \mid \gamma \in \Gamma',\ x \in X\}.$$
- For $\sigma \in S(M)$, we denote by D_σ the intersection of Γ and the subgroup generated by Dehn twists about all curves in σ. We write $\mathcal{G}_\sigma$ instead of $\mathcal{G}_{D_\sigma}$ for simplicity. If σ consists of one element $\alpha \in V(C)$, then we write D_α (resp. $\mathcal{G}_\alpha$) instead of D_σ (resp. $\mathcal{G}_\sigma$).

As in [27], we introduce two types of subgroupoids of infinite type, following the classification of subgroups of $\Gamma(M)$ in Theorem 2.1. Let $M(\mathcal{PMF})$ denote the space of all probability measures on $\mathcal{PMF}$.

THEOREM 3.2 ([27, Th. 4.41]). *Under the assumption* $(\star)$, *let* $Y \subset X$ *be a Borel subset with positive measure and let* $\mathcal{S}$ *be a subgroupoid of* $(\mathcal{G})_Y$ *of infinite type. If we have a* ρ*-invariant Borel map* $\varphi: Y \to M(\mathcal{PMF})$ *for* $\mathcal{S}$, *then there exists a Borel partition* $Y = Y_1 \sqcup Y_2$ *satisfying the following*:

(i) $\varphi(x)(\mathcal{MIN}) = 1$ *for a.e.* $x \in Y_1$;

(ii) $\varphi(x)(\mathcal{PMF} \setminus \mathcal{MIN}) = 1$ *for a.e.* $x \in Y_2$.

In this theorem, remark that both Y_1 and Y_2 are invariant for $\mathcal{S}$ and that if Y' is a Borel subset of Y with positive measure and $\psi: Y' \to M(\mathcal{PMF})$ is another ρ-invariant Borel map for $\mathcal{S}$, then

(i) $\psi(x)(\mathcal{MIN}) = 1$ for a.e. $x \in Y_1 \cap Y'$;

(ii) $\psi(x)(\mathcal{PMF} \setminus \mathcal{MIN}) = 1$ for a.e. $x \in Y_2 \cap Y'$,

where Y_1 and Y_2 are the same Borel subsets as in the theorem. Hence, it is natural to give the following

Definition 3.2. Under the assumption $(\star)$, let $Y \subset X$ be a Borel subset with positive measure and let $\mathcal{S}$ be a subgroupoid of $(\mathcal{G})_Y$ of infinite type.

(i) If there is a ρ-invariant Borel map $\varphi: Y \to M(\mathcal{PMF})$ for $\mathcal{S}$ such that

$$\varphi(x)(\mathcal{MIN}) = 1$$

for a.e. $x \in Y$, then we say that $\mathcal{S}$ is *IA* (= irreducible and amenable).

(ii) If there is a ρ-invariant Borel map $\varphi\colon Y \to M(\mathcal{PMF})$ for $\mathcal{S}$ such that

$$\varphi(x)(\mathcal{PMF} \setminus \mathcal{MIN}) = 1$$

for a.e. $x \in Y$, then we say that $\mathcal{S}$ is *reducible*.

It is shown that IA subgroupoids are in fact amenable (see Theorem 3.4 (iii)). Moreover, IA (resp. reducible) subgroupoids satisfy similar properties to the ones known for IA (resp. reducible) subgroups, which will be stated in subsequent theorems of this subsection. In fact, if X consists of a single point and $\mathcal{G}$ is isomorphic to Γ, then IA and reducible subgroupoids in Definition 3.2 coincides with IA and reducible subgroups given in Theorem 2.1.

We have explained in Section 1 that the key ingredient of the proof of Theorem 1.1 is to construct an essentially unique, almost $(\Gamma(M) \times \Gamma(M))$-equivariant Borel map $\Phi\colon \Sigma \to \mathrm{Aut}(C)$ for a self ME coupling (Σ, m) of $\Gamma(M)$. We give a rough outline of the construction of the map Φ in what follows. In [27], we develop the theory of recurrent subrelations of an equivalence relation arising from a standard action of the mapping class group. Thanks to it, we can classify such subrelations into two types, IA and reducible ones, as in Definition 3.2. The notion of normal subrelations also plays an important role in the ME classification theorem of [27]. In this subsection, we generalize various central results in [27] on such subrelations to the case where the action of the mapping class group is not necessarily essentially free. In this general case, although we need to consider discrete measured groupoids arising from group actions, the proof can be given along the same line.

In Section 4, using various results in this subsection, we characterize a reducible subgroupoid in terms of amenability and normal subgroupoids (see Propositions 4.1 and 4.2). Note that these properties are preserved under an isomorphism between two groupoids. As mentioned in Section 2.3, considering a self ME coupling of $\Gamma(M)$ is almost equivalent to considering an isomorphism f between two groupoids $\mathcal{G}_1$ and $\mathcal{G}_2$ arising from measure-preserving actions of $\Gamma(M)$. Thanks to the characterization of reducible subgroupoids, one sees that the image of a reducible subgroupoid of $\mathcal{G}_1$ via f is also reducible. Moreover, maximal reducible subgroupoids are mapped to maximal ones by f (see Corollary 4.5, Lemma 4.6 and Corollary 4.7).

Let $\mathcal{G}$ be the groupoid associated with a measure-preserving action of $\Gamma(M)$ on a standard finite measure space (X, μ). As a next stage, in Section 5, we study an amenable normal subgroupoid $\mathcal{S}$ of infinite type of a maximal reducible subgroupoid of $\mathcal{G}$. We show that $\mathcal{S}$ is contained in the groupoid $\mathcal{G}_\alpha$ generated by the action of the Dehn twist about some simple closed curve $\alpha \in V(C)$ on M up to a countable Borel partition of X (see Lemma 5.1). Conversely, for any $\alpha \in V(C)$, the subgroupoid $\mathcal{G}_\alpha$ is normal in some maximal reducible subgroupoid of $\mathcal{G}$. It follows that the subgroupoid generated by the Dehn twist about a simple closed curve can be characterized in terms of amenability and normal subgroupoids, and in the situation of the previous paragraph, we see that such subgroupoids are preserved by f. This implies that f induces a bijection of the set $V(C)$ of all isotopy classes of simple closed curves on M, which is shown to be an automorphism of the curve complex. Translating this fact into structural information on a self ME coupling (Σ, m) of $\Gamma(M)$, we can construct an almost $(\Gamma(M) \times \Gamma(M))$-equivariant Borel map Φ from Σ into $\mathrm{Aut}(C)$ as mentioned in Section 1.

Remark 3.1. When $\kappa(M) > 0$ and $M \neq M_{1,2}, M_{2,0}$, it is known that any isomorphism between finite index subgroups of $\Gamma(M)^\diamond$ is the restriction of a unique inner automorphism of $\Gamma(M)^\diamond$ (see §8.5 in [24]). A key ingredient of the proof of this fact is to show that such an isomorphism f maps sufficiently high powers of Dehn twists into powers of Dehn twists by characterizing a (power of) Dehn twist algebraically (see Theorem 7.5.B in [24]). It follows that f yields a bijection on the set $V(C)$, which is in fact an automorphism of the curve complex. This automorphism comes from an element g of $\Gamma(M)^\diamond$ by Theorem 2.8. After an easy computation shown in the proof of Theorem 8.5.A in [24], one can prove that f is the restriction of the inner automorphism of $\Gamma(M)^\diamond$ given by g. Our construction of the map Φ mentioned above heavily relies on this idea due to Ivanov.

Remark 3.2. Note that if we want to prove only Theorem 1.1, it is not necessary to generalize the results in [27] to the case where the action of the mapping class group is not necessarily essentially free because in general, when two discrete groups Λ_1 and Λ_2 are ME, there exists an ME coupling of Λ_1 and Λ_2 such that the $(\Lambda_1 \times \Lambda_2)$-action on it is essentially free, which induces weak orbit equivalence between standard actions of Λ_1 and Λ_2. However, in Theorem 1.4, we need to consider an ME coupling of $\Gamma(M)$ such that the $(\Gamma(M) \times \Gamma(M))$-action is not necessarily essentially free. Moreover, thanks to the generalization, we obtain information on stabilizers of measure-preserving actions of $\Gamma(M)$ (see Corollaries 3.11 and 3.19).

In the following theorems, we collect basic properties of IA and reducible subgroupoids. First, we treat IA subgroupoids. Let ∂C denote the boundary of the curve complex C for a surface M with $\kappa(M) > 0$. Let $\partial_2 C$ be the quotient space of $\partial C \times \partial C$ by the coordinate exchanging action of the symmetric group of two letters and let $M(\partial C)$ be the space of all probability measures on ∂C, which has the Borel structure introduced in the comment before Proposition 4.30 in [27]. Each element of $\partial_2 C$ can be naturally viewed as an atomic measure in $M(\partial C)$ so that each atom has measure 1 or $1/2$. Then $\partial_2 C$ is a Borel subset of $M(\partial C)$.

Under the assumption $(\star)$, let $Y \subset X$ be a Borel subset with positive measure and let $\mathcal{S}$ be a subgroupoid of $(\mathcal{G})_Y$ of infinite type. Note that if $\mathcal{S}$ is IA, then we can construct a ρ-invariant Borel map $Y \to M(\partial C)$ for $\mathcal{S}$ by using the natural $\Gamma(M)$-equivariant map $\mathcal{MIN} \to \partial C$ constructed in [29] (see also [19]).

PROPOSITION 3.3 ([27, Prop. 4.32 (ii), Corollary 4.43]). *Under the assumption* $(\star)$, *let* $Y \subset X$ *be a Borel subset with positive measure and let* $\mathcal{S}$ *be a subgroupoid of* $(\mathcal{G})_Y$ *of infinite type. Suppose that* $\mathcal{S}$ *admits a* ρ*-invariant Borel map* $\varphi\colon Y \to M(\partial C)$. *Then the cardinality of* $\operatorname{supp}(\varphi(x))$ *is at most two for a.e.* $x \in Y$, *where* $\operatorname{supp}(\nu)$ *denotes the support of a measure* ν. *Moreover*, $\mathcal{S}$ *is* IA.

THEOREM 3.4 ([27, §4.4.1, Lemma 4.58]). *Under the assumption* $(\star)$, *let* $Y \subset X$ *be a Borel subset with positive measure and let* $\mathcal{S}$ *be a subgroupoid of* $(\mathcal{G})_Y$ *of infinite type. Suppose that* $\mathcal{S}$ *is* IA. *Then*

(i) *there exists an essentially unique* ρ*-invariant Borel map* $\varphi_0\colon Y \to \partial_2 C$ *for* $\mathcal{S}$ *satisfying the following*: *If* Y' *is a Borel subset of* Y *with positive measure and* $\varphi\colon Y' \to M(\partial C)$ *is a* ρ*-invariant Borel map for* $\mathcal{S}$, *then*

$$\operatorname{supp}(\varphi(x)) \subset \operatorname{supp}(\varphi_0(x))$$

for a.e. $x \in Y'$;

(ii) *if* $\mathcal{T}$ *is a subgroupoid of* $(\mathcal{G})_Y$ *with* $\mathcal{S} \lhd \mathcal{T}$, *then* φ_0 *is* ρ*-invariant for* $\mathcal{T}$. *It follows from Proposition* 3.3 *that* $\mathcal{T}$ *is also* IA;

(iii) *the groupoid* $\mathcal{S}$ *is amenable.*

If X consists of a single point and $\mathcal{G}$ is isomorphic to Γ, then the above facts follow from the classification of subgroups of $\Gamma(M)$ described in Theorem 2.1. In this case, using properties of pseudo-Anosov elements, we can prove that $\mathcal{S}$ is virtually cyclic, which implies Theorem 3.4 (iii). To prove Theorem 3.4 (iii) in a general case, we need to use the amenability in a measurable sense of the action of $\Gamma(M)$ on ∂C (and on $\partial_2 C$). The assertion (ii) is a very important property of IA subgroupoids because it gives a sufficient condition for amenability of bigger subgroupoids. This property will be used in the algebraic characterization of various subgroupoids in Section 4. Although the following observation will not be used in the sequel, it proves the assertion (ii) in the case where X consists of a single point.

LEMMA 3.5. *Let* M *be a surface with* $\kappa(M) \geq 0$ *and let* Γ *and* N *be subgroups of* $\Gamma(M)$ *such that* N *is* IA *and is a normal subgroup of* Γ. *Then* Γ *is also* IA.

Proof. Since N is IA, there exists a pseudo-Anosov element $g \in N$ such that N fixes its pseudo-Anosov foliations $\{F_\pm(g)\}$. Let $h \in \Gamma$. Then hgh^{-1} is pseudo-Anosov and $h\{F_\pm(g)\} = \{F_\pm(hgh^{-1})\}$. On the other hand, since $hgh^{-1} \in N$, it fixes $\{F_\pm(g)\}$. Since each pseudo-Anosov element has exactly two fixed points on $\mathcal{PMF}$, we have $\{F_\pm(hgh^{-1})\} = \{F_\pm(g)\}$. Hence, h fixes $\{F_\pm(g)\}$. □

The ρ-invariant Borel map φ_0 in Theorem 3.4 (ii) plays the same role as the fixed point set $\{F_{\pm}(g)\}$ in the proof of Lemma 3.5.

Remark 3.3. Under the assumption $(\star)$, let $Y \subset X$ be a Borel subset with positive measure and let $\mathcal{S}$ be a subgroupoid of $(\mathcal{G})_Y$ of infinite type. It follows from Theorem 3.2 that there exists an essentially unique Borel partition $Y = Y_1 \sqcup Y_2 \sqcup Y_3$ satisfying the following:

- If Y_1 has positive measure, then $(\mathcal{S})_{Y_1}$ is IA;
- If Y_2 has positive measure, then $(\mathcal{S})_{Y_2}$ is reducible;
- If Y_3 has positive measure, then $(\mathcal{S})_{Y_3'}$ admits no ρ-invariant Borel map $Y_3' \to M(\mathcal{PMF})$ for any Borel subset Y_3' of Y_3 with positive measure.

If $\mathcal{S}$ is amenable and any restriction of $\mathcal{S}$ to a Borel subset of Y with positive measure is not reducible, then $\mathcal{S}$ is IA. The converse also holds by Theorem 3.2 and Theorem 3.4 (iii).

Next, we recall basic properties of reducible subgroupoids. We can define the canonical reduction system for a reducible subgroupoid as in the case of groups.

Definition 3.3. Under the assumption $(\star)$, let $Y \subset X$ be a Borel subset with positive measure and let $\mathcal{S}$ be a subgroupoid of $(\mathcal{G})_Y$ of infinite type. Let A be a Borel subset of Y with positive measure and let $\alpha \in V(C)$.

(i) We say that the pair (α, A) is *ρ-invariant* for $\mathcal{S}$ if there exists a countable Borel partition $A = \bigsqcup A_n$ of A such that the constant map $A_n \to \{\alpha\}$ is ρ-invariant for $\mathcal{S}$ for each n.

(ii) Suppose that (α, A) is ρ-invariant for $\mathcal{S}$. The pair (α, A) is said to be *purely ρ-invariant* if (β, B) is not ρ-invariant for $\mathcal{S}$ for any Borel subset B of A with positive measure and any $\beta \in V(C)$ with $i(\alpha, \beta) \neq 0$. (In [27], we call such a pair an essential ρ-invariant one for $\mathcal{S}$.)

Since we take a countable Borel partition in the definition of a ρ-invariant pair, it is easily shown that if there are $\alpha \in V(C)$ and a Borel subset $A_n \subset Y$ for $n \in \mathbb{N}$ with (α, A_n) ρ-invariant for $\mathcal{S}$, then the pair $(\alpha, \bigcup A_n)$ is also ρ-invariant for $\mathcal{S}$. It follows that for each $\alpha \in V(C)$, we can find an essentially maximal Borel subset $A_\alpha \subset Y$ such that (α, A_α) is ρ-invariant for $\mathcal{S}$ if there exists a ρ-invariant pair for $\mathcal{S}$. One can say the same thing for purely ρ-invariant pairs for $\mathcal{S}$.

THEOREM 3.6 ([27, §4.5, Lemma 4.60]). *Under the assumption $(\star)$, let $Y \subset X$ be a Borel subset with positive measure and let $\mathcal{S}$ be a subgroupoid of $(\mathcal{G})_Y$ of infinite type. Suppose that $\mathcal{S}$ is reducible. Then*

(i) *there exists a purely ρ-invariant pair for $\mathcal{S}$;*

(ii) *we can define an essentially unique ρ-invariant Borel map $\varphi: Y \to S(M)$ for $\mathcal{S}$ so that*

(a) *if $\sigma \in S(M)$ satisfies $\mu(\varphi^{-1}(\sigma)) > 0$ and $\alpha \in \sigma$, then $(\alpha, \varphi^{-1}(\sigma))$ is a purely ρ-invariant pair for $\mathcal{S}$;*

(b) *if (α, A) is a purely ρ-invariant pair for $\mathcal{S}$, then*

$$\mu(A \setminus \varphi^{-1}(\{\sigma \in S(M) \mid \alpha \in \sigma\})) = 0;$$

(iii) *if $\mathcal{T}$ is a subgroupoid of $(\mathcal{G})_Y$ with $\mathcal{S} \lhd \mathcal{T}$, then φ is ρ-invariant for $\mathcal{T}$. In particular, $\mathcal{T}$ is also reducible.*

We call φ in the above theorem the *canonical reduction system* (CRS) for $\mathcal{S}$. It is easy to see that if A is a Borel subset of Y with positive measure, then the CRS for $(\mathcal{S})_A$ is the restriction of φ to A (see Lemma 4.53 (iii) in [27]). If X consists of a single point and $\mathcal{G}$ is isomorphic to Γ, then the above definition of the CRS for $\mathcal{S}$ coincides with the one mentioned in Definition 2.1. As in Lemma 3.5, the following lemma proves the assertion (iii) in the case where X consists of a single point.

LEMMA 3.7. *Let M be a surface with $\kappa(M) \geq 0$ and let Γ and N be subgroups of $\Gamma(M)$ such that N is infinite reducible and is a normal subgroup of Γ. Then Γ is also reducible, and $\sigma(N) \subset \sigma(\Gamma)$.*

Proof. For $g \in \Gamma$, it follows that $g\sigma(N) = \sigma(gNg^{-1}) = \sigma(N)$, and thus Γ fixes $\sigma(N)$. The latter assertion easily follows from the definition of essential reduction classes for N and Γ. □

In the following two lemmas, we study the CRS's for certain reducible subgroupoids arising from measure-preserving actions of reducible subgroups.

LEMMA 3.8. *Under the assumption ($\star$), let G be an infinite reducible subgroup of Γ and let $\sigma \in S(M)$ be the* CRS *for G. Then $\mathcal{G}_G$ is reducible and its* CRS *$\varphi: X \to S(M)$ is constant with its value σ.*

Proof. It is clear that $\mathcal{G}_G$ is reducible and for any $\alpha \in \sigma$, the pair (α, X) is ρ-invariant for $\mathcal{G}_G$. Assume that there exists $\alpha \in \sigma$ such that the pair (α, X) is not purely ρ-invariant for $\mathcal{G}_G$. Then we have a Borel subset B of X with positive measure and $\beta \in V(C)$ with $i(\alpha, \beta) \neq 0$ such that (β, B) is a ρ-invariant pair for $\mathcal{G}_G$. It follows that there exists a Borel subset B' of B with positive measure such that $\rho(\gamma)\beta = \beta$ for a.e. $\gamma \in (\mathcal{G}_G)_{B'}$. We can find $g \in G$ of infinite order with $g\beta \neq \beta$ since $\alpha \in \sigma$. Since g has infinite order and the Γ-action on (X, μ) preserves the finite positive measure μ, the subgroupoid $(\mathcal{G}_{\langle g \rangle})_{B'}$ is of infinite type, where $\langle g \rangle$ denotes the cyclic subgroup generated by g. There exist a Borel subset $B_1' \subset B'$ with positive measure and $n \in \mathbb{Z} \setminus \{0\}$ such that $(g^n, x) \in (\mathcal{G}_{\langle g \rangle})_{B'}$ for a.e. $x \in B_1'$. Thus, $g^n\beta = \rho(g^n, x)\beta = \beta$ holds for a.e. $x \in B_1'$. Since G is a subgroup of $\Gamma(M; m)$, it follows from Theorem 2.2 that $g^k\beta = \beta$ for any $k \in \mathbb{Z} \setminus \{0\}$. This is a contradiction. Thus, (α, X) is a purely ρ-invariant pair for $\mathcal{G}_G$ and we see that σ is contained in $\varphi(x)$ for a.e. $x \in X$.

Next, assume that we have a Borel subset A of X with positive measure and $\beta \in \varphi(x) \setminus \sigma$ for any $x \in A$. For each $g \in G$, there are a Borel subset $A_1 \subset A$ with positive measure and $n \in \mathbb{Z} \setminus \{0\}$ such that $(g^n, x) \in (\mathcal{G}_G)_{A_1}$ and the equation $g^n \beta = \rho(g^n, x)\beta = \beta$ holds for any $x \in A_1$ because (β, A) is a ρ-invariant pair for $\mathcal{G}_G$. Thus, $g\beta = \beta$ for any $g \in G$ by Theorem 2.2. It follows from $\beta \notin \sigma$ that there exists $\gamma \in V(C)$ such that $i(\beta, \gamma) \neq 0$ and $h\gamma = \gamma$ for any $h \in G$. Thus, (γ, X) is a ρ-invariant pair for $\mathcal{G}_G$. This contradicts the assumption that $\beta \in \varphi(x)$ for any $x \in A$, that is, the pair (β, A) is purely ρ-invariant. □

LEMMA 3.9. *Under the assumption* $(\star)$, *let* $\alpha \in V(C)$ *and assume that the subgroup* D_α *is infinite. Let* Y *be a Borel subset of* X *with positive measure and let* $\mathcal{S}$ *be a subgroupoid of* $(\mathcal{G}_\alpha)_Y$ *of infinite type. Then* $\mathcal{S}$ *is reducible and its* CRS *for* $\mathcal{S}$ *is constant with its value* $\{\alpha\}$.

Proof. It is clear that $\mathcal{S}$ is reducible and the pair (α, Y) is ρ-invariant for $\mathcal{S}$. Let A be a Borel subset of Y with positive measure and $\beta \in V(C)$. Assume that the pair (β, A) is ρ-invariant for $\mathcal{S}$. Then there exists a Borel subset B of A with positive measure such that $\rho(\gamma)\beta = \beta$ for a.e. $\gamma \in (\mathcal{S})_B$. Since $\mathcal{S}$ is a subgroupoid of $(\mathcal{G}_\alpha)_Y$ of infinite type, there exist infinitely many $n \in \mathbb{Z}$ and a Borel subset B_n of B with positive measure such that $t^n \in \Gamma$ and $(t^n, x) \in (\mathcal{S})_B$ for any $x \in B_n$, where $t \in \Gamma(M)$ denotes the Dehn twist about α. Hence, $t^n\beta = \rho(t^n, x)\beta = \beta$ for a.e. $x \in B_n$. In particular, $t^n\beta = \beta$ for infinitely many $n \in \mathbb{Z}$. It follows from Lemma 4.2 in [22] that $i(\alpha, \beta) = 0$. Thus, the pair (α, Y) is a pure ρ-invariant one for $\mathcal{S}$.

If $\gamma \in V(C)$ satisfies $i(\alpha, \gamma) = 0$ and $\alpha \neq \gamma$, then there exists $\delta \in V(C)$ such that $i(\alpha, \delta) = 0$ and $i(\gamma, \delta) \neq 0$. Since the pair (δ, Y) is ρ-invariant for $\mathcal{S}$, the pair (γ, A') can not be a pure ρ-invariant one for $\mathcal{S}$ for any Borel subset A' of Y. □

The following proposition is also proved along the same line as in [27].

PROPOSITION 3.10 ([27, Prop. 4.61]). *Under the assumption* $(\star)$, *suppose that* Γ *is sufficiently large. Then* $(\mathcal{G})_Y$ *is neither* IA *nor reducible for any Borel subset* $Y \subset X$ *with positive measure.*

As an application of the above generalization of the results in [27], we obtain some information on stabilizers for a measure-preserving action of the mapping class group on a standard finite measure space.

COROLLARY 3.11. *Under the assumption* $(\star)$, *suppose that* Γ *is sufficiently large. Then for a.e.* $x \in X$, *the isotropy group*

$$\mathcal{G}_x^x = \{\gamma \in \mathcal{G} \mid r(\gamma) = s(\gamma) = x\}$$

is either trivial or sufficiently large.

This corollary follows from Lemma 2.14, Theorem 3.4 (ii), Theorem 3.6 (iii) and Proposition 3.10. Note that Γ is torsion-free and that

- for each pseudo-Anosov element $g \in \Gamma$, the subset of X consisting of all $x \in X$ such that $\mathcal{G}_x^x$ is IA and fixes the pair $\{F_{\pm}(g)\}$ of pseudo-Anosov foliations is Borel;
- for each $\sigma \in S(M)$, the subset of X consisting of all $x \in X$ such that $\mathcal{G}_x^x$ is reducible and its CRS is σ is Borel.

It follows from these remarks that for a Borel subset Y of X with positive measure, both subsets

$$Y_1 = \{x \in Y \mid \mathcal{G}_x^x \text{ is IA}\}, \quad Y_2 = \{x \in Y \mid \mathcal{G}_x^x \text{ is reducible}\}$$

are Borel, and $(\mathcal{G}_0)_{Y_1}$ is IA and $(\mathcal{G}_0)_{Y_2}$ is reducible, where

$$\mathcal{G}_0 = \{\gamma \in \mathcal{G} \mid r(\gamma) = s(\gamma)\}$$

is the isotropy groupoid of $\mathcal{G}$.

In order to analyze reducible subgroupoids $\mathcal{S}$ furthermore, in Theorems 3.13 and 3.15, we consider components of the surface obtained by cutting M along the CRS for $\mathcal{S}$. There are three types of components as in the case of subgroups of $\Gamma(M;m)$ mentioned in the comment right after Theorem 2.5.

If Γ is an infinite reducible subgroup of $\Gamma(M;m)$ with an integer $m \geq 3$ and $\sigma \in S(M)$ is the CRS for Γ, then we can classify each component Q of M_σ in terms of properties of the quotients $p_Q(\Gamma)$ by using Theorem 2.5, where $p_Q\colon \Gamma \to \Gamma(Q)$ is the natural homomorphism. More precisely, if $p_Q(\Gamma)$ is trivial, infinite amenable or nonamenable, then Q is said to be T, IA or IN, respectively. On the other hand, when we consider a reducible subgroupoid, we cannot construct such a quotient. However, fortunately, the properties of the quotient $p_Q(\Gamma)$ used in the classification of Q can be characterized in terms of fixed points for the action of $p_Q(\Gamma)$ on the space $M(\mathcal{PMF}(Q))$ of all probability measures on $\mathcal{PMF}(Q)$ as follows:

(a) Q is T for Γ if and only if either Q is a pair of pants or $p_Q(g)\alpha = \alpha$ for any $g \in \Gamma$ and any (or some) $\alpha \in V(C(Q))$.

(b) Q is IA for Γ if and only if the following three conditions are satisfied:
- Q is not a pair of pants;
- $p_Q(g)\alpha \neq \alpha$ for any nontrivial $g \in \Gamma$ and any (or some) $\alpha \in V(C(Q))$;
- There exists $\mu \in M(\mathcal{PMF}(Q))$ such that $p_Q(g)\mu = \mu$ for any $g \in \Gamma$ and $\mu(\mathcal{MIN}(Q)) = 1$.

(c) Q is IN for Γ if and only if the following two conditions are satisfied:
- Q is not a pair of pants;
- There exists no fixed point for the action of $p_Q(\Gamma)$ on $M(\mathcal{PMF}(Q))$.

Following this observation, we will introduce three types of components of the surface obtained by cutting M along the CRS for a reducible subgroupoid. Before stating the definition of the three types of components, we recall some notation.

Let L be a submanifold of the surface M which is a realization of some element of $S(M)$. Let Q be a component of M_L, where M_L denotes the surface obtained by cutting M along L. Let $p_L\colon M_L \to M$ denote the canonical map. For $\delta \in V(C(M))$, we define a finite subset $r(\delta, Q)$ of $V(C(Q))$ as follows. Let $\delta \in V(C(M))$ and represent the isotopy class δ by a circle D that intersects each of the components of L in the least possible number of points. Put $D_L = p_L^{-1}(D)$. The manifold D_L consists of some intervals or it is a circle (if $D \cap L = \varnothing$).

If either $D_L \cap Q = \varnothing$ or D_L is a circle which lies in Q and is peripheral for Q, then put $r(\delta, Q) = \varnothing$. If D_L is a nonperipheral circle lying in Q, put $r(\delta, Q) = \{\delta\}$. In the remaining cases, the intersection $D_L \cap Q$ consists of some intervals. For each such interval I, consider a regular neighborhood N_I in Q of the union of the interval I and those components of ∂Q on which the ends of I lie. Then N_I is a pair of pants. Let $r'(\delta, Q)$ be the set of isotopy classes of components of the manifolds $\partial N_I \setminus \partial Q$, where I runs through the set of all components of $D_L \cap Q$. Define $r(\delta, Q)$ as the resulting set of discarding from $r'(\delta, Q)$ the isotopy classes of trivial or peripheral circles of Q. We will regard $r(\delta, Q)$ as a subset of $V(C(M))$ using the embedding $V(C(Q)) \hookrightarrow V(C(M))$. It is clear that this definition depends only on δ and the isotopy class of Q.

Let $F\colon M \to M$ be a diffeomorphism such that $F(L) = L$ and the induced diffeomorphism $M_L \to M_L$ takes Q to Q. If $f \in \Gamma(M)$ denotes the isotopy class of F, then we have the equality $f(r(\delta, Q)) = r(f\delta, Q)$ by definition.

LEMMA 3.12 ([22, Lemma 7.9]). *Let L and Q be the same as above and let $\delta \in V(C(M))$. If $r(\delta, Q) = \varnothing$, then one of the following three cases occurs*:

(i) *There is a simple closed curve in the class δ which does not intersect Q*;

(ii) *δ is the isotopy class of one of the components of L*;

(iii) *Q is a pair of pants.*

We denote by $D = D(M)$ the set of all isotopy classes of subsurfaces in M and denote by $\mathcal{F}_0(D)$ the set of all finite subsets F of D (including the empty set) such that if $Q_1, Q_2 \in F$ and $Q_1 \neq Q_2$, then Q_1 and Q_2 can be realized disjointly on M.

THEOREM 3.13 ([27, Th. 5.6]). *Under the assumption* $(\star)$, *let $Y \subset X$ be a Borel subset with positive measure and let $\mathcal{S}$ be a subgroupoid of $(\mathcal{G})_Y$ of infinite type. Suppose that $\mathcal{S}$ is reducible and let $\varphi\colon Y \to S(M)$ be its* CRS. *Then there exist two essentially unique ρ-invariant Borel maps $\varphi_t, \varphi_i\colon Y \to \mathcal{F}_0(D)$ for $\mathcal{S}$ satisfying the following*:

(i) *Any element in* $\varphi_t(x) \cup \varphi_i(x)$ *is a component of* $M_{\varphi(x)}$ *for a.e.* $x \in Y$;

(ii) *Each component of* $M_{\varphi(x)}$ *belongs to* $\varphi_t(x) \cup \varphi_i(x)$, *and* $\varphi_t(x) \cap \varphi_i(x) = \varnothing$ *for a.e.* $x \in Y$;

(iii) *If* Q *is in* $F \in \mathcal{F}_0(D)$ *with* $\mu(\varphi_t^{-1}(F)) > 0$, *then either* Q *is a pair of pants or the pair* $(\alpha, \varphi_t^{-1}(F))$ *is* ρ*-invariant for* $\mathcal{S}$ *for any* $\alpha \in V(C(Q))$;

(iv) *If* Q *is in* $F \in \mathcal{F}_0(D)$ *with* $\mu(\varphi_i^{-1}(F)) > 0$, *then* Q *is not a pair of pants and* (α, A) *is not* ρ*-invariant for* $\mathcal{S}$ *for any* $\alpha \in V(C(M))$ *with* $r(\alpha, Q) \neq \varnothing$ *and any Borel subset* $A \subset \varphi_i^{-1}(F)$ *with positive measure.*

We call φ_t the *T system* for $\mathcal{S}$, and call φ_i the *I system* for $\mathcal{S}$. We often call elements in $\varphi_t(x)$ and $\varphi_i(x)$ *T* and *I subsurfaces* for $\mathcal{S}$ at $x \in Y$, respectively. When we identify a subsurface with a component of the surface obtained by cutting M along some curves, we call T and I subsurfaces *T* and *I components*, respectively. It is easy to see that if A is a Borel subset of Y with positive measure, then the T and I systems for $(\mathcal{S})_A$ are the restrictions of φ_t and φ_i to A, respectively (see Lemma 5.7 in [27]). It is shown that T components satisfy the following stronger property.

LEMMA 3.14 ([27, Lemma 5.4]). *Under the assumption* ($\star$), *let* $Y \subset X$ *be a Borel subset with positive measure and let* $\mathcal{S}$ *be a subgroupoid of* $(\mathcal{G})_Y$ *of infinite type. Suppose that* $\mathcal{S}$ *is reducible and let* $\varphi: Y \to S(M)$ *be its* CRS. *We assume the following*:

- φ *is constant with its value* $\sigma \in S(M)$ *and* Q *is a component of* M_σ;
- *We have* $\alpha \in V(C(M))$ *with* $r(\alpha, Q) \neq \varnothing$ *and a Borel subset* $A \subset Y$ *with positive measure such that* (α, A) *is* ρ*-invariant for* $\mathcal{S}$.

Then there exists a countable Borel partition $A = \bigsqcup A_n$ *such that* $\rho(\gamma)\beta = \beta$ *for any* $\beta \in V(C(Q))$ *and a.e.* $\gamma \in (\mathcal{S})_{A_n}$. *In particular*, *the pair* (β, A) *is* ρ*-invariant for* $\mathcal{S}$ *for any curve* $\beta \in V(C(Q))$.

If the cocycle $\rho: \mathcal{S} \to \Gamma$ is essentially valued in $\Gamma_\sigma = \{g \in \Gamma \mid g\sigma = \sigma\}$ for some σ and Q is a component of M_σ, then ρ_Q denotes the cocycle defined by the composition of ρ with $p_Q: \Gamma_\sigma \to \Gamma(Q)$. In the next theorem, we further divide I subsurfaces into two types, IA and IN ones.

THEOREM 3.15 ([27, Th. 5.9, §5.2]). *In Theorem* 3.13, *there exist two essentially unique* ρ*-invariant Borel maps* $\varphi_{ia}, \varphi_{in}: Y \to \mathcal{F}_0(D)$ *for* $\mathcal{S}$ *satisfying the following*:

(i) $\varphi_i(x) = \varphi_{ia}(x) \cup \varphi_{in}(x)$ *and* $\varphi_{ia}(x) \cap \varphi_{in}(x) = \varnothing$ *for a.e.* $x \in Y$.

(ii) *Let* Q *be a component in* $F \in \mathcal{F}_0(D)$ *with* $\mu(\varphi_{ia}^{-1}(F)) > 0$. *Then*

(a) *given any Borel subset A of $\varphi_{ia}^{-1}(F)$ with positive measure and any ρ_Q-invariant Borel map $\psi\colon A \to M(\mathcal{PMF}(Q))$ for $\mathcal{S}$, we have the equation $\psi(x)(\mathcal{MIN}(Q)) = 1$ for a.e. $x \in A$;*

(b) *we have an essentially unique ρ_Q-invariant Borel map $\psi_0\colon \varphi_{ia}^{-1}(F) \to \partial_2 C(Q)$ for $\mathcal{S}$ such that if A is a Borel subset of $\varphi_{ia}^{-1}(F)$ with positive measure and $\psi\colon A \to M(\partial C(Q))$ is a ρ_Q-invariant Borel map for $\mathcal{S}$, then*

$$\mathrm{supp}(\psi(x)) \subset \mathrm{supp}(\psi_0(x))$$

for a.e. $x \in A$.

(iii) *If Q is in $F \in \mathcal{F}_0(D)$ with $\mu(\varphi_{in}^{-1}(F)) > 0$, then $\mathcal{S}$ admits neither ρ_Q-invariant Borel maps $A \to M(\mathcal{PMF}(Q))$ nor $A \to \partial_2 C(Q)$ for any Borel subset $A \subset \varphi_{in}^{-1}(F)$ with positive measure.*

We call φ_{ia} and φ_{in} the IA and IN *systems* for $\mathcal{S}$, respectively. We often call elements in $\varphi_{ia}(x)$ and $\varphi_{in}(x)$ *IA* and *IN subsurfaces* (or *components*) at $x \in Y$, respectively. It is easy to see that if A is a Borel subset of Y with positive measure, then the IA and IN systems for $(\mathcal{S})_A$ are the restrictions of φ_{ia} and φ_{in} to A, respectively (see Lemma 5.10 in [27]). We recall some properties of IA components in the following lemma, which can be regarded as an analogue of Theorem 3.4 (ii). The assertion (ii) corresponds to the latter assertion of Lemma 3.7.

LEMMA 3.16 ([27, Lemma 5.13]). *Under the assumption ($\star$), let $Y \subset X$ be a Borel subset with positive measure and let $\mathcal{S}$ be a subgroupoid of $(\mathcal{G})_Y$ of infinite type. Let $\mathcal{H}$ be a subgroupoid of $(\mathcal{G})_Y$ with $\mathcal{S} \lhd \mathcal{H}$. Suppose that $\mathcal{S}$ is reducible (thus, so is $\mathcal{H}$) and all of the* CRS *and* T, IA *and* IN *systems for $\mathcal{S}$ are constant. Let Q be an* IA *component for $\mathcal{S}$ and let $\psi_0\colon Y \to \partial_2 C(Q)$ be the ρ_Q-invariant Borel map for $\mathcal{S}$ as in Theorem* 3.15 (*ii*) (*b*). *Then*

(i) *ψ_0 is ρ_Q-invariant for $\mathcal{H}$;*

(ii) *if we denote by $\psi\colon Y \to S(M)$ the* CRS *for $\mathcal{H}$, then $\sigma \subset \psi(x)$ for a.e. $x \in Y$, where $\sigma \in S(M)$ is the* CRS *for $\mathcal{S}$;*

(iii) *if we denote by $\psi_{ia}\colon Y \to \mathcal{F}_0(D)$ the* IA *system for $\mathcal{H}$, then $Q \in \psi_{ia}(x)$ for a.e. $x \in Y$.*

The following proposition implies that if a reducible subgroupoid has no IN component, then it is amenable as a groupoid.

PROPOSITION 3.17 ([27, Prop. 5.18]). *Under the assumption ($\star$), let $Y \subset X$ be a Borel subset with positive measure and let $\mathcal{S}$ be a subgroupoid of $(\mathcal{G})_Y$ of infinite type. Suppose that $\mathcal{S}$ is reducible and there exists $\sigma \in S(M)$ such that $\rho(\gamma)\sigma = \sigma$ for a.e. $\gamma \in \mathcal{S}$. Let $\{Q_i\}_i$ be the set of all components of M_σ which are not pairs of pants and let $\rho_\sigma\colon \mathcal{S} \to \prod_i \Gamma(Q_i)$ be the product $\prod_i \rho_{Q_i}$. Moreover,* *we assume that there exists a ρ_σ-invariant Borel map*

$$\psi\colon Y \to \prod_i \partial_2 C(Q_i).$$

Then the groupoid $\mathcal{S}$ is amenable.

Suppose that Γ is a subgroup of $\Gamma(M;m)$ with an integer $m \geq 3$ and that $\sigma \in S(M)$ is fixed by each element of Γ. Let $p_\sigma\colon \Gamma \to \prod_Q \Gamma(Q)$ be the product $\prod_Q p_Q$, where Q runs through all components in M_σ. Note that the kernel of p_σ is contained in the amenable subgroup of $\Gamma(M)$ generated by Dehn twists about all curves in σ by Lemma 2.4. It is then easily shown that if every component of M_σ is either T or IA, then Γ is amenable. This proves Proposition 3.17 in the case where X is a point.

3.2. *Groupoids associated with actions of hyperbolic groups.* In this subsection, we study subgroupoids of a groupoid defined by a measure-preserving action of a word-hyperbolic group. Let us mention that only Lemma 3.20 will be used in the rest of the paper.

Assumption 3.2. We call the following assumption $(\star)_{\mathrm{h}}$: Let Γ be an infinite subgroup of a hyperbolic group Γ_0. Let (X,μ) be a standard finite measure space and assume that we have a measure-preserving action of Γ on (X,μ). We denote by $\mathcal{G} = \Gamma \ltimes X$ and $\rho\colon \mathcal{G} \to \Gamma$ the associated groupoid and cocycle, respectively.

For a hyperbolic group Γ_0, let $\partial\Gamma_0$ be the boundary at infinity and let $M(\partial\Gamma_0)$ be the space of all probability measures on $\partial\Gamma_0$. We denote by $\partial_2\Gamma_0$ the quotient space of $\partial\Gamma_0 \times \partial\Gamma_0$ by the coordinate exchanging action of the symmetric group of two letters, which can be naturally viewed as a Borel subset of $M(\partial\Gamma_0)$ as in the case of the boundary of the curve complex. The following proposition can be shown along the same idea in Theorem 3.4.

PROPOSITION 3.18. *Under the assumption $(\star)_{\mathrm{h}}$, let Y be a Borel subset of X with positive measure and let $\mathcal{S}$ be a subgroupoid of $(\mathcal{G})_Y$ of infinite type. Assume that there is a ρ-invariant Borel map $Y \to M(\partial\Gamma_0)$ for $\mathcal{S}$. Then*

(i) *there exists an essentially unique ρ-invariant Borel map $\varphi_0\colon Y \to \partial_2\Gamma_0$ for $\mathcal{S}$ satisfying the following: If Y' is a Borel subset of Y with positive measure and $\varphi\colon Y' \to M(\partial\Gamma_0)$ is a ρ-invariant Borel map for $\mathcal{S}$, then*

$$\mathrm{supp}(\varphi(x)) \subset \mathrm{supp}(\varphi_0(x))$$

for a.e. $x \in Y'$;

(ii) *if $\mathcal{T}$ is a subgroupoid of $(\mathcal{G})_Y$ with $\mathcal{S} \lhd \mathcal{T}$, then φ_0 is ρ-invariant for $\mathcal{T}$;*

(iii) *the groupoid $\mathcal{S}$ is amenable.*

Using Lemma 2.14 and Proposition 3.18 (ii), (iii), we can show the following corollary in the same way as Corollary 3.11. Note that the set consisting of all points in $\partial_2\Gamma_0$ fixed by some infinite subgroup of Γ_0 is countable (see Chapitre 8 in [17]).

COROLLARY 3.19. *Under the assumption* $(\star)_{\mathrm{h}}$, *suppose that* Γ *is nonamenable. Then for a.e.* $x \in X$, *the isotropy group*

$$\mathcal{G}_x^x = \{\gamma \in \mathcal{G} \mid r(\gamma) = s(\gamma) = x\}$$

is either finite or nonamenable.

LEMMA 3.20. *Let* G_1 *and* G_2 *be infinite cyclic groups and suppose that we have a measure-preserving action of the free product* $G = G_1 * G_2$ *on a standard finite measure space* (X, μ). *Let* $\mathcal{G}$, $\mathcal{G}_1$ *and* $\mathcal{G}_2$ *be the groupoids arising from the actions of* G, G_1 *and* G_2, *respectively. Then the subgroupoid* $(\mathcal{G}_1)_A \vee (\mathcal{G}_2)_A$ *of* $(\mathcal{G})_A$ *generated by* $(\mathcal{G}_1)_A$ *and* $(\mathcal{G}_2)_A$ *is nonamenable for any Borel subset* $A \subset X$ *with positive measure.*

Proof. Suppose that $(\mathcal{G}_1)_A \vee (\mathcal{G}_2)_A$ is amenable. We have the natural cocycle $\rho\colon \mathcal{G} \to G$. It follows that there exists a ρ-invariant Borel map $\varphi_0\colon A \to \partial_2 G$ for $(\mathcal{G}_1)_A \vee (\mathcal{G}_2)_A$ as in Proposition 3.18 (i). Let $a_i^{\pm} \in \partial G$ be the two fixed points on the boundary ∂G of G for the action of the group G_i for $i = 1, 2$. Then the constant map $\varphi_i\colon A \to \partial_2 G$ with its value $\{a_i^{\pm}\}$ is ρ-invariant for the subgroupoid $(\mathcal{G}_i)_A$ of infinite type. It follows that φ_i has to satisfy the property in Proposition 3.18 (i). Thus, we have $\operatorname{supp}(\varphi_0(x)) \subset \operatorname{supp}(\varphi_i(x)) = \{a_i^{\pm}\}$ for $i = 1, 2$. This contradicts $\{a_1^{\pm}\} \cap \{a_2^{\pm}\} = \varnothing$. $\square$

4. **Characterizations of reducible subgroupoids**

The next two propositions characterize amenable and nonamenable reducible subgroupoids, respectively, in terms of amenability and normal subgroupoids. As in the previous section, we use the following notation under the assumption $(\star)$:

- For a subgroup Γ' of Γ, let $\mathcal{G}_{\Gamma'}$ denote the subgroupoid of $\mathcal{G}$ generated by the action of Γ':
$$\mathcal{G}_{\Gamma'} = \{(\gamma, x) \in \mathcal{G} \mid \gamma \in \Gamma',\ x \in X\}.$$
- For $\sigma \in S(M)$, we denote by D_σ the intersection of Γ and the subgroup generated by Dehn twists about all curves in σ. We write $\mathcal{G}_\sigma$ instead of $\mathcal{G}_{D_\sigma}$ for simplicity. If σ consists of one element $\alpha \in V(C)$, then we write D_α (resp. $\mathcal{G}_\alpha$) instead of D_σ (resp. $\mathcal{G}_\sigma$).
- For $\sigma \in S(M)$, we put
$$\Gamma_\sigma = \{g \in \Gamma \mid g\sigma = \sigma\}.$$

PROPOSITION 4.1. *Under the assumption* $(\star)$, *let* $Y \subset X$ *be a Borel subset with positive measure and let* $\mathcal{S}$ *be a subgroupoid of* $(\mathcal{G})_Y$ *of infinite type. Suppose that* $\mathcal{S}$ *is amenable. Consider the following two assertions*:

(i) $\mathcal{S}$ *is reducible.*

(ii) *For any Borel subset* A *of* Y *with positive measure*, *we have a Borel subset* B *of* A *with positive measure and the following three subgroupoids* $\mathcal{S}'$, $\mathcal{S}''$ *and* $\mathcal{T}$ *of* $(\mathcal{G})_B$:

 (a) *an amenable subgroupoid* $\mathcal{S}'$ *with* $(\mathcal{S})_B < \mathcal{S}'$;
 (b) *a subgroupoid* $\mathcal{S}''$ *of infinite type with* $\mathcal{S}'' < \mathcal{S}'$;
 (c) *a nonamenable subgroupoid* $\mathcal{T}$ *with* $\mathcal{S}'' \lhd \mathcal{T}$.

Then the assertion (ii) *implies the assertion* (i). *If* Γ *is a subgroup of finite index in* $\Gamma(M;m)$, *then the converse also holds.*

Before the proof, we explain a geometric meaning of the above subgroupoids when Γ is a finite index subgroup of $\Gamma(M;m)$ and X consists of a single point, that is, $\mathcal{G}$ is isomorphic to Γ. The subgroups Λ, Λ', Λ'' and Δ introduced below correspond to the subgroupoids $\mathcal{S}$, $\mathcal{S}'$, $\mathcal{S}''$ and $\mathcal{T}$ in Proposition 4.1, respectively. Given an infinite amenable reducible subgroup Λ of Γ, let $\sigma \in S(M)$ be its CRS, and classify each component of M_σ into T and IA ones. Let Λ' be the stabilizer in Γ of all of σ, the fixed points $\{F_\pm^Q\}$ of $p_Q(\Lambda)$ for all IA components Q and all $\alpha \in V(C(R))$ for all T components R.

If $|\sigma| = \kappa(M)+1$, then $\Lambda' = D_\sigma$. Choose one $\alpha \in \sigma$ and put $\sigma' = \sigma \setminus \{\alpha\} \in S(M)$. Put $\Lambda'' = D_{\sigma'}$, and let Δ be the stabilizer of σ' in Γ.

If $|\sigma| < \kappa(M)+1$, then put $\Lambda'' = D_\sigma$, and let Δ be the stabilizer of σ in Γ.

It is then easy to see that

- $\Lambda < \Lambda'$ and Λ' is amenable;
- $\Lambda'' < \Lambda'$ and Λ'' is infinite;
- $\Lambda'' \lhd \Delta$ and Δ is nonamenable.

Conversely, if Λ is an infinite amenable subgroup of Γ and if Λ', Λ'' and Δ are subgroups of Γ satisfying the above conditions, then Λ'' is either IA or reducible since Λ'' is amenable. If Λ'' were IA, then Δ would be IA by Lemma 3.5, and thus amenable. This is a contradiction. Hence, Λ'' is reducible. Since Λ' is also amenable, it is either IA or reducible. If Λ' were IA, then it would contradict the condition that Λ' contains an infinite reducible subgroup Λ'' because any IA subgroup has a finite index subgroup generated by a pseudo-Anosov element by Theorem 2.1. Therefore, Λ' is reducible, and so is Λ.

We give a proof of Proposition 4.1 along the same line as above.

Proof of Proposition 4.1. First, we show that the assertion (ii) implies the assertion (i). If $\mathcal{S}$ were not reducible, then since $\mathcal{S}$ is of infinite type and amenable, there would exist an invariant Borel subset A of Y for $\mathcal{S}$ with positive measure such that $(\mathcal{S})_A$ is IA (see Remark 3.3). It follows from our assumption that we have a Borel subset B of A with positive measure and subgroupoids $\mathcal{S}'$, $\mathcal{S}''$ and $\mathcal{T}$ satisfying the conditions in the assertion (ii). Since $(\mathcal{S})_A$ is IA and $\mathcal{S}'$ is amenable, it follows from Theorem 3.2 that $\mathcal{S}'$ is IA. Moreover, $\mathcal{S}''$ is also IA. Thus, $\mathcal{T}$ is also IA and amenable by Theorem 3.4 (ii) and Proposition 3.3. This is a contradiction.

Next, we assume that Γ is a subgroup of finite index in $\Gamma(M;m)$ and show that the assertion (i) implies the assertion (ii). Let A be a Borel subset of Y with positive measure. Then there exists a Borel subset B of A with positive measure satisfying the following conditions (see Lemma 3.14 for the second condition):

- All of the CRS and T and IA systems for $\mathcal{S}$ are constant on B. Let $\sigma \in S(M)$ and $\varphi_t, \varphi_{ia} \in \mathcal{F}_0(M)$ be their values on B, respectively. Note that the IN system for $\mathcal{S}$ is empty since $\mathcal{S}$ is amenable;
- For a.e. $\gamma \in (\mathcal{S})_B$ and any component Q in φ_t and $\alpha \in V(C(Q))$, we have $\rho_Q(\gamma)\alpha = \alpha$, where $\rho_Q\colon (\mathcal{S})_B \to \Gamma(Q)$ is the composition of ρ and the natural projection $\Gamma_\sigma \to \Gamma(Q)$.

For each $Q \in \varphi_{ia}$, we have the canonical ρ_Q-invariant Borel map $\psi_Q\colon B \to \partial_2 C(Q)$ for $(\mathcal{S})_B$ as in Theorem 3.15 (ii) (b). Let $\mathcal{S}'$ be a subgroupoid of $(\mathcal{G})_B$ consisting of all $\gamma \in (\mathcal{G})_B$ satisfying

$$\rho(\gamma)\sigma = \sigma, \quad \rho_Q(\gamma)\psi_Q(s(\gamma)) = \psi_Q(r(\gamma)), \quad \rho(\gamma)\alpha = \alpha$$

for any $Q \in \varphi_{ia}$, any $\alpha \in V(C(R))$ and any $R \in \varphi_t$ which is not a pair of pants. Note that $(\mathcal{S})_B < \mathcal{S}'$. It follows from Proposition 3.17 that $\mathcal{S}'$ is amenable.

If $|\sigma| < \kappa(M) + 1$, then put $\mathcal{S}'' = (\mathcal{G}_\sigma)_B$. Then $\mathcal{S}'' < \mathcal{S}'$. Since Γ is a subgroup of finite index in $\Gamma(M;m)$ and there exists a component of M_σ which is not a pair of pants, we see that $\mathcal{S}''$ is of infinite type and Γ_σ is nonamenable. Thus, the subgroupoid $\mathcal{T} = (\mathcal{G}_{\Gamma_\sigma})_B$ is nonamenable. Moreover, $\mathcal{S}'' \lhd \mathcal{T}$ since D_σ is a normal subgroup of Γ_σ by Lemma 2.4. This completes the construction of subgroupoids in the assertion (ii) in the case of $|\sigma| < \kappa(M) + 1$.

If $|\sigma| = \kappa(M) + 1$, then $\mathcal{S}' = (\mathcal{G}_\sigma)_B$ and it is amenable. Choose $\alpha_0 \in \sigma$. Let $\sigma' = \sigma \setminus \{\alpha_0\}$, which is an element of $S(M)$ since $\kappa(M) > 0$. Then $\mathcal{S}'' = (\mathcal{G}_{\sigma'})_B$ is a subgroupoid of infinite type with $\mathcal{S}'' < \mathcal{S}'$. Define $\mathcal{T} = (\mathcal{G}_{\Gamma_{\sigma'}})_B$. Then $\mathcal{T}$ is nonamenable and $\mathcal{S}'' \lhd \mathcal{T}$ since $D_{\sigma'}$ is a normal subgroup of $\Gamma_{\sigma'}$ by Lemma 2.4. This completes the construction of subgroupoids in the assertion (ii) in the case of $|\sigma| = \kappa(M) + 1$. □

PROPOSITION 4.2. *Under the assumption* $(\star)$, *let* $Y \subset X$ *be a Borel subset with positive measure and let* $\mathcal{S}$ *be a subgroupoid of* $(\mathcal{G})_Y$ *of infinite type. Suppose* *that* $(\mathcal{S})_{Y'}$ *is not amenable for any Borel subset* Y' *of* Y *with positive measure. Consider the following two assertions*:

(i) $\mathcal{S}$ *is reducible.*

(ii) *For any Borel subset* A *of* Y *with positive measure, we have a Borel subset* B *of* A *with positive measure and the following two subgroupoids* $\mathcal{S}'$ *and* $\mathcal{S}''$ *of* $(\mathcal{G})_B$:

 (a) *a subgroupoid* $\mathcal{S}'$ *with* $(\mathcal{S})_B < \mathcal{S}'$;

 (b) *an amenable subgroupoid* $\mathcal{S}''$ *of infinite type with* $\mathcal{S}'' \lhd \mathcal{S}'$.

Then the assertion (ii) *implies the assertion* (i). *If* Γ *is a subgroup of finite index in* $\Gamma(M;m)$, *then the converse also holds.*

As in the previous proposition, we first explain a geometric meaning of the above subgroupoids when Γ is a finite index subgroup of $\Gamma(M;m)$ and X consists of a single point, and $\mathcal{G}$ is isomorphic to Γ. The subgroups Λ, Λ' and Λ'' introduced below correspond to the subgroupoids $\mathcal{S}$, $\mathcal{S}'$ and $\mathcal{S}''$ in Proposition 4.2, respectively. Given a nonamenable reducible subgroup Λ of Γ, let $\sigma \in S(M)$ be its CRS. Define Λ' to be the stabilizer of σ in Γ and put $\Lambda'' = D_\sigma$. Then

- $\Lambda < \Lambda'$;
- $\Lambda'' \lhd \Lambda'$ and Λ'' is infinite amenable.

Conversely, if Λ is a nonamenable subgroup of Γ and if Λ' and Λ'' are subgroups of Γ satisfying the above conditions, then Λ'' is either IA or reducible. If Λ'' were IA, then Λ' would be IA by Lemmas 3.5, and thus amenable. This is a contradiction. Hence, Λ'' is reducible, and so is Λ' by Lemma 3.7. Therefore, Λ is also reducible.

Proof of Proposition 4.2. First, we show that the assertion (ii) implies the assertion (i). Suppose that $\mathcal{S}$ is not reducible. Then there exists a Borel subset A of Y with positive measure such that for any Borel subset B of A with positive measure, there is no ρ-invariant Borel map $B \to M(\mathcal{PMF})$ for $\mathcal{S}$ (see Remark 3.3). By assumption, we have a Borel subset B of A with positive measure and two subgroupoids $\mathcal{S}'$ and $\mathcal{S}''$ satisfying the conditions in the assertion (ii). Since $\mathcal{S}''$ is amenable, by Theorem 3.2, we have a Borel partition $B = B_1 \sqcup B_2$ (up to null sets) such that $(\mathcal{S}'')_{B_1}$ is IA and $(\mathcal{S}'')_{B_2}$ is reducible. It follows from Theorem 3.4 (ii) and Theorem 3.6 (iii) that $(\mathcal{S}')_{B_1}$ is IA and $(\mathcal{S}')_{B_2}$ is reducible. If B_1 has positive measure, then $(\mathcal{S})_{B_1}$ is nonamenable by the assumption on $\mathcal{S}$. Since $(\mathcal{S})_B < \mathcal{S}'$, the groupoid $(\mathcal{S}')_{B_1}$ is nonamenable, and this contradicts Theorem 3.4 (iii). On the other hand, if B_2 has positive measure, then $(\mathcal{S})_{B_2}$ has a ρ-invariant Borel map $B_2 \to S(M) \subset M(\mathcal{PMF})$. This is also a contradiction.

Next, we assume that Γ is a subgroup of finite index in $\Gamma(M;m)$ and show that the converse also holds. Let A be a Borel subset of Y with positive measure.

Then there exists a Borel subset B of A with positive measure such that the CRS for $\mathcal{S}$ is constant on B. We denote by $\sigma \in S(M)$ its value on B. Define the subgroupoid

$$\mathcal{S}' = \{\gamma \in (\mathcal{G})_B \mid \rho(\gamma)\sigma = \sigma\} = (\mathcal{G}_{\Gamma_\sigma})_B,$$

which satisfies $(\mathcal{S})_B < \mathcal{S}'$. Let $\mathcal{S}'' = (\mathcal{G}_\sigma)_B$. Then $\mathcal{S}''$ is of infinite type since Γ is a subgroup of finite index in $\Gamma(M;m)$. Since D_σ is a normal subgroup of Γ_σ and it is amenable by Lemma 2.4, we see that $\mathcal{S}'' \lhd \mathcal{S}'$ and $\mathcal{S}''$ is amenable. $\square$

Assumption 4.1. We refer the following assumption as ($\bullet$): For $i = 1, 2$, let Γ_i be a finite index subgroup of $\Gamma(M_i; m_i)$, where M_i is a surface with $\kappa(M_i) > 0$ and $m_i \geq 3$ is an integer. Consider a measure-preserving action of Γ_i on a standard finite measure space (X_i, μ_i) and let

$$\mathcal{G}^i = \mathcal{G}^i_\Gamma, \quad \rho_i : \mathcal{G}^i \to \Gamma_i$$

be the induced groupoid and cocycle, respectively. Suppose that we have a groupoid isomorphism

$$f : (\mathcal{G}^1)_{Y_1} \to (\mathcal{G}^2)_{Y_2},$$

where $Y_i \subset X_i$ is a Borel subset satisfying $\mathcal{G}^i Y_i = X_i$ up to null sets for $i = 1, 2$.

The following corollary is a consequence of Propositions 4.1 and 4.2 characterizing reducible subgroupoids.

COROLLARY 4.3. *Under the assumption* ($\bullet$), *let A_1 be a Borel subset of Y_1 with positive measure and let $\mathcal{S}^1$ be a subgroupoid of $(\mathcal{G}^1)_{A_1}$ of infinite type. Then $\mathcal{S}^1$ is reducible if and only if the image $f(\mathcal{S}^1)$ is reducible.*

Next, we characterize maximal reducible subgroupoids. In the assumption ($\star$), let Y be a Borel subset of X with positive measure and let $\varphi : Y \to S(M)$ be a Borel map. Then we define the reducible subgroupoid

$$\mathcal{S}_\varphi = \{\gamma \in (\mathcal{G})_Y \mid \rho(\gamma)\varphi(s(\gamma)) = \varphi(r(\gamma))\}.$$

PROPOSITION 4.4. *Under the assumption* ($\star$), *let Y be a Borel subset of X with positive measure and let $\varphi : Y \to S(M)$ be a Borel map. Assume that Γ is a subgroup of finite index in $\Gamma(M;m)$. Then the* CRS *for $\mathcal{S}_\varphi$ is φ and for a.e. $x \in Y$, each component of $M_{\varphi(x)}$ either is a pair of pants or is* IN *for $\mathcal{S}_\varphi$.*

Proof. We may assume that all of the CRS and T, IA and IN systems for $\mathcal{S}_\varphi$ and φ are constant. We denote the value of φ by the same symbol. Then note that $\mathcal{S}_\varphi$ is equal to $(\mathcal{G}_{\Gamma_\varphi})_Y$. It follows from Lemmas 2.6 and 3.8 that the CRS for $\mathcal{S}_\varphi$ is φ.

Let Q be a component of M_φ which is not a pair of pants. Let $g_1, g_2 \in \Gamma_\varphi$ be elements such that $p_Q(g_1), p_Q(g_2) \in \Gamma(Q)$ are pseudo-Anosov elements with $\{F_\pm(p_Q(g_1))\} \cap \{F_\pm(p_Q(g_2))\} = \varnothing$, where $p_Q : \Gamma_\varphi \to \Gamma(Q)$ is the natural homomorphism. Let G be the subgroup of Γ_φ generated by g_1 and g_2. Note that $(\mathcal{G}_G)_Y < \mathcal{S}_\varphi$.

If Q were T for $\mathcal{S}_\varphi$, then by Lemma 3.14, there would exist a Borel subset A of Y with positive measure such that $\rho_Q(\gamma)\alpha = \alpha$ for any $\alpha \in V(C(Q))$ and for a.e. $\gamma \in (\mathcal{S}_\varphi)_A$, where ρ_Q is the composition of ρ and p_Q. This contradicts the fact that $p_Q(g_1^n)\alpha \neq \alpha$ for any $\alpha \in V(C(Q))$ and all $n \in \mathbb{Z} \setminus \{0\}$.

If Q were IA for $\mathcal{S}_\varphi$, then we would have the canonical ρ_Q-invariant Borel map $\phi\colon Y \to \partial_2 C(Q)$ for $\mathcal{S}_\varphi$ as in Theorem 3.15 (ii) (b). For $i = 1, 2$, define a Borel map $\phi_i\colon Y \to \partial_2 C(Q)$ to be the constant map whose value is the image of $\{F_\pm(p_Q(g_i))\}$ in $\partial C(Q)$. Recall that the natural map $\mathcal{MIN} \to \partial C$ is injective on the set of all pseudo-Anosov foliations. It follows that ϕ_i is the canonical ρ_Q-invariant Borel map for $(\mathcal{G}_{G_i})_Y$ for $i = 1, 2$, where G_i is the cyclic subgroup generated by g_i. Since ϕ is ρ_Q-invariant for $(\mathcal{G}_{G_i})_Y$, we have the inclusion

$$\operatorname{supp}(\phi(x)) \subset \operatorname{supp}(\phi_i(x))$$

for a.e. $x \in Y$ and any $i = 1, 2$. This is a contradiction because $\{F_\pm(p_Q(g_1))\} \cap \{F_\pm(p_Q(g_2))\} = \emptyset$. ☐

In what follows, we regard $V(C)$ as a subset of $S(M)$ naturally.

COROLLARY 4.5. *Under the assumption* ($\star$), *let Y be a Borel subset of X with positive measure and let $\varphi\colon Y \to V(C)$ be a Borel map. Assume that Γ is a finite index subgroup of $\Gamma(M; m)$. If $\mathcal{S}$ is a reducible subgroupoid of $(\mathcal{G})_Y$ with $\mathcal{S}_\varphi < \mathcal{S}$, then $\mathcal{S} = \mathcal{S}_\varphi$.*

Proof. Let $\psi\colon Y \to S(M)$ be the CRS for $\mathcal{S}$. It is enough to show $\varphi = \psi$ up to null sets. Choose $\alpha \in V(C)$ and $\sigma \in S(M)$ such that $\mu(\varphi^{-1}(\alpha) \cap \psi^{-1}(\sigma)) > 0$ and put $A = \varphi^{-1}(\alpha) \cap \psi^{-1}(\sigma)$. It suffices to prove $\varphi = \psi$ a.e. on A, that is, $\sigma = \{\alpha\}$. We may assume that all of the T, IA and IN systems for $\mathcal{S}_\varphi$ on A are constant.

Choose $\beta \in \sigma$. Since β is in the CRS for $(\mathcal{S})_A$, the pair (β, A) is ρ-invariant for $\mathcal{S}_\varphi$. If we had a component Q of M_α which is not a pair of pants and satisfies $r(\beta, Q) \neq \emptyset$, then Q would be T for $\mathcal{S}_\varphi$ by Theorem 3.13. This contradicts Proposition 4.4. Thus, $r(\beta, Q) = \emptyset$ for each component Q of M_α which is not a pair of pants. It follows from Lemma 3.12 that β is a boundary component of Q, and thus $\alpha = \beta$. Therefore, $\sigma = \{\alpha\}$ and $\varphi = \psi$ a.e. on A. ☐

LEMMA 4.6. *Under the assumption* ($\star$), *let $Y \subset X$ be a Borel subset with positive measure and let $\mathcal{S}$ be a subgroupoid of $(\mathcal{G})_Y$ of infinite type. Suppose that $\mathcal{S}$ is reducible. Then there exists a Borel map $\psi\colon Y \to V(C)$ such that $\mathcal{S} < \mathcal{S}_\psi$.*

Proof. Let $\varphi\colon Y \to S(M)$ be the CRS for $\mathcal{S}$. Choose a countable Borel partition $Y = \bigsqcup Y_n$ of Y such that φ is constant on each Y_n. Let $\alpha_n \in V(C)$ be an element such that $\alpha_n \in \varphi(x)$ for a.e. $x \in Y_n$. Then the constant map

$Y_n \ni x \mapsto \alpha_n \in V(C)$ is ρ-invariant for $(\mathcal{S})_{Y_n}$. By using Lemma 3.1, we can construct a ρ-invariant Borel map $\psi\colon Y \to V(C)$ for $\mathcal{S}$. $\square$

The following is a consequence of Corollaries 4.3, 4.5 and Lemma 4.6.

COROLLARY 4.7. *Under the assumption* ($\bullet$), *let* A_1 *be a Borel subset of* Y_1 *with positive measure and let* $\varphi_1\colon A_1 \to V(C(M_1))$ *be a Borel map. Put* $A_2 = f(A_1)$ *and*

$$\mathcal{S}^1_{\varphi_1} = \{\gamma \in (\mathcal{G}^1)_{Y_1} \mid \rho_1(\gamma)\varphi_1(s(\gamma)) = \varphi_1(r(\gamma))\}.$$

Then there exists a Borel map $\varphi_2\colon A_2 \to V(C(M_2))$ *such that* $f(\mathcal{S}^1_{\varphi_1}) = \mathcal{S}^2_{\varphi_2}$, *where*

$$\mathcal{S}^2_{\varphi_2} = \{\gamma \in (\mathcal{G}^2)_{Y_2} \mid \rho_2(\gamma)\varphi_2(s(\gamma)) = \varphi_2(r(\gamma))\}.$$

5. An equivariant Borel map from a self ME coupling

In the next lemma, we study a normal amenable subgroupoid of a maximal reducible subgroupoid. As in the previous section, we regard the vertex set $V(C)$ as a subset of the simplex set $S(M)$ naturally.

LEMMA 5.1. *Under the assumption* ($\star$), *let* Y *be a Borel subset of* X *with positive measure and let* $\varphi\colon Y \to V(C)$ *be a Borel map. Assume that* Γ *is a finite index subgroup of* $\Gamma(M;m)$. *If* $\mathcal{S}$ *is an amenable subgroupoid of* $\mathcal{S}_\varphi$ *of infinite type with* $\mathcal{S} \lhd \mathcal{S}_\varphi$, *then there exists a countable Borel partition* $Y = \bigsqcup Y_n$ *of* Y *satisfying the following conditions*:

(i) *The map* φ *is constant a.e. on* Y_n. *Let* $\alpha_n \in V(C)$ *be its value*;

(ii) *For each* n, *we have* $(\mathcal{S})_{Y_n} < (\mathcal{G}_{\alpha_n})_{Y_n} < (\mathcal{S}_\varphi)_{Y_n}$.

Proof. Recall that $\mathcal{S}_\varphi$ is reducible and its CRS is given by φ (see Proposition 4.4). Since $\mathcal{S}$ is a subgroupoid of $\mathcal{S}_\varphi$, it is also reducible. Let $\psi\colon Y \to S(M)$ be the CRS for $\mathcal{S}$. Since $\mathcal{S}$ is normal in $\mathcal{S}_\varphi$, the map ψ is ρ-invariant for $\mathcal{S}_\varphi$ and satisfies $\psi(x) \subset \varphi(x)$ for a.e. $x \in Y$. Thus, $\psi(x) = \varphi(x)$ for a.e. $x \in Y$ by Lemma 3.16 because the cardinality of $\varphi(x)$ is one.

Let A be a Borel subset of Y with positive measure such that all of the CRS $\varphi = \psi$ and T, IA and IN systems for $\mathcal{S}$ and $\mathcal{S}_\varphi$ are constant on A. We denote by $\alpha \in V(C)$ the value of $\varphi = \psi$ on A. If Q is a component of M_α, then Q is not IN for $(\mathcal{S})_A$ since $\mathcal{S}$ is amenable. If Q were IA for $(\mathcal{S})_A$, then Q would be IA for $(\mathcal{S}_\varphi)_A$ by Lemma 3.16 (iii). This contradicts Proposition 4.4. Thus, each component of M_α is T for $(\mathcal{S})_A$.

It follows from Lemma 3.14 that we have a countable Borel partition $A = \bigsqcup A_n$ of A such that $\rho(\gamma)\beta = \beta$ for each component Q of M_α and $\beta \in V(C(Q))$ and for a.e. $\gamma \in (\mathcal{S})_{A_n}$ for any n. For a.e. $\gamma \in (\mathcal{S})_{A_n}$, consider the subgroup of Γ generated by $\rho(\gamma)$. If $\rho(\gamma)$ is nontrivial, then the CRS for the subgroup is $\{\alpha\}$ by Lemma 2.7. It follows from Theorem 2.5 that $\rho(\gamma)$ lies in the kernel of the natural homomorphism from Γ_α into $\prod_Q \Gamma(Q)$, where Q is taken over all components of M_α. Thus, $\rho(\gamma) \in D_\alpha$ by Lemma 2.4. Since A is any Borel subset of Y with positive measure such that all of $\varphi = \psi$ and T, IA and IN systems for $\mathcal{S}$ and $\mathcal{S}_\varphi$ are constant on A, we complete the proof. $\square$

Under the assumption (•), let $\alpha \in V(C(M_1))$. Define the constant map $\varphi_\alpha \colon Y_1 \ni x \mapsto \alpha \in V(C(M_1))$. It follows from Corollary 4.7 that we have a Borel map $\varphi_2 \colon Y_2 \to V(C(M_2))$ such that $f(\mathcal{S}^1_{\varphi_\alpha}) = \mathcal{S}^2_{\varphi_2}$, where we use the same notation as in the corollary. Since the intersection of Γ_1 and the subgroup of $\Gamma(M_1)$ generated by the Dehn twist about α is normal in

$$\Gamma_{1,\alpha} = \{g \in \Gamma_1 \mid g\alpha = \alpha\}$$

by Lemma 2.4 and $\mathcal{S}^1_{\varphi_\alpha} = (\mathcal{G}^1_{\Gamma_{1,\alpha}})_{Y_1}$, we see that $(\mathcal{G}^1_\alpha)_{Y_1} \lhd \mathcal{S}^1_{\varphi_\alpha}$. Thus, $f((\mathcal{G}^1_\alpha)_{Y_1}) \lhd \mathcal{S}^2_{\varphi_2}$. By Lemma 5.1, we have a countable Borel partition $Y_2 = \bigsqcup A_n$ such that

(i) the map φ_2 is constant on A_n for each n. Let $\beta_n \in V(C(M_2))$ be its value on A_n;

(ii) for each n, we have $(f((\mathcal{G}^1_\alpha)_{Y_1}))_{A_n} < (\mathcal{G}^2_{\beta_n})_{A_n} < (\mathcal{S}^2_{\varphi_2})_{A_n}$.

Therefore, for each $\alpha \in V(C(M_1))$, we can define a Borel map

$$\Psi(\cdot, \alpha) \colon Y_1 \to V(C(M_2))$$

by putting $\Psi(x, \alpha) = \beta_n$ if $x \in f^{-1}(A_n)$ (up to null sets). Note that this map does not depend on the decomposition $Y_2 = \bigsqcup A_n$.

LEMMA 5.2. *If $\alpha, \alpha' \in V(C(M_1))$ satisfy $i(\alpha, \alpha') = 0$, then $i(\Psi(x, \alpha), \Psi(x, \alpha')) = 0$ for a.e. $x \in Y_1$.*

Proof. Since $i(\alpha, \alpha') = 0$, we see that

$$(\mathcal{G}^1_\alpha)_A \lhd (\mathcal{G}^1_\alpha)_A \vee (\mathcal{G}^1_{\alpha'})_A$$

for any Borel subset A of Y_1 with positive measure (see Lemma 2.17). It follows from the construction of $\Psi(\cdot, \alpha)$ and $\Psi(\cdot, \alpha')$ that we have a countable Borel partition $Y_2 = \bigsqcup A_n$ and $\beta_n, \beta'_n \in V(C(M_2))$ such that

$$(f((\mathcal{G}^1_\alpha)_{Y_1}))_{A_n} < (\mathcal{G}^2_{\beta_n})_{A_n}, \quad (f((\mathcal{G}^1_{\alpha'})_{Y_1}))_{A_n} < (\mathcal{G}^2_{\beta'_n})_{A_n}$$

for each n. Using Lemma 3.9, we see that $(f((\mathcal{G}^1_\alpha)_{Y_1}))_{A_n}$ (resp. $(f((\mathcal{G}^1_{\alpha'})_{Y_1}))_{A_n}$) is a reducible subgroupoid of $(\mathcal{G}^2)_{A_n}$ and its CRS is given by the constant map $A_n \ni x \mapsto \beta_n$ (resp. β'_n) $\in V(C(M_2))$. It follows from the above normality that the constant map $A_n \ni x \mapsto \beta_n$ is ρ_2-invariant for $(f((\mathcal{G}^1_{\alpha'})_{Y_1}))_{A_n}$, which implies $i(\beta_n, \beta'_n) = 0$ by the pureness of the pair (β'_n, A_n) for $(f((\mathcal{G}^1_{\alpha'})_{Y_1}))_{A_n}$. $\square$

LEMMA 5.3. *Let M be a surface with $\kappa(M) \geq 0$ and let $\alpha, \alpha' \in V(C(M))$ with $i(\alpha, \alpha') \neq 0$. Then t^n_α and $t^m_{\alpha'}$ generate a free group of rank two for all sufficiently* *large $n, m \in \mathbb{N}$, where $t_\alpha, t_{\alpha'} \in \Gamma(M)$ denote the Dehn twists about α and α', respectively.*

Proof. We regard α and α' as elements in $\mathcal{PMF}$. Choose an open neighborhood U of α such that

$$\bar{U} \subset \{F \in \mathcal{PMF} \mid i(F, \alpha') \neq 0\},$$

where $\bar{K}$ denotes the closure of a subset K of $\mathcal{PMF}$. Choose an open neighborhood U' of α' such that

$$\bar{U}' \subset \{F \in \mathcal{PMF} \mid i(F, \alpha) \neq 0\}$$

and $\bar{U} \cap \bar{U}' = \varnothing$. It follows from Theorem 4.3 in [22] that there exist $n, m \in \mathbb{N}$ such that

$$t_\alpha^k(\bar{U}') \subset U \subset \bar{U}, \quad t_{\alpha'}^l(\bar{U}) \subset U' \subset \bar{U}'$$

for any $k, l \in \mathbb{Z}$ with $|k| \geq n$ and $|l| \geq m$.

The lemma follows from the above inclusions and the following ping-pong argument: We show that $a = t_\alpha^n$ and $b = t_{\alpha'}^m$ generate a free group of rank two. Let w be a nonempty reduced word consisting of $a^{\pm 1}$ and $b^{\pm 1}$. We prove that w is nontrivial in $\Gamma(M)$. It follows from the above inclusions that both a^k and b^l are nontrivial for any $k, l \in \mathbb{Z} \setminus \{0\}$. Therefore, by possibly replacing w by an appropriate conjugate and an inverse, it is enough to prove that $w = a^k w' b^l$ is nontrivial in $\Gamma(M)$, where $k, l \in \mathbb{Z} \setminus \{0\}$ and w' is a reduced word such that if w' is nonempty, then the first letter of w' is b or b^{-1} and the last letter of w' is a or a^{-1}. Then $w(x) \in U$ for any $x \in \bar{U} \setminus U$ by the above inclusions, and in particular $w(x) \neq x$. Thus, w is nontrivial in $\Gamma(M)$. $\square$

LEMMA 5.4. *If $\alpha, \alpha' \in V(C(M_1))$ satisfy $i(\alpha, \alpha') \neq 0$, then $i(\Psi(x, \alpha), \Psi(x, \alpha')) \neq 0$ for a.e. $x \in Y_1$.*

Proof. Let $\alpha, \alpha' \in V(C(M_1))$ with $i(\alpha, \alpha') \neq 0$. Assume that there exists a Borel subset A of Y_1 with positive measure satisfying the following conditions:

(i) $\Psi(\cdot, \alpha)$ and $\Psi(\cdot, \alpha')$ are constant on A. Let $\beta, \beta' \in V(C(M_2))$ be their values, respectively;

(ii) $i(\beta, \beta') = 0$ and

$$f((\mathcal{G}_\alpha^1)_A) < (\mathcal{G}_\beta^2)_{f(A)}, \quad f((\mathcal{G}_{\alpha'}^1)_A) < (\mathcal{G}_{\beta'}^2)_{f(A)}.$$

Since $i(\beta, \beta') = 0$, we see that $(\mathcal{G}_\beta^2)_{f(A)} \vee (\mathcal{G}_{\beta'}^2)_{f(A)}$ is amenable. On the other hand, $(\mathcal{G}_\alpha^1)_A \vee (\mathcal{G}_{\alpha'}^1)_A$ is nonamenable by Lemmas 3.20 and 5.3. This is a contradiction. $\square$

For each $\alpha \in V(C(M_1))$, we have a Borel subset A_α of Y_1 with full measure such that $\Psi(\cdot, \alpha)$ is defined on A_α. Put

$$A_1 = \bigcap_{\alpha \in V(C(M_1))} A_\alpha.$$

By Lemmas 5.2 and 5.4, for each pair $\{\alpha, \alpha'\}$ of elements in $V(C(M_1))$, we can take a Borel subset $A_{\alpha,\alpha'}$ of A_1 with full measure so that for any $x \in A_{\alpha,\alpha'}$, we have $i(\Psi(x,\alpha), \Psi(x,\alpha')) = 0$ if $i(\alpha,\alpha') = 0$ and $i(\Psi(x,\alpha), \Psi(x,\alpha')) \neq 0$ if $i(\alpha,\alpha') \neq 0$. Put

$$A = \bigcap_{\alpha,\alpha' \in V(C(M_1))} A_{\alpha,\alpha'}.$$

Then $\Psi(x,\alpha)$ is defined for any $x \in A$ and $\alpha \in V(C(M_1))$, and the conclusions in Lemmas 5.2 and 5.4 are satisfied for any $x \in A$.

Under the assumption ($\bullet$), suppose that the two surfaces M_1 and M_2 are equal. We denote the surface by M and put $C = C(M)$. Applying the above process to f and f^{-1}, we see that there exist a Borel subset A of Y_1 with full measure and a Borel map

$$\Psi\colon A \times V(C) \to V(C)$$

such that for each $x \in A$, the map $\Psi(x,\cdot)\colon V(C) \to V(C)$ defines an element of $\mathrm{Aut}(C)$, the automorphism group of the curve complex C. We define a Borel map $\Psi\colon A \to \mathrm{Aut}(C)$ by putting $\Psi(x) = \Psi(x,\cdot)$ for $x \in A$. For simplicity, we denote $\pi \circ \rho_i$ by ρ_i for $i = 1, 2$, where $\pi\colon \Gamma \to \mathrm{Aut}(C)$ is the natural homomorphism.

LEMMA 5.5. *The equality*

$$\Psi(r(\gamma)) = \rho_2(f(\gamma))\Psi(s(\gamma))\rho_1(\gamma^{-1})$$

holds for a.e. $\gamma \in (\mathcal{G}^1)_{Y_1}$.

Proof. Let A be a Borel subset of Y_1 and let $g_1 \in \Gamma_1$ and $g_2 \in \Gamma_2$ be elements satisfying the following conditions:

(a) $(g_1, x) \in (\mathcal{G}^1)_{Y_1}$ and $(g_2, f(x)) = f(g_1, x) \in (\mathcal{G}^2)_{Y_2}$ for any $x \in A$;

(b) The map Ψ is constant on A and $g_1 A$, respectively. Let $\psi, \psi' \in \mathrm{Aut}(C)$ be the values on A and $g_1 A$, respectively.

Note that Y_1 can be covered by countably many such Borel subsets A. For each $\alpha \in V(C)$, there exists a Borel subset B of A with positive measure such that $f((\mathcal{G}^1_\alpha)_B) < (\mathcal{G}^2_{\psi(\alpha)})_{f(B)}$. Note that for $\alpha \in V(C)$ and $g \in \Gamma(M)$, we have

$$g t_\alpha g^{-1} = t_{g\alpha}$$

by Lemma 4.1.C in [24], where $t_\beta \in \Gamma(M)$ denotes the Dehn twist about $\beta \in V(C)$. It follows that

$$(g_1, r(\gamma))\gamma(g_1^{-1}, g_1 s(\gamma)) \in (\mathcal{G}^1_{g_1\alpha})_{g_1 B},$$
$$(g_2, r(\delta))\delta(g_2^{-1}, g_2 s(\delta)) \in (\mathcal{G}^2_{g_2\psi(\alpha)})_{f(g_1 B)}$$

for $\gamma \in (\mathcal{G}^1_\alpha)_B$ and $\delta \in (\mathcal{G}_{\psi(\alpha)})_{f(B)}$. Therefore, $f((\mathcal{G}^1_{g_1\alpha})_{g_1 B}) < (\mathcal{G}^2_{g_2\psi(\alpha)})_{f(g_1 B)}$. Thus, $\psi'(g_1\alpha) = g_2\psi(\alpha)$. Since this equality holds for any $\alpha \in V(C)$, we have $\psi' = g_2\psi g_1^{-1}$. This implies the equation

$$\Psi(r(\gamma)) = \rho_2(f(\gamma))\Psi(s(\gamma))\rho_1(\gamma)^{-1}$$

for a.e. $\gamma = (g_1, x) \in (\mathcal{G}^1)_{Y_1}$ with $x \in A$. □

Definition 5.1. Let S be a Borel space and let m be a positive measure on S.

(i) Suppose that we are given a Borel space T, Borel actions of a discrete group G on S and on T and a Borel map $f\colon S \to T$. We say that the map f is *almost G-equivariant* if the equality

$$f(gx) = gf(x)$$

holds for any $g \in G$ and a.e. $x \in S$.

(ii) Suppose we have discrete groups Γ, Λ and G and homomorphisms $\pi\colon \Gamma \to G$ and $\tau\colon \Lambda \to G$. Then we denote by (G, π, τ) the Borel space G equipped with the $(\Gamma \times \Lambda)$-action given by

$$(\gamma, \lambda)g = \pi(\gamma)g\tau(\lambda)^{-1}$$

for $\gamma \in \Gamma$, $\lambda \in \Lambda$ and $g \in G$.

THEOREM 5.6. *For $i = 1, 2$, let Γ_i be a finite index subgroup of $\Gamma(M; m_i)$, where M is a surface with $\kappa(M) > 0$ and $m_i \geq 3$ is an integer. Suppose that we have an* ME *coupling (Σ, m) of Γ_1 and Γ_2. Then there exists an essentially unique*, almost *$(\Gamma_1 \times \Gamma_2)$-equivariant Borel map $\Phi\colon \Sigma \to (\mathrm{Aut}(C), \pi, \pi)$, where $\pi\colon \Gamma(M)^\diamond \to \mathrm{Aut}(C)$ is the natural homomorphism.*

Proof. As in Section 2.3, we can associate to Σ a measure-preserving action of Γ_i on a standard finite measure space (X_i, μ_i) for $i = 1, 2$ such that they satisfy the assumption (•). In this proof, we use the notation in (•). For the existence of Φ, it is enough to show that there exists a Borel map $\Phi\colon \Sigma \to \mathrm{Aut}(C)$ such that

$$\Phi((g_1, g_2)z) = \pi(g_2)\Phi(z)\pi(g_1)^{-1}$$

for any $g_1 \in \Gamma_1$, $g_2 \in \Gamma_2$ and a.e. $z \in \Sigma$. By Lemma 2.11, the space Σ is isomorphic to $X_1 \times \Gamma_2$ as a $(\Gamma_1 \times \Gamma_2)$-space. Here, the $(\Gamma_1 \times \Gamma_2)$-action on $X_1 \times \Gamma_2$ is given by the formula

$$(g_1, g_2)(x, \gamma) = (g_1 x, \alpha(g_1, x)\gamma g_2^{-1})$$

for $g_1 \in \Gamma_1$, $g_2, \gamma \in \Gamma_2$ and $x \in X_1$, where $\alpha\colon \Gamma_1 \times X_1 \to \Gamma_2$ is the associated cocycle. We identify Σ with $X_1 \times \Gamma_2$. For the proof of the theorem, it is enough to show that if we define $\Phi\colon \Sigma \to \mathrm{Aut}(C)$ by the formula

$$\Phi((g_1, g_2)(x, e)) = \pi(g_2)\Psi(x)\pi(g_1)^{-1}$$

for $g_1 \in \Gamma_1$, $g_2 \in \Gamma_2$ and $x \in Y_1$, then it is well-defined. In other words, it is enough to show that

$$\pi(g_2)\Psi(x)\pi(g_1)^{-1} = \pi(g_2')\Psi(x')\pi(g_1')^{-1}$$

for any $g_1, g_1' \in \Gamma_1$, $g_2, g_2' \in \Gamma_2$ and a.e. $x, x' \in Y_1$ satisfying

$$(g_1, g_2)(x, e) = (g_1', g_2')(x', e).$$

In what follows, we omit π for simplicity. Since

$$(x', e) = ((g_1')^{-1}g_1, (g_2')^{-1}g_2)(x, e) = ((g_1')^{-1}g_1x, \alpha((g_1')^{-1}g_1, x)g_2^{-1}g_2'),$$

we see that $x' = (g_1')^{-1}g_1x \in Y_1$. Since $\alpha(g, y) = \rho_2(f(g, y))$ for $g \in \Gamma_1$ and $y \in Y_1$ with $gy \in Y_1$, we have

$$\begin{aligned}\Psi(x') = \Psi((g_1')^{-1}g_1x) &= \rho_2(f((g_1')^{-1}g_1, x))\Psi(x)\rho_1((g_1')^{-1}g_1, x)^{-1}\\ &= (g_2')^{-1}g_2\Psi(x)g_1^{-1}g_1'\end{aligned}$$

by Lemma 5.5, which shows the claim. The uniqueness of Φ is a consequence of Theorem 2.9 and the following Lemma 5.7. □

Definition 5.2. Let $\pi\colon \Gamma \to G$ be a homomorphism between discrete groups. Then π is said to be ICC (= infinite conjugacy class) if the set $\{\pi(\gamma)g\pi(\gamma)^{-1} \mid \gamma \in \Gamma\}$ consists of infinitely many elements for any $g \in G \setminus \{e\}$.

LEMMA 5.7. *Let Γ, Λ and G be discrete groups and assume that*

$$\pi\colon \Gamma \to G, \quad \tau\colon \Lambda \to G$$

are homomorphisms such that either π or τ is ICC. *Suppose the following two conditions*:

(i) *We have an* ME *coupling (Σ, m) of Γ and Λ*;

(ii) *There exist two almost $(\Gamma \times \Lambda)$-equivariant Borel maps $\Phi, \Phi'\colon \Sigma \to (G, \pi, \tau)$.*

Then Φ and Φ' are essentially equal.

Proof. We may assume that π is ICC. Define a Borel map $\Phi_0\colon \Sigma \to G$ by $\Phi_0(x) = \Phi'(x)\Phi(x)^{-1}$ for $x \in \Sigma$. Then Φ_0 satisfies the equality

$$\Phi_0((\gamma, \lambda)x) = \pi(\gamma)\Phi_0(x)\pi(\gamma)^{-1}$$

for any $\gamma \in \Gamma$, $\lambda \in \Lambda$ and a.e. $x \in \Sigma$. Therefore, Φ_0 is Λ-invariant and induces an almost Γ-equivariant Borel map $\Lambda\backslash\Sigma \to G$, where the Γ-action on G is given by conjugation via π. By projecting the finite Γ-invariant measure on $\Lambda\backslash\Sigma$ to G, we obtain a finite measure on G which is invariant under the conjugation via π of each element of Γ. Since π is ICC, the support of this measure is equal to $\{e\}$, and this implies that $\Phi_0(x) = e$ for a.e. $x \in \Sigma$. □

LEMMA 5.8. *Let Γ, Λ and G be discrete groups and assume that*

$$\pi\colon \Gamma \to G, \quad \tau\colon \Lambda \to G$$

are homomorphisms. Suppose the following three conditions:

(i) *We have a normal subgroup Γ' of Γ (resp. Λ' of Λ) of finite index and an* ME *coupling (Σ, m) of Γ and Λ*;

(ii) *Either the restrictions $\pi\colon \Gamma' \to G$ or $\tau\colon \Lambda' \to G$ is* ICC;

(iii) *There exists an almost $(\Gamma' \times \Lambda')$-equivariant Borel map $\Phi\colon \Sigma \to (G, \pi, \tau)$.*

Then the map Φ is almost $(\Gamma \times \Lambda)$-equivariant.

Proof. We may assume that the restriction $\tau\colon \Lambda' \to G$ is ICC. For fixed $\gamma \in \Gamma$ and $\lambda \in \Lambda$, define a Borel map $\Phi_0\colon \Sigma \to G$ by the formula

$$\Phi_0(x) = \Phi((\gamma, \lambda)x)^{-1}\pi(\gamma)\Phi(x)\tau(\lambda)^{-1}$$

for $x \in \Sigma$.

Let $g \in \Gamma'$ and $h \in \Lambda'$. Since Γ' is normal in Γ and Λ' is normal in Λ, we have $g' \in \Gamma'$ and $h' \in \Lambda'$ such that $\gamma g = g'\gamma$ and $\lambda h' = h\lambda$. Then

$$\begin{aligned}
\Phi_0((g, h')x) &= \Phi((\gamma g, \lambda h')x)^{-1}\pi(\gamma)\pi(g)\Phi(x)\tau(h')^{-1}\tau(\lambda)^{-1} \\
&= \Phi((g'\gamma, h\lambda)x)^{-1}\pi(g')\pi(\gamma)\Phi(x)\tau(\lambda)^{-1}\tau(h)^{-1} \\
&= \tau(h)\Phi((\gamma, \lambda)x)^{-1}\pi(\gamma)\Phi(x)\tau(\lambda)^{-1}\tau(h)^{-1} \\
&= \tau(h)\Phi_0(x)\tau(h)^{-1}.
\end{aligned}$$

Since $g \in \Gamma'$ is arbitrary, the map Φ_0 induces a Borel map $\Gamma'\backslash\Sigma \to G$. The projected finite measure on G is invariant under the conjugation via τ of each element of Λ'. As in the proof of Lemma 5.7, we can show that $\Phi_0(x) = e$ for a.e. $x \in \Sigma$. □

COROLLARY 5.9. *Let M be a surface with $\kappa(M) > 0$ and let Γ and Λ be finite index subgroups of $\Gamma(M)^\diamond$. Suppose that we have an* ME *coupling (Σ, m) of Γ and Λ. Then there exists an essentially unique, almost $(\Gamma \times \Lambda)$-equivariant Borel map $\Sigma \to (\mathrm{Aut}(C), \pi, \pi)$.*

6. Measure equivalence rigidity

For an ME coupling (Σ, m) of discrete groups Γ and Λ, the *opposite coupling* $\check{\Sigma}$ of Λ and Γ is defined as the $(\Lambda \times \Gamma)$-space obtained by the canonical isomorphism between $\Gamma \times \Lambda$ and $\Lambda \times \Gamma$.

If (Σ, m) is an ME coupling of discrete groups Γ and Λ and (Ω, n) is an ME coupling of discrete groups Λ and Δ, then the *composed coupling* $\Sigma \times_\Lambda \Omega$ of Γ and Δ is defined as the $(\Gamma \times \Delta)$-space given by the quotient of $\Sigma \times \Omega$ by the diagonal Λ-action.

Definition 6.1. Let $\pi\colon \Gamma \to G$ be a homomorphism between discrete groups. We say that π is *almost an isomorphism* if $\pi(\Gamma)$ is a finite index subgroup of G and $\ker(\pi)$ is finite.

THEOREM 6.1. *Let Γ, Λ and G be discrete groups and let $\pi, \tau\colon \Gamma \to G$ be homomorphisms. Suppose that π is* ICC *and τ is almost an isomorphism and that we have an* ME *coupling (Σ, m) of Γ and Λ. Let $\Omega = \Sigma \times_\Lambda \Lambda \times_\Lambda \check{\Sigma}$ be the self* ME *coupling of Γ. Moreover, assume that there exists an almost $(\Gamma \times \Gamma)$-equivariant Borel map $\Phi\colon \Omega \to (G, \pi, \tau)$. Then we can find the following two maps*:

(a) *a homomorphism $\rho\colon \Lambda \to G$ which is almost an isomorphism*;

(b) *an almost $(\Gamma \times \Lambda)$-equivariant Borel map $\Phi_0\colon \Sigma \to (G, \tau, \rho)$.*

Before the proof, we give the following

LEMMA 6.2. *Let Γ, Λ and G be discrete groups and let (Σ, m) be an* ME *coupling of Γ and Λ. Let $Y \subset \Sigma$ be a fundamental domain for the Γ-action on Σ and let $\theta\colon \Lambda \times Y \to \Gamma$ be the associated cocycle. Suppose the following two conditions*:

(i) *We have a homomorphism $\pi\colon \Gamma \to G$ which is almost an isomorphism*;

(ii) *There exists a subgroup G_0 of G such that the cocycle $\pi \circ \theta\colon \Lambda \times Y \to G$ is cohomologous to a cocycle which is essentially valued in G_0.*

Then G_0 is a subgroup of finite index in G.

Proof. Take a standard Λ-action on a standard probability space X_0 and define a Γ-action and a $(G \times \Lambda)$-action on $\Sigma \times G \times X_0$ by

$$\gamma(z, g, x) = ((\gamma, e)z, \pi(\gamma)g, x)$$

$$(g_1, \lambda)(z, g, x) = ((e, \lambda)z, gg_1^{-1}, \lambda x)$$

for $g, g_1 \in G$, $\gamma \in \Gamma$, $\lambda \in \Lambda$, $z \in \Sigma$, and $x \in X_0$. Consider the $(G \times \Lambda)$-space $\widetilde{\Sigma}$ given by the quotient of $\Sigma \times G \times X_0$ by the Γ-action. Since $\ker(\pi)$ is finite, the action of Λ on $\widetilde{\Sigma}$ has a fundamental domain. Since the index $[G : \pi(\Gamma)]$ is finite, the fundamental domain has finite measure. Thus, the $(G\times\Lambda)$-space $\tilde{\check{\Sigma}}$ is an ME coupling of G and Λ.

Let $p\colon \Sigma\times G\times X_0\to\tilde{\check{\Sigma}}$ be the natural projection. Then $p(Y\times\{e\}\times X_0)$ is a fundamental domain for the G-action on $\tilde{\check{\Sigma}}$. Remark that p is injective on $Y\times\{e\}\times X_0$. The cocycle $\tilde{\theta}\colon \Lambda\times p(Y\times\{e\}\times X_0)\to G$ associated to it is given by

$$\tilde{\theta}(\lambda, p(y,e,x))=\pi\circ\theta(\lambda,y)$$

for $\lambda\in\Lambda$, $y\in Y$ and $x\in X_0$. By assumption, we can find a Borel map $\varphi\colon Y\to G$ such that

$$\theta'(\lambda,y)=\varphi(\lambda\cdot y)\pi\circ\theta(\lambda,y)\varphi(y)^{-1}\in G_0$$

for any $\lambda\in\Lambda$, and a.e. $y\in Y$. Define a Borel map $\tilde{\varphi}\colon p(Y\times\{e\}\times X_0)\to G$ by $\tilde{\varphi}(p(y,e,x))=\varphi(y)$ for $y\in Y$. Then

$$\begin{aligned}&\tilde{\varphi}(\lambda\cdot p(y,e,x))\tilde{\theta}(\lambda,p(y,e,x))\tilde{\varphi}(p(y,e,x))^{-1}\\ &=\varphi(\lambda\cdot y)\pi\circ\theta(\lambda,y)\varphi(y)^{-1}\in G_0,\end{aligned}$$

and thus $\tilde{\theta}$ is cohomologous to a cocycle which is essentially valued in G_0. The lemma now follows from Lemma 6.1 in [35]. □

Proof of Theorem 6.1. This proof is almost the same as the one given in Section 6.2 in [35]. One denotes the element corresponding to $(x,\lambda,y)\in\Sigma\times\Lambda\times\check{\Sigma}$ by $[x,\lambda,y]\in\Sigma\times_\Lambda\Lambda\times_\Lambda\check{\Sigma}$. As in Lemma 6.6 in [35], we can prove the following lemma by using the assumption that π is ICC.

LEMMA 6.3. *If one defines a Borel map* $\Psi\colon\Sigma^3\to G$ *by*

$$\Psi(x,y,z)=\Phi([x,e,z])\Phi([y,e,z])^{-1}$$

for $(x,y,z)\in\Sigma^3$, *then*

$$\Psi(x,y,z_1)=\Psi(x,y,z_2)$$

for m^4*-a.e.* $(x,y,z_1,z_2)\in\Sigma^4$.

Define a Borel map $F\colon\Sigma^2\to G$ by $F(x,y)=\Psi(x,y,z)$. It follows from Lemma 6.2 in [35] that for m-a.e. $x\in\Sigma$, the Borel map $\rho_x\colon\Lambda\to\Gamma$ given by

$$\rho_x(\lambda)=F(\lambda^{-1}x,y)F(x,y)^{-1}$$

is the same for m-a.e. $y\in\Sigma$ and defines a homomorphism. Moreover, the equality

$$\rho_y(\lambda)=F(x,y)^{-1}\rho_x(\lambda)F(x,y)$$

holds for any $\lambda\in\Lambda$ and m^2-a.e. $(x,y)\in\Sigma^2$. Note that we have the equality

$$\rho_x(\lambda)=\Phi([x,\lambda,z])\Phi([x,e,z])^{-1}$$

for any $\lambda \in \Lambda$ and m^2-a.e. $(x, z) \in \Sigma^2$. Let N be the normal subgroup of Λ that is the common kernel of ρ_x for m-a.e. $x \in \Sigma$.

Let $D \subset \Sigma$ be a fundamental domain for the Λ-action on Σ and put $\widetilde{\Omega} = D \times \Lambda \times D \subset \Sigma \times \Lambda \times \Sigma$. This inclusion induces a Borel isomorphism between $\widetilde{\Omega}$ and Ω. Define a Γ-action on $\widetilde{\Omega}$ induced by the second Γ-action on Ω and define a Λ-action on $\widetilde{\Omega}$ by the left multiplication on the second coordinate:

$$(\gamma, \lambda)(x, \lambda_1, y) = (x, \lambda\lambda_1\alpha(\gamma, y)^{-1}, \gamma \cdot y)$$

for $\gamma \in \Gamma$, $\lambda, \lambda_1 \in \Lambda$ and $x, y \in D$, where $\alpha\colon \Gamma \times D \to \Lambda$ is the cocycle associated to D.

Let $\widetilde{\Phi}\colon \widetilde{\Omega} \to G$ be the Borel map induced by Φ and the isomorphism between $\widetilde{\Omega}$ and Ω. Note that $\widetilde{\Phi}$ is almost Γ-equivariant in the following sense:

$$\widetilde{\Phi}((\gamma, e)\omega) = \widetilde{\Phi}(\omega)\tau(\gamma)^{-1}$$

for any $\gamma \in \Gamma$, and a.e. $\omega \in \widetilde{\Omega}$. Put $E_0 = \widetilde{\Phi}^{-1}(\{g_n\})$, where $\{g_n\} \subset G$ is a finite set of all representatives of $G/\tau(\Gamma)$. Remark that E_0 is invariant under the action of $\ker(\tau)$. If $E \subset E_0$ is a fundamental domain for the $\ker(\tau)$-action on E_0, then it is also a fundamental domain for the Γ-action on $\widetilde{\Omega}$. Since $\ker(\tau)$ is finite, the measure of E_0 is finite. The homomorphism ρ_x is given by

$$\rho_x(\lambda) = \widetilde{\Phi}(x, \lambda, z)\widetilde{\Phi}(x, e, z)^{-1}$$

for any $\lambda \in \Lambda$ and m^2-a.e. $(x, z) \in D^2$.

For $\lambda_0 \in \Lambda$, it is easy to see that $\lambda_0 \in N$ if and only if

$$\widetilde{\Phi}(x, \lambda_0\lambda_1, y) = \widetilde{\Phi}(x, \lambda_1, y)$$

for any $\lambda_1 \in \Lambda$ and m^2-a.e. $(x, y) \in D^2$. It follows that any element in N preserves E_0. Since the measure of E_0 is finite, we see that N is finite. Note that for any $\lambda \in \Lambda$ and a.e. $t = (x, \lambda_1, z) \in \widetilde{\Omega}$, we have

$$\begin{aligned}
\rho_x(\lambda) &= \rho_x(\lambda\lambda_1)\rho_x(\lambda_1)^{-1} \\
&= \widetilde{\Phi}(x, \lambda\lambda_1, z)\widetilde{\Phi}(x, e, z)^{-1}(\widetilde{\Phi}(x, \lambda_1, z)\widetilde{\Phi}(x, e, z)^{-1})^{-1} \\
&= \widetilde{\Phi}((e, \lambda)t)\widetilde{\Phi}(t)^{-1}.
\end{aligned}$$

Let $\theta\colon \Lambda \times E \to \Gamma$ be the cocycle associated to E. It follows from Fubini's theorem that there exists $x_0 \in D$ such that $\rho = \rho_{x_0}\colon \Lambda \to G$ is a homomorphism with its kernel N and for any $\lambda \in \Lambda$, $\gamma \in \Gamma$, a.e. $x \in D$, and a.e. $(\lambda_1, z) \in \Lambda \times D$, we have

$$\rho_x(\lambda) = F(x_0, x)^{-1}\rho(x)F(x_0, x)$$

and

$$\widetilde{\Phi}((\gamma, \lambda)(x_0, \lambda_1, z)) = \rho(\lambda)\widetilde{\Phi}(x_0, \lambda_1, z)\tau(\gamma)^{-1}.$$

We show that $\rho(\Lambda)$ is a subgroup of finite index in G. Define a Borel map $\varphi\colon E \to G$ by

$$\varphi(t) = F(x_0, x)\widetilde{\Phi}(t)$$

for $t = (x, \lambda_1, z) \in E$. Put

$$\theta'(\lambda, t) = \varphi(\lambda \cdot t)\tau \circ \theta(\lambda, t)\varphi(t)^{-1}$$

for $\lambda \in \Lambda$ and $t \in E$. Since $\lambda \cdot t = (\theta(\lambda, t), \lambda)t$, we see that

$$\widetilde{\Phi}(\lambda \cdot t) = \widetilde{\Phi}((\theta(\lambda, t), \lambda)t) = \widetilde{\Phi}((e, \lambda)t)\tau \circ \theta(\lambda, t)^{-1}$$

and

$$\begin{aligned}\theta'(\lambda, t) &= F(x_0, x)\widetilde{\Phi}(\lambda \cdot t)\tau \circ \theta(\lambda, t)\widetilde{\Phi}(t)^{-1}F(x_0, x)^{-1} \\ &= F(x_0, x)\widetilde{\Phi}((e, \lambda)t)\widetilde{\Phi}(t)^{-1}F(x_0, x)^{-1} \\ &= F(x_0, x)\rho_x(\lambda)F(x_0, x)^{-1} = \rho(\lambda) \in \rho(\Lambda).\end{aligned}$$

It follows from Lemma 6.2 that $\rho(\Lambda)$ is a subgroup of finite index in G.

Finally, we construct a Borel map $\Phi_0\colon \Sigma \to G$. Note that $\{x_0\} \times \Lambda \times D \subset \widetilde{\Omega}$ is a $(\Gamma \times \Lambda)$-invariant Borel subset isomorphic to Σ as a $(\Gamma \times \Lambda)$-space. It follows from the choice of x_0 that the composition of the restriction of $\widetilde{\Phi}$ to $\{x_0\} \times \Lambda \times D$ and the map $G \ni g \mapsto g^{-1} \in G$ is a desired map. □

Combining Corollary 5.9 and Theorem 6.1, we obtain Theorem 1.1.

Proof of Theorem 1.2. First, suppose that $\kappa(M^1) \geq \kappa(M^2)$. We may assume that $\kappa(M^1) \geq 2$. By Theorem 1.1, one can find an injective homomorphism $\Gamma(M^1; 3) \to \mathrm{Aut}(C(M^2))$ whose image is of finite index in $\mathrm{Aut}(C(M^2))$. By using Theorem 2.8 and restricting the homomorphism to some subgroup Γ_1 of finite index in $\Gamma(M^1; 3)$, one can construct an injective homomorphism from Γ_1 into $\Gamma(M^2)$ whose image is of finite index in $\Gamma(M^2)$. It follows from Theorem 2 in [39] that $M^1 = M_{0,6}$ and $M^2 = M_{2,0}$. Similarly, if we assume that $\kappa(M^1) \leq \kappa(M^2)$, then it can be shown that $M^1 = M_{0,6}$ and $M^2 = M_{2,0}$. □

7. Rigidity of a direct product of mapping class groups

We first review Monod and Shalom's result proved in Section 5.1 of [35]. Let $\mathscr{C}$ be the class mentioned in Section 1, i.e., the class consisting of discrete groups G which admit a mixing unitary representation π on a Hilbert space such that the second bounded cohomology group $H_b^2(G, \pi)$ of G with coefficient π does not vanish. In what follows, we fix a positive integer n. Let $\Gamma_1, \ldots, \Gamma_n$ be torsion-free discrete groups in $\mathscr{C}$ and let $\Lambda_1, \ldots, \Lambda_n$ be torsion-free discrete groups. Put

$\Gamma = \Gamma_1 \times \cdots \times \Gamma_n$ and $\Lambda = \Lambda_1 \times \cdots \times \Lambda_n$. We write

$$\Gamma_i' = \prod_{j \neq i} \Gamma_j, \quad \Lambda_i' = \prod_{j \neq i} \Lambda_j$$

for each $i \in \{1, \ldots, n\}$.

Suppose that we have an *ergodic* ME coupling (Σ, m) of Γ and Λ, that is, (Σ, m) is an ME coupling on which $\Gamma \times \Lambda$ acts ergodically. Then there exists a bijection $t\colon \{1, \ldots, n\} \to \{1, \ldots, n\}$ and fundamental domains $Y, X \subset \Sigma$ for the Γ-action and Λ-action on Σ, respectively, satisfying

$$\Lambda_{t(i)} Y \subset \Gamma_i Y, \quad \Gamma_i X \subset \Lambda_{t(i)} X$$

for any $i \in \{1, \ldots, n\}$. Let $\bar{\Sigma}_i$ be the space of ergodic components for the $(\Gamma_i' \times \Lambda_{t(i)}')$-action on (Σ, m) for $i \in \{1, \ldots, n\}$, which is naturally a $(\Gamma_i \times \Lambda_{t(i)})$-space. Define a measure μ_i (resp. ν_i) on $\bar{\Sigma}_i$ by projecting the restricted measure on $\Gamma_i Y$ (resp. $\Lambda_{t(i)} X$) through the natural map $\Gamma_i Y \to \bar{\Sigma}_i$ (resp. $\Lambda_{t(i)} X \to \bar{\Sigma}_i$). Then

(a) μ_i and ν_i are absolutely continuous with respect to each other;

(b) both μ_i and ν_i are invariant for the $(\Gamma_i \times \Lambda_{t(i)})$-action on $\bar{\Sigma}_i$;

(c) if $\bar{Y}$ (resp. $\bar{X}$) is the image of Y (resp. X) in $\bar{\Sigma}_i$, then it is a fundamental domain for the Γ_i-action on $(\bar{\Sigma}_i, \mu_i)$ (resp. the $\Lambda_{t(i)}$-action on $(\bar{\Sigma}_i, \nu_i)$). Moreover, both $\mu_i(\bar{Y})$ and $\nu_i(\bar{X})$ are finite.

These claims are shown in the proof of Theorem 1.16 in [35]. Let

$$c_i(x) = \frac{d\mu_i}{d\nu_i}(x), \quad x \in \bar{\Sigma}_i$$

be the Radon-Nikodym derivative, which is positive and finite a.e. on $\bar{\Sigma}_i$. It follows from the condition (b) that the function c_i is invariant for the $(\Gamma_i \times \Lambda_{t(i)})$-action. Put

$$\bar{\Sigma}_{i,k} = \{x \in \bar{\Sigma}_i \mid k < c_i(x) \leq k + 1\}$$

for each nonnegative integer k. Then $\bar{\Sigma}_i = \bigsqcup_{k=0}^{\infty} \bar{\Sigma}_{i,k}$ up to null sets. It follows from the condition (c) that $\bar{\Sigma}_{i,k}$ is an ME coupling of Γ_i and $\Lambda_{t(i)}$ with respect to μ_i for each k (if $\bar{\Sigma}_{i,k}$ has nonzero measure).

In this situation, we suppose the following condition: For each $i \in \{1, \ldots, n\}$ and $j \in \{1, 2\}$, let M_i^j be a surface with $\kappa(M_i^j) > 0$ and $M_i^j \neq M_{1,2}, M_{2,0}$. Assume that Γ_i (resp. Λ_i) is a torsion-free subgroup of finite index in $\Gamma(M_i^1)^\diamond$ (resp. $\Gamma(M_i^2)^\diamond$) for each i.

Remark that the mapping class group $\Gamma(M)$ of a surface M with $\kappa(M) \geq 0$ belongs to the class $\mathscr{C}$ (see Corollary B in [20]). Whether a discrete group belongs to $\mathscr{C}$ or not is preserved under ME, and in particular under commensurability up to finite kernels (see Corollary 7.6 in [35]). Note that M_i^1 and $M_{t(i)}^2$ are diffeomorphic for any i by Theorem 1.2 and let g_i be an isotopy class of a diffeomorphism $M_{t(i)}^2 \to M_i^1$. Let

$$\pi_g\colon \prod_{i=1}^n \Gamma(M_i^2)^\diamond \to \prod_{i=1}^n \mathrm{Aut}(C(M_i^1))$$

be the isomorphism defined by

$$\pi_g(\gamma_1, \dots, \gamma_n) = (\pi(g_1\gamma_{t(1)}g_1^{-1}), \dots, \pi(g_n\gamma_{t(n)}g_n^{-1}))$$

for $\gamma_i \in \Gamma(M_i^2)^\diamond$, where we denote by the same symbol π the natural homomorphism $\Gamma(M)^\diamond \to \mathrm{Aut}(C(M))$ for any surface M. By applying Corollary 5.9 to each ME coupling $\overline{\Sigma}_{i,k}$ of Γ_i and $\Lambda_{t(i)}$, one obtains an almost $(\Gamma_i \times \Lambda_{t(i)})$-equivariant Borel map $\Phi_i\colon \overline{\Sigma}_i \to (\mathrm{Aut}(C(M_i)), \pi, \pi_{g_i})$, where $\pi_{g_i}\colon \Gamma(M_{t(i)}^2)^\diamond \to \mathrm{Aut}(C(M_i^1))$ is the isomorphism defined by g_i. Define a Borel map $\Phi\colon \Sigma \to \prod_{i=1}^n \mathrm{Aut}(C(M_i^1))$ by

$$\Phi(x) = (\Phi_1(p_1(x)), \dots, \Phi_n(p_n(x)))$$

for $x \in \Sigma$, where $p_i\colon \Sigma \to \overline{\Sigma}_i$ denotes the natural projection. It is easy to see that

$$\Phi((\gamma, \lambda)x) = \pi(\gamma)\Phi(x)\pi_g(\lambda)^{-1}$$

for any $\gamma \in \Gamma$, $\lambda \in \Lambda$. and a.e. $x \in \Sigma$. Hence, we have shown the following

THEOREM 7.1. *For each $i \in \{1, \dots, n\}$ and $j \in \{1, 2\}$, let M_i^j be a surface with $\kappa(M_i^j) > 0$ and $M_i^j \neq M_{1,2}, M_{2,0}$. Assume that Γ_i (resp. Λ_i) is a torsion-free subgroup of finite index in $\Gamma(M_i^1)^\diamond$ (resp. $\Gamma(M_i^2)^\diamond$). Put $\Gamma = \Gamma_1 \times \cdots \times \Gamma_n$ and $\Lambda = \Lambda_1 \times \cdots \times \Lambda_n$. Suppose that we have an ergodic* ME *coupling (Σ, m) of Γ and Λ. Then we can find the following*:

(a) *a bijection t on the set $\{1, \dots, n\}$;*

(b) *an isotopy class g_i of a diffeomorphism $M_{t(i)}^2 \to M_i^1$ for each i;*

(c) *an almost $(\Gamma \times \Lambda)$-equivariant Borel map*

$$\Phi\colon \Sigma \to \left(\prod_{i=1}^n \mathrm{Aut}(C(M_i^1)), \pi, \pi_g\right).$$

COROLLARY 7.2. *Let M_i^j be the surfaces in Theorem* 7.1. *The conclusion of Theorem* 7.1 *holds even if Γ (resp. Λ) is a subgroup of finite index in $\Gamma(M_1^1)^\diamond \times \cdots \times \Gamma(M_n^1)^\diamond$ (resp. $\Gamma(M_1^2)^\diamond \times \cdots \times \Gamma(M_n^2)^\diamond$).*

Proof. It is easy to check that if M_i is a surface with $\kappa(M_i) > 0$ and if Γ_i is a finite index subgroup of $\Gamma(M_i)^\diamond$, then the natural homomorphism

$$\Gamma_1 \times \cdots \times \Gamma_n \to \mathrm{Aut}(C(M_1)) \times \cdots \times \mathrm{Aut}(C(M_n))$$

is almost an isomorphism and ICC. By Lemma 5.8 and Theorem 7.1, we obtain the corollary. □

The following corollary determines all isomorphisms between finite index subgroups of a direct product of mapping class groups.

COROLLARY 7.3. *For each $i \in \{1, \ldots, n\}$, let M_i be a surface with $\kappa(M_i) > 0$ and $M_i \neq M_{1,2}, M_{2,0}$ and let Γ be a finite index subgroup of*

$$G_0 = \Gamma(M_1)^\diamond \times \cdots \times \Gamma(M_n)^\diamond.$$

Suppose that we have an injective homomorphism $\tau\colon \Gamma \to G_0$ with the index $[G_0 : \tau(\Gamma)]$ finite. Then we can find a bijection t on the set $\{1, \ldots, n\}$ and an isotopy class g_i of a diffeomorphism $M_{t(i)} \to M_i$ for each i such that for any $\gamma = (\gamma_1, \ldots, \gamma_n) \in \Gamma$, we have

$$\tau(\gamma) = (g_1\gamma_{t(1)}g_1^{-1}, \ldots, g_n\gamma_{t(n)}g_n^{-1}).$$

Proof. We identify $\Gamma(M_i)^\diamond$ and $\mathrm{Aut}(C(M_i))$ via the natural isomorphism. Consider the ergodic ME coupling (G_0, π, τ) of G_0 and Γ. It follows from Corollary 7.2 that we can find the following:

(a) a bijection t on $\{1, \ldots, n\}$;

(b) an isotopy class g_i of a diffeomorphism $M_{t(i)} \to M_i$ for each i;

(c) an almost $(G_0 \times \Gamma)$-equivariant Borel map

$$\Phi\colon (G_0, \pi, \tau) \to (G_0, \pi, \pi_g).$$

Put $h = (h_1, \ldots, h_n) = \Phi(e)$ and define an automorphism π_{hg} of G_0 by

$$\pi_{hg}(s) = (h_1g_1s_{t(1)}g_1^{-1}h_1^{-1}, \ldots, h_ng_ns_{t(n)}g_n^{-1}h_n^{-1})$$

for $s = (s_1, \ldots, s_n) \in G_0$. Define a $(G_0 \times G_0)$-equivariant map

$$\Psi\colon (G_0, \pi, \pi_g) \to (G_0, \pi, \pi_{hg})$$

by $\Psi(s) = sh^{-1}$ for $s \in G_0$. It is easy to see that $\Psi \circ \Phi(e) = e$, and thus $\Psi \circ \Phi = \mathrm{id}$. Therefore, τ is the restriction of π_{hg}. □

COROLLARY 7.4. *For each $i \in \{1, \ldots, n\}$, let M_i be a surface with $\kappa(M_i) > 0$ and let Γ be a finite index subgroup of $G_0 = \Gamma(M_1)^\diamond \times \cdots \times \Gamma(M_n)^\diamond$. We put*

$$G = \mathrm{Aut}(\mathrm{Aut}(C(M_1)) \times \cdots \times \mathrm{Aut}(C(M_n)))$$

and denote by $\pi\colon G_0 \to G$ *the natural homomorphism. If* (Σ, m) *is a self* ME *coupling of* Γ, *then there exists an essentially unique*, *almost* $(\Gamma \times \Gamma)$*-equivariant Borel map* $\Phi\colon \Sigma \to (G, \pi, \pi)$.

Proof. By using Lemma 5.7 and Corollary 7.2, one can easily check that there exists an essentially unique, almost $(\Gamma \times \Gamma)$-equivariant Borel map from each ergodic component for the $(\Gamma \times \Gamma)$-action on (Σ, m) into (G, π, π). The corollary then follows from Corollary 3.6 in [10]. □

Theorem 1.3 follows from Theorem 6.1 and this corollary.

8. **Lattice embeddings of the mapping class group**

In this final section, we give another application of Corollary 7.4, following [11]. We prove Theorem 1.4 that describes all locally compact second countable (lcsc) groups containing a lattice isomorphic to a finite direct product of mapping class groups. We fix the notation as follows: Let n be a positive integer and let M_i be a surface with $\kappa(M_i) > 0$ for each $i \in \{1, \ldots, n\}$. Put

$$G_0 = \Gamma(M_1)^{\diamond} \times \cdots \times \Gamma(M_n)^{\diamond}, \quad G = \mathrm{Aut}(\mathrm{Aut}(C(M_1)) \times \cdots \times \mathrm{Aut}(C(M_n))).$$

Let $\pi\colon G_0 \to G$ be the natural homomorphism. Theorem 1.4 directly follows from the following

THEOREM 8.1. *Let* Γ *be a finite index subgroup of* G_0. *Suppose that we have an injective homomorphism* $\sigma\colon \Gamma \to H$ *into a* lcsc *group* H *such that* $\sigma(\Gamma)$ *is a lattice in* H. *Then there exist the following two maps*:

(i) *an almost* $(\Gamma \times \Gamma)$*-equivariant Borel map* $\Phi\colon (H, \sigma, \sigma) \to (G, \pi, \pi)$, *which satisfies* $\Phi(h_1h_2) = \Phi(h_1)\Phi(h_2)$ *for a.e.* $(h_1, h_2) \in H \times H$;

(ii) *a continuous homomorphism* $\Phi_0\colon H \to G$ *such that* $\Phi_0(h) = \Phi(h)$ *for a.e.* $h \in H$ *and* $\Phi_0(\sigma(\gamma)) = \pi(\gamma)$ *for any* $\gamma \in \Gamma$. *Moreover*, $\ker(\Phi_0)$ *is compact.*

Proof. First, we show the assertion (i). Applying Corollary 7.4 to the self ME coupling H of Γ with the Haar measure, one obtains an almost $(\Gamma \times \Gamma)$-equivariant Borel map

$$\Phi\colon (H, \sigma, \sigma) \to (G, \pi, \pi).$$

Define a Borel map $F\colon H \times H \to G$ by

$$F(h_1, h_2) = \Phi(h_1^{-1})^{-1}\Phi(h_1^{-1}h_2)\Phi(h_2)^{-1}$$

for $h_1, h_2 \in H$. Then for any $\gamma \in \Gamma$ and a.e. $(h_1, h_2) \in H \times H$, we have

$$F(h_1\sigma(\gamma), h_2) = F(h_1, h_2) = F(h_1, h_2\sigma(\gamma)^{-1}),$$

$$F(\sigma(\gamma)h_1, \sigma(\gamma)h_2) = \pi(\gamma)F(h_1, h_2)\pi(\gamma)^{-1}.$$

Thus, F induces a Borel map f from $X = (H/\sigma(\Gamma)) \times (H/\sigma(\Gamma))$ to G such that

$$f(\gamma x) = \pi(\gamma) f(x) \pi(\gamma)^{-1}$$

for any $\gamma \in \Gamma$ and a.e. $x \in X$, where the Γ-action on X is induced from the Γ-action on $H \times H$ given by $\gamma(h_1, h_2) = (\sigma(\gamma)h_1, \sigma(\gamma)h_2)$ for $\gamma \in \Gamma$ and $h_1, h_2 \in H$. By projecting the finite Γ-invariant measure on X to G through f, we obtain a finite measure μ on G invariant under the action of the diagonal subgroup of $\Gamma \times \Gamma$ on (G, π, π). It follows from Theorem 2.9 that the support of μ is $\{e\}$ and $F(h_1, h_2) = e$, that is, $\Phi(h_1^{-1}h_2) = \Phi(h_1^{-1})\Phi(h_2)$ for a.e. $(h_1, h_2) \in H \times H$.

Next, we show the assertion (ii). It follows from Theorems B.2 and B.3 in [44] that there exists a continuous homomorphism $\Phi_0 \colon H \to G$ such that $\Phi_0(h) = \Phi(h)$ for a.e. $h \in H$. For any $\gamma \in \Gamma$ and a.e. $h \in H$, we have

$$\pi(\gamma)\Phi(h) = \Phi(\sigma(\gamma)h) = \Phi_0(\sigma(\gamma)h) = \Phi_0(\sigma(\gamma))\Phi_0(h) = \Phi_0(\sigma(\gamma))\Phi(h),$$

which implies $\pi(\gamma) = \Phi_0(\sigma(\gamma))$ for any $\gamma \in \Gamma$. Since $\ker(\Phi_0)$ is essentially equal to $\Phi^{-1}(e)$, which has finite measure, we see that $\ker(\Phi_0)$ is compact. $\square$

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References

[1] C. Anantharaman-Delaroche and J. Renault, *Amenable Groupoids*, *Monogr. Enseign. Math.* **36**, Enseignement Math., Geneva, 2000. MR 2001m:22005 Zbl 0960.43003

[2] M. Bestvina and K. Fujiwara, Bounded cohomology of subgroups of mapping class groups, *Geom. Topol.* **6** (2002), 69–89. MR 2003f:57003 Zbl 1021.57001

[3] J. S. Birman, A. Lubotzky, and J. McCarthy, Abelian and solvable subgroups of the mapping class groups, *Duke Math. J.* **50** (1983), 1107–1120. MR 85k:20126 Zbl 0551.57004

[4] A. Connes, J. Feldman, and B. Weiss, An amenable equivalence relation is generated by a single transformation, *Ergodic Theory Dynam. Systems* **1** (1981), 431–450 (1982). MR 84h:46090 Zbl 0491.28018

[5] H. A. Dye, On groups of measure preserving transformation. I, *Amer. J. Math.* **81** (1959), 119–159. MR 24 #A1366 Zbl 0087.11501

[6] ———, On groups of measure preserving transformation. II, *Amer. J. Math.* **85** (1963), 551–576. MR 28 #1275 Zbl 0191.42803

[7] B. Farb and H. Masur, Superrigidity and mapping class groups, *Topology* **37** (1998), 1169–1176. MR 99f:57017 Zbl 0946.57018

[8] A. Fathi, F. Laudenbah, and V. Poénaru, *Travaux de Thurston sur les Surfaces. Séminaire Orsay*, *Astérisque* **66**, Société Mathématique de France, Paris, 1979. MR 82m:57003

[9] J. Feldman, C. E. Sutherland, and R. J. Zimmer, Subrelations of ergodic equivalence relations, *Ergodic Theory Dynam. Systems* **9** (1989), 239–269. MR 91c:28020 Zbl 0654.22003

[10] D. FISHER, D. W. MORRIS, and K. WHYTE, Nonergodic actions, cocycles and superrigidity, *New York J. Math.* **10** (2004), 249–269. MR 2005i:37003 Zbl 1074.37001

[11] A. FURMAN, Mostow-Margulis rigidity with locally compact targets, *Geom. Funct. Anal.* **11** (2001), 30–59. MR 2002d:22015 Zbl 0984.22008

[12] A. FURMAN, Gromov's measure equivalence and rigidity of higher rank lattices, *Ann. of Math.* **150** (1999), 1059–1081. MR 2001a:22017 Zbl 0943.22013

[13] ——, Orbit equivalence rigidity, *Ann. of Math.* **150** (1999), 1083–1108. MR 2001a:22018 Zbl 0943.22012

[14] ——, On Popa's cocycle superrigidity theorem, *Int. Math. Res. Not.* IMRN **2007**, no. 19, Art. ID rnm073, 46pp. MR 2009a:37007 Zbl 1134.46043

[15] D. GABORIAU, Examples of groups that are measure equivalent to the free group, *Ergodic Theory Dynam. Systems* **25** (2005), 1809–1827. MR 2006i:22024 Zbl 1130.37311

[16] D. GABORIAU, Invariants l^2 de relations d'équivalence et de groupes, *Publ. Math. Inst. Hautes Études Sci.* (2002), 93–150. MR 2004b:22009 Zbl 1022.37002

[17] E. GHYS and P. DE LA HARPE, *Sur les Groupes Hyperboliques d'après Mikhael Gromov*, *Progr. in Math.* **83**, Birkhäuser, Boston, MA, 1990. MR 92f:53050 Zbl 0731.20025

[18] M. GROMOV, Asymptotic invariants of infinite groups, in *Geometric Group Theory*, *Vol.* 2 (Sussex, 1991), *London Math. Soc. Lecture Note Ser.* **182**, Cambridge Univ. Press, Cambridge, 1993, pp. 1–295. MR 95m:20041 Zbl 0841.20039

[19] U. HAMENSTÄDT, Train tracks and the Gromov boundary of the complex of curves, in *Spaces of Kleinian Groups*, *London Math. Soc. Lecture Note Ser.* **329**, Cambridge Univ. Press, Cambridge, 2006, pp. 187–207. MR 2009a:30092 Zbl 1117.30036

[20] ——, Bounded cohomology and isometry groups of hyperbolic spaces, *J. Eur. Math. Soc.* (*JEMS*) **10** (2008), 315–349. MR 2009d:53050 Zbl 1139.22006

[21] W. J. HARVEY, Boundary structure of the modular group, in *Riemann Surfaces and Related Topics*: *Proceedings of the* 1978 *Stony Brook Conference*, *Ann. of Math. Stud.* **97**, Princeton Univ. Press, Princeton, N.J., 1981, pp. 245–251. MR 83d:32022 Zbl 0461.30036

[22] N. V. IVANOV, *Subgroups of Teichmüller Modular Groups*, *Transl*, *Math. Monogr.* **115**, Amer. Math. Soc., Providence, RI, 1992. MR 93k:57031 Zbl 0776.57001

[23] ——, Automorphism of complexes of curves and of Teichmüller spaces, *Internat. Math. Res. Notices* (1997), 651–666. MR 98j:57023

[24] ——, Mapping class groups, in *Handbook of Geometric Topology*, North-Holland, Amsterdam, 2002, pp. 523–633. MR 2003h:57022 Zbl 1002.57001

[25] V. A. KAIMANOVICH and H. MASUR, The Poisson boundary of the mapping class group, *Invent. Math.* **125** (1996), 221–264. MR 97m:32033 Zbl 0864.57014

[26] A. S. KECHRIS, *Classical Descriptive Set Theory*, *Grad. Texts Math.* **156**, Springer-Verlag, New York, 1995. MR 96e:03057 Zbl 0819.04002

[27] Y. KIDA, The mapping class group from the viewpoint of measure equivalence theory, *Mem. Amer. Math. Soc.* **196** (2008), viii+190. MR 2009h:37010 Zbl pre05498637

[28] ——, Orbit equivalence rigidity for ergodic actions of the mapping class group, *Geom. Dedicata* **131** (2008), 99–109. MR 2008k:37011 Zbl 05262505

[29] E. KLARREICH, The boundary at infinity of the curve complex and the relative Teichmüller space, preprint. Available at http://nasw.org/users/klarreich

[30] M. KORKMAZ, Automorphisms of complexes of curves on punctured spheres and on punctured tori, *Topology Appl.* **95** (1999), 85–111. MR 2000d:57025 Zbl 0926.57012

[31] M. KORKMAZ, On cofinite subgroups of mapping class groups, *Turkish J. Math.* **27** (2003), 115–123. MR 2004c:57003 Zbl 1031.57016

[32] F. LUO, Automorphisms of the complex of curves, *Topology* **39** (2000), 283–298. MR 2000j:57045 Zbl 0951.32012

[33] H. A. MASUR and Y. N. MINSKY, Geometry of the complex of curves. I. Hyperbolicity, *Invent. Math.* **138** (1999), 103–149. MR 2000i:57027 Zbl 0941.32012

[34] J. MCCARTHY and A. PAPADOPOULOS, Dynamics on Thurston's sphere of projective measured foliations, *Comment. Math. Helv.* **64** (1989), 133–166. MR 90e:57054 Zbl 0681.57002

[35] N. MONOD and Y. SHALOM, Orbit equivalence rigidity and bounded cohomology, *Ann. of Math.* **164** (2006), 825–878. MR 2007k:37007 Zbl 1129.37003

[36] D. S. ORNSTEIN and B. WEISS, Ergodic theory of amenable group actions. I. The Rohlin lemma, *Bull. Amer. Math. Soc.* **2** (1980), 161–164. MR 80j:28031 Zbl 0427.28018

[37] S. POPA, Strong rigidity of II_1 factors arising from malleable actions of w-rigid groups. II, *Invent. Math.* **165** (2006), 409–451. MR 2007h:46084 Zbl 1120.46044

[38] ———, Cocycle and orbit equivalence superrigidity for malleable actions of w-rigid groups, *Invent. Math.* **170** (2007), 243–295. MR 2008f:37010 Zbl 1131.46040

[39] K. J. SHACKLETON, Combinatorial rigidity in curve complexes and mapping class groups, *Pacific J. Math.* **230** (2007), 217–232. MR 2008g:57018 Zbl 1165.57017

[40] Y. SHALOM, Measurable group theory, in *European Congress of Mathematics*, Eur. Math. Soc., Zürich, 2005, pp. 391–423. MR 2006k:37007 Zbl 1137.37301

[41] M. TAKESAKI, *Theory of Operator Algebras.* III, *Encycl. Math. Sci.* **127**, Springer-Verlag, New York, 2003, *Operator Algebras and Non-commutative Geometry* **8**. MR 2004g:46080 Zbl 1059.46032

[42] S. VAES, *Rigidity Results for Bernoulli Actions and their von Neumann Algebras* (*after Sorin Popa*), *Astérisque* **311**, 2007, Séminaire Bourbaki. Vol. 2005/2006. MR 2359046

[43] S.-K. YEUNG, Representations of semisimple lattices in mapping class groups, *Int. Math. Res. Not.* (2003), 1677–1686. MR 2004d:53047 Zbl 1037.57011

[44] R. J. ZIMMER, *Ergodic Theory and Semisimple Groups*, *Monogr. Math.* **81**, Birkhäuser, Basel, 1984. MR 86j:22014 Zbl 0571.58015

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